Analysis of Algorithms and Asymptotics

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Analysis of Algorithms

- Correctness:

  The algorithm *terminates* with the *correct* answer

- Performance
  - Mainly Running time (Time complexity)
  - Use of other resources (space, …)

- Experimental vs. analytical evaluation of algorithms
- Other issues: simplicity, extensibility, …

Time Complexity

- Running time depends on the input
- Parameterize by the size *n* of the input, and express complexity as function *T(n)*

Worst Case: maximum time over all inputs of size *n*

Average Case: expected time, assuming a probability distribution over inputs of size *n*

Analysis

Cost of each operation depends on machine

Simplification 1: machine-independent analysis:
assume all operations unit cost →

can add the costs of the different steps

Asymptotic Analysis

Simplification 2: Look at *growth* of $T(n)$ as n goes to infinity; focus on dominant term

- Example: $3n^2 + 7n + 10$

  Dominant term: $3n^2$

- Simplification 3: Look at the *rate (order)* of growth: suppress the constant coefficient
  - Example: Quadratic complexity $\Theta(n^2)$

Example: Sorting

- Problem:
  - Input: Sequence of *n* numbers $\langle a_1, a_2, \ldots, a_n \rangle$
  - Output: A permutation (reordering) $\langle a_1^{'}, a_2^{'} \ldots, a_n^{'} \rangle$ of the input sequence such that $a_1^{'} \leq a_2^{'} \leq \cdots \leq a_n^{'}$

  Example: input 8, 3, 5, 9, 1 $\Longrightarrow$ output 1, 3, 5, 8, 9

- Algorithms
  - Insertion Sort, Merge Sort, Quicksort, Heapsort, ……..

Example: Insertion Sort

Input: A[1..n]

```
for  j =2 to n
 {  key = A[j]
    i = j-1
    while i > 0 and A[i] > key
       {  A[i+1] = A[i]
          i = i-1
        }
    A[i+1] = key
  }
```

Insert key(=A[j]) in the right place among the first j-1 elements

Pseudocode

Correctness

Loop Invariant:

At the start of each iteration of the for loop, the subarray A[1.. j-1] consists of the elements originally in these positions but in sorted order

Proof by Induction:

- Basis: Claim holds initially (j=2)
- Induction step: If it holds at the start of an iteration, it continues to hold at the start of the next iteration

At the end, sorted array

Time Analysis of Insertion Sort

	times executed
for j =2 **to** n	n
{ key = A[j]	n-1
i = j-1	n-1
while i > 0 and A[i] > key	$\sum_{j=2}^{n} t_j$
{ A[i+1] = A[i]	$\sum_{j=2}^{n} (t_j - 1)$
i = i-1	$\sum_{j=2}^{n} (t_j - 1)$
}	
A[i+1] = key	n-1
}	

t_j = #executions of while instruction for j

Analysis ctd.

Cost of each operation depends on machine

Simplification 1: machine-independent analysis – assume all operations unit cost

$$T(n) = n + 3(n-1) + \sum_{j=2}^{n} t_j + 2\sum_{j=2}^{n} (t_j - 1)$$

$$= 2n - 1 + 3\sum_{j=2}^{n} t_j$$

t_j is between 1 and j

Worst-case analysis

Input sequence sorted in reverse order

$t_j = j$ for all j

$$\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1$$

$$T(n) = 2n - 1 + 3\frac{n(n+1)}{2} - 3 = \frac{3}{2}n^2 + \frac{7}{2}n - 4$$

Asymptotic Analysis

Simplification 2: Look at growth of T(n) as n goes to infinity

Dominant term: $\frac{3}{2}n^2$

For $n = 100$, $\frac{3}{2}n^2 > 40\,(\frac{7}{2}n - 4)$; hence, $\frac{3}{2}n^2 > 97.5\%\ T(n)$.

