Technical University of Munich

Department of Mathematics

[Thesis Title]

Master's Thesis

von

[First Name Last Name]

Supervisor:	Prof. Dr. [First Name Last Name]
Advisor:	[First Name Last Name]
Submission Date:	[Day. Month. Year]

I hereby declare that this thesis is my own work and that no other sources have been used except those clearly indicated and referenced.

Place, Date

original, hand-written signature

Acknowledgements

[text of acknowledgements]

German Abstract

[abstract text]

English Abstract

[abstract text]

Contents

1. Introduction

2. Shape derivatives

We list here several assumptions to then formulate the shape optimization problem. For the sake of proving Gateaux differentiability, we only need to discuss concepts related to the $W^{1,\infty}$ topology.

Definition 2.0.1 (Admissible transformations)
Let $D \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Define $\Theta := \{\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ with $\theta|_D \in W^{2,\infty}(D; \mathbb{R}^n)$ and $\theta = 0$ on $\mathbb{R}^n \setminus D\}$

Assumption 2.0.2 (Assumptions for the shape optimization problem)
We assume the following:

- $\Omega_r \subset\subset D$ are bounded Lipschitz domains in $\mathbb{R}^n$
- $\tau : \mathbb{R}^n \to \mathbb{R}^n$ is a map in $\mathcal{T}^{1,\infty}$, with $\tau - \mathrm{Id} \in W^{2,\infty}(D)$, $\tau - \mathrm{Id} = 0$ on ∂D

A. Sobolev spaces

Theorem A.0.1 (Integration by parts)
Let Ω be a bounded Lipschitz domain. Let $1 < p < \infty$ and $f, g \in W^{1,p}(\Omega), W^{1,q}(\Omega)$, $q = p'$. Then:

$$\int_{\Omega} f \partial_i g = -\int_{\Omega} g \partial_i f + \int_{\partial\Omega} \mathrm{tr} u \nu_i d\mathcal{H}^{n-1}$$

Proof.

This follows from [10], theore, 18.1 at page 592, where g needs to be $C^1_c(\mathbb{R}^n)$. But [1], theorem 3.18 at page 54, says that (thanks to the smoothness of the boundary) the set of the restriction of such functions is dense in $W^{1,q}(\Omega)$, so that we can conclude by a density argument developed here.

□

Lemma A.0.2
$f \in L^\infty(\Omega; \mathbb{R}^N) \iff f_i \in L^\infty$, and two equivalent norms are $\|f\|_a := \||f|\|_\infty$, $\|f\|_b := \max_i \|f_i\|_\infty$, for $|\cdot|$ any finite dimensional norm.

Proof.

We choose $|\cdot| = |\cdot|_1$.

Consider $f_n \in X_a = \{[f], f : \Omega \to \mathbb{R}^n \text{ measurable }, \|f\|_a\}$, Cauchy. Then every component is Cauchy in the scalar L^∞, so that $f^i_n \to f^i$ in L^∞. The limit f is in X_a because the functions $|f_i|$ are essentially bounded, and so is $|f|$.

Then $\|f_n - f\|_a \leq \|f_n - f_m\|_a + \sum_i \|f^i_m - f^i\|_\infty$ for all n, m. Choose $m \geq n$ with $\|f^i_m - f^i\| \leq 1/(Nn)$ and conclude X_a is Banach.

We know from [10], theorem B.88 at page 671, and page 669, we know that $X_b = \{[f], f : \Omega \to \mathbb{R}^n \text{ measurable }, \|f\|_b\}$ is Banach.

Moreover $X_a = X_b$ as sets, so that the thesis follows.

□

Proposition A.0.3 (Characterization of $W^{1,\infty}$)
Let Ω be a bounded Lipschitz domain, or $\mathbb{R}^n$. Then $W^{1,\infty}(\Omega) = C^{0,1} \cap L^\infty(\Omega)$.

This means that $u \in W^{1,\infty}(\Omega)$ if and only if u has a (unique) representative that is bounded, Lipschitz continuous. Weak and classical derivatives coincide a.e.

Proof.

Extension

In the case Ω is bounded Lipschitz, then Ω is an extension domain for $W^{1,\infty}(\Omega)$, meaning that there is $E : W^{1,\infty}(\Omega) \to W^{1,\infty}(\mathbb{R}^n)$ linear bounded with $Eu = u$ a.e. on Ω (see [10], theorem 13.17 at page 425, 13.13 at page 424, and definition 9.57 at page 273).

The proof

Let $u \in W^{1,\infty}(\Omega)$. By [10], 11.50 at page 339, because Ω is an extension domain, we obtain that u has a representative $\bar{u}$ that is bounded Lipschitz. Let $\phi \in C_c^\infty(\Omega)$. By The Kirszbraun theorem (see e.g. [2]), we can extend $\bar{u}$ to a Lipschitz function e on $\mathbb{R}^n$. Then, for a large enough cube Q containing Ω, $\int_\Omega \bar{u}\partial_i\phi = \int_Q e\partial_i\phi = -\int_Q \partial_i e\phi$, by Fubini's theorem and integration by parts for AC functions.

Because $e = \bar{u}$ on Ω, we conclude $\int_\Omega \bar{u}\partial_i\phi = -\int_\Omega \partial_i\bar{u}\phi$, so that $\nabla\bar{u} = \nabla u$ almost everywhere.

Conversely, let u be bounded Lipschitz. The above reasoning shows that u has essentially bounded weak derivatives equal to the a.e. classical derivatives.

□

Corollary A.0.4 ($W^{k,\infty} = C_B^{k,1}$)
For a bounded Lipschitz domain Ω, or for $\Omega = \mathbb{R}^n$, then $W^{k,\infty} = C_B^{k,1}$ ($C^{k,1}$ bounded functions with bounded derivatives).

Proof.

We have already proved the case $k = 1$. We prove, for instance, the case $k = 2$. Then, $u \in W^{k,2} \implies u, \partial_i u \in W^{k,1}$ ([10], 11.7 at page 321), so that by proposition A.0.3, we find bounded Lipschitz h, g_i with $u = h$ a.e., $\partial_i u = \partial_i h$ a.e., $g_i = \partial_i u$ a.e..

Therefore h is continuous, with continuous weak derivatives g_i, which implies that $h \in C^1(\Omega)$ (see here and here).

Now, $\partial_i h = g_i$ a.e., so everywhere, so that:

- h is bounded Lipschitz and C^1
- $\partial_i h$ are bounded Lipschitz

$\square$

B. Bochner spaces

Here are some useful results about Bochner spaces.

Proposition B.0.1 (Bochner integral and bounded operators)
Let X, Y be separable Banach, let $T \in L(X, T)$ be a linear bounded operator. For $f \in L^1(I, X)$ define $Tf(t) := T(f(t))$. Then $Tf \in L^1(I, Y)$ with $T \int_I f = \int_I Tf$.

Proof.

First of all, a clarification on the definition. What is really happening is that from the time equivalence class f, we select a g, and then $Tf(t) := T(g(t))$. Tf is then the equivalence class of $t \mapsto Tf(t)$. The definition is well posed, because $g_1(t) = g_2(t) \implies T(g_1(t)) = T(g_2(t))$.

Let f_n be simple, $f_n \to f$ a.e., with $\lim_n \int_I f_n = \int_I f$ and $\|f_n\|_X \leq C \|f\|_X$ (see page 6, and corollary 2.7 at page 8 of [9]).

Measurability

For almost all t, $T(f_n(t)) \to T(f(t)) = Tf(t)$ in Y, so that Tf is measurable (strongly).

Integrability

By the assumptions, $\|Tf_n\| \leq \|T\| \|f_n\| \leq C \|f\| \in L^1(I)$, so that by dominated convergence (corollary 2.6 of [9]) Tf is integrable too. Thus $\int_T Tf = \lim_n \int_I Tf_n = \lim_n T \int_I f_n$, because f_n is simple. And now, by the choice of f_n, $\int_T Tf = \lim_n T \int_I f_n = T \lim_n \int_I f_n = T \int_I f$.

$\square$

Proposition B.0.2 (Derivations and bounded operators)
As before, let X, Y be separable Banach, let $T \in L(X, T)$ be a linear bounded operator.

For $k \geq 0$, $f \in H^k(I, X) \implies Tf \in H^k(I, Y)$, with weak derivatives $\partial_{t^i} Tf = T \partial_{t^i} f$, $0 \leq i \leq k$.

The map $f \mapsto Tf$, $H^k(I, X) \to H^k(I, Y)$ is linear bounded.

Proof.

The case $k = 0$ is proved above.

We prove now that $\partial_{t^i} Tf = T\partial_{t^i} f$ for $i = 1$. Note that $T\partial_t f \in L^2(I, Y)$, which qualifies as weak derivative.

In fact, for $\phi \in C_c^\infty(I)$, we have $\int_I \phi T\partial_i f = \int_I T(\phi\partial_t f) = T\int_I \phi\partial_t f = -T\int_I \phi' f = -\int_I \phi' Tf$.

Higher weak derivatives are treated analogously and the rest of the claims follow from the time stationarity of T and by $\|\partial_{t^i} Tf\| = \|T\partial_{t^i} f\| \leq \|T\| \, \|\partial_{t^i} f\|$.

□

Proposition B.0.3 (Continuous representatives)
Let X be separable Banach. $f \in L^1(I, X)$ has at most a continuous representative on $[0, T]$.

Proof.

Assume there exists two such continuous representatives, so that we get a function $\delta : [0, T] \to X$ that is zero almost everywhere and continuous. Hence, $[0, T] \ni t \mapsto \|\delta(t)\|$ is continuous in $\mathbb{R}$ and zero a.e., so that it must be zero everywhere. □

We now check that a vector valued test function has weak derivatives of all orders.

Proposition B.0.4 (Weak derivatives of test functions)
Let $\phi \in C^1([0, T], X)$, for X separable Banach. It means that the limit of the difference quotients exists for all points of I, that $t \mapsto \phi(t), \phi'(t)$ are continuous, and that they can be continuously extended to $[0, T]$.

Then these classical derivatives coincide a.e. with the weak derivatives of u.

Proof.

We rely on proposition 3.8 of [9] at page 26.

<u>Absolute continuity</u>

Consider $\epsilon > 0$. Divide $[a, b] \subset\subset (0, T)$ into a uniform partition t_i. By theorem 6 at page 146 of [5], we get that $\|\phi(t_i) - \phi(t_{i-1})\|_X \leq (t_i - t_{i-1}) \|\phi(\xi_i)\|_X \leq (b - a) \|\phi'\|_\infty / n$, and by choosing n small enough, we conclude that ϕ is (locally) absolutely continuous.

