Advanced Digital Image Processing

Chapter 4: : Image Enhancement in the Frequency Domain

Tien-Ying Kuo
E-mail: tykuo@ee.ntut.edu.tw

Department of Electrical Engineering
National Taipei University of Technology

Outline

Fourier Transform

1-D Fourier Transform

Definition

________:

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi\frac{ux}{M}} \qquad u = 0,1,2,\ldots,M-1,$$

$$f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi\frac{ux}{M}} \qquad x = 0,1,2,\ldots,M-1.$$

Fourier Transform

2-D Fourier Transform

Definition

________:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$$u = 0,1,\ldots,M-1, v = 0,1,\ldots,N-1$$

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$$x = 0,1,\ldots,M-1, y = 0,1,\ldots,N-1$$

Fourier Transform
DC Coefficient

Definition

The ____________ $F(0,0)$ is,

$$F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

Reason: The right side is the average of $f(x,y)$.
Therefore, F(0,0) is ____________ of the spectrum.

Property of Fourier Transform
Separability

Property

__________:

$$\begin{aligned} F(u,v) &= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} \\ &= \frac{1}{M}\sum_{x=0}^{M-1} e^{-j2\pi ux/M}\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi vy/N)} \\ &= \frac{1}{M}\sum_{x=0}^{M-1} F(x,v)e^{-j2\pi ux/M} \end{aligned}$$

where
$$F(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi vy/N}$$

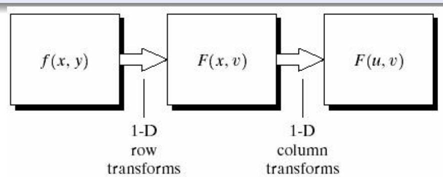


FIGURE 4.35 Computation of the 2-D Fourier transform as a series of 1-D transforms.

Property of Fourier Transform
Conjugate Symmetry & Rotation Invariant

Theorem

If f(x,y) is real ,

$$F(u,v) = F^*(-u,-v)$$

F(u,v) is ________________

Theorem

____________:

- *Rotating $f(x,y)$ by an angle θ_0 rotates $F(u,v)$ by the same angles.*
- *Rotating $F(x,y)$ by an angle θ_0 rotates $f(u,v)$ by the same angles.*

$$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\phi+\theta_0)$$

Property of Fourier Transform
Scaling

Theorem

______: Let $\triangle x, \triangle y$ be sampling space in spatial domain, $\triangle u, \triangle v$ are sampling space in feq, i.e.,

$$\begin{aligned} f(x,y) &\triangleq f(x_0 + x\triangle x, y_0 + y\triangle y) \\ F(u,v) &\triangleq F(u\triangle u, v\triangle v) \end{aligned}$$

Then, the samples are inversely related.

$$\begin{aligned} \triangle u &= \frac{1}{M\triangle x} \\ \triangle v &= \frac{1}{M\triangle y} \end{aligned}$$

Property of Fourier Transform
Periodicity

Theorem

__________________:

$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$$

______________:

$$f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$$

Property of Fourier Transform
Shifting in Frequency

Theorem

____

$$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

Example

When $u_0 = M/2$, $v_0 = N/2$,

$$e^{j2\pi(u_0x/M+v_0y/N)} = e^{j\pi(x+y)} = (-1)^{x+y}.$$

Therefore,

$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$$

It is common practice to multiply the image by $(-1)^{x+y}$ to shift the origin of $F(u,v)$ to frequency coordindates $(M/2, N/2)$. (M,N should be even number to make coordinate integer)

Property of Fourier Transform
Convolution Theorem

Definition

________:

$$f(x,y) * h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m, y-n)$$

Theorem

______________:

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

$$f(x,y)h(x,y) = F(u,v) * H(u,v)$$

Property of Fourier Transform
Correlation Theorem

Definition

________: $f(x,y) \circ h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^*(m,n)h(x+m, y+n)$

Theorem

__________________:

$$f(x,y) \circ h(x,y) \Leftrightarrow F^*(u,v)H(u,v)$$

$$f^*(x,y)h(x,y) = F(u,v) \circ H(u,v)$$

Theorem

________________:

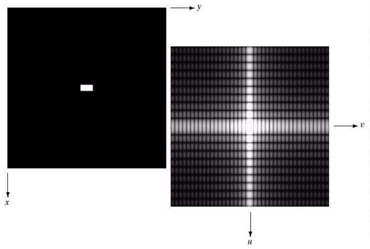
$$f(x,y) \circ f(x,y) \Leftrightarrow |F(u,v)|^2$$

$$|f(x,y)|^2 = F(u,v) \circ F(u,v)$$

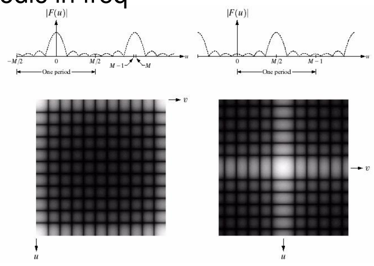
Property Demonstration
Symmetry+Periodity+Frequency Shifting

