

# Algebra

September 26, 2016

# 1 Group theory

## 1.1 Week 1

**Def 1.** A non-empty set  $G$  with a binary function  $f : G \times G \rightarrow G, (a, b) \mapsto ab$  is a **group** if it satisfies

1.  $(ab)c = a(bc)$ .
2.  $\exists 1 \in G$  s.t.  $1a = a1 = a, \forall a \in G$ .
3.  $\exists a^{-1} \in G$  s.t.  $aa^{-1} = a^{-1}a = 1$ .

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**Def 2.** Let  $G$  be a group. Then  $G$  is said to be **abelian** if  $\forall a, b \in G, ab = ba$ .

**Def 3.** Let  $G$  be a group and  $H \subseteq G, H \neq \emptyset$ . Then  $H$  is said to be a subgroup of  $G$ , denoted by  $H \leq G$ , if

1.  $\forall a, b \in H, ab \in H$ .
2.  $1 \in H$ .
3.  $\forall a \in H, a^{-1} \in H$ .

useful criterion:  $H \leq G \iff \forall a, b \in H, ab^{-1} \in H$ .

**pf:**

$\Rightarrow \quad b \in H \implies b^{-1} \in H$ , and  $a \in H$ , so  $ab^{-1} \in H$ .

$\Leftarrow \quad$

1.  $H \neq \emptyset \implies \exists a \in H \implies aa^{-1} = 1 \in H$ .
2.  $1, a \in H \implies 1a^{-1} = a^{-1} \in H$ .
3.  $a, b^{-1} \in H \implies a(b^{-1})^{-1} = ab \in H$ . □

**Eg 1.1.1.**  $(\mathbb{Z}, +, 0) \leq (\mathbb{Q}, +, 0) \leq (\mathbb{R}, +, 0) \leq (\mathbb{C}, +, 0) ; (\mathbb{Q}^\times, \times, 1) \leq (\mathbb{R}^\times, \times, 1) \leq (\mathbb{C}^\times, \times, 1)$

**Eg 1.1.2.**

- Special linear group  $\text{SL}(n, \mathbb{F}) = \{ A \in \text{GL}(n, \mathbb{F}) \mid \det A = 1 \}$
- Orthogonal group  $\text{O}(n) = \{ A \in \text{GL}(n, \mathbb{R}) \mid A^t A = I_n \}$
- Unitary group  $\text{U}(n) = \{ A \in \text{GL}(n, \mathbb{C}) \mid A^* A = I_n \}$
- Special orthogonal group  $\text{SO}(n) = \text{SL}(n, \mathbb{R}) \cap \text{O}(n)$
- Special unitary group  $\text{SU}(n) = \text{SL}(n, \mathbb{C}) \cap \text{U}(n)$

**Def 4.** Let  $f : G_1 \rightarrow G_2$ .  $f$  is called an **isomorphism** if

1.  $f$  is 1-1 and onto.
2.  $\forall a, b \in G_1, f(ab) = f(a)f(b)$ . (**homomorphism**)

, denoted by  $G_1 \cong G_2$ .

**Remark 1.** (practice)

1.  $f(1) = 1$ .

2.  $f(a^{-1}) = f(a)^{-1}$ .
3. If  $f$  is an isomorphism, then  $\exists f^{-1}$  is also a homomorphism.

**Eg 1.1.3.**

- $U(1) = \{ z \in \mathbb{C}^\times \mid \bar{z}z = 1 \}, z = \cos \theta + \sin \theta i$
- $SO(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$

notice that  $U(1) \cong SO(2)$ .  $S^1 = \{ (a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1 \}$ , 可被賦予群的結構.

**Eg 1.1.4.** Let  $A \in SU(2) \implies A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \alpha\bar{\alpha} + \beta\bar{\beta} = 1, \alpha, \beta \in \mathbb{C}$ .

Quaternion(四元數):  $\mathbb{H} = \{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \}$  with  $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j (\implies ij = -ji)$ .