For $n = 1000$, $\frac{3}{2}n^2 > 400\,(\frac{7}{2}n - 4)$; hence, $\frac{3}{2}n^2 > 99.7\%\ T(n)$

Asymptotic Analysis

Simplification 3: Look at the *rate (order)* of growth

- suppress the constant coefficient

quadratic complexity $\Theta(n^2)$

Applying asymptotic analysis:

Drop low order terms, ignore constants

$$\text{Example}: \quad 3n^4 + 10n^3 - 2n^2 + 22n + 7 \;=\; \Theta(n^4)$$

Average case analysis

Input probability distribution:

Assume all permutations are equally likely

Analysis of *expected* running time

$$\bar{t}_j = \frac{j+1}{2} \quad \text{for all j}$$

$$\bar{T}(n) = \frac{3}{4}n^2 + \frac{17}{4}n - 4 = \Theta(n^2)$$

Benefits of asymptotic analysis

- Machine independence – intrinsic complexity of algorithms
- Abstraction from details, concentrate on dominant factors
- A linear-time algorithm becomes faster than a quadratic algorithm eventually (for large enough n)

But .. caution:

- Eventually may be too late, if the constant of the linear-time algorithm that we ignored is huge,

  eg. $10^9 n > n^2 \quad \text{for} \quad n < 10^9$

- Some operations may be much more costly than others, and we may want to count them separately (for example, comparisons in sorting of complex objects)

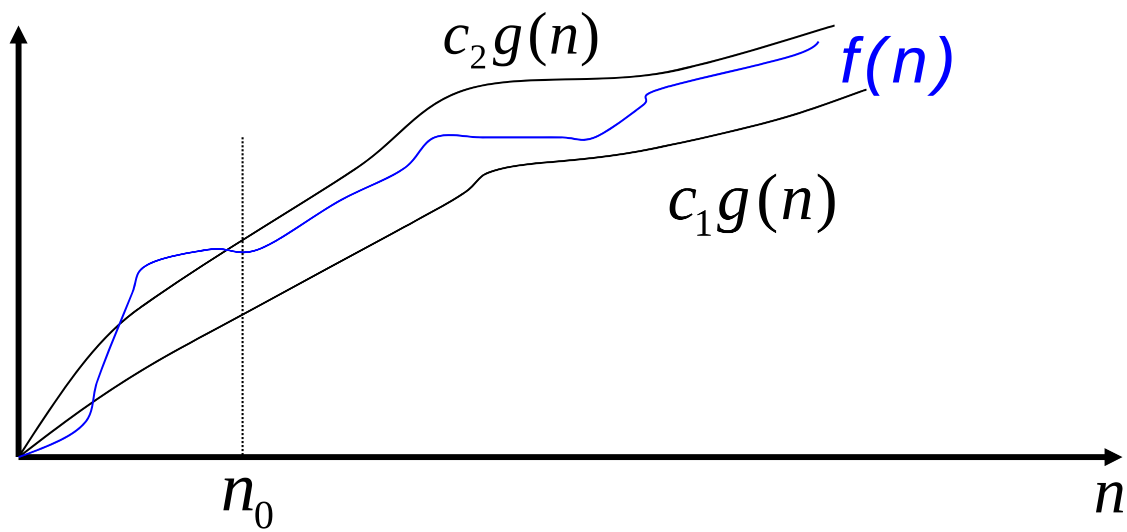
Asymptotic Notations: Theta, Big-Oh, Omega

Theta: $\Theta(g(n)) = \{\ f(n) \mid \exists \text{ constants } c_1, c_2 > 0 \text{ and } n_0 \text{ s.t. } \forall\, n \geq n_0 :$

