Technical University of Munich

Department of Mathematics

[Thesis Title]

Master's Thesis

von

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Submission Date:	[Day. Month. Year]

I hereby declare that this thesis is my own work and that no other sources have been used except those clearly indicated and referenced.

Place, Date

original, hand-written signature

Acknowledgements

[text of acknowledgements]

German Abstract

[abstract text]

English Abstract

[abstract text]

Contents

1. Introduction

For referencing a book or an article. Said $\int_\Omega$.

Moreover

$$\left(\bigcup_i B_i\right) \tag{1.0.1}$$

Moreover we get

$$1 + 1 = 2 \tag{1.0.2}$$

As seen in 1.0.1 and eq. (1.0.2). Interestingly enough:

Theorem 1.0.3
We are happy.

Here is a reference to theorem 1.0.3 and chapter 1

2. Chapter 2

3. Chapter 3

A. Sobolev spaces

Proposition A.0.1 (Characterization of $W^{1,\infty}$)
Let Ω be a bounded Lipschitz domain, or $\mathbb{R}^n$. Then $W^{1,\infty}(\Omega) = C^{0,1} \cap L^\infty(\Omega)$.

This means that $u \in W^{1,\infty}(\Omega)$ if and only if u has a (unique) representative that is bounded, Lipschitz continuous. Weak and classical derivatives coincide a.e.

Proof.

<u>Extension</u>

In the case Ω is bounded Lipschitz, then Ω is an extension domain for $W^{1,\infty}(\Omega)$, meaning that there is $E : W^{1,\infty}(\Omega) \to W^{1,\infty}(\mathbb{R}^n)$ linear bounded with $Eu = u$ a.e. on Ω (see [10], theorem 13.17 at page 425, 13.13 at page 424, and definition 9.57 at page 273).

<u>The proof</u>

Let $u \in W^{1,\infty}(\Omega)$. By [10], 11.50 at page 339, because Ω is an extension domain, we obtain that u has a representative $\bar{u}$ that is bounded Lipschitz. Let $\phi \in C_c^\infty(\Omega)$. By The Kirszbraun theorem (see e.g. [**kirszbraun**]), we can extend $\bar{u}$ to a Lipschitz function e on $\mathbb{R}^n$. Then, for a large enough cube Q containing Ω, $\int_\Omega \bar{u}\partial_i\phi = \int_Q e\partial_i\phi = -\int_Q \partial_i e\phi$, by Fubini's theorem and integration by parts for AC functions.

Because $e = \bar{u}$ on Ω, we conclude $\int_\Omega \bar{u}\partial_i\phi = -\int_\Omega \partial_i\bar{u}\phi$, so that $\nabla\bar{u} = \nabla u$ almost everywhere.

Conversely, let u be bounded Lipschitz. The above reasoning shows that u has essentially bounded weak derivatives equal to the a.e. classical derivatives.

□

B. Bochner spaces

Here are some useful results about Bochner spaces.

Proposition B.0.1 (Bochner integral and bounded operators)
Let X, Y be separable Banach, let $T \in L(X, T)$ be a linear bounded operator. For $f \in L^1(I, X)$ define $Tf(t) := T(f(t))$. Then $Tf \in L^1(I, Y)$ with $T \int_I f = \int_I Tf$.

Proof.

First of all, a clarification on the definition. What is really happening is that from the time equivalence class f, we select a g, and then $Tf(t) := T(g(t))$. Tf is then the equivalence class of $t \mapsto Tf(t)$. The definition is well posed, because $g_1(t) = g_2(t) \implies T(g_1(t)) = T(g_2(t))$.

Let f_n be simple, $f_n \to f$ a.e., with $\lim_n \int_I f_n = \int_I f$ adn $\|f_n\|_X \leq C \|f\|_X$ (see page 6, and corollary 2.7 at page 8 of [8]).

Measurability

For almost all t, $T(f_n(t)) \to T(f(t)) = Tf(t)$ in Y, so that Tf is measurable (strongly).

Integrability

By the assumptions, $\|Tf_n\| \leq \|T\| \|f_n\| \leq C \|f\| \in L^1(I)$, so that by dominated convergence (corollary 2.6 of [8]) Tf is integrable too. Thus $\int_T Tf = \lim_n \int_I Tf_n = \lim_n T \int_I f_n$, because f_n is simple. And now, by the choice of f_n, $\int_T Tf = \lim_n T \int_I f_n = T \lim_n \int_I f_n = T \int_I f$.

□

Proposition B.0.2 (Derivations and bounded operators)
As before, let X, Y be separable Banach, let $T \in L(X, T)$ be a linear bounded operator.

For $k \geq 0$, $f \in H^k(I, X) \implies Tf \in H^k(I, Y)$, with weak derivatives $\partial_{t^i} Tf = T \partial_{t^i} f$, $0 \leq i \leq k$.

The map $f \mapsto Tf$, $H^k(I, X) \to H^k(I, Y)$ is linear bounded.

Proof.

The case $k = 0$ is proved above.

We prove now that $\partial_{t^i} Tf = T\partial_{t^i} f$ for $i = 1$. Note that $T\partial_t f \in L^2(I, Y)$, which qualifies as weak derivative.

In fact, for $\phi \in C_c^\infty(I)$, we have $\int_I \phi T\partial_i f = \int_I T(\phi\partial_t f) = T\int_I \phi\partial_t f = -T\int_I \phi' f = -\int_I \phi' Tf$.

Higher weak derivatives are treated analogously and the rest of the claims follow from the time stationarity of T and by $\|\partial_{t^i} Tf\| = \|T\partial_{t^i} f\| \leq \|T\| \, \|\partial_{t^i} f\|$.

□

Proposition B.0.3 (Continuous representatives)
Let X be separable Banach. $f \in L^1(I, X)$ has at most a continuous representative on $[0, T]$.

Proof.

Assume there exists two such continuous representatives, so that we get a function $\delta : [0, T] \to X$ that is zero almost everywhere and continuous. Hence, $[0, T] \ni t \mapsto \|\delta(t)\|$ is continuous in $\mathbb{R}$ and zero a.e., so that it must be zero everywhere. □

We now check that a vector valued test function has weak derivatives of all orders.

Proposition B.0.4 (Weak derivatives of test functions)
Let $\phi \in C^1([0, T], X)$, for X separable Banach. It means that the limit of the difference quotients exists for all points of I, that $t \mapsto \phi(t), \phi'(t)$ are continuous, and that they can be continuously extended to $[0, T]$.

Then these classical derivatives coincide a.e. with the weak derivatives of u.

Proof.

We rely on proposition 3.8 of [8] at page 26.

<u>Absolute continuity</u>

Consider $\epsilon > 0$. Divide $[a, b] \subset\subset (0, T)$ into a uniform partition t_i. By theorem 6 at page 146 of [4], we get that $\|\phi(t_i) - \phi(t_{i-1})\|_X \leq (t_i - t_{i-1}) \|\phi(\xi_i)\|_X \leq (b - a) \|\phi'\|_\infty / n$, and by choosing n small enough, we conclude that ϕ is (locally) absolutely continuous.