Let  $x = a + bi + cj + dk, \bar{x} = a - bi - cj - dk$ , then  $N(x) = x\bar{x} = a^2 + b^2 + c^2 + d^2$ , For  $x \neq 0, N(x) \neq 0, x^{-1} = \frac{1}{N(x)}\bar{x}$

Now, for  $x = a + bi + cj + dk = (a + bi) + (c + di)j$ . So  $SU(2) \cong \{ x \in \mathbb{H}^\times \mid N(x) = 1 \}$ .  $S^3 = \{ (a, b, c, d) \in \mathbb{R}^4 \mid a^2 + b^2 + c^2 + d^2 = 1 \}$ , 可被賦予群的結構.

★ The only spheres with continuous group law are  $S^1, S^3$ .

**Def 5** (symplectic group).  $Sp(n, \mathbb{F}) = \{ A \in GL(2n, \mathbb{F}) \mid A^t J A = J \}$  where  $J = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}$ .

( $A^t J A = J$  preserving non-degenerate skew-symmetric forms)

$Sp(n) = \{ A \in GL(n, \mathbb{H}) \mid A^* A = I_n \}$ .

Ques: Find the smallest subgroup of  $SU(2)$  containing  $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ .

**Ex 1.1.1.** Let  $G$  be a semigroup. Then TFAE (the following are equivalent)

1.  $G$  is a group.
2. For all  $a, b \in G$  and the equations  $bx = a, yb = a$ , each of them has a solution in  $G$ .
3.  $\exists e \in G$  s.t.  $ae = a \forall a \in G$  and if we fix such  $e$ , then  $\forall b \in G \exists b' \in G$  s.t.  $bb' = e$ .

**Ex 1.1.2.** Let  $G$  be a group. Show that

1.  $\forall a \in G, a^2 = 1$ , then  $G$  is abelian.
2.  $G$  is abelian  $\iff \forall a, b \in G, (ab)^n = a^n b^n$  for three consecutive integer  $n$ .

**Ex 1.1.3.** Find a way to regard  $M_{n \times n}(\mathbb{H})$  as a subset of  $M_{2n \times 2n}(\mathbb{C})$ , which preserves addition and multiplication, and then there is a way to characterize  $GL(n, \mathbb{H})$ .

**Ex 1.1.4.** Show  $Sp(n) \cong U(2n) \cap Sp(n, \mathbb{C})$ .

## 1.2 Week 2

### 1.2.1 Permutation groups and Dihedral groups

**Def 6.** A permutation of a set  $B$  is a 1-1 and onto function from  $B$  to  $B$ .

Let  $S_B :=$  the set of permutations of  $B$ . Then  $(S_B, \cdot, \text{Id}_B)$  forms a group.

If  $B = \{a_1, \dots, a_n\}$ , then  $S_B \cong S_{\{1, \dots, n\}}$  and write  $S_n = S_{\{1, \dots, n\}}$ , called the symmetric group of degree  $n$ .

**Theorem 1** (Cayley theorem). Any group is isomorphic to a subgroup of some permutation group.

(Hint): Let  $G$  be a group. Set  $B = G$ . Consider  $a \in G$  as  $\sigma_a : G \rightarrow G, x \mapsto ax$ . Then  $\sigma_a \in S_G \implies G \leq S_G$ .

**Fact 1.2.1.**  $S_n$  is a finite group of order  $n!$ , i.e.  $|S_n| = n!$ .

**pf:** EASY =O

□

Cyclic notation:  $\sigma \in S_5$ , say  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$ . Write  $\sigma = (1\ 4)(2\ 3\ 5)$ .

$\Rightarrow$  Any permutation can be written as a product of disjoint cycles.

**Eg 1.2.1.** In  $S_7$ ,  $\sigma_1 = (1\ 2\ 3)(4\ 5\ 6)(7)$ ,  $\sigma_2 = (1\ 3\ 5\ 6)(2\ 4\ 7)$ .

Then  $\sigma_1\sigma_2 = (2\ 5\ 4\ 7\ 3\ 6)$ ,  $\sigma_1^{-1} = (1\ 3\ 2)(4\ 6\ 5)$ .

**Def 7.** A 2 cycle is called a **transposition**.

**Eg 1.2.2.**  $(1\ 2\ 3) = (1\ 3)(1\ 2)$ ,  $(1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2)$ .

Any permutation is a product of 2 cycles.

