Dynamic Programming

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Dynamic Programming

- General method that applies to a class of problems, often a class of optimization problems (but not only optimization)
- Problem reduced to smaller (possibly overlapping) problems

- Main principles:

1. Memoization: Do not solve the same problem instance repeatedly: Solve it once and record the result to reuse it if needed

   Bottom-up (iterative) version: Problems are solved from smaller to larger and solutions tabulated

   Top-down (recursive) version: Before initiating recursive call, check if solution was already computed previously.

Example: Fibonacci numbers

- $F_0 = 0$, $F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$
- Sequence: 0,1,1,2,3,5,8,13,21,…

- Recursive algorithm FIB(n)

```
if n =0 then return 0
   else if n=1 then return 1
      else return FIB(n-1) + FIB(n-2)
```

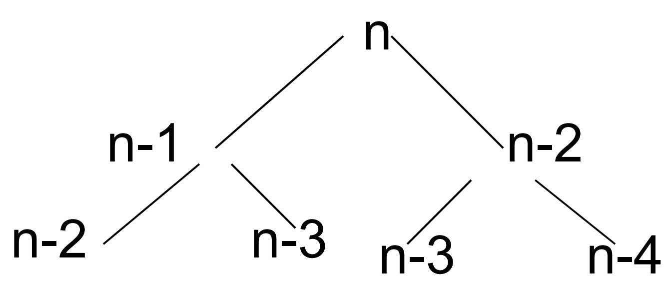
Complexity: T(n) = T(n-1)+T(n-2)+O(1)

Grows at least as fast as the Fibonacci numbers

$$F_n \approx \varphi^n / \sqrt{5}, \text{ where } \varphi = (1+\sqrt{5})/2 = \text{golden ratio}$$

Fibonacci numbers ctd.

- FIB is called only on n distinct arguments, but it is called repeatedly with the same arguments



Tabulate the result, so no need to evaluate it again on the same argument $\Rightarrow$ complexity O(n)

F[0]=0; F[1]=1;
for i=2 to n do F[i]=F[i-1]+F[i-2]
Return F[n]

(Actually, in this case we don't need an array: only need to remember the last two values)

Dynamic Programming for Optimization Problems

Main principles:

- Principle of Optimality (Optimal Substructure):

Problem can be reduced to a set of smaller subproblems;

Optimal solution for whole involves optimal solutions for subproblems

- Memoization:

Subproblems are solved from smaller to larger and solutions tabulated

Rod Cutting Problem

- Given rod of length n,
- prices p_i, i=1,…n, for rods of length i

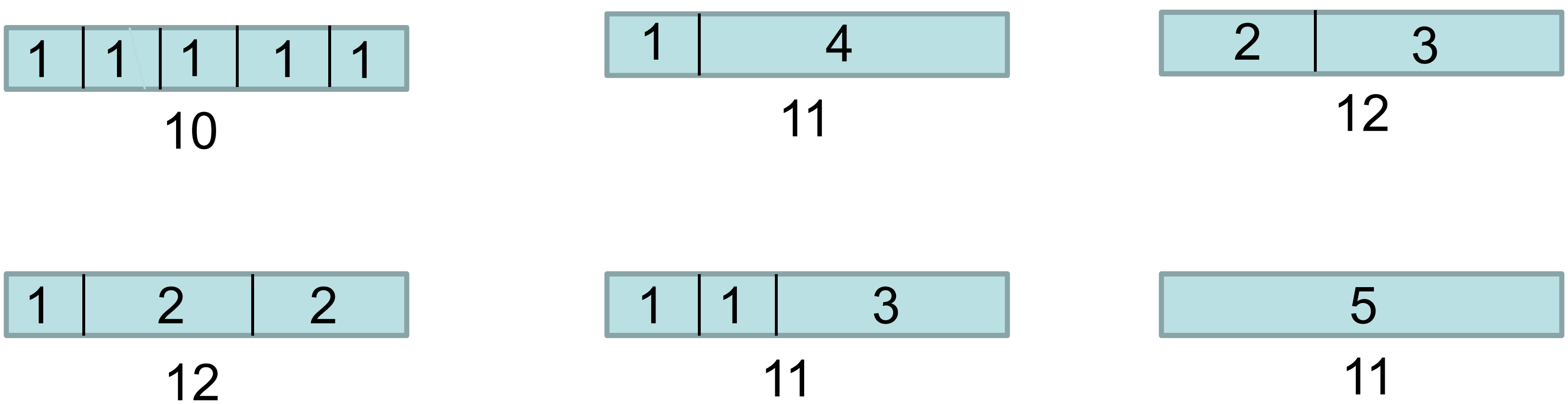
- Find the optimal way to cut the rod of length n into smaller rods that maximizes the total revenue
- i.e., determine $i_1, i_2, \ldots, i_k$ such that $i_1 + i_2 + \ldots + i_k = n$ and $p_{i1} + p_{i2} + \ldots + p_{ik}$ is maximized.

