Shortest Paths

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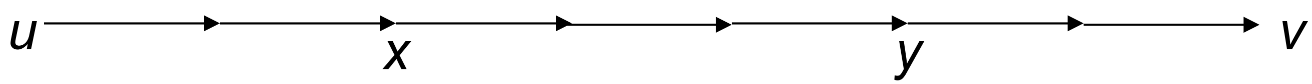
Shortest Paths

- Given graph (directed or undirected) $G=(N,E)$ with lengths (or weights or costs) on the edges $w: E \rightarrow R$
- Length of a path = sum of lengths of the edges.
- shortest s-t path = path with minimum length from s to t
- distance(s,t) = length of shortest s-t path

- If all lengths are ≥ 0, then we only have to consider simple paths (no need to repeat a node) $\Rightarrow$ distances between all pairs are well-defined
- If there are edges with negative length, there may be no shortest path

Properties of distances

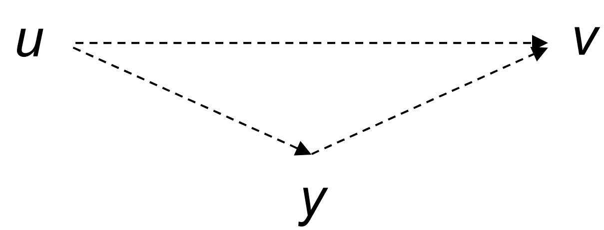
- Every subpath of a shortest path is a shortest path between the endnodes of the subpath



If shortest path from u to v, then x-y subpath is shortest path from x to y

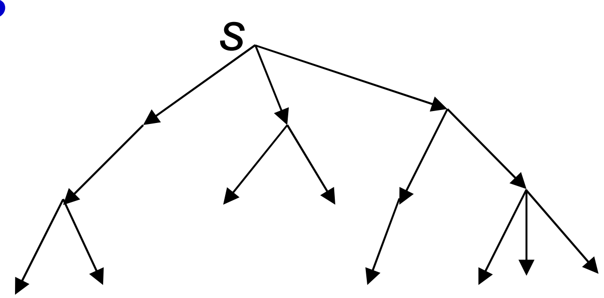
- dist(u,u)=0, for all nodes u
- dist(u,v) ≤ w(u,v), for all edges (u,v)
- dist(u,v) ≤ dist(u,y)+dist(y,v), for all nodes u,y,v

  (triangle inequality)



Shortest paths from a source node s

Want to compute distances from s to all the nodes and shortest path tree from s



Let d(v) =dist(s,v)

- d(s)=0
- d(v) ≤ d(u)+w(u,v) for all edges (u,v),

  with = for some node u $\Rightarrow$
- d(v) = min { d(u)+w(u,v) | all edges (u,v) }

Distances from a source node s

If we have a path from s to each node v of length $\delta(v)$ and the lengths $\delta(v)$ satisfy

- $\delta(s)=0$
- $\delta(v) \leq \delta(u)+w(u,v)$ for all edges (u,v)

then $\delta(v)=\text{dist}(u,v)$ for all v, and the paths are shortest paths

Proof: Show by induction on #edges of any path from s to any node v that length(path) $\geq \delta(v)$

Basis: #edges=0 $\Rightarrow$ v=s and length(path)=0=$\delta(s)$

Induction step: $s \longrightarrow \longrightarrow \longrightarrow \longrightarrow u \longrightarrow v$