- ↑: Real rectangle ⇒ Symmetric sinc
- ↑: Frequency shifting to $(M/2, N/2)$.
- ↓: Periodic in freq

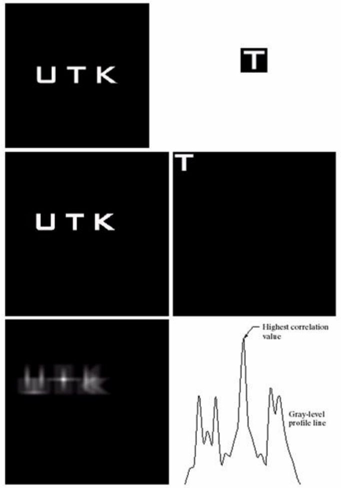
a b
FIGURE 4.3 (a) Image of a 20 × 40 white rectangle on a black background of size 512 × 512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.



a b
c d
FIGURE 4.34 (a) Fourier spectrum showing back-to-back half periods in the interval $[0, M-1]$. (b) Shifted spectrum showing a full period in the same interval. (c) Fourier spectrum of an image, showing the same back-to-back properties as (a), but in two dimensions. (d) Centered Fourier spectrum.



Property Demonstration
Convolution



a b
c d
e f
FIGURE 4.41 (a) Image. (b) Template. (c) and (d) Padded images. (e) Correlation function displayed as an image. (f) Horizontal profile line through the highest value in (e), showing the point at which the best match took place.

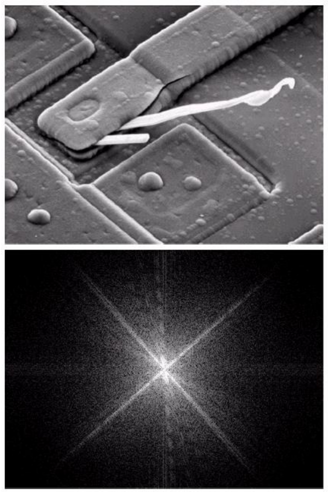
- Padding avoid aliasing during circular convoulation
- Convolution peak location show the best match

Association between Spatial and Frequency domains

Frequency is related to rate of change
⇒ Fourier Transform is related to intensity variations in an image

- ________ (at origin): Slowest varying frequncy component $((u, v) = (0, 0))$ is related to the average gray level of an image.
- ________ (near origin): Characterized by smooth area in gray level
- ________ (away from origin): Characterized by abrupt changes in gray level (edge, noise).

Example



a
b
FIGURE 4.4 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

- IC compoent $45°$ ⇒ DFT $45°$
- Long white protrusion ⇒ Vertical off-axis line
- Two white protrusions ⇒ Strong vertical off-axis line near origin

Filtering in Frequency Domain

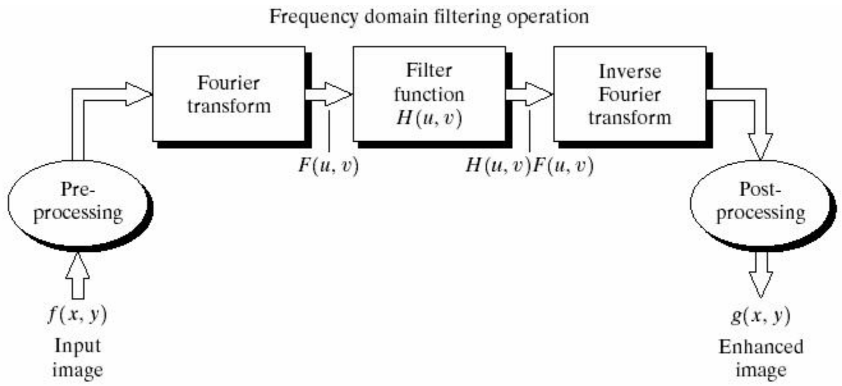


FIGURE 4.5 Basic steps for filtering in the frequency domain.

Notch Filter

Definition

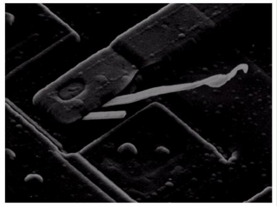
__________: Rejects (Passes) freqs in predefined neighborhoods about a center freq.