Example

- length i : 1 2 3 4 5
- price p_i : 2 5 7 9 11

- Rod of length 5

- Some possible cuttings and their revenue



Recurrence

- If first piece has length i, then remainder of length n-i should be cut in an optimal way that maximizes revenue

  (Principle of optimality)

- Let r_j= max revenue achievable from rod of length j=0,1…,n

- $r_n = \max\{ p_i + r_{n-i} \mid i = 1,\ldots,n \}$
- $r_0 = 0$

Recursive Algorithm

- CUT-ROD(p,n)

  if n=0 then return 0

  else { q= -∞

  for i=1 to n do q =max(q, p[i] + CUT-ROD(p,n-i)) }

  return q

Total number of calls: $$f(n) = 1 + \sum_{j=0}^{n-1} f(j)$$

$$\Rightarrow f(n) = 2^n$$

Only n+1 different arguments ⇒ Many duplicate calls

Dynamic Programming algorithm

- Compute in array r[0…n] the optimal revenues r_j according to the recurrence

```
r[0]=0
for j=1 to n do
   {  q = -∞
      for i = 1 to j do
           q = max(q, p[i] + r[j-i])
      r[j]=q
   }
return r[n]
```

Time Complexity: $\Theta(n^2)$

Reconstructing an optimal solution

- Record also in an array s[1…n] the best choice for the length of the first piece for each j=1,…,n

```
r[0]=0
for j=1 to n do
  {  q = -∞
       for i = 1 to j do
           if (q < p[i] + r[j-i]) then {q = p[i] + r[j-i] ; s[j]=i }
       r[j]=q
  }
j=n
while j>0 do
  { print s[j];  j=j-s[j] }
```

Top-Down DP Algorithm

- Like recursive algorithm but memoize in array r[0…n] the optimal revenues r_j and only make recursive call if value is not available
- MEMOIZED-CUT-ROD(p,n)

```
 for i=0 to n do r[i] = -∞    [ Initialization]
 return M-CUT-ROD(p,n)
```

- M- CUT-ROD(p,n)

```
if n=0 then return 0
   else {   q= -∞
            for i=1 to n do
               {  if (r[n-i] = -∞)  r[n-i] = M-CUT-ROD(p,n-i)
                  q =max(q, p[i] + r[n-i])
               }
        }
   return q
```

Time Complexity: $\Theta(n^2)$

0-1 Knapsack Problem

- n items, with given integer weights w_i, values v_i, $i=1,\ldots,n$
- Knapsack with weight capacity W

- Problem: Choose a subset of items that fits in the knapsack and has maximum value
  i.e, choose a subset S $\subseteq\{1,\ldots,n\}$ that maximizes $\sum_{i \in S} v_i$
  subject to $\sum_{i \in S} w_i \le W$

Example

Item	Weight	Value
1	4	25
2	6	30
3	2	10
4	5	27
5	7	35

Knapsack capacity: 13

Some feasible solutions, and their value

{1,2,3}: 65, {1,3,4}: 62, {1,3,5}: 70, {2,3,4}: 67,
{2,5}: 65, {4,5}: 62, ……

Optimal solution: {1,3,5}, value 70

Reduction to smaller subproblems

- Should we take the n-th item?

- If we take it, then, we have capacity $W-w_n$ left and can pick any subset from the first n-1 items → should pick the best

- If we don't take it, we have capacity W for the first n-1 items

Which of the two is better?

