

# Algebra

May 9, 2017

# 1 Group theory

## 1.1 Week 1

**Def 1.** A non-empty set  $G$  with a binary function  $f : G \times G \rightarrow G, (a, b) \mapsto ab$  is a **group** if it satisfies

1.  $(ab)c = a(bc)$ .
2.  $\exists 1 \in G$  s.t.  $1a = a1 = a, \forall a \in G$ .
3.  $\exists a^{-1} \in G$  s.t.  $aa^{-1} = a^{-1}a = 1$ .

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**Def 2.** Let  $G$  be a group. Then  $G$  is said to be **abelian** if  $\forall a, b \in G, ab = ba$ .

**Ex 1.1.1.** Let  $G$  be a semigroup. Then TFAE (the following are equivalent)

1.  $G$  is a group.
2. For all  $a, b \in G$  and the equations  $bx = a, yb = a$ , each of them has a solution in  $G$ .
3.  $\exists e \in G$  s.t.  $ae = a \forall a \in G$  and if we fix such  $e$ , then  $\forall b \in G \exists b' \in G$  s.t.  $bb' = e$ .

**Ex 1.1.2.** Let  $G$  be a group. Show that

1.  $\forall a \in G, a^2 = 1$ , then  $G$  is abelian.
2.  $G$  is abelian  $\iff \forall a, b \in G, (ab)^n = a^n b^n$  for three consecutive integer  $n$ .

**Def 3.** Let  $G$  be a group and  $H \subseteq G, H \neq \emptyset$ . Then  $H$  is said to be a subgroup of  $G$ , denoted by  $H \leq G$ , if

1.  $\forall a, b \in H, ab \in H$ .
2.  $1 \in H$ .
3.  $\forall a \in H, a^{-1} \in H$ .

useful criterion:  $H \leq G \iff \forall a, b \in H, ab^{-1} \in H$ .

*Proof.*

$\Rightarrow b \in H \implies b^{-1} \in H$ , and  $a \in H$ , so  $ab^{-1} \in H$ .

$\Leftarrow$  1.  $H \neq \emptyset \implies \exists a \in H \implies aa^{-1} = 1 \in H$ .

2.  $1, a \in H \implies 1a^{-1} = a^{-1} \in H$ .

3.  $a, b^{-1} \in H \implies a(b^{-1})^{-1} = ab \in H$ . □

**Eg 1.1.1.**  $(\mathbb{Z}, +, 0) \leq (\mathbb{Q}, +, 0) \leq (\mathbb{R}, +, 0) \leq (\mathbb{C}, +, 0) ; (\mathbb{Q}^\times, \times, 1) \leq (\mathbb{R}^\times, \times, 1) \leq (\mathbb{C}^\times, \times, 1)$

**Eg 1.1.2.**

- Special linear group  $\text{SL}(n, \mathbb{F}) = \{ A \in \text{GL}(n, \mathbb{F}) \mid \det A = 1 \}$
- Orthogonal group  $\text{O}(n) = \{ A \in \text{GL}(n, \mathbb{R}) \mid A^t A = I_n \}$
- Unitary group  $\text{U}(n) = \{ A \in \text{GL}(n, \mathbb{C}) \mid A^* A = I_n \}$
- Special orthogonal group  $\text{SO}(n) = \text{SL}(n, \mathbb{R}) \cap \text{O}(n)$

- Special unitary group  $SU(n) = SL(n, \mathbb{C}) \cap U(n)$

**Def 4.** Let  $f : G_1 \rightarrow G_2$ .  $f$  is called an **isomorphism** if

1.  $f$  is 1-1 and onto.
2.  $\forall a, b \in G_1, f(ab) = f(a)f(b)$ . (**homomorphism**)

, denoted by  $G_1 \cong G_2$ .

**Remark 1.** (practice)

1.  $f(1) = 1$ .
2.  $f(a^{-1}) = f(a)^{-1}$ .
3. If  $f$  is an isomorphism, then  $\exists f^{-1}$  is also a homomorphism.

**Eg 1.1.3.**

- $U(1) = \{ z \in \mathbb{C}^\times \mid \bar{z}z = 1 \}, z = \cos \theta + \sin \theta i$
- $SO(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$

notice that  $U(1) \cong SO(2)$ .  $S^1 = \{ (a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1 \}$ , 可被賦予群的結構.

**Eg 1.1.4.** Let  $A \in SU(2) \implies A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \alpha\bar{\alpha} + \beta\bar{\beta} = 1, \alpha, \beta \in \mathbb{C}$ .

Quaternion(四元數):  $\mathbb{H} = \{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \}$  with  $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j (\implies ij = -ji)$ .

Let  $x = a + bi + cj + dk, \bar{x} = a - bi - cj - dk$ , then  $N(x) = x\bar{x} = a^2 + b^2 + c^2 + d^2$ . For  $x \neq 0, N(x) \neq 0, x^{-1} = \frac{1}{N(x)}\bar{x}$

Now, for  $x = a + bi + cj + dk = (a + bi) + (c + di)j$ . So  $SU(2) \cong \{ x \in \mathbb{H}^\times \mid N(x) = 1 \}$ .  $S^3 = \{ (a, b, c, d) \in \mathbb{R}^4 \mid a^2 + b^2 + c^2 + d^2 = 1 \}$ , 可被賦予群的結構.

★ The only spheres with continuous group law are  $S^1, S^3$ .

**Ex 1.1.3.** Find a way to regard  $M_{n \times n}(\mathbb{H})$  as a subset of  $M_{2n \times 2n}(\mathbb{C})$ , which preserves addition and multiplication, and then there is a way to characterize  $GL(n, \mathbb{H})$ .

**Def 5** (symplectic group).  $Sp(n, \mathbb{F}) = \{ A \in GL(2n, \mathbb{F}) \mid A^t J A = J \}$  where  $J = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}$ .

( $A^t J A = J$  preserving non-degenerate skew-symmetric forms)

$Sp(n) = \{ A \in GL(n, \mathbb{H}) \mid A^* A = I_n \}$ .

**Ex 1.1.4.** Show  $Sp(n) \cong U(2n) \cap Sp(n, \mathbb{C})$ .

Ques: Find the smallest subgroup of  $SU(2)$  containing  $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ .

## 1.2 Week 2

### 1.2.1 Permutation groups and Dihedral groups

**Def 6.** A permutation of a set  $B$  is a 1-1 and onto function from  $B$  to  $B$ .

Let  $S_B :=$  the set of permutations of  $B$ . Then  $(S_B, \cdot, \text{Id}_B)$  forms a group.

If  $B = \{a_1, \dots, a_n\}$ , then  $S_B \cong S_{\{1, \dots, n\}}$  and write  $S_n = S_{\{1, \dots, n\}}$ , called the symmetric group of degree  $n$ .

**Theorem 1** (Cayley theorem). Any group is isomorphic to a subgroup of some permutation group.

(Hint): Let  $G$  be a group. Set  $B = G$ . Consider  $a \in G$  as  $\sigma_a : G \rightarrow G, x \mapsto ax$ . Then  $\sigma_a \in S_G \implies G \leq S_G$ .

**Fact 1.2.1.**  $S_n$  is a finite group of order  $n!$ , i.e.  $|S_n| = n!$ .

*Proof.* EASY =O □

Cyclic notation:  $\sigma \in S_5$ , say  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$ . Write  $\sigma = (1\ 4)(2\ 3\ 5)$ .

$\Rightarrow$  Any permutation can be written as a product of disjoint cycles.

**Eg 1.2.1.** In  $S_7$ ,  $\sigma_1 = (1\ 2\ 3)(4\ 5\ 6)(7)$ ,  $\sigma_2 = (1\ 3\ 5\ 6)(2\ 4\ 7)$ .

Then  $\sigma_1\sigma_2 = (2\ 5\ 4\ 7\ 3\ 6)$ ,  $\sigma_1^{-1} = (1\ 3\ 2)(4\ 6\ 5)$ .