length(s-u path) $\geq \delta(u)$ (by i.h.) $\Rightarrow$

length(s-v path) $\geq \delta(u)+w(u,v) \geq \delta(v)$

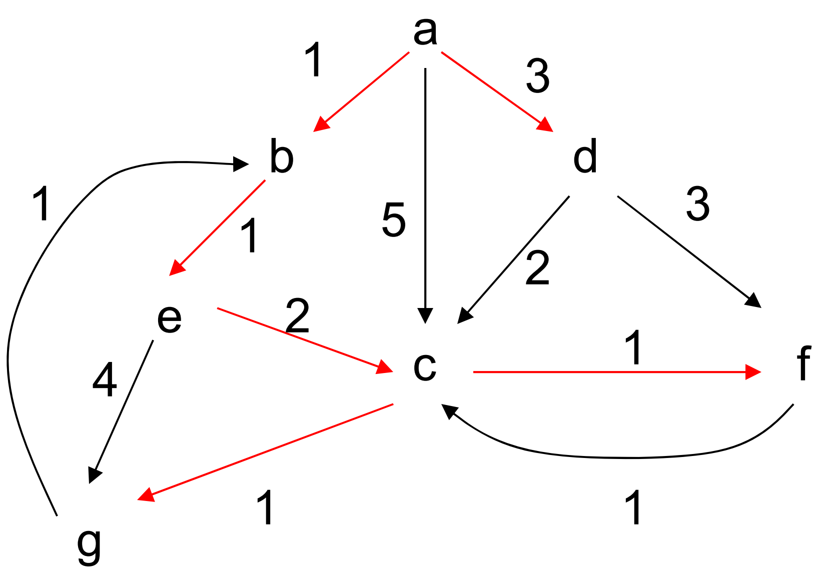
Shortest Paths in Graphs with Edge Lengths (Weights) ≥ 0

- Undirected graph $\Leftrightarrow$ Directed graph with two opposite arcs for each edge (digraph with same Adj lists)

- Dijkstra's algorithm
- Wave out of source s traveling along edges of the graph at unit speed
- d[v] = *tentative distance* of v from s: time it reaches v

- Done nodes v : d[v] = true distance
- Other nodes v: Min-Priority Queue Q with priority d[v]

Example



Dijkstra's algorithm

Dijkstra(G,w,s)

for each $v \in N-\{s\}$ do $\{d[v]=\infty;\ p[v]=\perp\}$

$d[s]=0;\ p[s]=\perp;$

$Q = N$ [alternatively, Q={s} and insert nodes when reached]

while $Q \neq \emptyset$ do

{ u=Extract-Min(Q) [Extract-Min operation; first time, u=s]

for each $v \in Adj[u]$ do

if $d[v] > d[u]+w(u,v)$ then

$\{d[v]= d[u]+w(u,v);\ p[v]=u\}$ [Decrease-Key(v)]

}

Shortest path tree: p[v] gives the parent of each node v = previous node in a shortest path from s to v

Correctness

- **Invariants:**

1. $\forall u$, $d[u] < \infty \Rightarrow \exists$ s-u path of length d[u]
2. Done (R-Q) nodes u have $d[u] \leq \text{Min}(Q)$

   ($\Rightarrow$ done nodes u will never change their d[u])

Theorem: $\forall v$: Final d[v] =length of shortest s-v path

Proof:

- $\geq$: length of s-v path in Dijkstra tree = d[v]
- $\leq$: By induction on length of shortest s-v path

  Consider shortest path s - - - u – v

  By i.h. $d[u] \leq$ length of s- - -u path

  Look at time u is processed:

  d[u] will not change thereafter and $d[v] \leq d[u]+w(u,v)$

Time Complexity

Operations:	Extract-Min	Decrease-Key
# of ops:	n	e
Time/Op.		
Array:	$O(n)$	$O(1)$
Heap:	$O(\log n)$	$O(\log n)$
Fibonacci Heap: (amortized)	$O(\log n)$	$O(1)$

Total (Worst-Case) Time Complexity:

Array: $O(n^2)$

Heap: $O((n+e)\log n)$

Fibonacci Heap: $O(e+n\log n)$

Shortest Paths in General Weighted Directed Graphs

- Graphs with + or – weights.

- If ∄ negative weight cycle, then distances are well-defined, shortest path between any two nodes is *simple*: does not repeat any node
- In particular for DAGs, distances always well-defined for both positive and negative weights

- If ∃ negative weight cycle, then weights of some paths can be made arbitrarily low by going repeatedly around the negative cycle

Directed Acyclic Graphs

Distances $d(v) = \mathrm{dist}(s,v)$ from a source node s satisfy

- $d(s)=0$
- $d(v) = \min \{ d(u)+w(u,v) \mid \text{all edges } (u,v) \}$

If the graph is acyclic, then the recurrence is not circular.

- Can compute d(v) for all v in topological order.