Useful formula:  $\sigma \in S_n$ ,  $\sigma(j_1 \dots j_m)\sigma^{-1} = (\sigma(j_1) \dots \sigma(j_m))$ .

**Eg 1.2.3.** Let  $\sigma = (1\ 2\ 3)(4\ 5\ 6\ 7)$ ,  $\sigma(2\ 3\ 4)\sigma^{-1} = (3\ 1\ 5)$ .

**pf:** Note that both sides are functions. For  $i \in \{1, \dots, n\}$ ,

Case 1:  $\exists k$  s.t.  $\sigma(j_k) = i$ , CONCON

Case 2: Otherwise, CONCON

□

**Fact 1.2.2.**  $S_n = \langle (1\ 2), \dots, (1\ n) \rangle$ .

**pf:**  $(1\ i)^{-1} = (1\ i)$  and  $(i\ j) = (1\ i)(1\ j)(1\ i)^{-1}$ .

□

**Def 8.** Let  $G$  be a group and  $S \subset G$ . The subgroup generated by  $S$  defined to be the smallest subgroup of  $G$  which contains  $S$ , denoted by  $\langle S \rangle$ .

**Def 9.**  $A_n = \{\text{even permutations of } S_n\} \leq S_n, |A_n| = \frac{n!}{2}$ .

**Remark 2.**  $\langle S \rangle = \bigcap_{S \subseteq H \leq G} H = \{a_1 a_2 \dots a_k \mid k \in \mathbb{N}, a_i \in S \cup S^{-1}\} \cup \{1\}$

The orthogonal transformations on  $\mathbb{R}^2$ :  $O(2)$ .

Let  $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in O(2)$ .

略... (這邊討論旋轉和反射的矩陣)

Case 1:  $A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$  is counterclockwise rotation w.r.t.  $\alpha$ .

Case 2:  $A = \begin{pmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{pmatrix}$  is the reflection.  $A^2 = I_2 \implies$  eigenvalues are  $\pm 1$ .

Easy to show that  $L_A(v) = v - 2\langle v, v_2 \rangle v_2$ .

$O(2) = \{\text{rotations}\} \cup \{\text{reflections}\}$ .

**Def 10.** The dihedral group  $D_n$  is the group of symmetries of a regular  $n$ -gon. In general,  $D_n = \langle T, R \mid T^n = 1, R^2 = 1, TR = RT^{-1} \rangle \leq O(2) \leq S_n, |D_n| = 2n$ .

**Def 11.** Let  $T$  be a linear transformation from  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ .

- $T$  is called a rotation if  $\exists$  a  $T$ -invariant subspace  $W \subseteq \mathbb{R}^n$  with  $\dim W = 2$  s.t.  $\begin{cases} T|_W \text{ is a rotation} \\ T|_{W^\perp} = \text{id}_{W^\perp} \end{cases}$
- $T$  is called a reflection if  $\exists$  a  $T$ -invariant subspace  $W \subseteq \mathbb{R}^n$  with  $\dim W = 1$  s.t.  $\begin{cases} T|_W = -\text{id}_W \\ T|_{W^\perp} = \text{id}_{W^\perp} \end{cases}$

Main result: the group of orthogonal transformations =  $\langle \text{rotations, reflections} \rangle$ .

**Prop 1.2.1.** For  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\exists$  a  $T$ -invariant subspace  $W \subseteq \mathbb{R}^n$  with  $1 \leq \dim W \leq 2$ .

**pf:** Let  $A = [T]_\alpha \in M_{n \times n}(\mathbb{R}) \subseteq M_{n \times n}(\mathbb{C})$ . Consider  $\widetilde{L}_A : \mathbb{C}^n \rightarrow \mathbb{C}^n, v \mapsto Av$ .

Then  $\exists$  an eigenvalue  $\lambda \in \mathbb{C}$  and an eigenvector  $v \in \mathbb{C}^n$  for  $\widetilde{L}_A$ . Let  $\lambda = \lambda_1 + \lambda_2 i, v = v_1 + v_2 i$ . By definition, we have

$$Av = \widetilde{L}_A(v) = \lambda v = (\lambda_1 + \lambda_2 i)(v_1 + v_2 i) \implies \begin{cases} Av_1 = \lambda_1 v_1 - \lambda_2 v_2 \\ Av_2 = \lambda_2 v_1 + \lambda_1 v_2 \end{cases},$$

so  $W = \langle v_1, v_2 \rangle$ . □

For  $n = 3, A \in O(3)$ , we have  $A \sim \begin{pmatrix} \cos \alpha & -\sin \alpha & \\ \sin \alpha & \cos \alpha & \\ & & \pm 1 \end{pmatrix}$ .