Let $M(b,i)$ = maximum value we can get with knapsack of capacity b, using a subset of the first i items only

$M(b,i) = \max\{ M(b-w_i, i-1)+v_i, M(b,i-1) \}$ if $b \geq w_i$ else $= M(b,i-1)$

$M(b,0) = 0$ for all b

Recursive Algorithm

KNAP(w,v,b,i)

[max value that can be obtained for capacity b from first i items only]

if $i=0$ or $b=0$ then return 0

else if $b< w_i$ then return KNAP(w,v,b,i-1)

else return max (KNAP(w,v,b-w_i,i-1)+v_i , KNAP(w,v,b,i-1))

- Main call: KNAP(w,v,W,n)

Time Complexity of Recursive algorithm

- A call for i items may generate two recursive calls with i-1 items.

- $T(i) = 2T(i-1) + O(1)$
- $\Rightarrow T(n) = \Theta(2^n)$

- But: if $W << 2^n$, then many of these recursive calls solve the same problem: at most nW different arguments.

Dynamic Programming Algorithm

```
for b=0 to W do M(b,0) =0
for i=1 to n do
   for b=0 to W do
      if b≥w_i and M(b-w_i, i-1)+v_i > M(b,i-1)
              then M(b,i) = M(b-w_i, i-1)+v_i
         else  M(b,i) = M(b,i-1)
Return M(W,n)
```

Running time: $O(nW)$

Reasonable if W is "small".

Not a polynomial-time algorithm if weights given in binary.

Pseudopolynomial algorithm: polynomial if numbers are given in unary notation

Recovering an optimal solution

- Record which case generates M(b,i) for every b,i

```
for b=0 to W do M(b,0) =0
for i=1 to n do
  for b=0 to W do
     if  b≥w_i and M(b-w_i, i-1)+v_i > M(b,i-1)
            then { M(b,i) = M(b-w_i, i-1)+v_i ; s(b,i)=1 }
       else { M(b,i) = M(b,i-1) ; s(b,i) = 0 }
Return M(W,n) and s
```

Optimal Solution

```
b=W; S = ∅
for i=n down to 1 do
  if s(b,i)=1 then { S = S ∪ {i}; b=b-w_i }
return S
```

Matrix Chain Multiplication

- Given a chain of *n* matrices $A_1, \quad A_2, \quad \cdots, \quad A_n$ of dimensions $[p_0 \times p_1], [p_1 \times p_2], \cdots\cdots, [p_{n-1} \times p_n]$ we want to multiply them using standard pairwise matrix multiplications

- Standard multiplication of a *pxq* times a *qxr* matrix has cost (#scalar multiplications) *pqr* (# additions also ≤ *pqr*)

- Many ways to parenthesize the chain
- Different ways may have drastically different costs

Example

$A_1,\ A_2,\ A_3$ of dimensions $2\times 100,\ 100\times 2,\ 2\times 100$

Solution 1: $(A_1 \times A_2) \times A_3$

Cost: $(2\cdot 100\cdot 2)+(2\cdot 2\cdot 100)=800$

Solution 2: $A_1 \times (A_2 \times A_3)$

Cost: $(100\cdot 2\cdot 100)+(2\cdot 100\cdot 100)=40{,}000$

Matrix Chain Multiplication Problem

- Input: Dimensions of the matrices $A_1, A_2, \cdots, A_n$
  i.e. n+1 numbers $p_0, p_1, p_2, \cdots\cdots, p_{n-1}, p_n$
- Output: A parenthesization of the product of the matrices that minimizes the cost (#scalar multiplications)

- Note: # parenthesizations is exponential in n
- Relation with #binary trees with n leaves

  Catalan number: $C(n-1) = \frac{1}{n}\binom{2n-2}{n-1} = \Omega(4^n / n^{3/2})$

- We cannot afford to try them all

Property1: Principle of Optimality (Optimal Substructure)

- If the last multiplication in the optimal solution is $(A_1 \times \cdots \times A_k) \times (A_{k+1} \times \cdots \times A_n)$ then both factors $(A_1 \times \cdots \times A_k)$ and $(A_{k+1} \times \cdots \times A_n)$ computed optimally

Recurrence relation for minimum cost of multiplying matrices A_i through A_j

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1} p_k p_j\} & \text{if } i < j \end{cases}$$

Minimum overall cost: $m[1,n]$

Recursive Matrix Chain

RMC(p,i,j)

```
If i=j then return 0
else {  m = ∞
        for k=i to j-1 do
          { q = RMC(p,i,k)+RMC(p,k+1,j)+ p_{i-1}p_k p_j
            if q < m then m=q
          }
     }
Return m
```

Main: RMC(p,1,n)

Recurrence relation for complexity of recursive algorithm

$$T(1) = \Theta(1)$$

$$T(n) = \sum_{k=1}^{n-1} (T(k) + T(n-k) + \Theta(1)) = 2\sum_{k=1}^{n-1} T(k) + \Theta(n)$$

$$\Rightarrow \quad T(n) = \Omega(3^n)$$

Too much!