<u>Weak derivative</u>

Therefore, ϕ is locally AC, differentialble everywhere and ϕ' is bounded, so that $\phi \in H^1(I, X)$ and weak and classical derivatives coincide.

$\square$

And now, introduce a time dependent version of the trace operator which is useful for our computations.

Definition B.0.5 (Time dependent trace)
Let Ω be a bounded Lipschitz domain. For $k \geq 0$ we define $\mathrm{tr} : H^k(I, H^1(\Omega)) \to H^k(I, H^{1/2}(\partial\Omega))$ by $\mathrm{tr}(u)(t) := \mathrm{tr}(u(t))$

Below are some properties of this operator.

Proposition B.0.6 (Properties of trace operator)
The trace operator just defined:

1. is well posed
2. is linear bounded
3. admits a linear bounded right inverse, for instance, $E(g)(t) := E(g(t))$ (for E a right inverse of the static trace)
4. tr and E, in the case of $k \in \mathbb{N}_0$, coincide (in the time a.e. sense) for the case $l \geq k$
5. for $k \geq 1$, $\mathrm{tr}u(0) = 0 \iff u(0) = 0$ (in the sense of continuous representatives)
6. it coincides with the trace treated for instance in [11]

Proof.

Proof of the proposition

We recall that the trace operator is bounded surjective onto $H^{1/2}(\partial\Omega)$, with a right inverse E (see theorem 3.37 at page 102 of [12]).

The first three points are consequences of this fact and of proposition B.0.1.

The fourth property follows by the definition of tr, E and the fact that $H^l \subseteq H^k$, for $k \leq l$.

Let now $k \geq 1$. We know that $H^1, H^{1/2}$ are separable and Banach (the latter is separable because the continuos image of H^1 separable, and Banach (see [7], page 20). Therefore, by [4], theorem 2 of page 286, we obtain the embeddings $H^k(I, H^1) \hookrightarrow C([0,T], H^1)$ and the same goes for $H^k(I, H^{1/2})$. The embedding is U, the unique continuous representative of a certain time equivalence class (proposition B.0.3). We also introduce brackets to indicate equivalence classes in time, so, $u = [Uu]$.

We want to prove $(Uu)(0) = 0 \iff U(\mathrm{tr}u)(0) = 0$. But we have $[t \mapsto U(\mathrm{tr}u)(t)] = tru := [t \mapsto \mathrm{tr}((Uu)(t))]$. So, $U(\mathrm{tr}u)(t) = \mathrm{tr}((Uu)(t))$ for all $t \in [0, T]$ by continuity.

For the last point, let $k = 0$. We have:

1. $H^1(\Omega) \cap C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ (see [1], theorem 3.18 at page 54, where being Ω bounded Lipschitz is important)
2. functions $\sum_{i \le m} \phi_i(t) f_i$ for $\phi_i \in C_c^\infty(I), f_i \in H^1(\Omega) \cap C^1(\overline{\Omega})$ are dense in $L^2(I, H^1)$ (see [8], page 39, lemma 1.9)

It follows by the third point that $C^1(\overline{\Omega \times I})$ is dense in $L^2(I, H^1)$, so that $u \mapsto u|_{I \times \partial\Omega}$ admits a unique extension by continuity to $L^2(I, H^1)$, so that this definition of trace coincides with the one from the literature in the case of the space $H^{1,0} := L^2(I, H^1)$ (see [11], theorem 4.1), we expand this argument below.

<u>Proof of leftover facts</u>

We call $C^k(\overline{\Omega}) := \{u \in C^k(\Omega) \text{ with } \partial_\alpha f \text{ extendable by continuity to } \overline{\Omega}\}$.

Consider $u(x, t) := \phi(t)v(x)$, for $\phi \in C^1([0, T]), v \in C^1(\overline{\Omega})$. Then, it has partial derivatives $u_t = \phi_t v, u_i = \phi u_i$. u and all its partial derivatives are continuous on $I \times \Omega$, meaning that $u \in C^1(\Omega \times I)$.

Moreover, $u, u_i, u_t \in C([0, T], C(\overline{\Omega}))$. We claim $C([0, T], C(\overline{\Omega})) = C(\overline{\Omega \times I})$. In fact, one direction is trivial, and so, let $f \in C([0, T], C(\overline{\Omega})) = C(\overline{\Omega})$. Fix $(t, x) \in \overline{\Omega \times I}$. Then, $|f(s, y) - f(t, x)| \le |f(t, y) - f(t, x)| + |f(t, y) - f(s, y)| \le |f(t, y) - f(t, x)| + \|f(t, \cdot) - f(s, \cdot)\|_\infty$. If now s is close to t, and y is close to x, then $|f(s, y) - f(t, x)|$ is small.

This shows $u, u_i, u_t \in C([0, T], C(\overline{\Omega})) \in C(\overline{\Omega \times I})$, i.e. $u \in C^1(\overline{Q \times I})$.

To conclude, let $u \in L^2(I, H^1)$. Approximate u by $u_k := \sum_{i \le m_k} \phi_i^k(t) f_i^k$ as in point 2, and approximate f_i^k by suitable $g_i^k \in H^1(\Omega) \cap C^1(\overline{\Omega})$, to obtain $u_k := \sum_{i \le m_k} \phi_i^k(t) g_i^k$

Then $\|u - w_k\|_{L^2(I,H^1)} \le \|u_k - w_k\|_{L^2(I,H^1)} + \|u_k - u\|_{L^2(I,H^1)}$. We only need to estimate $\|u_k - w_k\|_{L^2(I,H^1)} \le T \displaystyle\sum_{i \le m_k} \left\|\phi_i^k\right\|_\infty \left\|f_i^k - g_i^k\right\|_{H^1}$. By the first point, $\left\|f_i^k - g_i^k\right\|_{H^1}$ can be made as small as it is necessary to conclude.

<u>Last remarks</u>

Again with reference to [11], consider the anisotropic spaces $H^{r,s} := L^2(I, H^r) \cap H^s(I, L^2)$. We restrict to the case $r = 1$, $s \ge 0$. Denote the traces tr_s defined in theorem 4.1, mapping $H^{1,s}(\Omega \times I) \to H^{1/2,s/2}(\partial\Omega \times I)$. For $\partial\Omega$ Lipschitz this theorem is still valid, as $1/2 \leq 1$, see the discussion above lemma 2.4 in [3]. As stated in [11], tr_s is an extension of $u \mapsto u|_{I\times\partial\Omega}$, defined on the dense suspace $C^\infty(\overline{Q \times I})$ of $H^{1,s}$ (that this space is dense can be proved as in lemma 2.22 of [3]). So, let $C^\infty(\overline{Q \times I}) \ni u_n \to_{H^{r,s}} u \in H^{1,s}$.

We have $\mathrm{tr}_s u_n = \mathrm{tr}_0 u_n$. Then, $u_n \to_{H^{1,s}} u$, $u_n \to_{H^{1,0}} u$, so that $\mathrm{tr}_s u_n \to_{H^{1/2,s/2}} \mathrm{tr}_s u$ (hence $\mathrm{tr}_0 u_n \to_{H^{1/2,0}} \mathrm{tr}_s u$) and $\mathrm{tr}_0 u_n \to_{H^{1/2,0}} \mathrm{tr}_\sigma u$.

Thus $\mathrm{tr}_0 u = \mathrm{tr}_s u$.

Using what we derived before, we can conclude the characterization of the traces in the anisotropic settign define

$\square$

And now some sanity checks in the case of Gelfand triples.

Proposition B.0.7 (Sanity checks for Gelfand triples)
Consider the following Gelfand triples (the diagram commutes):

$$\begin{tikzcd}
V \arrow[dr, hook, "a"] & & & V^* \arrow[d, hook, "c^*"] \\
& H \arrow[r, "r"] & H^* \arrow[ur, hook, "a^*"] \arrow[dr, hook, "b^*"'] & \\
W \arrow[uu, hook, "c"] \arrow[ur, hook, "b"'] & & & W^*
\end{tikzcd}$$

Here $W \subseteq V \subseteq H$ are all separable Hilbert spaces, a, b, c the trivial injections, r the Riesz isomorphism of H. We denote by i_V the Gelfand triple embedding $V \hookrightarrow V^*$, so, $i_V = a^* r a$.

Then:

1. $H^1(I, V) \subseteq W(I, V)$ with continuous embedding. The $W(I, V)$ derivative of $u \in H^1(I, V)$ is $i_V u_t$.

2. for $u \in W(I, W)$ with $(i_W u)' \in L^2(I, H)$ (i.e. $(i_W u)_t = b^* r h$ for h in $L^2(I, H)$) we obtain $u \in W(I, V)$ (i.e. $cu \in W(I, V)$), with derivative $(i_V cu)' = a^* r h$, so that also $(i_V cu)' \in L^2(I, H)$. It also holds $(i_V cu)'|_W = (i_W u)'$. h is also the weak derivative $L^2(I, H)$ of bu.

3. let $u, v \in W(I, V)$ with $u - v \in W$. Then $u - v \in W(I, W)$ with derivative $(i_W(u - v))' = (i_V u)'|_W - (i_V v)'|_W$.

Proof.

We use several times that time integrals and bounded linear static operators commute, see proposition B.0.1. ϕ denotes $\phi \in C_c^\infty(I)$.

<u>First point</u>

We need to check that $a^*rau \in H^1(I, V^*)$. This follows from proposition B.0.2, so that $(a^*rau)_t = a^*rau_t$.

<u>Second point</u>

At first we claim that h is a weak derivative of $bu \in L^2(I, H)$. In fact, $b^*r \int_I bu\phi' = \int_I (i_W u)\phi' = \{\ u \in W(I, W)\ \} = -\int_I (i_W u)'\phi = -\int_I b^*rh\phi = b^*r(-\int_I h\phi)$. By density (definition of Gelfand triple), b^* is injective, r is too, and thus $\int_I bu\phi' = -\int_I h\phi$, which shows that bu has weak derivative h, in the $H^1(I, H)$ sense.

And now $\int_I i_V cu\phi' = \int_I a^*racu\phi' = a^*r \int_I bu\phi' = \{$ by what we just proved $\} = -a^*r \int_I h\phi$, proving that $(i_V cu)' = a^*rh$.

Morevoer $(i_V cu)'|_W = c^*a^*rh = b^*rh = \{$ assumption $\} = (i_W u)'$.

<u>Third point</u>

We check the derivative. We have $\int_i i_W(u-v)\phi' = \{\ u - v \in W \subseteq V\ \} = \int_I b^*ra(u-v) = c^* \int_I (i_V u - i_V v)\phi' = -\int c^*((i_V u)' - (i_V v)')\phi$.

