Examples of

Divide and Conquer

and the Master theorem

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Divide and Conquer

Reduce to any number of smaller instances:

1. Divide the given problem instance into subproblems
2. Conquer the subproblems by solving them recursively
3. Combine the solutions for the subproblems to a solution for the original problem

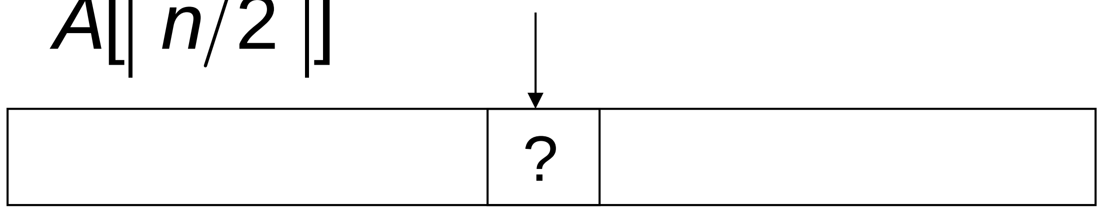
Search Problem

- Input: A set of numbers $a_1, a_2, \ldots, a_n$ and a number x
- Question: Is x one of the numbers a_i in the given set?

If sorted array A of numbers $a_1, a_2, \ldots, a_n$ then Binary Search

Binary Search

- Compare x to the middle element of the array $A[\lceil n/2 \rceil]$



  - If $x = A[\lceil n/2 \rceil]$ then done
  - If $x < A[\lceil n/2 \rceil]$ then

    recursively Search A$[1,\ldots,\lceil n/2 \rceil - 1]$
  - If $x > A[\lceil n/2 \rceil]$ then

    recursively Search A$[\lceil n/2 \rceil + 1,\ldots,n]$

Binary Search Analysis

$$T(n) = \begin{cases} T(n/2) + \Theta(1) & \text{for } n > 1 \\ \Theta(1) & \text{for } n = 1 \end{cases}$$

Can use Master Theorem

$$a = 1, b = 2, f(n) = \Theta(1), n^{\log_b a} = n^{\log_2 1} = n^0 = 1$$

$$\text{Case 2} \;\; \Rightarrow T(n) = \Theta(\log n)$$

Merge Sort

1. Divide: Divide the given n-element sequence to be sorted into two sequences of length n/2
2. Conquer: Sort recursively the two subsequences using Merge Sort
3. Combine: Merge the two sorted subsequences to produce the sorted answer

Merge

- Input: Sorted arrays K[1..n_1], L[1..n_2]
- Output: Merged sorted array M[1.. n_1+n_2]

$$
\begin{aligned}
&\mathrm{i} = 1, \quad \mathrm{j} = 1 \\
&\textbf{for } \mathrm{t} = 1 \text{ to } n_1 + n_2 \\
&\quad \{ \ \textbf{if } (\mathrm{i} \le n_1 \text{ and } (\ \mathrm{j} > n_2 \ \text{ or } \mathrm{K[i]} < \mathrm{L[j]})\ ) \\
&\qquad\qquad \textbf{then } \{\ \mathrm{M[t]} = \mathrm{K[i]}, \quad \mathrm{i} = \mathrm{i} + 1\ \} \\
&\qquad\qquad \textbf{else } \{\ \mathrm{M[t]} = \mathrm{L[j]}, \quad \mathrm{j} = \mathrm{j} + 1\ \} \\
&\quad \}
\end{aligned}
$$

Linear Time Complexity: $\Theta(n_1 + n_2)$

What if inputs, output in same array?

