Homework 5

Econ 357
Answer Key

Chapter 5: 3, 7 BONUS: 5 (on regret aversion).
Question 1 and 2 below
Chapter 6: 9, 21 (for b, find the equilibrium price and allocation).

5.3 (a) (Note: The left side of $v(g)$ should be $a_1, a_2, ...$). For this exercise, consider the case with two outcomes, $v(g) = (1+a_1)^{p_1}(1+a_2)^{p_2}$. For this to be a vNM utility function, we would need $v(g)$ to be able to be expressed as $p_1 u_1 + p_2 u_2$, where u_1 and u_2 are utility assignments to outcomes. Consider a degenerate gamble where $a_1 = a_2$. Because preferences can be represented by utility, if $a_1 = a_2$, we need $u_1 = u_2$. Denote this by u. For any degenerate gamble, we need

$$v(g) = (1+a_1)^{p_1+p_2} = (p_1+p_2)u = u.$$

This gives us a utility assignment of $u(a_i) = (1+a_i)$. Given this utility assignment, we can examine a different gamble, $p_1 = .5, p_2 = .5$ with $a_1 = 1, a_2 = 2$. If this is a vNM utility function, this gives us utility of $.5(1+1) + .5(2+1) = 2.5$. However, $v(g) = \left(\sqrt{2} * \sqrt{3}\right) = 2.45$, a contradiction, so this is not a vNM utility function.

(b) Any monotone (strictly increasing) transformation of a utility function preserves preferences, so take $u(g) = \ln(v(g)) = \ln\left(\Pi_i^n (1+a_i)^{p_1}\right) = \sum_{i=1}^n \ln(1+a_i)^{p_i} = \sum_{i=1}^n p_i \ln(1+a_i)$.

(c) $u'(w) = \frac{1}{1+w}$, and $u''(w) = -\frac{1}{(1+w)^2}$, so this function is strictly concave in w. The Arrow-Pratt coefficient of risk aversion is:

$$r_A(w) = \frac{1+w}{(1+w)^2} = \frac{1}{1+w}.$$

This decreases with w, so as w increases the consumer gets more risk neutral, and the consumer is very risk averse for low levels of w.

5.7 (a) $EV = .5 * 104000 + .5 * 4000 = 54000$

(b) $EU = .5\sqrt{104000} + .5\sqrt{4000} = 192.87$

(c) See attached.

(d) We need the certainty equivalent from part e to answer this question. This is $CE = 37198.04$. The Risk premium is the difference between the EV and the CE, $54000 - 37198.04 = 16801.96$.

(e) $\sqrt{CE} = 192.87 \longrightarrow CE = 37198.04$

(f) See attached.

5.5 (a) (Note, this question is supposed to say expected utility) In g^1, $h(g) = 2$, so we have that