<u>Weak derivative</u>

Therefore, ϕ is locally AC, differentialble everywhere and ϕ' is bounded, so that $\phi \in H^1(I, X)$ and weak and classical derivatives coincide.

□

And now, introduce a time dependent version of the trace operator which is useful for our computations, although "non-optimal" from a smoothness point of view.

Definition B.0.5 (Time dependent trace)
Let Ω be a bounded Lipschitz domain. For $k \geq 0$ we define $\mathrm{tr} : H^k(I, H^1(\Omega)) \to H^k(I, H^{1/2}(\partial\Omega))$ by $\mathrm{tr}(u)(t) := \mathrm{tr}(u(t))$

Below are some properties of this operator.

Proposition B.0.6 (Properties of trace operator)
The trace operator just defined:

1. is well posed
2. is linear bounded
3. admits a linear bounded right inverse, for instance, $E(g)(t) := E(g(t))$ (for E a right inverse of the static trace)
4. tr and E, in the case of $k \in \mathbb{N}_0$, coincide (in the time a.e. sense) for the case $l \geq k$
5. for $k \geq 1$, $\mathrm{tr}u(0) = 0 \iff u(0) = 0$
6. it coincides with the trace treated for instance in [11]

Proof.

Proof of the proposition

We recall that the trace operator is bounded surjective onto $H^{1/2}(\partial\Omega)$, with a right inverse E (see theorem 3.37 at page 102 of [12]).

The first three points are consequences of this fact and of proposition B.0.1.

The fourth property follows by the definition of tr, E and the fact that $H^l \subseteq H^k$, for $k \leq l$.

Let now $k \geq 1$. We know that $H^1, H^{1/2}$ are separable and Banach (the latter is separable because the continuos image of H^1 separable, and Banach because of the existence of E, for instance). Therefore, by [3], theorem 2 of page 286, we obtain the embeddings $H^k(I, H^1) \hookrightarrow C([0,T], H^1)$ and the same goes for $H^k(I, H^{1/2})$. The embedding is U, the unique continuous representative of a certain time equivalence class (proposition B.0.3). We also introduce brackets to indicate equivalence classes in time, so, $u = [Uu]$.

We want to prove $(Uu)(0) = 0 \iff U(\mathrm{tr}u)(0) = 0$. But we have $[t \mapsto U(\mathrm{tr}u)(t)] = tru := [t \mapsto \mathrm{tr}((Uu)(t))]$. So, $U(\mathrm{tr}u)(t) = \mathrm{tr}((Uu)(t))$ for all $t \in [0, T]$ by continuity.

For the last point, let $k = 0$. We have:

1. $H^1(\Omega) \cap C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ (see [1], theorem 3.18 at page 54, where being Ω bounded Lipschitz is important)

2. functions $\sum_{i \leq m} \phi_i(t) f_i$ for $\phi_i \in C_c^\infty(I), f_i \in H^1(\Omega) \cap C^1(\overline{\Omega})$ are dense in $L^2(I, H^1)$ (see [7], page 39, lemma 1.9)

It follows by the third point that $C^1(\overline{\Omega \times I})$ is dense in $L^2(I, H^1)$, so that $u \mapsto u|_{I \times \partial\Omega}$ admits a unique extension by continuity to $L^2(I, H^1)$, so that this definition of trace coincides with the one from the literature in the case of the space $H^{1,0} := L^2(I, H^1)$ (see [11], theorem 4.1), we expand this argument below.

<u>Proof of leftover facts</u>

We call $C^k(\overline{\Omega}) := \{u \in C^k(\Omega) \text{ with } \partial_\alpha f \text{ extendable by continuity to } \overline{\Omega}\}$.

Consider $u(x, t) := \phi(t)v(x)$, for $\phi \in C^1([0, T]), v \in C^1(\overline{\Omega})$. Then, it has partial derivatives $u_t = \phi_t v, u_i = \phi u_i$. u and all its partial derivatives are continuous on $I \times \Omega$, meaning that $u \in C^1(\Omega \times I)$.

Moreover, $u, u_i, u_t \in C([0, T], C(\overline{\Omega}))$. We claim $C([0, T], C(\overline{\Omega})) = C(\overline{\Omega \times I})$. In fact, one direction is trivial, and so, let $f \in C([0, T], C(\overline{\Omega})) = C(\overline{\Omega})$. Fix $(t, x) \in \overline{\Omega \times I}$. Then, $|f(s, y) - f(t, x)| \leq |f(t, y) - f(t, x)| + |f(t, y) - f(s, y)| \leq |f(t, y) - f(t, x)| + \|f(t, \cdot) - f(s, \cdot)\|_\infty$. If now s is close to t, and y is close to x, then $|f(s, y) - f(t, x)|$ is small.

This shows $u, u_i, u_t \in C([0, T], C(\overline{\Omega})) \in C(\overline{\Omega \times I})$, i.e. $u \in C^1(\overline{Q \times I})$.

To conclude, let $u \in L^2(I, H^1)$. Approximate u by $u_k := \sum_{i \leq m_k} \phi_i^k(t) f_i^k$ as in point 2, and approximate f_i^k by suitable $g_i^k \in H^1(\Omega) \cap C^1(\overline{\Omega})$, to obtain $u_k := \sum_{i \leq m_k} \phi_i^k(t) g_i^k$

Then $\|u - w_k\|_{L^2(I,H^1)} \leq \|u_k - w_k\|_{L^2(I,H^1)} + \|u_k - u\|_{L^2(I,H^1)}$. We only need to estimate $\|u_k - w_k\|_{L^2(I,H^1)} \leq T \sum_{i \leq m_k} \|\phi_i^k\|_\infty \|f_i^k - g_i^k\|_{H^1}$. By the first point, $\|f_i^k - g_i^k\|_{H^1}$ can be made as small as it is necessary to conclude.

<u>Last remarks</u>

Again with reference to [11], consider the anisotropic spaces $H^{r,s} := L^2(I, H^r) \cap H^s(I, L^2)$. We restrict to the case $r = 1$, $s \geq 0$. Denote the traces tr_s defined in theorem 4.1, mapping $H^{1,s}(\Omega \times I) \to H^{1/2,s/2}(\partial\Omega \times I)$. For $\partial\Omega$ Lipschitz this theorem is still valid, as

$1/2 \leq 1$, see the discussion above lemma 2.4 in [2]. As stated in [11], tr_s is an extension of $u \mapsto u|_{I\times\partial\Omega}$, defined on the dense suspace $C^\infty(\overline{Q \times I})$ of $H^{1,s}$ (that this space is dense can be proved as in lemma 2.22 of [2]). So, let $C^\infty(\overline{Q \times I}) \ni u_n \to_{H^{r,s}} u \in H^{1,s}$.

We have $\mathrm{tr}_s u_n = \mathrm{tr}_0 u_n$. Then, $u_n \to_{H^{1,s}} u$, $u_n \to_{H^{1,0}} u$, so that $\mathrm{tr}_s u_n \to_{H^{1/2,s/2}} \mathrm{tr}_s u$ (hence $\mathrm{tr}_0 u_n \to_{H^{1/2,0}} \mathrm{tr}_s u$) and $\mathrm{tr}_0 u_n \to_{H^{1/2,0}} \mathrm{tr}_\sigma u$.