- Input: Sorted array segments

  A[1..n_1], A[n_1+1.. n_1+n_2]

- Output: Merged sorted array A[1.. n_1+n_2]

Copy A[1..n_1] into new array K[1..n_1]

Copy A[n_1+1…n_1+n_2] into L[1..n_2]

Merge K[1..n_1] and L[1..n_2] into A[1.. n_1+n_2]

Linear Time Complexity: $\Theta(n_1 + n_2)$

Merge-Sort

Merge-Sort A[1…n]

If $n > 1$ then

1. Recursively merge-sort $A[1 \cdots \lfloor n/2 \rfloor]$

   and $A[\lfloor n/2 \rfloor + 1 \cdots n]$

2. Merge the two sorted subsequences

Analysis of Merge-Sort

$$T(n) = \begin{cases} T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n) & \text{for } n > 1 \\ \Theta(1) & \text{for } n = 1 \end{cases}$$

Assume for simplicity that n is a power of 2

$$T(n) = 2T(n/2) + cn$$

Can use Master Theorem

$$a = 2, b = 2, f(n) = \Theta(n), n^{\log_b a} = n^{\log_2 2} = n$$

$$\text{Case 2} \quad \Rightarrow T(n) = \Theta(n \log n)$$

Maximum Sum Subarray Problem

- Input: Array A[1…n] of integers (positive and negative)
- Problem: Compute a subarray A[i*…j*] with maximum sum

  i.e., if s(i,j) denotes the sum of the elements of a subarray A[i…j], $s(i,j) = \sum_{k=i}^{j} A[k]$

We want to compute indices i*≤j* such that

$$s(i^*, j^*) = \max\{s(i,j) \mid 1 \le i \le j \le n\}$$

Example: 3 -4 $\underbrace{5\ \ -2\ \ -2\ \ 6\ \ -3\ \ 5}_{\text{max sum = 9}}$ -3 2

Brute force solution

- Compute the sum of every subarray and pick the maximum
  Try every pair of indices i,j with $1 \leq i \leq j \leq n$, and for each one compute $$s(i,j) = \sum_{k=i}^{j} A[k]$$

- Time complexity $\Theta(n^3)$

- With a little more care, can improve to $\Theta(n^2)$:

  can compute the sums of all the subarrays in time $\Theta(n^2)$.

Brute force solution - improved

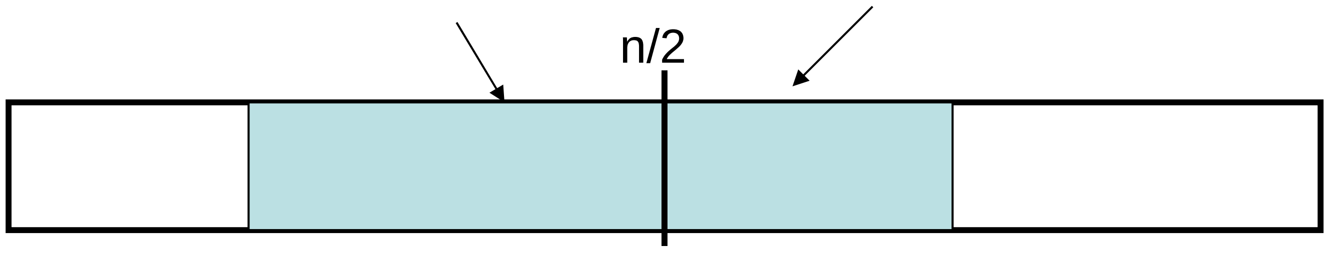
- With a little more care, can improve to $\Theta(n^2)$:
- Can compute the sums for all subarrays with same left end in O(n) time $\Rightarrow$ compute the sums of all the subarrays (there are n(n-1)/2 +n subarrays) in time $O(n^2)$

```
for i =1 to n
  {  s(i,i)=A[i]
     for j=i+1 to n
           s(i,j) = s(i,j-1)+A[j]
  }
```

Divide and Conquer

- A subarray A[i*…j*] with maximum sum is
  – Either contained entirely in the first half , i.e. j* $\leq$ n/2
  – Or contained entirely in the right half , i.e. i* $\geq$ n/2
  – Or overlaps both halfs: i* $\leq$ n/2 $\leq$ j*

- We can compute the best subarray of the first two types with recursive calls on the left and right half.
- The best subarray of the third type consists of the best subarray that ends at n/2 and the best subarray that starts at n/2. We can compute these in O(n) time.