Example

An example of notch filter is

$$H(u,v) = \begin{cases} 0 & \text{if } (u,v) = (M/2, N/2) \\ 1 & \text{otherwise} \end{cases}$$

FIGURE 4.6 Result of filtering the image in Fig. 4.4(a) with a notch filter that set to 0 the $F(0,0)$ term in the Fourier transform.



- Set $F(0,0) = 0$ and make prominent edges stand out.
- In reality, the displayed image cannot be with $F(0,0) = 0$ Reason: Avg value=0 ⇒ Negative pixel intensity somewhere)

HPF & LPF

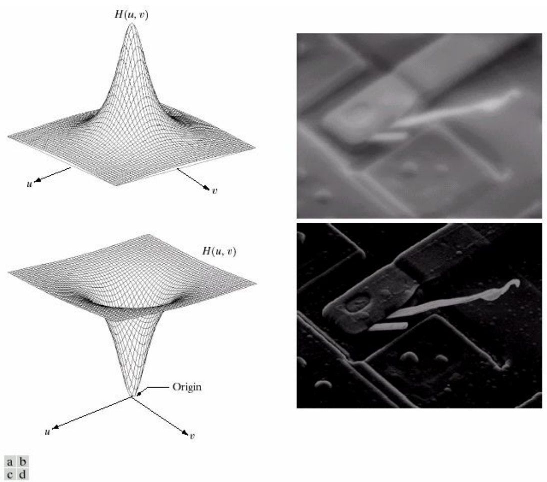
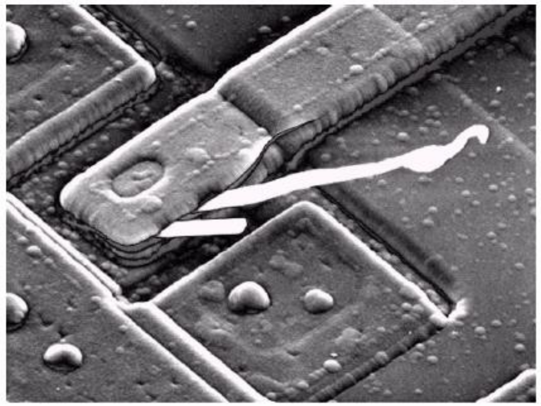


FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Modified HPF

Add a constant to the filter so that it will not completely eliminate $F(0,0)$.

FIGURE 4.8 Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).



Frequency Filtering versus Spatial Masking

Theorem

Masking in Time = Filtering in Fequency

That is, $h(x,y) \Leftrightarrow H(u,v)$

Proof.

By Convolution Theorem:

$$\begin{aligned} f(x,y) * h(x,y) &\Leftrightarrow F(u,v)H(u,v) \\ \delta(x,y) * h(x,y) &\Leftrightarrow \mathcal{F}[\delta(x,y)]H(u,v) \\ h(x,y) &\Leftrightarrow H(u,v) \end{aligned}$$

□

Frequency Filtering versus Spatial Masking
Gaussian Low-pass Filter

Example

1-D Gaussian LPF:

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2x^2} \Leftrightarrow H(u) = Ae^{-u^2/2\sigma^2}$$

- Fourier Transform of real Gaussian is still real Gaussian
- Spatial and Frequency behave reciprocally:
  $H(u)$ has broad profile (large σ), $h(x)$ has a narrow profile, vice versa.

Frequency Filtering versus Spatial Masking
Gaussian High-pass Filter

Example

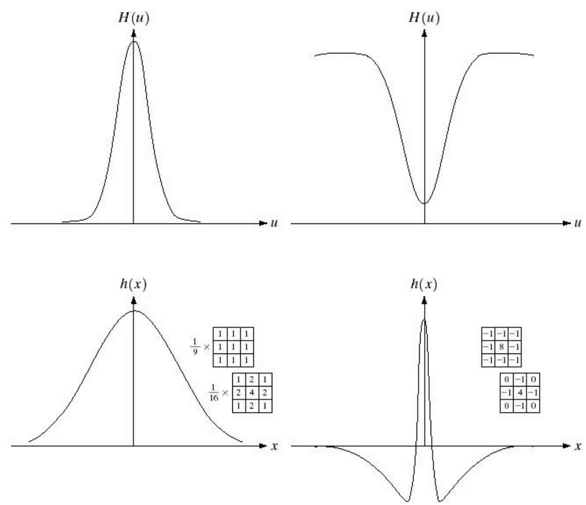
1-D Gaussian HPF:

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma B e^{-2\pi^2\sigma_2^2x^2}$$

$$\Leftrightarrow H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$

- Gaussian HPF is constructed as a difference of Gaussian LPF.