Thus $\mathrm{tr}_0 u = \mathrm{tr}_s u$.

Using what we derived before, we can conclude the characterization of the traces in the anisotropic settign define

□

And now some sanity checks in the case of Gelfand triples.

Proposition B.0.7
(Sanity checks for Gelfand triples)

Consider the following Gelfand triples:

$$\begin{tikzcd}
V \arrow[dr, hook, "a"] & & & V^* \arrow[d, hook, "c^*"] \\
 & H \arrow[r, "r"] & H^* \arrow[ur, hook, "a^*"] \arrow[dr, hook, "b^*"'] & \\
W \arrow[uu, hook, "c"] \arrow[ur, hook, "b"'] & & & W^*
\end{tikzcd}$$

Here $W \subseteq V \subseteq H$ are all separable Hilbert spaces, a, b, c the trivial injections, r the Riesz isomorphism of H. We denote by i_V the Gelfand triple embedding $V \hookrightarrow V^*$, so, $i_V = a^* r a$.

Then:

1. $H^1(I, V) \subseteq W(I, V)$ with continuous embedding. The $W(I, V)$ derivative of $u \in H^1(I, V)$ is $i_V u_t$.

2. for $u \in W(I, W)$ with $(i_W u)' \in L^2(I, H)$ (i.e. $(i_W u)_t = b^* r h$ for h in $L^2(I, H)$) we obtain $u \in W(I, V)$ (i.e. $cu \in W(I, V)$), with derivative $(i_V cu)' = a^* r h$, so that also $(i_V cu)' \in L^2(I, H)$. It also holds $(i_V cu)'|_W = (i_W u)'$. h is also the weak derivative $L^2(I, H)$ of bu.

3. let $u, v \in W(I, V)$ with $u - v \in W$. Then $u - v \in W(I, W)$ with derivative $(i_W(u - v))' = (i_V u)'|_W - (i_V v)'|_W$.

Proof.

We use several times that time integrals and bounded linear static operators commute, see proposition B.0.1. ϕ denotes $\phi \in C_c^\infty(I)$.

<u>First point</u>

We need to check that $a^* rau \in H^1(I, V^*)$. This follows from proposition B.0.2, so that $(a^* rau)_t = a^* rau_t$.

<u>Second point</u>

At first we claim that h is a weak derivative of $bu \in L^2(I, H)$. In fact, $b^* r \int_I bu\phi' = \int_I (i_W u)\phi' = \{\ u \in W(I, W)\ \} = -\int_I (i_W u)'\phi = -\int_I b^* rh = b^* r - \int_I h\phi$. By density (definition of Gelfand triple), b^* is injective, r is too, and thus $\int_I bu\phi' = -\int_I h\phi$, which shows that bu has weak derivative h.

And now $\int_I i_V cu\phi' = \int_I a^* racu\phi' = a^* r \int_I bu\phi' = \{$ by what we just proved $\} = -a^* r \int_I h\phi$, proving that $(i_V cu)' = a^* rh$.

Morevoer $(i_V cu)'|_W = c^* a^* rh = b^* rh = \{$ assumption $\} = (i_w u)'$.

<u>Third point</u>

We check the derivative. We have $\int_i i_W (u-v)\phi' = \{\ u - v \in W \subseteq V\ \} = \int_I b^* ra(u-v) = c^* \int_I (i_V u - i_V v)\phi' = -\int c^*((i_V u)' - (i_V v)')\phi$.

$\square$

C. Parabolic equations

Assumption C.0.1 (Basic assumption for parabolic problems)
Let $V \subseteq H$ be real separable Hilbert spaces, V dense in H. Then $H \hookrightarrow V^*$ is also dense, as stated in [13] at page 147. This embedding is $H \ni f \mapsto (f, \cdot)_H$. We thus obtain a Gelfand triple, and we have $W(I,V) \subseteq C(I,H)$.

Let $A : V \to V^*$ be linear bounded, $u \in W(I;V)$, $f \in L^2(I,V^*)$ and $u_0 \in H$.

We also assume that $\langle Av, v\rangle_{V^*,V} + \lambda \|v\|_H^2 \geq \alpha \|v\|_V^2$ for $\lambda \geq 0, \alpha > 0$.

We are interested in the following problem:

Problem C.0.2 (Abstract parabolic equation)

$$u_t + Au = f \text{ in } V^* \text{ and for a.e. } t \in (0,T) \tag{C.0.3}$$
$$u(0) = u_0 \tag{C.0.4}$$

Theorem C.0.5 (Basic well posedness of problem C.0.2)
Under assumption C.0.1, problem C.0.2 has a unique solution u. Moreover u satisfies the energy estimate:

$$\|u\|_{W(I,V)} + \|u\|_{C([0,T],H)} \leq c(\lambda, \alpha, \|A\|_{V^*}, T)(\|u_0\|_H + \|f\|_{L^2(I,V^*)}) \tag{C.0.6}$$

Proof.

See [5] at page 19, theorem 26. □

We can also obtain additional regularity. Here are further assumptions to make this possible.

Assumption C.0.7 (Assumptions for additional regularity)
We assume $u_0 \in V$, $f = f_1 + f_2 \in L^2(I,H) + H^1(I,V^*)$. We also need A to be symmetric (i.e. $\langle Au, v\rangle_{V^*,V} = \langle Av, u\rangle_{V^*,V}$).

Theorem C.0.8 (Regularity of time derivative)
Suppose assumption C.0.1 and assumption C.0.7. Then $u_t \in L^2(I,H)$ with the estimate:

$$\|u\|_{W(I,V)} + \|u\|_{C(I,H)} + \|u_t\|_{L^2(I,H)} \leq \tag{C.0.9}$$
$$c(\lambda, \alpha, \|A\|_{V^*}, T)(\|u_0\|_V + \|f_1\|_{L^2(I,H)} + \|f_2\|_{H^1(I,V^*)}) \tag{C.0.10}$$

That $u_t \in L^2(I,H)$ means precisely that there is $h \in L^2(I,H)$ with $a^* r h = (i_V u)'$, with the notation introduced in proposition B.0.8.

Proof.

We refer to page 26 of [5], theorem 28, and only prove the necessary modifications.