**Def 7.** A 2 cycle is called a **transposition**.

**Eg 1.2.2.**  $(1\ 2\ 3) = (1\ 3)(1\ 2)$ ,  $(1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2)$ .

Any permutation is a product of 2 cycles.

Useful formula:  $\sigma \in S_n$ ,  $\sigma(j_1 \dots j_m)\sigma^{-1} = (\sigma(j_1) \dots \sigma(j_m))$ .

**Eg 1.2.3.** Let  $\sigma = (1\ 2\ 3)(4\ 5\ 6\ 7)$ ,  $\sigma(2\ 3\ 4)\sigma^{-1} = (3\ 1\ 5)$ .

*Proof.* Note that both sides are functions. For  $i \in \{1, \dots, n\}$ ,

Case 1:  $\exists k$  s.t.  $\sigma(j_k) = i$ , CONCON

Case 2: Otherwise, CONCON □

**Fact 1.2.2.**  $S_n = \langle (1\ 2), \dots, (1\ n) \rangle$ .

*Proof.*  $(1\ i)^{-1} = (1\ i)$  and  $(i\ j) = (1\ i)(1\ j)(1\ i)^{-1}$ . □

**Def 8.** Let  $G$  be a group and  $S \subset G$ . The subgroup generated by  $S$  defined to be the smallest subgroup of  $G$  which contains  $S$ , denoted by  $\langle S \rangle$ .

**Ex 1.2.1.**

1.  $S_n = \langle (1\ 2), (2\ 3), \dots, (n-1\ n) \rangle, \quad n \geq 2.$
2.  $S_n = \langle (1\ 2), (1\ 2 \dots n) \rangle, \quad n \geq 2.$

**Def 9.**  $A_n = \{\text{even permutations of } S_n\} \leq S_n, |A_n| = \frac{n!}{2}.$

**Ex 1.2.2.**

1.  $A_n = \langle (1\ 2\ 3), (1\ 2\ 4), \dots, (1\ 2\ n) \rangle, n \geq 3.$
2.  $A_n = \langle (1\ 2\ 3), (2\ 3\ 4), \dots, (n-2\ n-1\ n) \rangle, n \geq 3.$

**Remark 2.**  $\langle S \rangle = \bigcap_{S \subseteq H \leq G} H = \{a_1 a_2 \dots a_k \mid k \in \mathbb{N}, a_i \in S \cup S^{-1}\} \cup \{1\}$

The orthogonal transformations on  $\mathbb{R}^2$ :  $O(2)$ .

Let  $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in O(2).$

略... (這邊討論旋轉和反射的矩陣)

Case 1:  $A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$  is counterclockwise rotation w.r.t.  $\alpha$ .

Case 2:  $A = \begin{pmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{pmatrix}$  is the reflection.  $A^2 = I_2 \implies$  eigenvalues are  $\pm 1$ .

Easy to show that  $L_A(v) = v - 2\langle v, v_2 \rangle v_2$ .

$O(2) = \{\text{rotations}\} \cup \{\text{reflections}\}.$

**Def 10.** The dihedral group  $D_n$  is the group of symmetries of a regular  $n$ -gon.

In general,  $D_n = \langle T, R \mid T^n = 1, R^2 = 1, TR = RT^{-1} \rangle \leq O(2) \leq S_n, |D_n| = 2n.$

**Def 11.** Let  $T$  be a linear transformation from  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ .

- $T$  is called a rotation if  $\exists$  a  $T$ -invariant subspace  $W \subseteq \mathbb{R}^n$  with  $\dim W = 2$  s.t.  $\begin{cases} T|_W \text{ is a rotation} \\ T|_{W^\perp} = \text{id}_{W^\perp} \end{cases}$
- $T$  is called a reflection if  $\exists$  a  $T$ -invariant subspace  $W \subseteq \mathbb{R}^n$  with  $\dim W = 1$  s.t.  $\begin{cases} T|_W = -\text{id}_W \\ T|_{W^\perp} = \text{id}_{W^\perp} \end{cases}$

Main result: the group of orthogonal transformations =  $\langle \text{rotations, reflections} \rangle$ .

**Prop 1.2.1.** For  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\exists$  a  $T$ -invariant subspace  $W \subseteq \mathbb{R}^n$  with  $1 \leq \dim W \leq 2$ .

*Proof.* Let  $A = [T]_\alpha \in M_{n \times n}(\mathbb{R}) \subseteq M_{n \times n}(\mathbb{C})$ . Consider  $\widetilde{L}_A : \mathbb{C}^n \rightarrow \mathbb{C}^n, v \mapsto Av$ .

Then  $\exists$  an eigenvalue  $\lambda \in \mathbb{C}$  and an eigenvector  $v \in \mathbb{C}^n$  for  $\widetilde{L}_A$ . Let  $\lambda = \lambda_1 + \lambda_2 i, v = v_1 + v_2 i$ . By definition, we have

$$Av = \widetilde{L}_A(v) = \lambda v = (\lambda_1 + \lambda_2 i)(v_1 + v_2 i) \implies \begin{cases} Av_1 = \lambda_1 v_1 - \lambda_2 v_2 \\ Av_2 = \lambda_2 v_1 + \lambda_1 v_2 \end{cases},$$

so  $W = \langle v_1, v_2 \rangle$ . □

**Ex 1.2.3.**

1. If  $T$  is orthogonal, then  $W^\perp$  is also  $T$ -invariant.
2. Use induction on  $n$  to show the main result.

For  $n = 3, A \in O(3)$ , we have  $A \sim \begin{pmatrix} \cos \alpha & -\sin \alpha & \\ \sin \alpha & \cos \alpha & \\ & & \pm 1 \end{pmatrix}$ .

### 1.2.2 Cyclic groups and internal direct product

**Def 12.** If  $G = \langle a \rangle = \{\dots, a^{-2}, a^{-1}, a, 1, a, a^2, \dots\} = \{a^n \mid n \in \mathbb{Z}\}$ , then  $G$  is a cyclic group generated by  $a$ .

**Eg 1.2.4.**  $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$ .

**Eg 1.2.5.** Let  $A = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} \in SO(2)$ . Then  $\langle A \rangle = \{I_2, A, A^2, \dots, A^{n-1}\}$  and  $A^n = I_2, A^m = A^r$  where  $m \equiv r \pmod{n}$ .

**Eg 1.2.6.**  $\mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \dots, \overline{(n-1)}\}$  with  $\bar{j} = \{m \in \mathbb{Z} \mid m \equiv j \pmod{n}\}$ .

Define  $\bar{i} + \bar{j} = \begin{cases} \overline{i+j} & \text{if } 0 \leq i+j \leq n \\ \overline{i+j-n} & \text{otherwise} \end{cases} \implies (\mathbb{Z}/n\mathbb{Z}, +, \bar{0}) \text{ forms a group.}$

**Remark 3.**  $\bar{i} \times \bar{j} = \overline{i \times j}$ .

- 略
- If  $\gcd(j, n) = d, \exists h, k \in \mathbb{Z}$  s.t.  $hj + kn = d$ .

**Def 13.**  $(\mathbb{Z}/n\mathbb{Z})^\times = \{j \in \mathbb{Z}/n\mathbb{Z} \mid \gcd(j, n) = 1\} \implies ((\mathbb{Z}/n\mathbb{Z})^\times, \times, \bar{1}) \text{ forms a group.}$

**Eg 1.2.7.** 略... 简化剩餘系, 原根 (generator)  $(1, 2, 4, p^k, 2p^k, p \text{ is an odd prime})$

**Def 14.**

- The **order** of a finite group  $G$  is the number of elements in  $G$ , denoted by  $|G|$ .
- Let  $a \in G$ , the order of  $a$  is defined to be the least positive integer  $n$  s.t.  $a^n = 1$ , denoted by  $\text{ord}(a) = n$ .
- If  $a^n \neq 1 \quad \forall n \in \mathbb{N}$ , then we call “ $a$  has infinite order”.