Frequency Filtering versus Spatial Masking
Gaussian Filters



a b
c d

FIGURE 4.9 (a) Gaussian frequency domain lowpass filter. (b) Gaussian frequency domain highpass filter. (c) Corresponding lowpass spatial filter. (d) Corresponding highpass spatial filter. The masks shown are used in Chapter 3 for lowpass and highpass filtering.

Frequency Filtering versus Spatial Masking

Pros and Cons

Frequency approach:

- More intuitive
- Computational complexity is better when mask size is larger.
  Ex: 1-D, Freq with FFT is faster than Spatial with mask if the number of points is greater than 32

Two-Dimensional Low-pass Filters

Three Types of LPFs:

- ______________________
- ____________________________
- ___________________________

Ideal Low-Pass Filter

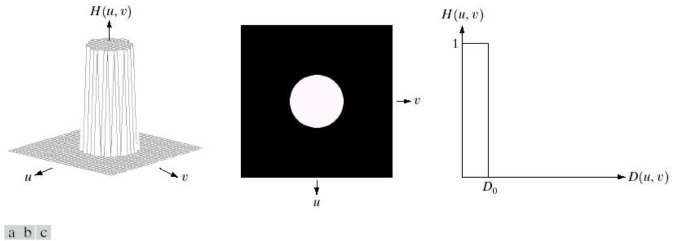
Definition

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

where

$$D(u,v) = \sqrt{(u-M/2)^2 + (v-N/2)^2}$$

and D_o is _________



Ideal Low-Pass Filter

Performance Metrics

The performace of different LPFs at the same cutoff frequency are compared by computing ______________________________,

$$\alpha = 100[\sum_u \sum_v P(u,v)/P_T]$$

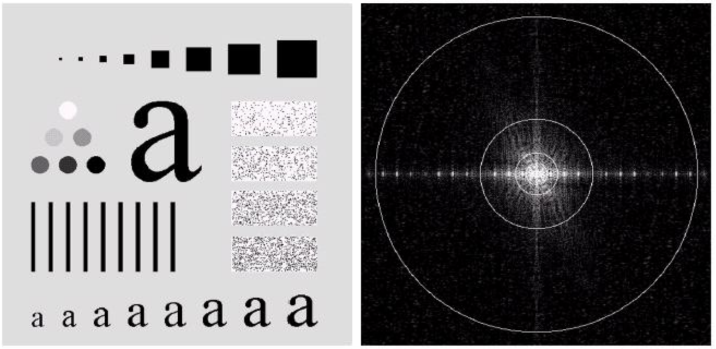
where

P_T is the _____________ before filtering

$$P_T = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} |F(u,v)|^2 = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} P(u,v)$$

Ideal Low-Pass Filter

Performance Metrics

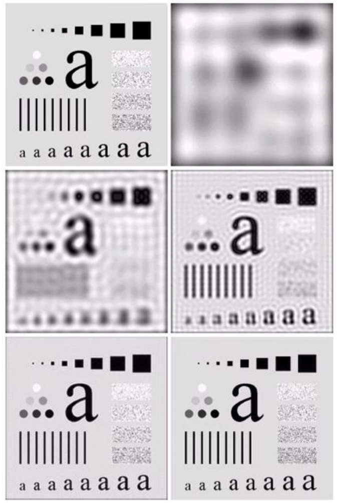


a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

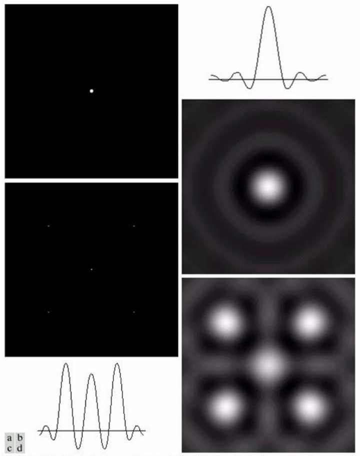
Ideal Low-Pass Filter

Performance Metrics



Ideal Low-Pass Filter

Rining and Blurred



a b
c d

FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

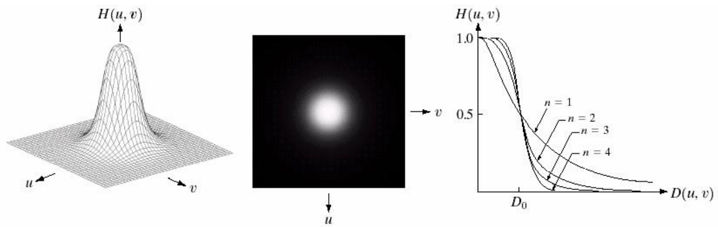
- h & H is reciprocal $\Rightarrow$ The narrower filter in freq, the more blurred and ringing image.

