Technical University of Munich

Department of Mathematics

[Thesis Title]

Master's Thesis

von

[First Name Last Name]

Supervisor:	Prof. Dr. [First Name Last Name]
Advisor:	[First Name Last Name]
Submission Date:	[Day. Month. Year]

I hereby declare that this thesis is my own work and that no other sources have been used except those clearly indicated and referenced.

Place, Date

original, hand-written signature

Acknowledgements

[text of acknowledgements]

German Abstract

[abstract text]

English Abstract

[abstract text]

Contents

1. Introduction

2. Continuous problem

2.1. Gateux differentiability

Assumption 2.1.1 (Geometry assumptions for the shape optimization problem)
Let $D \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain, and $\Omega_r \subset\subset D$ also bounded Lipschitz. Call $U_r := D \setminus \Omega_r$, another bounded Lipschitz domain.

Definition 2.1.2 (Admissible transformations)
Given D, we consider the set $\mathcal{T}$ as defined in definition C.1.6, based on the perturbations Θ, also defined in definition C.1.6.

We consider $\mathcal{T}_a := \mathcal{T} \cap \{\tau \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n), \|\tau - \mathrm{Id}\|_{W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)} < C(U_r)\}$, where $C(U_r)$ is given by theorem C.1.9.

This ensures that $T(U_r) \subset\subset D$ is also bounded Lipschitz, T the unique Lipschitz continuous representive of any $\tau \in \mathcal{T}_a$ (see corollary A.1.4).

Thanks to proposition C.1.8, a small Θ perturbation of $\tau \in \mathcal{T}_a$ done in the $W^{1,\infty}(D; \mathbb{R}^n)$ topology leaves us inside $\mathcal{T}_a$.

Note, we need T to have a Lipschitz inverse to conclude $T(U_r) \subset\subset D$: for $x \in D$, we have $0 < \delta = \inf_{d\in\partial D} |x - d| \leq \|T^{-1}\|_{W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)} \inf_{d\in\partial D} |T(x) - d|$.

We are interested in solving the following problem.

Problem 2.1.3 (Shape optimization problem)
Suppose assumption 2.1.1, assumption B.2.1.1, assumption B.2.2.1 hold. Using the notation $T = U(\tau)$ for "unique continuous representative" of τ, we want to solve:

$$\inf_{\tau\in\mathcal{T}_a} \frac{1}{2} \|v_\tau - w_\tau\|^2_{L^2(I,H_T)} =: J(\tau)$$

Here $v^\tau = v_0^\tau + \bar{u}$, and v_0^τ, w_τ are given in problem C.4.3 and $\bar{u}$ is given in problem B.2.1.12. The notation for the spaces also comes from problem C.4.3, where $\cdot_T$ means a space defined on the moving domain, $H = L^2, W = H_0^1, V = H_c^1$ (and the last space is defined in appendix B.2.1).

Observation 2.1.4 (Well posedness of J). We know from theorem B.2.1.11 that $v_0^T + \bar{u}$ doesn't depend on the particular choice of $\bar{u}$, therefore, for different τ's yielding the same domain, $J(\tau)$ doesn't change.

Moreover, assumption 2.1.1 ensures in particular assumption C.4.1, which enables the applicability of theorem C.4.2.

Now, let $U := T(U_r)$, for $T \in \mathcal{T}_a$ (we write $T \in \mathcal{T}_a$ for $\tau \in \mathcal{T}_a$, $U(\tau) = T$) and let $\delta\theta \in \Theta$.

We are interested in finding $J'(\tau) \in \Theta^*$ such that, for all $\delta\theta_k \to 0$ in Θ, we have:

$$\lim_k \frac{|J(\tau + \delta\theta_k) - J(\tau) - J'(\tau)(\delta\theta_k)|}{\|\delta\theta_k\|_\Theta}$$

For simplificity we have put $\|\theta\|_\Theta = \|\theta\|_{W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)} = \|\theta\|_{W^{1,\infty}(D;\mathbb{R}^n)}$.

Note, $\tau + \delta\theta_k \in \mathcal{T}_a$ for large enough k. In fact, $\tau + \delta\theta_k \in \mathcal{T}$ for large k as in proposition C.1.8, and the condition on $C(U_r)$ is satisfied too, because $\theta := \tau - \mathrm{Id}$ was already in the $W^{1,\infty}$ open ball of radius $C(U_r)$ centered at the origin, and so will be $\theta + \delta_k\theta$, always for large k.

Now, $\tau+\delta\theta_k = (\mathrm{Id}+\delta\theta_k\circ\tau^{-1})\circ\tau$, and $\mathrm{Id}+\delta\theta_k\circ\tau^{-1}$ is in $\mathcal{T}_a$ (it is in $\mathcal{T}$ by proposition C.1.8 and the reasoning above shows it is also in $\mathcal{T}_a$).

We are then equivalently interested in:

$$\lim_k \frac{|J((\mathrm{Id} + \delta\theta_k \circ \tau^{-1}) \circ \tau) - J(\tau) - J'(\tau)(\delta\theta_k)|}{\|\delta\theta_k\|_\Theta}$$

This amounts to setting the reference domain to $T(U_r)$ instead of U_r and perturbing the former, at least for the sake of computing derivatives.

We now introduce a Lagrangian.

For $k \geq K(T)$ we define $T_k := \mathrm{Id} + \delta\theta_k \circ \tau^{-1} \in \mathcal{T}_a$, as seen above.

With the help and the notation of theorem C.4.2 we can set:

$$
\begin{aligned}
&L_T(k, w, v_0, q, p) = \\
&\frac{1}{2}\int_I \int_{T(U_r)} |v_0 + \bar{u}_T \circ T_k - w|^2 |\det(DT_k)| + \\
&\int_I (w_t, q|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla w, \nabla q)_{H_T} - \int_I (g, \mathrm{tr}_U q)_{L^2(\Gamma_f)} + \\
&\int_I (v_{0t}, p|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla v_0, \nabla p)_{H_T} + \\
&\int_I ((\bar{u} \circ T_k)', p|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla(\bar{u} \circ T_k), \nabla p)_{H_T}
\end{aligned}
$$

for $w \in Q_0(I, V_T), v_0 \in Q_0(I, W_T), q \in Q^0(I, V_T), p \in Q^0(I, W_T)$. Remember, $A_T := (DT)^{-1}(DT)^{-t}|\det(DT)|$.

Note that to be precise, $\bar{u}$ is an extension of the Dirichlet datum f, on the moving domain $T_k(T(U_r))$. Because of this, let's fix $\bar{u}_T$ with this property on $T(U_r)$. We show that $\bar{u}_T \circ T_k^{-1}$ satisfies the conditions stated in problem B.2.1.12.

In particular:

- composition with T preserves the smoothness of the extension, as seen in proposition A.2.1, given that $\circ T_k^{-1}$ is a linear bounded operator between V_T and $V_{T_k \circ T}$
- the initial value is preserved, as seen in the proof of proposition C.3.1
- the trace on Σ_f is preserved, because the trace on $\Gamma_f = \partial D$ is preserved, see theorem C.2.1

Therefore we can state the following definition.

Definition 2.1.5 (Lagrangian)
For a fixed $T \in \mathcal{T}_a$ and $k \geq K(T)$, for $T_k := \mathrm{Id} + \delta\theta_k \circ \tau^{-1} \in \mathcal{T}_a$, we define:

$$
\begin{aligned}
&L_T(k, w, v_0, q, p) = \\
&\frac{1}{2}\int_I \int_{T(U_r)} |v_0 + \bar{u} - w|^2 |\det(DT_k)| + \\
&\int_I (w_t, q|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla w, \nabla q)_{H_T} - \int_I (g, \mathrm{tr}_U q)_{L^2(\Gamma_f)} + \\
&\int_I (v_{0t}, p|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla v_0, \nabla p)_{H_T} + \\
&\int_I (\bar{u}', p|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla \bar{u}, \nabla p)_{H_T}
\end{aligned}
$$

L_T is defined as a map $\{k \geq K(T)\} \times Q_0(I, V_T) \times Q_0(I, W_T) \times Q^0(I, V_T) \times Q^0(I, W_T) \to \mathbb{R}$.

We call $u = (w, v_0)$, $\pi = (q, p)$, $G(k, u, \pi) = L_T(k, w, v_0, q, p)$ to ease the notation.

We also call $b(k, u) = \frac{1}{2} \int_I \int_{T(U_r)} |v_0 + \bar{u}_T - w|^2 |\det(DT_k)|$ and $a(k, u, \pi) = G(k, u, \pi) - b(k, u)$, $E = W_0(I, V_T) \times W_0(I, W_T)$, $F = W^0(I, V_T) \times W^0(I, W_T)$.

Proposition 2.1.6 (Properties of the Lagrangian)
L_T satisfies the following properties:

1. $\psi \mapsto a(k, \phi, \psi)$ is linear, no matter what ϕ, k
2. G is Frechet differentiable with respect to ψ at $(k, \phi, 0)$ for all k, ϕ
3. $d_\psi G(k, \phi, 0)[\delta\psi] = 0$ for all $\delta\psi \in F$ admits a unique solution $\phi = u^k$
4. $[0, 1] \ni s \mapsto G(k, su^k + (1 - s)u^0, \psi)$ is $AC[0, 1]$, no matter what k, ψ
5. G is Frechet differentiable with respect to ϕ at (k, ψ, ϕ) for all k, ψ, ϕ
6. $[0, 1] \ni s \mapsto d_\phi G(k, su^k + (1 - s)u^0, \psi)[\delta\phi]$ is $L^1(0, 1)$, no matter what $k, \psi, \delta\phi$
7. there exists a unique solution $\psi = \pi^k$ to $\displaystyle\int_0^1 d_\phi G(k, su^k + (1 - s)u^0, \psi)[\delta\phi] ds = 0$ for all $\delta\psi$

In particular $\pi^k = (Q^k \circ \tau^k, P^k \circ \tau^k)$, where we introduced the average adjoint problems:

Problem 2.1.7 (Averaged adjoint equations)

$$
\begin{aligned}
-Q^k_t - \Delta Q^k = \frac{v_0^k - w^k + v_0^0 - w^0}{2} \circ T_k^{-1} + \bar{u}_T \circ T_k^{-1} \\
Q^k(T) = 0 \\
\partial_\nu Q^k = 0 \text{ on } \Sigma_f \\
Q^k = 0 \text{ on } \Sigma_m
\end{aligned}
$$

and

$$
\begin{aligned}
-P^k_t - \Delta P^k = -\frac{v_0^k - w^k + v_0^0 - w^0}{2} \circ T_k^{-1} - \bar{u}_T \circ T_k^{-1} \\
P^k(T) = 0 \\
P^k = 0 \text{ on } \Sigma_m \sqcup \Sigma_f
\end{aligned}
$$

Proof.

The first point is immediate.

Proof of 2

All the pieces are linear in ψ. We only check the boundedness of the various differentials. For simplicity, call $|\det(DT_k)| = d$, and note that $\|qd\|_{H_T} \le C(d)\,\|q\|_{H_T}$.

And now, for instance:

$$\begin{aligned}\int_I (w_t, \delta q|\det(DT_k)|)_{H_T} = \int_I (w_t, \delta q d)_{H_T} \le \\ C(d)\int_I \|w_t\|_{H_T}\,\|\delta q\|_{H_T} \le C(d)\,\|w_t\|_{L^2(I,H_T)}\,\|\delta q\|_{L^2(I,H_T)} \le \\ C(d)\,\|w\|_{Q(I,V_T)}\,\|\delta q\|_{Q(I,V_T)} \le C(d)\,\|w\|_{Q(I,V_T)}\,(\|\delta q\|_{Q(I,V_T)} + \|\delta p\|_{Q(I,V_T)}) = \\ C(d)\,\|w\|_{Q(I,V_T)}\,\|\delta\psi\|_F\end{aligned}$$

Or also:

$$\int_I (g, \mathrm{tr}_U \delta q)_{L^2(\Gamma_f)} \le \int_I \|g\|_{L^2(\Gamma_f)}\,\|\delta q\|_{V_T} \le \|g\|_{H^1(I,L^2(\Gamma_f))}\,\|\delta\psi\|_F$$

and:

$$\begin{aligned}\int_I (A_{T_k}\nabla\bar{u}_T, \nabla\delta p)_{H_T} \le C\,\|A_{T_k}\|_{L^\infty(D;\mathbb{R}^{n\times n})}\int_I \|\nabla\bar{u}_T\|_{L^2(I,H_T)}\,\|\nabla\delta p\|_{L^2(I,H_T)} \le \\ C(T)\,\|\delta\psi\|_F\,\|\bar{u}\|_{H^1(I,V_T)}\end{aligned}$$

Proof of 3

We get back the state equations, thanks to linearity, and by testing separately with $\delta\psi = (\delta q, 0)$ and $\delta\psi = (0, \delta p)$, so that a unique solution exists by theorem C.4.2.

Proof of 4

Every piece but b is linear or constant in the state ϕ. We only need to prove that $[0,1] \ni s \mapsto b(k, su^k + (1-s)u^0)$ is $AC[0,1]$. But by the structure of the cost function J, transported on $T(U_r)$, we see that the latter is a quadratic polynomial in s, hence, absolutely continuous.

Proof of 5

For the pieces with the gradients, it follows as above, by in case employing the simmetry of A_{T_k}.

Now, for instance the linear form $\delta v_0 \mapsto \int_I (\delta v_{0t}, p|\det(DT_k)|)_{H_T}$ is also bounded by $C(d)\left\|\delta v_0\right\|_{Q(I,W_T)}\left\|\delta q\right\|_{Q(I,V_T)}$ just like before.

What remains to check is the Frechet differentiability of b.

To do so, perturb ϕ by $\delta\phi$ and expanding the square:

$$
\begin{aligned}
\frac{1}{2}\int_I\int_{T(U_r)} |v_0+\delta v_0+\bar{u}_T-w-\delta w|^2|\det(DT_k)| = \\
\frac{1}{2}\int_I\int_{T(U_r)} |v_0+\bar{u}_T-w|^2|\det(DT_k)|+ \\
\frac{1}{2}\int_I\int_{T(U_r)} |\delta v_0-\delta w|^2|\det(DT_k)|+ \\
\int_I\int_{T(U_r)} (v_0+\bar{u}_T-w)(\delta v_0-\delta w)|\det(DT_k)|
\end{aligned}
$$

Now, $\displaystyle\int_I\int_{T(U_r)} |\delta v_0-\delta w|^2|\det(DT_k)| \leq C(T_k)\left\|\delta v_0-\delta w\right\|^2_{L^2(I,H_T)} \leq C(T_k)\left\|\phi\right\|^2_E$, so that this term is of higher term.

And $\int_I\int_{T(U_r)}(v_0+\bar{u}_T-w)(\delta v_0-\delta w)|\det(DT_k)|$ is linear and bounded by reasonings similar to the former ones.

<u>Proof of 6</u>

By the last point:

$$
\begin{aligned}
d_\phi G(k,\phi,\psi)[\delta\phi] = \\
\int_I ((v_0+\bar{u}_T-w)|\det(DT_k)|, \delta v_0-\delta w)_{H_T}+ \\
\int_I (\delta w_t, q|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla\delta w, \nabla q)_{H_T}+ \\
\int_I (\delta v_{0t}, p|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla\delta v_0, \nabla p)_{H_T}
\end{aligned}
$$

so that:

$$
\begin{aligned}
&d_\phi G(k, su^k + (1-s)u^0, \psi)[\delta\phi] = \\
&\int_I ((s(v_0^k + \bar{u}_T - w^k) + (1-s)(v_0^0 + \bar{u}_T - w^0))|\det(DT_k)|, \delta v_0 - \delta w)_{H_T} + \\
&\int_I (\delta w_t, q|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla\delta w, \nabla q)_{H_T} + \\
&\int_I (\delta v_{0t}, p|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla\delta v_0, \nabla p)_{H_T}
\end{aligned}
$$

which is a degree 1 polynomial in s, hence, $L^1(0,1)$.

<u>Proof of 7</u>

Rewriting the formula above and integrating in s, we come to:

$$
\begin{aligned}
&\int_0^1 d_\phi G(k, su^k + (1-s)u^0, \psi)[\delta\phi]ds = \\
&\int_I (((v_0^k + \bar{u}_T - w^k) + (v_0^0 + \bar{u}_T - w^0))/2|\det(DT_k)|, \delta v_0 - \delta w)_{H_T} + \\
&\int_I (\delta w_t, q|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla\delta w, \nabla q)_{H_T} + \\
&\int_I (\delta v_{0t}, p|\det(DT_k)|)_{H_T} + (A_{T_k}\nabla\delta v_0, \nabla p)_{H_T}
\end{aligned}
$$

As in proposition C.3.1, $\delta w_t = (\delta w \circ T_k^{-1})_t \circ T_k$, where $\delta w \circ T_k^{-1} \in Q_0(I, V_{T_k \circ T})$ by proposition C.3.1 (that can be applied thanks to the smallness of T_k).

Applying a change of variables we are left with:

$$
\begin{aligned}
&\int_0^1 d_\phi G(k, su^k + (1-s)u^0, \psi)[\delta\phi]ds = \\
&\int_I \left(\frac{v_0^k - w^k}{2} \circ T_k^{-1} + \frac{v_0^0 - w^0}{2} \circ T_k^{-1} + \bar{u}_T \circ T_k^{-1}, \delta v_0 \circ T_k^{-1} - \delta w \circ T_k^{-1}\right)_{H_{T_k \circ T}} + \\
&\int_I ((\delta w \circ T_k^{-1})_t, q \circ T_k^{-1})_{H_{T_k \circ T}} + (\nabla(\delta w \circ T_k^{-1}), \nabla(q \circ T_k^{-1}))_{H_{T_k \circ T}} + \\
&\int_I ((\delta v_0 \circ T_k^{-1})_t, p \circ T_k^{-1})_{H_{T_k \circ T}} + (\nabla\delta(v_0 \circ T_k^{-1}), \nabla(p \circ T_k^{-1}))_{H_{T_k \circ T}}
\end{aligned}
$$

Here, as we saw in proposition C.3.1, we have $\delta w \circ T_k^{-1}, w \circ T_k^{-1} \in Q_0(I, V_{T_k \circ T})$, $\delta v_0 \circ T_k^{-1}, v_0 \circ T_k^{-1} \in Q_0(I, W_{T_k \circ T})$, $q \circ T_k^{-1} \in Q^0(I, V_{T_k \circ T})$ and $p \circ T_k^{-1} \in Q^0(I, W_{T_k \circ T})$.

Because $\circ T_k^{-1}$ is a bijection of $Q_0(I, W_{T_k \circ T})$ and $Q_0(I, W_{T_k})$ as we saw in C.3.1 (and analogously of V), we have that $\int_0^1 d_\phi G(k, su^k + (1-s)u^0, \psi)[\delta\phi]ds = 0$ for all $\delta\phi \in E$ if and only if:

$$\int_I \left(\frac{v_0^k + w^k}{2} \circ T_k^{-1} - \frac{v_0^0 + w^0}{2} \circ T_k^{-1} + \bar{u}_T \circ T_k^{-1}, \delta V_0 - \delta W \right)_{H_{T_k \circ T}} + \int_I (\delta W_t, q \circ T_k^{-1})_{H_{T_k \circ T}} + (\nabla \delta W, \nabla (q \circ T_k^{-1}))_{H_{T_k \circ T}} + \int_I (\delta V_{0t}, p \circ T_k^{-1})_{H_{T_k \circ T}} + (\nabla \delta V_0, \nabla (p \circ T_k^{-1}))_{H_{T_k \circ T}} = 0$$

for all $\delta W, \in Q_0(I, V_{T_k \circ T})$, $\delta V_0 \in Q_0(I, W_{T_k \circ T})$.

