INSTITUTO TECNOLÓGICO DE AERONÁUTICA
CONSELHO NACIONAL DE DESENVOLVIMENTO CIENTÍFICO E TECNOLÓGICO - CNPq

PROGRAMA INSTITUCIONAL DE BOLSAS DE INICIAÇÃO CIENTÍFICA - PIBIC

Coordenação do Programa de Atividades de Iniciação Científica e Tecnológica do ITA
caic@ita.br

Tailoring quantum emission through Purcell effect using plasmon-polaritons

Leonardo Antônio Lugarini

RELATÓRIO FINAL DE ATIVIDADES

Orientador: André J. C. Chaves
Coorientador: Diego Rabelo da Costa

Setembro / 2023

INSTITUTO TECNOLÓGICO DE AERONÁUTICA
CONSELHO NACIONAL DE DESENVOLVIMENTO CIENTÍFICO E TECNOLÓGICO - CNPq

CNPq
CONSELHO NACIONAL DE DESENVOLVIM
CIENTÍFICO E TECNOLÓGICO

PROGRAMA INSTITUCIONAL DE BOLSAS DE INICIAÇÃO CIENTÍFICA - PIBIC

Relatório Final de Atividades
TAILORING QUANTUM EMISSION THROUGH PURCELL EFFECT USING PLASMON-POLARITONS

São José dos Campos, 01 / 10 / 2023

Nome do aluno	Leonardo Antônio Lugarini
Assinatura do aluno	Documento assinado digitalmente gov.br LEONARDO ANTONIO LUGARINI Data: 01/10/2023 22:02:35-0300 Verifique em https://validar.iti.gov.br
Telefone do aluno	(41) 996592226
E-mail do aluno	lugaleonardo@gmail.com

Nome do orientador	André Jorge Carvalho Chaves
Assinatura do orientador	Documento assinado digitalmente gov.br ANDRE JORGE CARVALHO CHAVES Data: 01/10/2023 19:33:56-0300 Verifique em https://validar.iti.gov.br
Telefone do orientador	(12) 98194-9481
Email do orientador	andrej6@gmail.com

Nome do coorientador	Diego Rabelo da Costa
Assinatura do coorientador	Documento assinado digitalmente gov.br DIEGO RABELO DA COSTA Data: 01/10/2023 21:16:41-0300 Verifique em https://validar.iti.gov.br
Telefone do coorientador	(85) 33669930
Email do coorientador	diego_rabelo@fisica.ufc.br

INSTITUTO TECNOLÓGICO DE AERONÁUTICA
PROGRAMA INSTITUCIONAL DE BOLSAS DE INICIAÇÃO CIENTÍFICA - PIBIC
Formulário de Aprovação de Relatório pelo Orientador

Relatório: ☐ Rel. Parcial ☒ Rel. Final

1- CONSIDERO O RELATÓRIO APROVADO COM BASE NOS SEGUINTES ASPECTOS

O relatório mostra o extenso trabalho do aluno no tema proposto. Foi derivado em detalhe o cálculo do efeito Purcell e foi demonstrado sua aplicação em exemplos envolvendo folhas de grafeno. Esse relatório será utilizado por outros alunos do grupo de pesquisa, que irão começar a estudar emissores de fótons únicos.

2- APRECIAÇÕES DO ORIENTADOR SOBRE O DESEMPENHO DO BOLSISTA NA EXECUÇÃO DO TRABALHO DE INICIAÇÃO CIENTÍFICA

O Bolsista teve um desempenho excepcional. Seu trabalho já foi exibido no XXV Simpósio de Aplicações Operacionais em Áreas de Defesa. Emissores quânticos baseados em materiais bidimensionais tem enorme promessa de aplicação espacial, em satélites para distribuição quântica de chaves criptográficas, por terem alto desempenho, operarem a temperatura ambiente, e serem resistentes à radiação espacial. Por todos esses fatores, o trabalho do Leonardo, que será seguido no Trabalho de Graduação, atrai bastante interesse das Forças Armadas.

Local e data: 01/10/2023 – São Jose dos Campos

Assinatura do Orientador gov.br Documento assinado digitalmente
ANDRE JORGE CARVALHO CHAVES
Data: 01/10/2023 19:34:43-0300
Verifique em https://validar.iti.gov.br

Contents

0.1 Initial Plan Summary

In our quest to boost quantum communication efficiency, we're exploring the Purcell Effect's potential to reduce quantum emitters' spontaneous emission time. We'll use dyadic Green functions, essential for understanding electromagnetic fields from electric dipoles. This theory sheds light on interactions between quantum emitters and their environments, emphasizing interfaces like graphene over typical ones like metal. For computation, we'll use Python to numerically integrate the Purcell factor, assessing the emitter's coupling impact on decay rates. This supports our analytical foundation and sets the stage for future empirical tests.

0.2 Summary of realized activities

Our project embarked with an in-depth exploration into the dynamics of quantum emitters in proximity to surfaces, chiefly orchestrated through the utilization of dyadic Green functions and the computational capabilities of Python as outlined in sections 0.3.2 and 0.3.10. Central to our investigation were the reflection coefficients, elucidated in section 0.3.1, serving as pivotal mediators in the interaction between an emitter and its electromagnetic environment, particularly near surface materials like graphene.

The empirical journey involved the collection and analysis of data to scrutinize the variations in emitted power from a quantum emitter near a graphene surface, simulated over a range of frequencies and distances. The visualization of this data, as depicted in the Results section, underscored regions where the influence of graphene was most pronounced, offering a vivid illustration of the quantum emitter-surface interplay.

In summary, our systematic approach, traversing from dyadic Green functions to effective data visualization, was aimed at unraveling the nuanced interactions of quantum emitters with surfaces, spotlighting the coupling with reflection coefficients as the key conduits in these interactions.

0.3 Problem Description

Quantum communication marks a significant advancement in information transmission and protection, with qubits as the primary information units (GISIN; THEW, 2007). Transmission is achievable via optical fibers or free space, extending to satellite communications (SIMON, 2017). A key aspect of quantum communication is Quantum Key Distribution (QKD), one of the more mature quantum technologies (LO; CURTY; TAMAKI, 2014). The no-cloning theorem underpins robust QKD protocols like BB84 (BENNETT; BRASSARD, 1984), bolstering security against eavesdropping (GROSSHANS; GRANGIER, 2002). The rise of quantum computing threatens existing cryptographic methods, prompting ongoing quantum cryptography exploration (MAVROEIDIS et al., 2018). In military contexts, securing information is crucial, and quantum cryptography offers promise for secure key transmissions via free space (GISIN et al., 2002).

At the heart of quantum communication is the single-photon source, realized through quantum emitters. Enhancing the photon emission rate can be achieved by leveraging the Purcell effect. Surface plasmon-polaritons, hybrid modes of electromagnetic fields and matter (plasmons), manifest at interfaces between dielectric and conductive media (MARADUDIN; SAMBLES; BARNES, 2014) or in two-dimensional materials (GONÇALVES; PERES, 2016), augmenting light-matter interaction and boosting quantum emitter efficiency.

Plasmons, collective electronic excitations within a material (GIULIANI; VIGNALE, 2005), exhibit a defining frequency, ω_p, the plasmon frequency, upon a charge density shift. Resonating at this frequency maintains these excitations, representing the plasmonic collective mode. Under certain conditions, light's electric field can trigger this mode, leading to surface plasmon-polariton modes (MARADUDIN; SAMBLES; BARNES, 2014).

Graphene's potential lies in hosting surface plasmon-polariton modes with significantly shortened wavelengths compared to electromagnetic waves in a vacuum. This feature aids in amplifying quantum emitter decay rates, bridging quantum communication and two-dimensional material science towards new technological horizons.

This study delves into the interaction between quantum emitters and graphene's surface plasmon-polaritons, aiming for an enhanced photon emission rate essential for efficient quantum communication. The subsequent sections delve into plasmon-polariton modes in graphene, detailing the Purcell effect calculation in dipole systems, followed by results, conclusion, acknowledgments, and bibliography.

Upcoming subsections outline equations examining the link between surface reflection coefficients and the dipole's evanescent electric field. Our objective extends beyond enhancing photon emission rate to unveiling new quantum communication applications, contributing to the evolving quantum technologies domain.

To initiate our study on quantum emitters, it's imperative to revisit Macroscopic Maxwell's equations, setting the stage for tackling the Purcell Effect equation (NOVOTNY; HECHT, 2012):

$$\nabla \times E(r,t) = -\frac{\delta B(r,t)}{\delta t},\ \nabla \times H(r,t) = \frac{\delta D(r,t)}{\delta t} + j(r,t),\ \nabla \cdot D(r,t) = \rho(r,t),\ \nabla \cdot B(r,t) = 0. \tag{1a}$$

Inside these macroscopic equations is implicitly contained the conservation of charge, given when taking the divergence of Equation 1a, the development is as follows:

$$\nabla \cdot \nabla \times H(r,t) = \nabla \cdot (\frac{\delta D(r,t)}{\delta t} + j(r,t)) = 0, \tag{2a}$$

$$\nabla \cdot \frac{\delta D(r,t)}{\delta t} + \nabla \cdot j(r,t) = \frac{\delta \nabla \cdot D(r,t)}{\delta t} + \nabla \cdot j(r,t) = 0,\ \nabla \cdot j(r,t) + \frac{\delta \rho(r,t)}{\delta t} = 0. \tag{2b}$$

For macroscopic polarization $\boldsymbol{P}$ and magnetization $\boldsymbol{M}$ we have the following equations:

$$D(r,t) = \epsilon_0 E(r,t) + P(r,t),\ H(r,t) = \mu_0^{-1} B(r,t) - M(r,t). \tag{3a}$$

From these equations it is possible to obtain the wave equations for $E(r,t)$ and $H(r,t)$, as follows:

$$\nabla \times E(r,t) = -\frac{\delta B(r,t)}{\delta t} = -\mu_0 \frac{\delta(H(r,t) + M(r,t))}{\delta t}, \tag{4a}$$

$$\nabla \times \nabla \times E(r,t) = -\mu_0 \frac{\delta(\nabla \times H(r,t))}{\delta t} - \mu_0 \frac{\delta(\nabla \times M(r,t))}{\delta t}, \tag{4b}$$

below for the magnetic vector:

$$\nabla \times H(r,t) = \frac{\delta D(r,t)}{\delta t} + j(r,t) = \frac{\delta(\epsilon_0 E(r,t) + P(r,t))}{\delta t} + j(r,t), \tag{5a}$$

$$\nabla \times \nabla \times H(r,t) = \epsilon_0 \frac{\delta(\nabla \times E(r,t))}{\delta t} + \frac{\delta(\nabla \times P(r,t))}{\delta t} + \nabla \times j(r,t), \tag{5b}$$

using 5a in 4b and assuming $c = (\mu_0 \epsilon_0)^{-1/2}$:

$$\nabla \times \nabla \times E = -\mu_0 \frac{\delta(\frac{\delta(\epsilon_0 E + P)}{\delta t} + j)}{\delta t} - \mu_0 \frac{\delta(\nabla \times M)}{\delta t}, \tag{6a}$$

$$\nabla \times \nabla \times E + \frac{1}{c^2}\frac{\delta^2(E)}{\delta t^2} = -\mu_0 \frac{\delta(j + \frac{\delta P}{\delta t} + \nabla \times M)}{\delta t}, \tag{6b}$$

using 4a in 5b:

$$\nabla \times \nabla \times H = \epsilon_0 \frac{\delta(-\mu_0 \frac{\delta(H+M)}{\delta t})}{\delta t} + \frac{\delta(\nabla \times P)}{\delta t} + \nabla \times j, \tag{7a}$$

$$\nabla \times \nabla \times H + \frac{1}{c^2}\frac{\delta^2(H)}{\delta t^2} = \nabla \times j + \nabla \times \frac{\delta P}{\delta t} - \frac{1}{c^2}\frac{\delta^2 M}{\delta t^2}, \tag{7b}$$

Constitutive relations:

$$D = \epsilon_0 \epsilon E,\ P = \epsilon_0 \chi_e E,\ B = \mu_0 \mu H,\ M = \chi_m H,\ j_c = \sigma E. \tag{8a}$$

Spectral representation of time-dependant fields:

$$\hat{E}(r,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(r,t)e^{i\omega t}\, dt\ ,\ E(r,t) = \int_{-\infty}^{\infty} \hat{E}(r,\omega)e^{-i\omega t}\, dt\ . \tag{9a}$$

Applying the Fourier transform to the Maxwell's equations, first to 1a:

$$\nabla \times E(r,t) = -\frac{\delta B(r,t)}{\delta t},\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \nabla \times E(r,t)e^{i\omega t}\, dt\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{\delta B(r,t)}{\delta t}e^{i\omega t}\, dt\ , \tag{10a}$$

$$\nabla \times \hat{E}(r,\omega) = \frac{1}{2\pi}[-e^{i\omega t}B(r,t)\Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} B(r,t)e^{i\omega t}\, dt\ ],\ \nabla \times \hat{E}(r,\omega) = i\omega \hat{B}(r,\omega). \tag{10b}$$

Applying the transform to 1a:

$$\nabla \times H = \frac{\delta D}{\delta t} + j(r,t),\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \nabla \times He^{i\omega t}\, dt\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\frac{\delta D}{\delta t} + j(r,t))e^{i\omega t}\, dt\ , \tag{11a}$$

$$\nabla \times \hat{H}(r,\omega) = \frac{1}{2\pi}[\int_{-\infty}^{\infty} \frac{\delta D(r,t)}{\delta t}e^{i\omega t}\, dt\ + \int_{-\infty}^{\infty} j(r,t)e^{i\omega t}\, dt\ ], \tag{11b}$$

$$\nabla \times \hat{H}(\omega) = \frac{1}{2\pi}e^{i\omega t}D\Big|_{-\infty}^{\infty} - i\omega\frac{1}{2\pi}\int_{-\infty}^{\infty} De^{i\omega t}\, dt\ + \hat{j}(r,\omega)],\ \nabla \times \hat{H}(r,\omega) = -i\omega \hat{D}(r,\omega) + \hat{j}(r,\omega). \tag{11c}$$