Butterworth LPF

Definition

Butterworth Low-pass filter (BLPF):

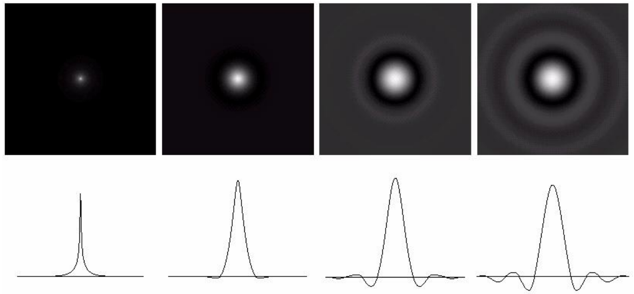
$$H(u,v) = \frac{1}{1+[D(u,v)/D_0]^{2n}},$$ where n is filter order



a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

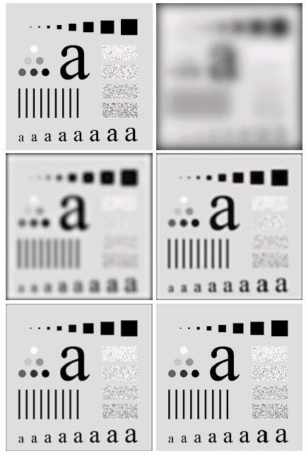
Butterworth LPF
Filter order



a b c d
FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

- filter order $= 1$: No rining
- filter order $\to \infty$: Larger ringing and blurring/Close to ideal LPF
- filter order $\to 1$: Close to Gaussian LPF
- BLPF behaves in between two extremes ILPF and GLPF

Butterworth LPF
Cut-off frequency



a b
c d
e f
FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

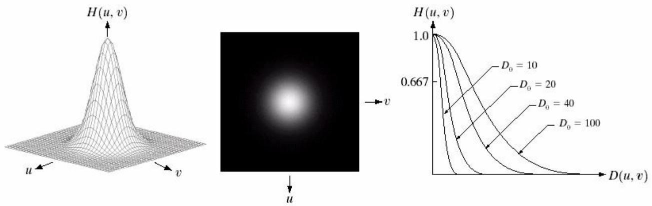
Gaussian LPF

Definition

Gaussian Low-pass filter (GLPF):

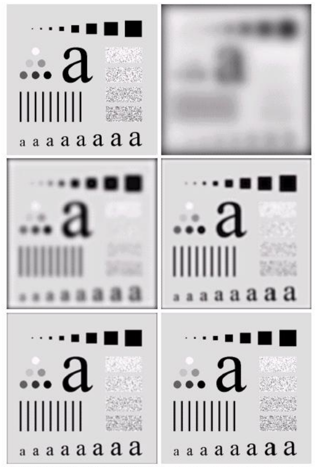
$$H(u,v) = e^{-D^2(u,v)/2\sigma^2} = e^{-D^2(u,v)/2D_o^2}$$

Note: Gaussian Filter (in freq domain) also means a spatial Gaussian filter.



a b c
FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Gaussian LPF

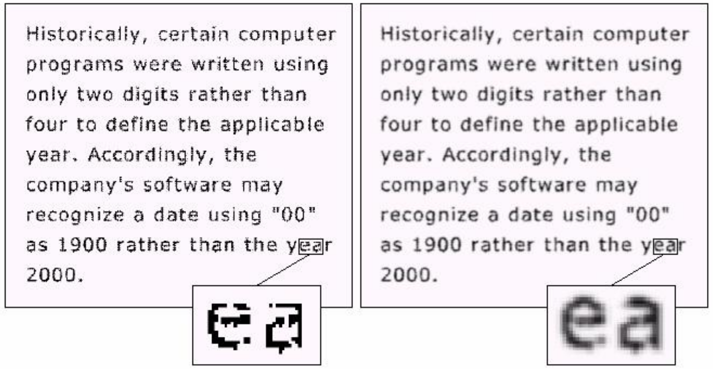


a b
c d
e f
FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

Compare to BLPF of order 2 at the same value of D_o, GLPF does not achieve as much smoothing as BLPF.
Reason: Because GLPF is not as tight as BLFP of order 2)

Gaussian LPF
Example

a b
FIGURE 4.19 (a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).