$$c_1 g(n) \leq f(n) \leq c_2 g(n)\ \}$$

$\text{Convention : We usually write } f(n) = \Theta(g(n))$

Caution: = here denotes membership, not equality

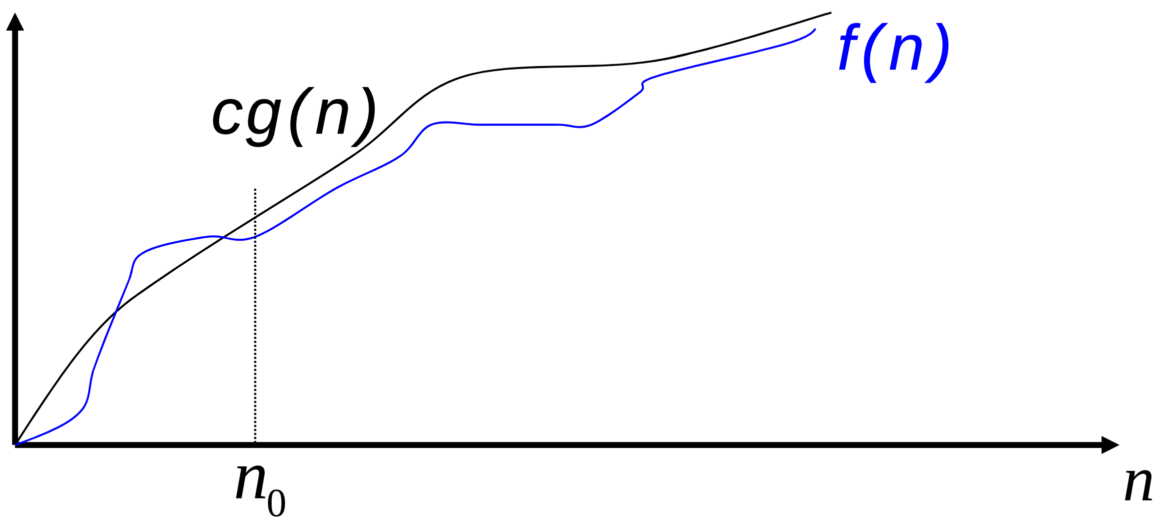


Asymptotic Notations: Theta, Big-Oh, Omega

Big-Oh: (Order)

$$O(g(n)) = \{\ f(n) \mid \exists \text{ constant } c > 0 \text{ and } n_0 \text{ s.t. } \forall\, n \geq n_0 : 0 \leq f(n) \leq c\, g(n)\ \}$$

Convention: We usually write $f(n) = O(g(n))$



Example:

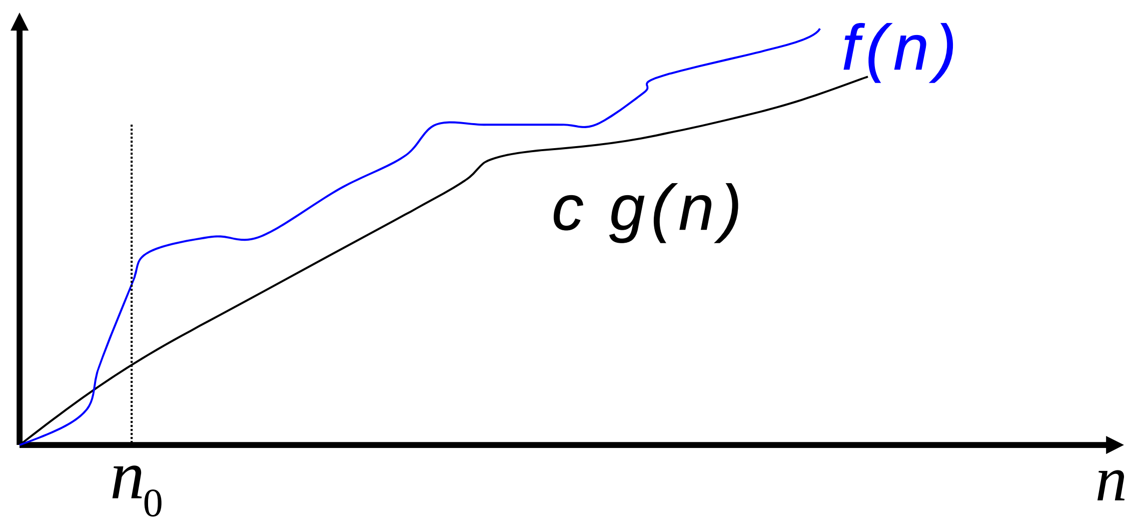
$5n = O(n^2)$

but not vice-versa

Asymptotic Notations: Theta, Big-Oh, Omega

Omega: $\Omega(g(n)) = \{\ f(n) \mid \exists \text{ constant } c > 0 \text{ and } n_0 \text{ s.t. } \forall\, n \geq n_0 :$
$c\, g(n) \leq f(n)\ \}$

$\text{Convention: We usually write } f(n) = \Omega(g(n))$



Example:

$5n^2 = \Omega(n)$

but not vice-versa

Asymptotic Notations: little-oh, little-omega

little-oh: $$o(g(n)) = \{\ f(n) \mid \forall \text{ constant } c > 0\ \exists n_0 \text{ s.t. } \forall n \geq n_0 : \\ 0 \leq f(n) \leq c\, g(n)\ \}$$

little-omega: $$\omega(g(n)) = \{\ f(n) \mid \forall \text{ constant } c > 0\ \exists n_0 \text{ s.t. } \forall n \geq n_0 : \\ 0 \leq c\, g(n) \leq f(n)\ \}$$

f(n)=o(g(n)) means that for large *n,* function *f* is smaller than any constant fraction of *g*

f(n)=ω*(g(n))* means that for large *n,* function *f* is larger than any constant multiple of *g,* i.e., *g=o(f(n))*