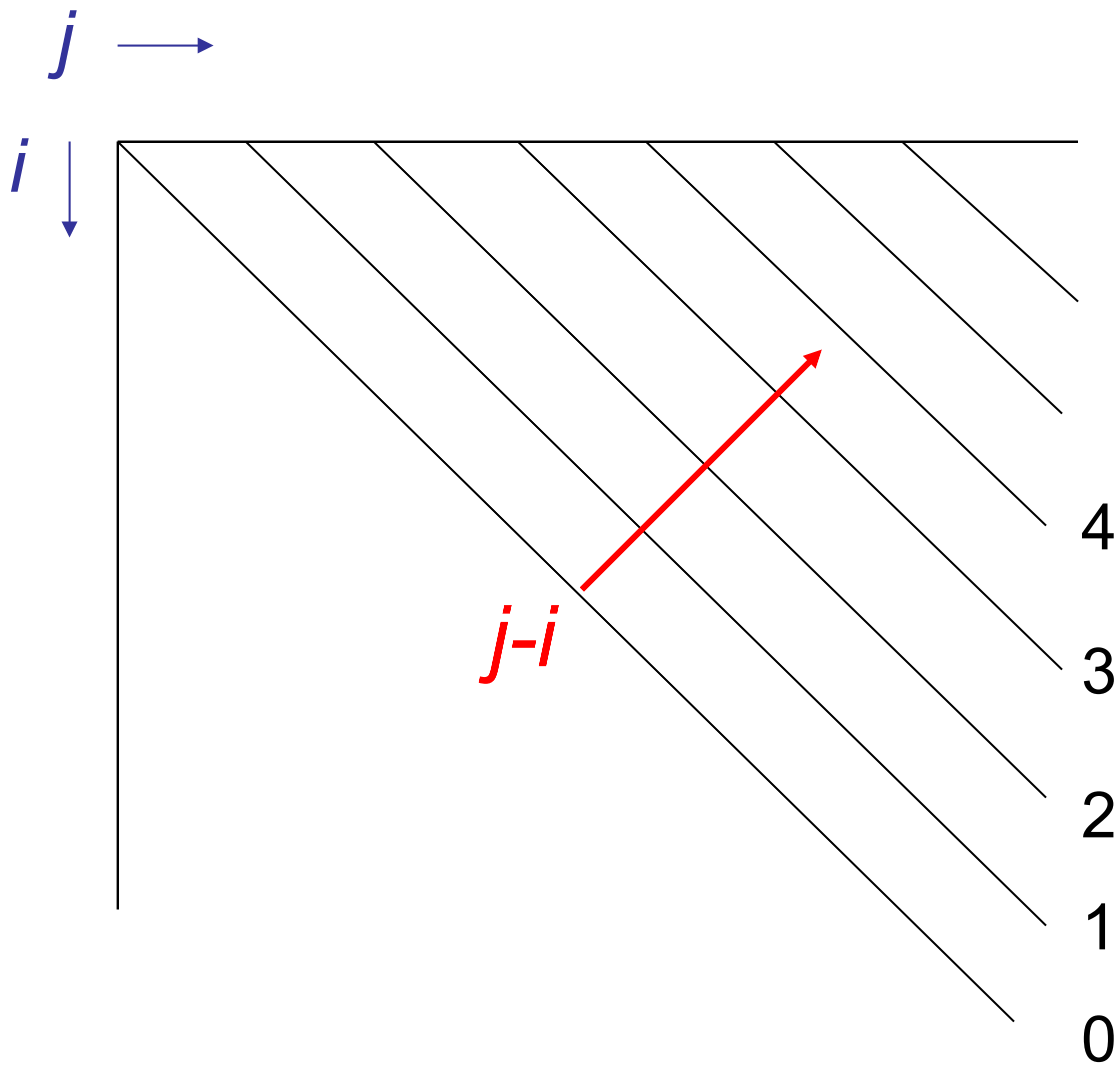
Same computations repeated over and over.