Gaussian LPF
Example

a b c
FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

Gaussian LPF
Example

a b c
FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)

High-Pass Filter

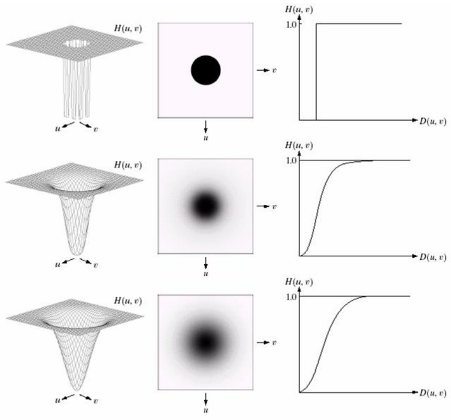
Definition

______________: Achieve image sharpening

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

- Ideal HPF
- Butterworth HPF
- Gaussian HPF

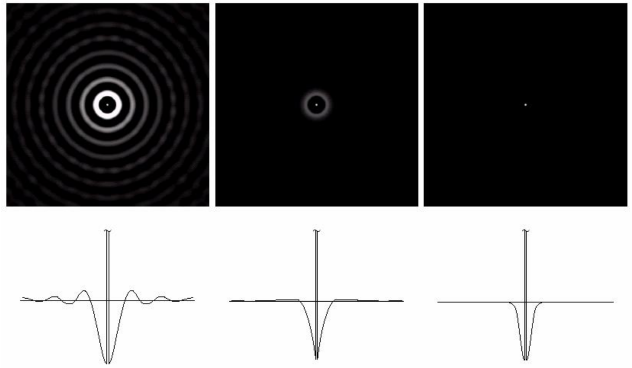
High-Pass Filter



a b c
d e f
g h i

FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

High-Pass Filter
Ringing



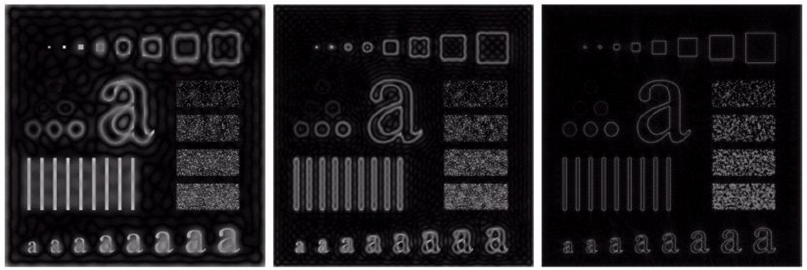
a b c

FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

Ideal HPF

Definition

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$



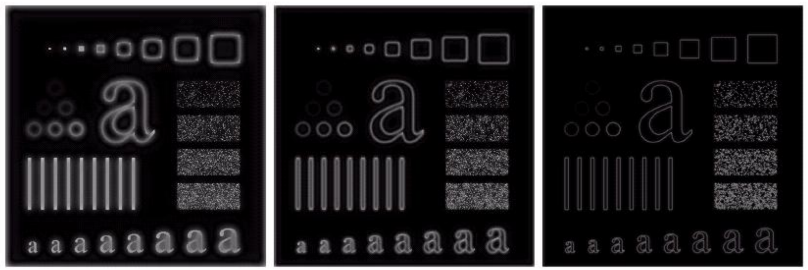
a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

Butterworth HPF

Definition

$$H(u,v) = \frac{1}{1 + [D_0/D(u,v)]^{2n}}, \text{ where } n \text{ is filter order}$$



a b c

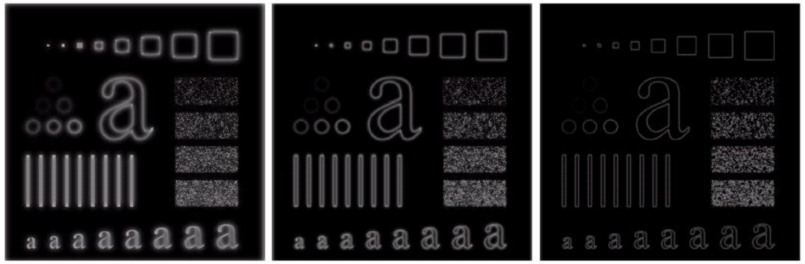
FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

Gaussian HPF

Definition

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_o^2}$$

Note: Another way: Diff of two GLPF (More control)



a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

Laplacian Filter in Frequency Domain

Definition

________________________________ is

$$H(u,v) = -[(u - M/2)^2 + (v - N/2)^2]$$

Proof.

$$\because \quad \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} \xrightarrow{\mathcal{F}} (ju)^2 F(u,v) + (jv)^2 F(u,v)$$

$$\therefore \quad \nabla^2 f(x,y) \xrightarrow{\mathcal{F}} -(u^2 + v^2) F(u,v)$$