Product rule for A

We have

$$\int_0^t \langle Au_n, u_n' \rangle_{V^*,V} = \sum_{k,l \leq n} \langle Aw_k^n, w_l^n \rangle_{V^*,V} \int_0^t g_k^n g_l^{n'} =$$
$$\sum_{k,l \leq n} \langle Aw_k^n, w_l^n \rangle_{V^*,V} \left(- \int_0^t g_k^{n'} g_l^n + g_k^n(t) g_l^n(t) - g_k^n(0) g_l^n(0) \right)$$

By linearity at first and then symmetry we get:

$$= \langle Au_n, u_n \rangle_{V^*,V}(t) - \langle Au_n, u_n' \rangle_{V^*,V}(0) - \int_0^t \langle Au_n', u_n \rangle_{V^*,V} =$$
$$= \langle Au_n, u_n \rangle_{V^*,V}(t) - \langle Au_n, u_n' \rangle_{V^*,V}(0) - \int_0^t \langle Au_n, u_n' \rangle_{V^*,V}$$

so that:

$$\int_0^t \langle Au_n, u_n' \rangle_{V^*,V} = \frac{1}{2} \left(\langle Au_n, u_n \rangle_{V^*,V}(t) - \langle Au_n, u_n' \rangle_{V^*,V}(0) \right)$$

Estimate for right hand side

We have:

$$\int_0^t \langle f_2, u_n' \rangle_{V^*,V} = \sum_{k\leq n} \int_0^t g_k^{n'} \langle f_2, w_k^n \rangle_{V^*,V}$$

By the smoothness of f_2 we have that $t \mapsto \langle f_2(t), w_k^n \rangle_{V^*,V}$ is $H^1(0,T)$, in particular $AC[0,t]$, so that we can integrate by parts:

$$= -\sum_{k\leq n} \int_0^t g_k^n \langle f_2', w_k^n \rangle_{V^*,V} + \sum_{k\leq n} g_k^n(t) \langle f_2(t), w_k^n \rangle_{V^*,V} - \sum_{k\leq n} g_k^n(0) \langle f_2(0), w_k^n \rangle_{V^*,V} =$$
$$- \int_0^t \langle f_2', u_n \rangle_{V^*,V} + \langle f_2, u_n \rangle_{V^*,V}(t) - \langle f_2, u_n \rangle_{V^*,V}(0) =$$

Here we have used proposition B.0.2 to take the derivative inside the bracket.

Going to the absolute values:

$$\left| \int_0^t \langle f_2, u_n' \rangle_{V^*,V} \right| \leq \int_0^T |\langle f_2', u_n \rangle_{V^*,V}| + \|f_2(t)\|_{V^*} \|u_n(t)\|_V + \|f_2(0)\|_{V^*} \|u_n(0)\|_V \leq$$
$$\frac{1}{2} \|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2} \|u_n\|^2_{L^2(I,V)} + \frac{\alpha}{4} \|u_n(t)\|^2_V +$$
$$+\frac{4}{\alpha} \|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2} \|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2} \|u_{n0}\|^2_V$$

Now, u_n converges weakly in $L^2(I,V)$ by estimate (59) of [5] and thus $\frac{1}{2}\|u_n\|_{L^2(I,V)}$ is bounded. The term $\frac{\alpha}{4}\|u_n(t)\|_V$ can be pulled to the left hand side, u_{0n} is V convergent hence bounded. Therefore as in [5] we are able to conclude that u_n' is bounded in $L^2(I,H)$. We want to conclude $u_t \in L^2(I,H)$. We know for sure that $\langle u_m', w_j \rangle_{V^*,V} = \langle f - Au_m, w_j \rangle_{V^*,V}$, so that muliplication by a test function and integration yields $\int_I \langle u_m', w_j\phi \rangle_{V^*,V} = \int_I \langle f - Au_m, w_j\phi \rangle_{V^*,V}$. Because $u_m \rightharpoonup u$ in $L^2(I,V)$ we observe that, by proposition B.0.1 applied on $A \in L(V,V^*)$, it holds $\int_I \langle u_m', w_j\phi \rangle_{V^*,V} \to \int_I \langle u', w_j\phi \rangle_{V^*,V}$.

What's more, is that $u_m' \rightharpoonup h$ in $L^2(I,H)$, so that $\int_I \langle h, w_j \rangle_{V^*,V}\phi = \int_I \langle u', w_j \rangle_{V^*,V}\phi$. It means that for almost all t, $\langle h, w_j \rangle_{V^*,V} = \langle u', w_j \rangle_{V^*,V}$. And now we can really say that $u' \in L^2(I,H)$, which even more precisely means $(i_V u)' = a^* r h$ almost everywhere.

We also obtain that u_t is bounded by $c(\alpha)(\|f_2\|_{L^\infty(I,V^*)} + \|f_2\|_{L^2(I,V^*)} + \|u_0\|_V + \|u\|_{L^2(I,V)})$.

Note that, by [3], theorem 2 of page 286, we can estimate $\|f_2\|_{L^\infty(I,V^*)}$ by $c(T)\|f_2\|_{H^1(I,V^*)}$, so that the claim for the time derivative u_t is proven.

□

For the case where $H = L^2$, $H^1 \supseteq V \supseteq H_0^1$, $f_2|_{H_0^1} = 0$ we have even more regularity available.

Theorem C.0.11 (Additional regularity)
Suppose assumption C.0.1 and assumption C.0.7.

Let additionally $H = L^2$, $H^1 \supseteq V \supseteq H_0^1$, $f_2|_{H_0^1} = 0$. Then $Au|_{H_0^1}$ extends to $\overline{Au_{H_0^1}} \in L^2(I,H)$ with:

$$\|u\|_{W(I,V)} + \|u\|_{C([0,T],H)} + \|u_t\|_{L^2(I,H)} + \left\|\overline{Au|_{H_0^1}}\right\|_{L^2(I,H)} \leq \tag{C.0.12}$$
$$c(\lambda, \alpha, \|A\|_{V^*}, T)(\|u_0\|_V + \|f_1\|_{L^2(I,H)} + \|f_2\|_{H^1(I,V^*)}) \tag{C.0.13}$$

Moreover $u_t + \overline{Au_{H_0^1}} = f_1$ in $L^2(0,T,L^2) \cong L^2(Q)$ and $\overline{Au|_{H_0^1}} = Au$ on H_0^1.

Proof.

For $v \in H_0^1$ we get $\langle Au, v\rangle_{V^*,V} = \langle f_1 - u_t, v\rangle_{V^*,V} = (f_1 - u_t, v)_H$, for almost all $t \in (0,T)$. From here we conclude that $Au(t)$ extends for a.a. t to an element of H with $(\overline{Au} - f_1 + u_t, v)_{L^2} = 0$ for all $v \in H_0^1$, almost all t. By density, $\overline{Au} - f_1 + u_t = 0$ in H for almost all t, so that $\overline{Au} = f_1 - u_t$ in $L^2(0,T,L^2) \cong L^2(Q)$.

This isometric isomorphism is stated in [13], page 144.

□

For our applications we also need to track the constants more precisely, which is accomplished in the next proposition.

Proposition C.0.14 (Tracking the costants)
With assumption C.0.1 there holds:

$$\|u\|^2_{C([0;T],H)} + \alpha \|u\|^2_{L^2(I,V)} \leq \exp(2\lambda T)(\|u_0\|^2_H + \alpha^{-1} \|f\|^2_{L^2(I,V^*)}) \tag{C.0.15}$$
$$\|u'\|_{L^2(I,V^*)} \leq \|A\|_{L(V,V^*)} \alpha^{-1/2} \sqrt{\exp(2\lambda T)} \|u_0\|_H + \tag{C.0.16}$$
$$\left(\|A\|_{L(V,V^*)} \alpha^{-1} \sqrt{\exp(2\lambda T)} + 1\right) \|f\|_{L^2(I,V^*)} \tag{C.0.17}$$

With additionally assumption C.0.7 we obtain:

$$C \|u'\|^2_{L^2(I,H)} \leq (1 + (1 + C_0)\alpha^{-1}) \|f_2\|^2_{H^1(I,V^*)} + \tag{C.0.18}$$
$$(1 + \|A\|_{L(V,V^*)}) \|u_0\|^2_V + C_0 \|u_0\|^2_H + \tag{C.0.19}$$
$$\|f_1\|^2_{L^2(I,H)} + C_0\alpha^{-1} \|f_1\|^2_{L^2(I,V^*)} \tag{C.0.20}$$

with $C > 0$ a number independent of the problem.