We wish to find a (unique) solution $(q^k, p^k) \in Q^0(I, V_T) \times Q^0(I, W_T)$ of this problem. We can equivalently (by proposition C.3.1) find $(Q^k, P^k) \in Q^0(I, V_{T_k \circ T}) \times Q^0(I, W_{T_k \circ T})$ satisfying:

$$\int_I \left(\frac{v_0^k - w^k}{2} \circ T_k^{-1} + \frac{v_0^0 - w^0}{2} \circ T_k^{-1} + \bar{u}_T \circ T_k^{-1}, \delta V_0 - \delta W \right)_{H_{T_k \circ T}} + \int_I (\delta W_t, Q^k)_{H_{T_k \circ T}} + (\nabla \delta W, \nabla Q^k)_{H_{T_k \circ T}} + \int_I (\delta V_{0t}, P^k)_{H_{T_k \circ T}} + (\nabla \delta V_0, \nabla P^k)_{H_{T_k \circ T}} = 0$$

for all $\delta W, \in Q_0(I, V_{T_k \circ T})$, $\delta V_0 \in Q_0(I, W_{T_k \circ T})$.

By testing first with $\delta W = 0$ and then with $\delta V_0 = 0$ we can equivalently look for:

$$\begin{aligned} &(Q^k, P^k) \in Q^0(I, V_{T_k \circ T}) \times Q^0(I, W_{T_k \circ T}) \text{ with} \\ &\int_I (\delta W_t, Q^k)_{H_{T_k \circ T}} + (\nabla \delta W, \nabla Q^k)_{H_{T_k \circ T}} = \\ &\int_I \left(\frac{v_0^k + w^k - v_0^0 - w^0}{2} \circ T_k^{-1} + \bar{u}_T \circ T_k^{-1}, \delta W \right)_{H_{T_k \circ T}} \\ &\int_I (\delta V_{0t}, P^k)_{H_{T_k \circ T}} + (\nabla \delta V_0, \nabla P^k)_{H_{T_k \circ T}} = \\ &- \int_I \left(\frac{v_0^k - w^k + v_0^0 - w^0}{2} \circ T_k^{-1} + \bar{u}_T \circ T_k^{-1}, \delta V_0 \right)_{H_{T_k \circ T}} \end{aligned}$$

An application of integration by parts in time (see proposition B.3.3) yields the problem:

$$
\begin{aligned}
&(Q^k, P^k) \in W^0(I, V_{T_k \circ T}) \times W^0(I, W_{T_k \circ T}) \text{ with} \\
&- \int_I (Q_t^k, \delta W)_{H_{T_k \circ T}} + (\nabla \delta W, \nabla Q^k)_{H_{T_k \circ T}} = \\
&\int_I \left(\frac{v_0^k - w^k + v_0^0 - w^0}{2} \circ T_k^{-1} + \bar{u}_T \circ T_k^{-1}, \delta W \right)_{H_{T_k \circ T}} \\
&- \int_I (P_t^k, \delta V_0)_{H_{T_k \circ T}} + (\nabla \delta V_0, \nabla P^k)_{H_{T_k \circ T}} = \\
&- \int_I \left(\frac{v_0^k - w^k + v_0^0 - w^0}{2} \circ T_k^{-1} + \bar{u}_T \circ T_k^{-1}, \delta V_0 \right)_{H_{T_k \circ T}}
\end{aligned}
$$

But this is the weak formulation (cfr. theorem C.4.2, problem B.2.1.12, problem B.2.2.2) of the problems:

$$
\begin{aligned}
-Q_t^k - \Delta Q^k = \frac{v_0^k - w^k + v_0^0 - w^0}{2} \circ T_k^{-1} + \bar{u}_T \circ T_k^{-1} \\
Q^k(T) = 0 \\
\partial_\nu Q^k = 0 \text{ on } \Sigma_f \\
Q^k = 0 \text{ on } \Sigma_m
\end{aligned}
$$

and

$$
\begin{aligned}
-P_t^k - \Delta P^k = -\frac{v_0^k - w^k + v_0^0 - w^0}{2} \circ T_k^{-1} - \bar{u}_T \circ T_k^{-1} \\
P^k(T) = 0 \\
P^k = 0 \text{ on } \Sigma_m \sqcup \Sigma_f
\end{aligned}
$$

Applying the time reversal $t \mapsto T - t$ (where $I = [0, T]$), these are a couple of standard heat equations for which we have available existence, uniqueness and stability results (see appendix B, and proposition B.3.5).

By calling then $\pi^k = (Q^k \circ T^k, P^k \circ T^k)$ we conclude the proof.

□

We now turn to the verification of Gateaux differentiability of J, applying the techniques proposed in [16].

Proposition 2.1.8 (Averaged adjoint method for Gateaux derivatives)
If $J'(\tau) \in \Theta^*$ satisfies:

$$\lim_k \frac{G(k, u^0, \pi^k) - G(0, u^0, \pi^k)}{t_k} = J'(\tau)[\delta\theta]$$

where $\delta\theta_k = t_k \delta\theta$ for $t_k \to 0$, then $J'(\tau)$ is the Gateaux derivative of J at τ.

Proof.

We have $G(k, u^k, \pi^k) - G(k, u^0, \pi^k) = \int_0^1 d_\phi G(k, su^k + (1-s)u^0, \pi^k)[u^k - u^0]ds = 0$ because $u^k - u^0 \in E$, and by absolute contintinuity and integrability of derivative as seen in proposition 2.1.6.

Moreover, calling $g_k = G(k, u^k, 0) - G(0, u^0, 0)$, we have:

- $g_0 = 0$
- $g_k = J((\mathrm{Id} + \delta\theta_k \circ \tau^{-1}) \circ \tau) - J(\tau)$, thanks again to a change of variables
- $g_k = G(k, u^k, \pi^k) - G(0, u^0, \pi^k)$ thanks to $\pi^k \in F$ and the state equations (note, this is possible because k, u^k appear, and $0, u^0$ appear, so that the indices don't mix)

And now, $J(\tau + \delta\theta_k) - J(\tau) = J((\mathrm{Id} + \delta\theta_k \circ \tau^{-1}) \circ \tau) - J(\tau) = g_k = G(k, u^k, \pi^k) - G(0, u^0, \pi^k) = G(k, u^k, \pi^k) - G(k, u^0, \pi^k) + G(k, u^0, \pi^k) - G(0, u^0, \pi^k) = G(k, u^0, \pi^k) - G(0, u^0, \pi^k)$.

□

Proposition 2.1.9 (Gateaux differentiability of J)
Given $\tau \in \mathcal{T}_a$, J is Gateaux differentiable at τ with respect to the $W^{1,\infty}$ topology. The Gateaux differential is:

$$\begin{aligned} J'(\tau)[\delta\theta] = \\ \int_I (w_t^\tau \operatorname{div}(\delta\theta \circ \tau^{-1}), q^\tau) + \int_I (A'(\delta\theta \circ \tau^{-1}) \nabla v^\tau, \nabla p^\tau) + \\ \int_I (v_t^\tau \operatorname{div}(\delta\theta \circ \tau^{-1}), p^\tau) + \int_I (A'(\delta\theta \circ \tau^{-1}) \nabla w^\tau, \nabla q^\tau) + \\ \frac{1}{2} \int_I \int_{T(U_r)} |v^\tau - w^\tau|^2 \operatorname{div}(\delta\theta \circ \tau^{-1}) \end{aligned}$$

where p^τ, q^τ solve:

$$\begin{aligned} -q_t^\tau - \Delta q^\tau &= v^\tau - w^\tau \\ q^\tau(T) &= 0 \\ \partial_\nu q^\tau &= 0 \text{ on } \Sigma_f \\ q^\tau &= 0 \text{ on } \Sigma_m \end{aligned}$$

and

$$\begin{aligned} -p_t^\tau - \Delta p^\tau &= -v^\tau + w^\tau \\ p^\tau(T) &= 0 \\ p^\tau &= 0 \text{ on } \Sigma_m \sqcup \Sigma_f \end{aligned}$$

and where $A'(\delta\theta) = -D\delta\theta - (D\delta\theta)^t + \operatorname{div}(\delta\theta)$.

Proof.

The shape derivative is linear and bounded

Linearity is immediate. For the boundedness:

$$|J'(\tau)[\delta\theta]| \leq$$
$$\left\|\operatorname{div}(\delta\theta \circ \tau^{-1})\right\|_{L^\infty(T(U_r))} \left(\int_I (\|q_t^\tau\|_{H^T} \|q^\tau\|_{H_T} + \|v_t^\tau\|_{H_T} \|p^\tau\|_{H_T}) + \frac{1}{2}\|v^\tau - w^\tau\|^2_{L^2(I,H_T)}\right) +$$
$$\left(\sum_{ij} \left\|(A'(\delta\theta \circ \tau^{-1})_{ij}\right\|_{L^\infty(T(U_r))}\right) \left(\int_I \|\nabla v^\tau\|_{H_T} \|\nabla p^\tau\|_{H_T} + \int_I \|\nabla w^\tau\|_{H_T} \|\nabla q^\tau\|_{H_T}\right)$$

and then, for C independent of $\delta\theta$:

$$|J'(\tau)[\delta\theta]| \leq$$
$$C\left(\left\|\operatorname{div}(\delta\theta \circ \tau^{-1})\right\|_{L^\infty(T(U_r))} + \left(\sum_{ij} \left\|(A'(\delta\theta \circ \tau^{-1})_{ij}\right\|_{L^\infty(T(U_r))}\right)\right) \leq$$
$$\left\|\delta\theta \circ \tau^{-1}\right\|_{W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)} \leq$$
$$C\|\delta\theta\|_{W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)}$$

where in the last step we applied point i) of lemme 2.2, [15]. This shows the boundedness.

Conclusion

Assume $p^k \rightharpoonup p^0$ in $Q(I, W_T)$ and $q^k \rightharpoonup q^0$ in $Q(I, V_T)$.

Now, using that $u^0 = (w^\tau, v_0^\tau)$:

$$
\begin{aligned}
G(k, u^0, \pi^k) - G(0, u^0, \pi^k) = \\
\frac{1}{2}\int_I \int_{T(U_r)} |v^\tau - w^\tau|^2 |\det(DT_k)| + \\
\int_I (w_t^\tau |\det(DT_k)|, q^k)_{H_T} + (A_{T_k} \nabla w^\tau, \nabla q^k)_{H_T} - \int_I (g, \mathrm{tr}_U q^k)_{L^2(\Gamma_f)} + \\
\int_I (v_t^\tau |\det(DT_k)|, p^k)_{H_T} + (A_{T_k} \nabla v^\tau, \nabla p^k)_{H_T} - \\
\frac{1}{2}\int_I \int_{T(U_r)} |v^\tau - w^\tau|^2 - \\
\int_I (w_t^\tau, q^k)_{H_T} + (\nabla w^\tau, \nabla q^k)_{H_T} + \int_I (g, \mathrm{tr}_U q^k)_{L^2(\Gamma_f)} - \\
\int_I (v_t^\tau, p^k)_{H_T} + (\nabla v^\tau, \nabla p^k)_{H_T}
\end{aligned}
$$

Grouping some terms and cancelling the boundary integral:

$$
\begin{aligned}
G(k, u^0, \pi^k) - G(0, u^0, \pi^k) = \\
\frac{1}{2}\int_I \int_{T(U_r)} |v^\tau - w^\tau|^2 (|\det(DT_k)| - 1) + \\
\int_I (w_t^\tau (|\det(DT_k)| - 1), q^k)_{H_T} + ((A_{T_k} - I) \nabla w^\tau, \nabla q^k)_{H_T} + \\
\int_I (v_t^\tau (|\det(DT_k)| - 1), p^k)_{H_T} + ((A_{T_k} - I) \nabla v^\tau, \nabla p^k)_{H_T}
\end{aligned}
$$

Now, the application $\delta\theta \mapsto \mathrm{Id} + \delta\theta \circ \tau^{-1}$ is Frechet differentiable at $\delta\theta = 0$, as a map of Θ into $\mathcal{V}^1$, with Frechet derivative $\delta \circ \theta\tau^{-1}$, which is linear and bounded by point i) of lemme 2.2, [15]. Note, we needed here $\tau \in \mathcal{T}^1$.

Also, the maps $\delta\eta \mapsto |\det(D\eta)|$ and $\eta \mapsto (D\eta)^{-1}(D\eta)^{-t}|\det D\eta|$ are Frechet differentiable at Id, from $\mathcal{V}^1$ into $L^\infty(\mathbb{R}^n; \mathbb{R})$ and $L^\infty(\mathbb{R}^n; \mathbb{R}^{n\times n})$, as stated in lemma 4.16, page

80 of [12]. Their Frechet derivatives are $\mathrm{div}(\beta)$ and $I - D\beta - (D\beta)^t$, respectively.

Therefore, composition with $\delta\theta \mapsto \mathrm{Id} + \delta\theta \circ \tau^{-1}$ yields two Frechet differentiable maps, whose derivatives at 0, in direction $\delta\theta \in \Theta$ are exactly:

- $\mathrm{div}(\delta\theta \circ \tau^{-1})$
- $A'(\delta\theta \circ \tau^{-1})$

These maps are:

- $\delta\theta_k \mapsto |\det(DT_k)|$
- $\delta\theta_k \mapsto A_{T_k}$

Therefore:

- $|\det(DT_k)| - 1 = |\det(DT_k)| - 1 - t_k \mathrm{div}(\delta\theta \circ \tau^{-1}) + t_k \mathrm{div}(\delta\theta \circ \tau^{-1}) = o_k^1 + t_k \mathrm{div}(\delta\theta \circ \tau^{-1})$
- $A_{T_k} - I = A_{T_k} - I - t_k A'(\delta\theta \circ \tau^{-1}) + t_k A'(\delta\theta \circ \tau^{-1}) = o_k^2 + t_k A'(\delta\theta \circ \tau^{-1})$

where $o_1^k \in L^\infty(\mathbb{R}^n; \mathbb{R})$ and $o_k^2 \in L^\infty(\mathbb{R}^n; \mathbb{R}^{n\times n})$ being higher order terms, in L^∞ and with respect to t_k.

We can then write $(G(k, u^0, \pi^k) - G(0, u^0, \pi^k))/t_k = a_k + o_k$.

Here:

$$
\begin{aligned}
a_k :=\ & \frac{1}{2}\int_I \int_{T(U_r)} |v^\tau - w^\tau|^2 \mathrm{div}(\delta\theta \circ \tau^{-1}) + \\
& \int_I (w_t^\tau \mathrm{div}(\delta\theta \circ \tau^{-1}), q^k)_{H_T} + (A'(\delta\theta \circ \tau^{-1})\nabla w^\tau, \nabla q^k)_{H_T} + \\
& \int_I (v_t^\tau \mathrm{div}(\delta\theta \circ \tau^{-1}), p^k)_{H_T} + (A'(\delta\theta \circ \tau^{-1})\nabla v^\tau, \nabla p^k)_{H_T}
\end{aligned}
$$

Thanks to the assumed weak convergence, $a_k \to J'(\tau)[\delta\theta]$.

So, we still have to show that:

$$
\begin{aligned}
&o_k := \\
&\frac{1}{2}\int_I\int_{T(U_r)} |v^\tau - w^\tau|^2 o_k^1 t_k^{-1} + \\
&\int_I (w_t^\tau o_k^1 t_k^{-1}, q^k)_{H_T} + (t_k^{-1} o_k^2 \nabla w^\tau, \nabla q^k)_{H_T} + \\
&\int_I (v_t^\tau o_k^1 t_k^{-1}, p^k)_{H_T} + (t_k^{-1} o_k^2 \nabla v^\tau, \nabla p^k)_{H_T}
\end{aligned}
$$

goes to zero. This is true because again we can write:

$$
\begin{aligned}
&|o_k| \leq \\
&\left\|o_k^1 t_k^{-1}\right\|_{L^\infty(T(U_r))} \left(\int_I (\|v_t^\tau\|_{H^T} \left\|p^k\right\|_{H_T} + \|v_t^\tau\|_{H_T} \left\|p^k\right\|_{H_T}) + \frac{1}{2}\left\|v^\tau - w^\tau\right\|^2_{L^2(I,H_T)}\right) + \\
&\left(\sum_{ij} \left\|((t_k^{-1} o_k^2)_{ij}\right\|_{L^\infty(T(U_r))}\right) \left(\int_I \|\nabla v^\tau\|_{H_T} \left\|\nabla p^k\right\|_{H_T} + \int_I \|\nabla w^\tau\|_{H_T} \left\|\nabla q^k\right\|_{H_T}\right)
\end{aligned}
$$

which goes to 0, thanks to the boundedness of the averaged adjoint states, which stems from their weak convergence.

We assumed $p^k \rightharpoonup p^0$ in $Q(I, W_T)$ and $q^k \rightharpoonup q^0$ in $Q(I, V_T)$. We now prove these claims.

Weak convergence of states

We show at first $v_0^k \rightharpoonup v_0^0$ in $Q(I, W_T)$ and $w^k \rightharpoonup w^0$ in $Q(I, V_T)$.

We do this by showing a bound, uniform in k.

To do this, recall that $V_0^k := v_0^k \circ T_k^{-1}$ and $W^k := w^k \circ T_k^{-1}$ satisfy, as seen in C.4.2:

$$
\begin{aligned}
&W^k \in Q_0(I, V_{T_k\circ T}), V_0^k \in Q_0(I, W_{T_k\circ T}) \\
&\int_I (W_t^k, Q)_{H_{T_k\circ T}} + (\nabla W^k, \nabla Q)_{H_{T_k\circ T}} = \int_I (g, \mathrm{tr}_{T_k\circ T(U)} Q)_{L^2(\Gamma_f)}, \quad \forall Q \in Q^0(I, V_{T_k\circ T}) \\
&\int_I (V_{0t}^k, P)_{H_{T_k\circ T}} + (\nabla V_0^k, \nabla P)_{H_{T_k\circ T}} = -\int_I (U_k', P)_{H_{T_k\circ T}} + (U^k, \nabla P)_{H_{T_k\circ T}}, \quad \forall P \in Q^0(I, W_{T_k\circ T})
\end{aligned}
$$

where $U^k := \bar{u} \circ T_k^{-1}$, where we used that pullbacks and time derivatives commute, see proposition C.3.1.

Thanks to theorem B.2.1.11 and theorem B.2.2.10 we obtain the stability estimates:

$$\begin{aligned} \left\|V^k\right\|^2_{C([0;T],H_{T_k\circ T})} + \left\|V^k\right\|^2_{L^2(I,H_{T_k\circ T})} + \\ \left\|\nabla V^k\right\|^2_{L^2(I,H_{T_k\circ T})} + \left\|(V^k)_t\right\|^2_{L^2(I,H_{T_k\circ T})} \leq \\ C\left\|U^k\right\|^2_{H^1(I,V_{T_k\circ T})} \\ \left\|W^k\right\|^2_{C([0;T],H_{T_k\circ T})} + \left\|W^k\right\|^2_{L^2(I,H_{T_k\circ T})} + \left\|\nabla W^k\right\|^2_{L^2(I,H_{T_k\circ T})} + \\ \left\|W^k_t\right\|^2_{L^2(I,H_{T_k\circ T})} \leq \\ C\left\|g\right\|^2_{H^1(I,L^2(\Gamma_f))} \end{aligned}$$

where C is independent of k.

