Solving Recurrences and the Master Theorem

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Recurrences

- Arise in the analysis of recursive algorithms that reduce an instance of a problem to smaller instances

$$T(n) = \begin{cases} \text{Function of T(smaller } n') & \text{if } n > cst. \\ c & \text{if } n \leq cst. \end{cases}$$

- Want to obtain tight asymptotic expression for T(n), i.e. T(n)=Θ(f(n))

- Means prove both upper bound O(f(n))

  and lower bound Ω(f(n))

Methods for Solving Recurrences

- Unfolding (expanding) the recursion

  if it is not too complicated

- Recursion-tree method

  Expand the recursion tree

- Substitution (Verification) method

  1. Guess the form of the solution
  2. Use induction to prove it formally

- Master Method

  – explicit solution for a class of common recurrences

Example: Insertion Sort (recursive form)

Method: Reduction to a smaller instance:

1. Sort recursively the first *n*-1 elements using Insertion-Sort
2. Insert the *n*-th element in the right position

Analysis of Running time

$$T(n) = \begin{cases} T(n-1) + \Theta(n) & \text{for } n > 1 \\ \Theta(1) & \text{for } n = 1 \end{cases}$$

Recursive equation / Recurrence

Unfolding (Expansion) of the recursion:

$$
\begin{aligned}
T(n) &= T(n-1) + cn \\
&= T(n-2) + c(n-1) + cn \\
&= T(n-3) + c(n-2) + c(n-1) + cn \\
&= \cdots \\
&= T(i) + c[(i+1) + (i+2) + \cdots + (n-1) + n] \\
&= \cdots \\
&= T(1) + c[2 + 3 + \cdots + (n-1) + n] \\
&= c\,[\,1 + 2 + \cdots + (n-2) + (n-1) + n\,] \\
&= cn\,(n+1)\,/\,2 \\
&= \Theta(n^2)
\end{aligned}
$$

Example: Recurrence of Merge-Sort

$$T(n) = \begin{cases} T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n) & \text{for } n > 1 \\ \Theta(1) & \text{for } n = 1 \end{cases}$$

Assume for simplicity that n is a power of 2

$$T(n) = 2T(n/2) + cn$$

Can solve it by unfolding, or by recursion tree

Unfolding the recursion

$$
\begin{aligned}
T(n) &= 2T(n/2) + cn \\
&= 2(2T(n/4) + c(n/2)) + cn \\
&= 4T(n/4) + 2cn \\
&= 4(2T(n/8) + c(n/4)) + 2cn \\
&= 8T(n/8) + 3cn \\
&\quad . \quad . \quad . \\
&= 2^i \cdot T(n/2^i) + i \cdot cn \\
&\quad . \quad . \quad . \\
&= n \cdot T\left(1\right) + \log n \cdot cn \\
&= \Theta(n \log n)
\end{aligned}
$$