Example: $5n = o(n^2)$, $5n^2 = \omega(n)$

Asymptotic Notations Summary

Notation	Growth rate of f(n) vs. g(n)
$f(n) = \omega(g(n))$	$>$
$f(n) = \Omega(g(n))$	$\geq$
$f(n) = \Theta(g(n))$	$=$
$f(n) = O(g(n))$	$\leq$
$f(n) = o(g(n))$	$<$

Asymptotic Notations Summary

Notation	Ratio *f(n)/g(n)* for large *n*
$f(n) = \omega(g(n))$	$f(n)/g(n) \to \infty$
$f(n) = \Omega(g(n))$	$c \le f(n)/g(n)$
$f(n) = \Theta(g(n))$	$c_1 \le f(n)/g(n) \le c_2$
$f(n) = O(g(n))$	$f(n)/g(n) \le c$
$f(n) = o(g(n))$	$f(n)/g(n) \to 0$

Example: Polynomials

- Polynomial: $$a_d n^d + a_{d-1} n^{d-1} + \cdots + a_1 n + a_0, \text{ where } a_d > 0$$ $$= \Theta(n^d)$$ $$\text{Ex}: 5n^3 + 4n^2 - 3n + 8 = \Theta(n^3)$$

Proof: $$\frac{f(n)}{n^d} = a_d + \frac{a_{d-1}}{n} + \cdots + \frac{a_0}{n^d} \to a_d + 0 + \cdots + 0 = a_d$$

$$(0 <) c < d \Leftrightarrow n^c = o(n^d)$$

$$\text{Ex}: n^{3.2} = o(n^{3.3})$$

Proof: $$\frac{n^c}{n^d} = \frac{1}{n^{d-c}} \to 0$$

Example: logarithms

- $\log_{10} n = \Theta(\log_2 n)$
- Proof: $\log_{10}n = \log_2 n / \log_2 10 = \log_2 n / 3.32$

- Same for any change of logarithm from one constant base a to another base b: $\log_a n = \Theta(\log_b n)$

- Notation: logn for $\log_2 n$; In n for $\log_e n$ (natural log)

Logs vs. powers/roots

- $\log n = o(n^c)$ for all $c>0$

- For example: $\log n = o(n^{0.4});\ \ \log n = o(\sqrt[20]{n})$

- Proof: Use L'Hospital's rule

$$\lim_{n\to\infty}\frac{\ln n}{n^c} = \lim_{n\to\infty}\frac{1/n}{cn^{c-1}} = \lim_{n\to\infty}\frac{1}{cn^c} = 0$$

Some common functions

$$\underbrace{n < n\log n < n^2 < n^3 < \cdots}_{\text{polynomial}} \ll \underbrace{2^n < 3^n < n!}_{\text{exponential}}$$

Properties

$$f(n) = o(g(n)) \;\Rightarrow\; f(n) = O(g(n))$$

$$f(n) = \omega(g(n)) \;\Rightarrow\; f(n) = \Omega(g(n))$$

$$f(n) = \Theta(g(n)) \;\Rightarrow\; f(n) = O(g(n)),\ f(n) = \Omega(g(n))$$

$$f(n) = \Theta(g(n)) \Leftarrow f(n) = O(g(n)),\ f(n) = \Omega(g(n))$$

Transitivity:

$$f = O(g) \text{ and } g = O(h) \;\Rightarrow\; f = O(h)$$

same for $o,\ \omega,\ \Omega,\ \Theta$

Sum: $f+g = \Theta(\max(f,g))$

Asymptotic notation in equations

$f(n) = 3n^2 + O(n)$ means

$f(n) = 3n^2 + h(n)$ for some function $h(n)$ that is $O(n)$

Can write equations like

$$3n^3 + O(n^2) + O(n) + O(1) = \Theta(n^3)$$

Caution: O(1)+O(1)+…+O(1) (n times) is not O(1)

O(n) + Ω(n) = ? It is not Θ(n)

…..