$\square$

C. Parabolic equations

Assumption C.0.1 (Basic assumption for parabolic problems)
Let $V \subseteq H$ be real separable Hilbert spaces, V dense in H. Then $H \hookrightarrow V^*$ is also dense, as stated in [14] at page 147. This embedding is $H \ni f \mapsto (f, \cdot)_H$. We thus obtain a Gelfand triple, and we have $W(I,V) \subseteq C(I,H)$.

Let $A : V \to V^*$ be linear bounded, $u \in W(I;V)$, $f \in L^2(I,V^*)$ and $u_0 \in H$.

We also assume that $\langle Av, v\rangle_{V^*,V} + \lambda \|v\|_H^2 \geq \alpha \|v\|_V^2$ for $\lambda \geq 0, \alpha > 0$.

We are interested in the following problem:

Problem C.0.2 (Abstract parabolic equation)

$$u_t + Au = f \text{ in } V^* \text{ and for a.e. } t \in (0,T) \tag{C.0.3}$$
$$u(0) = u_0 \tag{C.0.4}$$

Theorem C.0.5 (Basic well posedness of problem C.0.2)
Under assumption C.0.1, problem C.0.2 has a unique solution u. Moreover u satisfies the energy estimate:

$$\|u\|_{W(I,V)} + \|u\|_{C([0,T],H)} \leq c(\lambda, \alpha, \|A\|_{V^*}, T)(\|u_0\|_H + \|f\|_{L^2(I,V^*)}) \tag{C.0.6}$$

Proof.

See [6] at page 19, theorem 26. □

We can also obtain additional regularity. Here are further assumptions to make this possible.

Assumption C.0.7 (Assumptions for additional regularity)
We assume $u_0 \in V$, $f = f_1 + f_2 \in L^2(I,H) + H^1(I,V^*)$. We also need A to be symmetric (i.e. $\langle Au, v\rangle_{V^*,V} = \langle Av, u\rangle_{V^*,V}$).

Theorem C.0.8 (Regularity of time derivative)
Suppose assumption C.0.1 and assumption C.0.7. Then $u_t \in L^2(I,H)$ with the estimate:

$$\|u\|_{W(I,V)} + \|u\|_{C(I,H)} + \|u_t\|_{L^2(I,H)} \leq \tag{C.0.9}$$
$$c(\lambda, \alpha, \|A\|_{V^*}, T)(\|u_0\|_V + \|f_1\|_{L^2(I,H)} + \|f_2\|_{H^1(I,V^*)}) \tag{C.0.10}$$

That $u_t \in L^2(I,H)$ means precisely that there is $h \in L^2(I,H)$ with $a^* r h = (i_V u)'$, with the notation introduced in proposition B.0.7.

Proof.

We refer to page 26 of [6], theorem 28, and only prove the necessary modifications.

Product rule for A

We have

$$\int_0^t \langle Au_n, u_n' \rangle_{V^*,V} = \sum_{k,l \leq n} \langle Aw_k^n, w_l^n \rangle_{V^*,V} \int_0^t g_k^n g_l^{n'} =$$
$$\sum_{k,l \leq n} \langle Aw_k^n, w_l^n \rangle_{V^*,V} \left(-\int_0^t g_k^{n'} g_l^n + g_k^n(t) g_l^n(t) - g_k^n(0) g_l^n(0) \right)$$

By linearity at first and then symmetry we get:

$$= \langle Au_n, u_n \rangle_{V^*,V}(t) - \langle Au_n, u_n' \rangle_{V^*,V}(0) - \int_0^t \langle Au_n', u_n \rangle_{V^*,V} =$$
$$= \langle Au_n, u_n \rangle_{V^*,V}(t) - \langle Au_n, u_n' \rangle_{V^*,V}(0) - \int_0^t \langle Au_n, u_n' \rangle_{V^*,V}$$

so that:

$$\int_0^t \langle Au_n, u_n' \rangle_{V^*,V} = \frac{1}{2} \left(\langle Au_n, u_n \rangle_{V^*,V}(t) - \langle Au_n, u_n' \rangle_{V^*,V}(0) \right)$$

Estimate for right hand side

We have:

$$\int_0^t \langle f_2, u_n' \rangle_{V^*,V} = \sum_{k\leq n} \int_0^t g_k^{n'} \langle f_2, w_k^n \rangle_{V^*,V}$$

By the smoothness of f_2 we have that $t \mapsto \langle f_2(t), w_k^n \rangle_{V^*,V}$ is $H^1(0,T)$, in particular $AC[0,t]$, so that we can integrate by parts:

$$= -\sum_{k\leq n} \int_0^t g_k^n \langle f_2', w_k^n \rangle_{V^*,V} + \sum_{k\leq n} g_k^n(t) \langle f_2(t), w_k^n \rangle_{V^*,V} - \sum_{k\leq n} g_k^n(0) \langle f_2(0), w_k^n \rangle_{V^*,V} =$$
$$- \int_0^t \langle f_2', u_n \rangle_{V^*,V} + \langle f_2, u_n \rangle_{V^*,V}(t) - \langle f_2, u_n \rangle_{V^*,V}(0) =$$

Here we have used proposition B.0.2 to take the derivative inside the bracket.

Going to the absolute values:

$$\left| \int_0^t \langle f_2, u_n' \rangle_{V^*,V} \right| \leq \int_0^T |\langle f_2', u_n \rangle_{V^*,V}| + \|f_2(t)\|_{V^*} \|u_n(t)\|_V + \|f_2(0)\|_{V^*} \|u_n(0)\|_V \leq$$
$$\frac{1}{2} \|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2} \|u_n\|^2_{L^2(I,V)} + \frac{\alpha}{4} \|u_n(t)\|^2_V +$$
$$+\frac{4}{\alpha} \|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2} \|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2} \|u_{n0}\|^2_V$$

Now, u_n converges weakly in $L^2(I,V)$ by estimate (59) of [6] and thus $\frac{1}{2} \|u_n\|_{L^2(I,V)}$ is bounded. The term $\frac{\alpha}{4} \|u_n(t)\|_V$ can be pulled to the left hand side, u_{0n} is V convergent hence bounded. Therefore as in [6] we are able to conclude that u_n' is bounded in $L^2(I,H)$. We want to conclude $u_t \in L^2(I,H)$. We know for sure that $\langle u_m', w_j \rangle_{V^*,V} = \langle f - Au_m, w_j \rangle_{V^*,V}$, so that muliplication by a test function and integration yields $\int_I \langle u_m', w_j\phi \rangle_{V^*,V} = \int_I \langle f - Au_m, w_j\phi \rangle_{V^*,V}$. Because $u_m \rightharpoonup u$ in $L^2(I,V)$ we observe that, by proposition B.0.1 applied on $A \in L(V,V^*)$, it holds $\int_I \langle u_m', w_j\phi \rangle_{V^*,V} \to \int_I \langle u', w_j\phi \rangle_{V^*,V}$.

What's more, is that $u_m' \rightharpoonup h$ in $L^2(I,H)$, so that $\int_I \langle h, w_j \rangle_{V^*,V}\phi = \int_I \langle u', w_j \rangle_{V^*,V}\phi$. It means that for almost all t, $\langle h, w_j \rangle_{V^*,V} = \langle u', w_j \rangle_{V^*,V}$. And now we can really say that $u' \in L^2(I,H)$, which even more precisely means $(i_V u)' = a^* r h$ almost everywhere.

We also obtain that u_t is bounded by $c(\alpha)(\|f_2\|_{L^\infty(I,V^*)} + \|f_2\|_{L^2(I,V^*)} + \|u_0\|_V + \|u\|_{L^2(I,V)})$.

Note that, by [4], theorem 2 of page 286, we can estimate $\|f_2\|_{L^\infty(I,V^*)}$ by $c(T) \|f_2\|_{H^1(I,V^*)}$, so that the claim for the time derivative u_t is proven.

$\square$

For the case where $H = L^2$, $H^1 \supseteq V \supseteq H^1_0$, $f_2|_{H^1_0} = 0$ we have even more regularity available.

Theorem C.0.11 (Additional regularity)
Suppose assumption C.0.1 and assumption C.0.7.

Let additionally $H = L^2$, $H^1 \supseteq V \supseteq H^1_0$, $f_2|_{H^1_0} = 0$. Then $Au|_{H^1_0}$ extends to $\overline{Au_{H^1_0}} \in L^2(I,H)$ with:

$$\|u\|_{W(I,V)} + \|u\|_{C([0,T],H)} + \|u_t\|_{L^2(I,H)} + \left\|\overline{Au|_{H^1_0}}\right\|_{L^2(I,H)} \leq \tag{C.0.12}$$

$$c(\lambda, \alpha, \|A\|_{V^*}, T)(\|u_0\|_V + \|f_1\|_{L^2(I,H)} + \|f_2\|_{H^1(I,V^*)}) \tag{C.0.13}$$

Moreover $u_t + \overline{Au_{H^1_0}} = f_1$ in $L^2(0,T,L^2) \cong L^2(Q)$ and $\overline{Au|_{H^1_0}} = Au$ on H^1_0.

Proof.

For $v \in H^1_0$ we get $\langle Au, v\rangle_{V^*,V} = \langle f_1 - u_t, v\rangle_{V^*,V} = (f_1 - u_t, v)_H$, for almost all $t \in (0,T)$. From here we conclude that $Au(t)$ extends for a.a. t to an element of H with $(\overline{Au} - f_1 + u_t, v)_{L^2} = 0$ for all $v \in H^1_0$, almost all t. By density, $\overline{Au} - f_1 + u_t = 0$ in H for almost all t, so that $\overline{Au} = f_1 - u_t$ in $L^2(0,T,L^2) \cong L^2(Q)$.

This isometric isomorphism is stated in [14], page 144.

□

For our applications we also need to track the constants more precisely, which is accomplished in the next proposition.

Proposition C.0.14 (Tracking the costants)
With assumption C.0.1 there holds:

$$\|u\|^2_{C([0;T],H)} + \alpha\|u\|^2_{L^2(I,V)} \leq \exp(2\lambda T)(\|u_0\|^2_H + \alpha^{-1}\|f\|^2_{L^2(I,V^*)}) \tag{C.0.15}$$

$$\|u'\|_{L^2(I,V^*)} \leq \|A\|_{L(V,V^*)}\alpha^{-1/2}\sqrt{\exp(2\lambda T)}\|u_0\|_H + \tag{C.0.16}$$

$$\left(\|A\|_{L(V,V^*)}\alpha^{-1}\sqrt{\exp(2\lambda T)} + 1\right)\|f\|_{L^2(I,V^*)} \tag{C.0.17}$$

With additionally assumption C.0.7 we obtain:

$$C\|u'\|^2_{L^2(I,H)} \leq (1 + (1 + C_0)\alpha^{-1})\|f_2\|^2_{H^1(I,V^*)} + \tag{C.0.18}$$

$$(1 + \|A\|_{L(V,V^*)})\|u_0\|^2_V + C_0\|u_0\|^2_H + \tag{C.0.19}$$

$$\|f_1\|^2_{L^2(I,H)} + C_0\alpha^{-1}\|f_1\|^2_{L^2(I,V^*)} \tag{C.0.20}$$

with $C > 0$ a number independent of the problem.