Now, consider C.2.1. It says that for almost every time:

$$\begin{aligned} \left\|U^k\right\|_{V_{T_k\circ T}} \leq \\ \left(1+\|\det DT_k\|_{L^\infty(\mathbb{R}^n)}\right)^{1/2}\left\|(DT_k)^{-1}\right\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\bar{u}\|_{H^1(T(U_r));\mathbb{R}^n)} \end{aligned}$$

and the same goes for the first derivative.

This bound is uniform on k because of the continuity of the bound, with respect to k, as seen in 4.12, page IV.6, [15].

We conclude that $\left\|U^k\right\|^2_{H^1(I,V_{T_k\circ T})}$ is bounded and we thus have that $W^k \in Q_0(I,V_{T_k\circ T}), V_0^k \in Q_0(I,W_{T_k\circ T})$ are bounded.

Now, for almost all times, using 4.11, page IV.6 of [15], we obtain that, for instance:

$$\begin{aligned} \left\|v_0^k\right\|_{V_T} \leq \\ \left(1+\left\|\det(DT_k)^{-1}\right\|_{L^\infty(\mathbb{R}^n)}\right)^{1/2}\|DT_k\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\left\|V_0^k\right\|_{H^1(T_K(T(U_r)))} \end{aligned}$$

where we remember that H^1_0 was chosen to be normed with the full H^1 norm.

The same goes for w^k and the first derivatives in time, yielding that $w^k \in Q_0(I,V_T), v_0^k \in Q_0(I,W_T)$ are bounded.

We thus have $w^k \rightharpoonup w^? \in Q_0(I, V_T)$, $v_0^k \rightharpoonup v_0^? \in Q_0(I, W_T)$, in the weak topologies of, respectively, $Q(I, V_T), Q(I, W_T)$, and modulo subsequences. The initial values are preserved becase Q_0 is closed and convex in the Hilbert space Q (see B.3.3). The closedness follows from the fact that the embedding into continuous function is linear bounded, and evaluation at 0 is linear bounded from continuous functions.

We now prove that $w^? = w^0$, $v^? = v^0$, and this will yield the weak convergence of the whole sequence.

To prove e.g. that $v^? = v^0$, let us look at the weak formulations of v_0^k:

$$\int_I (v_{0t}^k, p|\det(DT_k)|)_{H_T} + (A_{T_k} \nabla v_0^k, \nabla p)_{H_T} + (\bar{u}', p|\det(DT_k)|)_{H_T} + (A_{T_k} \nabla \bar{u}, \nabla p)_{H_T} = 0$$

for all $p \in Q_0(I, W_T)$.

Let's analyize the first term, which is $\int_I (v_{0t}^k, p|\det(DT_k)|)_{H_T}) = (v_{0t}^k, p|\det(DT_k)|)_{L^2(I,H_T)}$.

We can write:

$$\begin{aligned} (v_{0t}^k, p|\det(DT_k)|)_{L^2(I,H_T)} = \\ (v_{0t}^k, p)_{L^2(I,H_T)} - (v_{0t}^k, p)_{L^2(I,H_T)} + (v_{0t}^k, p|\det(DT_k)|)_{L^2(I,H_T)} = \\ (v_{0t}^k, p)_{L^2(I,H_T)} + (v_{0t}^k, p(|\det(DT_k)| - 1))_{L^2(I,H_T)} \end{aligned}$$

Because $p \in Q(I, W_T)$, the first term converges to $(v_{0t}^?, p)_{L^2(I,H_T)}$, see proposition B.3.3 for details on why we can write the time derivative of the limit. The other term can be estimated as follows:

$$\begin{aligned} |(v_{0t}^k, p(|\det(DT_k)| - 1))_{L^2(I,H_T)}| \leq \\ \left\|v_{0t}^k\right\|_{L^2(I,H_T)} \|p(|\det(DT_k)| - 1))\|_{L^2(I,H_T)} \leq \left\|v_{0t}^k\right\|_{L^2(I,H_T)} \|p\|_{L^2(I,H_T)} \||\det(DT_k)| - 1\|_{L^\infty} \end{aligned}$$

Where the first term in the product is bounded by the weak convergence property, and the last one goes to 0 by continuity, see again 4.12, page IV.6 of [15].

In a similar fashion for the other pieces, and by passing to the limit:

$$\int_I (v_{0t}^?, p)_{H_T} + (\nabla v_0^?, \nabla p)_{H_T} + (\bar{u}', p)_{H_T} + (\nabla \bar{u}, \nabla p)_{H_T} = 0$$

By uniqueness, $v^? = v^0$.

Weak convergence of averages adjoint states

So, $v_0^k \rightharpoonup v_0^0, w^k \rightharpoonup w^0$ in the sense of the $Q(I, W_T)$ and $Q(I, V_T)$ weak convergence.

We now claim that $p^k \rightharpoonup p^0, q^k \rightharpoonup q^0$, in a similar style as before. To do so, remember that $P^k := p^k \circ T_k^{-1}$ and $Q^k := q^k \circ T_k^{-1}$ solve:

$$\begin{aligned} -Q_t^k - \Delta Q^k = \frac{v_0^k - w^k + v_0^0 - w^0}{2} \circ T_k^{-1} + \bar{u}_T \circ T_k^{-1} \\ Q^k(T) = 0 \\ \partial_\nu Q^k = 0 \text{ on } \Sigma_f \\ Q^k = 0 \text{ on } \Sigma_m \end{aligned}$$

and

$$\begin{aligned} -P_t^k - \Delta P^k = -\frac{v_0^k - w^k + v_0^0 - w^0}{2} \circ T_k^{-1} - \bar{u}_T \circ T_k^{-1} \\ P^k(T) = 0 \\ P^k = 0 \text{ on } \Sigma_m \sqcup \Sigma_f \end{aligned}$$

By proposition B.1.14, we will obtain a bound in Q of the tranported averaged adjoints as soon as we have a bound on $\frac{v_0^k - w^k + v_0^0 - w^0}{2} \circ T_k^{-1}$ in the $L^2(I, H)$ norm, and of $U^k := \bar{u}_T \circ T_k^{-1}$. The latter was proven above.

So, by theorem C.2.1 and 4.12 of [15] at page IV.6, it suffices to have an $L^2(I, H)$ bound on $\frac{v_0^k + w^k + v_0^0 + w^0}{2} \circ T_k^{-1} \circ T_k = \frac{v_0^k + w^k + v_0^0 + w^0}{2}$ which we have, since we just proved that v_0^k, w^k are weakly convergent in e.g. $L^2(I, H)$.

We conclude that Q^k, P^k are bounded in the $Q(I, V_{T_k \circ T})$ and $Q(I, W_{T_k \circ T})$ sense.

But what we want is a bound on q^k, p^k in the $Q(I, V_T)$ and $Q(I, W_T)$ sense. This can be accomplished in exactly the same way as before.

We conclude that there exist $q^?, p^?$ in $Q^0(I, V_T)$, $Q^0(I, W_T)$, that are, modulo subsequences, the weak limits of q^k, p^k.

To show e.g. $q^? = q^0$ and conclude the convergence of the full sequence, we analyze the weak formulation of q^k, which reads, after going to the moving domain and applying integration by parts in time (see B.3.3):

$$
\begin{aligned}
-\int_I \left(\frac{((v_0^k + \bar{u}_T - w^k) + (v_0^0 + \bar{u}_T - w^0))}{2} \right) | \det(DT_k)|, \delta w)_{H_T} + \\
- \int_I (q_t^k, \delta w | \det(DT_k)|)_{H_T} + (A_{T_k} \nabla \delta w, \nabla q^k)_{H_T}
\end{aligned}
$$

for all $\delta w \in Q_0(I, V_T)$.

We show the convergence of e.g. the member: $\int_I (v_0^k | \det(DT_k)|, \delta w)_{H_T}$. By splitting the scalar product as we saw above, we are left with checking that $\int_I (v_0^k, \delta w)_{H_T} \to \int_I (v_0^0, \delta w)_{H_T}$, which is true, since we proved that $v_0^k \rightharpoonup v_0^0$ in $Q(I, W_T)$.

We conclude upon convergence that:

$$
\begin{aligned}
-\int_I \left(\frac{((v_0^0 + \bar{u}_T - w^0) + (v_0^0 + \bar{u}_T - w^0))}{2} \right), \delta w)_{H_T} + \\
- \int_I (q_t^?, \delta w)_{H_T} + (\nabla \delta w, \nabla q^?)_{H_T}
\end{aligned}
$$

which is satisfied also by q^0, therefore $q^? = q^0$ and we have weak convergence of the entire sequence.

$\square$

A. Functional spaces

A.1. Sobolev spaces

Theorem A.1.1 (Integration by parts)
Let Ω be a bounded Lipschitz domain. Let $1 < p < \infty$ and $f, g \in W^{1,p}(\Omega), W^{1,q}(\Omega)$, $q = p'$. Then:

$$\int_\Omega f \partial_i g = -\int_\Omega g \partial_i f + \int_{\partial\Omega} \mathrm{tr} u \nu_i d\mathcal{H}^{n-1}$$

Proof.

This follows from [11], theorem 18.1 at page 592, where g needs to be $C_c^1(\mathbb{R}^n)$. But [1], theorem 3.18 at page 54, says that (thanks to the smoothness of the boundary) the set of the restriction of such functions is dense in $W^{1,q}(\Omega)$, so that we can conclude by a density argument developed here.

□

Lemma A.1.2
$f \in L^\infty(\Omega; \mathbb{R}^N) \iff f_i \in L^\infty$, and two equivalent norms are $\|f\|_a := \||f|\|_\infty$, $\|f\|_b := \max_i \|f_i\|_\infty$, for $|\cdot|$ any finite dimensional norm.

Proof.

We choose $|\cdot| = |\cdot|_1$.

Consider $f_n \in X_a = \{[f], f : \Omega \to \mathbb{R}^n \text{ measurable }, \|f\|_a\}$, Cauchy. Then every component is Cauchy in the scalar L^∞, so that $f_n^i \to f^i$ in L^∞. The limit f is in X_a because the functions $|f_i|$ are essentially bounded, and so is $|f|$.

Then $\|f_n - f\|_a \leq \|f_n - f_m\|_a + \sum_i \|f_m^i - f^i\|_\infty$ for all n, m. Choose $m \geq n$ with $\|f_m^i - f^i\| \leq 1/(Nn)$ and conclude X_a is Banach.

We know from [11], theorem B.88 at page 671, and page 669, we know that $X_b = \{[f], f : \Omega \to \mathbb{R}^n \text{ measurable }, \|f\|_b\}$ is Banach.

Moreover $X_a = X_b$ as sets, so that the thesis follows.

$\square$

Proposition A.1.3 (Characterization of $W^{1,\infty}$)
Let Ω be a bounded Lipschitz domain, or $\mathbb{R}^n$. Then $W^{1,\infty}(\Omega) = C^{0,1} \cap L^\infty(\Omega)$.

This means that $u \in W^{1,\infty}(\Omega)$ if and only if u has a (unique) representative that is bounded, Lipschitz continuous. Weak and classical derivatives coincide a.e.

Proof.

<u>Extension</u>

In the case Ω is bounded Lipschitz, then Ω is an extension domain for $W^{1,\infty}(\Omega)$, meaning that there is $E : W^{1,\infty}(\Omega) \to W^{1,\infty}(\mathbb{R}^n)$ linear bounded with $Eu = u$ a.e. on Ω (see [11], theorem 13.17 at page 425, 13.13 at page 424, and definition 9.57 at page 273).

<u>The proof</u>

Let $u \in W^{1,\infty}(\Omega)$. By [11], 11.50 at page 339, because Ω is an extension domain, we obtain that u has a representative $\bar{u}$ that is bounded Lipschitz. Let $\phi \in C_c^\infty(\Omega)$. By The Kirszbraun theorem (see e.g. [2]), we can extend $\bar{u}$ to a Lipschitz function e on $\mathbb{R}^n$. Then, for a large enough cube Q containing Ω, $\displaystyle\int_\Omega \bar{u}\partial_i\phi = \int_Q e\partial_i\phi = -\int_Q \partial_i e\phi$, by Fubini's theorem and integration by parts for AC functions.

Because $e = \bar{u}$ on Ω, we conclude $\displaystyle\int_\Omega \bar{u}\partial_i\phi = -\int_\Omega \partial_i\bar{u}\phi$, so that $\nabla\bar{u} = \nabla u$ almost everywhere.

Conversely, let u be bounded Lipschitz. The above reasoning shows that u has essentially bounded weak derivatives equal to the a.e. classical derivatives.

$\square$

Corollary A.1.4 ($W^{k,\infty} = C_B^{k,1}$)
For a bounded Lipschitz domain Ω, or for $\Omega = \mathbb{R}^n$, then $W^{k,\infty} = C_B^{k,1}$ ($C^{k,1}$ bounded functions with bounded classical derivatives up to order $k+1$).

Proof.

We have already proved the case $k = 1$. We prove, for instance, the case $k = 2$. Then, $u \in W^{k,2} \implies u, \partial_i u \in W^{k,1}$ ([11], 11.7 at page 321), so that by proposition A.1.3, we find bounded Lipschitz h, g_i with $u = h$ a.e., $\partial_i u = \partial_i h$ a.e., $g_i = \partial_i u$ a.e..

Therefore h is continuous, with continuous weak derivatives g_i, which implies that $h \in C^1(\Omega)$ (see here and here).

Now, $\partial_i h = g_i$ a.e., so everywhere, so that:

- h is bounded Lipschitz and C^1
- $\partial_i h$ are bounded Lipschitz

□

A.2. Bochner spaces

Here are some useful results about Bochner spaces.

Proposition A.2.1 (Bochner integral and bounded operators)
Let X, Y be separable Banach, let $T \in L(X, T)$ be a linear bounded operator. For $f \in L^1(I, X)$ define $Tf(t) := T(f(t))$. Then $Tf \in L^1(I, Y)$ with $T \int_I f = \int_I Tf$.

Proof.

First of all, a clarification on the definition. What is really happening is that from the time equivalence class f, we select a g, and then $Tf(t) := T(g(t))$. Tf is then the equivalence class of $t \mapsto Tf(t)$. The definition is well posed, because $g_1(t) = g_2(t) \implies T(g_1(t)) = T(g_2(t))$.

Let f_n be simple, $f_n \to f$ a.e., with $\lim_n \int_I f_n = \int_I f$ and $\|f_n\|_X \leq C \|f\|_X$ (see page 6, and corollary 2.7 at page 8 of [10]).

<u>Measurability</u>

For almost all t, $T(f_n(t)) \to T(f(t)) = Tf(t)$ in Y, so that Tf is measurable (strongly).

<u>Integrability</u>

By the assumptions, $\|Tf_n\| \leq \|T\| \|f_n\| \leq C \|f\| \in L^1(I)$, so that by dominated convergence (corollary 2.6 of [10]) Tf is integrable too. Thus $\int_T Tf = \lim_n \int_I Tf_n = \lim_n T \int_I f_n$, because f_n is simple. And now, by the choice of f_n, $\int_T Tf = \lim_n T \int_I f_n = T \lim_n \int_I f_n = T \int_I f$.

□

Proposition A.2.2 (Derivations and bounded operators)
As before, let X, Y be separable Banach, let $T \in L(X, T)$ be a linear bounded operator.

For $k \geq 0$, $f \in H^k(I,X) \implies Tf \in H^k(I,Y)$, with weak derivatives $\partial_{t^i} Tf = T\partial_{t^i} f$, $0 \leq i \leq k$.

The map $f \mapsto Tf$, $H^k(I,X) \to H^k(I,Y)$ is linear, and bounded by $\|T\|$.

Proof.

The case $k = 0$ is proved above.

We prove now that $\partial_{t^i} Tf = T\partial_{t^i} f$ for $i = 1$. Note that $T\partial_t f \in L^2(I,Y)$, which qualifies as weak derivative.

In fact, for $\phi \in C_c^\infty(I)$, we have $\int_I \phi T \partial_i f = \int_I T(\phi \partial_t f) = T \int_I \phi \partial_t f = -T \int_I \phi' f = -\int_I \phi' T f$.

Higher weak derivatives are treated analogously and the rest of the claims follow from the time stationarity of T and by $\|\partial_{t^i} Tf\| = \|T\partial_{t^i} f\| \leq \|T\| \, \|\partial_{t^i} f\|$.

□

Proposition A.2.3 (Continuous representatives)
Let X be separable Banach. $f \in L^1(I,X)$ has at most a continuous representative on $[0,T]$.

Proof.

Assume there exists two such continuous representatives, so that we get a function $\delta : [0,T] \to X$ that is zero almost everywhere and continuous. Hence, $[0,T] \ni t \mapsto \|\delta(t)\|$ is continuous in $\mathbb{R}$ and zero a.e., so that it must be zero everywhere. □

We now check that a vector valued test function has weak derivatives of all orders.

Proposition A.2.4 (Weak derivatives of test functions)
Let $\phi \in C^1([0,T],X)$, for X separable Banach. It means that the limit of the difference quotients exists for all points of I, that $t \mapsto \phi(t), \phi'(t)$ are continuous, and that they can be continuously extended to $[0,T]$.

Then these classical derivatives coincide a.e. with the weak derivatives of u.

Proof.

We rely on proposition 3.8 of [10] at page 26.

<u>Absolute continuity</u>

Consider $\epsilon > 0$. Divide $[a, b] \subset\subset (0, T)$ into a uniform partition t_i. By theorem 6 at page 146 of [6], we get that $\|\phi(t_i) - \phi(t_{i-1})\|_X \leq (t_i - t_{i-1}) \|\phi(\xi_i)\|_X \leq (b - a) \|\phi'\|_\infty / n$, and by choosing n small enough, we conclude that ϕ is (locally) absolutely continuous.

Weak derivative

Therefore, ϕ is locally AC, differentialble everywhere and ϕ' is bounded, so that $\phi \in H^1(I, X)$ and weak and classical derivatives coincide.

□

And now, introduce a time dependent version of the trace operator which is useful for our computations.

Definition A.2.5 (Time dependent trace)
Let Ω be a bounded Lipschitz domain. For $k \geq 0$ we define $\mathrm{tr} : H^k(I, H^1(\Omega)) \to H^k(I, H^{1/2}(\partial\Omega))$ by $\mathrm{tr}(u)(t) := \mathrm{tr}(u(t))$

Below are some properties of this operator.

Proposition A.2.6 (Properties of trace operator)
The trace operator just defined:

1. is well posed
2. is linear bounded
3. admits a linear bounded right inverse, for instance, $E(g)(t) := E(g(t))$ (for E a right inverse of the static trace)
4. tr and E, in the case of $k \in \mathbb{N}_0$, coincide (in the time a.e. sense) for the case $l \geq k$
5. for $k \geq 1$, $\mathrm{tr} u(0) = 0 \iff u(0) = 0$ (in the sense of continuous representatives)
6. it coincides with the trace treated for instance in [13]

Proof.

Proof of the proposition

We recall that the trace operator is bounded surjective onto $H^{1/2}(\partial\Omega)$, with a right inverse E (see theorem 3.37 at page 102 of [14]).

The first three points are consequences of this fact and of proposition A.2.1.

The fourth property follows by the definition of tr, E and the fact that $H^l \subseteq H^k$, for $k \leq l$.