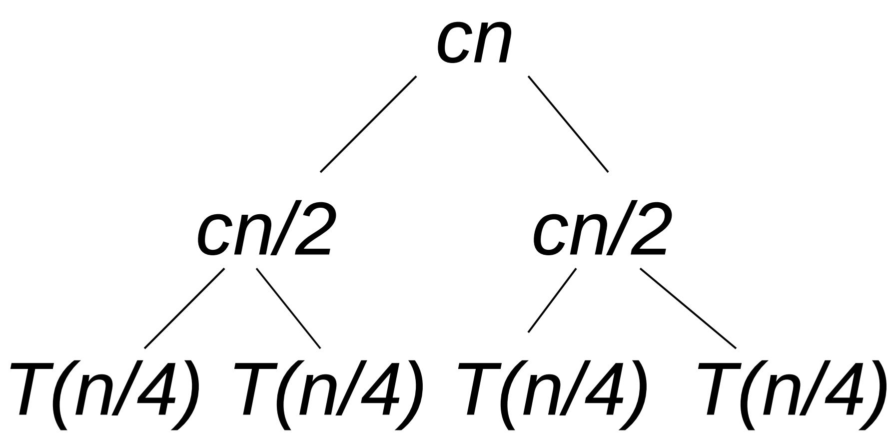
Recursion Tree

$T(n)$

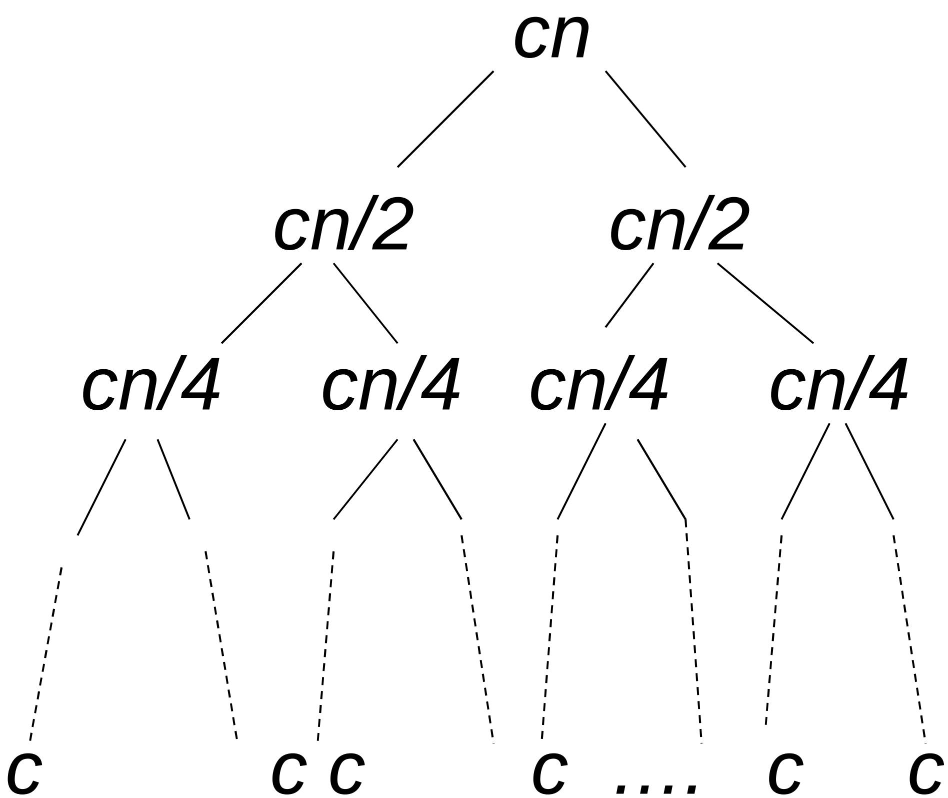
Recursion Tree

$$
\begin{array}{ccc}
 & cn & \\
\diagup & & \diagdown \\
T(n/2) & & T(n/2)
\end{array}
$$