Here $C_0 = 2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\exp(2\lambda T)$.

Proof.

No regularity

From page 21 of [6] we obtain that $\|u\|^2_{C([0;T],H)} + \alpha\|u\|^2_{L^2(I,V)} \leq \exp(2\lambda T)(\|u_0\|^2_H + \alpha^{-1}\|f\|^2_{L^2(I,V^*)})$.

In particular, $\sqrt{\alpha}\|u\|_{L^2(I,V)} \leq \sqrt{\exp(2\lambda T)}(\|u_0\|_H + \alpha^{-1/2}\|f\|_{L^2(I,V^*)})$, or $\|u\|_{L^2(I,V)} \leq \alpha^{-1/2}\sqrt{\exp(2\lambda T)}(\|u_0\|_H + \alpha^{-1/2}\|f\|_{L^2(I,V^*)})$.

Moreover $\|u'\|_{L^2(I,V^*)} \leq \|Au\|_{L^2(I,V^*)} + \|f\|_{L^2(I,V^*)} \leq \|A\|\,\|u\|_{L^2(I,V)} + \|f\|_{L^2(I,V^*)}$.

All in all, we obtain:

$$\|u\|^2_{C([0;T],H)} + \alpha\|u\|^2_{L^2(I,V)} \leq \exp(2\lambda T)(\|u_0\|^2_H + \alpha^{-1}\|f\|^2_{L^2(I,V^*)})$$

and:

$$\|u'\|_{L^2(I,V^*)} \leq \|A\|_{L(V,V^*)}\,\alpha^{-1/2}\sqrt{\exp(2\lambda T)}(\|u_0\|_H + \alpha^{-1/2}\|f\|_{L^2(I,V^*)}) + \|f\|_{L^2(I,V^*)}$$

More regularity

We tie back to page 25 of [6]. In particular:

$$\int_0^t \|u_n'\|^2_H + \int_0^t \langle Au_n, u_n'\rangle_{V^*,V} = \int_0^t (f_1, u_n')_H + \int_0^t \langle f_2, u_n'\rangle_{V^*,V}$$

Then:

$$\int_0^t \langle Au_n, u_n'\rangle_{V^*,V} \geq \frac{\alpha}{2}\|u_n(t)\|^2_V - \frac{\lambda}{2}\|u_n(t)\|^2_H - \frac{\|A\|}{2}\|u_{n0}\|_V$$

whereas, as in the proof of theorem C.0.8:

$$\left|\int_0^t \langle f_2, u_n'\rangle_{V^*,V}\right| \leq \frac{1}{2}\|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)} + \frac{\alpha}{4}\|u_n(t)\|^2_V + \\ + \frac{4}{\alpha}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2}\|u_{n0}\|^2_V$$

Also:

$$\int_0^t (f_1, u_n')_H \leq \frac{1}{2}\|f_1\|^2_{L^2(I,H)} + \frac{1}{2}\int_0^t \|u_n'\|^2_H$$

Putting all together:

$$\begin{aligned}\int_0^t \|u_n'\|^2_H + \frac{\alpha}{2}\|u_n(t)\|^2_V - \frac{\lambda}{2}\|u_n(t)\|^2_H - \frac{\|A\|}{2}\|u_{n0}\|_V \\ \frac{1}{2}\|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)} + \frac{\alpha}{4}\|u_n(t)\|^2_V + \\ +\frac{4}{\alpha}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2}\|u_{n0}\|^2_V + \\ +\frac{1}{2}\|f_1\|^2_{L^2(I,H)} + \frac{1}{2}\int_0^t \|u_n'\|^2_H\end{aligned}$$

which brings us to:

$$\begin{aligned}\frac{1}{2}\int_0^t \|u_n'\|^2_H + \frac{\alpha}{4}\|u_n(t)\|^2_V - \frac{\lambda}{2}\|u_n(t)\|^2_H \leq \\ \frac{1}{2}\|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)} + \\ +\frac{8+\alpha}{2\alpha}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1+\|A\|}{2}\|u_{n0}\|^2_V + \\ +\frac{1}{2}\|f_1\|^2_{L^2(I,H)}\end{aligned}$$

and thus, because norms are lower semicontinuous and because we have weak convergence of the time derivative, and V-strong convergence of the initial data:

$$\begin{aligned}\frac{1}{2}\int_0^T \|u'\|^2_H \leq \frac{1}{2}\|f_2'\|^2_{L^2(I,V^*)} + \frac{8+\alpha}{2\alpha}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1+\|A\|}{2}\|u_0\|^2_V + \frac{1}{2}\|f_1\|^2_{L^2(I,H)} + \\ \mathrm{limsup}_n\left(\frac{\lambda}{2}\|u_n\|^2_{C([0,T],H)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)}\right)\end{aligned}$$

Using a purely numeric constant C without dependences on the problem we can write:

$$\int_0^T \|u'\|_H^2 \leq \|f_2'\|^2_{L^2(I,V^*)} + C(1+\alpha^{-1})\|f_2\|^2_{L^\infty(I,V^*)} + C(1+\|A\|)\|u_0\|_V^2 + \|f_1\|^2_{L^2(I,H)} + C\text{limsup}_n\left(\frac{\lambda}{2}\|u_n\|^2_{C([0,T],H)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)}\right)$$

For the last term, employing the exact argument as in the first part of the proof:

$$\begin{aligned}
\text{limsup}_n\left(\frac{\lambda}{2}\|u_n\|^2_{C([0,T],H)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)}\right) &\leq \\
2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\text{limsup}_n\left(\|u_n\|^2_{C([0,T],H)} + \alpha\|u_n\|^2_{L^2(I,V)}\right) &\leq \\
2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\exp(2\lambda T)(\|u_0\|_H^2 + \alpha^{-1}\|f_1+f_2\|^2_{L^2(I,V^*)}) &\leq \\
2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\exp(2\lambda T)(\|u_0\|_H^2 + 2\alpha^{-1}\|f_1\|^2_{L^2(I,V^*)} + 2\alpha^{-1}\|f_2\|^2_{L^2(I,V^*)}) &\leq \\
CC_0(\|u_0\|_H^2 + \alpha^{-1}\|f_1\|^2_{L^2(I,V^*)} + \alpha^{-1}\|f_2\|^2_{L^2(I,V^*)})&
\end{aligned}$$

where $C_0 = 2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\exp(2\lambda T)$ and C is a purely numeric constant without dependences on the problem.

Therefore:

$$C\int_0^T \|u'\|_H^2 \leq \|f_2'\|^2_{L^2(I,V^*)} + (1+\alpha^{-1})\|f_2\|^2_{L^\infty(I,V^*)} + (1+\|A\|)\|u_0\|_V^2 + \|f_1\|^2_{L^2(I,H)} + C_0(\|u_0\|_H^2 + \alpha^{-1}\|f_1\|^2_{L^2(I,V^*)} + \alpha^{-1}\|f_2\|^2_{L^2(I,V^*)})$$

The embedding $H^1(I,V^*) \hookrightarrow C([0,T],V^*)$ has norm that only depends on T, which follows from the equality $f_2(t) = f_2(s) + \int_s^t f_2'$, for $0 \leq s \leq t \leq T$, a bound being $1+T$.

Thus:

$$\begin{aligned}
C\int_0^T \|u'\|_H^2 \leq (1+(1+C_0)\alpha^{-1})\|f_2\|^2_{H^1(I,V^*)} &+ \\
(1+\|A\|)\|u_0\|_V^2 + C_0\|u_0\|_H^2 &+ \\
\|f_1\|^2_{L^2(I,H)} + C_0\alpha^{-1}\|f_1\|^2_{L^2(I,V^*)}&
\end{aligned}$$

$\square$

C.1. Application to inhomogeneous parabolic problems

C.1.1. Inhomogeneous Dirichlet problem

We make the following assumption.

Assumption C.1.1.1 (Assumptions for problem C.1.1.2)
We assume $\Omega \subset\subset D$ to be bounded Lipschitz domains, so that $U := D \setminus \Omega$ is bounded Lipschitz too and the trace operator is bounded surjective onto $H^{1/2}(\partial U)$, with a right inverse E (see theorem 3.37 at page 102 of [12]). For such a choice we also have $H^1_0 = H^1 \cap \ker \operatorname{tr}$, see [10], page 595, theorem 18.7.

Moreover, we select $f \in H^1(I, H^{1/2}(\Gamma_f))$, $f(0) = 0$.

Note that, given a bounded extension operator $E : H^{1/2}(\partial U) \to H^1(U)$, we obtain by proposition B.0.2 that $Ef \in H^2(I, H^1(U))$. We have defined $\operatorname{tr} u(t) := tr(u(t))$ and analogously $Eu(t) := E(u(t))$ (see proposition B.0.6).

Call $H = L^2(U)$, $V = \{v \in H^1(U), \operatorname{tr} u = 0 \text{ on } \Gamma_m\} =: H^1_c$. V is a closed subspace of H^1, which is Hilbert separable, hence also Hilbert separable. We norm it with the full H^1 norm. Because $H^1_0(U)$ is dense in H, so is V and we obtain a Gelfand triple. That V is a closed subspace of H^1 follows from the observation that if $u_n \to u$ in the V norm, then $\operatorname{tr} u_n \to \operatorname{tr} u$ in $L^2(\partial U)$. We can take an almost everywhere pointwise convergent sequence, so that $\operatorname{tr} u_n \to \operatorname{tr} u$ a.e., and by the fact that Γ_m has positive Hausdorff measure, we conclude $\operatorname{tr} u = 0$ on Γ_m.

We define $A := H^1 \to H^{1*}$ by $(Au)v := \int_u \nabla u \nabla v$. This operator can be the recast to $V \to H^{-1}$ and $V \to V^*$.