**Prop 1.2.2.** Let  $G = \langle a \rangle$  with  $\text{ord}(a) = n$ . Then

1.  $a^m = 1 \iff n \mid m$ .

*Proof.*

$\Leftarrow$ : Let  $m = dn$ , then  $a^m = (a^n)^d = 1$ .

$\Rightarrow$ : Let  $m = qn + r, 0 \leq r < n$ . If  $r \neq 0$ , then  $a^r = a^{m-qn} = (a^m)(a^n)^{-q} = 1$ . But  $r < n$ , which is a contradiction. Hence  $r = 0 \implies n \mid m$ .  $\square$

2.  $\text{ord}(a^r) = n / \gcd(r, n)$ .

*Proof.* Let  $\gcd(r, n) = d, n = dn', r = dr'$  with  $\gcd(n', r') = 1$ . Plan to show “ $\text{ord}(a^r) = n'$ .”

- $(a^r)^{n'} = a^{r'dn'} = (a^n)^{r'} = 1 \implies \text{ord}(a^r) \mid n'$ .
- $1 = (a^r)^{\text{ord}(a^r)} = a^{r \text{ord}(a^r)} \implies n \mid r \text{ord}(a^r) \implies n' \mid r' \text{ord}(a^r) \implies n' \mid \text{ord}(a^r)$ .

□

**Prop 1.2.3.** Any subgroup of a cyclic group is cyclic.

*Proof.* Let  $G = \langle a \rangle$  and  $H \leq G$ . If  $H = \{1\}$ , then  $H = \langle 1 \rangle$ , done!

Otherwise,  $d = \min\{m \in \mathbb{N} \mid a^m \in H\}$ , by well-ordering axiom. Claim  $H = \langle a^d \rangle$ .

⊃:  $a^d \in H$  by the definition of  $d$ .

⊆:  $\forall a^m \in H$ , write  $m = qd + r, 0 \leq r < d$ . If  $r \neq 0$ , then  $a^r = a^{m-qd} = a^m(a^d)^{-q} \in H$ , which is a contradiction. Hence  $r = 0 \implies d \mid m$ .

□

**Ex 1.2.4.**

1.  $\text{ord}(a) = \text{ord}(a^{-1}) = n$ .
2.  $\langle a^r \rangle = \langle a^{\gcd(n, r)} \rangle$ .
3.  $\langle a^{r_1} \rangle = \langle a^{r_2} \rangle \iff \gcd(n, r_1) = \gcd(n, r_2)$ .
4.  $\forall m \mid n, \exists ! H \leq \langle a \rangle$  s.t.  $|H| = m$ . Conversely, if  $H \leq \langle a \rangle$ , then  $|H| \mid n$ .

**Prop 1.2.4.** Let  $G = \langle a \rangle$ . Then

1.  $\text{ord}(a) = n \implies G \cong \mathbb{Z}/n\mathbb{Z}$
2.  $\text{ord}(a) = \infty \implies G \cong \mathbb{Z}$

**Ex 1.2.5.** Show Prop 1.2.4.

**Def 15.** Let  $G_1, G_2 \leq G$ .  $G$  is the internal direct product of  $G_1, G_2$  if  $G_1 \times G_2 \rightarrow G, (g_1, g_2) \mapsto g_1 g_2$  is an isom.

**Remark 4.** In this case, we find that

- $G = G_1 G_2 = \{g_1 g_2 \mid g_1 \in G_1, g_2 \in G_2\}$ .
- $G_1 \cap G_2 = \{1\}$ . (consider  $a \neq 1 \in G_1 \cap G_2$ , then  $(1, a) \mapsto a, (a, 1) \mapsto a$ , but the function is 1-1, which is a contradiction.)
- If  $a \in G$  with  $a = g_1 g_2 = g'_1 g'_2$ , then  $(g'_1)^{-1} g_1 = (g'_2) g_2^{-1} \in G_1 \cap G_2 = \{1\} \implies \begin{cases} g_1 = g'_1 \\ g_2 = g'_2 \end{cases}$ .
- For  $g_1 \in G_1, g_2 \in G_2, (g_1, g_2) = (g_1, 1)(1, g_2) = (1, g_2)(g_1, 1) \implies g_1 g_2 = g_2 g_1$ .

**Ex 1.2.6.** TFAE

1.  $G$  is the internal direct product of  $G_1, G_2$ .
2.  $\forall a \in G, \exists ! g_1 \in G_1, g_2 \in G_2$  s.t.  $a = g_1 g_2$ ;  $\forall g_1 \in G_1, g_2 \in G_2, g_1 g_2 = g_2 g_1$ .
3.  $G_1 \cap G_2 = \{1\}$ ;  $G = G_1 G_2$ ;  $\forall g_1 \in G_1, g_2 \in G_2, g_1 g_2 = g_2 g_1$ .

**Eg 1.2.8.**

1.  $G = \mathbb{Z}/6\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ ,  $G_1 = \{\bar{0}, \bar{3}\}$ ,  $G_2 = \{\bar{0}, \bar{2}, \bar{4}\}$ . We have  $G \cong G_1 \times G_2$ .
2.  $G = S_3$ ,  $G_1 = \langle (1\ 2) \rangle$ ,  $G_2 = \langle (1\ 2\ 3) \rangle$ . We have  $G_1 \times G_2 \not\cong G$  since  $(1\ 2)(1\ 2\ 3) \neq (1\ 2\ 3)(1\ 2)$ .

**Eg 1.2.9.**  $G = S_3$ ,  $G_1 = \langle (1\ 2) \rangle$ ,  $G_2 = \langle (2\ 3) \rangle$ ,  $G_1 G_2 = \{1, (1\ 2), (2\ 3), (1\ 2\ 3)\} \not\leq G$  since  $(1\ 3\ 2) = (1\ 2\ 3)^{-1} \notin G_1 G_2$ .

**Prop 1.2.5.** Let  $H, K \leq G$ . Then  $HK \leq G \iff HK = KH$ .

*Proof.*

$$\Rightarrow: \begin{cases} H \leq HK \\ K \leq HK \end{cases} \implies KH \subseteq HK ; \forall hk \in HK, \exists h'k' \in HK \text{ s.t. } (hk)(h'k') = 1 \implies hk = (k')^{-1}(h')^{-1} \in KH \implies HK \subseteq KH.$$

$$\Leftarrow: \text{ For } h_1k_1, h_2k_2 \in HK, (h_1k_1)(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1h'k' \in HK.$$

□



## 1.3 Week 3

### 1.3.1 Coset and Quotient Group

Let  $f : G_1 \rightarrow G_2$  be a group homo. Define  $\text{Im } f := f(G_1)$ .

Notice that  $\text{Im } f \leq G_2$ .

*Proof.* Let  $z_1 = f(a_1), z_2 = f(a_2)$ , then  $z_1 z_2^{-1} = f(a_1) f(a_2)^{-1} = f(a_1) f(a_2^{-1}) = f(a_1 a_2^{-1}) \in \text{Im } f$ .  $\square$

**Def 16.**  $\ker f := \{x \in G_1 \mid f(x) = 1\} \leq G_1$ .

**Fact 1.3.1.**

1.  $x \in (\ker f)a \iff f(x) = f(a)$ .
2.  $\ker f = \{1\} \iff f$  is 1-1.

**Def 17.** Let  $H \leq G, \forall a \in G, Ha$  is called a **right coset** of  $H$  in  $G$ .

**Fact 1.3.2.**

1. For 2 right cosets  $Ha, Hb$ , either  $Ha = Hb$  or  $Ha \cap Hb = \phi$  must hold.
2.  $\{Ha : a \in G\}$  forms a partition of  $G$ .