### 1.2.2 Cyclic groups and internal direct product

**Def 12.** If  $G = \langle a \rangle = \{\dots, a^{-2}, a^{-1}, a, 1, a, a^2, \dots\} = \{a^n \mid n \in \mathbb{Z}\}$ , then  $G$  is a cyclic group generated by  $a$ .

**Eg 1.2.4.**  $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$ .

**Eg 1.2.5.** Let  $A = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} \in SO(2)$ . Then  $\langle A \rangle = \{I_2, A, A^2, \dots, A^{n-1}\}$  and  $A^n = I_2, A^m = A^r$  where  $m \equiv r \pmod{n}$ .

**Eg 1.2.6.**  $\mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \dots, \overline{(n-1)}\}$  with  $\bar{j} = \{m \in \mathbb{Z} \mid m \equiv j \pmod{n}\}$ .

Define  $\bar{i} + \bar{j} = \begin{cases} \overline{i+j} & \text{if } 0 \leq i+j \leq n \\ \overline{i+j-n} & \text{otherwise} \end{cases} \implies (\mathbb{Z}/n\mathbb{Z}, +, \bar{0}) \text{ forms a group.}$

**Remark 3.**  $\bar{i} \times \bar{j} = \overline{i \times j}$ .

- 略

- If  $\gcd(j, n) = d, \exists h, k \in \mathbb{Z}$  s.t.  $hj + kn = d$ .

**Def 13.**  $(\mathbb{Z}/n\mathbb{Z})^\times = \{j \in \mathbb{Z}/n\mathbb{Z} \mid \gcd(j, n) = 1\} \implies ((\mathbb{Z}/n\mathbb{Z})^\times, \times, \bar{1})$  forms a group.

**Eg 1.2.7.** 略... 简化剩余系, 原根 (generator)  $(1, 2, 4, p^k, 2p^k, p \text{ is an odd prime})$

**Def 14.**

- The **order** of a finite group  $G$  is the number of elements in  $G$ , denoted by  $|G|$ .
- Let  $a \in G$ , the order of  $a$  is defined to be the least positive integer  $n$  s.t.  $a^n = 1$ , denoted by  $\text{ord}(a) = n$ .
- If  $a^n \neq 1 \quad \forall n \in \mathbb{N}$ , then we call “ $a$  has infinite order”.

**Prop 1.2.2.** Let  $G = \langle a \rangle$  with  $\text{ord}(a) = n$ . Then

1.  $a^m = 1 \iff n \mid m$ .

**pf:**

$\Leftarrow$ : Let  $m = dn$ , then  $a^m = (a^n)^d = 1$ .

$\Rightarrow$ : Let  $m = qn + r, 0 \leq r < n$ . If  $r \neq 0$ , then  $a^r = a^{m-qn} = (a^m)(a^n)^{-q} = 1$ . But  $r < n$ , which is a contradiction. Hence  $r = 0 \implies n \mid m$ .  $\square$

2.  $\text{ord}(a^r) = n / \gcd(r, n)$ .

**pf:** Let  $\gcd(r, n) = d, n = dn', r = dr'$  with  $\gcd(n', r') = 1$ . Plan to show “ $\text{ord}(a^r) = n'$ ”

- $(a^r)^{n'} = a^{r'dn'} = (a^n)^{r'} = 1 \implies \text{ord}(a^r) \mid n'$ .
- $1 = (a^r)^{\text{ord}(a^r)} = a^{r \text{ord}(a^r)} \implies n \mid r \text{ord}(a^r) \implies n' \mid r' \text{ord}(a^r) \implies n' \mid \text{ord}(a^r)$ .