The problem under consideration is the following. For $U = D \setminus \Omega$ we have:

Problem C.1.1.2 (Inhomogeneous heat equation, Dirichlet conditions)

$$u_t - \Delta u = 0 \text{ in } (0,T) \times U \tag{C.1.1.3}$$
$$u(\Sigma_f) = f \tag{C.1.1.4}$$
$$u(\Sigma_m) = 0 \tag{C.1.1.5}$$
$$u(0) = 0 \tag{C.1.1.6}$$

By this we mean:

$$u \in W(I, H^1_c) \tag{C.1.1.7}$$
$$u_t|_{H^{-1}} + Au = 0 \text{ in } H^{-1} \text{ and for a.e. } t \in (0,T) \tag{C.1.1.8}$$
$$\mathrm{tr} u = f \text{ on } \Sigma_f \tag{C.1.1.9}$$
$$u(0) = 0 \tag{C.1.1.10}$$

Theorem C.1.1.11 (Well posedness and regularity for problem C.1.1.2)
Given assumption C.1.1.1, the solution u to problem C.1.1.2 is unique with $u_t \in L^2(I, H)$.

The problem is equivalent to:

Problem C.1.1.12 (Equivalent formulation with extension)

$$u_0 \in W(I, H^1_0) \tag{C.1.1.13}$$
$$u_0' + Au_0 = -((\bar{u}', \cdot)_H + A\bar{u}) \text{ in } H^{-1} \text{ and for a.e. } t \in (0,T) \tag{C.1.1.14}$$
$$u_0(0) = 0 \tag{C.1.1.15}$$

with $\bar{u}$ any given $\bar{u} \in H^1(I, H^1_c(U))$ such that $\mathrm{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$. This means that u solves problem C.1.1.2 $\implies$ $u - \bar{u}$ solves problem C.1.1.12, and if $u_0(\bar{u})$ solves problem C.1.1.12, then $\bar{u} + u_0(\bar{u})$ solves problem C.1.1.2.

Furthermore:

$$\|u\|^2_{C([0;T],H)} + \|u\|^2_{L^2(I,H)} + \|\nabla u\|^2_{L^2(I,H)} + \|u'\|^2_{L^2(I,H)} \leq C(T)\,\|\bar{u}\|^2_{H^1(I,V)} \tag{C.1.1.16}$$

with $C > 1$, only dependent on T, smoothly, exploding for large T.

Proof.

Extension of the boundary data

Let $\bar{u} \in H^1(I, H^1_c(U))$ be such that $\mathrm{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$. We can choose for instance $E\tilde{f}$, see proposition B.0.6, where $\tilde{f} = 0$ on Σ_m, $\tilde{f} = f$ on Σ_f. $\tilde{f} \in H^1(I, H^{1/2}(\partial U))$, because Γ_f and Γ_m have positive distance (see the definition of the norm in [7], page 20).

Reformulation (first part)

Consider the following commutative diagram, where $V = H^1_c$, $W = H^1_0$:

$$\begin{tikzcd}
H^1_c \arrow[dr, hook, "a"] & & & (H^1_c)^* \arrow[dd, hook, "c^*"] \\
& H \arrow[r, "r"] & H^* \arrow[ur, hook, "a^*"] \arrow[dr, hook, "b^*"'] & \\
H^1_0 \arrow[uu, hook, "c"] \arrow[ur, hook, "b"'] & & & H^{-1}
\end{tikzcd}$$

Here, a, b, c are the trivial injections, r the Riesz isomorphism $h \mapsto (h, \cdot)_H$.

Now $(i_W(u-\bar{u}))'+A(u-\bar{u}) = (i_V u)'|_{H^{-1}} - (i_V \bar{u})'|_{H^{-1}} + Au - A\bar{u} = \{$ proposition B.0.7 $\} = (i_V u)'|_{H^{-1}} - (i_V \bar{u}_t)|_{H^{-1}} + Au - A\bar{u} = -(i_V \bar{u}_t)|_{H^{-1}} - A\bar{u}$ if u solves problem C.1.1.2, where $\bar{u}_t$ is the weak derivative of $\bar{u}$ in the $H^1(I, V)$ sense. Call $u_0 = u - \bar{u}$. By again proposition B.0.7, $u_0 \in W(I, H^1_0)$.

This motivates us to consider the problem:

$$u_0 \in W(I, H^1_0) \tag{C.1.1.17}$$
$$u_0' + Au_0 = -(f_1 + f_2) \text{ in } H^{-1} \text{ and for a.e. } t \in (0, T) \tag{C.1.1.18}$$
$$u_0(0) = 0 \tag{C.1.1.19}$$

Here, $f_1 := (i_V \bar{u}_t)|_{H^{-1}} = c^* a^* r a \bar{u}_t = b^* r(a\bar{u}_t) \in L^2(I, H)$.

Moreover, $A \in L(V, H^{-1})$, so, by proposition B.0.2, $f_2 := A\bar{u} \in H^1(I, H^{-1})$.

Existence

By theorem C.0.11 we get a solution of the above problem with $u_0' \in L^2(I, H)$.

And now, let $u := \bar{u} + cu_0 = \bar{u} + u_0$. We claim it is a solution. The initial and boundary conditions are surely satisfied. We check it is in $W(I, V)$ and is satisfies the partial differential equation.

By proposition B.0.7, we have both $\bar{u}, cu_0 \in W(I, V)$. The derivative of $\bar{u}$ becomes $i_V \bar{u}_t$, see proposition B.0.7. Therefore $(i_V(\bar{u} + cu_0))'|_{H^1_0} = c^*(i_V(\bar{u} + cu_0))' = c^* i_V \bar{u}_t + c^* i_V (cu_0)' = b^* r(a\bar{u}_t) + i_W(u_0)'$ by proposition B.0.7.

Using the pde of u_0, ... $= b^* r(a\bar{u}_t) - Au_0 - f_1 - f_2 = -A(u_0 + \bar{u})$.

Uniqueness

For two solutions u_1, u_2 of $C.1.1.2$ we can form $d := u_1 - u_2 \in W(I, H^1_0)$ by proposition B.0.7. Clearly, $d(0) = 0$. Moreover, $(i_{H^1_0} d)' = \{$ proposition B.0.7 $\} = (i_V u_1)'|_{H^1_0} - (i_V u_2)'|_{H^1_0} = A(u_1 - u_2)$.

By uniqueness stated in theorem C.0.5 we obtain $d = 0$ in $L^2(I, H)$, so that the solution is unique and doesn't depend on the choice of the extension of the Dirichlet datum.

Reformulation (part 2)

Therefore $u = \bar{u} + u_0$ above is the unique solution of problem C.1.1.2. So, given any $\bar{u} \in H^1(I, H^1_c(U))$ such that $\mathrm{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$, we can construct u_0 as above and get $u = \bar{u} + u_0$ solving problem C.1.1.2.

Viceversa, let u solve problem C.1.1.2. Call $u_0 = u - \bar{u}$. Then, as seen above, $u_0 \in W(I, H^1_0)$ and $(i_V(u - \bar{u}))'|_{H^{-1}} + A(u - \bar{u}) = (i_V u)'|_{H^{-1}} - (i_V \bar{u})'|_{H^{-1}} + Au - A\bar{u} = \{$ proposition B.0.7 $\} = (i_V u)'|_{H^{-1}} - (i_V \bar{u}_t)|_{H^{-1}} + Au - A\bar{u} = -(i_V \bar{u}_t)|_{H^{-1}} - A\bar{u}$ if u solves problem C.1.1.2, where $\bar{u}_t$ is the weak derivative of $\bar{u}$ in the $H^1(I, V)$ sense. Call $u_0 = u - \bar{u}$. By again proposition B.0.7, $(i_W(u-\bar{u}))' + A(u-\bar{u}) = (i_V u)'|_{H^{-1}} - (i_V \bar{u})'|_{H^{-1}} + Au - A\bar{u} = (i_V u)'|_{H^{-1}} - (i_V \bar{u}_t)|_{H^{-1}} + Au - A\bar{u} = -(i_V \bar{u}_t)|_{H^{-1}} - A\bar{u} = -b^* r(a\bar{u}_t) - A\bar{u}$. Moreover, $u_0(0) = 0$, so that u_0 solves problem C.1.1.12.

Regularity

Let $u = \bar{u} + u_0$ be the unique solution, as before, of problem C.1.1.2. From proposition B.0.7 we know $(i_V(\bar{u}))' = i_V(\bar{u}_t) = a^* r(a\bar{u}_t)$, and $i_V(cu_0)' = a^* r(u_0')$, for $u_0' \in L^2(I, H)$ the representative of $(i_W(u_0))'$, equivalently, the weak derivative of u_0 in the $H^1(I, H)$ sense. It follows that $(i_V u)' = a^* r(a\bar{u}_t + u_0')$, proving the additional time smoothness claim.

Stability

Let $\bar{u} \in H^1(I, H^1_c(U))$ such that $\mathrm{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$. Consider u_0. Then, by C.0.14:

$$\|u_0\|^2_{C([0;T],H)} + \alpha \|u_0\|^2_{L^2(I,H^1_0)} \leq \exp(2\lambda T)\alpha^{-1} \|(\bar{u}', \cdot)_H + A\bar{u}\|^2_{L^2(I,H^{-1})}$$

$$C \|u_0'\|^2_{L^2(I,H)} \leq (1 + (1 + C_0)\alpha^{-1}) \|A\bar{u}\|^2_{H^1(I,H^{-1})} + \|(\bar{u}', \cdot)_H\|^2_{L^2(I,H)} + C_0\alpha^{-1} \|(\bar{u}', \cdot)_H\|^2_{L^2(I,H^{-1})}$$

$C_0 = 2^{-1} \max(1, \lambda) \max(1, \alpha^{-1}) \exp(2\lambda T)$.

We norm H^1_0 with the full H^1 norm too. Then:

$$\|(\bar{u}', \cdot)_H + A\bar{u}\|_{L^2(I,H^{-1})} \leq \sup_{\|v\|_{L^2(I,H^1_0)}=1} \|\bar{u}'\|_{L^2(I,H)} \|v\|_{L^2(I,H)} + \|\nabla\bar{u}\|_{L^2(I,H)} \|\nabla v\|_{L^2(I,H)} \leq C(\|\bar{u}'\|_{L^2(I,H)} + \|\nabla\bar{u}\|_{L^2(I,H)})$$

By proposition B.0.2, $\|A\bar{u}\|_{H^1(I,H^{-1})} \leq \|A\|_{L(V,H^{-1})} \|\bar{u}\|_{H^1(I,V)}$ (we could apply it since H^{-1} is separable, as a dual of a reflexive Banach space).

Finally, $\|(\bar{u}', \cdot)_H\|^2_{L^2(I,H^{-1})} \leq \|\bar{u}'\|^2_{L^2(I,H)}$.