Here $C_0 = 2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\exp(2\lambda T)$.

Proof.

No regularity

From page 21 of [5] we obtain that $\|u\|^2_{C([0;T],H)} + \alpha\|u\|^2_{L^2(I,V)} \leq \exp(2\lambda T)(\|u_0\|^2_H + \alpha^{-1}\|f\|^2_{L^2(I,V^*)})$.

In particular, $\sqrt{\alpha}\|u\|_{L^2(I,V)} \leq \sqrt{\exp(2\lambda T)}(\|u_0\|_H + \alpha^{-1/2}\|f\|_{L^2(I,V^*)})$, or $\|u\|_{L^2(I,V)} \leq \alpha^{-1/2}\sqrt{\exp(2\lambda T)}(\|u_0\|_H + \alpha^{-1/2}\|f\|_{L^2(I,V^*)})$.

Moreover $\|u'\|_{L^2(I,V^*)} \leq \|Au\|_{L^2(I,V^*)} + \|f\|_{L^2(I,V^*)} \leq \|A\|\,\|u\|_{L^2(I,V)} + \|f\|_{L^2(I,V^*)}$.

All in all, we obtain:

$$\|u\|^2_{C([0;T],H)} + \alpha\|u\|^2_{L^2(I,V)} \leq \exp(2\lambda T)(\|u_0\|^2_H + \alpha^{-1}\|f\|^2_{L^2(I,V^*)})$$

and:

$$\|u'\|_{L^2(I,V^*)} \leq \|A\|_{L(V,V^*)}\,\alpha^{-1/2}\sqrt{\exp(2\lambda T)}(\|u_0\|_H + \alpha^{-1/2}\|f\|_{L^2(I,V^*)}) + \|f\|_{L^2(I,V^*)}$$

More regularity

We tie back to page 25 of [5]. In particular:

$$\int_0^t \|u_n'\|^2_H + \int_0^t \langle Au_n, u_n'\rangle_{V^*,V} = \int_0^t (f_1, u_n')_H + \int_0^t \langle f_2, u_n'\rangle_{V^*,V}$$

Then:

$$\int_0^t \langle Au_n, u_n'\rangle_{V^*,V} \geq \frac{\alpha}{2}\|u_n(t)\|^2_V - \frac{\lambda}{2}\|u_n(t)\|^2_H - \frac{\|A\|}{2}\|u_{n0}\|_V$$

whereas, as in the proof of theorem C.0.8:

$$\left|\int_0^t \langle f_2, u_n'\rangle_{V^*,V}\right| \leq \frac{1}{2}\|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)} + \frac{\alpha}{4}\|u_n(t)\|^2_V + \\ + \frac{4}{\alpha}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2}\|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2}\|u_{n0}\|^2_V$$

Also:

$$\int_0^t (f_1, u_n')_H \leq \frac{1}{2} \|f_1\|^2_{L^2(I,H)} + \frac{1}{2} \int_0^t \|u_n'\|^2_H$$

Putting all together:

$$\begin{aligned}\int_0^t \|u_n'\|^2_H + \frac{\alpha}{2} \|u_n(t)\|^2_V - \frac{\lambda}{2} \|u_n(t)\|^2_H - \frac{\|A\|}{2} \|u_{n0}\|_V \\ \frac{1}{2} \|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2} \|u_n\|^2_{L^2(I,V)} + \frac{\alpha}{4} \|u_n(t)\|^2_V + \\ + \frac{4}{\alpha} \|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2} \|f_2\|^2_{L^\infty(I,V^*)} + \frac{1}{2} \|u_{n0}\|^2_V + \\ + \frac{1}{2} \|f_1\|^2_{L^2(I,H)} + \frac{1}{2} \int_0^t \|u_n'\|^2_H\end{aligned}$$

which brings us to:

$$\begin{aligned}\frac{1}{2} \int_0^t \|u_n'\|^2_H + \frac{\alpha}{4} \|u_n(t)\|^2_V - \frac{\lambda}{2} \|u_n(t)\|^2_H \leq \\ \frac{1}{2} \|f_2'\|^2_{L^2(I,V^*)} + \frac{1}{2} \|u_n\|^2_{L^2(I,V)} + \\ + \frac{8+\alpha}{2\alpha} \|f_2\|^2_{L^\infty(I,V^*)} + \frac{1+\|A\|}{2} \|u_{n0}\|^2_V + \\ + \frac{1}{2} \|f_1\|^2_{L^2(I,H)}\end{aligned}$$

and thus, because norms are lower semicontinuous and because we have weak convergence of the time derivative, and V-strong convergence of the initial data:

$$\begin{aligned}\frac{1}{2} \int_0^T \|u'\|^2_H \leq \frac{1}{2} \|f_2'\|^2_{L^2(I,V^*)} + \frac{8+\alpha}{2\alpha} \|f_2\|^2_{L^\infty(I,V^*)} + \frac{1+\|A\|}{2} \|u_0\|^2_V + \frac{1}{2} \|f_1\|^2_{L^2(I,H)} + \\ \operatorname{limsup}_n \left(\frac{\lambda}{2} \|u_n\|^2_{C([0,T],H)} + \frac{1}{2} \|u_n\|^2_{L^2(I,V)} \right)\end{aligned}$$

Using a purely numeric constant C without dependences on the problem we can write:

$$\int_0^T \|u'\|_H^2 \leq \|f_2'\|^2_{L^2(I,V^*)} + C(1+\alpha^{-1})\|f_2\|^2_{L^\infty(I,V^*)} + C(1+\|A\|)\|u_0\|_V^2 + \|f_1\|^2_{L^2(I,H)} + C\text{limsup}_n\left(\frac{\lambda}{2}\|u_n\|^2_{C([0,T],H)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)}\right)$$

For the last term, employing the exact argument as in the first part of the proof:

$$\begin{aligned}\text{limsup}_n\left(\frac{\lambda}{2}\|u_n\|^2_{C([0,T],H)} + \frac{1}{2}\|u_n\|^2_{L^2(I,V)}\right) \leq \\ 2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\text{limsup}_n\left(\|u_n\|^2_{C([0,T],H)} + \alpha\|u_n\|^2_{L^2(I,V)}\right) \leq \\ 2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\exp(2\lambda T)(\|u_0\|_H^2 + \alpha^{-1}\|f_1+f_2\|^2_{L^2(I,V^*)}) \leq \\ 2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\exp(2\lambda T)(\|u_0\|_H^2 + 2\alpha^{-1}\|f_1\|^2_{L^2(I,V^*)} + 2\alpha^{-1}\|f_2\|^2_{L^2(I,V^*)}) \leq \\ CC_0(\|u_0\|_H^2 + \alpha^{-1}\|f_1\|^2_{L^2(I,V^*)} + \alpha^{-1}\|f_2\|^2_{L^2(I,V^*)})\end{aligned}$$

where $C_0 = 2^{-1}\max(1,\lambda)\max(1,\alpha^{-1})\exp(2\lambda T)$ and C is a purely numeric constant without dependences on the problem.