$$v(g) = 1 - 2 = -1.$$

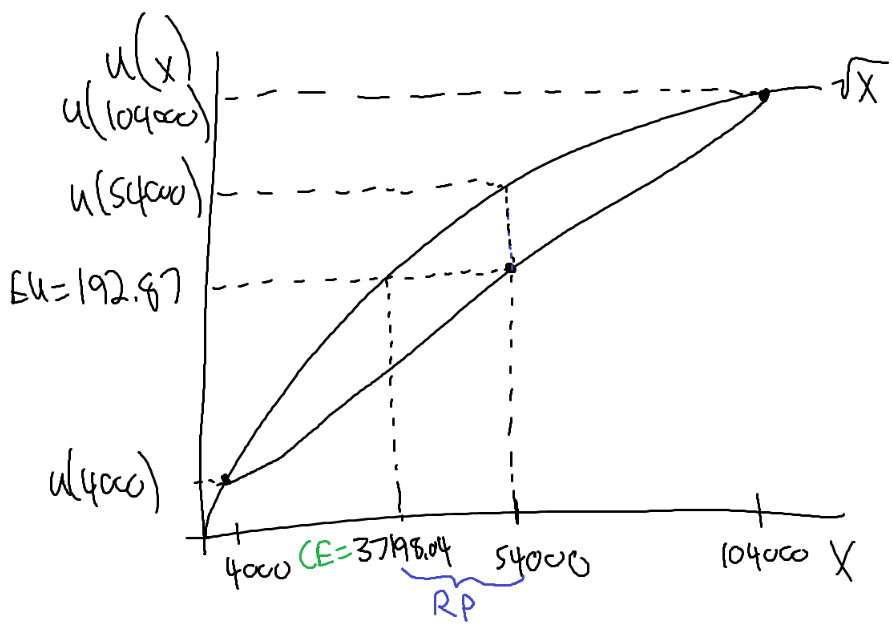
In g^2, $h(g) = 5$, so we have that

$$v(g) = \frac{16}{6} - 5 = -\frac{7}{3}.$$

(b) With any deterministic outcome, denote the outcome that occurs with probability 1 a. Because only outcome a occurs with any probability, $h(g) = a$. Thus we have that

$$v(g) = a - a = 0 \ \forall a$$

(c) If the preference relation satisfies monotonicity and outcomes are deterministic, if $a_1 > a_2$ then $v(a_1) > v(a_2)$. That is, receiving a_1 with certainty should give strictly higher utility than receiving a_2 with certainty because it is strictly higher wealth. However, as we demonstrated in the previous question, this only gives weakly higher utility, violating monotonicity here.



1. In an economy with two consumers (A and B), each consumer has a utility function and endowments:

$$u^A\left(x_1^A, x_2^A\right) = x_1^A x_2^A \text{ and } e^A = (8, 2)$$

$$u^B\left(x_1^B, x_2^B\right) = x_1^B x_2^B \text{ and } e^B = (2, 8)$$

(a) Draw an Edgeworth box for this economy.

(b) Find the core of this economy.

Pareto: $\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B}$. Using feasibility constraints, we have

$$\frac{x_2^A}{x_1^A} = \frac{10 - x_2^A}{10 - x_1^A}$$

$$x_1^A = x_2^A$$

This implies also that we have $x_1^B = x_2^B$ in a Pareto optimal bundle.
For the core, we need $u^A \geq 16$ and $u^B \geq 16$. This means we have

$$\left(x_1^A\right)^2 \geq 16 \longrightarrow x_1^A \geq 4$$

.

For B's constraint, we have

$$\left(10 - x_1^A\right)^2 \geq 16 \longrightarrow x_1^A \leq 6$$

So the core is any x_1^A such that $4 \leq x_1^A \leq 6$ and such that the bundle is Pareto ($\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B}$) and feasible.

(c) Show that $x^A = (4, 4)$, $x^B = (6, 6)$ is in the core.
$MRS^A = 1 = MRS^B$, and $x_1^A \geq 4$, so this allocation is in the core.

(d) Now replicate the economy once so that there are two consumers of each type, for a total of four consumers of the economy. Each A type has the same initial endowment, as does each B type (So there are now a total of 20 of each good in the economy). Show that in this new economy, giving each A type consumer $(4, 4)$ and each B type consumer $(6, 6)$ is **not** in the core of the economy. (Hint: Try to find a blocking coalition.)

Consider the coalition of two consumers of type A and two consumers of type B. The total endowment for this economy is $e = (18, 12)$. Suppose that we give the single consumer of type B $(6, 6)$, and we allocate each type A consumer $(6, 3)$. This gives each A type $u\left(x_1^A, x_2^A\right) = 18$ and each B type $u\left(x_1^B, x_2^B\right) = 36$. For this coalition, this allocation gives each consumer at least as much utility as giving each type A consumer $(4, 4)$ and each B type consumer $(6, 6)$, while giving each A type consumer more utility $18 > 16$, so this is a blocking coalition for this allocation.

2. In a two-good, two-consumer economy, utility functions are:

$$u^A\left(x_1^A, x_2^A\right) = x_1^A\left(x_2^A\right)^2$$

$$u^B\left(x_1^B, x_2^B\right) = \left(x_1^B\right)^2 x_2^B$$

There is a total endowment of $(10, 20)$ to be divided among consumers in this question.

(a) A social planner wants to allocate goods to maximize consumer A's utility while holding constant consumer B's utility at $u^B = 8000/27$. Find the assignment of goods to consumers that solves the planner's problem and show the solution is Pareto efficient. (Note: formulate the planner's problem and the associated Lagrangian using the information presented here. Everything should be formulated in terms of x^A by utilizing the feasibility condition to transform u^B.)