**Theorem 2** (Lagrange). Let  $|G| < \infty$  and  $H \leq G, |H| \mid |G|$ .

*Proof.*  $\square$

**Remark 5.**  $r$  is called the **index** of  $H$  in  $G$ , denoted by  $[G : H]$ . (The concept of index can be extended to infinite  $G, H$ .)

**Ex 1.3.1.** no subgroup of  $A_4$  has order 6. (converse of Lagrange thm. is false.)

**Coro 1.3.1.** If  $|G| = p$  is a prime in  $\mathbb{Z}$ , then  $G$  is cyclic.

*Proof.*  $\square$

**Coro 1.3.2.** If  $|G| < \infty, a \in G$ , then  $a^{|G|} = 1$ .

*Proof.*  $\square$

**Remark 6.**

1. Let  $H \leq G, a \in G, aH$  is called a **left coset**.
2.  $\{\text{right cosets of } H\} \leftrightarrow \{\text{right cosets of } H\}$  by  $Ha \mapsto a^{-1}H$ .

Ques: How to make  $\{aH : a \in G\}$  to be a group? For  $aH, bH$ , we must have  $(aH)(bH) = abH$ .

In general,  $(aH)(bH) = abH$  is not well-defined.

**Ex 1.3.1.** Let  $H = \langle (1\ 2) \rangle \leq S_3, a_1 = (1\ 3), a_2 = (1\ 2\ 3), b_1 = (1\ 3\ 2), b_2 = (2\ 3)$ . 出慘點

If we hope  $a_1b_1H = a_2b_2H$ , then we need  $(a_1b_1)^{-1}a_2b_2 \in H$ .

$$b_1^{-1}a_1^{-1}a_2b_2 = b_1^{-1}b_2b_2^{-1}a_1^{-1}a_2b_2$$

Notice that  $b_1^{-1}b_2, a_1^{-1}a_2 \in H$ , so we need  $b_2^{-1}a_1^{-1}a_2b_2 \in H$ .

**Def 18.** Let  $H \leq G$ .  $H$  is said to be **normal subgroup** of  $G$  if  $\forall g \in G, h \in H, g^{-1}hg \in H$  (or  $g^{-1}Hg \subseteq H$ ), denoted by  $H \triangleleft G$ .

**Def 19.** Let  $H \triangleleft G$ . The set  $\{aH \mid a \in G\}$  forms a group under  $(aH)(bH) = abH, a, b \in G$ . We call it the **quotient group** of  $G$  by  $H$ , denoted by  $G/H$ .

(Note: The identity is  $H = hH$  and  $(aH)^{-1} = a^{-1}H$ .)

**Remark 7.** Define  $q : G \rightarrow G/H, a \mapsto aH$ , called the quotient homomorphism.

**Ex 1.3.2.** Let  $H \leq G$ . Then TFAE

- (a)  $H \triangleleft G$ .
- (b)  $\forall x \in G, xHx^{-1} = H$ .
- (c)  $\forall x \in G, xH = Hx$ .
- (d)  $\forall x, y \in G, (xH)(yH) = (xy)H$ .

Ques: How to find a normal subgroup of  $G$ ?

**Prop 1.3.1.**

- 1. If  $G$  is abelian, then  $\forall H \leq G \rightsquigarrow H \triangleleft G$ . (done by (c))
- 2. If  $H \leq G$  with  $[G : H] = 2$ , then  $H \triangleleft G$ .

**Ex 1.3.2.**  $n \leq 3, [S_n : A_n] = 2 \implies A_n \triangleleft S_n$ .

*Proof.* We can write  $G = H \cup Ha = H \cup aH \implies aH = Ha, \forall a \notin H$ . □

**Def 20.** Define the center of  $G$  to be  $Z_G = \{a \in G \mid ax = xa, \forall x \in G\} \leq G$ .

**Prop 1.3.2.**

- 1.  $Z_G \triangleleft G$ . (by (c) and def.)
- 2. If  $G/Z_G$  is cyclic, then  $G$  is abelian.

*Proof.* Let  $G/Z_G = \langle aZ_G \rangle$ , (let  $\bar{a} := aZ_G$ ) for some  $a \in G$ . For  $x_1, x_2 \in G$ , let  $x_1 = a^{k_1}z_1, x_2 = a^{k_2}z_2$ , then  $x_1x_2 = a^{k_1+k_2}z_1z_2 = x_2x_1$ . ( $z_i$  可以各種交換) □

**Def 21.** The commutator of  $G$  is define to be  $[G, G] = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$ .

**Prop 1.3.3.**  $[G, G] \triangleleft G ; [G, G] = 1 \iff G$  is abelian.

*Proof.*  $\forall x \in G, a \in [G, G], xax^{-1} = xax^{-1}a^{-1}a$  and  $xax^{-1}a^{-1}, a \in [G, G]$ . □

**Ex 1.3.3.**

- 1. If  $H \leq S_n$  and  $\exists \sigma \in H$  is odd, then  $[H : H \cap A_n] = 2$ .

2. For  $n \geq 3$ ,  $[S_n, S_n] = A_n$ .

**Ex 1.3.4.** Let  $H \leq G$ . Then  $H \triangleleft G$  and  $G/H$  is abelian  $\iff [G, G] \leq H$ . (hint:  $G/[G, G]$  is "max" among all abelian quotient groups)

### 1.3.2 Isomorphism theorems & Factor theorem

**Theorem 3** (1st isomorphism theorem). Let  $f : G_1 \rightarrow G_2$  be a group homo. Then  $G_1/\ker f \cong \text{Im } f$ .

*Proof.* Define  $\varphi : a \ker f \mapsto f(a)$ .

- well-defined:  $a \ker f = b \ker f \implies a^{-1}b \in \ker f \implies f(a^{-1}b) = 1 \implies f(a)^{-1}f(b) = 1 \implies f(a) = f(b)$ .
- group homo:  $\varphi((a \ker f)(b \ker f)) = \varphi(ab \ker f) = f(ab) = f(a)f(b) = \varphi(a \ker f)\varphi(b \ker f)$ .
- onto: by def. of  $\text{Im } f$ .
- 1-1:  $f(a) = f(b) \implies a \ker f = b \ker f$  (easy).

□

**Theorem 4** (Factor theorem). Let  $f : G_1 \rightarrow G_2$  be a group homo. and  $H \triangleleft G_1, H \leq \ker f$ . Then  $\exists$  a group homo.  $\varphi : G/H \rightarrow G_2$  s.t.

$$\begin{array}{ccc} G_1 & \xrightarrow{q} & G/H \\ & \searrow f & \downarrow \varphi \\ & & G_2 \end{array}$$

**Eg 1.3.3.** Let  $G = \langle a \rangle$  with  $\text{ord}(a) = n$ . Then  $G \cong \mathbb{Z}/n\mathbb{Z}$ . (1st isom. thm.)

**Eg 1.3.4.**  $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}, 4\mathbb{Z} \leq 2\mathbb{Z}$ , so by factor thm.,  $\mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ .

**Eg 1.3.5.**  $\det : \text{GL}(n, \mathbb{F}) \rightarrow \mathbb{F}^\times \implies \text{GL}(n, \mathbb{F})/\text{SL}(n, \mathbb{F}) \cong \mathbb{F}^\times$

**Eg 1.3.6.**  $\text{sgn} : S_n \rightarrow \{\pm 1\} \implies S_n/A_n \cong \{\pm 1\}$

**Theorem 5** (2nd isomorphism theorem). Let  $H \leq G, K \triangleleft G$ . Then  $HK/K \cong H/H \cap K$ .

*Proof.* First,  $\begin{cases} H \leq G \\ K \triangleleft G \end{cases} \implies HK = KH \implies HK \leq G; K \triangleleft G \implies K \triangleleft HK$ .