Therefore:

$$C\int_0^T \|u'\|_H^2 \leq \|f_2'\|^2_{L^2(I,V^*)} + (1+\alpha^{-1})\|f_2\|^2_{L^\infty(I,V^*)} + (1+\|A\|)\|u_0\|_V^2 + \|f_1\|^2_{L^2(I,H)} + C_0(\|u_0\|_H^2 + \alpha^{-1}\|f_1\|^2_{L^2(I,V^*)} + \alpha^{-1}\|f_2\|^2_{L^2(I,V^*)})$$

The embedding $H^1(I,V^*) \hookrightarrow C([0,T],V^*)$ has norm that only depends on T, which follows from the equality $f_2(t) = f_2(s) + \int_s^t f_2'$, for $0 \leq s \leq t \leq T$, a bound being $1+T$.

Thus:

$$\begin{aligned}C\int_0^T \|u'\|_H^2 \leq (1+(1+C_0)\alpha^{-1})\|f_2\|^2_{H^1(I,V^*)} + \\ (1+\|A\|)\|u_0\|_V^2 + C_0\|u_0\|_H^2 + \\ \|f_1\|^2_{L^2(I,H)} + C_0\alpha^{-1}\|f_1\|^2_{L^2(I,V^*)}\end{aligned}$$

□

C.1. Application to inhomogeneous parabolic problems

C.1.1. Inhomogeneous Dirichlet problem

We make the following assumption.

Assumption C.1.1.1 (Assumptions for problem C.1.1.2)
We assume $\Omega \subset\subset D$ to be bounded Lipschitz domains, so that $U := D \setminus \Omega$ is bounded Lipschitz too and the trace operator is bounded surjective onto $H^{1/2}(\partial U)$, with a right inverse E (see theorem 3.37 at page 102 of [12]). For such a choice we also have $H_0^1 = H^1 \cap \ker \text{tr}$, see [10], page 595, theorem 18.7.

Moreover, we select $f \in H^1(I, H^{1/2}(\Gamma_f))$, $f(0) = 0$.

Note that, given a bounded extension operator $E : H^{1/2}(\partial U) \to H^1(U)$, we obtain by proposition B.0.2 that $Ef \in H^2(I, H^1(U))$. We have defined $\text{tr}u(t) := tr(u(t))$ and analogously $Eu(t) := E(u(t))$.

Call $H = L^2(U)$, $V = \{v \in H^1(U), \text{tr}u = 0 \text{ on } \Gamma_m\} =: H_c^1$. V is a closed subspace of H^1, which is Hilbert separable, hence also Hilbert separable. We norm it with the full H^1 norm. Because $H_0^1(U)$ is dense in H, so is V and we obtain a Gelfand triple.

We define $A := H^1 \to H^{1*}$ by $(Au)v := \int_u \nabla u \nabla v$. This operator can be the restricted to $V \to H^{-1}$ and $V \to V^*$.

The problem under consideration is the following. For $U = D \setminus \Omega$ we have:

Problem C.1.1.2 (Inhomogeneous heat equation, Dirichlet conditions)

$$u_t - \Delta u = 0 \text{ in } (0, T) \times U \tag{C.1.1.3}$$
$$u(\Sigma_f) = f \tag{C.1.1.4}$$
$$u(\Sigma_m) = 0 \tag{C.1.1.5}$$
$$u(0) = 0 \tag{C.1.1.6}$$

By this we mean:

$$u \in W(I, H_c^1) \tag{C.1.1.7}$$
$$u_t|_{H^{-1}} + Au = 0 \text{ in } H^{-1} \text{ and for a.e. } t \in (0, T) \tag{C.1.1.8}$$
$$\text{tr}u = f \text{ on } \Sigma_f \tag{C.1.1.9}$$
$$u(0) = 0 \tag{C.1.1.10}$$

Note that $u_t \in L^2(I, V^*)$, so $u_t \in V^*$ for a.a. t and so, we can say that $u_t \in H^{-1}$ for a.a. t.

Theorem C.1.1.11 (Well posedness and regularity for problem C.1.1.2)
Given assumption C.1.1.1, the solution u to problem C.1.1.2 is unique with $u_t \in L^2(I, H)$. The uniqueness holds in more generally in $L^2(I, V) \cap H^1(I, H^{-1})$.

The problem is equivalent to:

Problem C.1.1.12 (Equivalent formulation with extension)

$$u_0 \in W(I, H_0^1) \tag{C.1.1.13}$$
$$u_0' + Au_0 = -((\bar{u}', \cdot)_H + A\bar{u}) \text{ in } H^{-1} \text{ and for a.e. } t \in (0, T) \tag{C.1.1.14}$$
$$u_0(0) = 0 \tag{C.1.1.15}$$

with $\bar{u}$ any given $\bar{u} \in H^1(I, H_c^1(U))$ such that $\text{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$. This means that u solves problem C.1.1.2 $\implies$ $u - \bar{u}$ solves problem C.1.1.12, and if $u_0(\bar{u})$ solves problem C.1.1.12, then $\bar{u} + u_0(\bar{u})$ solves problem C.1.1.2.

Furthermore:

$$\|u\|^2_{C([0;T],H)} + \|u\|^2_{L^2(I,H)} + \|\nabla u\|^2_{L^2(I,H)} + \|u'\|^2_{L^2(I,H)} \leq C(T) \|\bar{u}\|^2_{H^1(I,V)} \tag{C.1.1.16}$$

with $C > 1$, only dependent on T, smoothly, exploding for large T.

There holds the estimate:

Proof.

Extension of the boundary data

Let $\bar{u} \in H^1(I, H_c^1(U))$ be such that $\text{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$. We can choose for instance $E\tilde{f}$, see proposition B.0.7, where $\tilde{f} = 0$ on Σ_m, $\tilde{f} = f$ on Σ_f. $\tilde{f} \in H^1(I, H^{1/2}(\partial U))$, because Γ_f and Γ_m have positive distance (see the definition of the norm in [6], page 20).

Reformulation (first part)

Consider the following commutative diagram, where $V = H_c^1$:

$$
\begin{tikzcd}
H_c^1 \arrow[dr, "a"] & & & (H_c^1)^* \arrow[ddd, hook, "c^*"] \\
& H \arrow[r, "r"] & H^* \arrow[ur, hook, "a^*"] \arrow[dr, hook, "b^*"'] & \\
& & & \\
H_0^1 \arrow[uuu, hook, "c"] \arrow[uur, hook, "b"'] & & & H^{-1}
\end{tikzcd}
$$

Here, a, b, c are the trivial injections, r the Riesz isomorphism $h \mapsto (h, \cdot)_H$.