Single Source Shortest Paths in a DAG

Input: Weighted DAG G=(N,E), weights w: E→ℝ, source s

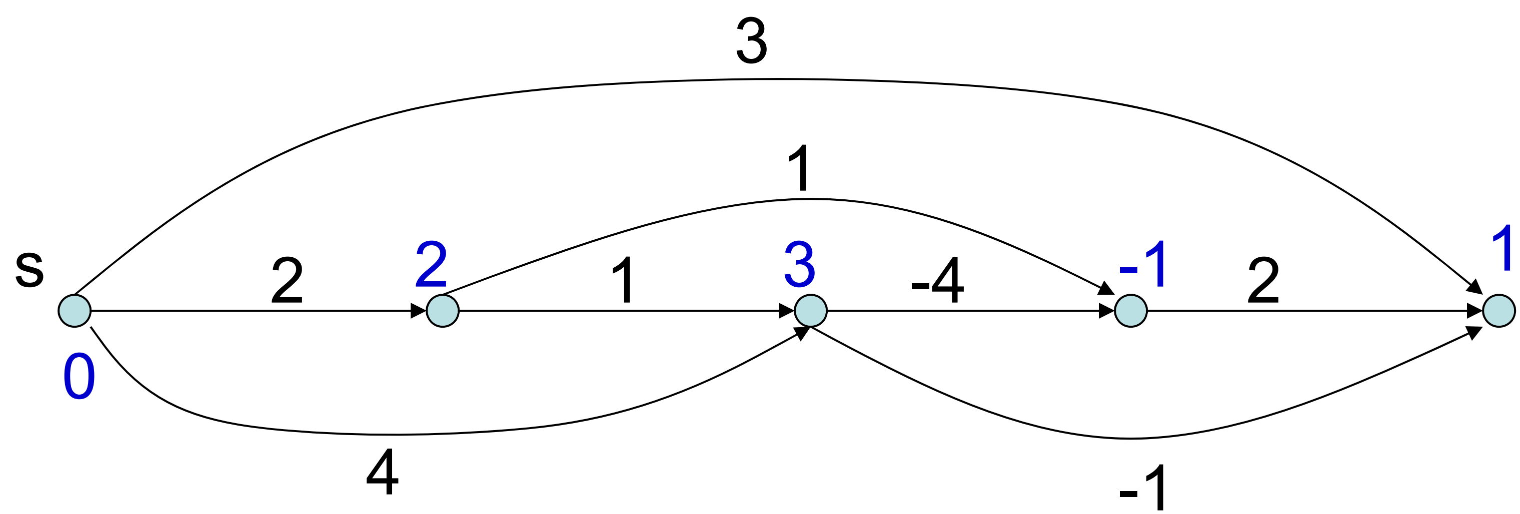
Output: Weight d[v] of shortest path from s to each node v and shortest path tree

1. Initialization: for each v∈N-{s} do {d[v]=∞; p[v]=⊥}
   d[s]=0; p[s]= ⊥ ;
2. Sort topologically the nodes
3. For each node u in topological order do
   for each v in Adj[u] do
   if d[v]> d[u]+w(u,v) then {d[v]=d[u]+w(u,v); p[v]=u}

Time Complexity: O(n+e)

Similarly: Longest paths, other problems on DAGs….

Example



Longest Paths in a DAG

Input: Weighted DAG $G=(N,E)$, weights $w: E \rightarrow \mathbb{R}$

Output: Maximum weight $d[v]$ of a path starting from each node v

1. Sort topologically the nodes: $v_1,\ldots,v_n$
2. For i=n down to 1 do

   if $Adj[v_i] = \emptyset$ then $d[v_i]=0$

   else $d[v_i]=\max(0, \{ d[v_j]+w(v_i,v_j) \mid v_j \in Adj[v_i] \})$

Time Complexity: $O(n+e)$

Dynamic Programming, DAGs

Shortest path in DAG: prototypical Dynamic Programming

$d(v) = \min \{ d(u)+w(u,v) \mid u \in Pred(v) \}$: "recursive calls d(u)"

Example: LCS, edit distance can be expressed as a longest / shortest path problem in a DAG

- Many Problems on DAGs can be solved by DP:

- Compute a function value at u from values at Pred(u)
  $\rightarrow$ evaluate in topological order of nodes
- or from values at Adj(u) : use reverse topological order

HW Exercises:

1. Count the # of paths that start at each node of a DAG
2. Count the # paths from s to t

Note: # paths is generally exponential, so cannot list them all

Dynamic Programming, Trees, Graphs

- Rooted trees = special case of DAGs
  - Many problems on trees can be solved this way: process tree bottom-up (or recursively top-down)
  - Sometimes, have to extend problem so that DP will go through (from value at children, compute value at parent)

- Problems on directed graphs: Often decompose into SCCs, process SCCs in topological order.