Define  $\varphi : H \rightarrow HK/K, h \mapsto hK$ . which is a group homo.

- onto:  $\forall (hk)K, hK = hK$ , so  $\varphi(h) = hK = hkK$ .
- Find  $\ker \varphi$ :  $a \in \ker \varphi \iff \begin{cases} a \in H \\ aK = K \end{cases} \iff a \in H \cap K$ , so  $\ker \varphi = H \cap K$ .

Then by 1st isom. thm.

□

**Eg 1.3.7.**  $G = \text{GL}(2, \mathbb{C}), H = \text{SL}(2, \mathbb{C}), K = \mathbb{C}^\times I_2 = Z_G \triangleleft G$ .

By 2nd isom. thm.,  $G/K \cong H/\{\pm I_2\}$ . ( $G = HK, \{\pm I_2\} = H \cap K$ )

projective linear group:  $\text{PGL}(2, \mathbb{C}) = G/K$ .

projective special linear group:  $\text{PSL}(2, \mathbb{C}) = H/H \cap K$ .

齊次座標...OTL

**Ex 1.3.5.**

1. Let  $H_1 \triangleleft G_1, H_2 \triangleleft G_2$ . Then  $(H_1 \times H_2) \triangleleft (G_1 \times G_2)$  and  $G_1 \times G_2 / H_1 \times H_2 \cong G_1 / H_1 \times G_2 / H_2$ .
2. Let  $H \triangleleft G, K \triangleleft G$  s.t.  $G = HK$ . Then  $G/H \cap K \cong G/H \times G/K$ .

**Ex 1.3.6.** Let  $H \triangleleft G$  with  $[G : H] = p$ , which is a prime in  $\mathbb{Z}$ . Then  $\forall K \leq G$ , either (1)  $K \leq H$  or (2)  $G = HK$  and  $[K : K \cap H] = p$ .

**Theorem 6** (3rd isomorphism theorem). Let  $K \triangleleft G$ .

1. There is a 1-1 correspondence between  $\{H \leq G \mid K \leq H\}$  and  $\{\text{subgroups of } G/K\}$ . ( $H \triangleleft G$  ... normal)

*Proof.* Define  $\varphi : H \mapsto H/K$ . ( $H/K \leq G/K$ )

- 1-1: Assume  $H_1/K = H_2/K$ . For  $a \in H_1$ ,  $aK \in H_1/K = H_2/K$ . so  $\exists b \in H_2$  s.t.  $aK = bK \implies b^{-1}a \in K \leq H_2 \implies a \in bH_2 = H_2$ . So  $H_1 \leq H_2$ . By symmetry,  $H_2 \leq H_1$ , and thus  $H_1 = H_2$ .
- onto: Given a subgroup  $Q$  of  $G/K$ , consider  $H = q^{-1}(Q)$  where  $q : G \rightarrow G/K$ .
  - $H \leq G$ :  $\forall a, b \in H, q(a), q(b) \in Q \implies q(a)q(b)^{-1} \in Q \implies q(ab^{-1}) \in Q \implies ab^{-1} \in H \implies H \leq G$ .
  - $K \leq H$ :  $\forall a \in K, q(a) = aK = K \in Q \implies a \in H \implies K \leq H$ .
  - $Q = H/K$ :  $\forall aK \in Q, aK = q(a) \implies a \in H \implies aK \in H/K \implies Q \subseteq H/K$ .  
And  $\forall aK \in H/K (a \in H), q(a) \in Q \implies H/K \subseteq Q$ . So  $Q = H/K$ .
- $H \triangleleft G, K \leq H \iff \forall g \in G, gHg^{-1} = H, K \leq H \iff \forall \bar{g} \in G/K, \bar{g}(H/K)\bar{g}^{-1} = H/K \iff H/K \triangleleft G/K$ .  $\square$

2. If  $H \triangleleft G$  with  $K \leq H$ , then  $(G/K)/(H/K) \cong G/H$ .

*Proof.* Define  $\varphi : G \rightarrow (G/K)/(H/K)$  with  $\varphi : a \mapsto aK(H/K)$ .

- onto: ... easy.
- Find  $\ker \varphi$ :  $a \in \ker \varphi \iff aK(H/K) = H/K \iff aK \in H/K \iff a \in H$ .

By 1st isom. thm.,  $(G/K)/(H/K) \cong G/H$ .  $\square$

**Fig 1.3.8.**  $m\mathbb{Z} + n\mathbb{Z}/m\mathbb{Z} \cong n\mathbb{Z}/m\mathbb{Z} \cap n\mathbb{Z}$ . ( $m\mathbb{Z} + n\mathbb{Z} = \gcd(m, n)\mathbb{Z}, m\mathbb{Z} \cap n\mathbb{Z} = \text{lcm}(m, n)\mathbb{Z}$ )

Ques:  $G/K \cong G'/K'$  and  $K \cong K' \not\Rightarrow G \cong G'$ .

**Fig 1.3.9.**  $Q_8$  and  $D_4$  交給陳力

Extension problem: given two groups  $A, B$ , how to find  $G$  and  $K \triangleleft G$ , s.t.  $K \cong A, G/K \cong B$ ?  
( $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ , short exact sequence)

(e.g.  $G = A \times B, K = A \times \{1\}$ )

## 1.4 Week 4

### 1.4.1 Universal property and direct sum & product

In general, let  $f_1 : G_1 \rightarrow G, f_2 : G_2 \rightarrow G$  are group homo.  $f_1 \times f_2 : G_1 \times G_2 \rightarrow G, (a, b) \mapsto f_1(a)f_2(b)$ . But we have  $(a, b) = (a, 1)(1, b) = (1, b)(a, 1)$ , so  $f_1(a)f_2(b) = f_2(b)f_1(a) \implies$  need  $G$  to be abelian.

So we intend to define the direct sum in the category of abelian group.

Notation: For abelian groups, we use “+” to denote the group operation and “0” to denote the identity.

**Def 22.** Given a non-empty family of abelian groups  $\{G_s \mid s \in \Lambda\}$ , a (external) direct sum of  $\{G_s \mid s \in \Lambda\}$  is an abelian group  $\bigoplus_{s \in \Lambda} G_s$  with the embedding mappings  $i_{s_0} : G_{s_0} \rightarrow \bigoplus_{s \in \Lambda} G_s, \forall s_0 \in \Lambda$  satisfying the universal property:

for any abelian group  $H$  and group homo.  $\varphi_s : G_s \rightarrow H \forall s \in \Lambda, \exists!$  group homo.  $\varphi : \bigoplus_{s \in \Lambda} G_s \rightarrow H$  s.t. 又一個 $\tau$ 圖

**Theorem 7.**  $\bigoplus_{s \in \Lambda} G_s$  exists and is unique up to isomorphisms.

*Proof.* Existence:  $\bigoplus_{s \in \Lambda} G_s = \{(g_s)_{s \in \Lambda} \mid g_s \in G_s, \text{ almost all of the } g_s \text{' are } 0\}$  and

$$i_{s_0} : G_{s_0} \rightarrow \bigoplus_{s \in \Lambda} G_s, a_{s_0} \mapsto (g_s)_{s \in \Lambda} \text{ with } g_{s_0} = a_{s_0}, g_s = 0, \forall s \neq s_0.$$

group operation:  $(g_s)_{s \in \Lambda} + (g'_s)_{s \in \Lambda} := (g_s + g'_s)_{s \in \Lambda} \in \bigoplus_{s \in \Lambda} G_s$ . 這邊也一個 $\tau$ 圖