Now $(i_W(u-\bar{u}))'+A(u-\bar{u}) = (i_V u)'|_{H^{-1}}-(i_V\bar{u})'|_{H^{-1}}+Au-A\bar{u} = \{$ proposition B.0.8 $\} = (i_V u)'|_{H^{-1}}-(i_V\bar{u}_t)|_{H^{-1}}+Au-A\bar{u} = -(i_V\bar{u}_t)|_{H^{-1}}-A\bar{u}$ if u solves problem C.1.1.2, where $\bar{u}_t$ is the weak derivative of $\bar{u}$. Call $u_0 = u-\bar{u}$. By again proposition B.0.8, $u_0 \in W(I, H_0^1)$.

This motivates us to consider the problem:

$$u_0 \in W(I, H_0^1) \tag{C.1.1.17}$$
$$u_0' + Au_0 = -(f_1+f_2) \text{ in } H^{-1} \text{ and for a.e. } t \in (0,T) \tag{C.1.1.18}$$
$$u_0(0) = 0 \tag{C.1.1.19}$$

Here, $f_1 := (i_V\bar{u}_t)|_{H^{-1}} = c^*a^*ra\bar{u}_t = b^*r(a\bar{u}_t) \in L^2(I, H)$.

Moreover, $A \in L(V, H^{-1})$, so, by proposition B.0.2, $f_2 := A\bar{u} \in H^1(I, H^{-1})$.

Existence

By theorem C.0.11 we get a solution of the above problem with $u_0' \in L^2(I, H)$.

And now, let $u := \bar{u} + cu_0 = \bar{u} + u_0$. We claim it is a solution. The initial and boundary conditions are surely satisfied. We check it is in $W(I, V)$ and is satisfies the partial differential equation.

By proposition B.0.8, we have both $\bar{u}, cu_0 \in W(I, V)$. The derivatives are $i_V\bar{u}_t$ and $a^*r(bu_0)'$, with $(bu_0)'$ the $L^2(I, H)$ weak derivative of bu_0, which exists by proposition B.0.8. Therefore $(i_V(\bar{u}+cu_0))'|_{H_0^1} = c^*(i_V(\bar{u}+cu_0))' = c^*i_V\bar{u}_t + c^*i_V(bu_0)' = b^*r(a\bar{u}_t) + i_W(u_0)'$ by proposition B.0.8.

Using the pde of u_0, ... $= b^*r(a\bar{u}_t) - Au_0 - f_1 - f_2 = -A(u_0 + \bar{u})$.

Uniqueness

For two solutions u_1, u_2 of $C.1.1.2$ we can form $d := u_1 - u_2 \in W(I, H_0^1)$ by proposition B.0.8. Clearly, $d(0) = 0$. Moreover, $(i_{H_0^1}d)' = \{$ proposition B.0.8 $\} = (i_V u_1)'|_{H_0^1} - (i_V u_2)'|_{H_0^1} = A(u_1 - u_2)$.

By uniqueness stated in theorem C.0.5 we obtain $d = 0$ in $L^2(I, H)$, so that the solution is unique and doesn't depend on the choice of the extension of the Dirichlet datum.

Reformulation (part 2)

Therefore $u = \bar{u} + u_0$ above is the unique solution of problem C.1.1.2. So, given any $\bar{u} \in H^1(I, H^1_c(U))$ such that $\mathrm{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$, we can construct u_0 as above and get $u = \bar{u} + u_0$ solving problem C.1.1.2.

Viceversa, let u solve problem C.1.1.2. Call $u_0 = u - \bar{u}$. Then, as seen above, $u_0 \in W(I, H^1_0)$ and $(i_V(u - \bar{u}))'|_{H^{-1}} + A(u - \bar{u}) = (i_V u)'|_{H^{-1}} - (i_V \bar{u})'|_{H^{-1}} + Au - A\bar{u} = \{$ proposition B.0.8 $\} = (i_V u)'|_{H^{-1}} - (i_V \bar{u}_t)|_{H^{-1}} + Au - A\bar{u} = -(i_V \bar{u}_t)|_{H^{-1}} - A\bar{u}$ if u solves problem C.1.1.2, where $\bar{u}_t$ is the weak derivative of $\bar{u}$. Call $u_0 = u - \bar{u}$. By again proposition B.0.8, $(i_W(u - \bar{u}))' + A(u - \bar{u}) = (i_V u)'|_{H^{-1}} - (i_V \bar{u})'|_{H^{-1}} + Au - A\bar{u} = (i_V u)'|_{H^{-1}} - (i_V \bar{u}_t)|_{H^{-1}} + Au - A\bar{u} = -(i_V \bar{u}_t)|_{H^{-1}} - A\bar{u} = -b^* r(a\bar{u}_t)$. Moreover, $u_0(0) = 0$, so that u_0 solves problem C.1.1.12.

Stability

Let $\bar{u} \in H^1(I, H^1_c(U))$ such that $\mathrm{tr}\bar{u} = f$ on Σ_f, and with $\bar{u}(0) = 0$. Consider u_0. Then, by C.0.14:

$$\|u_0\|^2_{C([0;T],H)} + \alpha \|u_0\|^2_{L^2(I,H^1_0)} \le \exp(2\lambda T)\alpha^{-1} \|(\bar{u}', \cdot)_H + A\bar{u}\|^2_{L^2(I,H^{-1})}$$

$$\begin{aligned} C\|u_0'\|^2_{L^2(I,H)} \le (1 + (1 + C_0)\alpha^{-1}) \|A\bar{u}\|^2_{H^1(I,H^{-1})} + \\ \|(\bar{u}', \cdot)_H\|^2_{L^2(I,H)} + C_0\alpha^{-1} \|(\bar{u}', \cdot)_H\|^2_{L^2(I,H^{-1})} \end{aligned}$$

$C_0 = 2^{-1} \max(1, \lambda) \max(1, \alpha^{-1}) \exp(2\lambda T)$.

We norm H^1_0 with the full H^1 norm too. Then:

$$\begin{aligned} &\|(\bar{u}', \cdot)_H + A\bar{u}\|_{L^2(I,H^{-1})} \le \\ \sup_{\|v\|_{L^2(I,H^1_0)}=1} &\|\bar{u}'\|_{L^2(I,H)} \|v\|_{L^2(I,H)} + \|\nabla\bar{u}\|_{L^2(I,H)} \|\nabla v\|_{L^2(I,H)} \le \\ &C(\|\bar{u}'\|_{L^2(I,H)} + \|\nabla\bar{u}\|_{L^2(I,H)}) \end{aligned}$$

By proposition B.0.2, $\|A\bar{u}\|_{H^1(I,H^{-1})} \le \|A\|_{L(V,H^{-1})} \|\bar{u}\|_{H^1(I,V)}$ (we could apply it since H^{-1} is separable, as a dual of a reflexive Banach space).

Finally, $\|(\bar{u}', \cdot)_H\|^2_{L^2(I,H^{-1})} \le \|\bar{u}'\|_{L^2(I,H)}$.

