

常见公式

leolinuxer

July 1, 2020

1 初等数学

1.1 排列组合

$$A_n^m = n(n-1)(n-2)\cdots(n-m+1) = \frac{n!}{(n-m)!}$$

$$C_n^m = \frac{A_n^m}{m!} = \frac{n!}{m!(n-m)!}$$

$$C_n^m = C_n^{(n-m)}$$

二项式定理：

$$(a+b)^n = \sum_{i=0}^n C_n^i a^{n-i} b^i$$

2 三角函数

和差角

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

和差化积

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin(\alpha) - \sin(\beta) = 2 \sin\left(\frac{\alpha - \beta}{2}\right) \cos\left(\frac{\alpha + \beta}{2}\right)$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\tan(\alpha) + \tan(\beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha) \cos(\beta)}$$

$$\tan(\alpha) - \tan(\beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha) \cos(\beta)}$$

积化和差

$$\sin(\alpha) \cos(\beta) = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}$$

$$\cos(\alpha) \sin(\beta) = \frac{\sin(\alpha + \beta) - \sin(\alpha - \beta)}{2}$$

$$\cos(\alpha) \cos(\beta) = \frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2}$$

$$\sin(\alpha) \sin(\beta) = -\frac{\cos(\alpha + \beta) - \cos(\alpha - \beta)}{2}$$

二倍角公式

$$\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha)$$

$$\cos(2\alpha) = 2 \cos^2(\alpha) - 1 = 1 - 2 \sin^2(\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

$$\tan(2\alpha) = \frac{2 \tan(\alpha)}{1 - \tan^2(\alpha)}$$

万能公式设

$$\tan\left(\frac{\alpha}{2}\right) = t, \quad \alpha \neq 2k\pi + \pi, k \in \mathbb{Z}$$

$$\sin(\alpha) = \frac{2 \sin(\alpha/2) \cos(\alpha/2)}{\sin^2(\alpha/2) + \cos^2(\alpha/2)} = \frac{2t}{1 + t^2}$$

$$\cos(\alpha) = \frac{\cos^2(\alpha/2) - \sin^2(\alpha/2)}{\sin^2(\alpha/2) + \cos^2(\alpha/2)} = \frac{1 - t^2}{1 + t^2}$$

$$\tan(\alpha) = \frac{2 \tan(\alpha/2)}{1 - \tan^2(\alpha/2)} = \frac{2t}{1 - t^2}$$

就是说 $\sin(\alpha), \cos(\alpha), \tan(\alpha)$ 都可以用 $\tan(\alpha/2)$ 来表示, 当要求一串函数式最值的时候, 就可以用万能公式, 推导成只含有一个变量的函数, 最值就很好求了.

$f(x) = \sin(2x) + 2\sin(x)$, 求最大值

$$\begin{aligned}|f(x)| &= |2\sin(x)\cos(x) + 2\sin(x)| \\&= 2|\sin(x)(\cos(x) + 1)| \\&= 2|2\sin(\frac{x}{2})\cos(\frac{x}{2}) \cdot 2\cos^2(\frac{x}{2})| \\&= \frac{8}{\sqrt{3}}\sqrt{(3\sin^2(\frac{x}{2}))(\cos^2(\frac{x}{2}))(\cos^2(\frac{x}{2}))(\cos^2(\frac{x}{2}))} \\&\leq \frac{8}{\sqrt{3}}\sqrt{[\frac{(3\sin^2(\frac{x}{2})) + (\cos^2(\frac{x}{2})) + (\cos^2(\frac{x}{2})) + (\cos^2(\frac{x}{2}))}{4}]^4} \\&= \frac{8}{\sqrt{3}} \cdot \frac{9}{16} \\&= \frac{3\sqrt{3}}{2}\end{aligned}$$

另, 采用万能公式计算:

$$\begin{aligned}f(x) &= \sin(2x) + 2\sin(x) \\f(t) &= 2 \cdot \frac{2t}{1+t^2} \cdot \frac{1-t^2}{1+t^2} + 2 \cdot \frac{2t}{1+t^2} = \cdots = \frac{8t}{(1+t^2)^2} \\f'(t) &= \frac{8t}{(1+t^2)^2} - \frac{32t^2}{(1+t^2)^3} = \frac{8-32t^2}{(1+t^2)^3}\end{aligned}$$