The social planner problem is the following:

$$\max_{x^A \geq 0} u^A \text{ s.t. } u^B \geq 8000/27$$

This gives us the problem:

$$\max_{x^A \geq 0} x_1^A \left(x_2^A\right)^2 \text{ s.t. } \left(10 - x_1^A\right)^2 \left(20 - x_2^A\right) \geq 8000/27$$

The Lagrangian is then

$$\mathcal{L} = x_1^A \left(x_2^A\right)^2 - \lambda \left[8000/27 - \left(10 - x_1^A\right)^2 \left(20 - x_2^A\right)\right]$$

The FOC are:

$x_1^A : \left(x_2^A\right)^2 - 2\lambda \left(10 - x_1^A\right)\left(20 - x_2^A\right) = 0$

$x_2^A : 2x_1^A x_2^A - \lambda \left(10 - x_1^A\right)^2 = 0$

$\lambda : \left(10 - x_1^A\right)^2 \left(20 - x_2^A\right) = 8000/27$

Solving for λ yields:

$$\frac{x_2^A}{2\left(20 - x_2^A\right)} = \frac{2x_1^A}{10 - x_1^A} \longrightarrow 10x_2^A - x_1^A x_2^A = 80x_1^A - 4x_1^A x_2^A$$

$$\longrightarrow x_2^A = \frac{80x_1^A}{10 + 3x_1^A}$$

Plugging into the λ constraint yields:

$$\left(10 - x_1^A\right)^2 \left(20 - \frac{80x_1^A}{10 + 3x_1^A}\right) = 8000/27$$

$$\left(10 - x_1^A\right)^2 \left(\frac{200 - 20x_1^A}{10 + 3x_1^A}\right) = 8000/27$$

$$\frac{\left(10 - x_1^A\right)^3}{10 + 3x_1^A} = 400/27$$

$$1000 - 300x_1^A + 30\left(x_1^A\right)^2 - \left(x_1^A\right)^3 = 4000/27 + \frac{400x_1^A}{9}$$

Simplifying this using Wolfram alpha, this yields:

$$x_1^A = 10/3$$

This gives us $x_2^A = 40/3$, which gives $x_1^B = 20/3$ and $x_2^B = 20/3$.

To show that this is Pareto efficient,

$$\text{MRS}^A = \frac{x_2^A}{2x_1^A} = 2$$

$$\text{MRS}^B = \frac{2x_2^B}{x_1^B} = 2$$

(b) Suppose, instead, that the planner just divides the endowments so that $e^A = (10, 0)$ and $e^B = (0, 20)$ and then lets the consumer transact through perfectly competitive markets. Find the Walrasian equilibrium and show that the WEA is the same solution as in part (i).

For this problem, because only relative prices matter, we will assume that $p_2 = 1$ and that $p_1 = p$ consumers in this economy solving their UMP, we have that

$$\text{MRS}^A = \text{MRS}^B = p$$

$$\frac{x_2^A}{2x_1^A} = \frac{2x_2^B}{x_1^B} = p$$

For consumer A, we have that $x_2^A = 2x_1^A p$. Consumer A has a budget constraint $px_1^A + x_2^A = 10p$. Using these two gives us:

$$x_1^A = \frac{10}{3} \text{ and therefore } x_2^A = \frac{20p}{3}$$

For consumer B we have that $x_2^B = \frac{px_1^B}{2}$. Consumer B has budget constraint $px_1^B + x_2^B = 20$. Using these two gives us:

$$x_1^B = \frac{40}{3p} \text{ and therefore } x_2^B = \frac{20}{3}$$

For good 1, the excess demand function is

$$z_1(p) = \frac{10}{3} + \frac{40}{3p} - 10$$

In a Walrasian equilibrium, we need prices such that $z_1(p) = 0$. This gives us

$$\frac{40}{3p} = \frac{20}{3} \longrightarrow p = 2$$

Solving $z_2(p) = 0$ gives the same price.

This gives us WEA of $\left(x_1^A\left(p, pe^A\right), x_2^A\left(p, pe^A\right)\right) = \left(\frac{10}{3}, \frac{40}{3}\right)$ and $\left(x_1^B\left(p, pe^B\right), x_2^B\left(p, pe^B\right)\right) = \left(\frac{20}{3}, \frac{20}{3}\right)$, the exact same as in part (i).