Let now $k \geq 1$. We know that $H^1, H^{1/2}$ are separable and Banach (the latter is separable because the continuos image of H^1 separable, and Banach (see [8], page 20). Therefore, by [5], theorem 2 of page 286, we obtain the embeddings $H^k(I, H^1) \hookrightarrow C([0,T], H^1)$ and the same goes for $H^k(I, H^{1/2})$. The embedding is U, the unique continuous representative of a certain time equivalence class (proposition A.2.3). We also introduce brackets to indicate equivalence classes in time, so, $u = [Uu]$.

We want to prove $(Uu)(0) = 0 \iff U(\mathrm{tr}u)(0) = 0$. But we have $[t \mapsto U(\mathrm{tr}u)(t)] = tru := [t \mapsto \mathrm{tr}((Uu)(t))]$. So, $U(\mathrm{tr}u)(t) = \mathrm{tr}((Uu)(t))$ for all $t \in [0,T]$ by continuity.

For the last point, let $k = 0$. We have:

1. $H^1(\Omega) \cap C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ (see [1], theorem 3.18 at page 54, where being Ω bounded Lipschitz is important)

2. functions $\sum_{i \leq m} \phi_i(t) f_i$ for $\phi_i \in C_c^\infty(I), f_i \in H^1(\Omega) \cap C^1(\overline{\Omega})$ are dense in $L^2(I, H^1)$ (see [9], page 39, lemma 1.9)

It follows by the third point that $C^1(\overline{\Omega \times I})$ is dense in $L^2(I, H^1)$, so that $u \mapsto u|_{I \times \partial\Omega}$ admits a unique extension by continuity to $L^2(I, H^1)$, so that this definition of trace coincides with the one from the literature in the case of the space $H^{1,0} := L^2(I, H^1)$ (see [13], theorem 4.1), we expand this argument below.

<u>Proof of leftover facts</u>

We call $C^k(\overline{\Omega}) := \{u \in C^k(\Omega)$ with $\partial_\alpha f$ extendable by continuity to $\overline{\Omega}\}$.

Consider $u(x,t) := \phi(t)v(x)$, for $\phi \in C^1([0,T]), v \in C^1(\overline{\Omega})$. Then, it has partial derivatives $u_t = \phi_t v, u_i = \phi u_i$. u and all its partial derivatives are continuous on $I \times \Omega$, meaning that $u \in C^1(\Omega \times I)$.

Moreover, $u, u_i, u_t \in C([0,T], C(\overline{\Omega}))$. We claim $C([0,T], C(\overline{\Omega})) = C(\overline{\Omega \times I})$. In fact, one direction is trivial, and so, let $f \in C([0,T], C(\overline{\Omega})) = C(\overline{\Omega})$. Fix $(t,x) \in \overline{\Omega \times I}$. Then, $|f(s,y) - f(t,x)| \leq |f(t,y) - f(t,x)| + |f(t,y) - f(s,y)| \leq |f(t,y) - f(t,x)| + \|f(t,\cdot) - f(s,\cdot)\|_\infty$. If now s is close to t, and y is close to x, then $|f(s,y) - f(t,x)|$ is small.

This shows $u, u_i, u_t \in C([0,T], C(\overline{\Omega})) \in C(\overline{\Omega \times I})$, i.e. $u \in C^1(\overline{Q \times I})$.

To conclude, let $u \in L^2(I, H^1)$. Approximate u by $u_k := \sum_{i \leq m_k} \phi_i^k(t) f_i^k$ as in point 2, and approximate f_i^k by suitable $g_i^k \in H^1(\Omega) \cap C^1(\overline{\Omega})$, to obtain $u_k := \sum_{i \leq m_k} \phi_i^k(t) g_i^k$

Then $\|u - w_k\|_{L^2(I,H^1)} \leq \|u_k - w_k\|_{L^2(I,H^1)} + \|u_k - u\|_{L^2(I,H^1)}$. We only need to estimate $\|u_k - w_k\|_{L^2(I,H^1)} \leq T \sum\limits_{i \leq m_k} \left\|\phi_i^k\right\|_\infty \left\|f_i^k - g_i^k\right\|_{H^1}$. By the first point, $\left\|f_i^k - g_i^k\right\|_{H^1}$ can be made as small as it is necessary to conclude.

Last remarks

Again with reference to [13], consider the anisotropic spaces $H^{r,s} := L^2(I, H^r) \cap H^s(I, L^2)$. We restrict to the case $r = 1$, $s \geq 0$. Denote the traces tr_s defined in theorem 4.1, mapping $H^{1,s}(\Omega \times I) \to H^{1/2,s/2}(\partial\Omega \times I)$. For $\partial\Omega$ Lipschitz this theorem is still valid, as $1/2 \leq 1$, see the discussion above lemma 2.4 in [4]. As stated in [13], tr_s is an extension of $u \mapsto u|_{I\times\partial\Omega}$, defined on the dense suspace $C^\infty(\overline{Q \times I})$ of $H^{1,s}$ (that this space is dense can be proved as in lemma 2.22 of [4]). So, let $C^\infty(\overline{Q \times I}) \ni u_n \to_{H^{r,s}} u \in H^{1,s}$.

We have $\mathrm{tr}_s u_n = \mathrm{tr}_0 u_n$. Then, $u_n \to_{H^{1,s}} u$, $u_n \to_{H^{1,0}} u$, so that $\mathrm{tr}_s u_n \to_{H^{1/2,s/2}} \mathrm{tr}_s u$ (hence $\mathrm{tr}_0 u_n \to_{H^{1/2,0}} \mathrm{tr}_s u$) and $\mathrm{tr}_0 u_n \to_{H^{1/2,0}} \mathrm{tr}_\sigma u$.

Thus $\mathrm{tr}_0 u = \mathrm{tr}_s u$.

Using what we derived before, we can conclude the characterization of the traces in the anisotropic settign define

□

And now some sanity checks in the case of Gelfand triples.

Proposition A.2.7 (Sanity checks for Gelfand triples)
Consider the following Gelfand triples (the diagram commutes):

$$\begin{tikzcd}
V \arrow[dr, hook, "a"] & & & V^* \arrow[dd, hook, "c^*"] \\
& H \arrow[r, "r"] & H^* \arrow[ur, hook, "a^*"] \arrow[dr, hook, "b^*"] & \\
W \arrow[uu, hook, "c"] \arrow[ur, hook, "b"'] & & & W^*
\end{tikzcd}$$

Here $W \subseteq V \subseteq H$ are all separable Hilbert spaces, a, b, c the trivial injections, r the Riesz isomorphism of H. We denote by i_V the Gelfand triple embedding $V \hookrightarrow V^*$, so, $i_V = a^* r a$.

Then:

1. $H^1(I, V) \subseteq W(I, V)$ with continuous embedding. The $W(I, V)$ derivative of $u \in H^1(I, V)$ is $i_V u_t$.

2. for $u \in W(I, W)$ with $(i_W u)' \in L^2(I, H)$ (i.e. $(i_W u)_t = b^* r h$ for h in $L^2(I, H)$) we obtain $u \in W(I, V)$ (i.e. $cu \in W(I, V)$), with derivative $(i_V cu)' = a^* r h$, so that also $(i_V cu)' \in L^2(I, H)$. It also holds $(i_V cu)'|_W = (i_W u)'$. h is also the weak derivative $L^2(I, H)$ of bu.

3. let $u, v \in W(I, V)$ with $u - v \in W$. Then $u - v \in W(I, W)$ with derivative $(i_W(u-v))' = (i_V u)'|_W - (i_V v)'|_W$.

Proof.

We use several times that time integrals and bounded linear static operators commute, see proposition A.2.1. ϕ denotes $\phi \in C_c^\infty(I)$.

First point

We need to check that $a^* rau \in H^1(I, V^*)$. This follows from proposition A.2.2, so that $(a^* rau)_t = a^* rau_t$.

Second point

At first we claim that h is a weak derivative of $bu \in L^2(I, H)$. In fact, $b^* r \int_I bu\phi' = \int_I (i_W u)\phi' = \{\ u \in W(I, W)\ \} = -\int_I (i_W u)'\phi = -\int_I b^* rh\phi = b^* r(-\int_I h\phi)$. By density (definition of Gelfand triple), b^* is injective, r is too, and thus $\int_I bu\phi' = -\int_I h\phi$, which shows that bu has weak derivative h, in the $H^1(I, H)$ sense.

And now $\int_I i_V cu\phi' = \int_I a^* racu\phi' = a^* r \int_I bu\phi' = \{$ by what we just proved $\} = -a^* r \int_I h\phi$, proving that $(i_V cu)' = a^* rh$.

Morevoer $(i_V cu)'|_W = c^* a^* rh = b^* rh = \{$ assumption $\} = (i_W u)'$.

Third point

We check the derivative. We have $\int_i i_W(u-v)\phi' = \{\ u - v \in W \subseteq V\ \} = \int_I b^* ra(u-v) = c^* \int_I (i_V u - i_V v)\phi' = -\int c^*((i_V u)' - (i_V v)')\phi$.

$\square$

B. Parabolic equations

B.1. Abstract theory

Assumption B.1.1 (Basic assumption for parabolic problems)
Let $V \subseteq H$ be real separable Hilbert spaces, V dense in H. Then $H \hookrightarrow V^*$ is also dense, as stated in [17] at page 147. This embedding is $H \ni f \mapsto (f, \cdot)_H$. We thus obtain a Gelfand triple, and we have $W(I,V) \subseteq C(I,H)$.

Let $A : V \to V^*$ be linear bounded, $u \in W(I;V)$, $f \in L^2(I, V^*)$ and $u_0 \in H$.

We also assume that $\langle Av, v\rangle_{V^*,V} + \lambda \|v\|_H^2 \geq \alpha \|v\|_V^2$ for $\lambda \geq 0, \alpha > 0$.

We are interested in the following problem:

Problem B.1.2 (Abstract parabolic equation)

$$u_t + Au = f \text{ in } V^* \text{ and for a.e. } t \in (0,T) \tag{B.1.3}$$
$$u(0) = u_0 \tag{B.1.4}$$

Theorem B.1.5 (Basic well posedness of problem B.1.2)
Under assumption B.1.1, problem B.1.2 has a unique solution u. Moreover u satisfies the energy estimate:

$$\|u\|_{W(I,V)} + \|u\|_{C([0,T],H)} \leq c(\lambda, \alpha, \|A\|_{V^*}, T)(\|u_0\|_H + \|f\|_{L^2(I,V^*)}) \tag{B.1.6}$$

Proof.

See [7] at page 19, theorem 26. □

We can also obtain additional regularity. Here are further assumptions to make this possible.

Assumption B.1.7 (Assumptions for additional regularity)
We assume $u_0 \in V$, $f = f_1 + f_2 \in L^2(I,H) + H^1(I,V^*)$. We also need A to be symmetric (i.e. $\langle Au, v\rangle_{V^*,V} = \langle Av, u\rangle_{V^*,V}$).

Theorem B.1.8 (Regularity of time derivative)
Suppose assumption B.1.1 and assumption B.1.7. Then $u_t \in L^2(I,H)$ with the estimate:

$$\|u\|_{W(I,V)} + \|u\|_{C(I,H)} + \|u_t\|_{L^2(I,H)} \leq \tag{B.1.9}$$
$$c(\lambda, \alpha, \|A\|_{V^*}, T)(\|u_0\|_V + \|f_1\|_{L^2(I,H)} + \|f_2\|_{H^1(I,V^*)}) \tag{B.1.10}$$

That $u_t \in L^2(I,H)$ means precisely that there is $h \in L^2(I,H)$ with $a^* r h = (i_V u)'$, with the notation introduced in proposition A.2.7.

Proof.

We refer to page 26 of [7], theorem 28, and only prove the necessary modifications.

Product rule for A

We have

$$\int_0^t \langle Au_n, u_n' \rangle_{V^*,V} = \sum_{k,l \leq n} \langle Aw_k^n, w_l^n \rangle_{V^*,V} \int_0^t g_k^n g_l^{n'} =$$
$$\sum_{k,l \leq n} \langle Aw_k^n, w_l^n \rangle_{V^*,V} \left(- \int_0^t g_k^{n'} g_l^n + g_k^n(t) g_l^n(t) - g_k^n(0) g_l^n(0) \right)$$

By linearity at first and then symmetry we get:

$$= \langle Au_n, u_n \rangle_{V^*,V}(t) - \langle Au_n, u_n' \rangle_{V^*,V}(0) - \int_0^t \langle Au_n', u_n \rangle_{V^*,V} =$$
$$= \langle Au_n, u_n \rangle_{V^*,V}(t) - \langle Au_n, u_n' \rangle_{V^*,V}(0) - \int_0^t \langle Au_n, u_n' \rangle_{V^*,V}$$

so that:

$$\int_0^t \langle Au_n, u_n' \rangle_{V^*,V} = \frac{1}{2} \left(\langle Au_n, u_n \rangle_{V^*,V}(t) - \langle Au_n, u_n' \rangle_{V^*,V}(0) \right)$$

Estimate for right hand side

We have:

$$\int_0^t \langle f_2, u_n' \rangle_{V^*,V} = \sum_{k \le n} \int_0^t g_k^{n'} \langle f_2, w_k^n \rangle_{V^*,V}$$

By the smoothness of f_2 we have that $t \mapsto \langle f_2(t), w_k^n \rangle_{V^*,V}$ is $H^1(0,T)$, in particular $AC[0,t]$, so that we can integrate by parts:

$$= -\sum_{k \le n} \int_0^t g_k^n \langle f_2', w_k^n \rangle_{V^*,V} + \sum_{k \le n} g_k^n(t) \langle f_2(t), w_k^n \rangle_{V^*,V} - \sum_{k \le n} g_k^n(0) \langle f_2(0), w_k^n \rangle_{V^*,V} =$$
$$- \int_0^t \langle f_2', u_n \rangle_{V^*,V} + \langle f_2, u_n \rangle_{V^*,V}(t) - \langle f_2, u_n \rangle_{V^*,V}(0)$$

Here we have used proposition A.2.2 to take the derivative inside the bracket.

Note that by the smoothness of f_2, we can write, for instance, $\langle f_2, u_n \rangle_{V^*,V}(0) = \langle f_2(0), u_n \rangle_{V^*,V}$.

NB: here I need $f_2(0) \in V$ probably, so that we are using the compatibility condition!.

Going to the absolute values:

$$\left| \int_0^t \langle f_2, u_n' \rangle_{V^*,V} \right| \le \int_0^T |\langle f_2', u_n \rangle_{V^*,V}| + \|f_2(t)\|_{V^*} \|u_n(t)\|_V + \|f_2(0)\|_{V^*} \|u_n(0)\|_V \le$$
$$\frac{1}{2} \|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2} \|u_n\|^2_{L^2(I,V)} + \frac{\alpha}{4} \|u_n(t)\|_V^2 +$$
$$+ \frac{4}{\alpha} \|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2} \|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2} \|u_{n0}\|_V^2$$

Now, u_n converges weakly in $L^2(I,V)$ by estimate (59) of [7] and thus $\frac{1}{2} \|u_n\|_{L^2(I,V)}$ is bounded. The term $\frac{\alpha}{4} \|u_n(t)\|_V$ can be pulled to the left hand side, u_{0n} is V convergent hence bounded. Therefore as in [7] we are able to conclude that u_n' is bounded in $L^2(I,H)$. We want to conclude $u_t \in L^2(I,H)$. We know for sure that $\langle u_m', w_j \rangle_{V^*,V} = \langle f - Au_m, w_j \rangle_{V^*,V}$, so that muliplication by a test function and integration yields $\int_I \langle u_m', w_j \phi \rangle_{V^*,V} = \int_I \langle f - Au_m, w_j \phi \rangle_{V^*,V}$. Because $u_m \rightharpoonup u$ in $L^2(I,V)$ we observe that, by proposition A.2.1 applied on $A \in L(V,V^*)$, it holds $\int_I \langle u_m', w_j \phi \rangle_{V^*,V} \to \int_I \langle u', w_j \phi \rangle_{V^*,V}$.

What's more, is that $u_m' \rightharpoonup h$ in $L^2(I,H)$, so that $\int_I \langle h, w_j \rangle_{V^*,V} \phi = \int_I \langle u', w_j \rangle_{V^*,V} \phi$. It means that for almost all t, $\langle h, w_j \rangle_{V^*,V} = \langle u', w_j \rangle_{V^*,V}$. And now we can really say that $u' \in L^2(I,H)$, which even more precisely means $(i_V u)' = a^* r h$ almost everywhere.

We also obtain that u_t is bounded by $c(\alpha)(\|f_2\|_{L^\infty(I,V^*)} + \|f_2\|_{L^2(I,V^*)} + \|u_0\|_V + \|u\|_{L^2(I,V)})$.

Note that, by [5], theorem 2 of page 286, we can estimate $\|f_2\|_{L^\infty(I,V^*)}$ by $c(T)\,\|f_2\|_{H^1(I,V^*)}$, so that the claim for the time derivative u_t is proven.

□

For the case where $H = L^2$, $H^1 \supseteq V \supseteq H_0^1$, $f_2|_{H_0^1} = 0$ we have even more regularity available.

Theorem B.1.11 (Additional regularity)
Suppose assumption B.1.1 and assumption B.1.7.

Let additionally $H = L^2$, $H^1 \supseteq V \supseteq H_0^1$, $f_2|_{H_0^1} = 0$. Then $Au|_{H_0^1}$ extends to $\overline{Au_{H_0^1}} \in L^2(I,H)$ with:

$$\|u\|_{W(I,V)} + \|u\|_{C([0,T],H)} + \|u_t\|_{L^2(I,H)} + \left\|\overline{Au|_{H_0^1}}\right\|_{L^2(I,H)} \leq \tag{B.1.12}$$

$$c(\lambda,\alpha,\|A\|_{V^*},T)(\|u_0\|_V + \|f_1\|_{L^2(I,H)} + \|f_2\|_{H^1(I,V^*)}) \tag{B.1.13}$$

Moreover $u_t + \overline{Au_{H_0^1}} = f_1$ in $L^2(0,T,L^2) \cong L^2(Q)$ and $\overline{Au|_{H_0^1}} = Au$ on H_0^1.

Proof.

For $v \in H_0^1$ we get $\langle Au, v\rangle_{V^*,V} = \langle f_1 - u_t, v\rangle_{V^*,V} = (f_1 - u_t, v)_H$, for almost all $t \in (0,T)$. From here we conclude that $Au(t)$ extends for a.a. t to an element of H with $(\overline{Au} - f_1 + u_t, v)_{L^2} = 0$ for all $v \in H_0^1$, almost all t. By density, $\overline{Au} - f_1 + u_t = 0$ in H for almost all t, so that $\overline{Au} = f_1 - u_t$ in $L^2(0,T,L^2) \cong L^2(Q)$.

This isometric isomorphism is stated in [17], page 144.

□

For our applications we also need to track the constants more precisely, which is accomplished in the next proposition.

Proposition B.1.14 (Tracking the costants)
With assumption B.1.1 there holds:

$$\|u\|^2_{C([0;T],H)} + \alpha\,\|u\|^2_{L^2(I,V)} \leq \exp(2\lambda T)(\|u_0\|^2_H + \alpha^{-1}\,\|f\|^2_{L^2(I,V^*)}) \tag{B.1.15}$$

$$\|u'\|_{L^2(I,V^*)} \leq \|A\|_{L(V,V^*)}\,\alpha^{-1/2}\sqrt{\exp(2\lambda T)}\,\|u_0\|_H + \tag{B.1.16}$$

$$\left(\|A\|_{L(V,V^*)}\,\alpha^{-1}\sqrt{\exp(2\lambda T)} + 1\right)\|f\|_{L^2(I,V^*)} \tag{B.1.17}$$

With additionally assumption B.1.7 we obtain:

$$C \left\| u' \right\|_{L^2(I,H)}^2 \leq (1 + (1 + C_0)\alpha^{-1}) \left\| f_2 \right\|_{H^1(I,V^*)}^2 + \tag{B.1.18}$$
$$(1 + \left\| A \right\|_{L(V,V^*)}) \left\| u_0 \right\|_V^2 + C_0 \left\| u_0 \right\|_H^2 + \tag{B.1.19}$$
$$\left\| f_1 \right\|_{L^2(I,H)}^2 + C_0 \alpha^{-1} \left\| f_1 \right\|_{L^2(I,V^*)}^2 \tag{B.1.20}$$

with $C > 0$ a number independent of the problem.