Applying the transform to 1a:

$$\nabla \cdot D(r,t) = \rho(r,t),\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \nabla \cdot D(r,t)e^{i\omega t}\, dt\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(r,t)e^{i\omega t}\, dt\ ,\ \nabla \cdot \hat{D}(r,\omega) = \hat{\rho}(r,\omega). \tag{12a}$$

Applying the transform to 1a:

$$\nabla \cdot B(r,t) = 0,\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \nabla \cdot B(r,t)e^{i\omega t}\, dt\ = 0,\ \nabla \cdot \hat{B}(r,\omega) = 0. \tag{13a}$$

Therefore, we have the following Maxwell's transformed equations, obtained in the (r,ω) domain.

$$\nabla \times \hat{E}(r,\omega) = i\omega \hat{B}(r,\omega),\ \nabla \times \hat{H}(r,\omega) = -i\omega \hat{D}(r,\omega) + \hat{j}(r,\omega),\ \nabla \cdot \hat{D}(r,\omega) = \hat{\rho}(r,\omega),\ \nabla \cdot \hat{B}(r,\omega) = 0 \tag{14a}$$

Solving for complex dielectric constant, using 8a, 8a, 14a and 14a.

$$\nabla \times E(r) = i\omega B(r) = i\omega\mu_0\mu H(r),\ \mu^{-1}\nabla \times E(r) = i\omega\mu_0 H(r), \tag{15a}$$

$$\nabla \times \mu^{-1}\nabla \times E(r) = i\omega\mu_0 \nabla \times H(r). \tag{15b}$$

Using that $j = j_s + j_c$ and 8a:

$$\nabla \times H(r) = -i\omega D(r) + j(r) = -i\omega\epsilon_0\epsilon E(r) + j_s + j_c, \tag{16a}$$

$$\nabla \times H(r) = (-i\omega\epsilon_0\epsilon + \sigma)E(r) + j_s, \tag{16b}$$

$$\nabla \times \epsilon^{-1}\nabla \times H(r) = (-i\omega\epsilon_0 + \epsilon^{-1}\sigma)\nabla \times E(r) + \nabla \times \epsilon^{-1}j_s. \tag{16c}$$

Substituting 16b in 15b, knowing that $c = (\mu_0\epsilon_0)^{-1/2}$ and $k_0 = \frac{\omega}{c}$:

$$\nabla \times \mu^{-1}\nabla \times E(r) = i\omega\mu_0\nabla \times H(r) = i\omega\mu_0(-i\omega\epsilon_0\epsilon E(r) + \sigma E(r) + j_s), \tag{17a}$$

$$\nabla \times \mu^{-1}\nabla \times E(r) - \frac{\omega^2}{c^2}[\epsilon + i\frac{\sigma}{\omega\epsilon_0}]E(r) = \nabla \times \mu^{-1}\nabla \times E(r) - k_0^2\boldsymbol{\epsilon}E(r) = i\omega\mu_0 j_s. \tag{17b}$$

Substituting 15a in 16c:

$$\nabla \times \epsilon^{-1}\nabla \times H(r) - \nabla \times \epsilon^{-1}j_s = (-i\omega\epsilon_0 + \epsilon^{-1}\sigma)(i\omega\mu_0\mu H(r)) + = \omega^2\mu_0\epsilon_0[\mu + i\frac{\epsilon^{-1}\mu}{\omega\epsilon_0}]H(r), \tag{18a}$$

$$\nabla \times \epsilon^{-1}\nabla \times H(r) - k_0^2\boldsymbol{\mu}H(r) = \nabla \times \epsilon^{-1}j_s. \tag{18b}$$

For a interface between two homogeneous domains D_i and D_j, using 17b,18b, 8a, 14a and the identity $\nabla \times \nabla \times = -\nabla^2 + \nabla(\nabla \cdot)$ the wave equations in the domain are as follows, first for the electric field:

$$\nabla \times \nabla \times E_i - k_0^2 \mu_i \epsilon_i E_i = i\omega\mu_0\mu_i j_i, \tag{19a}$$

$$-\nabla^2 E_i + \nabla(\nabla \cdot E_i) - k_i^2 E_i = -\nabla^2 E_i + \nabla(\frac{\nabla \cdot D_i}{\epsilon_0 \epsilon_i}) - k_i^2 E_i = i\omega\mu_0\mu_i j_i, \tag{19b}$$

$$-\nabla^2 E_i + \frac{\nabla \cdot \rho_i}{\epsilon_0 \epsilon_i} - k_i^2 E_i = i\omega\mu_0\mu_i j_i,\ (\nabla^2 + k_i^2) E_i = -i\omega\mu_0\mu_i j_i + \frac{\nabla \cdot \rho_i}{\epsilon_0 \epsilon_i}. \tag{19c}$$

Now for the magnetic field, using 8a and 14a:

$$\nabla \times \nabla \times H_i - k_0^2 \epsilon_i \mu_i H_i = -\nabla^2 H_i + \nabla(\frac{\nabla B_i}{\mu_i \mu_0}) - k_i^2 H_i = \nabla \times j_i,\ (\nabla^2 + k_i^2) H_i = -\nabla \times j_i. \tag{20a}$$

0.3.1 Fresnel reflection and transmission coefficients

For the Fresnel reflection and transmission coefficients (NOVOTNY; HECHT, 2012), we consider an arbitrarily polarized plane wave $E_1 \exp(k_1 \cdot r - i\omega t)$ and the two planes related to the wave, a perpendicular (p) and a parallel (s), so, the resultant wave form is the sum of (s) and (p). Considering the dielectric constants of the medium of incidence and transmittance as ϵ_1 and ϵ_2, and also the magnetic permeability for both mediuns (μ_i). Therefore, we can consider the wavevectors ($\boldsymbol{k}$) as $|\boldsymbol{k_i}| = \frac{\omega}{c}\sqrt{\epsilon_i \mu_i}$, $\boldsymbol{k_i} = (k_x, k_y, k_z)$.

From the translational symmetry along the transverse axis, it is know that the transverse components of the wave vector are conserved:

$$k_{z_i} = \sqrt{(k_i)^2 - ((k_x)^2 + (k_y)^2)},\ k_{transverse} = k_{||} = \sqrt{(k_x)^2 + (k_y)^2)} = k_1 \sin(\theta_1). \tag{21a}$$

The amplitudes of the reflected and transmitted polarized waves are as follow:

$$E_{1r}^{(s)} = E_1^{(s)} r^s(k_x, k_y),\ E_{1r}^{(p)} = E_1^{(p)} r^p(k_x, k_y),\ E_2^{(s)} = E_1^{(s)} t^s(k_x, k_y),\ E_2^{(p)} = E_1^{(p)} t^p(k_x, k_y), \tag{22a}$$

assuming that θ_i is the incidence angle, that the reflection angle is given by $\theta_r = \pi - \theta_i$, and that the transmitted angle is θ_t. Using the boundaries conditions at the interface (NOVOTNY; HECHT, 2012), for E and H related as follows, $|H| = \sqrt{\frac{\epsilon}{\mu}}|E|$, we have:

$$E_1^{(s)} + E_{1r}^{(s)} = E_2^{(s)},\ H_1^{(s)} \cos\theta_i + H_{1r}^{(s)} \cos\theta_r = H_2^{(s)} \cos\theta_t, \tag{23a}$$

$$H_1^{(p)} + H_{1r}^{(p)} = H_2^{(p)},\ E_1^{(p)} \cos\theta_i + E_{1r}^{(p)} \cos\theta_r = E_2^{(p)} \cos\theta_t. \tag{23b}$$

Using the relations for $|H|$ and θ_r, and using the definition that $E_{1r}^{(p)}$ is reflected to the negative direction that the inflected field:

$$E_1^{(s)} + E_{1r}^{(s)} = E_2^{(s)},\ \sqrt{\epsilon_1 \mu_2}(E_1^{(s)} \cos\theta_i - E_{1r}^{(s)} \cos\theta_i) = \sqrt{\epsilon_2 \mu_1}(E_2^{(s)} \cos\theta_t), \tag{24a}$$

$$\sqrt{\epsilon_1 \mu_2}(E_1^{(p)} + E_{1r}^{(p)}) = \sqrt{\epsilon_2 \mu_1}(E_2^{(p)}),\ E_1^{(p)} \cos\theta_i - E_{1r}^{(p)} \cos\theta_i = E_2^{(p)} \cos\theta_t. \tag{24b}$$

Using the relations for amplitudes:

$$1 + r^s = t^s,\ \sqrt{\epsilon_1 \mu_2}(1 - r^s) \cos\theta_i = \sqrt{\epsilon_2 \mu_1}(t^s) \cos\theta_t, \tag{25a}$$

$$\sqrt{\epsilon_1 \mu_2}(1 + r^p) = \sqrt{\epsilon_2 \mu_1}(t^p),\ (1 - r^p) \cos\theta_i = t^p \cos\theta_t. \tag{25b}$$

Solving for (s) first, knowing that $k_{z_{1,2}} = k_{1,2} \cos(\theta_{i,t})$ and $k_i = \frac{\omega}{c}\sqrt{\epsilon_i \mu_i}$:

$$\sqrt{\epsilon_1 \mu_2}(1 - r^s)\frac{k_{z_1}}{k_1} = \sqrt{\epsilon_2 \mu_1}(1 + r^s)\frac{k_{z_2}}{k_2},\ \sqrt{\epsilon_1 \mu_2}(1 - r^s) k_{z_1} k_2 = \sqrt{\epsilon_2 \mu_1}(1 + r^s) k_{z_2} k_1, \tag{26a}$$

$$\sqrt{\epsilon_1 \mu_2}(1 - r^s) k_{z_1} \sqrt{\epsilon_2 \mu_2} = \sqrt{\epsilon_2 \mu_1}(1 + r^s) k_{z_2} \sqrt{\epsilon_1 \mu_1},\ \mu_2(1 - r^s) k_{z_1} = \mu_1(1 + r^s) k_{z_2}, \tag{26b}$$

$$\mu_2 k_{z_1} - \mu_1 k_{z_2} = r^s(\mu_2 k_{z_1} + \mu_1 k_{z_2}),\ r^s = \frac{\mu_2 k_{z_1} - \mu_1 k_{z_2}}{\mu_2 k_{z_1} + \mu_1 k_{z_2}}. \tag{26c}$$

For t^s:

$$t^s = 1 + r^s \ , t^s = 1 + \frac{\mu_2 k_{z_1} - \mu_1 k_{z_2}}{\mu_2 k_{z_1} + \mu_1 k_{z_2}}, \ t^s = \frac{2\mu_2 k_{z_1}}{\mu_2 k_{z_1} + \mu_1 k_{z_2}}. \tag{27a}$$

Now, solving for (p):

$$(1 - r^p)\frac{k_{z_1}}{k_1} = t^p \frac{k_{z_2}}{k_2}, \ (1 - r^p) k_{z_1}\sqrt{\epsilon_2 \mu_2} = t^p k_{z_2}\sqrt{\epsilon_1 \mu_1}. \tag{28a}$$

Using Eq. 25b and 28a:

$$\sqrt{\epsilon_1 \mu_2}(1 + r^p) = \sqrt{\epsilon_2 \mu_1}(\frac{(1 - r^p) k_{z_1}\sqrt{\epsilon_2 \mu_2}}{k_{z_2}\sqrt{\epsilon_1 \mu_1}}), \ \epsilon_1 (1 + r^p) k_{z_2} = \epsilon_2 (1 - r^p) k_{z_1}, \ r^p = \frac{\epsilon_2 k_{z_1} - \epsilon_1 k_{z_2}}{\epsilon_2 k_{z_1} + \epsilon_1 k_{z_2}}. \tag{29a}$$

For t^p:

$$\sqrt{\epsilon_2 \mu_1}(t^p) = \sqrt{\epsilon_1 \mu_2}(1 + r^p), \ t^p = (1 + (\frac{\epsilon_2 k_{z_1} - \epsilon_1 k_{z_2}}{\epsilon_2 k_{z_1} + \epsilon_1 k_{z_2}}))\sqrt{\frac{\epsilon_1 \mu_2}{\epsilon_2 \mu_1}} = (\frac{2\epsilon_2 k_{z_1}}{\epsilon_2 k_{z_1} + \epsilon_1 k_{z_2}})\sqrt{\frac{\epsilon_1 \mu_2}{\epsilon_2 \mu_1}}. \tag{30a}$$

Therefore, the Fresnel reflection and transmission coefficients are given by:

$$r^s(k_x, k_y) = \frac{\mu_2 k_{z1} - \mu_1 k_{z2}}{\mu_2 k_{z1} + \mu_1 k_{z2}}, \ t^s(k_x, k_y) = \frac{2\mu_2 k_{z1}}{\mu_2 k_{z1} + \mu_1 k_{z2}}, \tag{31a}$$

$$r^p(k_x, k_y) = \frac{\epsilon_2 k_{z1} - \epsilon_1 k_{z2}}{\epsilon_2 k_{z1} + \epsilon_1 k_{z2}}, \ t^p(k_x, k_y) = \frac{2\epsilon_2 k_{z1}}{\epsilon_2 k_{z1} + \epsilon_1 k_{z2}}\sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \epsilon_2}}. \tag{31b}$$

0.3.2 Dyadic Green's Function

Next, the Dyadic Green's functions (NOVOTNY; HECHT, 2012) are introduced to better understand the behavior of the electromagnetic fields due to a point source. So, at first, we star with a equation of a a linear operator acting on a vector field $\boldsymbol{A}$:

$$\mathcal{D} A(r) = B(r). \tag{32a}$$

Allowing a particular solution to exist, we consider the inhomogeneity of a Dirac function :

$$\mathcal{D}\ G_i(r, r') = n_i\ \delta(r - r'), \ \ (i = x, y, z), \ \mathcal{D}\ \overleftrightarrow{G}(r, r') = \overleftrightarrow{I}\ \delta(r - r'), \tag{33a}$$

$$\mathcal{D}\ \overleftrightarrow{G}(r, r')\ B(r') = B(r')\ \delta(r - r'), \ \int_V \mathcal{D}\ \overleftrightarrow{G}(r, r')\ B(r')\, dV' \ = \int_V B(r')\ \delta(r - r')\, dV' \ , \tag{33b}$$

$$B = \int_V \mathcal{D}\ \overleftrightarrow{G}(r, r')\ B(r')\, dV' \ , \ \mathcal{D}\ A = \int_V \mathcal{D}\ \overleftrightarrow{G}(r, r')\ B(r')\, dV' \ , \ A = \int_V \overleftrightarrow{G}(r, r')\ B(r')\, dV' \ . \tag{33c}$$