6.9 $e^A = (10, 0), e^B = (0, 10)$

a To have a Pareto allocation, we need $x^A = y^A$ and $x^B = y^B$, such that $x^A + x^B = 10$ and $y^A + y^B = 10$.

For a WEA, let p_y be the numeraire good ($p_y = 1$) and let $p_x = p$. Person A needs $x^A = y^A$ with a budget constraint $px^A + y^A = 10p$. This yields:

$$x^A = \frac{10p}{1+p} = y^A$$

For person B they need $x^B = y^B$ such that $px^B + y^B = 10$. This yields:

$$x^B = \frac{10}{1+p} = y^B$$

.

For the market for good x to clear ($z_x(p) = 0$), we need:

$$x^A + x^B = 10 \longrightarrow \frac{10p}{1+p} + \frac{10}{1+p} = 10$$

This clears at any prices $p > 0$. Thus every bundle such that $x^A = y^A$, $x^B = y^B$, $x^A + x^B = 10$ and $y^A + y^B = 10$ is a Walrasian equilibrium allocation.

b To have a Pareto allocation, we need $x^A = y^A$. In terms of x^B and y^B using feasibility, we need $10 - x^B = 10 - y^B$, which simplifies to $x^B = y^B$.

For a WEA, let p_y be the numeraire good ($p_y = 1$) and let $p_x = p$. Person A needs $x^A = y^A$ with a budget constraint $px^A + y^A = 10p$. This yields:

$$x^A = \frac{10p}{1+p} = y^A$$

For person B, person B solves their UMP at $MRS^B = p$, which gives us $y^B = px^B$. Along with their budget constraint $px^B + y^B = 10$, this solves to:

$$x^B = \frac{5}{p}$$

this also gives us $y^B = 5$. For the market for good y to clear ($z_y(p) = 0$), we need:

$$\frac{10p}{1+p}+5=10$$

$$10p+5+5p=10+10p$$

$$p=1$$

.

This yields a Walrasian equilibrium allocation of $\left(x^A,y^A\right)=(5,5)$, $\left(x^B,y^B\right)=(5,5)$.

6.21 $e^A=e^B=(.5,.5)$, $u^A\left(x_1^A,x_2^A\right)=\ln\left(x_1^A\right)+\ln\left(x_2^A\right)$, $u^B\left(x_1^B,x_2^B\right)=\left(x_1^B\right)^{\frac{1}{4}}\left(x_2^B\right)^{\frac{3}{4}}$, $y_2=\sqrt{y_1}$

a. Firm problem: $\pi=\sqrt{y_1}-p_1y_1$
FOC: $\frac{1}{2}\left(y_1\right)^{\frac{-1}{2}}-p_1=0$

$$y_1\left(p_1\right)=\frac{1}{4p_1^2}$$

This gives us $y_2=\frac{1}{2p_1}$, and equilibrium profit $\pi=\frac{1}{2p_1}-\frac{1}{4p_1}=\frac{1}{4p_1}$. (Note that y_1 is an input and y_2 is an output here.)
Consumer problems: For consumer A, we have

$$\max_{x\geq 0}\ln\left(x_1^A\right)+\ln\left(x_2^A\right)\text{ s.t. }p_1x_1^A+x_2^A\leq\frac{1}{2}p_1+\frac{1}{2}$$

This is maximized when $\mathrm{MRS}^A=p_1$. So we have that $\frac{x_2^A}{x_1^A}=p_1$, which gives us $x_2^A=p_1x_1^A$. Plugging this into the budget constraint yields:

$$2p_1x_1^A=\frac{1}{2}p_1+\frac{1}{2}$$

$$x_1^A\left(p_1,p_1\cdot e^A\right)=\frac{1}{4}+\frac{1}{4p_1}$$

This gives $x_2^A\left(p_1,p_1\cdot e^A\right)=\frac{p_1}{4}+\frac{1}{4}$.
For consumer B, we have

$$\max_{x\geq 0}\left(x_1^B\right)^{\frac{1}{4}}\left(x_2^B\right)^{\frac{3}{4}}\text{ s.t. }p_1x_1^B+x_2^B\leq\frac{1}{2}p_1+\frac{1}{2}+\frac{1}{4p_1}$$