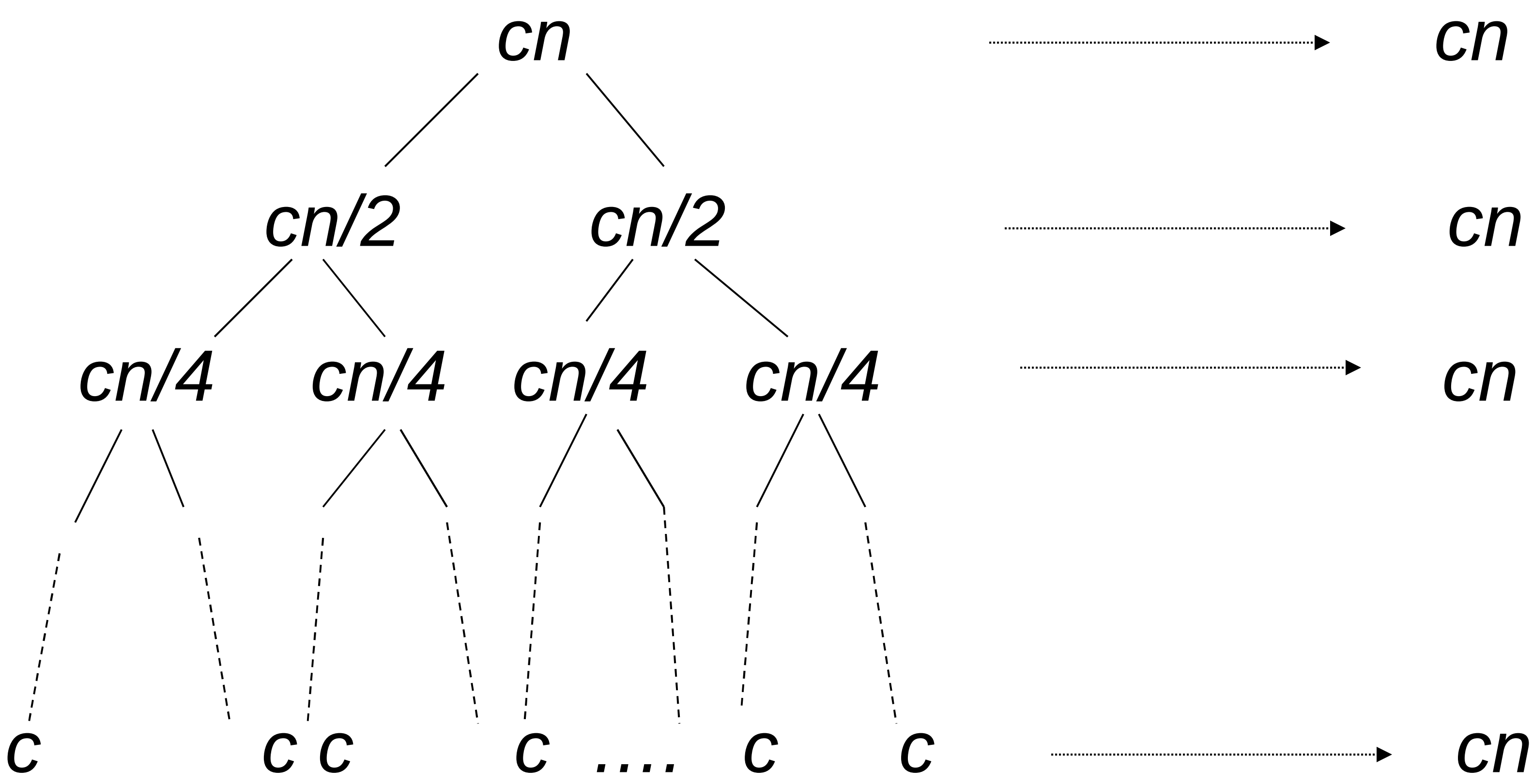
Recursion Tree



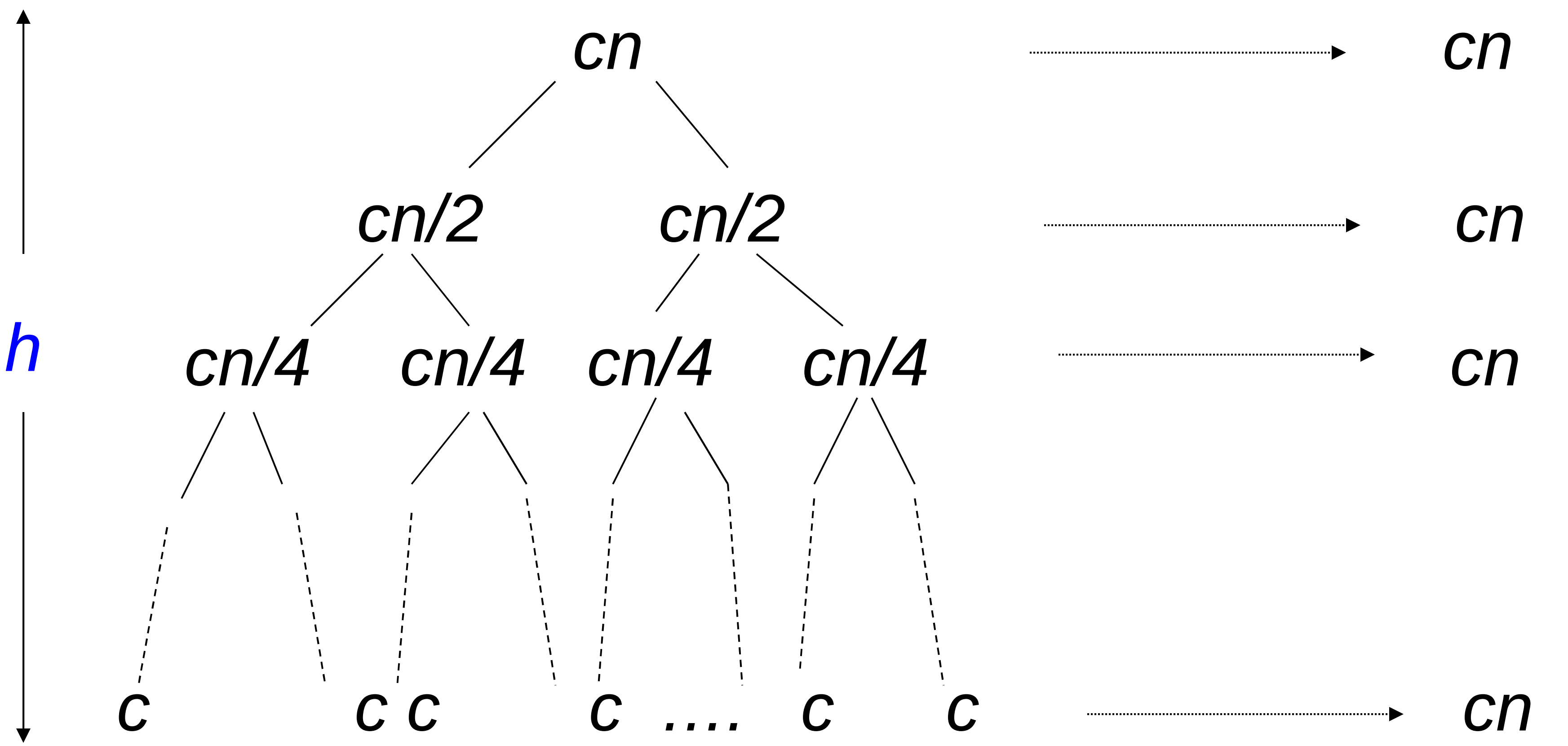
Recursion Tree



Recursion Tree

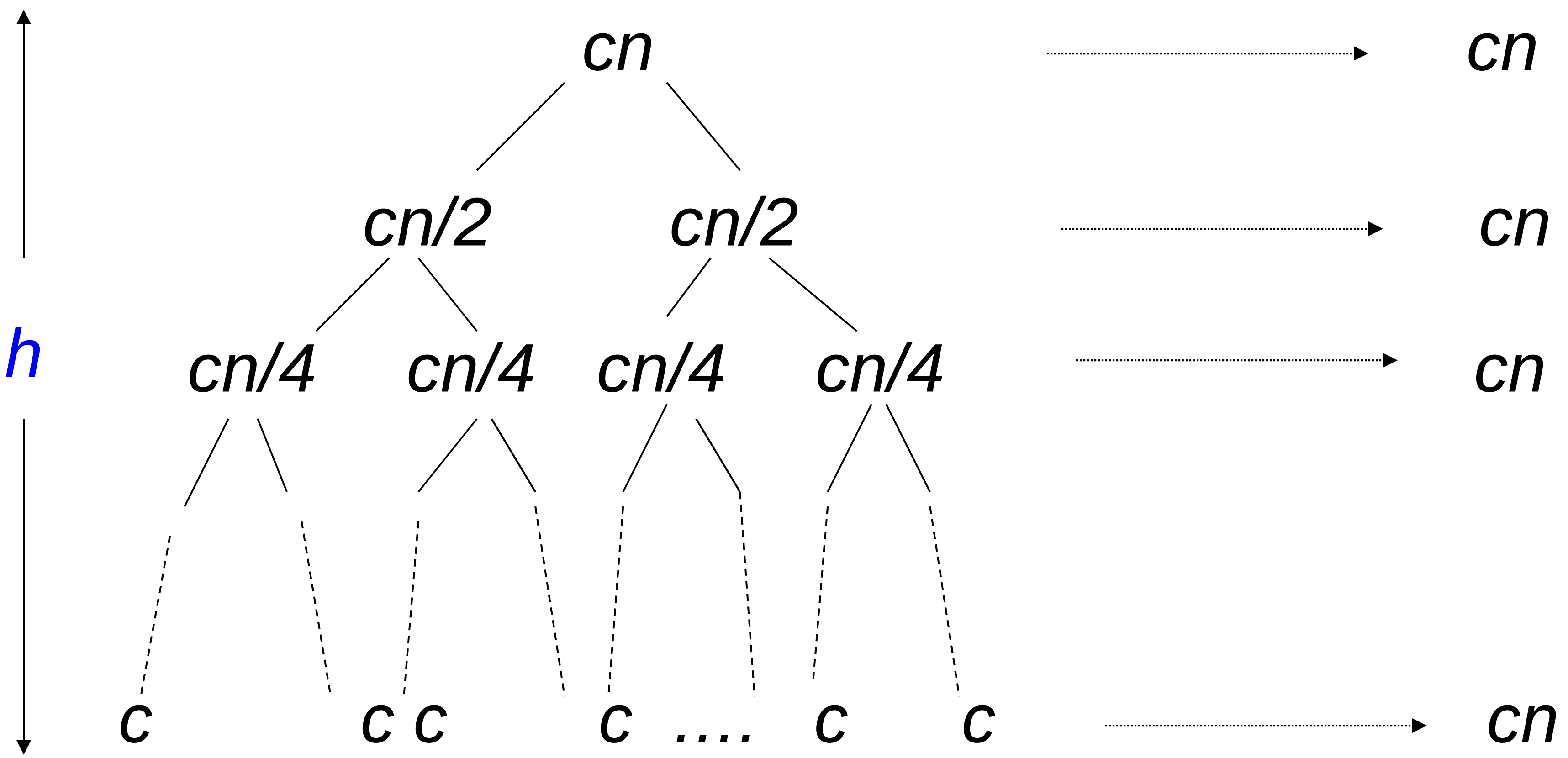


Recursion Tree



#levels: $h = \log n + 1$

Recursion Tree



#levels: $h = \log n + 1$

Total: $cn(\log n + 1)$

What if n is not a power of 2?

$$T(n) = \begin{cases} T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn & \text{for } n > 1 \\ c & \text{for } n = 1 \end{cases}$$

$T(n)$ is monotonic in n, i.e.,

$n \le m \quad \Rightarrow \quad T(n) \le T(m)$

Formal Proof: By induction on n

Basis: $n = 1$: $T(1) = c \le T(m)$

Induction step: $n > 1$

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn \\ &\le T(\lfloor m/2 \rfloor) + T(\lceil m/2 \rceil) + cm \\ &= T(m) \end{aligned}$$

What if n is not a power of 2?

n lies between two succesive powers of 2:

$$2^k \le n \le 2^{k+1}, \qquad k = \lfloor \log n \rfloor$$

$T(n)$ is monotonic in $n \Rightarrow$

$$T(2^k) \le T(n) \le T(2^{k+1})$$

$$c2^k(k+1) \le T(n) \le c2^{k+1}(k+2)$$

$$T(n) = \Theta(n \log n)$$

Substitution method - example

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$$

Guess: $T(n) = \Theta(n)$

Verify upper bound:

Guess that $T(n) \le cn$ and show it inductively

$$\begin{aligned} T(n) &\le c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1 \\ &= cn + 1 \end{aligned}$$

Induction does not go through.