Deriving the Green function for the electric field considering the time harmonic vector potential A and the scalar potential ϕ :

$$E(r) = i\omega A(r) - \nabla\phi(r), \ H(r) = \frac{1}{\mu_0 \mu}\nabla \times A(r), \ \nabla \times \hat{H}(r, \omega) = -i\omega \hat{D}(r, \omega) + \hat{j}(r, \omega), \tag{34a}$$

$$\nabla \times \nabla \times A(r) = -i\omega\mu_0\mu\epsilon_0\epsilon E(r) + \mu_0\mu j(r) = \mu_0\mu j(r) - i\omega\mu_0\mu\epsilon_0\epsilon(i\omega A(r) - \nabla\phi(r)), \tag{34b}$$

$$-\nabla^2 A(r) + \nabla(\nabla \cdot A(r)) = \mu_0\mu j(r) + \omega^2\mu_0\mu\epsilon_0\epsilon A(r) + i\omega\mu_0\mu\epsilon_0\epsilon\nabla\phi(r). \tag{34c}$$

Setting the gauge condition as :

$$\nabla \cdot A(r) = i\omega\mu_0\mu\epsilon_0\epsilon\phi(r). \tag{35a}$$

Therefore, the former equation can be written (g(r) being a source and G_0 the Green function for the operator) as follows:

$$(\nabla^2 + \mu_0\mu\epsilon_0\epsilon\ \omega^2)A(r) = (\nabla^2 + k^2)A(r) = -\mu_0\mu j(r), \tag{36a}$$

$$(\nabla^2 + k^2) f(r) = -g(r), \ (\nabla^2 + k^2) G_0(r, r') = -\delta(r - r'). \tag{36b}$$

Solving the equation, starting with Equation 36a:

$$(\nabla^2 + k^2)A(r) = -\mu_0\mu j(r),\ (\nabla^2 + k^2)G_0(r,r')\ A(r) = -\mu_0\mu j(r)\ G_0(r,r'), \quad (37a)$$

$$\int_V (\nabla^2 + k^2)G_0(r,r')\ A(r')\,dV' \ = \int_V (-\delta(r-r'))\ A(r')\,dV' \ = -\mu_0\mu \int_V j(r')\ G_0(r,r')\,dV'\ , \quad (37b)$$

$$A(r) = \mu_0\mu \int_V j(r')\ G_0(r,r')\,dV'\ . \quad (37c)$$

To solve for the Green function, the Fourier transform will be used:

$$\mathcal{F}((\nabla^2 + k^2)G_0(r,r')) = \mathcal{F}(-\delta(r-r'))\ ,\ G_0(p)(k^2-p^2) = -e^{-ip\cdot r'}\ ,\ G_0(p) = \frac{e^{-ip\cdot r'}}{(p^2-k^2)}\ , \quad (38a)$$

$$G_0(r) = \frac{1}{(2\pi)^3}\iiint_V \frac{e^{-ip\cdot r'}}{(p^2-k^2)}e^{ip\cdot r}\,d^3p \ = \frac{1}{(2\pi)^3}\iiint \frac{1}{(p^2-k^2)}e^{ip\cdot(r-r')}p^2\sin\theta\,dpd\theta d\phi\ , \quad (38b)$$

using $k^2 = z$, $r - r' = r$, $p \cdot r = pru$ and $u = \cos\theta$, changing to spherical coordinates:

$$G_0(r) = \frac{1}{(2\pi)^2}\iint \frac{1}{(p^2-z)}e^{ipr\cos\theta}p^2\sin\theta\,dpd\theta \ = \frac{1}{(2\pi)^2}\iint \frac{p^2}{(z-p^2)}e^{ipru}\,dudp\ , \quad (39a)$$

$$G_0(r) = \frac{1}{(2\pi)^2}\int \frac{p^2}{(z-p^2)}\frac{(e^{ipr}-e^{-ipr})}{ipr}\,dp \ = \frac{1}{(2\pi)^2 ir}\int_0^\infty \frac{p}{(z-p^2)}(e^{ipr}-e^{-ipr})\,dp\ , \quad (39b)$$

it is noticeable that the function is even for p :

$$G_0(r) = \frac{1}{2(2\pi)^2 ir}\int_{-\infty}^\infty \frac{p}{(z-p^2)}(e^{ipr}-e^{-ipr})\,dp\ . \quad (40a)$$

For the two poles, $p = \pm\sqrt{z}$, it becomes necessary to analyze the divergence of the exponential for a complex value of p, $p = Re(p) + iIm(p)$, assuming that $r > 0$:

$$\exp(\pm ipr) = \exp(i(Re(p) + iIm(p))r) = \exp(\pm(-Im(p))r)\ \exp(\pm iRe(p)r). \quad (41a)$$

Therefore, for $\exp(ipr)$ the imaginary part of p has to be positive, $Im(p) > 0$. And, for $\exp(-ipr)$, it is necessary that $Im(p) < 0$. Now, to use Cauchy's Residual Theorem, first we suppose that $\sqrt{z}$ has a positive imaginary part, so that $\exp(ipr)$ converges for $\sqrt{z}$ and $\exp(-ipr)$ converges for $-\sqrt{z}$.

$$G_0 = \frac{1}{2(2\pi)^2 ir}\int_{-\infty}^\infty \frac{p}{(z-p^2)}(e^{ipr}-e^{-ipr})\,dp \ = \frac{1}{2(2\pi)^2 ir}\int_{-\infty}^\infty \frac{p\exp(ipr)}{(z-p^2)} - \frac{p\exp(-ipr)}{(z-p^2)}\,dp \ = \quad (42a)$$

$$= \frac{2\pi i}{2(2\pi)^2 ir}[(\frac{\sqrt{z}\exp(i\sqrt{z}r)}{(\sqrt{z}+(\sqrt{z}))}) - (\frac{-\sqrt{z}\exp(-i(-\sqrt{z})r)}{(\sqrt{z}-(-\sqrt{z}))})] = \frac{\sqrt{z}}{4\pi r}[(\frac{\exp(i\sqrt{z}r)}{(2\sqrt{z})}) + (\frac{\exp(i(\sqrt{z})r)}{(2\sqrt{z})})] \quad (42b)$$

$$G_0(r) = \frac{1}{4\pi r}\exp(i\sqrt{z}r) \quad (42c)$$

By choosing an negative imaginary part for $\sqrt{z}$ the result is analogous, remembering the convention for clockwise motion in the imaginary field.

$$G_0(r) = \frac{1}{2(2\pi)^2 ir}\int_{-\infty}^\infty \frac{p}{(z-p^2)}(e^{ipr}-e^{-ipr})\,dp \ = \frac{1}{2(2\pi)^2 ir}\int_{-\infty}^\infty \frac{p\exp(ipr)}{(z-p^2)} - \frac{p\exp(-ipr)}{(z-p^2)}\,dp\ , \quad (43a)$$

$$G_0(r) = \frac{2\pi i}{2(2\pi)^2 ir}[-(\frac{-\sqrt{z}\exp(i(-\sqrt{z})r)}{(\sqrt{z}-(-\sqrt{z}))}) + (\frac{\sqrt{z}\exp(-i(\sqrt{z})r)}{(\sqrt{z}+(\sqrt{z}))})], \quad (43b)$$

$$G_0(r) = \frac{1}{4\pi r}[(\frac{\sqrt{z}\exp(-i\sqrt{z}r)}{(2\sqrt{z})}) + (\frac{\sqrt{z}\exp(-i(\sqrt{z})r)}{(2\sqrt{z})})] = \frac{1}{4\pi r}\exp(-i\sqrt{z}r). \quad (43c)$$

So, we have two solutions for the Green function, assuming again that $z = k^2$ and $r > 0 \to r = |r-r'|$:

$$G_0(r,r') = \frac{e^{\pm ik|r-r'|}}{4\pi|r-r'|}. \quad (44a)$$

Now, for the Dyadic Green function of electric and magnetic fields is necessary to start with Eq. (19a) and $i = (x, y, z)$:

$$\nabla \times \nabla \times E(r) - k^2 E(r) = i\omega\mu_0\mu j(r), \quad (45a)$$

$$\nabla \times \nabla \times G_i(r, r') - k^2 G_i(r, r') = \delta(r - r')\mathbf{n}_i, \ \nabla \times \nabla \times \overleftrightarrow{G}(r, r') - k^2 \overleftrightarrow{G}(r, r') = \overleftrightarrow{I}\delta(r - r'). \quad (45b)$$

The particular and homogeneous solutions, respectively, of the electric field are given by:

$$E_p(r) = i\omega\mu\mu_0 \int_V \overleftrightarrow{G}(r, r')j(r')\, dV' \ , \ E_h(r) = E_0(r) + i\omega\mu\mu_0 \int_V \overleftrightarrow{G}(r, r')j(r')\, dV' \ , \ (r \notin V). \quad (46a)$$

For the corresponding magnetic field:

$$E = E_0 + i\omega\mu\mu_0 \int_V \overleftrightarrow{G}(r, r')j(r')\, dV' \ , \ \nabla \times E = \nabla \times E_0 + i\omega\mu\mu_0 \nabla \times \int_V \overleftrightarrow{G}(r, r')j(r')\, dV' \ , (47a)$$

$$i\omega B = i\omega B_0 + i\omega\mu\mu_0 \int_V [\nabla \times \overleftrightarrow{G}(r, r')]j(r')\, dV' \ , \ i\omega\mu_0\mu H = i\omega\mu_0\mu H_0 + i\omega\mu\mu_0 \int_V [\nabla \times \overleftrightarrow{G}]j\, dV' \ , (47b)$$

$$H(r) = H_0(r) + \int_V [\nabla \times \overleftrightarrow{G}(r, r')]j(r')\, dV' \ , (r \notin V). (47c)$$

Using (35a) in (34a), for the gauge condition:

$$E(r) = i\omega A(r) - \nabla\phi(r) = i\omega A(r) - \frac{w}{ik^2}\nabla\nabla A(r) = i\omega[1 + \frac{1}{k^2}\nabla\nabla\cdot]A(r). \quad (48a)$$

For a point source $j(r) = (i\omega\mu_0\mu)^{-1}\delta(r - r')n_x$ (NOVOTNY; HECHT, 2012) into Eq. (37c):

$$A(r) = \mu_0\mu \int_V j(r')\ G_0(r, r')\, dV' \ = \mu_0\mu \int (i\omega\mu_0\mu)^{-1}\delta(r - r')n_x\ G_0(r, r')\, dV' \ , \quad (49a)$$

$$A(r) = \mu_0\mu(i\omega\mu_0\mu)^{-1} \int \delta(r - r')\ G_0(r, r')n_x\, dV' \ = (i\omega)^{-1}G_0(r, r')n_x, \quad (49b)$$

using (34a):

$$E = i\omega[1 + \frac{1}{k^2}\nabla\nabla\cdot]\ A = i\omega[1 + \frac{1}{k^2}\nabla\nabla\cdot]\ (i\omega)^{-1}G_0(r, r')n_x,\ G_x(r, r') = [1 + \frac{1}{k^2}\nabla\nabla\cdot]\ G_0(r, r')n_x, \ (50a)$$

with similar results for G_y and G_z, the dyadic Green's function is:

$$\overleftrightarrow{G}(r, r') = [\overleftrightarrow{I} + \frac{1}{k^2}\nabla\nabla]\ G_0(r, r'). \quad (51a)$$

0.3.3 Electric dipole fields in a homogeneous space

For the Green function derived we have:

$$E(r) = \omega^2\mu\mu_0 \overleftrightarrow{G}_0(r, r_0)\boldsymbol{\mu},\ H(r) = -i\omega[\nabla \times \overleftrightarrow{G}_0(r, r_0)]\boldsymbol{\mu}. \quad (52a)$$

Given (44a) and (51a) we define $\boldsymbol{R} = \vec{r} - \vec{r_0}$ and $R = |r - r_0|$, so:

$$\overleftrightarrow{G}(r, r_0) = [\overleftrightarrow{I} + \frac{1}{k^2}\nabla\nabla]\frac{\exp(ikR)}{4\pi R} = \frac{\exp(ikR)}{4\pi R}\overleftrightarrow{I} + \frac{1}{k^2}\nabla\nabla(\frac{\exp(ikR)}{4\pi R}), \quad (53a)$$

$$\overleftrightarrow{G} = \frac{\exp(ikR)}{4\pi R}\overleftrightarrow{I} + \frac{1}{k^2}\nabla(\frac{ik\nabla(R)\exp(ikR)4\pi R - \exp(ikR)4\pi\nabla(R)}{(4\pi R)^2}). \quad (53b)$$

For the value of $\nabla(R)$:

$$\nabla(R) = \nabla(|r - r_0|) = \nabla(\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}), \quad (54a)$$

$$\nabla(R) = (\frac{(x - x_0)}{R}, \frac{(y - y_0)}{R}, \frac{(z - z_0)}{R}) = \frac{\vec{r} - \vec{r_0}}{R} = \frac{\boldsymbol{R}}{R}. \quad (54b)$$

For the next step we can derive the value of $\nabla(\boldsymbol{R})$:

$$\nabla(\boldsymbol{R}) = \nabla[(x-x_0),(y-y_0),(z-z_0)] = [(1,0,0);(0,1,0);(0,0,1)] = \overleftrightarrow{I}. \tag{55a}$$