This is maximized when $\text{MRS}^B = p_1$. So we have that $\frac{x_2^B}{3x_1^B} = p_1$, which gives us $x_2^B = 3x_1^B p_1$. Plugging this into the budget constraint yields:

$$4p_1 x_1^B = \frac{1}{2}p_1 + \frac{1}{2} + \frac{1}{4p_1}$$

$$x_1^B\left(p, p \cdot e^i\right) = \frac{1}{8} + \frac{1}{8p_1} + \frac{1}{16p_1^2}$$

This gives $x_2^B\left(p, p \cdot e^i\right) = \frac{3p_1}{8} + \frac{3}{8} + \frac{3}{16p_1}$

b. We need $z_1(p_1) = 0$:

$$\frac{1}{4} + \frac{1}{4p_1} + \frac{1}{8} + \frac{1}{8p_1} + \frac{1}{16p_1^2} = 1 - \frac{1}{4p_1^2}$$

$$4p_1^2 + 4p_1 + 2p_1^2 + 2p_1 + 1 = 16p_1^2 - 4$$

$$0 = 10p_1^2 - 6p_1 - 5$$

The solution to this quadratic (using Wolfram alpha) is $p_1 = 1.07$.

c. With constant returns to scale, Note that if $p_1 < 1$, the firm transforms all good 1 into good 2. If $p_1 > 1$, then the firm will not transform any of good 1 into good 2 (otherwise they get negative profits). Thus production can only occur if p_1 is exactly 1.

Consumer A's demand is calculated in exactly the same way. We then have $x_1^A\left(p_1, p_1 \cdot e^A\right) = \frac{1}{4} + \frac{1}{4p_1}$ and $x_2^A\left(p_1, p_1 \cdot e^A\right) = \frac{p_1}{4} + \frac{1}{4}$.

For Consumer B, their problem is the same except their budget has no firm profit, so this means that we have $x_1^B\left(p, p \cdot e^i\right) = \frac{1}{8} + \frac{1}{8p_1}$ and $x_2^B\left(p, p \cdot e^i\right) = \frac{3p_1}{8} + \frac{3}{8}$

Suppose that we have no production, this means that in order to have $z_1(p) = 0$:

$$\frac{1}{4} + \frac{1}{4p_1} + \frac{1}{8} + \frac{1}{8p_1} = 1$$

$$\frac{3}{8p_1} = \frac{5}{8}$$

$$p_1 = \frac{3}{5}$$

However this cannot be a Walrasian equilibrium. The firm would transform good 1 into good 2 at this price. If we had $p_1 > 1$ we would have a WE.

This means that we have production, but we know that production where $p_1 < 1$ cannot be a WE since the firm would use all of the input good available, which is not optimal for any consumer. Thus, we know that the Walrasian equilibrium is $p_1 = 1$. We need to find the amount of x_1 the firm demands in this equilibrium:

$$\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}=1-y_1^*$$

$$y_1^*=\frac{1}{4}$$

This means that we have WEA of: $y_2^*=\frac{1}{4}$, $x_1^A=\frac{1}{2}$, $x_2^A=\frac{1}{2}$, $x_1^B=\frac{1}{4}$, $x_2^B=\frac{3}{4}$