Uniqueness: Assume  $\exists$  another  $G$  satisfies the universal property, 一個大 $\tau$ 圖  $(G, \bigoplus_{s \in \Lambda} G_s)$  互相有唯一一個映射可以 keep  $i_{s_0}, \varphi \circ \psi = \text{id}_G, \psi \circ \varphi = \text{id}_{\bigoplus_{s \in \Lambda} G_s}$   $\square$

**Def 23.** Given a non-empty family of groups  $\{G_s \mid s \in \Lambda\}$ , a direct product of  $\{G_s \mid s \in \Lambda\}$  is a group  $\prod_{s \in \Lambda} G_s$  with projections  $p_{s_0} : \prod_{s \in \Lambda} G_s \rightarrow G_{s_0}, \forall s_0 \in \Lambda$  satisfying the following universal property:

for any group  $H$  with group homo.  $\varphi_s : H \rightarrow G_s, \forall s \in \Lambda, \exists!$   $\varphi : H \rightarrow \prod_{s \in \Lambda} G_s$  s.t. 又一個 $\tau$ 圖

**Theorem 8.**  $\prod_{s \in \Lambda} G_s$  exists and is unique up to isomorphisms.

*Proof.* Existence:  $\prod_{s \in \Lambda} G_s = \{(g_s)_{s \in \Lambda} \mid g_s \in G_s\}$  and

$$p_{s_0} : \prod_{s \in \Lambda} G_s \rightarrow G_{s_0}, (g_s)_{s \in \Lambda} \mapsto g_{s_0}, \forall s_0 \in \Lambda$$

- group operation:  $(g_s)_{s \in \Lambda} \cdot (g'_s)_{s \in \Lambda} := (g_s g'_s)_{s \in \Lambda} \in \prod_{s \in \Lambda} G_s$ .
- Define  $\varphi$ : 這邊也一個 $\tau$ 圖 which is uniquely defined.

Uniqueness: Assume  $\exists$  another  $G$  satisfies the universal property, 一個大 $\tau$ 圖  $(G, \prod_{s \in \Lambda} G_s)$  互相有唯一一個映射可以 keep  $i_{s_0}, \varphi \circ \psi = \text{id}_G, \psi \circ \varphi = \text{id}_{\prod_{s \in \Lambda} G_s}$   $\square$

**Ex 1.4.1.** Google the definition of the **direct limit** and show the existence and uniqueness.

**Ex 1.4.2.** Google the definition of the **inverse limit** and show the existence and uniqueness.

Motivation:  $\zeta_m$  is called an  $m$ -th root of unity if  $\zeta_m^m = 1$ .

$$\varinjlim_n \mathbb{Z}/2^n\mathbb{Z} \cong \{2^n\text{-th roots of unity} : n \in \mathbb{N}\}$$

$$\varinjlim_n \mathbb{Z}/2^n\mathbb{Z} = \left( \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2^n\mathbb{Z} \right) / \langle i_k(a) - i_j(f_{kj}(a)) \mid k \leq j, a \in \mathbb{Z}/2^k\mathbb{Z} \rangle$$

where  $f_{kj} : \mathbb{Z}/2^k\mathbb{Z} \rightarrow \mathbb{Z}/2^j\mathbb{Z}$ .

Inverse limit:

$$\varprojlim_n \mathbb{Z}/2^n\mathbb{Z} = \left\{ (n_1, n_2, \dots) \in \prod_n \mathbb{Z}/2^n\mathbb{Z} \mid \forall i < j, n_i \equiv n_j \pmod{2^{i+1}} \right\}$$

#### 1.4.2 Rings and fields

**Def 24.** A **ring** is a non-empty set  $R$  with two operations  $R \times R \rightarrow R$

$$(a, b) \mapsto a + b \quad \text{and} \quad (a, b) \mapsto ab$$

satisfying

1.  $(R, +, 0)$  is an abelian group.
2.  $(R, \cdot)$  is a semigroup. (if it is a monoid, then it is called “a ring with 1.”)
3. (Distributive laws)  $\forall a, b, c \in R, \begin{cases} a(b + c) = ab + ac \\ (b + c)a = ba + ca \end{cases}$

**Eg 1.4.1.**  $\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/n\mathbb{Z}, M_{n \times n}(\mathbb{F})$

**Eg 1.4.2.** Let  $G$  be an abelian group. Define (endomorphism, automorphism)

$$\text{End}(G) := \{ \text{group homo. } G \rightarrow G \} \quad \text{Aut}(G) := \{ \text{group isom. } G \rightarrow G \}$$

A natural ring structure on  $\text{End}(G)$  is:

$$\forall a \in G, \begin{cases} (f + g)(a) := f(a) + g(a) \\ (f \cdot g)(a) := f(g(a)) \end{cases}$$

**Eg 1.4.3.**  $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} \subset \mathbb{R}$ .

**Def 25.** Let  $R$  be a ring with 1.

- (a)  $\forall a \in R, a \neq 0$ ,  $a$  is called a unit if  $\exists a^{-1} \in R$ .
- (b)  $(R^\times = \{\text{units in } R\}, \cdot, 1)$  forms a group.
- (c)  $R$  is called a division ring if  $R \setminus \{0\} = R^\times$ .
- (d)  $R$  is said to be commutative if  $ab = ba, \forall a, b \in R$ .
- (e)  $R$  is a field if  $R$  is a commutative division ring.
- (f)  $a \neq 0$  is called a left zero divisor if  $\exists b \in R, b \neq 0$  s.t.  $ab = 0$ .
- (g)  $a$  is called a zero divisor if  $a$  is either a left or right zero divisor.
- (h)  $R$  is called an integral domain if  $R$  is a commutative ring without zero divisors.

Fact:

1. fields  $\implies$  integral domains.
2. finite + integral domain  $\implies$  fields.

*Proof.* Let  $R = \{0, a_1, \dots, a_n\}$ , for  $a \in R, a \neq 0, aa_i = aa_j \implies a(a_i - a_j) = 0 \implies i = j$ .  
So  $\{0, aa_1, \dots, aa_n\} = R \implies \exists a_i$  s.t.  $aa_i = 1$ .  $\square$

**Prop 1.4.1.** TFAE

1.  $\mathbb{Z}/n\mathbb{Z}$  is an integral domain.
2.  $\mathbb{Z}/n\mathbb{Z}$  is a field.
3.  $n = p$  is a prime.

easy to prove.

**Def 26.**

- $f : R_1 \rightarrow R_2$  is called a ring homomorphism if  $\forall a, b \in R, \begin{cases} f(a+b) = f(a) + f(b) \\ f(ab) = f(a)f(b) \end{cases}$ .
- $\text{Im } f$  is a subring of  $R_2$ .
- $\ker f = \{x \in R_1 \mid f(x) = 0\}$  is an additive group of  $R_1$  and  $\forall r \in R_1, x \in \ker f, f(rx) = f(r)f(x) = f(r)0 = 0 \implies rx \in \ker f, xr \in \ker f$ .
- $R_1/\ker f$  is an additive group and  $R_1/\ker f \cong \text{Im } f$  (additive isomorphism).

**Def 27.** Let  $I$  be an additive subgroup of  $R$ .  $I$  is called an ideal if  $\forall r \in R, x \in I, rx \in I, xr \in I$ .  
( $R/I, +, \cdot$ ) forms a quotient ring under

$$\forall r_1, r_2 \in R, (r_1 + I)(r_2 + I) = r_1 r_2 + I$$

well-defined: easy to show.

**Ex 1.4.3.** State and show the isomorphism theorems and the factor theorem.

**Prop 1.4.2.** If  $R$  is a ring with 1, then  $\exists!$  ring homo.  $\varphi : \mathbb{Z} \rightarrow R$  s.t.  $\varphi(1) = 1$ .