So, defining $\boldsymbol{RR}$ as the outer product of $\boldsymbol{R}$ with itself, we have:

$$\overleftrightarrow{G} = \frac{\exp(ikR)}{4\pi R}\overleftrightarrow{I} + \frac{1}{k^2}\nabla(\frac{(ikR-1)\exp(ikR)\boldsymbol{R}}{4\pi(R)^3}), \tag{56a}$$

$$\overleftrightarrow{G} = \frac{\exp(ikR)}{4\pi R}\overleftrightarrow{I} + \exp(ikR)\frac{((ikR-1)\nabla(\boldsymbol{R}) - k^2R\nabla(R)\boldsymbol{R})4\pi R^3}{k^2(4\pi(R)^3)^2} - \frac{(ikR-1)\boldsymbol{R}(12\pi R^2)\nabla(R)}{k^2(4\pi(R)^3)^2}, \tag{56b}$$

$$\overleftrightarrow{G} = \frac{\exp(ikR)}{4\pi R}\overleftrightarrow{I} + \exp(ikR)\frac{((ikR-1)\overleftrightarrow{I} - k^2\boldsymbol{RR})R^3 - (3ikR^2-3R)\boldsymbol{RR}}{k^2 4\pi R^6}, \tag{56c}$$

$$\overleftrightarrow{G} = \frac{\exp(ikR)}{4\pi R}((1+\frac{(ikR-1)}{k^2R^2})\overleftrightarrow{I} + \frac{(3-3ikR-k^2R^2)}{k^2R^2}\frac{\boldsymbol{RR}}{R^2}). \tag{56d}$$

For the cross product:

$$\nabla\times\overleftrightarrow{G} = \nabla\times([\overleftrightarrow{I} + \frac{1}{k^2}\nabla\nabla]\frac{\exp(ikR)}{4\pi R}) = \nabla\times(\frac{\exp(ikR)}{4\pi R}\overleftrightarrow{I}) + \nabla\times(\frac{1}{k^2}\nabla\nabla\frac{\exp(ikR)}{4\pi R}). \tag{57a}$$

Considering $\nabla(\frac{\exp(ikR)}{4\pi R}) = \Omega(R)$, we have:

$$\nabla\times\overleftrightarrow{G} = (\nabla(\frac{\exp(ikR)}{4\pi R}))\times\overleftrightarrow{I} + \frac{1}{k^2}\nabla\times\nabla(\Omega(R))). \tag{58a}$$

Knowing that the curl of a gradient is zero, we have:

$$\nabla\times\overleftrightarrow{G} = (\nabla(\frac{\exp(ikR)}{4\pi R}))\times\overleftrightarrow{I} = (\frac{(ikR-1)\exp(ikR)\boldsymbol{R}}{4\pi R^3})\times\overleftrightarrow{I} = \frac{\exp(ikR)}{4\pi R}\frac{k(\boldsymbol{R}\times\overleftrightarrow{I})}{R}(i-\frac{1}{kR}). \tag{59a}$$

For these Green functions we have terms affected by the magnitude of the wave number, separating terms for near field(NF), intermediate field (IF) and far field(FF):

$$\overleftrightarrow{G} = \overleftrightarrow{G}_{NF} + \overleftrightarrow{G}_{IF} + \overleftrightarrow{G}_{FF}, \tag{60a}$$

where we have, for the near field ($R \ll \lambda$, $(kR)^{-3}$ dominant term) :

$$\overleftrightarrow{G}_{NF} = \frac{\exp(ikR)}{4\pi}[(\frac{1}{R} + \frac{i}{kR^2} - \frac{1}{k^2R^3})\overleftrightarrow{I} + (\frac{3}{k^2R^3} - \frac{3i}{kR^2} - \frac{1}{R})\frac{\boldsymbol{RR}}{R^2}], \tag{61a}$$

$$\overleftrightarrow{G}_{NF} = \frac{\exp(ikR)}{4\pi}[(-\frac{1}{k^2R^3})\overleftrightarrow{I} + (\frac{3}{k^2R^3})\frac{\boldsymbol{RR}}{R^2}] = \frac{\exp(ikR)}{4\pi R}\frac{1}{k^2R^2}[-\overleftrightarrow{I} + \frac{3\boldsymbol{RR}}{R^2}], \tag{61b}$$

for the intermediate field ($R \approx \lambda$, $(kR)^{-2}$ dominant term) :

$$\overleftrightarrow{G}_{IF} = \frac{\exp(ikR)}{4\pi}[(\frac{1}{R} + \frac{i}{kR^2} - \frac{1}{k^2R^3})\overleftrightarrow{I} + (\frac{3}{k^2R^3} - \frac{3i}{kR^2} - \frac{1}{R})\frac{\boldsymbol{RR}}{R^2}], \tag{62a}$$

$$\overleftrightarrow{G}_{IF} = \frac{\exp(ikR)}{4\pi}[(\frac{i}{kR^2})\overleftrightarrow{I} + (-\frac{3i}{kR^2})\frac{\boldsymbol{RR}}{R^2}] = \frac{\exp(ikR)}{4\pi R}(\frac{i}{kR})[\overleftrightarrow{I} - \frac{3\boldsymbol{RR}}{R^2}], \tag{62b}$$

and, for the far field ($R \gg \lambda$, $(kR)^{-1}$ dominant term) :

$$\overleftrightarrow{G}_{FF} = \frac{\exp(ikR)}{4\pi}[(\frac{1}{R} + \frac{i}{kR^2} - \frac{1}{k^2R^3})\overleftrightarrow{I} + (\frac{3}{k^2R^3} - \frac{3i}{kR^2} - \frac{1}{R})\frac{\boldsymbol{RR}}{R^2}], \tag{63a}$$

$$\overleftrightarrow{G}_{FF} = \frac{\exp(ikR)}{4\pi}[(\frac{1}{R})\overleftrightarrow{I} + (-\frac{1}{R})\frac{\boldsymbol{RR}}{R^2}] = \frac{\exp(ikR)}{4\pi R}[\overleftrightarrow{I} - \frac{\boldsymbol{RR}}{R^2}]. \tag{63b}$$

Now, for the dipole in the z direction ($\boldsymbol{\mu} = |\boldsymbol{\mu}|\cdot\boldsymbol{n_z}$)

$$E(r) = |\boldsymbol{\mu}|\omega^2\mu\mu_0\frac{\exp(ikR)}{4\pi R}\begin{bmatrix} xz\frac{(3-3ikR-k^2R^2)}{k^2R^4} \\ yz\frac{(3-3ikR-k^2R^2)}{k^2R^4} \\ (1+\frac{(ikR-1)}{k^2R^2}) + zz\frac{(3-3ikR-k^2R^2)}{k^2R^4}\end{bmatrix}, \tag{64a}$$

$$E(r) = |\boldsymbol{\mu}|\omega^2\mu\mu_0\frac{\exp(ikR)}{4\pi R}(\begin{bmatrix} 0 \\ 0 \\ 1+\frac{(ikR-1)}{k^2R^2}\end{bmatrix} + z\frac{(3-3ikR-k^2R^2)}{k^2R^4}\begin{bmatrix} x \\ y \\ z\end{bmatrix}). \tag{64b}$$

For future use we consider E_z as follows:

$$E_z = |\boldsymbol{\mu}|\frac{\exp(ikr)}{4\pi\epsilon_0\epsilon r}k^2(1+\frac{(ikr-1)}{k^2r^2}+\cos^2\upsilon\frac{(3-3ikr-k^2r^2)}{k^2r^2}), \tag{65a}$$

$$E_z = |\boldsymbol{\mu}|\frac{\exp(ikr)}{4\pi\epsilon_0\epsilon r}\frac{(k^2r^2\sin^2\upsilon+ikr(1-3\cos^2\upsilon)+3\cos^2\upsilon-1}{r^2}), \tag{65b}$$

$$E_z = \frac{|\boldsymbol{\mu}|}{4\pi\epsilon_0\epsilon}\frac{\exp(ikr)}{r}(k^2\sin^2\upsilon+\frac{1}{r^2}(3\cos^2\upsilon-1)-\frac{ik}{r}(3\cos^2\upsilon-1)). \tag{65c}$$

Using polar coordinates to describe the electric field:

$$(x,y,z) = r(\sin\upsilon\cos\varphi, \sin\upsilon\sin\varphi, \cos\upsilon), \tag{66a}$$

and the change of coordinates matrix is:

$$[\dots]_{polar} = \begin{bmatrix} \sin\upsilon\cos\varphi & \sin\upsilon\sin\varphi & \cos\upsilon \\ \cos\upsilon\cos\varphi & \cos\upsilon\sin\varphi & -\sin\upsilon \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix}[\dots]_{cartesian}. \tag{67a}$$

So, for the vectors in the electric field equation:

$$\begin{bmatrix}0\\0\\1\end{bmatrix}_{cartesian} = \begin{bmatrix}\cos\upsilon\\-\sin\upsilon\\0\end{bmatrix}_{polar}, \quad \begin{bmatrix}x\\y\\z\end{bmatrix}_{cartesian} = \begin{bmatrix}r\\0\\0\end{bmatrix}_{polar}. \tag{68a}$$

Therefore:

$$E_{polar} = \frac{|\boldsymbol{\mu}|\exp(ikr)}{4\pi\epsilon_0\epsilon r}k^2(\begin{bmatrix}\cos\upsilon(1+\frac{(ikr-1)}{k^2r^2})\\-\sin\upsilon(1+\frac{(ikr-1)}{k^2r^2})\\0\end{bmatrix}+\begin{bmatrix}\cos\upsilon(\frac{(3-3ikr-k^2r^2)}{k^2r^2})\\0\\0\end{bmatrix}), \tag{69a}$$

$$E = \frac{|\boldsymbol{\mu}|\exp(ikr)k^2}{4\pi\epsilon_0\epsilon r}\begin{bmatrix}\cos\upsilon(1+\frac{(ikr-1)}{k^2r^2}+\frac{(3-3ikr-k^2r^2)}{k^2r^2})\\-\sin\upsilon(1+\frac{(ikr-1)}{k^2r^2})\\0\end{bmatrix} = \frac{|\boldsymbol{\mu}|\exp(ikr)k^2}{4\pi\epsilon_0\epsilon r}\begin{bmatrix}\cos\upsilon(\frac{2}{k^2r^2}-\frac{2i}{kr})\\\sin\upsilon(\frac{1}{k^2r^2}-\frac{i}{kr}-1)\\0\end{bmatrix}. \tag{69b}$$

Now, for H(r):

$$H = -i|\boldsymbol{\mu}|\omega[\nabla\times\overleftrightarrow{G}_0]\boldsymbol{n_z} = |\boldsymbol{\mu}|\omega\frac{\exp(ikR)}{4\pi R^2}(k+\frac{i}{R})(\boldsymbol{R}\times\overleftrightarrow{I})\boldsymbol{n_z} = |\boldsymbol{\mu}|\omega\frac{\exp(ikR)}{4\pi R^2}(k+\frac{i}{R})\begin{bmatrix}y\\-x\\0\end{bmatrix}. \tag{70a}$$

Note that:

$$\begin{bmatrix}y\\-x\\0\end{bmatrix}_{cartesian} = \begin{bmatrix}0\\0\\-r\sin\upsilon\end{bmatrix}_{polar}, \tag{71a}$$

so,

$$H_{polar}(r,\upsilon,\varphi) = \frac{|\boldsymbol{\mu}|}{4\pi\epsilon_0\epsilon}\frac{\exp(ikr)}{r}k^2(-\frac{i}{kr}-1)\begin{bmatrix}0\\0\\\sin\upsilon\end{bmatrix}. \tag{72a}$$

0.3.4 Rate of energy dissipation in inhomogeneous environments

For the time dependent Maxwell's Equations we can write:

$$H \cdot (\nabla \times E) - E \cdot (\nabla \times H) = -H \cdot \frac{\partial B}{\partial t} - E \cdot \frac{\partial D}{\partial t} - j \cdot E, \tag{73a}$$

$$\int_V \nabla \cdot (E \times H)\, dV \; = - \int_V [H \cdot \frac{\partial B}{\partial t} + E \cdot \frac{\partial D}{\partial t} + j \cdot E]\, dV \; . \tag{73b}$$

Applying the divergence theorem:

$$\int_{\partial V} (E \times H) \cdot n\, da \; = - \int_V [H \cdot \frac{\partial B}{\partial t} + E \cdot \frac{\partial D}{\partial t} + j \cdot E]\, dV \; , \tag{74a}$$

the Poynting vector is defined as $S = (E \times H)$ and the energy dissipation is given by the $j \cdot E$ factor, so we have the average dissipation given by:

$$\int_{\partial V} \langle S \rangle \cdot n\, da \; = -\frac{1}{T} \iint_0^T Re(j(r,t)) \cdot Re(E(r,t))\, dtdV \; , \tag{75a}$$

$$\int_{\partial V} \langle S \rangle \cdot n\, da \; = -\frac{1}{T} \iint_0^T (\frac{1}{2}(j(t)e^{i\omega t} + j^*(t)e^{-i\omega t}) \cdot \frac{1}{2}(E(t)e^{i\omega t} + E^*(t)e^{-i\omega t}))\, dtdV \; , \tag{75b}$$

$$\int_{\partial V} \langle S \rangle \cdot n\, da \; = -\frac{1}{T} \iint_0^T (\frac{1}{2}Re(j^* \cdot E)) + (\frac{1}{2}Re(j(r) \cdot E(r)e^{2i\omega t}))\, dtdV \; . \tag{75c}$$

The double frequency term has an average of zero, therefore the rate of energy dissipation:

$$\frac{dW}{dt} = \int_{\partial V} \langle S \rangle \cdot n\, da \; = -\frac{1}{2} \int_V (Re(j^* \cdot E))\, dV \; . \tag{76a}$$

From the current density defined as $j(r) = -i\omega \boldsymbol{\mu} \delta(r - r_0)$:

$$\frac{dW}{dt} = -\frac{\omega}{2} \int_V (Re(i\boldsymbol{\mu}^* \delta(r - r_0) \cdot E))\, dV \; = \frac{\omega}{2} \int_V (Im(\boldsymbol{\mu}^* \cdot E))\delta(r - r_0)\, dV \; = \frac{\omega}{2} Im(\boldsymbol{\mu}^* \cdot E(r_0)). \tag{77a}$$

From the result (52a) and the dipole in z-direction, we have:

$$\frac{dW}{dt} = \frac{\omega}{2} Im(\boldsymbol{\mu}^* \cdot \omega^2 \mu \mu_0 \overleftrightarrow{G}_0(r_0, r_0) \cdot \boldsymbol{\mu}) = \frac{\omega^3 |\boldsymbol{\mu}|^2}{2c^2 \epsilon_0 \epsilon} Im(\boldsymbol{n_z} \cdot \overleftrightarrow{G}_0(r_0, r_0) \cdot \boldsymbol{n_z}). \tag{78a}$$