We can then say:

$$\begin{aligned} \|u_0\|^2_{C([0;T],H)} + C\alpha \|u_0\|^2_{L^2(I,H_0^1)} &\leq \exp(2\lambda T)\alpha^{-1} \|\bar{u}\|^2_{H^1(I,V)} \\ C \|u_0'\|^2_{L^2(I,H)} &\leq ((1 + (1 + C_0)\alpha^{-1}) \|A\|^2_{L(V,H^{-1})} + 1 + C_0\alpha^{-1}) \|\bar{u}\|^2_{H^1(I,V)} \end{aligned}$$

Now, $\langle Av, v\rangle_{H^{-1},H_0^1} + 1 \cdot \|v\|^2_H = 1 \cdot \|v\|^2_{H_0^1}$, so that $\alpha = \lambda = 1$. Moreover, $\langle Au, v\rangle_{H^{-1},H_0^1} \leq \|u\|_V \|v\|_{H_0^1}$, i.e. $\|A\|_{L(V,H^{-1})} \leq 1$.

Therefore $\|u_0\|^2_{C([0;T],H)} + \|u_0\|^2_{L^2(I,H_0^1)} + \|u_0'\|^2_{L^2(I,H)} \leq C(T) \|\bar{u}\|^2_{H^1(I,V)}$ with $C > 1$, only dependent on T, smoothly, exploding for large T.

Now, let's analyse the norms of $\bar{u}$. Because $\bar{u} \in H^1(I,V)$, then, $\bar{u} \in C([0,T],V) \hookrightarrow C([0,T],H)$, where the embedding is non-expansive by the choice of the norm of V. Therefore $\|\bar{u}\|_{C([0;T],H)} \leq \|\bar{u}\|_{C([0;T],V)} \leq (1+T) \|\bar{u}\|_{H^1(I,V)}$. We can therefore conclude that $\|u\|^2_{C([0;T],H)} + \|u\|^2_{L^2(I,H_0^1)} + \|u'\|^2_{L^2(I,H)} \leq C(T) \|\bar{u}\|^2_{H^1(I,V)}$ with $C > 1$, only dependent on T, smoothly, exploding for large T.

□

C.1.2. Inhomogeneous Neumann-Dirichlet problem

We make the following assumption.

Assumption C.1.2.1 (Assumptions for problem C.1.1.2)
We keep assumption C.1.1.1 (apart from the Dirichlet datum). We consired $g \in H^1(I, L^2(\Gamma_f))$, $g(0) = 0$.

Again, call $H = L^2(U)$, $V = \{v \in H^1(U), \mathrm{tr} u = 0 \text{ on } \Gamma_m\} =: H_c^1$. H, V induce a Gelfand triple as seen before.

The problem under consideration is:

Problem C.1.2.2 (Inhomogeneous heat equation, Neumann conditions)

$$u_t - \Delta u = 0 \text{ in } (0,T) \times U \tag{C.1.2.3}$$
$$\partial_\nu u(\Sigma_f) = g \tag{C.1.2.4}$$
$$u(\Sigma_m) = 0 \tag{C.1.2.5}$$
$$u(0) = 0 \tag{C.1.2.6}$$

By this we mean:

$$u \in W(I, H_c^1) \tag{C.1.2.7}$$
$$u_t + Au = G \text{ in } V^* \text{ and for a.e. } t \in (0,T) \tag{C.1.2.8}$$
$$u(0) = 0 \tag{C.1.2.9}$$

where $\langle G(t), v\rangle_{V^*,V} := \int_{\Gamma_f} g(t)\mathrm{tr}v d\sigma$, σ the 1-codimensional Hausdorff measure, and A was introduced before in $L(V, H^{-1})$.

By proposition B.0.2, $G \in H^1(I, V^*)$. In fact, define $T : L^2(\Sigma_f) \to V^*$ by $\langle Tg, v\rangle_{V^*,V} := \int_{\Gamma_f} g\mathrm{tr}v d\sigma$. Then, $\langle Tg, v\rangle_{V^*,V} \leq \|g\|_{L^2(\Gamma_f)} \|v\|_V$ by trace theory. Now, $G(t) = Tg(t)$.

Moreover, $\langle Av, v\rangle_{V^*,V} + 1 \cdot \|v\|_H = 1 \cdot \|V\|$, so that we can immediately conclude:

Theorem C.1.2.10 (Well posedness and regularity for problem C.1.2.2)
Given assumption C.1.2.1, the solution u to problem C.1.2.2 is unique with $u_t \in L^2(I, H)$.

Furthermore:

$$\|u\|^2_{C([0;T],H)} + \|u\|^2_{L^2(I,H)} + \|\nabla u\|^2_{L^2(I,H)} + \|u'\|^2_{L^2(I,H)} \leq C(T) \|g\|^2_{H^1(I,L^2(\Gamma_f))} \tag{C.1.2.11}$$

with $C > 1$, only dependent on T, smoothly, exploding for large T.

Proof.

It is an application of theorem C.0.5, theorem C.0.8 and proposition C.0.14. □

C.2. Reformulation of parabolic equations

We just saw that the two parabolic equations of interest can be recasted into the problem of finding $u \in W(I, V)$, $u(0) = 0$, $u_t + Au = f$ for a.e. t in V^*, with notation from preceding sections.

In particular, $f \in L^2(I, V^*)$ and so is Au (because $A \in L(V, V^*)$, and by B.0.2).

Call then $E(u) := u_t + Au - f \in L^2(I, V^*)$ and $W_0(I, V)$ the $W(I, V)$ functions with zero initial value. Then, the differential equation reads $\langle E(u)(t), v\rangle_{V^*,V} = 0$ for all $v \in V$, for a.a. t, equivalently, $E(u) = 0$ for a.a. t. Thus, we are interested in the abstract problem:

Problem C.2.1 (Even more abstract parabolic equation)
Given a function $E : W(I, V) \to L^2(I, V^*)$, find $u \in W_0(I, V)$, such that $E(u) = 0$ for a.a. t.

We can view $L^2(I, V^*) \cong L^2(I, V)^*$.

Hence $\langle E(u), v\rangle_{L^2(I,V)^*,L^2(I,V)} = \int_I \langle E(u)(t), v(t)\rangle_{V^*,V} dt$ (see [8], theorem 1.31 at page 39).

Now, by B.0.4, we get $C_c^\infty(I, V) \subseteq H^1(I, V) \subseteq W(I, V)$. Actually, $C_c^\infty(I, V) \subseteq W^0(I, V)$, the functions of $W(I, V)$ having zero terminal values.

We have therefore $C_c^\infty(I, V) \subseteq W^0(I, V) \subseteq L^2(I, V)$, which implies that $W^0(I, V) \subseteq L^2(I, V)$ is dense in $L^2(I, V)$. We can then formulate:

Proposition C.2.2 (Equivalent testing)
Let $E : W(I, V) \to L^2(I, V^*)$, and $u \in W_0(I, V)$.

Then:

$$\begin{aligned} E(u) &= 0 \\ &\Longleftrightarrow \\ \langle E(u), v\rangle_{L^2(I,V)^*,L^2(I,V)} = 0 \quad \forall v &\in L^2(I, V) \\ &\Longleftrightarrow \\ \langle E(u), v\rangle_{L^2(I,V)^*,L^2(I,V)} = 0 \quad \forall v &\in W^0(I, V) \end{aligned}$$

D. Facts about domains transformations

Throughout, D is a bounded Lipschitz domain. We define as in [13] the following spaces of transformations:

Definition D.0.1 (Spaces of transformations)
We define:

- $\mathcal{M} := \{\tau : \mathbb{R}^n \to \mathbb{R}^n \text{ measurable }\}/\sim$, the quotient being the almost everywhere equal relation
- $\mathcal{V}^k = \{\tau \in \mathcal{M}, \tau - \mathrm{Id} \in W^{k,\infty}(\mathbb{R}^n, \mathbb{R}^n)\}$, $k \geq 1$
- $U : \mathcal{V}^k \to \mathrm{C}^{0,1}(\mathbb{R}^n; \mathbb{R}^n)$, and $U : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{C}^{0,1}(\mathbb{R}^n; \mathbb{R}^n)$ the application "unique Lipschitz continuous representative" (see proposition A.0.3)
- $\circ : \mathcal{V}^1 \times \mathcal{V}^1 \to \mathcal{V}^1$ and $\circ : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \times \mathcal{V}^1 \to W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ by $\tau_1 \circ \tau_2 = [U(\tau_1) \circ \tau]$, for any $[\tau] = \tau_2$, $[]$ being an equivalence class according to $\sim$.
- $\mathcal{T}^k = \{\tau \in \mathcal{V}^k \text{ with an } \eta \in \mathcal{V}^k, \tau \circ \eta = \eta \circ \tau = \mathrm{Id}\}$. Any such η is unique, we denote it by τ^{-1} and we have that $U(\tau)$ is a Lipschitz homeomorphism with $U(\tau^{-1}) = U(\tau)^{-1}$

Observation D.0.2 (A technicality). Technically, in the original definition of [13], τ need not to be a continuous function, although this is suggested e.g. in remarque 2.1 at page II-4.

Going to equivalence classes of τ makes the identification with continuous functions more precise, as we now show.

<u>One implication</u>

Let $\tau : \mathbb{R}^n \to \mathbb{R}^n$ with $[\tau - \mathrm{Id}] \in W^{k,\infty}$. Then τ is equal a.e. to a (Lebesgue) measurable function, hence also (Lebesgue) measurable, and thus $[\tau] \in \mathcal{V}^k$ as we have defined it.

Now, suppose τ is a bijection, and $[\tau^{-1} - \mathrm{Id}] \in W^{k,\infty}$ too. Then $\tau = \mathrm{Id} + g = G, \tau^{-1} = \mathrm{Id} + h = H$ almost everywhere. Here, G, H are at least Lipschitz. But then $\tau \circ H = \mathrm{Id}$

a.e., and since H is Lipschitz, we can conclude also $G \circ H = \mathrm{Id}$ a.e., so, everywhere. With a symmetric reasoning, we are lead to $G = H^{-1}$, so that G is bi-Lipschitz.

Thus, $[\tau] \circ [\tau^{-1}] := [U(\tau) \circ U(\tau^{-1})] = [G \circ G^{-1}] = \mathrm{Id}$ and an analogous reasoning leads to $[\tau] \in \mathcal{T}^k$ as we have defined it.

<u>The other implication</u>

It is immediate for $\mathcal{V}^k$ and for $\mathcal{T}^k$, in the equivalence class of $\tau \in \mathcal{T}^k$ there is a unique $U(\tau)$ at least bi-Lipschitz, hence invertible, with $[U(\tau)] = \tau$.

This shows that:

1. $\{\tau : \mathbb{R}^n \to \mathbb{R}^n \text{ with } [\tau - \mathrm{Id}] \in W^{k,\infty}\}/ \sim = \mathcal{V}^k$
2. $\{\tau : \mathbb{R}^n \to \mathbb{R}^n \text{ bijection with } [\tau^{\pm 1} - \mathrm{Id}] \in W^{k,\infty}\}/ \sim = \mathcal{T}^k$

We need to check the well-posedness of $\circ$.

