A Finite Presentation of CNOT-dihedral Operators

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Outline

Quantum circuits

Clifford+T and universal gate sets

CNOT-dihedral gates

Rules for CNOT-dihedral gates

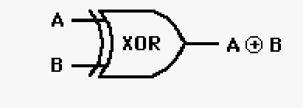
Normal forms

Phase polynomials and uniqueness

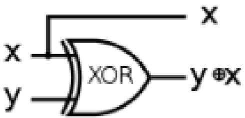
Open questions

Classical circuits

- Logical gates (boolean functions $\mathbb{B}^n \to \mathbb{B}^m$ with cartesian product):

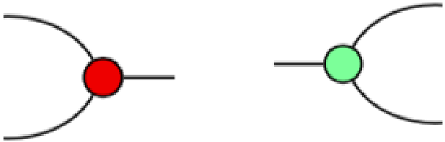


- Reversible gates (information is physical):

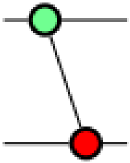


Quantum circuits

- Logical gates (linear maps $(\mathbb{CB})^{\otimes n} \to (\mathbb{CB})^{\otimes m}$ with tensor product):



- Reversible gates (unitaries):



Towards an algebraic theory of quantum circuits

Axiomatisations of logical gates (linear maps): ZX, ZW.
Clifford+T is a universal set of (unitary) gates for quantum computing, generated by $\{CNOT, X, H, T\}$.
Want an algebraic theory of reversible (unitary) quantum circuits.

- Lafont (2003): affine circuits in the classical case induces rules on $\{CNOT, X\}$,
- Selinger (2015): generators and relations for the Clifford groupoid $\{CNOT, X, H, T^2\}$,
- Amy et al. (2013, 2016): phase polynomials, T-count optimization $\{CNOT, T\}$,
- The present paper gives generators and relations for the CNOT-dihedral groupoid $\{CNOT, X, T\}$.

CNOT-dihedral gates

In the paper, the CNOT-dihedral operators are defined as follows.

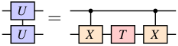
Definition 3.1. The *generators* are the scalar $\omega = e^{i\pi/4}$ and the gates X, T, and CNOT defined below.


$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$


$= \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$


$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Definition 3.2. The *derived generators* are the gates U and V defined below.



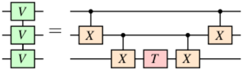
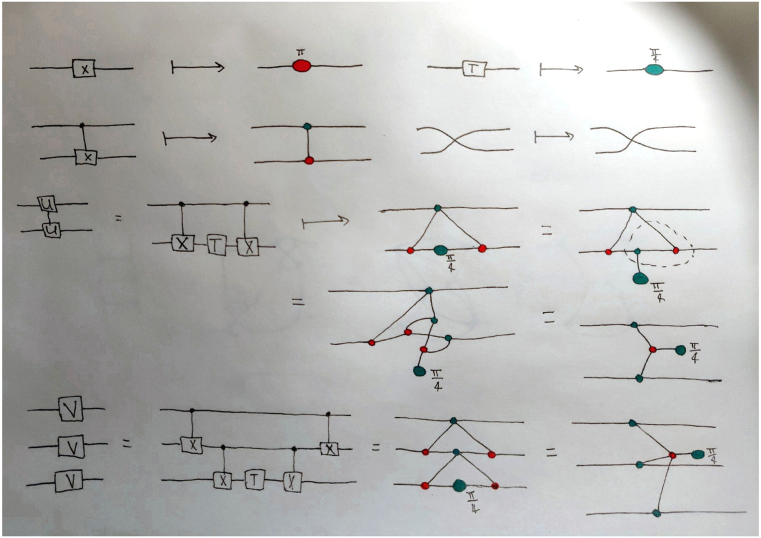


Figure: Amy, Chen and Ross, p. 86.

- The gates X, $CNOT$ and $SWAP$ are called *affine*.
- The gates ω, T, U and V are called *diagonal*.

CNOT-dihedral gates in ZX

The gates can be expressed in the ZX-calculus.