Due to the product between $\boldsymbol{\mu}$ and the electric field, aswell as the $\overleftrightarrow{G}_0(r_0, r_0)$ relation with n_z, we have to only analyze the imaginary part of E_z (65c). Moreover, there is a singularity when R goes to zero, so we will take the limit of the function:

$$\frac{dW}{dt} = \frac{\omega |\boldsymbol{\mu}|}{2} \lim_{R \to 0} Im(E_z), \tag{79a}$$

$$\frac{8\pi\epsilon_0\epsilon}{\omega |\boldsymbol{\mu}|^2} \frac{dW}{dt} = \lim_{R \to 0} Im(\frac{\exp(ikR)}{R}(k^2 \sin^2 \upsilon + \frac{1}{R^2}(3\cos^2 \upsilon - 1) - \frac{ik}{R}(3\cos^2 \upsilon - 1))). \tag{79b}$$

Expanding $\exp(ikR)$ as a Taylor series:

$$\exp(ikR) = \sum_{n=0}^{\infty} \frac{(ikR)^n}{n!} = 1 + ikR + \frac{(ikR)^2}{2} + \frac{(ikR)^3}{6} + \ldots, \tag{80a}$$

we have:

$$\frac{8\pi\epsilon_0\epsilon}{\omega|\boldsymbol{\mu}|^{\mathbf{2}}}\frac{dW}{dt} = \lim_{R\to\ 0} Im\left(\sum_{n=0}^{\infty}\frac{(ik)^n(R)^{n-1}}{n!}(k^2\sin^2\upsilon + \frac{1}{R^2}(3\cos^2\upsilon - 1) - \frac{ik}{R}(3\cos^2\upsilon - 1))\right), \quad (81a)$$

$$= \lim_{R\to\ 0} Im\left(\frac{(i)^n(k)^{n+2}(R)^{n-1}}{n!}\sin^2\upsilon + \frac{(ik)^n(R)^{n-3}}{n!}(3\cos^2\upsilon - 1) - \frac{(ik)^{n+1}(R)^{n-2}}{n!}(3\cos^2\upsilon - 1))\right), \quad (81b)$$

$$= \lim_{R\to\ 0} Im\left(\sum_{n=0}^{\infty}\frac{(i)^n(k)^{n+2}(R)^{n-1}}{n!}\sin^2\upsilon + \frac{(3\cos^2\upsilon - 1)}{R^3} - \sum_{n=0}^{\infty}\frac{(n+1)(ik)^{n+2}(R)^{n-1}}{(n+2)!}(3\cos^2\upsilon - 1))\right), \quad (81c)$$

$$= \lim_{R\to\ 0} Im(\frac{(3\cos^2\upsilon - 1)}{R^3} + \sum_{n=0}^{\infty}(\frac{(i)^n(k)^{n+2}(R)^{n-1}}{n!}\sin^2\upsilon - \frac{(n+1)(ik)^{n+2}(R)^{n-1}}{(n+2)!}(3\cos^2\upsilon - 1))), \quad (81d)$$

$$\frac{8\pi\epsilon_0\epsilon}{\omega|\boldsymbol{\mu}|^{\mathbf{2}}}\frac{dW}{dt} = \lim_{R\to\ 0} Im(\sum_{n=0}^{\infty}(\frac{(i)^n(k)^{n+2}(R)^{n-1}}{n!}\sin^2\upsilon + \frac{(i)^n(k)^{n+2}(R)^{n-1}}{(n+2)(n!)}(3\cos^2\upsilon - 1))), \quad (81e)$$

$$\frac{8\pi\epsilon_0\epsilon}{\omega|\boldsymbol{\mu}|^{\mathbf{2}}}\frac{dW}{dt} = \lim_{R\to\ 0} Im(\sum_{n=0}^{\infty}(\frac{(i)^n(k)^{n+2}(R)^{,n-1}}{n!}(\frac{(n-1)\sin^2\upsilon + 2}{(n+2)})). \quad (81f)$$

For the image, we take the even values of n ($n = 2p + 1$):

$$\frac{8\pi\epsilon_0\epsilon}{\omega|\boldsymbol{\mu}|^{\mathbf{2}}}\frac{dW}{dt} = \lim_{R\to\ 0}\sum_{p=0}^{\infty}(\frac{(-1)^p(k)^{2p+3}(R)^{2p}}{(2p+1)!}(\frac{(2p)\sin^2\upsilon + 2}{(2p+3)}), \quad (82a)$$

$$\frac{8\pi\epsilon_0\epsilon}{\omega|\boldsymbol{\mu}|^{\mathbf{2}}}\frac{dW}{dt} = (\frac{2}{3}k^3 + \sum_{p=1}^{\infty}(\frac{(-1)^p(k)^{2p+3}(\lim_{R\to\ 0}(R)^{2p})}{(2p+1)!}(\frac{(2p)\sin^2\upsilon + 2}{(2p+3)})). \quad (82b)$$

For the limit, where R goes to zero, in the dipole origin we have:

$$P_0 = \frac{dW}{dt} = \frac{|\boldsymbol{\mu}|^{\mathbf{2}}}{12\pi}\frac{\omega}{\epsilon_0\epsilon}k^3. \quad (83a)$$

Considering now the electric field at the origin (r_0) as a superposition of the field from the dipole and the reflected field, and using (77a):

$$E(r_0) = E_0(r_0) + E_{ref}(r_0), \quad \frac{dW}{dt} = \frac{\omega}{2}Im(\boldsymbol{\mu}^* \cdot (E_0(r_0) + E_{ref}(r_0))), \quad (84a)$$

$$P = \frac{\omega}{2}Im(\boldsymbol{\mu}^* \cdot E_0(r_0)) + \frac{\omega}{2}Im(\boldsymbol{\mu}^* \cdot E_{ref}(r_0)). \quad (84b)$$

For the energy dissipation of E_0 we have (83a), so we define the normalized rate of energy dissipation in function of the reflected field:

$$\frac{P}{P_0} = 1 + \frac{12\pi}{|\boldsymbol{\mu}|^{\mathbf{2}}}\frac{\epsilon_0\epsilon}{\omega}\frac{1}{k^3}\frac{\omega}{2}Im(\boldsymbol{\mu}^* \cdot E_{ref}(r_0)) = 1 + \frac{6\pi\epsilon_0\epsilon}{|\boldsymbol{\mu}|^{\mathbf{2}}}\frac{1}{k^3}Im(\boldsymbol{\mu}^* \cdot E_{ref}(r_0)) \quad (85a)$$

0.3.5 Angular spectrum representation of the dipole field

Representing a dipole field as a angular spectrum, at first, the potential of a dipole is given by Eq. (49b) and (44a), where $-ik\mathcal{Z}_{\mu\epsilon} = -ik\sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}$ is a multiplicative factor:

$$\mathbf{A}(x,y,z) = A(x,y,z)\cdot n_z = \frac{-ik\mathcal{Z}_{\mu\epsilon}}{4\pi}\frac{e^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\cdot n_z. \quad (86a)$$

Now, to find an angular spectrum representation for the electric and magnetic field, it is first necessary to find the angular spectrum of the function:

$$G_0(r) = \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}. \quad (87a)$$

First, we suppose a representation of G_0 in the half-space $z > 0$ (MANDEL; WOLF, 1995):

$$\frac{e^{ik|r|}}{|r|} = \iint_{-\infty}^{\infty} a(k_x, k_y) e^{ik(k_x x + k_y y + k_z |z|)} \, dk_x dk_y \ , \tag{88a}$$

for the value of k_z we consider:

$$k_z = \begin{cases} \sqrt{1 - k_x^2 - k_y^2}, & k_x^2 + k_y^2 \leq 1 \\ i\sqrt{k_x^2 + k_y^2 - 1}, & k_x^2 + k_y^2 > 1 \end{cases} . \tag{89a}$$

Assuming that the function $a(k_x, k_y)$ and the Equation (88a) remain valid as the value of z approaches 0 ($|r| \neq 0$). Therefore, in the limit of $z \to 0$:

$$\frac{e^{ik\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \iint_{-\infty}^{\infty} a(k_x, k_y) e^{ik(k_x x + k_y y)} \, dk_x dk_y \ . \tag{90a}$$

Changing the variables, $k_x = \omega_x \frac{2\pi}{k}$ and $k_y = \omega_y \frac{2\pi}{k}$:

$$\frac{e^{ik\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \iint_{-\infty}^{\infty} a(\omega_x, \omega_y) (\frac{2\pi}{k})^2 e^{i(\omega_x x + \omega_y y)} \, d\omega_x d\omega_y \ . \tag{91a}$$

Applying the inverse Fourier transform:

$$a(k_x, k_y) = (\frac{k}{2\pi})^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{ik\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} e^{-ik(k_x x + k_y y)} \, dx \, dy \ . \tag{92a}$$

For the following variable changes: $x = R\cos\alpha,\ y = R\sin\alpha,\ k_x = \rho\cos\beta,\ k_y = \rho\sin\beta$.
Now, Eq. (92a) becomes:

$$a = (\frac{k}{2\pi})^2 \iint e^{ikR} e^{-ikR\rho(\cos\alpha\cos\beta + \sin\alpha\sin\beta)} \, d\alpha \, dR \ = (\frac{k}{2\pi})^2 \int_0^{\infty} \int_0^{2\pi} e^{ikR} e^{-ikR\rho\cos(\alpha-\beta)} \, d\alpha \, dR \ . \tag{93a}$$

Analyzing the α integral (I_α):

$$I_\alpha = \int_0^{2\pi} e^{-ikR\rho\cos(\alpha-\beta)} \, d\alpha \ . \tag{94a}$$

This integral is similar to the Bessel Function (for n = 0) (WATSON, 1966):

$$J_n(z) = \frac{1}{2\pi} \int_a^{2\pi+a} e^{i(n\theta - z\sin\theta)} \, d\theta, \ J_0(z) = \frac{1}{2\pi} \int_a^{2\pi+a} e^{-i(z\sin\theta)} \, d\theta \ . \tag{95a}$$

Returning to Eq. (94a), using ($z = kR\rho$), ($\alpha - \beta = \theta - \frac{\pi}{2}$) and ($a = \frac{\pi}{2} - \beta$):

$$I_\alpha = \int_0^{2\pi} e^{-ikR\rho\cos(\alpha-\beta)} \, d\alpha, \ I_\theta = \int_{(\frac{\pi}{2}-\beta)}^{2\pi+(\frac{\pi}{2}-\beta)} e^{-iz\cos(-(\frac{\pi}{2}-\theta))} \, d\theta \ = \int_a^{2\pi+a} e^{-iz\sin(\theta)} \, d\theta \ . \tag{96a}$$

Using Eq. (95a) and ($z = kR\rho$), knowing that J_0 is the Bessel function of first kind and zero order:

$$\int_0^{2\pi} e^{(-ikR\rho\cos(\alpha-\beta))} \, d\alpha \ = 2\pi J_0(kR\rho). \tag{97a}$$

Now, returning to Eq. (93a):

$$a(\rho, \beta) = \frac{k^2}{2\pi} \int_0^{\infty} e^{ikR} J_0(kR\rho) \, dR \ . \tag{98a}$$

For the integral in Eq. (98a), the Hankel transforms are suitable for use (BATEMAN, 1954) ($y > 0$):

$$\int_0^{\infty} \cos(ax) J_0(xy) \, dx \ = \begin{cases} 0, & y < a \\ \sqrt{\frac{1}{(y^2-a^2)}}, & a < y \end{cases}, \ \int_0^{\infty} \sin(ax) J_0(xy) \, dx \ = \begin{cases} \sqrt{\frac{1}{(a^2-y^2)}}, & y < a \\ 0, & a < y \end{cases} . \tag{99a}$$

Using the fact that $e^{iax} = \cos(ax) + i\sin(ax)$:

$$\int_0^\infty e^{(iax)} J_0(xy)\, dx \;= \begin{cases} i\sqrt{\frac{1}{(a^2-y^2)}}, & 0 < y < a \\ \sqrt{\frac{1}{(y^2-a^2)}}, & a < y < \infty \end{cases} . \tag{100a}$$

Using $a = k$, $x = R$, $y = k\rho$, and the conditions of Eq.(**??**):

$$\rho^2 = k_x^2 + k_y^2,\ (0 < k\rho < k) \rightarrow k_x^2 + k_y^2 < 1,\ (k < k\rho < \infty) \rightarrow k_x^2 + k_y^2 > 1, \tag{101a}$$

given the condition above and the changes of variables:

$$\int_0^\infty e^{(ikR)} J_0(kR\rho)\, dR \;= \begin{cases} \frac{i}{k\sqrt{(1-(k_x^2+k_y^2))}}, & k_x^2 + k_y^2 < 1 \\ \frac{1}{k\sqrt{((k_x^2+k_y^2)-1)}}, & k_x^2 + k_y^2 > 1 \end{cases} . \tag{102a}$$

Returning to Eq. (98a) and using the definition in Eq. (89a):

$$a(k_x, k_y) = \begin{cases} \frac{k}{2\pi}\frac{i}{k_z}, & k_x^2 + k_y^2 < 1 \\ \frac{k}{2\pi}\frac{i}{k_z}, & k_x^2 + k_y^2 > 1 \end{cases} ,\ a(k_x, k_y) = \frac{ik}{2\pi k_z}. \tag{103a}$$

So, the resulting representation for the spherical wave is given by:

$$\frac{e^{ik|r|}}{|r|} = \frac{ik}{2\pi}\iint_{-\infty}^{\infty} \frac{1}{k_z} e^{ik(k_x x + k_y y + k_z|z|)}\, dk_x dk_y \ . \tag{104a}$$

Assuming that k is constant and $r = r - r_0$, a change of variables is performed $k_i = kk_i$ for i = (x, y, z):

$$\frac{e^{ik|r|}}{|r|} = \frac{ik^2}{2\pi}\iint_{-\infty}^{\infty} \frac{1}{k_z} e^{i(k_x x + k_y y + k_z|z|)}\, \frac{dk_x}{k}\frac{dk_y}{k} \;= \frac{i}{2\pi}\iint_{-\infty}^{\infty} \frac{1}{k_z} e^{i(k_x x + k_y y + k_z|z|)}\, dk_x dk_y \ , \tag{105a}$$

$$\frac{e^{ik|r-r_0|}}{|r-r_0|} = \frac{i}{2\pi}\iint_{-\infty}^{\infty} \frac{1}{k_z} e^{i(k_x(x-x_0) + k_y(y-y_0) + k_z|z-z_0|)}\, dk_x dk_y \ . \tag{105b}$$