Proposition D.0.3
$\circ : \mathcal{V}^1 \times \mathcal{V}^1 \to \mathcal{V}^1$ and $\circ : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \times \mathcal{V}^1 \to W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ are well defined.

Proof.

We start by $\circ : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \times \mathcal{V}^1 \to \mathcal{V}^1$. Aside from its measurability, which follows by the continuity of $U(\theta)$, we have $\theta \circ \tau = [U(\theta) \circ U(\tau)]$ for instance, the latter being is a bounded Lipschitz map, so it remains in $W^{1,\infty}$. For the second claim, just write $\eta \circ \tau - \mathrm{Id} = (\eta - \mathrm{Id}) \circ \tau + \tau - \mathrm{Id}$ and use the first part.

□

Proposition D.0.4
$\circ : \mathcal{M} \times \mathcal{T}^1 \to \mathcal{M}$ given by $[f] \circ [U(\psi)] = [f \circ U(\psi)]$ is well defined.

It is also well defined from $\{[f], f : \mathbb{R}^n \to \mathbb{R}^k \text{ continuous }\} \times \mathcal{M}$ into measurable functions, by $f \circ [\phi] := [U(f) \circ \phi]$.

Proof.

Suppose $f, g \in [f]$. Then $f = g$ everywhere but on the null set E. Because $\psi \in \mathcal{T}^1$ we know that $U(\psi)$ is a Lipschitz homeomorphism, so that $U(\psi)^{-1}(E)$ has zero measure by the lemma of Vitali. For the same reason, the composition is measurable, see [13], remarque 2.2, page II-7.

□

Proposition D.0.5 (Chain rule for $k = 1$)
Let $f \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ or $\mathcal{V}^1$, together with $\psi \in \mathcal{T}^1$. Then:

- $f \circ \psi$ has essentially bounded weak derivatives, and $D(f \circ \psi) = Df \circ \psi D\psi$. The equality holds a.e. also for the classical derivatives.
- $(D(\psi^{-1}))^{-1} = D\psi \circ \psi^{-1}$, where $(D(\psi^{-1}))^{-1} := [(DU(\psi^{-1}))^{-1}]$, the representative being a.e. invertible. The equality holds a.e. also for the classical derivatives.
- $|\det(D\psi)|$ is an essentially bounded measurable function with $|\det(D\psi)| \geq \delta > 0$ a.e..

Proof.

Weak derivatives

We notice that $f \circ \phi$ has a unique Lipschitz representative, that is $U(f) \circ U(\phi)$. The desired formula follows as in [13], lemme 2.1 at page II-6, for the classical derivatives (or from [15], page 53, theorem 2.2.2 in the case of left composition by $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ vector fields).

For the weak derivatives:

- $U(f)$ is Lipschitz, so that $DU(f)$, the classical derivative, is also the weak derivative Df (note that f need not to be essentially bounded to state this). The latter is a measurable function, as a.e. limit of difference quotients.
- $DU(f) \circ U(\psi)$, as pointed out in [13], remarque 2.2, page II-7, is measurable. It is also essentially bounded.
- By proposition D.0.4 we observe that $DU(f) \circ U(\psi)$ represents $Df \circ \psi$
- $D\psi = [DU(\psi)]$ as seen above
- the product of equivalence classes is always defined as the product of their representatives

Therefore $Df \circ \psi D\psi = [DU(f) \circ U(\psi) DU(\psi)]$.

And now, because $f \circ \phi$ is Lipschitz, it has weak derivatives, $D(f \circ \phi)$, equal to the classical derivatives $DU(f \circ \phi) = D(U(f) \circ U(\psi)) = DU(f) \circ U(\psi) DU(\psi)$, where the last equality holds a.e., as mentioned at the beginning of the proof.

This let us conclude the first claim.

Inverse Jacobian

For the second one, put $f = \psi^{-1}$. Then, for the classical derivatives, $I = DU(\psi) \circ$

$U(\psi)^{-1}DU(\psi^{-1})$ a.e., so that both $DU(\psi)\circ U(\psi)^{-1}, DU((\psi)^{-1})$ are invertible as matrices, a.e..

Determinant

We have defined $|\det(D\psi)| := [|\det DU(\psi)|]$, see proposition D.0.4. The claim follows as in lemme 4.2, pag. IV-7 of [13], and because det is a polynomial of essentially bounded functions.

□

We go on to define the space of admissible transformations.

Definition D.0.6 (Admissible transformations)
We define $\Theta := \{\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ with $\theta|_D \in W^{2,\infty}(D;\mathbb{R}^n)$ and $\theta = 0$ on $\mathbb{R}^n \setminus D\}$, a vector space.

We also define $\mathcal{T} := \{\tau \in \mathcal{T}^1, \tau^{\pm 1}|_{\mathbb{R}^n\setminus D} = \mathrm{Id}, \tau^{\pm 1}|_D \in W^{2,\infty}(D;\mathbb{R}^n)\}$.

Proposition D.0.7 (Some group properties of $\mathcal{T}$)
Let $\eta, \tau \in \mathcal{T}, \theta \in \Theta$. Then:

- $\eta \circ \tau \in \mathcal{T}$
- $\theta \circ \tau \in \Theta$
- Id is the neutral element
- $\eta^{-1} \in \mathcal{T}$

Proof.

Stability under composition (regularity)

We start by showing that $\tau \circ \eta|_D \in W^{k,2}$.

By $\tau \in \mathcal{V}^1$, we sure have that $\tau \circ \eta \in \mathcal{V}^1$ by proposition D.0.4. So, $U(\tau)\circ U(\eta) = \mathrm{Id}+\theta$, a bounded Lipschitz function, which is $W^{1,\infty}(D)$. Morevoer $D(U(\tau)\circ U(\eta)) = D(U(\tau))\circ U(\eta)DU(\eta)$ almost everywhere in D. Actually, η happens to be in $W^{2,\infty}(D)$, so that $U(\eta) \in C^{1,1}(D)$, see proposition A.0.3., and the equality holds everywhere.

This is a product of a bounded Lipschitz function and a bounded Lipschitz function, so it is also bounded Lipschitz. Therefore, $D(U(\tau)\circ U(\eta)) \in C^{0,1}(D)$ and thus $U(\tau)\circ U(\eta) \in C^{1,1}_B(D)$, so that by $A.0.4$, $[U(\tau)\circ U(\eta)|_D] = \tau \circ \eta|_D \in W^{2,\infty(D)}$.

This same proof shows that for $\theta \in \Theta$, $\tau \in \mathcal{T}$, $\theta \circ \tau \in \Theta$, because $\theta \circ \tau$ was already a $W^{1,\infty}$ function by proposition D.0.4, and because τ fixed $\mathbb{R}^n \setminus D$.

Stability under inversion

It is trivial, because the definition of $\mathcal{T}$ is symmetric with respect to inversion.

Stability unded composition ($\mathcal{T}^1$)

$\eta \circ \tau$ is surely in $\mathcal{V}^1$ by proposition D.0.4. Now, by the above point, $\tau^{-1} \circ \tau^{-1}$ is in $\mathcal{V}^1$ too, and the composition yields: $(\eta \circ \tau) \circ (\tau^{-1} \circ \tau^{-1}) = [U(\eta) \circ U(\tau)] \circ [(U\tau)^{-1} \circ (U\tau)^{-1}] = \mathrm{Id}$.

$\square$

Proposition D.0.8 (Gateaux differentiability)
Consider $J : \mathcal{T} \to E$ for E some Banach space.

Let $\tau \in \mathcal{T}$. Then:

$$\forall \delta\theta \in \Theta \text{ exists } \lim_{t \to 0} \frac{J(\tau + t\delta\theta) - J(\tau)}{t} \iff \forall \delta\theta \in \Theta \text{ exists } \lim_{t \to 0} \frac{J((\mathrm{Id} + t\delta\theta) \circ \tau) - J(\tau)}{t}$$

In case of existence, we have:

$$\lim_{t \to 0} \frac{J((\mathrm{Id} + t\delta\theta) \circ \tau) - J(\tau)}{t} = \lim_{t \to 0} \frac{J(\tau + t\delta\theta \circ \tau) - J(\tau)}{t}$$

Proof.

It suffices to show that $\Theta \circ \tau = \Theta$.

In fact, $\delta\theta \circ \tau \in \Theta$ for $\theta \in \Theta$, as we verified in proposition D.0.7. Using the same result, $\tau^{-1} \in \mathcal{T}$, $(\delta\theta \circ \tau) \circ \tau^{-1} \in \Theta$ and $(\delta\theta \circ \tau) \circ \tau^{-1} = [U(\delta\theta) \circ U(\tau)] \circ [U(\tau)^{-1}] = [(U(\delta\theta) \circ U(\tau)) \circ (U(\tau)^{-1})] = \delta\theta$.

$\square$

Theorem D.0.9 (Change of variables)
Let U be open and $T = U(\tau)$ for $\tau \in \mathcal{T}^1$, and let $p \in [1, \infty]$. Then:

1. $f \in L^p(T(U)) \iff f \circ T \in L^p(U)$ and there holds, for $f \in L^p(T(U))$:

$$\|f\|_{L^p(T(Q))} \leq \left(\|\det DT\|_{L^\infty(\mathbb{R}^n)} \right)^{1/p} \|f \circ T\|_{L^p(Q)}$$

2. $f \in W^{1,p}(T(U)) \iff f \circ T \in W^{1,p}(U)$ and there holds, for $f \in W^{1,p}(T(U))$:

$$Df \circ T = (Df)^{-t} D(f \circ T)$$

$$\|Df\|_{L^p(T(Q);\mathbb{R}^n)} \leq \left(\|\det DT\|_{L^\infty(\mathbb{R}^n)}\right)^{1/p} \left\|(DT)^{-1}\right\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \|D(f\circ T)\|_{L^p(Q;\mathbb{R}^n)}$$

3. if $p \in (1,\infty)$, $f \in W_0^{1,p}(T(U)) \iff f\circ T \in W_0^{1,p}(U)$
4. therefore, composition by T is a linear isomorphism between $W^{k,p}(T(U)) \to W^{k,p}(U)$ for $k = 0,1$, and between $W_0^{1,p}(T(U)) \to W_0^{1,p}(U)$ for $k = 0,1$, $p \in (1,\infty)$
5. for D a bounded Lipschitz domain and $\mathcal{T}, \Theta$ defined before, we get, for $f \in H^1(D)$, that $\mathrm{tr} f = \mathrm{tr}(f\circ T)$
6. if moreover, $\Omega, T(\Omega) \subset\subset D$ are also bounded Lipschitz domains, letting $U := D\backslash\Omega$, for $f \in H^1(T(U))$ and $\mathrm{tr}_{T(U)} f = 0$ on $\partial T(\Omega)$, then $\mathrm{tr}_U f\circ T = 0$ on $\partial\Omega$ and $\mathrm{tr}_{T(U)} f = \mathrm{tr}_U f\circ T$ on ∂D

Proof.