Here $C_0 = 2^{-1} \max(1, \lambda) \max(1, \alpha^{-1}) \exp(2\lambda T)$.

Proof.

No regularity

From page 21 of [7] we obtain that $\left\| u \right\|_{C([0;T],H)}^2 + \alpha \left\| u \right\|_{L^2(I,V)}^2 \leq \exp(2\lambda T)(\left\| u_0 \right\|_H^2 + \alpha^{-1} \left\| f \right\|_{L^2(I,V^*)}^2)$.

In particular, $\sqrt{\alpha} \left\| u \right\|_{L^2(I,V)} \leq \sqrt{\exp(2\lambda T)}(\left\| u_0 \right\|_H + \alpha^{-1/2} \left\| f \right\|_{L^2(I,V^*)})$, or $\left\| u \right\|_{L^2(I,V)} \leq \alpha^{-1/2}\sqrt{\exp(2\lambda T)}(\left\| u_0 \right\|_H + \alpha^{-1/2} \left\| f \right\|_{L^2(I,V^*)})$.

Moreover $\left\| u' \right\|_{L^2(I,V^*)} \leq \left\| Au \right\|_{L^2(I,V^*)} + \left\| f \right\|_{L^2(I,V^*)} \leq \left\| A \right\| \left\| u \right\|_{L^2(I,V)} + \left\| f \right\|_{L^2(I,V^*)}$.

All in all, we obtain:

$$\left\| u \right\|_{C([0;T],H)}^2 + \alpha \left\| u \right\|_{L^2(I,V)}^2 \leq \exp(2\lambda T)(\left\| u_0 \right\|_H^2 + \alpha^{-1} \left\| f \right\|_{L^2(I,V^*)}^2)$$

and:

$$\left\| u' \right\|_{L^2(I,V^*)} \leq \left\| A \right\|_{L(V,V^*)} \alpha^{-1/2}\sqrt{\exp(2\lambda T)}(\left\| u_0 \right\|_H + \alpha^{-1/2} \left\| f \right\|_{L^2(I,V^*)}) + \left\| f \right\|_{L^2(I,V^*)}$$

More regularity

We tie back to page 25 of [7]. In particular:

$$\int_0^t \left\| u_n' \right\|_H^2 + \int_0^t \langle Au_n, u_n' \rangle_{V^*,V} = \int_0^t (f_1, u_n')_H + \int_0^t \langle f_2, u_n' \rangle_{V^*,V}$$

Then:

$$\int_0^t \langle Au_n, u_n' \rangle_{V^*,V} \geq \frac{\alpha}{2} \left\| u_n(t) \right\|_V^2 - \frac{\lambda}{2} \left\| u_n(t) \right\|_H^2 - \frac{\left\| A \right\|}{2} \left\| u_{n0} \right\|_V$$

whereas, as in the proof of theorem B.1.8:

$$\left|\int_0^t \langle f_2, u_n'\rangle_{V^*,V}\right| \leq \frac{1}{2}\|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)} + \frac{\alpha}{4}\|u_n(t)\|^2_V + \\ + \frac{4}{\alpha}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2}\|u_{n0}\|^2_V$$

Also:

$$\int_0^t (f_1, u_n')_H \leq \frac{1}{2}\|f_1\|^2_{L^2(I,H)} + \frac{1}{2}\int_0^t \|u_n'\|^2_H$$

Putting all together:

$$\int_0^t \|u_n'\|^2_H + \frac{\alpha}{2}\|u_n(t)\|^2_V - \frac{\lambda}{2}\|u_n(t)\|^2_H - \frac{\|A\|}{2}\|u_{n0}\|_V \\ \frac{1}{2}\|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)} + \frac{\alpha}{4}\|u_n(t)\|^2_V + \\ + \frac{4}{\alpha}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2}\|u_{n0}\|^2_V + \\ + \frac{1}{2}\|f_1\|^2_{L^2(I,H)} + \frac{1}{2}\int_0^t \|u_n'\|^2_H$$

which brings us to:

$$\frac{1}{2}\int_0^t \|u_n'\|^2_H + \frac{\alpha}{4}\|u_n(t)\|^2_V - \frac{\lambda}{2}\|u_n(t)\|^2_H \leq \\ \frac{1}{2}\|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)} + \\ + \frac{8+\alpha}{2\alpha}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1+\|A\|}{2}\|u_{n0}\|^2_V + \\ + \frac{1}{2}\|f_1\|^2_{L^2(I,H)}$$

and thus, because norms are lower semicontinuous and because we have weak convergence of the time derivative, and V-strong convergence of the initial data:

$$\frac{1}{2}\int_0^T \|u'\|^2_H \leq \frac{1}{2}\|f_2'\|^2_{L^2(I,V^*)} + \frac{8+\alpha}{2\alpha}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1+\|A\|}{2}\|u_0\|^2_V + \frac{1}{2}\|f_1\|^2_{L^2(I,H)} + \\ \text{limsup}_n \left(\frac{\lambda}{2}\|u_n\|^2_{C([0,T],H)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)}\right)$$

Using a purely numeric constant C without dependences on the problem we can write:

$$\int_0^T \|u'\|_H^2 \leq \|f_2'\|_{L^2(I,V^*)}^2 + C(1+\alpha^{-1})\|f_2\|_{L^\infty(I,V^*)}^2 + C(1+\|A\|)\|u_0\|_V^2 + \|f_1\|_{L^2(I,H)}^2 + \\ C\mathrm{limsup}_n\left(\frac{\lambda}{2}\|u_n\|_{C([0,T],H)}^2 + \frac{1}{2}\|u_n\|_{L^2(I,V)}^2\right)$$

For the last term, employing the exact argument as in the first part of the proof:

$$\begin{aligned}\mathrm{limsup}_n\left(\frac{\lambda}{2}\|u_n\|_{C([0,T],H)}^2 + \frac{1}{2}\|u_n\|_{L^2(I,V)}^2\right) \leq \\ 2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\mathrm{limsup}_n\left(\|u_n\|_{C([0,T],H)}^2 + \alpha\|u_n\|_{L^2(I,V)}^2\right) \leq \\ 2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\exp(2\lambda T)(\|u_0\|_H^2 + \alpha^{-1}\|f_1+f_2\|_{L^2(I,V^*)}^2) \leq \\ 2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\exp(2\lambda T)(\|u_0\|_H^2 + 2\alpha^{-1}\|f_1\|_{L^2(I,V^*)}^2 + 2\alpha^{-1}\|f_2\|_{L^2(I,V^*)}^2) \leq \\ CC_0(\|u_0\|_H^2 + \alpha^{-1}\|f_1\|_{L^2(I,V^*)}^2 + \alpha^{-1}\|f_2\|_{L^2(I,V^*)}^2)\end{aligned}$$

where $C_0 = 2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\exp(2\lambda T)$ and C is a purely numeric constant without dependences on the problem.

Therefore:

$$C\int_0^T \|u'\|_H^2 \leq \|f_2'\|_{L^2(I,V^*)}^2 + (1+\alpha^{-1})\|f_2\|_{L^\infty(I,V^*)}^2 + (1+\|A\|)\|u_0\|_V^2 + \|f_1\|_{L^2(I,H)}^2 + \\ C_0(\|u_0\|_H^2 + \alpha^{-1}\|f_1\|_{L^2(I,V^*)}^2 + \alpha^{-1}\|f_2\|_{L^2(I,V^*)}^2)$$

The embedding $H^1(I,V^*) \hookrightarrow C([0,T],V^*)$ has norm that only depends on T, which follows from the equality $f_2(t) = f_2(s) + \int_s^t f_2'$, for $0 \leq s \leq t \leq T$, a bound being $1+T$.

Thus:

$$\begin{aligned}C\int_0^T \|u'\|_H^2 \leq (1+(1+C_0)\alpha^{-1})\|f_2\|_{H^1(I,V^*)}^2 + \\ (1+\|A\|)\|u_0\|_V^2 + C_0\|u_0\|_H^2 + \\ \|f_1\|_{L^2(I,H)}^2 + C_0\alpha^{-1}\|f_1\|_{L^2(I,V^*)}^2\end{aligned}$$

□

B.2. Application to inhomogeneous parabolic problems

B.2.1. Inhomogeneous Dirichlet problem

We make the following assumption.

Assumption B.2.1.1 (Assumptions for problem B.2.1.2)
We assume $\Omega \subset\subset D$ to be bounded Lipschitz domains, so that $U := D \setminus \Omega$ is bounded Lipschitz too and the trace operator is bounded surjective onto $H^{1/2}(\partial U)$, with a right inverse E (see theorem 3.37 at page 102 of [14]). For such a choice we also have $H_0^1 = H^1 \cap \ker \operatorname{tr}$, see [11], page 595, theorem 18.7.

Moreover, we select $f \in H^1(I, H^{1/2}(\Gamma_f))$, $f(0) = 0$.

Note that, given a bounded extension operator $E : H^{1/2}(\partial U) \to H^1(U)$, we obtain by proposition A.2.2 that $Ef \in H^2(I, H^1(U))$. We have defined $\operatorname{tr}u(t) := tr(u(t))$ and analogously $Eu(t) := E(u(t))$ (see proposition A.2.6).

Call $H = L^2(U)$, $V = \{v \in H^1(U), \operatorname{tr}u = 0 \text{ on } \Gamma_m\} =: H_c^1$. V is a closed subspace of H^1, which is Hilbert separable, hence also Hilbert separable. We norm it with the full H^1 norm. Because $H_0^1(U)$ is dense in H, so is V and we obtain a Gelfand triple. That V is a closed subspace of H^1 follows from the observation that if $u_n \to u$ in the V norm, then $\operatorname{tr}u_n \to \operatorname{tr}u$ in $L^2(\partial U)$. We can take an almost everywhere pointwise convergent sequence, so that $\operatorname{tr}u_n \to \operatorname{tr}u$ a.e., and by the fact that Γ_m has positive Hausdorff measure, we conclude $\operatorname{tr}u = 0$ on Γ_m.

We define $A := H^1 \to H^{1*}$ by $(Au)v := \int_u \nabla u \nabla v$. This operator can be the recast to $V \to H^{-1}$ and $V \to V^*$.

The problem under consideration is the following. For $U = D \setminus \Omega$ we have:

Problem B.2.1.2 (Inhomogeneous heat equation, Dirichlet conditions)

$$u_t - \Delta u = 0 \text{ in } (0,T) \times U \tag{B.2.1.3}$$

$$u(\Sigma_f) = f \tag{B.2.1.4}$$

$$u(\Sigma_m) = 0 \tag{B.2.1.5}$$

$$u(0) = 0 \tag{B.2.1.6}$$

By this we mean:

$$u \in W(I, H_c^1) \tag{B.2.1.7}$$
$$u_t|_{H^{-1}} + Au = 0 \text{ in } H^{-1} \text{ and for a.e. } t \in (0,T) \tag{B.2.1.8}$$
$$\mathrm{tr} u = f \text{ on } \Sigma_f \tag{B.2.1.9}$$
$$u(0) = 0 \tag{B.2.1.10}$$

Theorem B.2.1.11 (Well posedness and regularity for problem B.2.1.2)
Given assumption B.2.1.1, the solution u to problem B.2.1.2 is unique with $u_t \in L^2(I,H)$.

The problem is equivalent to:

Problem B.2.1.12 (Equivalent formulation with extension)

$$u_0 \in W(I, H_0^1) \tag{B.2.1.13}$$
$$u_0' + Au_0 = -((\bar{u}', \cdot)_H + A\bar{u}) \text{ in } H^{-1} \text{ and for a.e. } t \in (0,T) \tag{B.2.1.14}$$
$$u_0(0) = 0 \tag{B.2.1.15}$$

with $\bar{u}$ any given $\bar{u} \in H^1(I, H_c^1(U))$ such that $\mathrm{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$. This means that u solves problem B.2.1.2 $\implies$ $u - \bar{u}$ solves problem B.2.1.12, and if $u_0(\bar{u})$ solves problem B.2.1.12, then $\bar{u} + u_0(\bar{u})$ solves problem B.2.1.2.

Furthermore:

$$\|u\|^2_{C([0;T],H)} + \|u\|^2_{L^2(I,H)} + \|\nabla u\|^2_{L^2(I,H)} + \|u'\|^2_{L^2(I,H)} \leq C(T)\, \|\bar{u}\|^2_{H^1(I,V)} \tag{B.2.1.16}$$

with $C > 1$, only dependent on T, smoothly, exploding for large T.

Proof.

Extension of the boundary data

Let $\bar{u} \in H^1(I, H_c^1(U))$ be such that $\mathrm{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$. We can choose for instance $E\tilde{f}$, see proposition A.2.6, where $\tilde{f} = 0$ on Σ_m, $\tilde{f} = f$ on Σ_f. $\tilde{f} \in H^1(I, H^{1/2}(\partial U))$, because Γ_f and Γ_m have positive distance (see the definition of the norm in [8], page 20).

Reformulation (first part)

Consider the following commutative diagram, where $V = H_c^1$, $W = H_0^1$:

$$
\begin{tikzcd}
H^1_c \arrow[dr, hook, "a"] & & & (H^1_c)^* \arrow[dd, hook, "c^*"] \\
& H \arrow[r, "r"] & H^* \arrow[ur, hook, "a^*"] \arrow[dr, hook, "b^*"'] & \\
H^1_0 \arrow[uu, hook, "c"] \arrow[ur, hook, "b"'] & & & H^{-1}
\end{tikzcd}
$$

Here, a, b, c are the trivial injections, r the Riesz isomorphism $h \mapsto (h, \cdot)_H$.

Now $(i_W(u-\bar{u}))' + A(u-\bar{u}) = (i_V u)'|_{H^{-1}} - (i_V \bar{u})'|_{H^{-1}} + Au - A\bar{u} = \{$ proposition A.2.7 $\} = (i_V u)'|_{H^{-1}} - (i_V \bar{u}_t)|_{H^{-1}} + Au - A\bar{u} = -(i_V \bar{u}_t)|_{H^{-1}} - A\bar{u}$ if u solves problem B.2.1.2, where $\bar{u}_t$ is the weak derivative of $\bar{u}$ in the $H^1(I, V)$ sense. Call $u_0 = u - \bar{u}$. By again proposition A.2.7, $u_0 \in W(I, H^1_0)$.

This motivates us to consider the problem:

$$u_0 \in W(I, H^1_0) \tag{B.2.1.17}$$

$$u_0' + Au_0 = -(f_1 + f_2) \text{ in } H^{-1} \text{ and for a.e. } t \in (0, T) \tag{B.2.1.18}$$

$$u_0(0) = 0 \tag{B.2.1.19}$$

Here, $f_1 := (i_V \bar{u}_t)|_{H^{-1}} = c^* a^* r a \bar{u}_t = b^* r(a\bar{u}_t) \in L^2(I, H)$.

Moreover, $A \in L(V, H^{-1})$, so, by proposition A.2.2, $f_2 := A\bar{u} \in H^1(I, H^{-1})$.

Existence

By theorem B.1.11 we get a solution of the above problem with $u_0' \in L^2(I, H)$.

And now, let $u := \bar{u} + cu_0 = \bar{u} + u_0$. We claim it is a solution. The initial and boundary conditions are surely satisfied. We check it is in $W(I, V)$ and is satisfies the partial differential equation.

By proposition A.2.7, we have both $\bar{u}, cu_0 \in W(I, V)$. The derivative of $\bar{u}$ becomes $i_V \bar{u}_t$, see proposition A.2.7. Therefore $(i_V(\bar{u} + cu_0))'|_{H^1_0} = c^*(i_V(\bar{u} + cu_0))' = c^* i_V \bar{u}_t + c^* i_V (cu_0)' = b^* r(a\bar{u}_t) + i_W(u_0)'$ by proposition A.2.7.

Using the pde of u_0, ... $= b^* r(a\bar{u}_t) - Au_0 - f_1 - f_2 = -A(u_0 + \bar{u})$.

Uniqueness

For two solutions u_1, u_2 of $B.2.1.2$ we can form $d := u_1 - u_2 \in W(I, H^1_0)$ by proposition A.2.7. Clearly, $d(0) = 0$. Moreover, $(i_{H^1_0} d)' = \{$ proposition A.2.7 $\} = (i_V u_1)'|_{H^1_0} - (i_V u_2)'|_{H^1_0} = A(u_1 - u_2)$.

By uniqueness stated in theorem B.1.5 we obtain $d = 0$ in $L^2(I, H)$, so that the solution is unique and doesn't depend on the choice of the extension of the Dirichlet datum.

Reformulation (part 2)

Therefore $u = \bar{u} + u_0$ above is the unique solution of problem B.2.1.2. So, given any $\bar{u} \in H^1(I, H^1_c(U))$ such that $\mathrm{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$, we can construct u_0 as above and get $u = \bar{u} + u_0$ solving problem B.2.1.2.

Viceversa, let u solve problem B.2.1.2. Call $u_0 = u - \bar{u}$. Then, as seen above, $u_0 \in W(I, H^1_0)$ and $(i_V(u - \bar{u}))'|_{H^{-1}} + A(u - \bar{u}) = (i_V u)'|_{H^{-1}} - (i_V \bar{u})'|_{H^{-1}} + Au - A\bar{u} = \{$ proposition A.2.7 $\} = (i_V u)'|_{H^{-1}} - (i_V \bar{u}_t)|_{H^{-1}} + Au - A\bar{u} = -(i_V \bar{u}_t)|_{H^{-1}} - A\bar{u}$ if u solves problem B.2.1.2, where $\bar{u}_t$ is the weak derivative of $\bar{u}$ in the $H^1(I, V)$ sense. Call $u_0 = u - \bar{u}$. By again proposition A.2.7, $(i_W(u - \bar{u}))' + A(u - \bar{u}) = (i_V u)'|_{H^{-1}} - (i_V \bar{u})'|_{H^{-1}} + Au - A\bar{u} = (i_V u)'|_{H^{-1}} - (i_V \bar{u}_t)|_{H^{-1}} + Au - A\bar{u} = -(i_V \bar{u}_t)|_{H^{-1}} - A\bar{u} = -b^* r(a\bar{u}_t) - A\bar{u}$. Moreover, $u_0(0) = 0$, so that u_0 solves problem B.2.1.12.

Regularity

Let $u = \bar{u} + u_0$ be the unique solution, as before, of problem B.2.1.2. From proposition A.2.7 we know $(i_V(\bar{u}))' = i_V(\bar{u}_t) = a^* r(a\bar{u}_t)$, and $i_V(cu_0)' = a^* r(u_0')$, for $u_0' \in L^2(I, H)$ the representative of $(i_W(u_0))'$, equivalently, the weak derivative of u_0 in the $H^1(I, H)$ sense. It follows that $(i_V u)' = a^* r(a\bar{u}_t + u_0')$, proving the additional time smoothness claim.