General Digraphs, General Weights

- Suppose there are edges with negative weight
- Shortest paths, distances well defined for all pairs of nodes if there is no negative cycle.

- Easier question: Is there a cycle that contains a negative edge?
- Can be answered in O(n+e) time:

  - Compute the SCCs,

  - Check if any negative edge (u,v) is contained inside a SCC (i.e. u,v in same SCC)

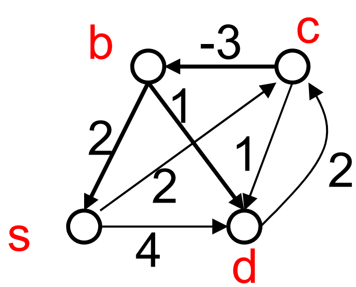
- Determining if there is a negative cycle harder: no O(n+e) algorithm known

Single-source shortest paths – general weights: Bellman-Ford Algorithm

- Input: Digraph G(N,E), weights w: $E \rightarrow \mathbb{R}$, source s
- Output: indicates if $\exists$ reachable negative cycle, and if $\nexists$, distances from s to all other nodes

```
for each v∈N-{s} do {d[v]=∞; p[v]=⊥}
d[s]=0; p[s]= ⊥ ;
repeat n-1 times
   for each edge (u,v) ∈ E do
      if d[v]> d[u]+w(u,v) then {d[v]=d[u]+w(u,v); p[v]=u}
for each edge (u,v) ∈ E do
     if d[v]> d[u]+w(u,v) then return "negative cycle"
return "no reachable negative cycle"
```

Example



s	b	c	d	iteration
0	∞	∞	∞	0
0	∞	2	4	1
0	-1	2	3	2
0	-1	2	0	3
0	-1	2	0	4

If edge d$\rightarrow$c had weight 1 instead of 2, then last iteration would give value 1 for c, indication of a negative cycle.

Negative cycle: c$\rightarrow$b$\rightarrow$d$\rightarrow$c

Correctness

1. If there is no reachable negative cycle, then d[v] = distance(s,v) for all reachable nodes v, and algorithm responds correctly "no reachable negative cycle"

Proof: By induction on # k of iterations of the loop show

Invariant: d[v] ≤ min weight of any s-v path with ≤k edges

Basis: k=0. By initialization

Induction step: Shortest s-v path with ≤k+1 edges

$$s \underbrace{\dashrightarrow}_{\leq k \text{ edges}} u \longrightarrow v$$

In (k+1)th iteration: d[v] ≤ d[u]+w(u,v) ≤ weight of s-v path

At the end, d[v] ≤weight of shortest simple s-v path = dist(s,v)

Correctness ctd.

2. If the algorithm returns "no reachable negative cycle", then $\nexists$ reachable negative cycle

- Proof: Reachable Cycle C: $v_1 \to v_2 \to \cdots v_m \to v_1$

  reachable $\Rightarrow d[v_i] < \infty, \quad \forall i$

  $d[v_i] \le d[v_{i-1}] + w(v_{i-1}, v_i), \quad \forall i$

  $\Rightarrow \sum_i d[v_i] \le \sum_i d[v_{i-1}] + \sum_i w(v_{i-1}, v_i)$

  $\Rightarrow 0 \le w(C)$

Time Complexity : O(ne)

Detection of Negative Cycles

- Input: Digraph G(N,E), weights w: $E \rightarrow \mathbb{R}$
- Output: ∃ negative weight cycle ?