令 $f'(t) = 0$, 有 $t = \frac{\sqrt{3}}{3}$

所以 $f(t)_{max} = f(\frac{\sqrt{3}}{3}) = \frac{3\sqrt{3}}{2}$

3 复数

复数的代数表示式:

$$z = a + bi, \quad a, b \in R$$

复数的三角表示式:

$$z = r(\cos \theta + i \sin \theta)$$

复数的指数表示式 (欧拉公式):

$$z = e^{i\theta} = \cos \theta + i \sin \theta$$

令 $\theta = \pi$, 可以得到欧拉魔幻等式:

$$e^{i\pi} + 1 = 0$$

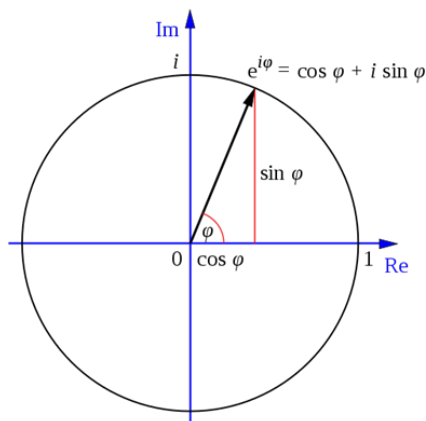
利用指数表示式，有：

$$e^{i\theta_1} \cdot e^{i\theta_2} = (\cos \theta_1 + i \sin \theta_1) \cdot (\cos \theta_2 + i \sin \theta_2) = e^{i(\theta_1 + \theta_2)}$$

同时，有：

$$(e^{i\theta})^n = e^{in\theta}$$

圆上的欧拉公式如下图：



4 极限和微积分相关

4.1 一些级数的和

$$\sum_{k=1}^n k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

4.2 圆周率 π 的相关公式

圆周率 π 的公式：

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots$$

$$\frac{\pi}{4} = \int_0^1 \frac{1}{x^2 + 1} dx$$

4.3 关于 e

$$e = \left(1 + \frac{1}{n}\right)^n \quad n \rightarrow \infty$$

4.4 积分运算的规则

$$\int_a^b [cf(x) + dg(x)]dx = c \int_a^b f(x)dx + d \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = - \int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^b f(u)du = \int_a^b f(t)dt$$

利用上式参数变换的思想，有：

$$\int_1^b f(x)dx = \int_0^{b-1} du \quad x = x' + 1$$

例：若 $n > 0$ ，证明 $(1+x)^n$ 从 -1 到 z 的积分等于 $\frac{(1+z)^{n+1}}{n+1}$

证明：令 $x' = 1 + x$ ，那么有：

$$\int_{-1}^z (1+x)^n dx = \int_0^{z+1} x'^n dx' = \left[\frac{x'^{n+1}}{n+1} \right]_0^{z+1} = \frac{(1+z)^{n+1}}{n+1}$$

$$\int_a^b f(x)dx \leq \int_a^b |f(x)|dx$$

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

4.5 常见导数公式

原函数	导函数
链式法则 $y = f[g(x)]$	$y' = f'[g(x)] \times g'(x)$
反函数求导法则: 若 $y = f(x)$ 的反函数是 $x = g(y)$	$y' = \frac{1}{x'}$
$y = uv$	$y' = u'v + uv'$
$y = \frac{u}{v}$	$y' = \frac{u'v - uv'}{v^2}$
$y = c$	$y' = 0$
$y = n^x$	$y' = n^x \ln n$
$y = \log_a x$	$y' = \frac{1}{x \ln a}$
$y = \ln x$	$y' = \frac{1}{x}$
$y = x^n$	$y' = nx^{n-1}$
$y = \sin x$	$y' = \cos x$
$y = \cos x$	$y' = -\sin x$
$y = \tan x$	$y' = \frac{1}{\cos^2 x} = \sec^2 x$
$y = \cot x$	$y' = -\frac{1}{\sin^2 x} = -\csc^2 x$
$y = \arcsin x$	$y' = \frac{1}{\sqrt{1-x^2}}$ (利用反函数求导法则)
$y = \arccos x$	$y' = -\frac{1}{\sqrt{1-x^2}}$
$y = \arctan x$	$y' = \frac{1}{1+x^2}$
$y = \operatorname{arccot} x$	$y' = -\frac{1}{1+x^2}$
$y = \operatorname{arcsec} x$	$y' = \frac{1}{x\sqrt{x^2-1}}$
$y = \operatorname{arccsc} x$	$y' = -\frac{1}{x\sqrt{x^2-1}}$
$y = shx = \frac{e^x - e^{-x}}{2}$	$y' = chx$ 双曲函数
$y = chx = \frac{e^x + e^{-x}}{2}$	$y' = shx$
$y = thx = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	$y' = \frac{1}{ch^2 x}$
$y = \operatorname{arsh} x = \ln(x + \sqrt{x^2 + 1})$	$y' = \frac{1}{\sqrt{x^2 + 1}}$
$y = \operatorname{arch} x = \ln(x + \sqrt{x^2 - 1})$	$y' = \frac{1}{\sqrt{x^2 - 1}}$
$y = \operatorname{arth} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$	$y' = \frac{1}{1-x^2}$

4.6 泰勒公式

$$f(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{k-1} \frac{x^k}{k} \quad (|x| < 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$$