Substitution method - example

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$$

Guess: $T(n) \le cn - b$

Verify upper bound:

$$\begin{aligned} T(n) &\le c\lfloor n/2 \rfloor - b + c\lceil n/2 \rceil - b + 1 \\ &= cn - 2b + 1 \\ &\le cn - b, \quad \text{holds provided } b \ge 1 \end{aligned}$$

Choose *c, b* large enough to handle also the boundary condition:

For example, if T(1)=3, then let b=1, c=4

Master Method

- Applies to class of recurrences

$T(n) = aT(n/b) + f(n)$, where constants $a \geq 1$, $b > 1$

- Arise often in divide and conquer

  - Divide the given instance of size n into

    a subinstances of size n/b

  - Conquer recursively the subinstances
  - Combine the solutions for the subinstances

$f(n)$ = time of divide and combine steps

Master Method: $T(n) = aT(n/b) + f(n)$

1. $f(n) = O(n^{\log_b a - \varepsilon}),\ \varepsilon > 0 \quad \Rightarrow \quad \Theta(n^{\log_b a})$

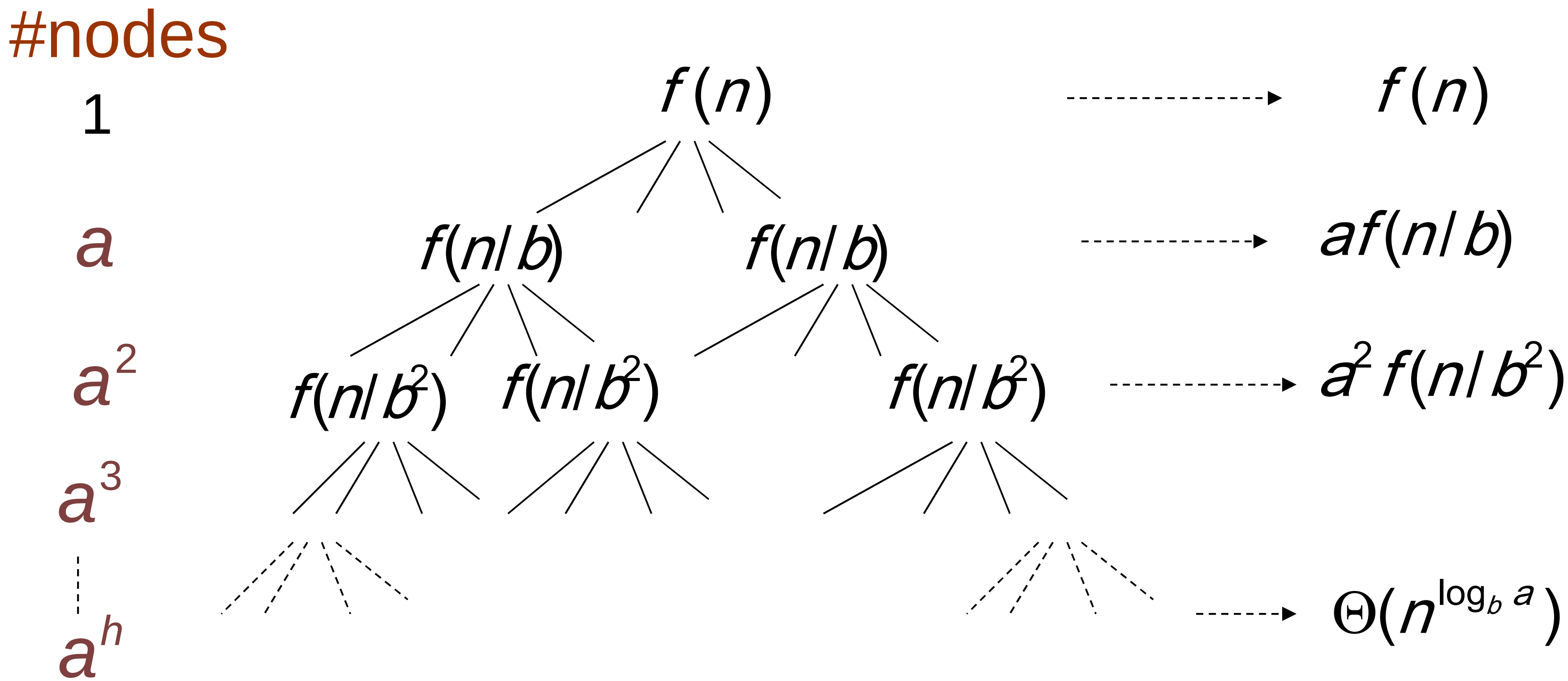
2. $f(n) = \Theta(n^{\log_b a}) \quad \Rightarrow \quad \Theta(n^{\log_b a} \log n)$