Divide and Conquer analysis

- Recurrence: $T(n) = 2T(n/2) + \Theta(n)$
- Solution: $T(n) = \Theta(n \log n)$

- It is possible to do better: can compute the maximum sum subarray in $\Theta(n)$ time.

  (Not divide and conquer)

For a nice paper on this problem see

J. Bentley, Programming Pearls, Addison-Wesley,

chapter 8 (Algorithm Design Techniques)

Also in Communications of the ACM, 27(9), 1984.

Multiplication of Big integers

Count # bit operations

- Given integers A, B with n bits each, can + , - in O(n) time.
- Ordinary multiplication: n^2 time (n additions)

- D&C: partition into high n/2 and low n/2 bits

A	A_h	A_l
B	B_h	B_l

$$A = A_h \cdot 2^{n/2} + A_l$$

$$B = B_h \cdot 2^{n/2} + B_l$$

$$A \cdot B = (A_h \cdot 2^{n/2} + A_l) \cdot (B_h \cdot 2^{n/2} + B_l)$$

$$= A_h B_h \cdot 2^n + A_h B_l \cdot 2^{n/2} + A_l B_h 2^{n/2} + A_l B_l$$

Multiplication of Big integers

$$A \cdot B = (A_h \cdot 2^{n/2} + A_l) \cdot (B_h \cdot 2^{n/2} + B_l)$$
$$= A_h B_h \cdot 2^n + A_h B_l \cdot 2^{n/2} + A_l B_h 2^{n/2} + A_l B_l$$

4 multiplications of n/2-bit numbers: $A_h B_h$, $A_h B_l$, $A_l B_h$, $A_l B_l$,

additions and shifts.

Note: multiplications by powers of 2 are just shifts

Recurrence: $T(n) = 4T(n/2) + cn$

(last term for additions and shifts)

Solution: $T(n) = O(n^2)$

Multiplication of Big integers – Karatsuba'60

$$A \cdot B = (A_h \cdot 2^{n/2} + A_l) \cdot (B_h \cdot 2^{n/2} + B_l)$$
$$= A_h B_h \cdot 2^n + A_h B_l \cdot 2^{n/2} + A_l B_h 2^{n/2} + A_l B_l$$

$$(A_h + A_l)(B_l + B_h) = A_h B_l + A_h B_h + A_l B_l + A_l B_h \Rightarrow$$
$$A \cdot B = A_h B_h \cdot 2^n + [(A_h + A_l)(B_h + B_l) - A_h B_h - A_l B_l] \cdot 2^{n/2} + A_l B_l$$

3 multiplications of n/2-bit numbers:

$A_h B_h$, $A_l B_l$, $(A_h + A_l)(B_h + B_l)$

\+ additions,subtractions and shifts.

Recurrence: T(n) = 3T(n/2) + cn

Solution: $T(n) = n^{\log_2 3} = n^{1.585}$

Multiplication of Big integers – Karatsuba'60

Recursive Algorithm MULT(A,B)