*Proof.* Let  $\varphi : \mathbb{Z} \rightarrow R$  is a ring homo. s.t.  $\varphi(1) = 1$ . Then  $\forall n \in \mathbb{Z}, \varphi(n) = \varphi(1) + \dots + \varphi(1) = n1$ .  
Now  $\forall n, m \in \mathbb{Z}, \varphi(n)\varphi(m) = (n1)(m1) = n(m1) = (nm)1$  by the distributive law. So  $\varphi$  is well-defined and unique.  $\square$

**Def 28.** In Prop 1.4.2,  $\ker \varphi = m\mathbb{Z}$  for some  $m > 0$ . We call  $m$  the characteristic of  $R$ , denoted by  $\text{char } R = m$ .

**Prop 1.4.3.**

1. If  $R$  is an integral domain, then  $\text{char } R = 0$  or  $p$ , where  $p$  is a prime. (try to prove this)
2. In the case of  $\text{char } R = p, \forall a, b \in R, (a+b)^p = a^p + b^p$ .

*Proof.*

$$(a+b)^p = a^p + \binom{p}{1}a^{p-1}b + \dots + b^p = a^p + b^p$$

because  $p \mid \binom{p}{i} \implies \binom{p}{i}a^{p-i}b^i = 0$ .  $\square$

**Ex 1.4.4.** Let  $F$  be a field. Show that

1. if  $\text{char } F = 0$ , then  $\mathbb{Q} \hookrightarrow \text{subfield of } F$ .
2. if  $\text{char } F = p$ , then  $\mathbb{Z}/p\mathbb{Z} \hookrightarrow \text{subfield of } F$ .

**Theorem 9.** If  $F$  is a finite field, then  $|F| = p^n$  for some  $n \in \mathbb{N}$  and  $p$  is a prime.

*Proof.* By Ex. 1.4.4,  $\text{char } F = p$ ,  $p$  is a prime and  $\mathbb{Z}/p\mathbb{Z} \hookrightarrow F$ .

We have  $\mathbb{Z}/p\mathbb{Z} \times F \rightarrow F, (r, v) \mapsto rv$ .  $F$  can be regarded as a vector space over  $\mathbb{Z}/p\mathbb{Z}$ .

Let  $\dim_{\mathbb{Z}/p\mathbb{Z}} F = n$ , then  $F \cong (\mathbb{Z}/p\mathbb{Z})^n \implies |F| = p^n$ . □

**Theorem 10.** Let  $F$  be a field. Then any finite subgroup  $G$  of  $(F^\times, \cdot, 1)$  is cyclic.

*Proof.* Let  $|G| = n$ . Define  $h$  to be the max order of an element in  $G$ , say  $a^h = 1$ .

If  $h = n$ , then  $|\langle a \rangle| = h = n = |G|$  and  $\langle a \rangle \subseteq G$ , so  $G = \langle a \rangle$ .

Otherwise,  $h < n$ . We know that  $x^h - 1$  has at most  $h$  roots. So  $\exists b \in G$  is not a root of  $x^h - 1$ . Let  $\text{ord}(b) = h'$ , so  $h' \mid n$  and  $h' \nmid h$ . So  $\exists$  a prime  $p$  s.t.  $p^r \mid h'$  but  $p^r \nmid h$ .

Write  $h = mp^s$ ,  $s < r$  and  $\gcd(m, p) = 1 \implies \text{ord}(a^{p^s}) = m$ .

Write  $h' = qp^r \implies \text{ord}(b^q) = p^r$ .

Since  $\gcd(m, p^r) = 1$ ,  $\text{ord}(a^{p^s} b^q) = mp^r > mp^s = h$ , which is a contradiction. □

**Ex 1.4.5.**

1. Let  $a, b \in G$  with  $ab = ba$  and  $\text{ord}(a) = m, \text{ord}(b) = n$ . If  $\gcd(m, n) = 1$ , then  $\text{ord}(ab) = mn$ .  
In general, is the order of  $ab$  equal to  $\text{lcm}(m, n)$ ?
2. Let  $G$  be a finite group and  $H, K \leq G$ . Then  $|HK| = \frac{|H||K|}{|H \cap K|}$ .



## 1.5 Week 5

### 1.5.1 Group actions I

**Def 29.** A group  $G$  is said to act on a nonempty set  $X$  if  $\exists$  a map  $G \times X \rightarrow X$  with  $(g, x) \mapsto gx$  s.t.

1.  $1x = x$
2.  $(g_1g_2)x = g_1(g_2x) \quad \forall g_1, g_2 \in G$

**Prop 1.5.1.**  $\{\text{actions of } G\} \leftrightarrow \{\text{group homo. } G \rightarrow S_X\}$

*Proof.* Given an action  $(g, x) \mapsto gx$ , consider  $\varphi : G \rightarrow S_X$  s.t.  $\varphi : g \mapsto (\tau_g : x \mapsto gx)$ .

- 1-1:  $gx = gy \implies g^{-1}(gx) = y \implies x = y$ .
- onto:  $\forall y \in X$ , let  $x = g^{-1}y$ , then  $y = gx$ .
- group homo.:  $\varphi(gg') = (\tau_{gg'} : x \mapsto gg'x) = \tau_g \circ \tau_{g'} = \varphi(g)\varphi(g')$ .

Conversely, given a group homo.  $\varphi : G \rightarrow S_X$ , consider  $(g, x) \mapsto \varphi(g)(x)$ .

- $1x = \varphi(1)(x) = \text{Id}(x) = x$ .
- $g_1g_2x = \varphi(g_1g_2)(x) = \varphi(g_1) \circ \varphi(g_2)(x) = g_1(g_2x)$ . □

**Def 30.** A representation of  $G$  on a vector space  $V$  is a group action of  $G$  on  $V$  linearly. i.e.  $\exists$  group homo.  $\varphi : G \rightarrow \text{GL}(V)$ .

**Eg 1.5.1.**

$$\mathbb{Z}/m\mathbb{Z} \rightarrow \text{SO}(2), \quad \bar{k} \mapsto \begin{pmatrix} \cos \frac{2k\pi}{m} & -\sin \frac{2k\pi}{m} \\ \sin \frac{2k\pi}{m} & \cos \frac{2k\pi}{m} \end{pmatrix}$$

**Eg 1.5.2.**

$$S_n \rightarrow \text{GL}(n, \mathbb{R}), \quad \sigma \mapsto (\tau_\sigma : e_i \mapsto e_{\sigma(i)})$$

**Remark 8.**

1. An action  $G \times X \rightarrow X$  is said to be faithful if the corresponding group homo.  $\varphi : G \hookrightarrow S_X$ , denoted by  $G \curvearrowright X$ .
2. In general,  $\ker \varphi = \{g \in G \mid gx = x \quad \forall x \in X\} = \bigcap_{x \in X} \{g \mid gx = x\}$ .  
Define  $G_x = \{g \mid gx = x\} \leq G$  is the isotropy subgroup of  $G$  at  $x$ . (the stabilizer of  $G$  at  $x$ )
3.  $\varphi : G \rightarrow S_X \implies G/\ker \varphi \hookrightarrow S_X$ . So  $G/\ker \varphi \times X \rightarrow X$  is faithful.
4. Let  $\mathcal{C}(X) = \{f : X \rightarrow \mathbb{C}\}$ . If  $G \curvearrowright X$ , then  $G \curvearrowright \mathcal{C}(X)$  by  $G \times \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  with  $(g, f) \mapsto gf(x) = f(g^{-1}x)$ .  
The reason:  $(g_1g_2)f(x) = f((g_1g_2)^{-1}x) = f(g_2^{-1}g_1^{-1}x) = g_1(g_2f)(x)$ .

**Def 31.** Let  $G \curvearrowright X$  and  $x \in X$ .

- The **orbit** of  $x$  is defined to be  $Gx = \{gx \mid g \in G\}$ .
- $G \curvearrowright X$  is said to be transitive if  $\exists$  only one orbit. i.e.  $\forall x, y \in X, \exists g \in G$  s.t.  $y = gx$ .