- One method:

1. Compute the strongly connected components
2. For each scc C, pick arbitrary "source" node s in C and run Bellman-Ford from s restricted to C

- ∃ negative cycle iff ∃ negative cycle inside a scc
- Time Complexity: O(ne)

Detection of Negative Cycles: Method 2

1. Add new source node s, and 0 weight edges from s to all the nodes of G $\rightarrow$ graph G'
2. Apply Bellman-Ford to (G',w,s)

- $\exists$ a reachable negative cycle in (G',w,s) iff (G,w) has a negative cycle

- If no negative cycle, then final d values satisfy

  $d[v] \leq d[u] + w(u,v), \ \forall\ (u,v) \in E$

  i.e. $d[v] - d[u] \leq w(u,v), \forall\ (u,v) \in E$

One application: Difference Constraints

- Variables $x_1, \ldots, x_n$
- Set of linear inequalities of the form

  $x_j - x_i \leq b_k$ (a constant)
- Consistent if there is a solution in Reals

- Constraint graph G with one node v_i for each variable x_i
- Edges (x_i, x_j) with weight b_k

- Set is consistent iff no negative cycle in G
- Distances from new source s satisfy the inequalities.

All Pairs Shortest Paths

All-Pairs Shortest Paths

- Input: Digraph G(N,E), weights w: E→$\mathbb{R}$
- Find: Shortest paths between all pairs of nodes or determine ∃ negative weight cycle

- To test if a weighted graph (G,w) has a negative cycle:
  1. Add new source node s, and 0 weight edges from s to all the nodes of G → graph G'
  2. Apply Bellman-Ford to (G',w,s):

     ∃ a reachable negative cycle in (G',w,s) iff (G,w) has a negative cycle

Weight Transformation

- If there are no negative cycles, we can use the computed distances from s in G',w',s as 'node potentials' to make all edge weights ≥ 0, without affecting the shortest paths for any pair of nodes:
- $w'(u,v) = w(u,v) + d[u] – d[v], \quad \forall (u,v) \in E$

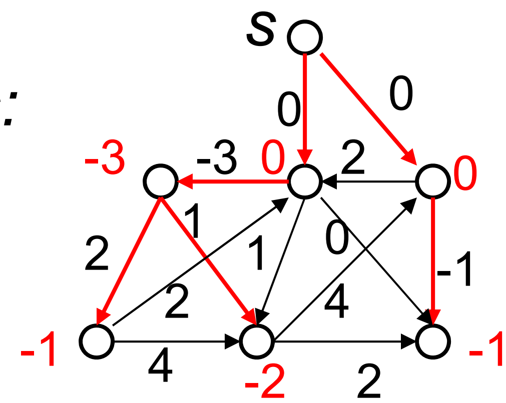
Properties:

- $w'(u,v) \geq 0, \ \forall (u,v) \in E$ (since $d[v] \leq d[u] + w(u,v)$)
- $\forall$ pair of nodes x,y and path p from x to y,

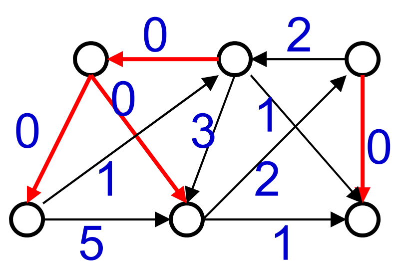
  $w'(p) = w(p) + d[x] - d[y]$
- p is shortest x-y path in (G,w) $\Leftrightarrow$ shortest in (G,w')

Example

Distances from s:



Modified weights:



Johnson's Algorithm for the All-pairs Shortest Path Problem

1. Use Bellman-Ford to determine if $\exists$ negative weight cycle and transform the weights to ≥ 0
2. Apply Dijkstra from every node

Time Complexity: $O(n^2 \log n + en)$

Floyd-Warshall Algorithm

- Alternative DP algorithm for all-pair shortest paths
- Simple, good for dense graphs

- Nodes N = { 1,…,n }, weights $w_{ij}=w(i, j)$

  $w_{ij}=\infty$ if no edge (i, j), $w_{ij} = 0$ if $i=j$

- Dynamic Programming: Compute the length $d_{ij}^{(k)}$

  of the shortest path from i to j that does not use any intermediate nodes beyond k, for k=0,1,2,...

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}) & \text{if } k \geq 1 \end{cases}$$

Floyd-Warshall Algorithm

$D^{(0)} = W$

for k = 1 to n do

  for i = 1 to n do

    for j = 1 to n do

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

return $D^{(n)}$

Time Complexity $O(n^3)$

Can drop the superscripts $\rightarrow$ Space $O(n^2)$

To recover the shortest paths: record best k for each i,j

Graph has a negative cycle $\Leftrightarrow \exists$ negative entry on the diagonal of the final matrix