$\square$

**Prop 1.2.3.** Any subgroup of a cyclic group is cyclic.

**pf:** Let  $G = \langle a \rangle$  and  $H \leq G$ . If  $H = \{1\}$ , then  $H = \langle 1 \rangle$ , done!

Otherwise,  $d = \min\{m \in \mathbb{N} \mid a^m \in H\}$ , by well-ordering axiom. Claim  $H = \langle a^d \rangle$ .

$\supset$ :  $a^d \in H$  by the definition of  $d$ .

$\subset$ :  $\forall a^m \in H$ , write  $m = qd + r, 0 \leq r < d$ . If  $r \neq 0$ , then  $a^r = a^{m-qd} = a^m(a^d)^{-q} \in H$ , which is a contradiction. Hence  $r = 0 \implies d \mid m$ .

$\square$

**Prop 1.2.4.** Let  $G = \langle a \rangle$ . Then

1.  $\text{ord}(a) = n \implies G \cong \mathbb{Z}/n\mathbb{Z}$
2.  $\text{ord}(a) = \infty \implies G \cong \mathbb{Z}$

**Def 15.** Let  $G_1, G_2 \leq G$ .  $G$  is the internal direct product of  $G_1, G_2$  if  $G_1 \times G_2 \rightarrow G, (g_1, g_2) \mapsto g_1 g_2$  is an isom.

**Remark 4.** In this case, we find that

- $G = G_1 G_2 = \{g_1 g_2 \mid g_1 \in G_1, g_2 \in G_2\}$ .
- $G_1 \cap G_2 = \{1\}$ . (consider  $a \neq 1 \in G_1 \cap G_2$ , then  $(1, a) \mapsto a, (a, 1) \mapsto a$ , but the function is 1-1, which is a contradiction.)
- If  $a \in G$  with  $a = g_1 g_2 = g'_1 g'_2$ , then  $(g'_1)^{-1} g_1 = (g'_2) g_2^{-1} \in G_1 \cap G_2 = \{1\} \implies \begin{cases} g_1 = g'_1 \\ g_2 = g'_2 \end{cases}$ .
- For  $g_1 \in G_1, g_2 \in G_2, (g_1, g_2) = (g_1, 1)(1, g_2) = (1, g_2)(g_1, 1) \implies g_1 g_2 = g_2 g_1$ .

**Eg 1.2.8.**

1.  $G = \mathbb{Z}/6\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}, G_1 = \{\bar{0}, \bar{3}\}, G_2 = \{\bar{0}, \bar{2}, \bar{4}\}$ . We have  $G \cong G_1 \times G_2$ .
2.  $G = S_3, G_1 = \langle (1\ 2) \rangle, G_2 = \langle (1\ 2\ 3) \rangle$ . We have  $G_1 \times G_2 \not\cong G$  since  $(1\ 2)(1\ 2\ 3) \neq (1\ 2\ 3)(1\ 2)$ .

**Eg 1.2.9.**  $G = S_3, G_1 = \langle (1\ 2) \rangle, G_2 = \langle (2\ 3) \rangle, G_1 G_2 = \{1, (1\ 2), (2\ 3), (1\ 2\ 3)\} \not\subseteq G$  since  $(1\ 3\ 2) = (1\ 2\ 3)^{-1} \notin G_1 G_2$ .

**Prop 1.2.5.** Let  $H, K \leq G$ . Then  $HK \leq G \iff HK = KH$ .

**pf:**

$$\begin{aligned} \Rightarrow: \begin{cases} H \leq HK \\ K \leq HK \end{cases} &\implies KH \subseteq HK; \forall hk \in HK, \exists h'k' \in HK \text{ s.t. } (hk)(h'k') = 1 \implies hk = \\ &(k')^{-1}(h')^{-1} \in KH \implies HK \subseteq KH. \end{aligned}$$

$$\Leftarrow: \text{For } h_1 k_1, h_2 k_2 \in HK, (h_1 k_1)(h_2 k_2)^{-1} = h_1 k_1 k_2^{-1} h_2^{-1} = h_1 h' k' \in HK.$$

□

**Ex 1.2.1.**

1.  $S_n = \langle (1\ 2), (2\ 3), \dots, (n-1\ n) \rangle, \quad n \geq 2$ .
2.  $S_n = \langle (1\ 2), (1\ 2 \dots n) \rangle, \quad n \geq 2$ .