Write $A = A_h 2^{n/2} + A_l$ and $B = B_h 2^{n/2} + B_l$

Compute $a = A_h + A_l$ and $b = B_h + B_l$

$C = \text{MULT}(a,b)$

$D_h = \text{MULT}(A_h, B_h)$

$D_l = \text{MULT}(A_l, B_l)$

Return $D_h \cdot 2^n + [C - D_h - D_l] \cdot 2^{n/2} + D_l$

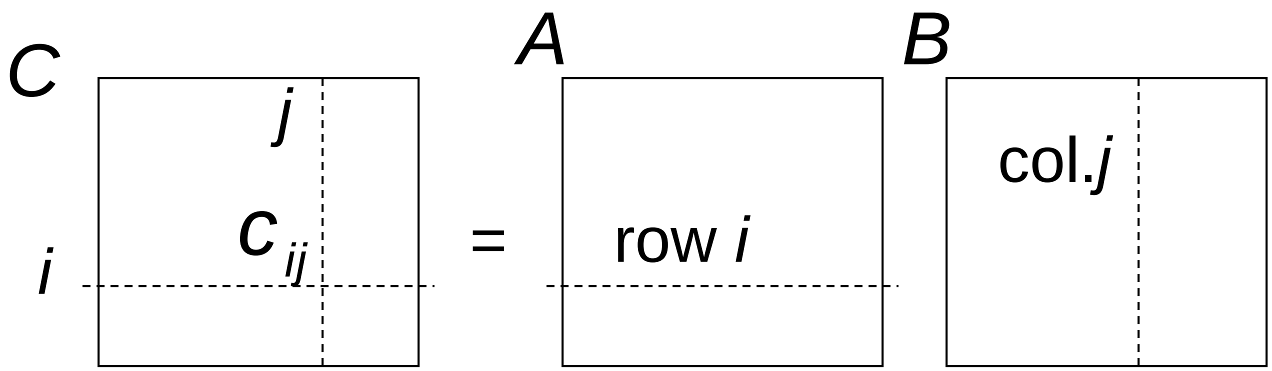
Time: $T(n) = n^{\log_2 3} = n^{1.585}$

FFT-based method: n logn log logn

Matrix Multiplication

Input: Matrices $A = [a_{ij}], B = [b_{ij}], \quad i, j = 1, \ldots, n$

Output: $C = [c_{ij}] = A \cdot B$



$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Standard Matrix Multiplication algorithm

for $i = 1$ to n

  for $j = 1$ to n

    { $c_{ij} = 0$

      for $k = 1$ to n

        $c_{ij} = c_{ij} + a_{ik} b_{kj}$

    }

Time Complexity: $\Theta(n^3)$

Divide and Conquer

Partition matrices *A,B,C* into 4 *n/2* x *n/2* submatrices

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$C_{11} = A_{11} B_{11} + A_{12} B_{21}$

$C_{12} = A_{11} B_{12} + A_{12} B_{22}$

$C_{21} = A_{21} B_{11} + A_{22} B_{21}$

$C_{22} = A_{21} B_{12} + A_{22} B_{22}$

8 recursive multiplications of

n/2 x *n/2* matrices

4 additions (direct – no recursion)

$$T(n) = 8T(n/2) + \Theta(n^2)$$

$$a = 8,\ b = 2,\ f(n) = \Theta(n^2),\ \ n^{\log_b a} = n^3$$

$$\text{Case 1} \ \Rightarrow\ T(n) = \Theta(n^3)$$

Same as standard MM algorithm

Strassen's algorithm

- $P = (A_{11} + A_{22})(B_{11} + B_{22})$
  $Q = (A_{21} + A_{22})B_{11}$
  $R = A_{11}(B_{12} - B_{22})$
  $S = A_{22}(B_{21} - B_{11})$
  $T = (A_{11} + A_{12})B_{22}$
  $U = (A_{21} - A_{11})(B_{11} + B_{12})$
  $V = (A_{12} - A_{22})(B_{21} + B_{22})$.
- $C_{11} = P + S - T + V$
  $C_{12} = R + T$
  $C_{21} = Q + S$
  $C_{22} = P + R - Q + U$

Strassen's algorithm

- Can multiply 2x2 matrices with 7 multiplications, and 18 additions and subtractions. The method does not assume commutativity of multiplication
- Method applies to multiplication of 2x2 block matrices.
- Can be used in divide and conquer scheme with 7 recursive multiplications of *n/2* x *n/2* submatrices.

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

Strassen's Algorithm

$$T(n) = 7\,(n/2) + \Theta(n^2)$$

$$a = 7,\, b = 2,\, f(n) = \Theta(n^2),\, n^{\log_b a} = n^{\log_2 7} \approx n^{2.81}$$

Case 1 $\Rightarrow$ $T(n) = \Theta(n^{\log 7})$

Best current (theoretical) result: $\Theta(n^{2.373})$