0.3.6 Angular spectrum representation of the dyadic Green's function

Moving on, it becomes necessary to represent the Dyadic's Green function as a angular spectrum. So, the Green's function $G_0(r, r_0)$ defines the electric field (Eq. 46a) for a electric dipole (μ) with current density given by ($j(r) = -i\omega\delta(r - r_0)\boldsymbol{\mu}$ (NOVOTNY; HECHT, 2012)) in a space (ϵ_1 and μ_1):

$$E(r) = i\omega\mu_1\mu_0 \int_V \overleftrightarrow{G}(r, r') j(r')\, dV' \;= i\omega\mu_1\mu_0 \int_V \overleftrightarrow{G}(r, r')(-i\omega)\delta(r - r')\boldsymbol{\mu}\, dV' \ , \tag{106a}$$

$$E(r) = i\omega(-i\omega)\mu_1\mu_0 \int_V \overleftrightarrow{G}(r, r')\delta(r - r')\, dV' \ \boldsymbol{\mu} = \omega^2\mu_1\mu_0\overleftrightarrow{G_0}(r, r_0)\boldsymbol{\mu}. \tag{106b}$$

Having the electric field described by the Green's function, now is necessary to look into the vector potential A(r), given in Eq. (37c), using the positive dyadic function (44a):

$$A(r) = \mu_1\mu_0 \int_V (-i\omega)\delta(r - r')\boldsymbol{\mu} G_0(r, r')\, dV' \;= -i\omega\mu_1\mu_0 \int_V \delta(r - r') G_0(r, r')\, dV' \ \boldsymbol{\mu}, \tag{107a}$$

$$A = -i\omega\mu_1\mu_0 G_0\boldsymbol{\mu} = -i\omega\mu_1\mu_0(\frac{e^{ik_1|r-r_0|}}{4\pi|r-r_0|})\boldsymbol{\mu} = \frac{-i\omega^2\mu_1\epsilon_1 i\mu_0}{c^2\epsilon_0\epsilon_1\mu_0 i\omega}(\frac{e^{ik_1|r-r_0|}}{4\pi|r-r_0|})\boldsymbol{\mu} = \frac{k_1^2 e^{ik_1|r-r_0|}}{i\omega\epsilon_0\epsilon_1 4\pi|r-r_0|}\boldsymbol{\mu}. \tag{107b}$$

Defining $e^{i(k_x(x-x_0)+k_y(y-y_0)+k_{z1}|z-z_0|)} = \phi(x, y, z)$ such as:

$$\frac{\partial^2\phi}{\partial x^2} = -k_x^2\phi,\ \frac{\partial^2\phi}{\partial y^2} = -k_y^2\phi,\ \frac{\partial^2\phi}{\partial z^2} = -k_{z1}^2\phi,\ \frac{\partial^2\phi}{\partial xy} = -k_x k_y\phi,\ \frac{\partial^2\phi}{\partial xz} = \mp k_x k_{z1}\phi,\ \frac{\partial^2\phi}{\partial yz} = \mp k_y k_{z1}\phi. \tag{108a}$$

Using Eq. (105b):

$$A(r) = \frac{k_1^2}{8\pi^2\omega\epsilon_0\epsilon_1}\iint_{-\infty}^{\infty}\frac{1}{k_{z1}}\phi(x,y,z)\,dk_x dk_y\boldsymbol{\mu}. \tag{109a}$$

Knowing the relation between E(r) and A(r) given by (48a), and assuming the x, y and z different directions for $\boldsymbol{\mu}$. For x-direction at first, we have:

$$E_x(r) = \frac{|\boldsymbol{\mu_x}|i}{8\pi^2\epsilon_0\epsilon_1}[k_1^2 + \nabla\nabla\cdot]\hat{x}\iint\frac{1}{k_{z1}}\phi\,dk_x dk_y = \frac{|\boldsymbol{\mu_x}|i}{8\pi^2\epsilon_0\epsilon_1}[k_1^2 + \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial yx}, \frac{\partial^2}{\partial zx}]\iint\frac{1}{k_{z1}}\phi\,dk_x dk_y, \tag{110a}$$

$$E_x(r) = \frac{|\boldsymbol{\mu_x}|i}{8\pi^2\epsilon_0\epsilon_1}\iint\frac{[k_1^2 - k_x^2, -k_x k_y, \mp k_x k_{z1}]}{k_{z1}}\phi\,dk_x dk_y, \tag{110b}$$

similarly, for the y and z directions:

$$\frac{E_y}{|\boldsymbol{\mu_y}|} = \iint\frac{i[-k_x k_y, k_1^2 - k_y^2, \mp k_y k_{z1}]}{8\pi^2\epsilon_0\epsilon_1 k_{z1}}\phi\,dk_x dk_y,\ \frac{E_z}{|\boldsymbol{\mu_z}|} = \iint\frac{i[\mp k_x k_{z1}, \mp k_y k_{z1}, k_1^2 - k_{z1}^2]}{8\pi^2\epsilon_0\epsilon_1 k_{z1}}\phi\,dk_x dk_y \tag{111a}$$

Therefore, for E(r):

$$E(r) = \frac{i\boldsymbol{\mu}}{8\pi^2\epsilon_0\epsilon_1}\iint_{-\infty}^{\infty}\frac{1}{k_{z1}}\begin{bmatrix} k_1^2 - k_x^2 & -k_x k_y & \mp k_x k_{z1} \\ -k_x k_y & k_1^2 - k_y^2 & \mp k_y k_{z1} \\ \mp k_x k_{z1} & \mp k_y k_{z1} & k_1^2 - k_{z1}^2 \end{bmatrix}\phi(x,y,z)\,dk_x dk_y\ . \tag{112a}$$

From Eq. (106b) and ($\boldsymbol{\mu} \neq 0$):

$$\frac{\omega^2\mu_1\epsilon_1}{c^2}\overleftrightarrow{G_0}(r,r_0) = \frac{i}{8\pi^2}\iint_{-\infty}^{\infty}\frac{1}{k_{z1}}\begin{bmatrix} k_1^2 - k_x^2 & -k_x k_y & \mp k_x k_{z1} \\ -k_x k_y & k_1^2 - k_y^2 & \mp k_y k_{z1} \\ \mp k_x k_{z1} & \mp k_y k_{z1} & k_1^2 - k_{z1}^2 \end{bmatrix}\phi(x,y,z)\,dk_x dk_y\ , \tag{113a}$$

$$\overleftrightarrow{G_0}(r,r_0) = \frac{i}{8\pi^2 k_1^2}\iint_{-\infty}^{\infty}\frac{1}{k_{z1}}\begin{bmatrix} k_1^2 - k_x^2 & -k_x k_y & \mp k_x k_{z1} \\ -k_x k_y & k_1^2 - k_y^2 & \mp k_y k_{z1} \\ \mp k_x k_{z1} & \mp k_y k_{z1} & k_1^2 - k_{z1}^2 \end{bmatrix}\phi(x,y,z)\,dk_x dk_y\ . \tag{113b}$$

Defining $\overleftrightarrow{\boldsymbol{M}}$ as:

$$\overleftrightarrow{\boldsymbol{M}} = \frac{1}{k_1^2 k_{z1}}\begin{bmatrix} k_1^2 - k_x^2 & -k_x k_y & \mp k_x k_{z1} \\ -k_x k_y & k_1^2 - k_y^2 & \mp k_y k_{z1} \\ \mp k_x k_{z1} & \mp k_y k_{z1} & k_1^2 - k_{z1}^2 \end{bmatrix}. \tag{114a}$$

So, the result for the dyadic Green's function is:

$$\overleftrightarrow{G_0}(r,r_0) = \frac{i}{8\pi^2}\iint_{-\infty}^{\infty}\overleftrightarrow{\boldsymbol{M}}e^{i(k_x(x-x_0)+k_y(y-y_0)+k_{z1}|z-z_0|)}\,dk_x dk_y\ . \tag{115a}$$

0.3.7 Decomposition of the dyadic Green's function

Giving this function, it is possible to decompose it in (s) and (p) polarized parts (NOVOTNY; HECHT, 2012), using the matrix for this:

$$\overleftrightarrow{\boldsymbol{M}} = \overleftrightarrow{\boldsymbol{M}}^{(\boldsymbol{s})} + \overleftrightarrow{\boldsymbol{M}}^{(\boldsymbol{p})}. \tag{116a}$$

For a electric dipole in the z-direction a fully (p) polarized field is formed, so, a magnetic dipole can be put in the z-direction to create a fully (s) polarized field:

$$E(r) = E^{(s)}(r) + E^{(p)}(r),\ H(r) = H^{(s)}(r) + H^{(p)}(r). \tag{117a}$$

We define the vector dipole for the electric dipole (A^e) and, for the magnetic dipole, it is defined a vector analogous to the electric dipole (A^h). For the (p) polarization, the results were already calculated:

$$E^{(p)}(r) = i\omega(1 + \frac{1}{k^2}\nabla\nabla\cdot)A^e(r),\ H^{(p)}(r) = \frac{1}{\mu_0\mu_1}\nabla \times A^e(r). \tag{118a}$$

Now, for the magnetic dipole, we start with an analogous definition for $H^{(s)}$, using an analogous gauge condition (35a):

$$H^{(s)} = i\omega A^h - \nabla\phi(r),\ \nabla \cdot A^h = i\omega\mu_0\mu_1\epsilon_0\epsilon_1\phi = i\omega A^h - (\frac{\nabla\nabla \cdot A^h}{i\omega\mu_0\mu_1\epsilon_0\epsilon_1}) = i\omega(1 + \frac{1}{k_1^2}\nabla\nabla\cdot)A^h. \tag{119a}$$

For the (s) polarized electric field we have, supposing that ($\sigma = 0 \to j(r) = 0$):

$$\nabla \times H^{(s)}(r) = -i\omega\epsilon_0\epsilon_1 E^{(s)}(r) + j(r),\ \nabla \times (i\omega(1 + \frac{1}{k_1^2}\nabla\nabla\cdot)A^h(r)) = -i\omega\epsilon_0\epsilon_1 E^{(s)}(r), \tag{120a}$$

$$\nabla \times A^h(r) + (\frac{1}{k_1^2}\nabla \times (\nabla(\nabla \cdot A^h(r)))) = -\epsilon_0\epsilon_1 E^{(s)}(r). \tag{120b}$$

Knowing that the curl of a gradient is zero, we have:

$$-\epsilon_0\epsilon_1 E^{(s)}(r) = \nabla \times A^h(r),\ E^{(s)}(r) = -\frac{1}{\epsilon_0\epsilon_1}\nabla \times A^h(r). \tag{121a}$$

Therefore:

$$E = i\omega(1 + \frac{1}{k_1^2}\nabla\nabla\cdot)A^e - \frac{1}{\epsilon_0\epsilon_1}\nabla \times A^h,\ H = i\omega(1 + \frac{1}{k_1^2}\nabla\nabla\cdot)A^h + \frac{1}{\mu_0\mu_1}\nabla \times A^e. \tag{122a}$$

Defining the angular representation of the polarized vector potentials as (NOVOTNY; HECHT, 2012):

$$A^{e,h}(x, y, z) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \widehat{A}^{e,h}(k_x, k_y)e^{i(k_x(x-x_0)+k_y(y-y_0)+k_{z1}|z-z_0|)}\, dk_x dk_y\ . \tag{123a}$$

For 122a, knowing that ($A^{e,h} = A^{e,h} \cdot n_z$) :

$$E = (i\omega + \frac{i\omega}{k_1^2}\nabla\nabla\cdot)(A_z^e\hat{\mathbf{z}}) - \frac{1}{\epsilon_0\epsilon_1}\nabla \times (A_z^e\hat{\mathbf{z}}) = \frac{i\omega}{k_1^2}[\frac{\partial^2}{\partial xz}, \frac{\partial^2}{\partial yz}, k_1^2 + \frac{\partial^2}{\partial z^2}]A_z^e - \frac{i}{\epsilon_0\epsilon_1}[\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}, 0]A_z^h, \tag{124a}$$

introducing 123a:

$$E = \frac{i\omega}{2\pi k_1^2}\iint_{-\infty}^{\infty} \widehat{A}^e[\mp k_x k_{z1}, \mp k_y k_{z1}, k_1^2 - k_{z1}^2]\phi\, dk_x dk_y\ + \frac{i}{2\pi\epsilon_0\epsilon_1}\iint_{-\infty}^{\infty} \widehat{A}^h[-k_y, k_x, 0]\phi\, dk_x dk_y\ , \tag{125a}$$

decomposing into the x,y,z directions:

$$E_{x,y} = \mu_{x,y}\iint \frac{\mp i\omega\widehat{A}^e k_{x,y}k_{z1}}{2\pi k_1^2} - \frac{i\widehat{A}^h k_{y,x}}{2\pi\epsilon_0\epsilon_1}\phi\, dk_x dk_y\ ,\ E_z = \mu_z\iint \frac{i\omega\widehat{A}^e(k_1^2 - k_{z1}^2)}{2\pi k_1^2}\phi\, dk_x dk_y\ . \tag{126a}$$

Comparing to 112a, in the z direction first:

$$\iint \frac{i\omega^2\mu_0\mu_1[\mp k_x k_{z1}, \mp k_y k_{z1}, k_1^2 - k_{z1}^2]}{8\pi^2 k_1^2 k_{z1}}\phi(x, y, z)\, dk_x dk_y = \iint \frac{i\omega\widehat{A}^e(k_1^2 - k_{z1}^2)}{2\pi k_1^2}\phi(x, y, z)\, dk_x dk_y\ , \tag{127a}$$

$$\widehat{A}^e(k_x, k_y) = \frac{\omega\mu_0\mu_1}{4\pi}\frac{[\mp k_x k_{z1}, \mp k_y k_{z1}, k_1^2 - k_{z1}^2]}{k_{z1}(k_x^2 + k_y^2)}. \tag{127b}$$