Stability

Let $\bar{u} \in H^1(I, H^1_c(U))$ such that $\mathrm{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$. Consider u_0. Then, by B.1.14:

$$\|u_0\|^2_{C([0;T], H)} + \alpha \|u_0\|^2_{L^2(I, H^1_0)} \leq \exp(2\lambda T)\alpha^{-1} \|(\bar{u}', \cdot)_H + A\bar{u}\|^2_{L^2(I, H^{-1})}$$

$$\begin{aligned} C \|u_0'\|^2_{L^2(I,H)} \leq (1 + (1 + C_0)\alpha^{-1}) \|A\bar{u}\|^2_{H^1(I,H^{-1})} + \\ \|(\bar{u}', \cdot)_H\|^2_{L^2(I,H)} + C_0\alpha^{-1} \|(\bar{u}', \cdot)_H\|^2_{L^2(I,H^{-1})} \end{aligned}$$

$C_0 = 2^{-1} \max(1, \lambda) \max(1, \alpha^{-1}) \exp(2\lambda T)$.

We norm H^1_0 with the full H^1 norm too. Then:

$$\begin{aligned} \|(\bar{u}', \cdot)_H + A\bar{u}\|_{L^2(I,H^{-1})} \leq \\ \sup_{\|v\|_{L^2(I,H^1_0)} = 1} \|\bar{u}'\|_{L^2(I,H)} \|v\|_{L^2(I,H)} + \|\nabla\bar{u}\|_{L^2(I,H)} \|\nabla v\|_{L^2(I,H)} \leq \\ C(\|\bar{u}'\|_{L^2(I,H)} + \|\nabla\bar{u}\|_{L^2(I,H)}) \end{aligned}$$

By proposition A.2.2, $\|A\bar{u}\|_{H^1(I,H^{-1})} \leq \|A\|_{L(V,H^{-1})} \|\bar{u}\|_{H^1(I,V)}$ (we could apply it since H^{-1} is separable, as a dual of a reflexive Banach space).

Finally, $\|(\bar{u}', \cdot)_H\|^2_{L^2(I,H^{-1})} \leq \|\bar{u}'\|^2_{L^2(I,H)}$.

We can then say:

$$\begin{aligned} \|u_0\|^2_{C([0;T],H)} + C\alpha \|u_0\|^2_{L^2(I,H_0^1)} &\leq \exp(2\lambda T)\alpha^{-1} \|\bar{u}\|^2_{H^1(I,V)} \\ C\|u_0'\|^2_{L^2(I,H)} &\leq ((1+(1+C_0)\alpha^{-1}) \|A\|^2_{L(V,H^{-1})} + 1 + C_0\alpha^{-1}) \|\bar{u}\|^2_{H^1(I,V)} \end{aligned}$$

Now, $\langle Av, v\rangle_{H^{-1},H_0^1} + 1 \cdot \|v\|^2_H = 1 \cdot \|v\|^2_{H_0^1}$, so that $\alpha = \lambda = 1$. Moreover, $\langle Au, v\rangle_{H^{-1},H_0^1} \leq \|u\|_V \|v\|_{H_0^1}$, i.e. $\|A\|_{L(V,H^{-1})} \leq 1$.

Therefore $\|u_0\|^2_{C([0;T],H)} + \|u_0\|^2_{L^2(I,H_0^1)} + \|u_0'\|^2_{L^2(I,H)} \leq C(T) \|\bar{u}\|^2_{H^1(I,V)}$ with $C > 1$, only dependent on T, smoothly, exploding for large T.

Now, let's analyse the norms of $\bar{u}$. Because $\bar{u} \in H^1(I,V)$, then, $\bar{u} \in C([0,T],V) \hookrightarrow C([0,T],H)$, where the embedding is non-expansive by the choice of the norm of V. Therefore $\|\bar{u}\|_{C([0;T],H)} \leq \|\bar{u}\|_{C([0;T],V)} \leq (1+T) \|\bar{u}\|_{H^1(I,V)}$. We can therefore conclude that $\|u\|^2_{C([0;T],H)} + \|u\|^2_{L^2(I,H_0^1)} + \|u'\|^2_{L^2(I,H)} \leq C(T) \|\bar{u}\|^2_{H^1(I,V)}$ with $C > 1$, only dependent on T, smoothly, exploding for large T.

□

B.2.2. Inhomogeneous Neumann-Dirichlet problem

We make the following assumption.

Assumption B.2.2.1 (Assumptions for problem B.2.1.2)
We keep assumption B.2.1.1 (apart from the Dirichlet datum). We consired $g \in H^1(I, L^2(\Gamma_f))$, $g(0) = 0$.

Again, call $H = L^2(U)$, $V = \{v \in H^1(U), \mathrm{tr} u = 0 \text{ on } \Gamma_m\} =: H_c^1$. H, V induce a Gelfand triple as seen before.

The problem under consideration is:

Problem B.2.2.2 (Inhomogeneous heat equation, Neumann conditions)

$$u_t - \Delta u = 0 \text{ in } (0,T) \times U \tag{B.2.2.3}$$
$$\partial_\nu u(\Sigma_f) = g \tag{B.2.2.4}$$
$$u(\Sigma_m) = 0 \tag{B.2.2.5}$$
$$u(0) = 0 \tag{B.2.2.6}$$

By this we mean:

$$u \in W(I, H_c^1) \tag{B.2.2.7}$$
$$u_t + Au = G \text{ in } V^* \text{ and for a.e. } t \in (0,T) \tag{B.2.2.8}$$
$$u(0) = 0 \tag{B.2.2.9}$$

where $\langle G(t), v\rangle_{V^*,V} := \int_{\Gamma_f} g(t)\mathrm{tr}v d\sigma$, σ the 1-codimensional Hausdorff measure, and A was introduced before in $L(V, H^{-1})$.

By proposition A.2.2, $G \in H^1(I, V^*)$. In fact, define $T : L^2(\Sigma_f) \to V^*$ by $\langle Tg, v\rangle_{V^*,V} := \int_{\Gamma_f} g\mathrm{tr}v d\sigma$. Then, $\langle Tg, v\rangle_{V^*,V} \leq \|g\|_{L^2(\Gamma_f)} \|v\|_V$ by trace theory. Now, $G(t) = Tg(t)$.

Moreover, $\langle Av, v\rangle_{V^*,V} + 1 \cdot \|v\|_H = 1 \cdot \|V\|$, so that we can immediately conclude:

Theorem B.2.2.10 (Well posedness and regularity for problem B.2.2.2)
Given assumption B.2.2.1, the solution u to problem B.2.2.2 is unique with $u_t \in L^2(I, H)$.

Furthermore:

$$\|u\|^2_{C([0;T],H)} + \|u\|^2_{L^2(I,H)} + \|\nabla u\|^2_{L^2(I,H)} + \|u'\|^2_{L^2(I,H)} \leq C(T) \|g\|^2_{H^1(I,L^2(\Gamma_f))} \tag{B.2.2.11}$$

with $C > 1$, only dependent on T, smoothly, exploding for large T.

Proof.

It is an application of theorem B.1.5, theorem B.1.8 and proposition B.1.14. □

B.3. Reformulation of parabolic equations

We just saw that the two parabolic equations of interest can be recasted into the problem of finding $u \in W(I, V)$, $u(0) = 0$, $u_t + Au = f$ for a.e. t in V^*, with notation from preceding sections.

In particular, $f \in L^2(I, V^*)$ and so is Au (because $A \in L(V, V^*)$, and by A.2.2).

Call then $E(u) := u_t + Au - f \in L^2(I, V^*)$ and $W_0(I, V)$ the $W(I, V)$ functions with zero initial value. Then, the differential equation reads $\langle E(u)(t), v\rangle_{V^*,V} = 0$ for all $v \in V$, for a.a. t, equivalently, $E(u) = 0$ for a.a. t. Thus, we are interested in the abstract problem:

Problem B.3.1 (Even more abstract parabolic equation)
Given a function $E : W(I, V) \to L^2(I, V^*)$, find $u \in W_0(I, V)$, such that $E(u) = 0$ for a.a. t.

We can view $L^2(I, V^*) \cong L^2(I, V)^*$.

Hence $\langle E(u), v\rangle_{L^2(I,V)^*,L^2(I,V)} = \int_I \langle E(u)(t), v(t)\rangle_{V^*,V} dt$ (see [9], theorem 1.31 at page 39).

We are now ready to restric both state and adjoint space.

Definition B.3.2 ($Q(I, V)$)
We define $Q(I, V) = H^{1,1} = L^2(I, V) \cap H^1(I, H)$, with the norm $\|v\|_Q^2 = \|v\|_{L^2(I,V)}^2 + \|v_t\|_{L^2(I,H)}^2$.

Proposition B.3.3 (Properties of Q)
There holds:

- $Q = Q(I, V)$ is Hilbert with $(v, w)_{L^2(I,V)} + (v_t, w_t)_{L^2(I,H)}$
- $Q(I, V)$ is dense in $L^2(I, V)$
- $Q(I, V) \hookrightarrow C([0, T], H)$
- $Q_0(I, V)$ is dense in $L^2(I, V)$, $Q_0(I, V)$ the space of $Q(I, V)$ function with zero initial value
- $Q(I, V) = W(I, V) \cap H^1(I, H)$, $Q_0(I, V) = W_0(I, V) \cap H^1(I, H)$ as sets. There holds that the $W(I, V)$ derivative is represented by the $H^1(I, H)$ derivative and $\langle u', v\rangle_{V^*,V} = (u', v)_H$, with the suitable interpretations of u' (on the left, we have $i_V(u)'$, i_V the Gelfand triple embedding; on the right we have u, seen in H, and then weakly differentiated in the $H^1(I, H)$ sense)
- integration by parts in time holds: $\int_I (v_t, w)_H = -\int_I (w_t, v)_H + (v(T), w(T))_H - (v(0), w(0))_H$
- if q_n is bounded in $Q(I, V)$, then there exists a weakly convergent subsequence q_k such that $q_k \rightharpoonup q$ in $L^2(I, H)$, $\partial_i q_k \rightharpoonup \partial_i q$ in $L^2(I, H)$ and $q'_k \rightharpoonup q'$ in $L^2(I, H)$

Proof.

<u>Completeness</u>

We have the inclusions $L^2(I,H) \subseteq L^2(I,V) \cap H^1(I,H) \subseteq H^1(I,V)$.

If q_n is Cauchy in Q, then it is Cauchy in the individual norms of $L^2(I,V), H^1(I,H)$, so that q_n convergens to two limits, one in $L^2(I,V)$ and one in $H^1(I,H)$. The convergence is common in $L^2(I,H)$, which implies that the two limits coincide at $q \in Q(I,V)$. The convergence in $Q(I,V)$ of q_n to Q follows from the individual convergences of $q_n, \nabla q_n, q_{nt}$ in $L^2(I,H), L^2(I,H), L^2(I,H)$.

<u>Density</u>

We have $C_c^\infty(I,V) \subseteq Q(I,V) \subseteq L^2(I,V)$. The first inclusion holds because of proposition A.2.4, so that $C_c^\infty(I,V) \subseteq H^1(I,V)$. Moreover $H^1(I,V) \subseteq Q(I,V)$ trivially, where the $H^1(I,H)$ derivative is the $H^1(I,V)$ derivative. $C_c^\infty(I,V)$ is dense in $L^2(I,V)$ by [9], page 39, lemma 1.9.

<u>Continuity</u>

Follows from the embedding $H^1(I,H) \hookrightarrow C([0,T],H)$, as seen in [5], theorem 2 of page 286.

<u>More density</u>

We can therefore speak of initial values. In particular, $C_c^\infty(I,V) \subseteq Q_0(I,V) \subseteq L^2(I,V)$, and as before, the density result follows.

<u>Relationship with $W(I,V)$</u>

Consider the chain:

$$V \xhookrightarrow{a} H \xrightarrow{r} H^* \xhookrightarrow{a^*} V^*$$

where a is the trivial embedding and r the Riesz isomorphism.

We claim that for $v \in Q(I,V)$, then $(a^*rav)' = a^*r(av)'$, where $av \in H^1(I,H)$. In fact, for $\phi \in C_c^\infty(I)$, we get $\int_I a^*rav\phi' = \{\text{ A.2.1 }\} = a^*r\int_I av\phi' = -a^*r\int_I (av)'\phi = -\int_I a^*r(av)'\phi$.

Now, let $u \in W(I,V)$, with $(a^*rau)' = a^*rh$, $h \in L^2(I,H)$. Then $a^*r\int_I h\phi = \{\text{ A.2.1 }\} = \int_I a^*rh\phi = \int_I (a^*rau)'\phi = a^*r(-\int au\phi')$.

We know that a^* is injective and so is r, so that $\int_I h\phi = -\int_I au\phi'$ as we wanted.

<u>Integration by parts</u>

We note that for $v, w \in Q(I,V) \subseteq W(I,V)$, we have $\int_I \langle (a^* rav)', w\rangle_{V^*,V} = \int_I \langle a^* r(av)', w\rangle_{V^*,V} = \int_I ((av)', aw)_H$. We can now apply theorem 3.11 at page 148 of [17] to conclude that $\int_I (v_t, w)_H = -\int_I ((aw)_t, av)_H + ((av)(T), (aw)(T))_H - ((av)(0), (aw)(0))_H$

<u>Weak convergence</u>

At first we note that ∂_i, ∂_t are linear bounded operators from $Q(I,V)$ to $L^2(I,H)$.

Remember that in any case, V is a closed subspace of H^1. Then, $\partial_i : V \to H$ is linear and bounded, because V is bounded by the full H^1 norm, as we declared already.

Therefore, by proposition A.2.2, ∂_i extends to a linear bounded map from $L^2(I,V)$ to $L^2(I,H)$, therefore, to a linear bounded map on $Q(I,V)$, in the sense of:

$$Q(I,V) \xhookrightarrow{i} L^2(I,V) \xrightarrow{\partial_i} L^2(I,H)$$

Because q_n is bounded in the Hilbert space $Q(I,V)$, it has a weakly convergent subsequence $q_k \rightharpoonup q \in Q(I,V)$. Therefore, $\partial_i(i(q_k)) \rightharpoonup \partial_i(i(q))$ in $L^2(I,H)$. By the Hilbert space property of $L^2(I,H)$ we conclude that $(\partial_i q_k, p)_{L^2(I,H)} \to (\partial_i q, p)_{L^2(I,H)}$ for all $p \in L^2(I,H)$.

For the time derivative we can draw a similar diagram:

$$Q(I,V) \xhookrightarrow{j} H^1(I,H) \xrightarrow{\partial_t} L^2(I,H)$$

We therefore obtain $(\partial_t q_k, p)_{L^2(I,H)} \to (\partial_t q, p)_{L^2(I,H)}$ for all $p \in L^2(I,H)$.

The convergence $(q_k, p)_{L^2(I,H)} \to (q, p)_{L^2(I,H)}$ for all $p \in L^2(I,H)$ follows analogously.

$\square$

We can therefore restrict the testing space.

Proposition B.3.4 (Equivalent testing)
Let $E : W(I,V) \to L^2(I,V^*)$, and $u \in W_0(I,V)$.

Then:

$$
\begin{aligned}
E(u) &= 0 \\
&\Longleftrightarrow \\
\langle E(u), v\rangle_{L^2(I,V)^*,L^2(I,V)} = 0 \quad \forall v &\in L^2(I,V) \\
&\Longleftrightarrow \\
\langle E(u), v\rangle_{L^2(I,V)^*,L^2(I,V)} = 0 \quad \forall v &\in W^0(I,V) \\
&\Longleftrightarrow \\
\langle E(u), v\rangle_{L^2(I,V)^*,L^2(I,V)} = 0 \quad \forall v &\in Q^0(I,V)
\end{aligned}
$$

We have also seen that with smoothness assumption on data (assumption B.2.1.1 and assumption B.2.2.1) we obtain that the solutions of problem B.2.1.2, problem B.2.2.2 have $Q_0(I,V)$ smoothness.

We can therefore formulate the two partial differential equations directly on $Q_0(I,V)$ as follows.

$$
\begin{aligned}
w \in W_0(I,H_c^1), \bar{u}+v_0 \in W_0(I,H_c^1), v_0 \in W_0(I,H_0^1) \\
w' + Aw = (g,\cdot)_{L^2(\Gamma_f)} \text{ in } H_c^{1*} \text{ and for a.e. } t \in (0,T) \\
v_0' + Av_0 = -((\bar{u}',\cdot)_H + A\bar{u}) \text{ in } H^{-1} \text{ and for a.e. } t \in (0,T)
\end{aligned}
$$

with $\bar{u}$ any given $\bar{u} \in H^1(I,H_c^1)$ such that $\mathrm{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$.

We are working under B.2.1.1, B.2.2.1.

Thanks to proposition B.3.5, this is equivalent to:

$$
\begin{aligned}
w \in W_0(I,H_c^1), \bar{u}+v_0 \in W_0(I,H_c^1), v_0 \in W_0(I,H_0^1) \\
\int_I \langle w', q\rangle_{H_c^{1*},H_c^1} + (\nabla w, \nabla q)_H = \int_I (g, \mathrm{tr}q)_{L^2(\Gamma_f)}, \quad \forall q \in Q^0(I,H_c^1) \\
\int_I \langle v_0', p\rangle_{H^{-1},H_0^1} + (\nabla v_0, \nabla p)_H = -\int_I (\bar{u}', p)_H + (\nabla \bar{u}, \nabla p)_H, \quad \forall p \in Q^0(I,H_0^1)
\end{aligned}
$$

By regularity, see theorem B.2.1.11, theorem B.2.2.10, and thanks to proposition B.3.3 this implies:

$$
\begin{aligned}
w \in Q_0(I,H_c^1), \bar{u}+v_0 \in Q_0(I,H_c^1), v_0 \in Q_0(I,H_0^1) \\
\int_I (w', q)_H + (\nabla w, \nabla q)_H = \int_I (g, \mathrm{tr}q)_{L^2(\Gamma_f)}, \quad \forall q \in Q^0(I,H_c^1) \\
\int_I (v_0', p)_H + (\nabla v_0, \nabla p)_H = -\int_I (\bar{u}', p)_H + (\nabla \bar{u}, \nabla p)_H, \quad \forall p \in Q^0(I,H_0^1)
\end{aligned}
$$

where now the derivatives are in the $H^1(I,H)$ sense. Indeed we have proved in proposition B.3.3 that $u \in W(I,V)$ with $L^2(I,H)$ derivative, is actually $Q(I,V)$, with weak derivative in the $H^1(I,H)$ sense equal to the $L^2(I,H)$ representative of u' in the $W(I,V)$ sense. There, we also proved the representation of the duality bracket.