We need to prove only the last points, for the other are proved in [13], pages IV.4, IV.5, IV.6.

Static strace

To do so, let $f_n \in C(\overline{D}) \cap H^1(D)$ converging in $H^1(D)$ to f. By point 4, we have $f_n\circ T \to f\circ T$ in $H^1(D)$ (rememeber, $T(D) = D$ by invertibility of T and the fact that $T(x) = x$ outside of D). Therefore we have:

$$\mathrm{tr} f \leftarrow_{L^2(\partial D)} \mathrm{tr}(f_n) = f_n|_{\partial D} = (f_n\circ T)|_{\partial D} = \mathrm{tr}(f_n\circ T) \to_{L^2(\partial D)} \mathrm{tr}(f\circ T)$$

Zero moving trace

First of all, as T is a homeomorphism of $\mathbb{R}^n$, $TU = D\setminus T(\Omega)$, $T\partial U = \partial D \sqcup \partial\Omega$, $T\partial\Omega = \partial T\Omega$.

Now, an application of theorem A.0.1 yields that the extension to 0 in $T\Omega$ of f, call it $\bar{f}$, is $H^1(D)$, with $\partial_i \bar{f} = \partial_i f$ in TU, 0 in $T(\Omega)$.

We claim that $\mathrm{tr}_D \bar{f} = \mathrm{tr}_{T(U)} f|_{\partial D}$. In fact, approximate $\bar{f}$ by restrictions to D of $C_c^\infty(\mathbb{R}^n)$ functions f_n, which also approximate f on $T(U)$, by the observation that $\left\|f_n|_{T(U)}\right\|_{H^1(T(U))} \leq \|f_n|_D\|_{H^1(D)}$. Then:

$$\mathrm{tr}_{T(U)}(f_n|_{T(U)})|_{\partial D} = (f_n|_{T(U)})|_{\partial T(U)}|_{\partial D} = f_n|_{\partial D} = \mathrm{tr}_D(f_n|_D)$$

Now, by what we observed before, $\mathrm{tr}_{T(U)}(f_n|_{T(U)}) \to \mathrm{tr}_{T(U)}(f)$ in $L^2(\partial T(U))$, so that $\mathrm{tr}_{T(U)}(f_n|_{T(U)})|_{\partial D} \to \mathrm{tr}_{T(U)}(f)|_{\partial D}$. On the other hand $\mathrm{tr}_D(f_n|_D) \to \mathrm{tr}_D \bar{f}$, which yields the claim.

Using this: $\operatorname{tr}_{T(U)}(f)|_{\partial D} = \operatorname{tr}_D \bar{f} = \{$ point 5 $\} = \operatorname{tr}_D(\bar{f} \circ T) = \operatorname{tr}_D(\overline{f \circ T}) = \operatorname{tr}_U(f \circ T)|_{\partial D}$, where we used that $\bar{f} \circ T$ is zero in $T^{-1}T\Omega = \Omega$ (because again T maps null sets into null sets), so it is the zero extension $\overline{f \circ T}$ of $f \circ T$. Both $\bar{f} \circ T$ and $f \circ T$ are H^1 functions by point 2.

We can now also say that $\operatorname{tr}_U f \circ T = 0$ on $\partial\Omega$.

$$(\eta\phi_n)|_{\partial\Omega} = \operatorname{tr}_U(\phi_n|_U)_{\partial\Omega} \to \operatorname{tr}_U(f \circ T)_{\partial\Omega}$$

<u>Multiplication by a $W^{1,\infty}$ function</u>

We claim that, for $\psi \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R})$ and $f \in H^1(U)$, then $f\psi$ has the same trace as f as long as $\psi = 1$ in a neighbourhood of ∂U.

Note that $f\psi \in H^1(U)$ still. Now: approximate f by restriction of test functions f_n. Then $f_n\psi$ is $C(\overline{U}) \cap H^1(U)$ (thanks also to corollary A.0.4), so that $\operatorname{tr}_U(f_n\psi) = \operatorname{tr}_U(f_n)$. Because $f_n\psi \to f\psi$ is $H^1(U)$ the claim is valid.

This last convergence follows from $\|(f_n - f)\psi\|_{L^2} \leq \|(f_n - f)\|_{L^2} \|\psi\|_{L^\infty}$, the chain rule $\partial_i(f_n\psi) = \partial_i f_n\psi + \partial_i\psi f_n$ (see corollary 4.1.18 here) and again $\|\partial_i(f_n - f)\psi\|_{L^2} \leq \|\partial_i(f_n - f)\|_{L^2} \|\psi\|_{L^\infty}$, $\|(f_n - f)\partial_i\psi\|_{L^2} \leq \|(f_n - f)\|_{L^2} \|\partial_i\psi\|_{L^\infty}$.

<u>Reducing to a function of 0 trace</u>

Let η be a smooth cut-off function which is 1 close to ∂D and 0 close to $\partial T\Omega$, $\beta = 0$ close to ∂D and 1 close to $\partial T\Omega$. This can be accomplished by e.g. building a suitable partition of unity of the compact sets $\partial\Omega$ and ∂D. (can I do this? Yes, see bachelor's thesis, take $K = \partial\Omega$ etc. Also, be careful with all of these equalities...).

$f\beta$ has zero trace, as it can be verified by approximating f by smooth functions again:

$$\operatorname{tr}_{T(U)} f\beta \leftarrow_{L^2(\partial T(U))} \operatorname{tr}_{T(U)} f_n\beta$$

where the latter quantity is $\operatorname{tr}_{T(U)} f_n$ on ∂D and 0 on $\partial T(U)$. By restricting the convergence to first ∂D and then to $\partial T(U)$, and using almost everywhere convergent subsequences, we conclude that $\operatorname{tr}_{T(U)} f\beta = \operatorname{tr}_{T(U)} f$ on $\partial T(U)$ and $\operatorname{tr}_{T(U)} f\beta = 0$ on ∂D, i.e. $f\beta$ has zero trace.

<u>Domain transformation</u>

But zero trace functions in $H^1(T(U))$, since $T(U)$ is assumed to be bounded Lipschitz, are exactly the functions $H^1_0(T(U))$ (theorem 18.7 at page 595 of [10]).

By then point 4, $(f\beta) \circ T \in H^1_0(U)$.

Because T is bi-Lipschitz, we can write $(f\beta) \circ T = f \circ T\beta \circ T$ almost everywhere.

We have that $\beta \circ T + \eta \circ T$ is $W^{1,\infty}$ and 1 near ∂U.

So, $\mathrm{tr}_U f \circ T = tr_U f \circ T(\beta \circ T + \eta \circ T) = \mathrm{tr}_U(f \circ T \beta \circ T) + \mathrm{tr}_U(f \circ T \eta \circ T)$.

Approximate $f \circ T$ by g_n smooth as seen above. Then, $\mathrm{tr}_U(g_n \eta \circ T)$ is 0 on ∂U and $\mathrm{tr}_U g_n$ on ∂D. By selecting an almost everywhere convergent subsequence, we conclude $\mathrm{tr}_U(f \circ T \eta \circ T) = 0$ on $\partial \Omega$.

Hence $\mathrm{tr}_U f \circ T|_{\partial\Omega} = tr_U f \circ T(\beta \circ T)|_{\partial\Omega} = 0$.

$\square$

Bibliography

[1] R.A. Adams. *Sobolev Spaces. Adams*. Pure and applied mathematics. Academic Press, 1975. URL: https://books.google.it/books?id=JxzpSAAACAAJ.

[2] Daniel Azagra, Erwan Le Gruyer, and Carlos Mudarra. *Kirszbraun's theorem via an explicit formula*. 2018. DOI: 10.48550/ARXIV.1810.10288. URL: https://arxiv.org/abs/1810.10288.

[3] Martin Costabel. "Boundary Integral Operators for the Heat Equation". In: *Integral Equations and Operator Theory* 13 (July 1990), pp. 498–552. DOI: 10.1007/BF01210400.

[4] L.C. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010. ISBN: 9780821849743. URL: https://books.google.it/books?id=Xnu0o_EJrCQC.

[5] Thomas Muirhead Flett. "Mean value theorems for vector-valued functions". In: *Tohoku Mathematical Journal, Second Series* 24.2 (1972), pp. 141–151.

[6] G. Gilardi. "Equazioni paraboliche astratte: impostazione variazionale".

[7] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 2011. ISBN: 9781611972023. URL: https://books.google.it/books?id=lJTtujsZSgkC.

[8] M. Hinze et al. *Optimization with PDE Constraints*. Mathematical Modelling: Theory and Applications. Springer Netherlands, 2008. ISBN: 9781402088391. URL: https://books.google.it/books?id=PFbqxa2uDS8C.

[9] M. Kreuter. "Sobolev Spaces of Vector-Valued Functions". MA thesis. Ulm University, 2015.

[10] G. Leoni. *A First Course in Sobolev Spaces*. Graduate studies in mathematics. American Mathematical Society, 2017. ISBN: 9781470429218. URL: https://books.google.it/books?id=qoA8DwAAQBAJ.

[11] J.L. Lions, P. Kenneth, and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications: Volume II*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012. ISBN: 9783642652172. URL: https://books.google.it/books?id=xD71CAAAQBAJ.

[12] W. McLean and W.C.H. McLean. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, 2000. ISBN: 9780521663755. URL: https://books.google.it/books?id=RILqjEeMfK0C.

[13] François Murat and Jacques Simon. "Etude de problemes d'optimal design". In: *Optimization Techniques Modeling and Optimization in the Service of Man Part 2*. Ed. by Jean Cea. Berlin, Heidelberg: Springer Berlin Heidelberg, 1976, pp. 54–62. ISBN: 978-3-540-38150-1.

[14] F. Tröltzsch and J. Sprekels. *Optimal Control of Partial Differential Equations: Theory, Methods, and Applications*. Graduate studies in mathematics. American Mathematical Society, 2010. ISBN: 9780821849040. URL: `https://books.google.it/books?id=04yDAwAAQBAJ`.

[15] W.P. Ziemer. *Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation*. Graduate texts in mathematics. World Publishing Corporation, 1991. ISBN: 9787506210225. URL: `https://books.google.it/books?id=gR3HQwAACAAJ`.