For the x-direction:

$$\frac{i\omega^2\mu_0\mu_1[k_1^2 - k_x^2, -k_x k_y, \mp k_x k_{z1}]}{8\pi^2 k_1^2 k_{z1}} = \frac{i\omega^2\mu_0\mu_1[k_x^2 k_{z1}^2, k_y k_x k_{z1}^2, \mp k_x k_{z1} k_1^2 \pm k_x k_{z1}^3]}{8\pi^2 k_1^2 k_{z1}(k_x^2 + k_y^2)} - \frac{i\widehat{A}^h k_y}{2\pi\epsilon_0\epsilon_1}, \tag{128a}$$

$$-\frac{\widehat{A}^h k_y 4\pi k_1^2 k_{z1}(k_x^2 + k_y^2)}{\epsilon_0\epsilon_1} = \omega^2\mu_0\mu_1[(k_y^2 + k_{z1}^2)(k_x^2 + k_y^2) -, \tag{128b}$$

$$-k_x^2 k_{z1}^2(-k_x k_y)(k_x^2 + k_y^2) - k_y k_x k_{z1}^2(\mp k_x k_{z1})(k_x^2 + k_y^2) \pm k_x k_{z1} k_1^2 \mp k_x k_{z1}^3], \tag{128c}$$

$$\widehat{A}^h = \frac{\omega^2\epsilon_0\epsilon_1\mu_0\mu_1[-k_y^2 k_1^2, k_x k_y k_1^2, 0]}{4\pi k_y k_1^2 k_{z1}(k_x^2 + k_y^2)} = \frac{\omega^2\epsilon_0\epsilon_1\mu_0\mu_1[-k_y, k_x, 0]}{4\pi k_{z1}(k_x^2 + k_y^2)} = \frac{k_1^2}{4\pi}\frac{[-k_y, k_x, 0]}{k_{z1}(k_x^2 + k_y^2)}. \tag{128d}$$

Now, to derive the decomposed matrices (116a), we can analyze the (s) part of (122a) and using (115a) for the (s) electric field:

$$E^{(s)}(r) = -\frac{1}{\epsilon_0\epsilon_1}\nabla\times\boldsymbol{A^h(r)},\ \omega^2\mu_1\mu_0\overleftrightarrow{G_0}^{(s)}(r,r_0)\boldsymbol{\mu} = -\frac{1}{\epsilon_0\epsilon_1}\nabla\times(0,0,A^h(r)), \tag{129a}$$

$$\frac{i\omega^2\mu_1\mu_0}{8\pi^2}\iint_{-\infty}^{\infty}\overleftrightarrow{\boldsymbol{M}}^{(s)}\phi(x,y,z)\,dk_x dk_y\ \boldsymbol{\mu} = -\frac{1}{\epsilon_0\epsilon_1}\left(\frac{\partial A^h}{\partial y},-\frac{\partial A^h}{\partial x},0\right). \tag{129b}$$

For the x direction in the dipole:

$$A_x^h = \iint_{-\infty}^{\infty}\frac{-k_y k_1^2}{8\pi^2 k_{z1}(k_x^2+k_y^2)}\phi\,dk_x dk_y\ ,\ \left(\frac{\partial A_x^h}{\partial y},-\frac{\partial A_x^h}{\partial x},0\right) = \frac{i}{8\pi^2}\iint_{-\infty}^{\infty}\frac{k_1^2(-k_y^2,k_xk_y,0)}{k_{z1}(k_x^2+k_y^2)}\phi\,dk_x dk_y\ . \tag{130a}$$

Therefore, for x:

$$\overleftrightarrow{\boldsymbol{M}}_{\boldsymbol{x}}^{(s)} = \frac{1}{\omega^2\mu_1\mu_0\epsilon_0\epsilon_1}\frac{k_1^2(k_y^2,-k_xk_y,0)}{k_{z1}(k_x^2+k_y^2)} = \frac{(k_y^2,-k_xk_y,0)}{k_{z1}(k_x^2+k_y^2)}. \tag{131a}$$

For the y direction in the dipole:

$$A_y^h = \frac{1}{8\pi^2}\iint_{-\infty}^{\infty}\frac{k_x k_1^2}{k_{z1}(k_x^2+k_y^2)}\phi(x,y,z)\,dk_x dk_y\ , \tag{132a}$$

$$\left(\frac{\partial A_y^h}{\partial y},-\frac{\partial A_y^h}{\partial x},0\right) = \frac{i}{8\pi^2}\iint_{-\infty}^{\infty}\frac{k_1^2(k_xk_y,-k_x^2,0)}{k_{z1}(k_x^2+k_y^2)}\phi(x,y,z)\,dk_x dk_y\ . \tag{132b}$$

Therefore, for y:

$$\overleftrightarrow{\boldsymbol{M}}_{\boldsymbol{y}}^{(s)} = \frac{1}{\omega^2\mu_1\mu_0\epsilon_0\epsilon_1}\frac{k_1^2(-k_xk_y,k_x^2,0)}{k_{z1}(k_x^2+k_y^2)} = \frac{(-k_xk_y,k_x^2,0)}{k_{z1}(k_x^2+k_y^2)}. \tag{133a}$$

For the z direction: $\overleftrightarrow{\boldsymbol{M}}_{\boldsymbol{z}}^{(s)} = (0,0,0)$. So, the (s) polarized matrix is:

$$\overleftrightarrow{\boldsymbol{M}}^{(s)} = \frac{1}{k_{z1}(k_x^2+k_y^2)}\begin{bmatrix} k_y^2 & -k_xk_y & 0\\ -k_xk_y & k_x^2 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{134a}$$

Now that we have the (s) matrix, we can derive the (p) using (116a) and (114a):

$$\overleftrightarrow{\boldsymbol{M}} = \overleftrightarrow{\boldsymbol{M}}^{(s)} + \overleftrightarrow{\boldsymbol{M}}^{(p)}, \tag{135a}$$

$$\frac{1}{k_1^2k_{z1}}\begin{bmatrix} k_1^2-k_x^2 & -k_xk_y & \mp k_xk_{z1}\\ -k_xk_y & k_1^2-k_y^2 & \mp k_yk_{z1}\\ \mp k_xk_{z1} & \mp k_yk_{z1} & k_1^2-k_{z1}^2\end{bmatrix} = \frac{1}{k_{z1}(k_x^2+k_y^2)}\begin{bmatrix} k_y^2 & -k_xk_y & 0\\ -k_xk_y & k_x^2 & 0\\ 0 & 0 & 0\end{bmatrix} + \overleftrightarrow{\boldsymbol{M}}^{(p)}, \tag{135b}$$

$$\begin{bmatrix} (k_1^2-k_x^2)(k_x^2+k_y^2) & -k_xk_y(k_x^2+k_y^2) & \mp k_xk_{z1}(k_x^2+k_y^2)\\ -k_xk_y(k_x^2+k_y^2) & (k_1^2-k_y^2)(k_x^2+k_y^2) & \mp k_yk_{z1}(k_x^2+k_y^2)\\ \mp k_xk_{z1}(k_x^2+k_y^2) & \mp k_yk_{z1}(k_x^2+k_y^2) & (k_1^2-k_{z1}^2)(k_x^2+k_y^2)\end{bmatrix} =, \tag{135c}$$

$$= \begin{bmatrix} k_y^2k_1^2 & -k_xk_yk_1^2 & 0\\ -k_xk_yk_1^2 & k_x^2k_1^2 & 0\\ 0 & 0 & 0\end{bmatrix} + k_1^2k_{z1}(k_x^2+k_y^2)\overleftrightarrow{\boldsymbol{M}}^{(p)}, \tag{135d}$$

$$\begin{bmatrix} k_x^2k_{z1}^2 & k_xk_yk_{z1}^2 & \mp k_xk_{z1}(k_x^2+k_y^2)\\ k_xk_yk_{z1}^2 & k_y^2k_{z1}^2 & \mp k_yk_{z1}(k_x^2+k_y^2)\\ \mp k_xk_{z1}(k_x^2+k_y^2) & \mp k_yk_{z1}(k_x^2+k_y^2) & (k_x^2+k_y^2)^2\end{bmatrix} = k_1^2k_{z1}(k_x^2+k_y^2)\overleftrightarrow{\boldsymbol{M}}^{(p)}. \tag{135e}$$

Therefore, we have $\overleftrightarrow{\boldsymbol{M}}^{(p)}$ given by:

$$\overleftrightarrow{\boldsymbol{M}}^{(p)} = \frac{1}{k_1^2(k_x^2+k_y^2)}\begin{bmatrix} k_x^2k_{z1} & k_xk_yk_{z1} & \mp k_x(k_x^2+k_y^2)\\ k_xk_yk_{z1} & k_y^2k_{z1} & \mp k_y(k_x^2+k_y^2)\\ \mp k_x(k_x^2+k_y^2) & \mp k_y(k_x^2+k_y^2) & \frac{(k_x^2+k_y^2)^2}{k_{z1}}\end{bmatrix}. \tag{136a}$$

0.3.8 Dyadic Green's functions for the reflected and transmitted fields

For a dipole located above a planar interface (NOVOTNY; HECHT, 2012), we can calculate the reflected and transmitted Green's functions using the former Fresnel coefficients (31a, 31a, 31b and 31b), and choosing a coordinate system where the z-direction starts in z_0 (so that z is the height of the dipole above the surface), we have: $\phi' = e^{i(k_x(x-x_0)+k_y(y-y_0)+k_{z1}(z+z_0))}$, so, following:

$$\overleftrightarrow{G}_{ref}(r, r_0) = \frac{i}{8\pi^2} \iint_{-\infty}^{\infty} [\overleftrightarrow{\boldsymbol{M}}^{(s)}_{\boldsymbol{ref}} + \overleftrightarrow{\boldsymbol{M}}^{(p)}_{\boldsymbol{ref}}]\phi'(x, y, z)\, dk_x dk_y \ . \tag{137a}$$

The reflection matrices are given by:

$$\overleftrightarrow{\boldsymbol{M}}^{(s)}_{\boldsymbol{ref}} = \frac{r^s(k_x, k_y)}{k_{z1}(k_x^2 + k_y^2)} \begin{bmatrix} k_y^2 & -k_x k_y & 0 \\ -k_x k_y & k_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{138a}$$

For the (p) we use the same definition of the reflected field used in the Fresnel derivation (resulting in a negative signal for the reflection) and the new coordinate system for z as the height of the dipole above z_0:

$$\overleftrightarrow{\boldsymbol{M}}^{(p)}_{\boldsymbol{ref}} = \frac{-r^p(k_x, k_y)}{k_1^2(k_x^2 + k_y^2)} \begin{bmatrix} k_x^2 k_{z1} & k_x k_y k_{z1} & k_x(k_x^2 + k_y^2) \\ k_x k_y k_{z1} & k_y^2 k_{z1} & k_y(k_x^2 + k_y^2) \\ -k_x(k_x^2 + k_y^2) & -k_y(k_x^2 + k_y^2) & -\frac{(k_x^2+k_y^2)^2}{k_{z1}} \end{bmatrix}. \tag{139a}$$

So, the total electric field after the reflection is given by the sum of $\overleftrightarrow{G}_0$ and $\overleftrightarrow{G}_{ref}$:

$$E(r) = \omega^2 \mu_1 \mu_0 [\overleftrightarrow{G}_0(r, r_0) + \overleftrightarrow{G}_{ref}(r, r_0)]\boldsymbol{\mu}. \tag{140a}$$

0.3.9 Spontaneous decay rates near planar interfaces

The normalized rate of energy dissipation is defined in (85a) for the reflected field at the dipole origin:

$$E_{ref}(r_0) = \omega^2 \mu_0 \mu_1 \overleftrightarrow{G}_{ref}(r_0, r_0)\boldsymbol{\mu}. \tag{141a}$$

Where $\overleftrightarrow{G}_{ref}(r, r_0)$ is defined in (137a), (143a) and (145a). To define the value of the reflected Green function the following substitutions are made: $k_x = k_\rho \cos\alpha$, $k_y = k_\rho \sin\alpha$, $k_1^2 = k_\rho^2 + k_{z1}^2$, $dk_x dk_y = k_\rho dk_\rho d\alpha$.

Now, for the Green function at the dipole:

$$\overleftrightarrow{G}_{ref}(r_0, r_0) = \frac{i}{8\pi^2} \int_0^{2\pi} \int_0^{\infty} [\overleftrightarrow{\boldsymbol{M}}^{(s)}_{\boldsymbol{ref}} + \overleftrightarrow{\boldsymbol{M}}^{(p)}_{\boldsymbol{ref}}] e^{2i(k_{z1})z_0} k_\rho\, dk_\rho d\alpha \ . \tag{142a}$$

For the substitution in the matrices, first for the (s) matrix:

$$\overleftrightarrow{\boldsymbol{M}}^{(s)}_{\boldsymbol{ref}} = \frac{r^s}{k_{z1}} \begin{bmatrix} \sin^2\alpha & -\sin\alpha\cos\alpha & 0 \\ -\sin\alpha\cos\alpha & \cos^2\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{143a}$$

Now, for the (p) matrix:

$$\overleftrightarrow{\boldsymbol{M}}^{(p)}_{\boldsymbol{ref}} = \frac{-r^p}{k_1^2} \begin{bmatrix} \cos^2\alpha k_{z1} & \sin\alpha\cos\alpha k_{z1} & k_\rho\cos\alpha \\ \sin\alpha\cos\alpha k_{z1} & \sin^2\alpha k_{z1} & k_\rho\sin\alpha \\ -k_\rho\cos\alpha & -k_\rho\sin\alpha & -\frac{k_\rho^2}{k_{z1}} \end{bmatrix}. \tag{144a}$$

For the sum of both matrices:

$$\overleftrightarrow{\boldsymbol{M}}_{\boldsymbol{ref}} = \begin{bmatrix} \frac{r^s \sin^2\alpha k_1^2 - r^p \cos^2\alpha k_{z1}^2}{k_1^2 k_{z1}} & \frac{-(r^s k_1^2 + r^p k_{z1}^2)\sin\alpha\cos\alpha}{k_1^2 k_{z1}} & \frac{-r^p k_\rho \cos\alpha}{k_1^2} \\ \frac{-(r^s k_1^2 + r^p k_{z1}^2)\sin\alpha\cos\alpha}{k_1^2 k_{z1}} & \frac{r^s \cos^2\alpha k_1^2 - r^p \sin^2\alpha k_{z1}^2}{k_1^2 k_{z1}} & \frac{-r^p k_\rho \sin\alpha}{k_1^2} \\ \frac{r^p k_\rho \cos\alpha}{k_1^2} & \frac{r^p k_\rho \sin\alpha}{k_1^2} & \frac{r^p k_\rho^2}{k_1^2 k_{z1}} \end{bmatrix} . \tag{145a}$$