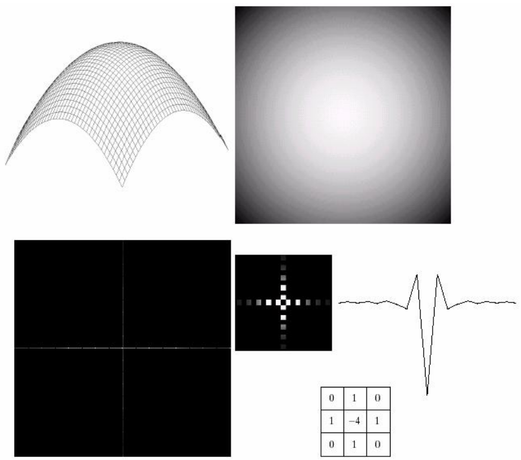
Considering the origin shifting,

$$\nabla^2 f(x,y) \xrightarrow{\mathcal{F}} -((u - M/2)^2 + (v - N/2)^2) F(u,v)$$

□

Laplacian Filter in Frequency Domain

versus Laplacian Filter in Spatial Domain



a b c d e f

FIGURE 4.27 (a) 3-D plot of Laplacian in the frequency domain. (b) Image representation of (a). (c) Laplacian in the spatial domain obtained from the inverse DFT of (b). (d) Zoomed section of the origin

Original plus Laplacian in Frequency Domain

Definition

The ________________________ in frequency domain is

$$G(u,v) = 1 - [(u - M/2)^2 + (v - N/2)^2]$$

Proof.

Becaues Laplaican filter $H(u,v)$ is negative, the enahcned image $g(x,y)$ is

$$g(x,y) = f(x,y) - \nabla^2 f(x,y)$$

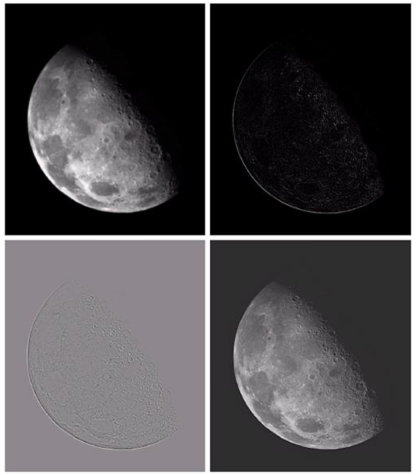
□

Original plus Laplacian in Frequency Domain
Example

All are processed in freq domain

a b
c d

FIGURE 4.28
(a) Image of the North Pole of the moon.
(b) Laplacian filtered image.
(c) Laplacian image scaled.
(d) Image enhanced by using Eq. (4.4-12). (Original image courtesy of NASA.)



Unsharp Masking

Definition

________________ in Frequency Domain:

$$H_{hp}(x,y) = 1 - H_{lp}(x,y)$$

Proof.

Unsharp masking in Spaital Domain:

$$f_{hp}(x,y) = f(x,y) - f_{lp}(x,y)$$

□

High-boost Filtering

Definition

________________ in Frequency Domain:

$$H_{hp}(x,y) = (A-1) - H_{lp}(x,y)$$

Proof.

High-boost filtering in Spaital Domain:

$$\begin{aligned} f_{hp}(x,y) &= Af(x,y) - f_{lp}(x,y) \\ &= (A-1)f(x,y) + [f(x,y) - f_{lp}(x,y)] \\ &= (A-1)f(x,y) + f_{hp}(x,y) \end{aligned}$$

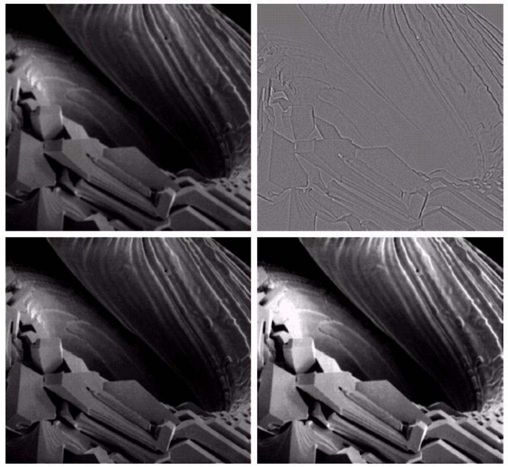
□

High-boost Filtering
Example

All are processed in freq domain

a b
c d

FIGURE 4.29
Same as Fig. 3.43, but using frequency domain filtering. (a) Input image. (b) Laplacian of (a). (c) Image obtained using Eq. (4.4-17) with $A = 2$. (d) Same as (c), but with $A = 2.7$. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)