Conversely, a solution $w \in Q_0(I,H_c^1), \bar{u}+v_0 \in Q_0(I,H_c^1), v_0 \in Q_0(I,H_0^1)$ to the above problem satisfies $w \in W_0(I,H_c^1), \bar{u}+v_0 \in W_0(I,H_c^1), v_0 \in W_0(I,H_0^1)$, see proposition B.3.3, and the proof of theorem B.2.1.11 . And by proposition B.3.3 we can get to:

$$
\begin{aligned}
w \in W_0(I,H_c^1), \bar{u}+v_0 \in W_0(I,H_c^1), v_0 \in W_0(I,H_0^1) \\
\int_I \langle w', q\rangle_{H_c^{1*},H_c^1} + (\nabla w, \nabla q)_H = \int_I (g, \mathrm{tr} q)_{L^2(\Gamma_f)}, \quad \forall q \in Q^0(I,H_c^1) \\
\int_I \langle v_0', p\rangle_{H^{-1},H_0^1} + (\nabla v_0, \nabla p)_H = -\int_I (\bar{u}', p)_H + (\nabla \bar{u}, \nabla p)_H, \quad \forall p \in Q^0(I,H_0^1)
\end{aligned}
$$

By proposition B.3.4 we obtain back:

$$
\begin{aligned}
w \in W_0(I,H_c^1), \bar{u}+v_0 \in W_0(I,H_c^1), v_0 \in W_0(I,H_0^1) \\
w' + Aw = (g,\cdot)_{L^2(\Gamma_f)} \text{ in } H_c^{1*} \text{ and for a.e. } t \in (0,T) \\
v_0' + Av_0 = -((\bar{u}',\cdot)_H + A\bar{u}) \text{ in } H^{-1} \text{ and for a.e. } t \in (0,T)
\end{aligned}
$$

Therefore:

Proposition B.3.5 (Equivalent formulation)
Under assumption B.2.1.1, assumption B.2.2.1, problem B.2.1.2, problem B.2.2.2 can be equivalently formulated as:

$$
\begin{aligned}
w \in Q_0(I,H_c^1), \bar{u}+v_0 \in Q_0(I,H_c^1), v_0 \in Q_0(I,H_0^1) \\
\int_I (w', q)_H + (\nabla w, \nabla q)_H = \int_I (g, \mathrm{tr} q)_{L^2(\Gamma_f)}, \quad \forall q \in Q^0(I,H_c^1) \\
\int_I (v_0', p)_H + (\nabla v_0, \nabla p)_H = -\int_I (\bar{u}', p)_H + (\nabla \bar{u}, \nabla p)_H, \quad \forall p \in Q^0(I,H_0^1)
\end{aligned}
$$

Existence, uniqueness and stability proved already in theorem B.2.1.11, theorem B.2.2.10 carry over to this new formulation.

C. Domains transformations

C.1. Transforming domains

Proposition C.1.1 (Measurability of composition)
Define $\mathcal{M} := \{\tau : \mathbb{R}^n \to \mathbb{R}^n \text{ Lebesgue measurable }\}/\sim$, the quotient being the almost everywhere equal relation (according to the Lebesgue measure).

Consider also U from $\mathcal{M}_c := \{\tau : \mathbb{R}^n \to \mathbb{R}^n \text{ continuous }\}/\sim$, the application "unique continuous representative", and $\mathcal{M}_{BL} := \{\tau : \mathbb{R}^n \to \mathbb{R}^n \text{ Lipschitz homeomorphism}\}/\sim \subseteq \mathcal{M}_c$.

We can then define $\circ : \mathcal{M} \times \mathcal{M}_{BL}, \mathcal{M}_c \times \mathcal{M} \to \mathcal{M}$ by, respectively, $[f] \circ g := [f \circ U(g)], f \circ [g] := [U(f) \circ g]$. These definitions are well posed.

Proof.

$\underline{\mathcal{M}_c}$

$U(f)$ is Borel measurable, so the preimage of a Borel set is Borel measurable, and g is Lebesgue measurable, so his preimage of such Borel set is Lebesgue measurable (see here for the different notions of measurability).

This shows that $U(f) \circ g$ is measurable.

To complete the well posedness, if $h = g$ a.e., then clearly $U(f) \circ g = U(f) \circ h$ a.e..

$\underline{\mathcal{M}_{BL}}$

Consider $f \circ U(g)$. We need to prove it is measurable and that is only depends on $[f]$.

For the measurability: the preimage of a Borel set, by f, is Lebesgue measurable L. $U(g)$ has a Lipschitz inverse, which will map this set to a Lebesgue set. Indeed, $L = B \cup N$, with B Borel and N Lebesgue measurable and null (see here). Image and unions commute, so, $U(g)^{-1}(L) = U(g)^{-1}(B) \cup U(g)^{-1}(N)$. The first set is Lebesgue measurable by measurability of $U(g)$, the second one is null, because Lipschitz maps map null sets into null sets, see 9.54 at page 271 of [11].

□

Throughout, D is a bounded Lipschitz domain. We define as in [15] the following spaces of transformations:

Definition C.1.2 (Spaces of transformations)
We define:

- $\mathcal{V}^k = \{\tau \in \mathcal{M}, \tau - \mathrm{Id} \in W^{k,\infty}(\mathbb{R}^n, \mathbb{R}^n)\}$, $k \geq 1$
- $\mathcal{T}^k = \{\tau \in \mathcal{V}^k$ with an $\eta \in \mathcal{V}^k, \tau \circ \eta = \eta \circ \tau = \mathrm{Id}\}$. Any such η is unique, we denote it by τ^{-1} and we have that $U(\tau)$ is a Lipschitz homeomorphism with $U(\tau^{-1}) = U(\tau)^{-1}$

Observation C.1.3 (A technicality). Technically, in the original definition of [15], τ need not to be a continuous function, although this is suggested e.g. in remarque 2.1 at page II-4.

Going to equivalence classes of τ makes the identification with continuous functions more precise, as we now show.

<u>One implication</u>

Let $\tau : \mathbb{R}^n \to \mathbb{R}^n$ with $[\tau - \mathrm{Id}] \in W^{k,\infty}$. Then τ is equal a.e. to a (Lebesgue) measurable function, hence also (Lebesgue) measurable, and thus $[\tau] \in \mathcal{V}^k$ as we have defined it (this is proved here; note that $\{g \neq f\}$ is measurable as the Lebesgue measure is complete).

Now, suppose τ is a bijection, and $[\tau^{-1} - \mathrm{Id}] \in W^{k,\infty}$ too. Then $\tau = \mathrm{Id} + g = G, \tau^{-1} = \mathrm{Id} + h = H$ almost everywhere. Here, G, H are at least Lipschitz. But then $\tau \circ H = \mathrm{Id}$ a.e., and since H is Lipschitz, we can conclude also $G \circ H = \mathrm{Id}$ a.e., so, everywhere. With a symmetric reasoning, we are lead to $G = H^{-1}$, so that G is bi-Lipschitz.

Thus, $[\tau] \circ [\tau^{-1}] := [U(\tau) \circ U(\tau^{-1})] = [G \circ G^{-1}] = \mathrm{Id}$ and an analogous reasoning leads to $[\tau] \in \mathcal{T}^k$ as we have defined it.

<u>The other implication</u>

It is immediate for $\mathcal{V}^k$ and for $\mathcal{T}^k$, in the equivalence class of $\tau \in \mathcal{T}^k$ there is a unique $U(\tau)$ at least bi-Lipschitz, hence invertible, with $[U(\tau)] = \tau$.

This shows that:

1. $\{\tau : \mathbb{R}^n \to \mathbb{R}^n$ with $[\tau - \mathrm{Id}] \in W^{k,\infty}\}/ \sim = \mathcal{V}^k$
2. $\{\tau : \mathbb{R}^n \to \mathbb{R}^n$ bijection with $[\tau^{\pm 1} - \mathrm{Id}] \in W^{k,\infty}\}/ \sim = \mathcal{T}^k$

We need to check the well-posedness of $\circ$.

Proposition C.1.4
$\circ : \mathcal{V}^1 \times \mathcal{V}^1 \to \mathcal{V}^1$ and $\circ : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \times \mathcal{V}^1 \to W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ are well defined.

Proof.

We start by $\circ : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \times \mathcal{V}^1 \to \mathcal{V}^1$. We have $\theta \circ \tau = [U(\theta) \circ U(\tau)]$ for instance (see proposition C.1.1); the latter is a bounded Lipschitz map, so it remains in $W^{1,\infty}$.

For the second claim, just write $\eta \circ \tau - \mathrm{Id} = (\eta - \mathrm{Id}) \circ \tau + \tau - \mathrm{Id}$ and use the first part.

□

Proposition C.1.5 (Chain rule for $k = 1$)
Let $f \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ or $\mathcal{V}^1$, together with $\psi \in \mathcal{T}^1$. Then:

- $f \circ \psi$ has essentially bounded weak derivatives, and $D(f \circ \psi) = Df \circ \psi D\psi$. The equality holds a.e. also for the classical derivatives.
- $D(\psi^{-1}) = (D\psi)^{-1} \circ \psi^{-1}$, where $(D(\psi^{-1}))^{-1} := [(DU(\psi^{-1}))^{-1}]$, the representative being a.e. invertible. The equality holds a.e. also for the classical derivatives.
- $|\det(D\psi)|$ is an essentially bounded measurable function with $|\det(D\psi)| \geq \delta > 0$ a.e..

Proof.

Weak derivatives

We notice that $f \circ \phi$ has a unique Lipschitz representative, that is $U(f) \circ U(\phi)$. The desired formula follows as in [15], lemme 2.1 at page II-6, for the classical derivatives, because Lipschitz function are almost everywhere differentiable by the Rademacher theorem (see here). The chain rule holds for functions differentiable only at one point.

Now, to identify the weak derivatives:

- $U(f)$ is Lipschitz, so that $DU(f)$, the classical derivative, is also the weak derivative Df (note that f need not to be essentially bounded to state this). The latter is a measurable function, as a.e. limit of difference quotients.
- $DU(f) \circ U(\psi)$, is measurable, see proposition C.1.1. It is also essentially bounded.
- By proposition C.1.1 we observe that $DU(f) \circ U(\psi)$ represents $Df \circ \psi$
- $D\psi = [DU(\psi)]$ as seen above
- the product of equivalence classes is always defined as the product of their representatives

Therefore $Df \circ \psi D\psi = [DU(f) \circ U(\psi) DU(\psi)]$.

And now, because $f \circ \phi$ is Lipschitz, it has weak derivatives, $D(f \circ \phi)$, equal to the classical derivatives $DU(f \circ \phi) = D(U(f) \circ U(\psi)) = DU(f) \circ U(\psi) DU(\psi)$, where the last equality holds a.e., as mentioned at the beginning of the proof.

This let us conclude the first claim.

Inverse Jacobian

For the second one, put $f = \psi^{-1}$. Then, for the classical derivatives, $I = DU(\psi) \circ U(\psi)^{-1} DU(\psi^{-1})$ a.e., so that both $DU(\psi) \circ U(\psi)^{-1}, DU((\psi)^{-1})$ are invertible as matrices, a.e..

Determinant

We have defined $|\det(D\psi)| := [|\det DU(\psi)|]$, see proposition C.1.1. The claim follows as in lemme 4.2, pag. IV-7 of [15], and because det is a polynomial of essentially bounded functions.

$\square$

We go on to define the space of admissible transformations.

Definition C.1.6 (Admissible transformations)
We define $\Theta := \{\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \text{ with } \theta = 0 \text{ on } \mathbb{R}^n \setminus D\}$, a Banach subspace of $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$.

We also define $\mathcal{T} := \{\tau \in \mathcal{T}^1, \tau^{\pm 1}|_{\mathbb{R}^n \setminus D} = \mathrm{Id}\}$.

Proposition C.1.7 (Some group properties of $\mathcal{T}$)
Let $\eta, \tau \in \mathcal{T}, \theta \in \Theta$. Then:

- $\eta \circ \tau \in \mathcal{T}$
- $\theta \circ \tau \in \Theta$
- Id is the neutral element
- $\eta^{-1} \in \mathcal{T}$

Proof.

Stability under inversion

It is trivial, because the definition of $\mathcal{T}$ is symmetric with respect to inversion.

Stability under composition ($\mathcal{T}^1$)

$\eta \circ \tau$ is surely in $\mathcal{V}^1$ by proposition C.1.1. Now, by the above point, $\tau^{-1} \circ \eta^{-1}$ is in $\mathcal{V}^1$ too, and the composition yields: $(\eta \circ \tau) \circ (\tau^{-1} \circ \eta^{-1}) = [U(\eta) \circ U(\tau)] \circ [(U\tau)^{-1} \circ (U\eta)^{-1}] = \mathrm{Id}$.

$\square$

Proposition C.1.8 (Small perturbations of $\mathcal{T}$)
Let $\theta \in \Theta$ with small enough $\|\theta\|_{W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)}$. Then, $\mathrm{Id} + \theta \in \mathcal{T}$.

Let $\delta\theta \in \Theta$ with small enough $\|\delta\theta\|_{W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)}$, and $\tau \in \mathcal{T}$. Then, $\tau + \delta\theta \in \mathcal{T}$.

Proof.

Perturbation of identity

We only need to check the properties of the inverse map.

$U(\tau)^{-1}$ exists and is Lipschitz, see the proof of lemme 2.4 of [15], page II-16. We can therefore define τ^{-1} and we obtain that it is $\mathcal{V}^1$. So, $\tau \in \mathcal{T}^1$. The fact that $\tau = \mathrm{Id}$ outside of D automatically implies $\tau^{-1} = \mathrm{Id}$ outside of D, which can be seen more precisely by going to the smooth representatives of τ, τ^{-1}.

Perturbation, not of identity

We solve the equation $\tau + \delta\theta = \eta \circ \tau$, i.e., we define $\eta := \mathrm{Id} + \delta\theta \circ \tau^{-1}$. Because $\tau^{-1} \in \mathcal{T}$ and $\delta\theta \in \Theta$ we observe that $\delta\theta \circ \tau^{-1} \in \Theta$, thanks to proposition C.1.7.

We only need to prove that $\delta\theta \circ \tau^{-1}$ is small, and then use the first part.

But by proposition C.1.5 this follows immediately. One can alternatively apply point i) of lemme 2.2, [15].

$\square$

Theorem C.1.9
Small perturbations of identity, Lipschitz property Let $U \subset\subset D$ be Lipschitz bounded. There exists $0 < C(U) < 1$ such that, for $\tau \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)$ and $\|\tau - \mathrm{Id}\|_{W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)} \leq C(U)$, then $T(U)$ is also bounded Lipschitz, where T is $U(\tau)$, the unique Lipschitz continuous representative of τ (see proposition A.1.3).

This result can be applied to, e.g., $\tau \in \mathcal{T}$ which is a small perturbation of identity in the $W^{1,\infty}$ topology.

Proof.

It is done in [3], lemma 3, page 629. $\square$

C.2. Transforming Sobolev spaces

Theorem C.2.1 (Change of variables)
Let U be open and $T = U(\tau)$ for $\tau \in \mathcal{T}^1$, and let $p \in [1, \infty]$. Then:

1. $f \in L^p(T(U)) \iff f \circ T \in L^p(U)$ and there holds, for $f \in L^p(T(U))$:

$$\|f\|_{L^p(T(Q))} \leq \left(\|\det DT\|_{L^\infty(\mathbb{R}^n)}\right)^{1/p} \|f \circ T\|_{L^p(Q)}$$

2. $f \in W^{1,p}(T(U)) \iff f \circ T \in W^{1,p}(U)$ and there holds, for $f \in W^{1,p}(T(U))$:

$$Df \circ T = (Df)^{-t} D(f \circ T)$$

$$\|Df\|_{L^p(T(Q);\mathbb{R}^n)} \leq \left(\|\det DT\|_{L^\infty(\mathbb{R}^n)}\right)^{1/p} \left\|(DT)^{-1}\right\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \|D(f \circ T)\|_{L^p(Q;\mathbb{R}^n)}$$

3. add the rest of this proposition

4. if $p \in (1, \infty)$, $f \in W_0^{1,p}(T(U)) \iff f \circ T \in W_0^{1,p}(U)$

5. therefore, composition by T is a linear isomorphism between $W^{k,p}(T(U)) \to W^{k,p}(U)$ for $k = 0, 1$, and between $W_0^{1,p}(T(U)) \to W_0^{1,p}(U)$ for $k = 0, 1$, $p \in (1, \infty)$

6. for D a bounded Lipschitz domain and $\mathcal{T}, \Theta$ defined before, we get, for $f \in H^1(D)$, that $\mathrm{tr} f = \mathrm{tr}(f \circ T)$

7. if moreover, $\Omega, T(\Omega) \subset\subset D$ are also bounded Lipschitz domains, letting $U := D \backslash \Omega$, another bounded Lipschitz domain, for $f \in H^1(T(U))$ and $\mathrm{tr}_{T(U)} f = 0$ on $\partial T(\Omega)$, then $\mathrm{tr}_U f \circ T = 0$ on $\partial\Omega$ and $\mathrm{tr}_{T(U)} f = \mathrm{tr}_U f \circ T$ on ∂D

8. so, $\circ T$ is a linear isomorphism of $H_c^1(U)$ and $H_c^1(T(U))$ (H_c^1 is defined in appendix B.2.1)

Proof.

We need to prove only the last points, for the other are proved in [15], pages IV.4, IV.5, IV.6.

<u>Static strace</u>

To do so, let $f_n \in C(\overline{D}) \cap H^1(D)$ converging in $H^1(D)$ to f. By point 4, we have $f_n \circ T \to f \circ T$ in $H^1(D)$ (rememeber, $T(D) = D$ by invertibility of T and the fact that $T(x) = x$ outside of D). Therefore we have:

$$\mathrm{tr} f \leftarrow_{L^2(\partial D)} \mathrm{tr}(f_n) = f_n|_{\partial D} = (f_n \circ T)|_{\partial D} = \mathrm{tr}(f_n \circ T) \rightarrow_{L^2(\partial D)} \mathrm{tr}(f \circ T)$$

Zero moving trace

First of all, as T is a homeomorphism of $\mathbb{R}^n$, $TU = D \setminus T(\Omega)$, $T\partial U = \partial D \sqcup \partial\Omega$, $T\partial\Omega = \partial T\Omega$.

Now, an application of theorem A.1.1 yields that the extension to 0 in $T\Omega$ of f, call it $\bar{f}$, is $H^1(D)$, with $\partial_i \bar{f} = \partial_i f$ in TU, 0 in $T(\Omega)$.

We claim that $\mathrm{tr}_D \bar{f} = \mathrm{tr}_{T(U)} f|_{\partial D}$. In fact, approximate $\bar{f}$ by restrictions to D of $C_c^\infty(\mathbb{R}^n)$ functions f_n (see theorem 3.18 of [1], page 54), which also approximate f on $T(U)$, by the observation that $\left\|f_n|_{T(U)}\right\|_{H^1(T(U))} \leq \|f_n|_D\|_{H^1(D)}$. Then:

$$\mathrm{tr}_{T(U)}(f_n|_{T(U)})|_{\partial D} = (f_n|_{T(U)})|_{\partial T(U)}|_{\partial D} = f_n|_{\partial D} = \mathrm{tr}_D(f_n|_D)$$

Now, by what we observed before, $\mathrm{tr}_{T(U)}(f_n|_{T(U)}) \to \mathrm{tr}_{T(U)}(f)$ in $L^2(\partial T(U))$, so that $\mathrm{tr}_{T(U)}(f_n|_{T(U)})|_{\partial D} \to \mathrm{tr}_{T(U)}(f)|_{\partial D}$. On the other hand $\mathrm{tr}_D(f_n|_D) \to \mathrm{tr}_D \bar{f}$, which yields the claim.

Using this: $\mathrm{tr}_{T(U)}(f)|_{\partial D} = \mathrm{tr}_D \bar{f} = \{$ point 5 $\} = \mathrm{tr}_D(\bar{f} \circ T) = \mathrm{tr}_D(\overline{f \circ T}) = \mathrm{tr}_U(f \circ T)|_{\partial D}$, where we used that $\bar{f} \circ T$ is zero in $T^{-1}T\Omega = \Omega$ (because again T maps null sets into null sets), so it is the zero extension $\overline{f \circ T}$ of $f \circ T$, and applied the same reasoning as above to conclude $\mathrm{tr}_D(\overline{f \circ T}) = \mathrm{tr}_U(f \circ T)|_{\partial D}$. Both $\bar{f} \circ T$ and $f \circ T$ are H^1 functions by point 2.