   $f(n) = \Theta(n^{\log_b a} \log^k n) \quad \Rightarrow \quad \Theta(n^{\log_b a} \log^{k+1} n)$

3. $$\left.\begin{array}{l} f(n) = \Omega(n^{\log_b a + \varepsilon}),\ \varepsilon > 0 \\ af(n/b) \le cf(n),\ \text{some } c < 1 \end{array}\right\} \Rightarrow \quad \Theta(f(n))$$

Key Criterion: Compare $f(n)$ with $n^{\log_b a}$

Recursion tree



Height $h = \log_b n$

$\#\text{leaves} = a^{\log_b n} = n^{\log_b a}$

Total: $\Theta(n^{\log_b a}) + \sum_{i=0}^{\log_b n - 1} a^i f(n/b^i)$

Which term dominates?

Case 1: $f(n) \ll n^{\log_b a}$

$$f(n) = O(n^{\log_b a-\varepsilon}) \Rightarrow \exists c > 0 \text{ s.t. } f(n) \le cn^{\log_b a-\varepsilon} \text{ for large } n$$

$$a^i f(n/b^i) \le ca^i (n/b^i)^{\log_b a-\varepsilon} = c\, n^{\log_b a-\varepsilon} a^i \frac{b^{i\varepsilon}}{b^{i \log_b a}} = c\, n^{\log_b a-\varepsilon} b^{i\varepsilon}$$

$$\sum_{i=0}^{\log_b n-1} a^i f(n/b^i) \le c\, n^{\log_b a-\varepsilon} \sum_{i=0}^{\log_b n-1} b^{i\varepsilon} = c\, n^{\log_b a-\varepsilon} \frac{b^{\varepsilon \log_b n} - 1}{b^{\varepsilon} - 1}$$

$$= c\, n^{\log_b a-\varepsilon} \frac{n^{\varepsilon} - 1}{b^{\varepsilon} - 1} = O(n^{\log_b a})$$

Total: $\Theta(n^{\log_b a})$ (first term dominates)

Case 2: $f(n) = \Theta(n^{\log_b a})$

Upper bound:

$$\sum_{i=0}^{\log_b n - 1} a^i f(n/b^i) \le \sum_{i=0}^{\log_b n - 1} a^i c\,(n/b^i)^{\log_b a} = \sum_{i=0}^{\log_b n - 1} c\,a^i \frac{n^{\log_b a}}{b^{i \log_b a}}$$

$$= \sum_{i=0}^{\log_b n - 1} c\,n^{\log_b a} = O(n^{\log_b a} \log n)$$

Lower bound:

similar (reverse inequality)