There is a better way.

Property 2: Many common subproblems → Memoization

- Only $O(n^2)$ subproblems [i,j]
- Solve each subproblem once and record the optimal cost in a table m[i,j]
- Compute the table in increasing order of the problem size = difference j-i
- Record for each pair i,j also the k that gave the optimal value, to trace back the optimal solution
- Time per pair i,j is $\Theta(j-i)$
- Total Complexity: $\Theta(n^3)$

Order of computation



Matrix Chain DP Algorithm

```
For i=1 to n do m[i,i]=0
For d =1 to n-1 do      /* d=difference j-i */
 for i=1 to n-d do
    {  j=i+d
       m[i,j]=∞
       for k=i to j-1 do
           { q = m[i,k]+m[k+1,j]+ p_{i-1}p_k p_j
              if q < m[i,j] then { m[i,j]=q; s[i,j]=k }
           }
     }
Return m and s
```

Complexity: $O(n^3)$

Recovering the optimal expression

```
Print-Opt(s,i,j)
If i=j then print "Ai"
  else {  print "("
          Print-Opt(s,i,s[i,j])
          print "×"
          Print-Opt(s,s[i,j]+1,j)
          print ")"
        }
```

Main: Print-Opt(s,1,n)

Once we have computed s, this takes O(n) time.

Longest Common Subsequence (LCS)

- Given two sequences x[1…*m*], y[1…*n*], find a longest common subsequence (gaps allowed)

x = A B C B D A B

y = B D C A B A

- Applications: computational biology, diff

- Naive way: Take every subsequence of x check against y $\rightarrow$ Time $(2^m n)$

Optimal Substructure

- Subproblems i,j : Compute LCS($x_1 \ldots x_i$, $y_1 \ldots y_j$) and its length $c[i,j]$, for i=1,…,m; j=1,…,n
- Overall LCS has length $c[m,n]$
- Recurrence

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max\{c[i, j-1], c[i-1, j]\} & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

Proof: Look at LCS $z_1 z_2 \cdots z_k$ of $(x_1 \cdots x_i)$ and $(y_1 \cdots y_j)$

Case 1: $x_i = y_j \Rightarrow z_k = x_i$ and $z_1 \cdots z_{k-1}$ = LCS($x_1 \cdots x_{i-1}$, $y_1 \cdots y_{j-1}$)

Case 2: $x_i \neq y_j$, $z_k \neq x_i \Rightarrow z_1 \cdots z_k$ = LCS($x_1 \cdots x_{i-1}$, $y_1 \cdots y_j$)

Case 3: $x_i \neq y_j$, $z_k \neq y_j \Rightarrow z_1 \cdots z_k$ = LCS($x_1 \cdots x_i$, $y_1 \cdots y_{j-1}$)