4.7 利用积分求无穷级数的和

对于函数 $f(x)$ ，它在 $x \in [a, b]$ 内的面积，用积分可以表示为：

$$S = \int_a^b f(x) dx$$

同时，我们将 $x = a$ 到 $x = b$ 的区间分割成 n 个小区间，在每个分点上做垂线，它的高是该垂线的长度，这就构成了 n 个矩形；当 $n \rightarrow \infty$ 时，这些矩形的面积之和就等于 S 。这些矩形的面积可以组成一个序列：

$$S_1, S_2, S_3, \dots$$

使得当 n 无限增加、 S_n 中最宽的矩形的宽度趋于零时，该序列趋于极限 A ，

$$S_n \rightarrow A$$

这里， S_n 可以简写成：

$$S_n = \sum_{j=1}^n f(x_j) \Delta x$$

也就是说，我们可以看出极限和积分的关系为：

$$A = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(x_j) \Delta x = \int_a^b f(x) dx$$

根据这个思想，我们可以求解一些无穷级数的和。

例题：证明当 $n \rightarrow \infty$ 时，

$$S = \frac{1^k + 2^k + \dots + n^k}{n^{k+1}} \rightarrow \frac{1}{k+1}$$

证明：考虑如下积分：

$$S = \int_0^1 x^k dx$$

将其在 $[0, 1]$ 的区间内，平均分成 n 份，则每份的宽度为 $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ ，且每份的高度分别为：

- $x_1 = \Delta x, y_1 = (\Delta x)^k = \left(\frac{1}{n}\right)^k$
- $x_2 = 2\Delta x, y_2 = (2\Delta x)^k = \left(\frac{2}{n}\right)^k$
- \dots
- $x_n = n\Delta x, y_n = (n\Delta x)^k = \left(\frac{n}{n}\right)^k$

所以，面积的和为：

$$S_n = \Delta x((\Delta x)^k + (2\Delta x)^k + \cdots + (n\Delta x)^k) \quad (1)$$

$$= \frac{1}{n}((\frac{1}{n})^k + (\frac{2}{n})^k + \cdots + (\frac{n}{n})^k) \quad (2)$$

$$= \int_0^1 x^k dx = [\frac{x^{k+1}}{k+1}]_0^1 = \frac{1}{k+1} \quad (3)$$

即：

$$S = \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} \rightarrow \frac{1}{k+1} = \int_0^1 x^k dx = \frac{1}{k+1}$$

例题：证明当 $n \rightarrow \infty$ 时，

$$\frac{1}{\sqrt{n}}(\frac{1}{\sqrt{1+n}} + \frac{1}{\sqrt{2+n}} + \cdots + \frac{1}{\sqrt{n+n}}) \rightarrow 2(\sqrt{2}-1)$$

证明：考虑如下积分：

$$S = \int_0^2 \frac{1}{\sqrt{x}} dx \quad (\text{注意，积分区间是}[0, 2])$$

将其在 $[0, 2]$ 的区间内，平均分成 $2n$ 份，则每份的宽度为 $\Delta x = \frac{2-0}{2n} = \frac{1}{n}$ ，且每份的高度分别为：

- $x_1 = \Delta x, y_1 = \frac{1}{\sqrt{\Delta x}}$
- $x_2 = 2\Delta x, y_1 = \frac{1}{\sqrt{2\Delta x}}$
- $x_3 = 3\Delta x, y_1 = \frac{1}{\sqrt{3\Delta x}}$
- \dots
- $x_n = n\Delta x, y_1 = \frac{1}{\sqrt{n\Delta x}}$
- $x_{n+1} = (n+1)\Delta x, y_1 = \frac{1}{\sqrt{(n+1)\Delta x}}$
- \dots
- $x_{2n} = (2n)\Delta x, y_1 = \frac{1}{\sqrt{(2n)\Delta x}}$

所以，面积的和为：

$$S_n = \Delta x \left(\frac{1}{\sqrt{\Delta x}} + \frac{1}{\sqrt{2\Delta x}} + \cdots + \frac{1}{\sqrt{n\Delta x}} + \frac{1}{\sqrt{(n+1)\Delta x}} + \cdots + \frac{1}{\sqrt{(2n)\Delta x}} \right) \quad (4)$$

$$= \sqrt{\Delta x} \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}} \right) \quad (5)$$

$$= \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}} \right) \quad (6)$$

$$= \int_0^2 \frac{1}{\sqrt{x}} dx = \int_0^2 x^{-\frac{1}{2}} dx = [2x^{\frac{1}{2}}]_0^2 = 2\sqrt{2} \quad (7)$$

也就是说，级数：

$$\frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}} \right) = \int_0^2 \frac{1}{\sqrt{x}} dx = 2\sqrt{2}$$

所以，题目中的级数：

$$\frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{1+n}} + \frac{1}{\sqrt{2+n}} + \cdots + \frac{1}{\sqrt{n+n}} \right) = \int_0^2 \frac{1}{\sqrt{x}} dx - \int_1^2 \frac{1}{\sqrt{x}} dx = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$$

References