The integral for α can be evaluated analytically, we can see from the following table of integrals (BATEMAN, 1954). Now, applying these results in (142a):

$$\overleftrightarrow{G}_{ref}(r_0, r_0) = \frac{i}{8\pi^2} \int_0^\infty e^{2ik_{z1}z_0} k_\rho \int_0^{2\pi} [\overleftrightarrow{\boldsymbol{M}}_{\boldsymbol{ref}}]\, d\alpha dk_\rho \ , \tag{146a}$$

$$\overleftrightarrow{G}_{ref} = \frac{i}{8\pi^2} \int_0^\infty e^{2ik_{z1}z_0} k_\rho \begin{bmatrix} \frac{(r^s k_1^2 - r^p k_{z1}^2)\pi}{k_1^2 k_{z1}} & 0 & 0 \\ 0 & \frac{(r^s k_1^2 - r^p k_{z1}^2)\pi}{k_1^2 k_{z1}} & 0 \\ 0 & 0 & \frac{2r^p k_\rho^2 \pi}{k_1^2 k_{z1}} \end{bmatrix} dk_\rho \ , \tag{146b}$$

$$\overleftrightarrow{G}_{ref} = \frac{i}{8\pi k_1^2} \int_0^\infty \frac{k_\rho}{k_{z1}} \begin{bmatrix} r^s k_1^2 - r^p k_{z1}^2 & 0 & 0 \\ 0 & r^s k_1^2 - r^p k_{z1}^2 & 0 \\ 0 & 0 & 2r^p k_\rho^2 \end{bmatrix} e^{2ik_{z1}z_0}\, dk_\rho \ . \tag{146c}$$

Performing the following substitutions:

$$s = \frac{k_\rho}{k_1}, \ s_z = \sqrt{1-s^2} = \frac{k_{z1}}{k_1}. \tag{147a}$$

We have for the green function:

$$\overleftrightarrow{G}_{ref}(r_0, r_0) = \frac{i}{8\pi} \int_0^\infty \frac{sk_1}{s_z} \begin{bmatrix} r^s - r^p s_z^2 & 0 & 0 \\ 0 & r^s - r^p s_z^2 & 0 \\ 0 & 0 & 2r^p s^2 \end{bmatrix} e^{2ik_1 z_0 s_z}\, ds \ . \tag{148a}$$

So, the electric field is given by:

$$E_{ref}(r_0) = \frac{ik_1\omega^2\mu_0\mu_1}{8\pi} \int_0^\infty \frac{s}{s_z} \begin{bmatrix} (r^s - r^p s_z^2)\mu_x \\ (r^s - r^p s_z^2)\mu_y \\ 2r^p s^2 \mu_z \end{bmatrix} e^{2ik_1 z_0 s_z}\, ds \ . \tag{149a}$$

For the normalized rate of energy dissipation (85a)

$$\frac{P}{P_0} = 1 + \frac{6\pi\epsilon_0\epsilon_1}{|\boldsymbol{\mu}|^2 k_1^3} Im(\boldsymbol{\mu}^* E_{ref}) = 1 + \frac{3(\omega^2\mu_0\mu_1\epsilon_0\epsilon_1)}{4|\boldsymbol{\mu}|^2 k_1^2} Im(\boldsymbol{\mu}^* \int_0^\infty \frac{is}{s_z} \begin{bmatrix} (r^s - r^p s_z^2)\mu_x \\ (r^s - r^p s_z^2)\mu_y \\ 2r^p s^2 \mu_z \end{bmatrix} e^{2ik_1 z_0 s_z}\, ds\ ), \tag{150a}$$

$$\frac{P}{P_0} = 1 + \frac{3}{4|\boldsymbol{\mu}|^2} Re((\int_0^\infty \frac{s}{s_z} [\mu_x \ \mu_y \ \mu_z] \cdot \begin{bmatrix} (r^s - r^p s_z^2)\mu_x \\ (r^s - r^p s_z^2)\mu_y \\ 2r^p s^2 \mu_z \end{bmatrix} e^{2ik_1 z_0 s_z}\, ds\ )), \tag{150b}$$

$$\frac{P}{P_0} = 1 + \frac{3}{4|\boldsymbol{\mu}|^2} Re((\int_0^\infty \frac{s}{s_z}((r^s - r^p s_z^2)(\mu_x^2 + \mu_y^2) + 2r^p s^2 \mu_z^2) e^{2ik_1 z_0 s_z}\, ds\ )). \tag{150c}$$

Therefore, the result for the normalized rate of energy dissipation is the following integral:

$$\frac{P}{P_0} = 1 + \frac{3(\mu_x^2 + \mu_y^2)}{4|\boldsymbol{\mu}|^2} \int_0^\infty Re(\frac{s}{s_z}(r^s - r^p s_z^2) e^{2ik_1 z_0 s_z})\, ds \ + \frac{3(\mu_z^2)}{2|\boldsymbol{\mu}|^2} \int_0^\infty Re(\frac{s^3}{s_z} r^p e^{2ik_1 z_0 s_z})\, ds \ . \tag{151a}$$

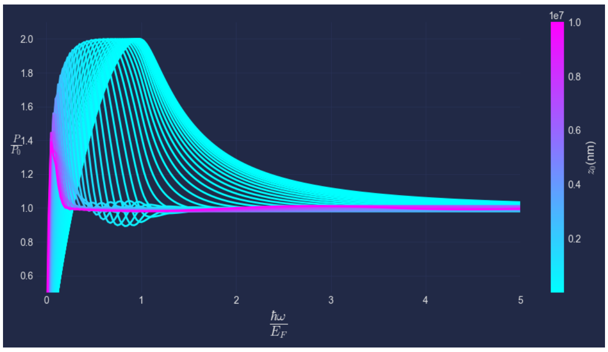
0.3.10 Numerical integration using Python

In our study, to understand the enhanced emission rates of quantum emitters, we numerically evaluated the Purcell effect using integration of Eq. 151a. We utilized the Python library SciPy, specifically the quad function, for single-variable definite integrations. However, poles from plasmon-polaritons in the integrands led to convergence issues. To address these singularities, we applied the Cauchy principal value method, setting the quad function's weight to 'cauchy'. This adjustment provided accurate integral evaluations, effectively depicting the Purcell effect in our system.

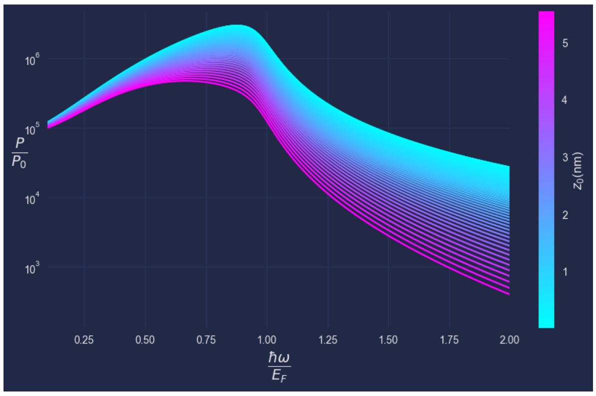
0.4 Results

For the results, we examined a quantum emitter, defined as an electric dipole with a moment μ solely in the z-direction. The presence of a 2D material like graphene or a metal surface impacts the vacuum electromagnetic field, changing spontaneous emission via the Purcell Effect. The dipole is perpendicular to the graphene sheet.

Figure 1 shows the Purcell factor P/P0 against the emitter frequency for varying distances z_0. There's a significant peak value with the emitter over graphene. The Purcell effect decreases with greater distance or frequency, but is still high across the studied ranges. The effect in metals is 5 orders of magnitude less than in graphene, due to plasmon-polaritons coupling.



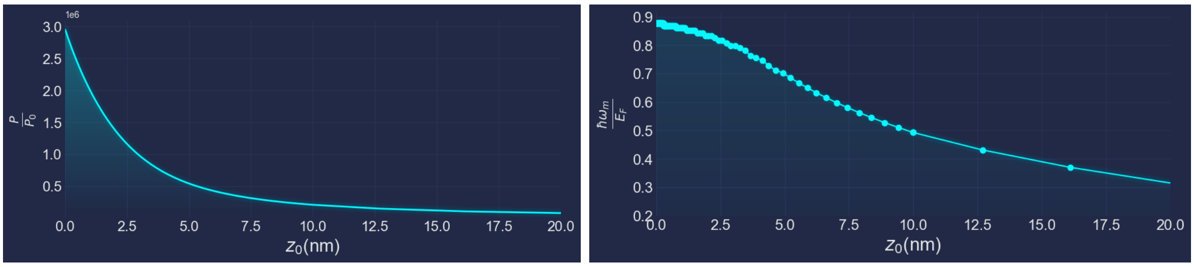
(a) Purcell factor, P/P_0, as a function of ω for different distances z_0 to the metal interface. Note the order of magnitude of P/P0 values related to the graphene.



(b) Purcell factor, P/P_0, as a fucntion of quantum emitter frequency (ω) for different distances (z_0) to the graphene interface (logscale in y axis). Note the spike of emission near $\hbar\omega \approx E_f$.

Figure 1 – Relation between Purcell Effect and the frequency, varying distance.

Exploring the distance dependency in greater detail, Figure 2 illustrates how the peak Purcell factor value changes with frequency as the emitter's distance from the surface is adjusted. Notably, the data shows a significant tenfold decrease in the Purcell factor when the emitter is moved from directly above the graphene sheet to a distance of 20 nm.



(a) Maximum Purcell factor for each distance to the graphene layer.

(b) Frequency where the Purcell factor is maximum for each distance to the graphene layer.

Figure 2 – Relation between distance and the Purcell Effect, and the frequency.

The observed increase in the Purcell factor can be attributed to graphene's ability to support plasmon-polariton modes, as shown in Figure 1. This coupling of the quantum emitter with graphene is responsible for the behavior described in this section, highlighting graphene's potential in enhancing photon emission rates—a critical factor for advancing quantum communication technologies.

This examination underscores the significant impact of graphene's unique properties on the Purcell effect, offering a pathway to optimize photon emission rates essential for quantum communication and cryptography. The integration of two-dimensional materials with quantum emitters opens the door to numerous applications, pushing the boundaries of quantum technologies.

0.5 Conclusion

In summary, our study explored the Purcell effect's role in boosting photon emission rates in quantum emitters coupled to surface plasmon-polaritons in graphene. We found that these hybridized field-matter modes can enhance photon emission rates by up to five orders of magnitude.

These findings highlight the potential of two-dimensional materials like graphene to improve single-photon sources, which are crucial for various applications, including quantum communication, bioluminescence detection, DNA sequencing, remote sensing, and diffuse optical tomography.

Looking ahead, we plan to investigate the coupling of quantum emitters with anisotropic surface modes, such as phonon-polaritons in hBN, plasmon-polaritons, and exciton-polaritons in phosphorene. We will also examine the impact of optical cavities on photon emission rates and explore coupling with Tamm-polariton modes at the metal-photonic crystal interface.

The utilization of quantum effects in plasmon-polaritons of graphene and other 2D materials holds great promise for advancing various quantum technologies, including cybersecurity and emerging technologies. Progress in this research field is crucial for driving innovation, contributing to the development of high-tech products, and advancing science and technology.

0.6 Acknowledgments

The author Leonardo A. Lugarini wishes to express sincere gratitude to the CNPq agency for their invaluable support. The scholarship granted through the PIBIC program has been instrumental in advancing this research.

Bibliography

BATEMAN, H. *Tables of Integrals Transforms, vol. II.* [S.l.]: McGRAW-HILL BOOK COMPANY, INC., 1954.

BENNETT, C.; BRASSARD, G. Quantum cryptography: Public key distribution and coin tossing, proc. of ieee int. conf. on comput. sys. and sign. proces. 1984.

GISIN, N. et al. Quantum cryptography. *Rev. Mod. Phys.*, American Physical Society, v. 74, p. 145–195, Mar 2002. Disponível em: <https://link.aps.org/doi/10.1103/RevModPhys.74.145>.

GISIN, N.; THEW, R. Quantum communication. *Nature photonics*, Nature Publishing Group UK London, v. 1, n. 3, p. 165–171, 2007.

GIULIANI, G.; VIGNALE, G. *Quantum theory of the electron liquid.* [S.l.]: Cambridge university press, 2005.

GONÇALVES, P. A. D.; PERES, N. M. *An introduction to graphene plasmonics.* [S.l.]: World Scientific, 2016.

GROSSHANS, F.; GRANGIER, P. Continuous variable quantum cryptography using coherent states. *Phys. Rev. Lett.*, American Physical Society, v. 88, p. 057902, Jan 2002. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevLett.88.057902>.

LO, H.-K.; CURTY, M.; TAMAKI, K. Secure quantum key distribution. *Nature Photonics*, v. 8, n. 8, p. 595–604, Aug 2014. ISSN 1749-4893.

MANDEL, L.; WOLF, E. *Optical coherence and quantum optics.* [S.l.]: Press Syndicate of the University of Cambridge, 1995.

MARADUDIN, A. A.; SAMBLES, J. R.; BARNES, W. L. *Modern plasmonics.* [S.l.]: Elsevier, 2014.

MAVROEIDIS, V. et al. The impact of quantum computing on present cryptography. *International Journal of Advanced Computer Science and Applications*, The Science and Information Organization, v. 9, n. 3, 2018. Disponível em: <http://dx.doi.org/10.14569/IJACSA.2018.090354>.

NOVOTNY, L.; HECHT, B. *Principles of nano-optics.* [S.l.]: Cambridge university press, 2012.

SIMON, C. Towards a global quantum network. *Nature Photonics*, v. 11, n. 11, p. 678–680, Nov 2017. ISSN 1749-4893. Disponível em: <https://doi.org/10.1038/s41566-017-0032-0>.

WATSON, G. N. *A TREATISE ON THE THEORY OF BESSEL FUNCTIONS, second edition.* [S.l.]: The Syndics of the Cambridge University Press, 1966.