Example

y

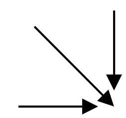
j

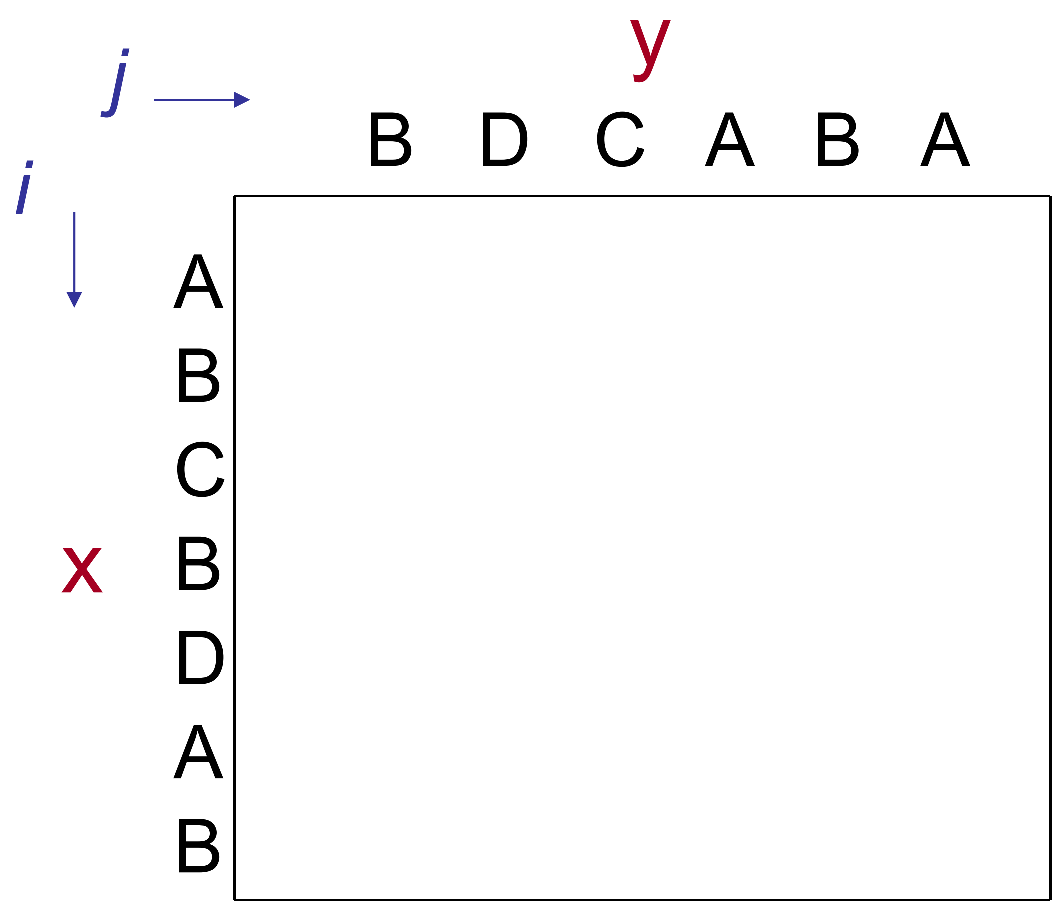
i

			B	D	C	A	B	A
		0	0	0	0	0	0	0
	A	0	0	0	0	1	1	1
	B	0	1	1	1	1	2	2
	C	0	1	1	2	2	2	2
x	B	0	1	1	2	2	3	3
	D	0	1	2	2	2	3	3
	A	0	1	2	2	3	3	4
	B	0	1	2	2	3	4	4

Order of computation

- Any order that is consistent with dependencies of recurrence: left to right, up to down, diagonally





Complexity
O(*mn*)

“Sparse” LCS

- If not many symbol repetitions → can improve
- $O((m+n+p)\log(m+n))$, where $p = |\{ (i,j) : x_i = y_j \}|$
- Worst case $p=mn$, but could be much smaller
- If distinct symbols (even in one string) $\Rightarrow p \le m+n$

- Does not process all (i,j) pairs, only the matches

  $M = \{ (i,j) : x_i = y_j \}$

  - Only places where $c[i,j]$ may increase in its row and column

Sparse LCS Algorithm

1. Compute the set $M = \{ (i,j) : x_i = y_j \}$
2. Sort M in increasing i, decreasing j
3. Maintain set $S=\{ j_1, j_2, \ldots\}$, initially $S = \varnothing$
4. For each (i,j) in M (in order of step 2)

   if $j \notin S$ then { Insert(S,j); Delete(S, successor(j)) }
5. Return $|S|$

At the end, if S has k elements then LCS of length k

Recovering the LCS: For every insertion of an element $j \in S$, record corresponding i

Example

y

			1	2	3	4	5	6	
i \ *j*			**B**	**D**	**C**	**A**	**B**	**A**	**S**
		0	0	0	0	0	0	0	
x	A	0	0	0	0	1	1	1	{4}
x	B	0	1	1	1	1	2	2	{1, 5}
x	C	0	1	1	2	2	2	2	{1, 3}
x	B	0	1	1	2	2	3	3	{1, 3, 5}
x	D	0	1	2	2	2	3	3	{1, 2, 5}
x	A	0	1	2	2	3	3	4	{1, 2, 4, 6}
x	B	0	1	2	2	3	4	4	{1, 2, 4, 5}

Invariant

- After processing (i,j)

$$\forall j', \ \#\text{elements} \le j' \text{ in } S = \begin{cases} c(i-1, j') & \text{if } j' < j \\ c(i, j') & \text{if } j' \ge j \end{cases}$$

- After processing *i-th* row ,

$\exists$ CS between $x[1\ldots i]$ and y of length $r \Leftrightarrow$

S has at least r elements, there is a CS of length r that ends with $y[j_r]$, and j_r is as small as possible with this property