Rules for CNOT-dihedral gates

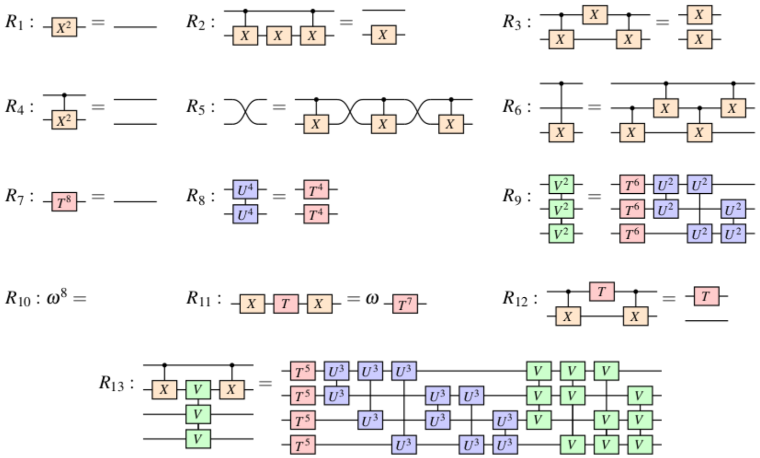
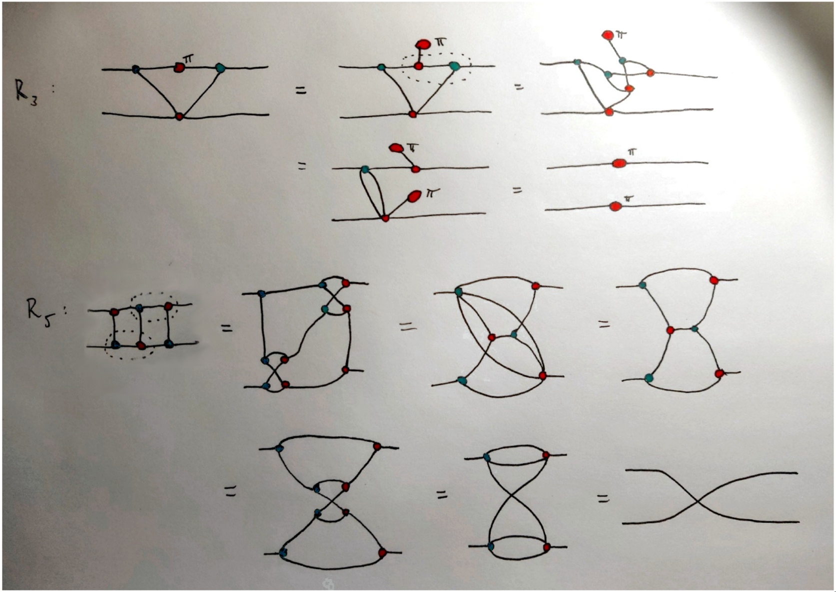


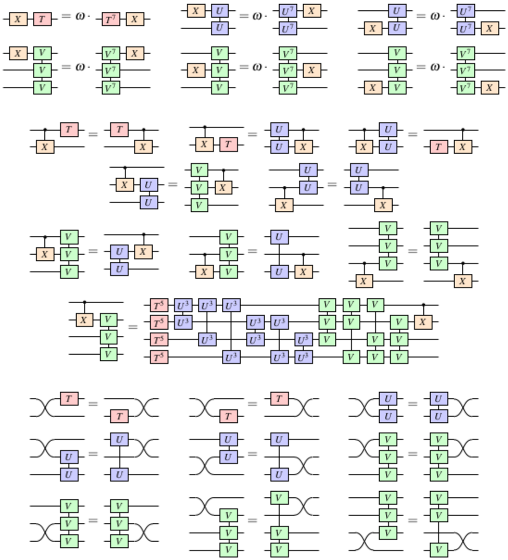
Figure 1: The relations. R_1 through R_6 are affine relations. R_7 through R_{10} are diagonal relations. R_{11} through R_{12} are commutation relations.

Figure: Amy, Chen and Ross, p. 87.

Derivation of R3 and R5 in ZX

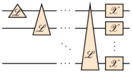


Commutation rules



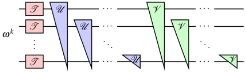
Normal forms

Definition 4.3. An *affine normal form* is a circuit A of the form



such that $\deg_X(A) \in \mathbb{Z}_2$ and $\deg_{\mathrm{CNOT}}(A) \in \mathbb{Z}_2$.