We can now also say that $\mathrm{tr}_U f \circ T = 0$ on $\partial\Omega$.

$$(\eta\phi_n)|_{\partial\Omega} = \mathrm{tr}_U(\phi_n|_U)_{\partial\Omega} \to \mathrm{tr}_U(f \circ T)_{\partial\Omega}$$

Multiplication by a $W^{1,\infty}$ function

We claim that, for $\psi \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R})$ and $f \in H^1(U)$, then $f\psi$ has the same trace as f as long as $\psi = 1$ in a neighbourhood of ∂U.

Note that $f\psi \in H^1(U)$ still. Now: approximate f by restriction of test functions f_n. Then $f_n\psi$ is $C(\overline{U}) \cap H^1(U)$ (thanks also to corollary A.1.4), so that $\mathrm{tr}_U(f_n\psi) = \mathrm{tr}_U(f_n)$. Because $f_n\psi \to f\psi$ is $H^1(U)$ the claim is valid.

This last convergence follows from $\|(f_n - f)\psi\|_{L^2} \leq \|(f_n - f)\|_{L^2} \|\psi\|_{L^\infty}$, the chain rule $\partial_i(f_n\psi) = \partial_i f_n \psi + \partial_i \psi f_n$ (see corollary 4.1.18 here) and again $\|\partial_i(f_n - f)\psi\|_{L^2} \leq \|\partial_i(f_n - f)\|_{L^2} \|\psi\|_{L^\infty}$, $\|(f_n - f)\partial_i\psi\|_{L^2} \leq \|(f_n - f)\|_{L^2} \|\partial_i\psi\|_{L^\infty}$.

Reducing to a function of 0 trace

Let η be a smooth cut-off function which is 1 close to ∂D and 0 close to $\partial T\Omega$, $\beta = 0$ close to ∂D and 1 close to $\partial T\Omega$. This can be accomplished by e.g. building a suitable partition of unity of the compact sets $\partial\Omega$ and ∂D. (can I do this? Yes, see bachelor's thesis, take $K = \partial\Omega$ etc. Also, be careful with all of these equalities...).

$f\beta$ has zero trace, as it can be verified by approximating f by smooth functions again:

$$\mathrm{tr}_{T(U)} f\beta \leftarrow_{L^2(\partial T(U))} \mathrm{tr}_{T(U)} f_n\beta$$

where the latter quantity is $\mathrm{tr}_{T(U)} f_n$ on $\partial T(U)$ and 0 on ∂D. By restricting the convergence to first ∂D and then to $\partial T(U)$, and using almost everywhere convergent subsequences, we conclude that $\mathrm{tr}_{T(U)} f\beta = \mathrm{tr}_{T(U)} f$ on $\partial T(U)$ and $\mathrm{tr}_{T(U)} f\beta = 0$ on ∂D, i.e. $f\beta$ has zero trace.

Domain transformation

But zero trace functions in $H^1(T(U))$, since $T(U)$ is assumed to be bounded Lipschitz, are exactly the functions $H_0^1(T(U))$ (theorem 18.7 at page 595 of [11]).

By then point 4, $(f\beta) \circ T \in H_0^1(U)$.

Because T is bi-Lipschitz, we can write $(f\beta) \circ T = f \circ T \beta \circ T$ almost everywhere.

We have that $\beta \circ T + \eta \circ T$ is $W^{1,\infty}$ and 1 near ∂U.

So, $\mathrm{tr}_U f \circ T = \mathrm{tr}_U f \circ T(\beta \circ T + \eta \circ T) = \mathrm{tr}_U(f \circ T\beta \circ T) + \mathrm{tr}_U(f \circ T\eta \circ T)$.

Approximate $f \circ T$ by g_n smooth as seen above. Then, $\mathrm{tr}_U(g_n\eta \circ T)$ is 0 on $\partial\Omega$ and $\mathrm{tr}_U g_n$ on ∂D. By selecting an almost everywhere convergent subsequence, we conclude $\mathrm{tr}_U(f \circ T\eta \circ T) = 0$ on $\partial\Omega$.

Hence $\mathrm{tr}_U f \circ T|_{\partial\Omega} = tr_U f \circ T(\beta \circ T)|_{\partial\Omega} = 0$.

$\square$

C.3. Transforming Bochner spaces

Proposition C.3.1 (Isomorphism between Q spaces)

$$\circ T : Q(I, V_T) \to Q(I, V)$$

is a linear isomorphism, and so is:

$$\circ T : Q_0(I, V_T) \to Q_0(I, V)$$

In particular, $(u \circ T)' = u' \circ T$ (under suitable identifications, see the proof).

Proof.

Existence of derivative

Consider the following commutative diagram:

$$
\begin{tikzcd}
V_T \arrow[r, hook, "a_T"] \arrow[d, "\circ T_H =: t_V"'] & H_T \arrow[r, "r_T"] \arrow[d, "\circ T_H =: t_H"] & H_T^* \arrow[r, hook, "a_T^*"] & V_T^* \\
V \arrow[r, hook, "a"] & H \arrow[r, "r"] & H^* \arrow[r, hook, "a^*"] & V^*
\end{tikzcd}
$$

We know that $f \in Q(I, V_T)$ satisfies $a_T f \in H^1(I, H_T)$. Therefore, by proposition A.2.2, $t_H a_T f = a t_V f \in H^1(I, H)$, and $t_v \in L^2(I, V)$, so that $t_V : Q(I, V_T) \to Q(I, V)$ is well defined.

Boundedness

We have that $t_V : V_T \to V$ is linear bounded, so that $t_V : L^2(I, V_T) \to L^2(I, V)$ is also linear and bounded. Still by proposition A.2.2, we have $(a t_V f)' = (t_H a_T f)' = t_H (a_T f)'$.

Thus $\|t_V f\|_{L^2(I,V)} \leq C(T) \|f\|_{L^2(I,V_T)}$, together with $\|(a t_V f)'\|_{L^2(I,H)} \leq C(T) \|(a_T f)'\|_{L^2(I,H_T)}$.

By noting that $(t_V)^{-1} = (\circ T)^{-1}$ is bijective, and by the bounded inverse theorem:

$$\circ T : Q(I, V_T) \to Q(I, V)$$

is a linear isomorphism.

Zero initial value

Consider $a t_V f$. It has a unique $C([0, T], H)$ representative, $U(a t_V f)$. Also, $a_T f$ has a unique continuous representative $U(a_T f)$. Now, $a t_V f = t_H a_T f$, so that $[U(a t_V f)] = [t_H U(a_T f)]$. By continuity, $U(a t_V f) = t_H U(a_T f)$ on $[0, T]$ and thus, whenever $U(a_T f)(0) = 0$, so is $U(a t_V f)$, informally, also $t_V f(0) = 0$.

So, $t_V(Q_0(I, V_T)) \subseteq Q_0(I, V)$.

$(t_V)^{-1} = (\circ T)^{-1}$ and we can conclude that $t_V^{-1}(Q_0(I, V)) \subseteq Q_0(I, V_T)$.

□

C.4. Transforming partial differential equations

We consider again the two parabolic equations of interest, namely, problem B.2.1.2 and problem B.2.2.2.

We continue from proposition B.3.5.

$$
\begin{aligned}
w \in Q_0(I, H^1_c), v_0 \in Q_0(I, H^1_0) \\
\int_I (w', q)_H + (\nabla w, \nabla q)_H = \int_I (g, \operatorname{tr} q)_{L^2(\Gamma_f)}, \quad \forall q \in Q^0(I, H^1_c) \\
\int_I (v_0', p)_H + (\nabla v_0, \nabla p)_H = - \int_I (\bar{u}', p)_H + (\nabla \bar{u}, \nabla p)_H, \quad \forall p \in Q^0(I, H^1_0)
\end{aligned}
$$

We are working under the assumption:

Assumption C.4.1
We have $T = U(\tau), \tau \in \mathcal{T}$, $U \subset\subset D$ bounded Lipschitz domains and we also assume that $T(U)$ is bounded Lipschitz.

Suppose the problem is formulated on $T(U)$. To ease the notation, call $H^1_c(T(U)) = V_T, H^1_c(U) = V, H^1_0(U) = W$ and analogously for the other spaces.

We write the problem as:

$$
\begin{aligned}
w^T \in Q_0(I, V_T), v_0^T \in Q_0(I, W_T) \\
\int_I (w^T_t, q)_{H_T} + (\nabla w^T, \nabla q^T)_{H_T} = \int_I (g, \operatorname{tr}_{T(U)} q^T)_{L^2(\Gamma_f)}, \quad \forall q^T \in Q^0(I, V) \\
\int_I (v^T_{0t}, p)_{H_T} + (\nabla v^T_0, \nabla p^T)_{H_T} = - \int_I (\bar{u}', p^T)_{H_T} + (\nabla \bar{u}, \nabla p^T)_{H_T}, \quad \forall p^T \in Q^0(I, W_T)
\end{aligned}
$$

Applying a change of variables, we get equivalently:

$$
\begin{aligned}
& w^T \in Q_0(I, V_T), v_0^T \in Q_0(I, W_T) \\
& \int_I (w_t^T \circ T, q^T \circ T|\det(DT)|)_H + (A_T \nabla(w^T \circ T), \nabla(q^T \circ T))_H = \\
& \int_I (g, \mathrm{tr}_{T(U)} q^T)_{L^2(\Gamma_f)}, \quad \forall q^T \in Q^0(I, V_T) \\
& \int_I (v_{0t}^T \circ T, p^T \circ T|\det(DT)|)_H + (A_T \nabla(v_0^T \circ T), \nabla(p^T \circ T))_H = \\
& -\int_I ((\bar{u}') \circ T, p^T \circ T|\det(DT)|)_H + (A_T \nabla(\bar{u} \circ T), \nabla(p^T \circ T))_H, \quad \forall p^T \in Q^0(I, W_T)
\end{aligned}
$$

Here $A_T = |\det(DT)| DT^{-1}(DT)^{-t}$.

Now, we note that:

- $\mathrm{tr}_{T(U)} q = \mathrm{tr}_U(q \circ T)$ on Σ_f by theorem C.2.1
- $w_t^T \circ T = (w^T \circ T)_t$ by the proof of proposition C.3.1 and analogously for v_0
- by proposition C.3.1,$\circ T$ is a bijection between $Q^0(I, V_T)$ and $Q^0(I, V)$ and analogously for W
- $\bar{u} \in H^1(I, V_T)$ and that $\bar{u}'$ denoted the weak derivative in the $H^1(I, V_T)$ sense, so that A.2.1 yields $\bar{u} \circ T \in H^1(I, V)$ and $(\bar{u} \circ T)' = \bar{u}' \circ T$

We therefore get, equivalently:

$$
\begin{aligned}
& w^T \in Q_0(I, V_T), v_0^T \in W_0(I, W_T) \\
& \int_I ((w^T \circ T)_t, q|\det(DT)|)_H + (A_T \nabla(w^T \circ T), \nabla q)_H = \\
& \int_I (g, \mathrm{tr}_U q)_{L^2(\Gamma_f)}, \quad \forall q \in Q^0(I, V) \\
& \int_I ((v_0^T \circ T)_t, p|\det(DT)|)_H + (A_T \nabla(v_0^T \circ T), \nabla p)_H = \\
& -\int_I ((\bar{u} \circ T)', p|\det(DT)|)_H + (A_T \nabla(\bar{u} \circ T), \nabla p)_H, \quad \forall p \in W^0(I, W)
\end{aligned}
$$

and by proposition C.3.1, we also get $w^T \circ T \in Q_0(I, V), v_0^T \circ T \in Q_0(I, W)$.

Note, here one could absorb det into the test functions, complicate the PDE but get rid of the time coefficients. From here one could prove a continuity result for the states and then get Frechet differentiability from Gateaux differentiability.

On the other hand, consider:

$$
\begin{aligned}
& w \in Q_0(I,V), v_0 \in Q_0(I,W) \\
\int_I (w_t, q|\det(DT)|)_H + (A_T \nabla w, \nabla q)_H = & \int_I (g, \mathrm{tr}_U q)_{L^2(\Gamma_f)}, \quad \forall q \in Q^0(I,V) \\
& \int_I (v_{0t}, p|\det(DT)|)_H + (A_T \nabla v_0, \nabla p)_H = \\
- \int_I ((\bar{u} \circ T)', p|\det(DT)|)_H + & (A_T \nabla (\bar{u} \circ T), \nabla p)_H, \quad \forall p \in Q^0(I,W)
\end{aligned}
$$

Then, we note the following:

- by proposition C.3.1, $w \circ T^{-1} \in Q_0(I, V_T), v_0 \circ T^{-1} \in Q_0(I, W_T)$, and as seen above, $((w \circ T^{-1}) \circ T)_t = (w \circ T^{-1})_t \circ T$ and the same goes for $v_0 \circ T^{-1}$

Therefore we obtain, equivalently:

$$
\begin{aligned}
& w \circ T^{-1} \in Q_0(I, V_T), v_0 \circ T^{-1} \in Q_0(I, W_T) \\
& \int_I ((w \circ T^{-1})_t, q^T)_{H_T} + (\nabla (w \circ T^{-1}), \nabla q^T)_{H_T} = \\
& \int_I (g, \mathrm{tr}_{T(U)} q^T)_{L^2(\Gamma_f)}, \quad \forall q^T \in Q^0(I, V_T) \\
& \int_I ((v_0 \circ T^{-1})_t, p^T)_{H_T} + (\nabla (v_0 \circ T^{-1}), \nabla p^T)_{H_T} = \\
- & \int_I (\bar{u}', p^T)_{H_T} + (\nabla \bar{u}, \nabla p^T)_{H_T}, \quad \forall p^T \in Q^0(I, W_T)
\end{aligned}
$$

and $w \circ T^{-1} \in Q_0(I, V_T), v_0 \circ T^{-1} \in Q_0(I, W_T)$.

These findings can be summarized as follows.

Theorem C.4.2 (Equivalent formulations with transported domain)
Let assumption B.2.1.1, assumption B.2.2.1, assumption C.4.1 hold.

Consider the following problems.

Problem C.4.3 (Joint parabolic problem, moving domain)

$$
\begin{aligned}
&w^T \in Q_0(I, V_T), v_0^T \in Q_0(I, W_T)\\
\int_I (w_t^T, q^T)_{H_T} + (\nabla w^T, \nabla q^T)_{H_T} &= \int_I (g, \mathrm{tr}_{T(U)} q^T)_{L^2(\Gamma_f)}, \quad \forall q^T \in Q^0(I, V_T)\\
\int_I (v_{0t}^T, p^T)_{H_T} + (\nabla v_0^T, \nabla p^T)_{H_T} &= -\int_I (\bar{u}', p^T)_{H_T} + (\nabla \bar{u}, \nabla p^T)_{H_T}, \quad \forall p^T \in Q^0(I, W_T)
\end{aligned}
$$

Problem C.4.4 (Joint parabolic problem, reference domain)

$$
\begin{aligned}
&w \in Q_0(I, V), v_0 \in Q_0(I, W)\\
\int_I (w_t, q|\det(DT)|)_H + (A_T \nabla w, \nabla q)_H &= \int_I (g, \mathrm{tr}_U q)_{L^2(\Gamma_f)}, \quad \forall q \in Q^0(I, V)\\
\int_I (v_{0t}, p|\det(DT)|)_H + (A_T \nabla v_0, \nabla p)_H &=\\
-\int_I ((\bar{u} \circ T)', p|\det(DT)|)_H + (A_T \nabla (\bar{u} \circ T), \nabla p)_H, &\quad \forall p \in Q^0(I, W)
\end{aligned}
$$

We have the following:

- consider $w^T \in Q_0(I, V_T), v_0^T \in Q_0(I, W_T)$. They solve problem C.4.3 $\iff$ $w^T \circ T, v_0^T \circ T$ solve problem C.4.4
- consider $w \in Q_0(I, V), v_0^T \in Q_0(I, W)$. They solve problem C.4.4 $\iff$ $w \circ T^{-1}, v_0 \circ T^{-1}$ solve problem C.4.3

Here, $A_T := (DT)^{-1}(DT)^{-t}|\det(DT)|$.

Bibliography

[1] R.A. Adams. *Sobolev Spaces. Adams*. Pure and applied mathematics. Academic Press, 1975. URL: https://books.google.it/books?id=JxzpSAAACAAJ.

[2] Daniel Azagra, Erwan Le Gruyer, and Carlos Mudarra. *Kirszbraun's theorem via an explicit formula*. 2018. DOI: 10.48550/ARXIV.1810.10288. URL: https://arxiv.org/abs/1810.10288.

[3] J F Almagro Bello et al. "The Differentiability of the Drag with Respect to the Variations of a Lipschitz Domain in a Navier–Stokes Flow". In: *Siam Journal on Control and Optimization* 35 (1997), pp. 626–640.

[4] Martin Costabel. "Boundary Integral Operators for the Heat Equation". In: *Integral Equations and Operator Theory* 13 (July 1990), pp. 498–552. DOI: 10.1007/BF01210400.

[5] L.C. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010. ISBN: 9780821849743. URL: https://books.google.it/books?id=Xnu0o_EJrCQC.

[6] Thomas Muirhead Flett. "Mean value theorems for vector-valued functions". In: *Tohoku Mathematical Journal, Second Series* 24.2 (1972), pp. 141–151.

[7] G. Gilardi. "Equazioni paraboliche astratte: impostazione variazionale".

[8] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 2011. ISBN: 9781611972023. URL: https://books.google.it/books?id=lJTtujsZSgkC.

[9] M. Hinze et al. *Optimization with PDE Constraints*. Mathematical Modelling: Theory and Applications. Springer Netherlands, 2008. ISBN: 9781402088391. URL: https://books.google.it/books?id=PFbqxa2uDS8C.

[10] M. Kreuter. "Sobolev Spaces of Vector-Valued Functions". MA thesis. Ulm University, 2015.

[11] G. Leoni. *A First Course in Sobolev Spaces*. Graduate studies in mathematics. American Mathematical Society, 2017. ISBN: 9781470429218. URL: https://books.google.it/books?id=qoA8DwAAQBAJ.

[12] Florian Lindemann. "Theoretical and Numerical Aspects of Shape Optimization with Navier-Stokes Flows". In: 2012.

[13] J.L. Lions, P. Kenneth, and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications: Volume II*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012. ISBN: 9783642652172. URL: `https://books.google.it/books?id=xD71CAAAQBAJ`.

[14] W. McLean and W.C.H. McLean. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, 2000. ISBN: 9780521663755. URL: `https://books.google.it/books?id=RILqjEeMfK0C`.

[15] François Murat and Jacques Simon. "Etude de problemes d'optimal design". In: *Optimization Techniques Modeling and Optimization in the Service of Man Part 2*. Ed. by Jean Cea. Berlin, Heidelberg: Springer Berlin Heidelberg, 1976, pp. 54–62. ISBN: 978-3-540-38150-1.

[16] Kevin Sturm and Antoine Laurain. "Distributed shape derivative via averaged adjoint method and applications". In: *ESAIM Mathematical Modelling and Numerical Analysis* 50 (Sept. 2015). DOI: `10.1051/m2an/2015075`.

[17] F. Tröltzsch and J. Sprekels. *Optimal Control of Partial Differential Equations: Theory, Methods, and Applications*. Graduate studies in mathematics. American Mathematical Society, 2010. ISBN: 9780821849040. URL: `https://books.google.it/books?id=04yDAwAAQBAJ`.