

# A Finite Presentation of CNOT-dihedral Operators

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# Outline

Quantum circuits

Clifford+T and universal gate sets

CNOT-dihedral gates

Rules for CNOT-dihedral gates

Normal forms

Phase polynomials and uniqueness

Open questions

## Phase polynomials

Action of a diagonal gate:

$$D |x\rangle = \omega^{p_D(x)} |x\rangle, \quad p_D : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_8,$$

$$p_D(x) = \sum_{i=1}^k a_i g_i(x).$$

where  $a_i \in \mathbb{Z}_8$  and  $g_i : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$  are terms on at most  $n$  variables.

$$\omega^k |x\rangle = \omega^k |x\rangle$$

$$T^k |x_1\rangle = \omega^{kx_1} |x_1\rangle$$

$$U^k |x_1 x_2\rangle = \omega^{k(x_1 \oplus x_2)} |x_1 x_2\rangle$$

$$V^k |x_1 x_2 x_3\rangle = \omega^{k(x_1 \oplus x_2 \oplus x_3)} |x_1 x_2 x_3\rangle$$

Therefore for  $D$  in normal form we have

$$p_D(x) = a_0 + \sum_i a_i x_i + \sum_{i < j} b_{ij} (x_i \oplus x_j) + \sum_{i < j < k} c_{ijk} (x_i \oplus x_j \oplus x_k)$$

Where  $a_i \in \mathbb{Z}_8$ ,  $b_{ij} \in \mathbb{Z}_4$ ,  $c_{ijk} \in \mathbb{Z}_2$  (from the bounds on  $\deg_X(D)$  for  $X = T, U, V$  obtained in normal form).

## Uniqueness

Suppose  $D$  and  $D'$  are distinct diagonal normal forms, then by construction  $p_D(x) \neq p_{D'}(x)$  as polynomials. However this does not mean that  $\exists y \in \mathbb{Z}_2^n$  s.t.  $p_D(y) \neq p_{D'}(y)$ .

Counterexample:  $4x_1 + 4x_2 + 4(x_1 \oplus x_2) = 0 \pmod 8$  for any  $x_1, x_2 \in \mathbb{Z}_2$

To show there exists such a  $y$  need a technical lemma: construct a multilinear polynomial  $q : \mathbb{Z}_8^n \rightarrow \mathbb{Z}_8$  such that  $p_D(y) - p_{D'}(y) = q(y) \forall y \in \mathbb{Z}_2^n$ .

Do this by translating from mod 2 to mod 8:

$$x_i \oplus x_j = x_i + x_j - 2x_i x_j$$

$$x_i \oplus x_j \oplus x_k = x_i + x_j + x_k - 2x_i x_j - 2x_i x_k - 2x_j x_k + 4x_i x_j x_k$$

Then  $p_D(x) - p_{D'}(x) \neq 0 \implies q(x) \neq 0$ , because of the bounds on  $a_i$ ,  $b_{ij}$  and  $c_{ijk}$ . Pick a non-zero term  $dx_{i_1} \dots x_{i_k}$  in  $q(x)$  then letting  $y$  be the vector with 1's in  $i_j$ th positions and zero everywhere else we obtain  $q(y) = d \neq 0$  and so  $D|y\rangle \neq D'|y\rangle$ .

## Open questions

- ▶ Interaction between ZX and CNOT-dihedral rules (rule 13?)
- ▶ The paper doesn't contain an algorithm for normalizing CNOT-dihedral circuits, it uses the properties of the ambient symmetric monoidal structure non-constructively. What could be a rewrite system? Algebraic vs combinatorial description.
- ▶ Complexity of CNOT-dihedral circuits.