We can then say:

$$\|u_0\|^2_{C([0;T],H)} + C\alpha \|u_0\|^2_{L^2(I,H^1_0)} \leq \exp(2\lambda T)\alpha^{-1} \|\bar{u}\|^2_{H^1(I,V)}$$
$$C \|u_0'\|^2_{L^2(I,H)} \leq ((1 + (1 + C_0)\alpha^{-1}) \|A\|^2_{L(V,H^{-1})} + 1 + C_0\alpha^{-1}) \|\bar{u}\|^2_{H^1(I,V)}$$

Now, $\langle Av, v\rangle_{H^{-1},H^1_0} + 1 \cdot \|v\|^2_H = 1 \cdot \|v\|^2_{H^1_0}$, so that $\alpha = \lambda = 1$. Moreover, $\langle Au, v\rangle_{H^{-1},H^1_0} \leq \|u\|_V \|v\|_{H^1_0}$, i.e. $\|A\|_{L(V,H^{-1})} \leq 1$.

Therefore $\|u_0\|^2_{C([0;T],H)} + \|u_0\|^2_{L^2(I,H^1_0)} + \|u_0'\|^2_{L^2(I,H)} \leq C(T) \|\bar{u}\|^2_{H^1(I,V)}$ with $C > 1$, only dependent on T, smoothly, exploding for large T.

Now, let's analyse the norms of $\bar{u}$. Because $\bar{u} \in H^1(I,V)$, then, $\bar{u} \in C([0,T],V) \hookrightarrow C([0,T],H)$, where the embedding is non-expansive by the choice of the norm of V. Therefore $\bar{u}_{C([0;T],H)} \leq \bar{u}_{C([0;T],V)} \leq (1+T)\bar{u}_{H^1(I,V)}$. We can therefore conclude that $\|u\|^2_{C([0;T],H)} + \|u\|^2_{L^2(I,H^1_0)} + \|u'\|^2_{L^2(I,H)} \leq C(T) \|\bar{u}\|^2_{H^1(I,V)}$ with $C > 1$, only dependent on T, smoothly, exploding for large T.

□

C.1.2. Inhomogeneous Neumann-Dirichlet problem

We make the following assumption.

Assumption C.1.2.1 (Assumptions for problem C.1.1.2)
We keep assumption C.1.1.1 (apart from the Dirichlet datum). We consired $g \in H^1(I, L^2(\Gamma_f))$, $g(0) = 0$.

Again, call $H = L^2(U)$, $V = \{v \in H^1(U), \mathrm{tr} u = 0 \text{ on } \Gamma_m\} =: H^1_c$. H, V induce a Gelfand triple as seen before.

The problem under consideration is:

Problem C.1.2.2 (Inhomogeneous heat equation, Neumann conditions)

$$u_t - \Delta u = 0 \text{ in } (0,T) \times U \tag{C.1.2.3}$$
$$\partial_\nu u(\Sigma_f) = g \tag{C.1.2.4}$$
$$u(\Sigma_m) = 0 \tag{C.1.2.5}$$
$$u(0) = 0 \tag{C.1.2.6}$$

By this we mean:

$$u \in W(I, H_c^1) \tag{C.1.2.7}$$
$$u_t + Au = G \text{ in } V^* \text{ and for a.e. } t \in (0,T) \tag{C.1.2.8}$$
$$u(0) = 0 \tag{C.1.2.9}$$

where $\langle G(t), v\rangle_{V^*,V} := \int_{\Gamma_f} g(t)\mathrm{tr}v d\sigma$, σ the 1-codimensional Hausdorff measure, and A was introduced before in $L(V, H^{-1})$.

By proposition B.0.2, $G \in H^1(I, V^*)$. In fact, define $T : L^2(\Sigma_f) \to V^*$ by $\langle Tg, v\rangle_{V^*,V} := \int_{\Gamma_f} g\mathrm{tr}v d\sigma$. Then, $\langle Tg, v\rangle_{V^*,V} \leq \|g\|_{L^2(\Gamma_f)} \|v\|_V$ by trace theory. Now, $G(t) = Tg(t)$.

Moreover, $\langle Av, v\rangle_{V^*,V} + 1 \cdot \|v\|_H = 1 \cdot \|V\|$, so that we can immediately conclude:

Theorem C.1.2.10 (Well posedness and regularity for problem C.1.2.2)
Given assumption C.1.2.1, the solution u to problem C.1.2.2 is unique with $u_t \in L^2(I, H)$.

Furthermore:

$$\|u\|^2_{C([0;T],H)} + \|u\|^2_{L^2(I,H)} + \|\nabla u\|^2_{L^2(I,H)} + \|u'\|^2_{L^2(I,H)} \leq C(T) \|g\|^2_{H^1(I,L^2(\Gamma_f))} \tag{C.1.2.11}$$

with $C > 1$, only dependent on T, smoothly, exploding for large T.

Proof.

It is an application of theorem C.0.5, theorem C.0.8 and proposition C.0.14. □

C.2. Reformulation of parabolic equations

We just saw that the two parabolic equations of interest can be recasted into the problem of finding $u \in W(I, V)$, $u(0) = 0$, $u_t + Au = f$ for a.e. t in V^*, with notation from preceding sections.

In particular, $f \in L^2(I, V^*)$ and so is Au (because $A \in L(V, V^*)$, and by B.0.2).

Call then $E(u) := u_t + Au - f \in L^2(I, V^*)$ and $W_0(I, V)$ the $W(I, V)$ functions with zero initial value. Then, the differential equation reads $\langle E(u)(t), v\rangle_{V^*,V} = 0$ for all $v \in V$, for a.a. t, equivalently, $E(u) = 0$ for a.a. t. Thus, we are interested in the abstract problem:

Problem C.2.1 (Even more abstract parabolic equation)
Given a function $E : W(I,V) \to L^2(I,V^*)$, find $u \in W_0(I,V)$, such that $E(u) = 0$ for a.a. t.

We can view $L^2(I,V^*) \cong L^2(I,V)^*$.

Hence $\langle E(u), v\rangle_{L^2(I,V)^*,L^2(I,V)} = \int_I \langle E(u)(t), v(t)\rangle_{V^*,V} dt$ (see [7], theorem 1.31 at page 39).

Now, by proposition B.0.4, we get $C_c^\infty(I,V) \cap H^1(I,V) \subseteq W(I,V)$. Actually, $C_c^\infty(I,V) \subseteq W^0(I,V)$, the functions of $W(I,V)$ having zero terminal values.

We have therefore $C_c^\infty(I,V) \subseteq W^0(I,V) \subseteq L^2(I,V)$, which implies that $W^0(I,V) \subseteq L^2(I,V)$ is dense in $L^2(I,V)$. We can then formulate:

Proposition C.2.2 (Equivalent testing)
Let $E : W(I,V) \to L^2(I,V^*)$, and $u \in W_0(I,V)$.

Then:

$$
\begin{aligned}
E(u) &= 0 \\
&\Longleftrightarrow \\
\langle E(u), v\rangle_{L^2(I,V)^*,L^2(I,V)} = 0 \quad \forall v \in L^2(I,V) \\
&\Longleftrightarrow \\
\langle E(u), v\rangle_{L^2(I,V)^*,L^2(I,V)} = 0 \quad \forall v \in W^0(I,V)
\end{aligned}
$$

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