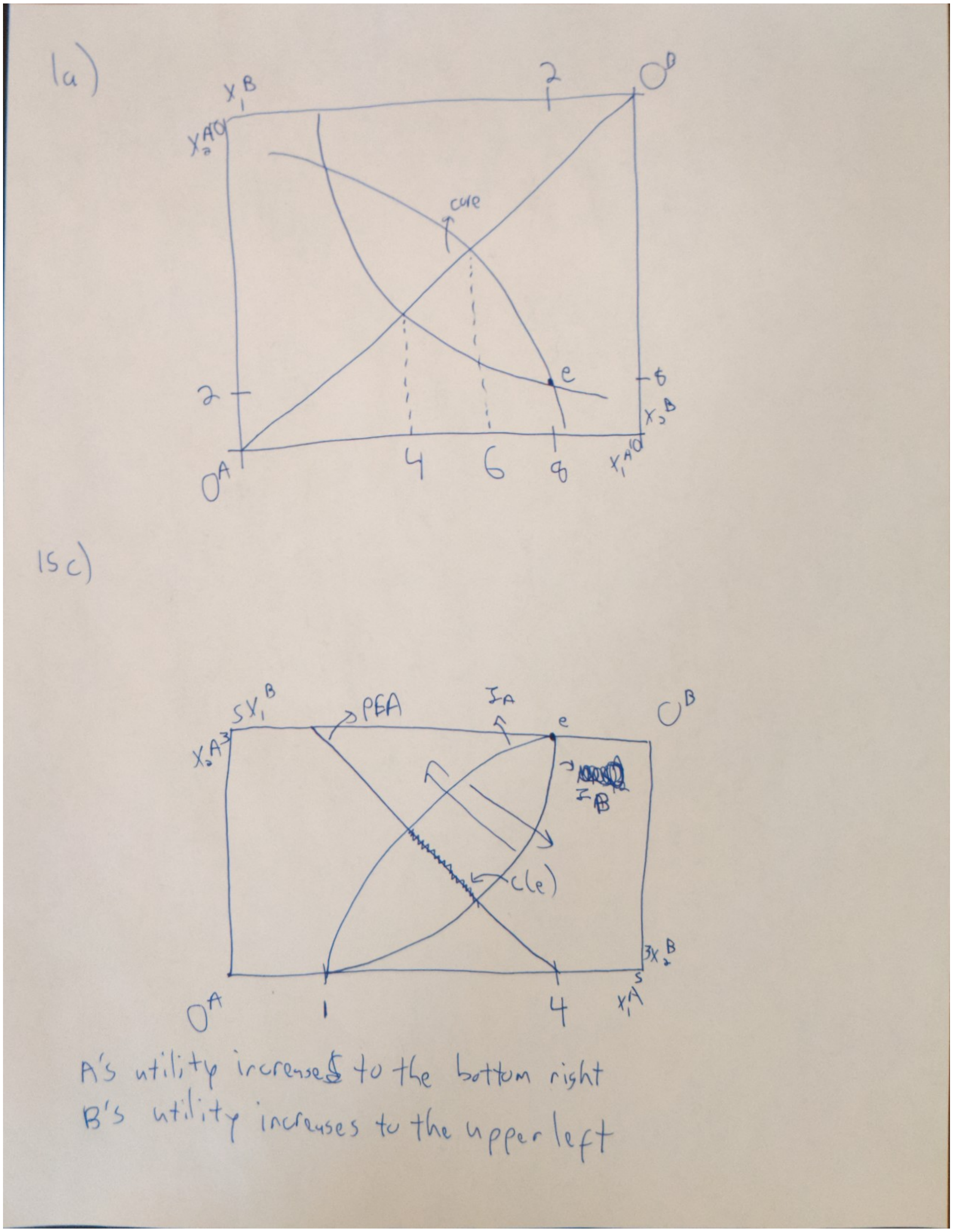
d. We essentially solved this problem in the previous question. Consumer A and consumer B maximize their utility in the same way as in the previous question since there was no profit, so we have $x_1^A\left(p_1, p_1\cdot e^A\right)=\frac{1}{4}+\frac{1}{4p_1}$, $x_2^A\left(p_1, p_1\cdot e^A\right)=\frac{p_1}{4}+\frac{1}{4}$, $x_1^B\left(p, p\cdot e^i\right)=\frac{1}{8}+\frac{1}{8p_1}$, and $x_2^B\left(p, p\cdot e^i\right)=\frac{3p_1}{8}+\frac{3}{8}$.
We need the market to clear in an equilibrium, which means that we need:

$$\frac{1}{4}+\frac{1}{4p_1}+\frac{1}{8}+\frac{1}{8p_1}=1$$

This yields the solution:

$$p_1=\frac{3}{5}$$

For this WE price, the WEA are $x_1^A=\frac{2}{3}$, $x_2^A=\frac{2}{5}$, $x_1^B=\frac{1}{3}$, and $x_2^B=\frac{3}{5}$.



(Ignore 15c picture: This is a problem on the recommended exercises)