Definition 4.8. A *diagonal normal form* is a circuit D of the form



such that $k \in \mathbb{Z}_8$, $\deg_T(D) \in \mathbb{Z}_8$, $\deg_U(D) \in \mathbb{Z}_4$, and $\deg_V(D) \in \mathbb{Z}_2$.

- Degrees in the normal forms reflect the degree reduction rules.
- Every affine circuit admits a unique normal form (Lafont, 2003).
- Every diagonal circuit admits a normal form (Lemma 5.3.)
- Every CNOT-dihedral circuit C admits a normal form $C = DA$, where D is a diagonal circuit in normal form and A is an affine circuit in normal form.

Phase polynomials

Action of a diagonal gate:

$$D\,|x\rangle = \omega^{p_D(x)}\,|x\rangle\,, \quad p_D : \mathbb{Z}_2^n \to \mathbb{Z}_8,$$

$$\omega^k\,|x\rangle = \omega^k\,|x\rangle$$

$$T^k\,|x_1\rangle = \omega^{kx_1}\,|x_1\rangle$$

$$U^k\,|x_1x_2\rangle = \omega^{k(x_1\oplus x_2)}\,|x_1x_2\rangle$$

$$V^k\,|x_1x_2x_3\rangle = \omega^{k(x_1\oplus x_2\oplus x_3)}\,|x_1x_2x_3\rangle$$

Therefore for D in normal form we have

$$p_D(x) = a_0 + \sum_i a_i x_i + \sum_{i<j} b_{ij}(x_i \oplus x_j) + \sum_{i<j<k} c_{ijk}(x_i \oplus x_j \oplus x_k)$$

Where $a_i \in \mathbb{Z}_8$, $b_{ij} \in \mathbb{Z}_4$, $c_{ijk} \in \mathbb{Z}_2$ (from the bounds on $deg_X(D)$ for $X = T, U, V$ obtained in normal form).

Uniqueness

Suppose D and D' are distinct diagonal normal forms, we want to show that $\exists y \in \mathbb{Z}_2^n$ such that $D\,|y\rangle \neq D'\,|y\rangle$.
By construction $p_D(x) \neq p_{D'}(x)$ as polynomials. However this does not mean that $\exists y \in \mathbb{Z}_2^n$ s.t. $p_D(y) \neq p_{D'}(y)$.
Counterexample: $4x_1 + 4x_2 + 4(x_1 \oplus x_2) = 0 \bmod 8$ for any $x_1, x_2 \in \mathbb{Z}_2$
The existence of such a y comes from the bounds on the coefficients a_i, b_{ij} and c_{ijk}.
Construct a multilinear polynomial $q : \mathbb{Z}_8^n \to \mathbb{Z}_8$ such that $p_D(y) - p_{D'}(y) = q(y)\ \forall y \in \mathbb{Z}_2^n$.
Do this by translating from mod 2 to mod 8:

$$x_i \oplus x_j = x_i + x_j - 2x_ix_j$$

$$x_i \oplus x_j \oplus x_k = x_i + x_j + x_k - 2x_ix_j - 2x_ix_k - 2x_jx_k + 4x_ix_jx_k$$

Then $p_D(x) - p_{D'}(x) \neq 0 \implies q(x) \neq 0$, because of the bounds on a_i, b_{ij} and c_{ijk}. Pick a non-zero term $dx_{i_1} \ldots x_{i_k}$ in $q(x)$ and let y have 1's in i_jth positions and zero everywhere else. Then $q(y) = d \neq 0$ and so $D\,|y\rangle \neq D'\,|y\rangle$.

Open questions

- Interaction between ZX and CNOT-dihedral rules (rule 13?).
- Phase polynomials representations for graph rewriting.
- Algebraic vs combinatorial description: the paper doesn't contain an algorithm for normalizing CNOT-dihedral circuits, it uses the properties of the ambient symmetric monoidal structure non-constructively. What would be a rewrite system?
- Complexity of CNOT-dihedral circuits: word problem for CNOT-dihedral circuits? Classical simulation of the normal form?