**Ex 1.2.2.**

1.  $A_n = \langle (1\ 2\ 3), (1\ 2\ 4), \dots, (1\ 2\ n) \rangle, n \geq 3$ .
2.  $A_n = \langle (1\ 2\ 3), (2\ 3\ 4), \dots, (n-2\ n-1\ n) \rangle, n \geq 3$ .

**Ex 1.2.3.**

1. If  $T$  is orthogonal, then  $W^\perp$  is also  $T$ -invariant.
2. Use induction on  $n$  to show the main result.

**Ex 1.2.4.**

1.  $\text{ord}(a) = \text{ord}(a^{-1}) = n$ .
2.  $\langle a^r \rangle = \langle a^{\gcd(n, r)} \rangle$ .
3.  $\langle a^{r_1} \rangle = \langle a^{r_2} \rangle \iff \gcd(n, r_1) = \gcd(n, r_2)$ .
4.  $\forall m \mid n, \exists! H \leq \langle a \rangle$  s.t.  $|H| = m$ . Conversely, if  $H \leq \langle a \rangle$ , then  $|H| \mid n$ .

**Ex 1.2.5.** Show Prop 1.2.4.

**Ex 1.2.6.** TFAE

1.  $G$  is the internal direct product of  $G_1, G_2$ .
2.  $\forall a \in G, \exists! g_1 \in G_1, g_2 \in G_2$  s.t.  $a = g_1 g_2$  ;  $\forall g_1 \in G_1, g_2 \in G_2, g_1 g_2 = g_2 g_1$ .
3.  $G_1 \cap G_2 = \{1\}$  ;  $G = G_1 G_2$  ;  $\forall g_1 \in G_1, g_2 \in G_2, g_1 g_2 = g_2 g_1$ .



## 1.3 Week 3

### 1.3.1 Coset and Quotient Group

Let  $f : G_1 \rightarrow G_2$  be a group homo. Define  $\text{Im } f := f(G_1)$ .

Notice that  $\text{Im } f \leq G_2$ .

**pf:** Let  $z_1 = f(a_1), z_2 = f(a_2)$ , then  $z_1 z_2^{-1} = f(a_1) f(a_2)^{-1} = f(a_1) f(a_2^{-1}) = f(a_1 a_2^{-1}) \in \text{Im } f$ .  $\square$

**Def 16.**  $\text{Ker } f := \{x \in G_1 \mid f(x) = 1\} \leq G_1$ .

**Fact 1.3.1.**

1.  $x \in (\text{Ker } f)a \iff f(x) = f(a)$ .
2.  $\text{Ker } f = \{1\} \iff f$  is 1-1.

**Def 17.** Let  $H \leq G, \forall a \in G, Ha$  is called a **right coset** of  $H$  in  $G$ .

**Fact 1.3.2.**

1. For 2 right cosets  $Ha, Hb$ , either  $Ha = Hb$  or  $Ha \cap Hb = \phi$  must hold.
2.  $\{Ha : a \in G\}$  forms a partition of  $G$ .

**Theorem 2** (Lagrange). Let  $|G| < \infty$  and  $H \leq G, |H| \mid |G|$ .

**pf:**

$\square$

**Remark 5.**  $r$  is called the **index** of  $H$  in  $G$ , denoted by  $[G : H]$ . (The concept of index can be extended to infinite  $G, H$ .)

**Coro 1.3.1.** If  $|G| = p$  is a prime in  $\mathbb{Z}$ , then  $G$  is cyclic.

**pf:**

$\square$

**Coro 1.3.2.** If  $|G| < \infty, a \in G$ , then  $a^{|G|} = 1$ .

**pf:**

$\square$

**Remark 6.**

1. Let  $H \leq G, a \in G, aH$  is called a **left coset**.
2.  $\{\text{right cosets of } H\} \leftrightarrow \{\text{right cosets of } H\}$  by  $Ha \mapsto a^{-1}H$ .

Ques: How to make  $\{aH : a \in G\}$  to be a group? For  $aH, bH$ , we must have  $(aH)(bH) = abH$ . In general,  $(aH)(bH) = abH$  is not well-defined.