High-Frequency Emphasis Filtering

Definition

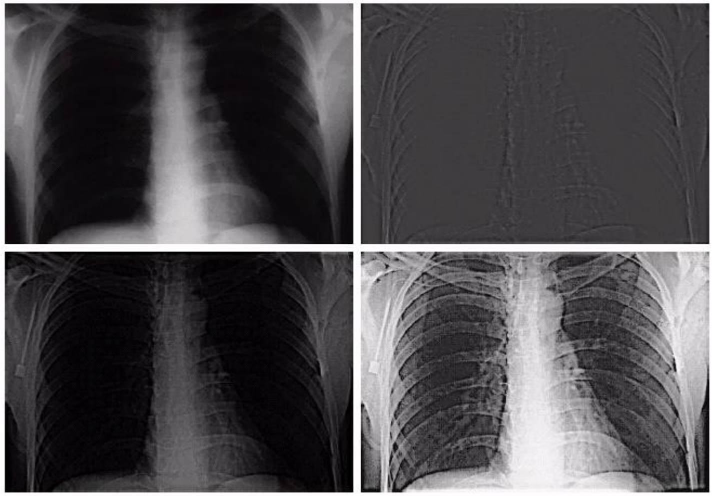
____________________ is to enhance the high-frequency components of an image.
This can be done by

- Multiplying a HPF by a constant
- Add an offset so that zero frequency term is not eliminated by the filter

$$H_{hfe}(u,v) = a + bH_{hp}(u,v)$$

- $a \geq 0$ from 0.25 to 0.5, $b > a$ from 1.5 to 2.0.
- If $b > 1$, the high frequencies are emphasized
- $a = (A-1)$ and $b > 1$: High-boost filtering

High-Frequency Emphasis Filtering
Example



a b
c d

FIGURE 4.30 (a) A chest X-ray image. (b) Result of Butterworth highpass filtering. (c) Result of high-frequency emphasis filtering. (d) Result of performing histogram equalization on (c). (Original image courtesy Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

Homomorphic System

Definition

__________________ is to design filter $H(u,v)$ for the logarithm of the input image.

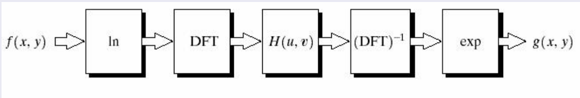


FIGURE 4.31 Homomorphic filtering approach for image enhancement.

Why

Recall

$$f(x,y) = i(x,y)r(x,y)$$

where $i(x,y)$ and $r(x,y)$ are illumination and reflection

$$\mathcal{F}\{\ln f(x,y)\}H(u,v) = \mathcal{F}\{\ln i(x,y)\}H(u,v) + \mathcal{F}\{\ln r(x,y)\}H(u,v)$$

$$g(x,y) = e^{\mathcal{F}^{-1}\{\mathcal{F}\{\ln i(x,y)\}H(u,v)\}} e^{\mathcal{F}^{-1}\{\mathcal{F}\{\ln r(x,y)\}H(u,v)\}}$$

Homomorphic filter function $H(u,v)$

- __________: **Slow spatial variations**
  $\Rightarrow$ Low frequency of the Fourier transform of the logarithm of an image.
- __________: **Vary abruptly** (junctions of dissimilar objects)
  $\Rightarrow$ High frequency of the Fourier transform of the logarithm of an image.

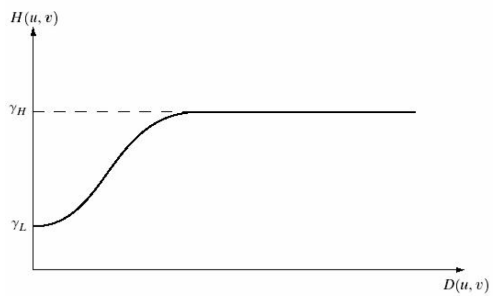


FIGURE 4.32 Cross section of a circularly symmetric filter function. $D(u,v)$ is the distance from the origin of the centered transform.

Homomorphic filter function $H(u,v)$

The curve shape of ______________________________ can be approximated by modifed Gaussian HPF

$$H(u,v) = (r_H - r_L)[1 - e^{-c(D^2(u,v)/D_0^2)}] + r_L$$

And choose,

- $r_L < 1$ to decrease the illumination contribution
- $r_H > 1$ to amplify reflectance
- c is to control the sharpness of the slope of the filter function as it transistions between r_L and r_H.

Note: $r_L < 1$ and $r_H > 1$ achieve simultaneous **dynamic range compression** and **contrast enhancement**

Homomorphic filtering: Example

a b

FIGURE 4.33
(a) Original image. (b) Image processed by homomorphic filtering (note details inside shelter). (Stockham.)