Total: $\Theta(n^{\log_b a} \log n)$

Case 3: $f(n) \gg n^{\log_b a}$

$$af(n/b) \le cf(n) \text{ for some } c<1 \;\Rightarrow$$

$$a^i f(n/b^i) \le c^i f(n) \text{ (proof by induction on } i\text{)} \;\Rightarrow$$

$$\sum_{i=0}^{\log_b n-1} a^i f(n/b^i) \le \sum_{i=0}^{\log_b n-1} c^i f(n) = f(n)\sum_{i=0}^{\log_b n-1} c^i$$

$$\le f(n)\sum_{i=0}^{\infty} c^i = \frac{1}{1-c} f(n) = O(f(n)) \quad \text{since } c<1$$

The first term is $\Theta(n^{\log_b a}) = O(f(n))$

Total: $\Theta(f(n))$

Master Method

- Same result applies to the floor and ceiling variants:

$$T(n) = aT(\lfloor n/b \rfloor) + f(n)$$

$$T(n) = aT(\lceil n/b \rceil) + f(n)$$

Examples

1. Merge Sort: $T(n) = 2T(n/2) + \Theta(n)$

$a = 2,\ b = 2,\ f(n) = \Theta(n),\ n^{\log_b a} = n$

Case 2 $\Rightarrow T(n) = \Theta(n \log n)$

2. Binary search: $T(n) = T(n/2) + 1$

$a = 1,\ b = 2,\ f(n) = 1,\ n^{\log_b a} = n^0 = 1$

Case 2 $\Rightarrow T(n) = \Theta(\log n)$

Examples ctd.

3. $T(n) = 2T(n/2) + 1$

$$a = 2, b = 2, f(n) = 1, n^{\log_b a} = n$$

$$\text{Case 1} \Rightarrow T(n) = \Theta(n)$$

4. $T(n) = 3T(n/2) + n^2$

$$a = 3, b = 2, f(n) = n^2, n^{\log_b a} = n^{\log 3} \approx n^{1.58}$$

$$af(n/b) = 3(n/2)^2 = (3/4)n^2 = (3/4)f(n)$$

$$\text{Case 3} \Rightarrow T(n) = \Theta(n^2)$$

Changing variables

$$T(n) = 2T(\sqrt{n}) + \log n$$

$$m = \log n$$

$$T(2^m) = 2T(2^{m/2}) + m$$

Rename: $S(m) = T(2^m)$

$$S(m) = 2S(m/2) + m$$

Solution: $S(m) = \Theta(m \log m)$

$$T(n) = \Theta(\log n \log \log n)$$

Sometimes the master method does not apply: in-between cases

$$T(n) = 4T(n/2) + n^2 / \log n$$

$$a = 4,\ b = 2,\ f(n) = n^2 / \log n,\ \ n^{\log_b a} = n^2$$

- $f(n) = O(n^{\log_b a})$ but not $O(n^{\log_b a - \varepsilon})$ for any constant $\varepsilon > 0$

  because $\dfrac{f(n)}{n^{\log_b a-\varepsilon}} = \dfrac{n^2/\log n}{n^2/n^{\varepsilon}} = \dfrac{n^{\varepsilon}}{\log n} \to \infty$

  $\Rightarrow$ not Case 1

- $f(n)$ not $\Theta(n^{\log_b a})$ because $\dfrac{f(n)}{n^{\log_b a}} = \dfrac{1}{\log n} \to 0$

  $\Rightarrow$ not Case 2 (nor 3)

Solution: $\Theta(n^2 \log\log n)$

Master Theorem Summary

- *T(n) = aT(n/b) + f(n)*

Compare $f(n)$ to $n^{\log_b a}$

	Case 1	Case 2	Case 3
f(n):	$O(n^{\log_b a}/n^{\varepsilon})$	$\Theta(n^{\log_b a})$	$\Omega(n^{\log_b a} \cdot n^{\varepsilon})$ + extra condition
T(n):	$\Theta(n^{\log_b a})$	$\Theta(n^{\log_b a} \log n)$	$\Theta(f(n))$