**Ex 1.3.1.** Let  $H = \langle (1\ 2) \rangle \leq S_3, a_1 = (1\ 3), a_2 = (1\ 2\ 3), b_1 = (1\ 3\ 2), b_2 = (2\ 3)$ . 出慘點

If we hope  $a_1 b_1 H = a_2 b_2 H$ , then we need  $(a_1 b_1)^{-1} a_2 b_2 \in H$ .

$$b_1^{-1} a_1^{-1} a_2 b_2 = b_1^{-1} b_2 b_2^{-1} a_1^{-1} a_2 b_2$$

Notice that  $b_1^{-1} b_2, a_1^{-1} a_2 \in H$ , so we need  $b_2^{-1} a_1^{-1} a_2 b_2 \in H$ .

**Def 18.** Let  $H \leq G$ .  $H$  is said to be **normal subgroup** of  $G$  if  $\forall g \in G, h \in H, g^{-1} h g \in H$  (or  $g^{-1} H g \subseteq H$ ), denoted by  $H \triangleleft G$ .

**Def 19.** Let  $H \triangleleft G$ . The set  $\{aH \mid a \in G\}$  forms a group under  $(aH)(bH) = abH, a, b \in G$ . We call it the **quotient group** of  $G$  by  $H$ , denoted by  $G/H$ .  
(Note: The identity is  $H = eH$  and  $(aH)^{-1} = a^{-1}H$ .)

**Remark 7.** Define  $q : G \rightarrow G/H, a \mapsto aH$ , called the quotient homomorphism.

Ques: How to find a normal subgroup of  $G$ ?

**Prop 1.3.1.**

1. If  $G$  is abelian, then  $\forall H \leq G \rightsquigarrow H \triangleleft G$ . (done by (c))
2. If  $H \leq G$  with  $[G : H] = 2$ , then  $H \triangleleft G$ .

**Eg 1.3.2.**  $n \leq 3, [S_n : A_n] = 2 \implies A_n \triangleleft S_n$ .

**pf:** We can write  $G = H \cup Ha = H \cup aH \implies aH = Ha, \forall a \notin H$ . □

**Def 20.** Define the center of  $G$  to be  $Z_G = \{a \in G \mid ax = xa, \forall x \in G\} \leq G$ .

**Prop 1.3.2.**

1.  $Z_G \triangleleft G$ . (by (c) and def.)
2. If  $G/Z_G$  is cyclic, then  $G$  is abelian.

**pf:** Let  $G/Z_G = \langle aZ_G \rangle$ , (let  $\bar{a} := aZ_G$ ) for some  $a \in G$ . For  $x_1, x_2 \in G$ , let  $x_1 = a^{k_1}z_1, x_2 = a^{k_2}z_2$ , then  $x_1x_2 = a^{k_1+k_2}z_1z_2 = x_2x_1$ . ( $z_i$  可以各種交換) □

**Def 21.** The commutator of  $G$  is define to be  $[G, G] = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$ .

**Prop 1.3.3.**  $[G, G] \triangleleft G ; [G, G] = 1 \iff G$  is abelian.

**pf:**  $\forall x \in G, a \in [G, G], xax^{-1} = xax^{-1}a^{-1}a$  and  $xax^{-1}a^{-1}, a \in [G, G]$ . □

**Ex 1.3.1.** no subgroup of  $A_4$  has order 6. (converse of Lagrange thm. is false.)

**Ex 1.3.2.** Let  $H \leq G$ . Then TFAE

- (a)  $H \triangleleft G$ .
- (b)  $\forall x \in G, xHx^{-1} = H$ .
- (c)  $\forall x \in G, xH = Hx$ .
- (d)  $\forall x, y \in G, (xH)(yH) = (xy)H$ .

**Ex 1.3.3.**

1. If  $H \leq S_n$  and  $\exists \sigma \in H$  is odd, then  $[H : H \cap A_n] = 2$ .
2. For  $n \geq 3, [S_n, S_n] = A_n$ .

**Ex 1.3.4.** Let  $H \leq G$ . Then  $H \triangleleft G$  and  $G/H$  is abelian  $\iff [G, G] \leq H$ . (hint:  $G/[G, G]$  is "max" among all abelian quotient groups)