The set of orbits forms a partition:  $x \sim y \iff \exists g \in G$  s.t.  $y = gx$ .

**Prop 1.5.2.** Let  $G \curvearrowright X$  and  $x \in X$ . Then  $|Gx| = [G : G_x]$ .

In particular,  $|G| < \infty \implies |G| = |Gx||G_x| \quad \forall x \in X$ .

*Proof.* Define  $\psi : Gx \rightarrow \{\text{left coset of } G_x\}$  as  $\psi : gx \mapsto gG_x$ .

- well-defined and 1-1:  $g_1x = g_2x \iff g_2^{-1}g_1x = x \iff g_2^{-1}g_1 \in G_x \iff g_2^{-1}g_1G_x = G_x \iff g_1G_x = g_2G_x$ .
- onto:  $\forall g \in G, \psi(gx) = gG_x$ . □

### 1.5.2 Action by left multiplication

- The action  $G \times G \rightarrow G, (g, x) \mapsto gx$  is associated with  $\varphi : G \hookrightarrow S_G$ . It is faithful (Cayley theorem) and transitive.
- Let  $H \leq G$  and  $X := \{\text{left coset of } H\}$ . The group action  $(g, xH) \mapsto gxH \rightsquigarrow \varphi : G \rightarrow S_X$ .

$$\ker \varphi = \bigcap_{x \in G} \underbrace{xHx^{-1}}_{\text{a conjugate of } H} \leq H$$

which is the largest normal subgroup in  $G$  contained in  $H$ .

*Proof.* If  $\begin{cases} N \triangleleft G \\ N \leq H \end{cases}, \forall x \in G, xNx^{-1} \leq xHx^{-1} \implies N = N(xx^{-1}) = xNx^{-1} \leq xHx^{-1}$ . □

**Prop 1.5.3.** Let  $H \leq G$  with  $[G : H] = p$  being the smallest prime dividing  $|G|$ . Then  $H \triangleleft G$ .

*Proof.* Let  $X = \{a_1H, \dots, a_pH\}$  (all left coests of  $H$ ) and  $\varphi : G \rightarrow S_p$  be the associated group homo. for the group action  $(g, a_iH) \mapsto ga_iH$ .

By the 1st isom. thm.,  $G/\ker \varphi \hookrightarrow S_p$ .

By Lagrange thm.  $|G/\ker \varphi| \mid |S_p| = p!$  and  $|G/\ker \varphi| \mid |G| \implies |G/\ker \varphi| \mid p$ .

So  $|G/\ker \varphi| = 1$  or  $p$ .

If  $|G/\ker \varphi| = 1 \implies G = \ker \varphi \leq H \leq G$ , which is a contradiction.

So  $|G/\ker \varphi| = p \implies [G : \ker \varphi] = p \implies [G : H][H : \ker \varphi] = p \implies [H : \ker \varphi] = 1 \implies H = \ker \varphi \triangleleft G$ . □

### 1.5.3 Action by conjugation

- The action  $G \times G \rightarrow G, (g, x) \mapsto gxg^{-1}$  is associated with the group homo.  $\varphi : G \rightarrow S_G, g \mapsto (\tau_g : x \mapsto gxg^{-1})$ .

$$\text{Inn}(G) := \{ \tau_g \mid g \in G \}$$

**Fact 1.5.1.**  $\tau_g$  is an automorphism. (isom.  $G \rightarrow G$ )

So  $\varphi : G \twoheadrightarrow \text{Inn}(G) \leq \text{Aut}(G) \leq S_G$ .

$\ker \varphi = \{ g \in G \mid gxg^{-1} = x \quad \forall x \in G \} = Z_G$ .

By the 1st isom. thm.,  $G/\ker \varphi \cong \text{Inn}(G)$ .

- The conjugacy class:  $Gx = \{ gxg^{-1} \mid g \in G \} = \text{Cl}(x)$ .
- The centralizer of  $x$  in  $G$ :  $G_x = \{ g \in G \mid gxg^{-1} = x \} = Z_G(x)$ .

$$|\text{Cl}(x)| = [G : Z_G(x)], \text{ if } |G| < \infty, |G| = |\text{Cl}(x)| |Z_G(x)|$$

- For  $H \triangleleft G$ , define  $G \times H \rightarrow H$   $(g, h) \mapsto ghg^{-1}$  with the group homo.  $\varphi : G \rightarrow \text{Aut}(H)$ .

$$\ker \varphi = \{ g \in G \mid gxg^{-1} = x \quad \forall x \in H \} = Z_G(H) \implies G/Z_G(H) \leq \text{Aut}(H)$$

- The normalizer of  $H$  in  $G$ :  $N_G(H) = \{ g \in G \mid gHg^{-1} = H \}$

**Theorem 11** (Normalizer-Centralizer theorem). If  $H \leq G$  then  $N_G(H)/Z_G(H) \hookrightarrow \text{Aut}(H)$ .

*Proof.* Define  $\varphi = g \mapsto (h \mapsto ghg^{-1}) \in \text{Aut}(H)$ . Then  $\ker \varphi = Z_G(H)$ , so  $N_G(H)/Z_G(H) \cong \text{Im } \varphi \leq \text{Aut}(H)$ .  $\square$

## 1.6 Week 6

### 1.6.1 Group actions II

**Def 32.** Let  $G \curvearrowright X$  and  $|X| < \infty$ . Write  $\text{Fix } G := \{x \in X \mid gx = x \ \forall g \in G\}$ .

- $x \in \text{Fix } G, Gx = \{x\}$ .
- $x \notin \text{Fix } G, |Gx| = [G : G_x]$ .

Let  $\{G_{x_1}, \dots, G_{x_n}\}$  be the set of distinct orbits. After rearrangement, assume  $x_1, \dots, x_r \in \text{Fix } G, x_{r+1}, \dots, x_n \notin \text{Fix } G$ . Then

$$|X| = |\text{Fix } G| + \sum_{i=r+1}^n [G : G_{x_i}]$$

**Theorem 12** (class equation). Let  $|G| < \infty$ . Then either  $G = Z_G$  or  $\exists a_1, \dots, a_m \in G \setminus Z_G$  s.t.

$$|G| = |Z_G| + \sum_{i=1}^m [G : G_{a_i}]$$

*Proof.* Consider the action  $(g, x) \mapsto gxg^{-1}$ , then

$$\text{Fix } G = \{x \in G \mid gxg^{-1} = x \ \forall g \in G\} = Z_G$$

It follows from the above argument.  $\square$

**Def 33.**  $G$  is called a  $p$ -group if  $|G| = p^n$ , where  $p$  is a prime,  $n \in \mathbb{N}$ .

**Prop 1.6.1.** If  $G$  is a  $p$ -group, then  $Z_G \neq \{1\}$ .

*Proof.* Let  $|G| = p^n$ . If  $G = Z_G$ , then done. Otherwise, by the class equation (use action by conjugation),  $|G| = |Z_G| + \sum_{i=1}^n [G : G_{a_i}]$ ,  $a_i \notin Z_G$ .

$$G_{a_i} = Z_G(a_i), \text{ so } a_i \notin Z_G \implies Z_G(a_i) \subsetneq G \implies p \mid [G : Z_G(a_i)] = \frac{|G|}{|Z_G(a_i)|}.$$

$$\text{So } |Z_G| = |G| - \sum_{i=1}^n [G : Z_G(a_i)] \implies p \mid |Z_G| \implies Z_G \neq \{1\}. \quad \square$$

**Prop 1.6.2.** If  $|G| = p^2$ , then  $G$  is abelian. ( $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  and  $\mathbb{Z}/p^2\mathbb{Z}$ )

*Proof.* Assume that  $G$  is not abelian. By prop 1.6.1,  $|Z_G| = p \implies |G/Z_G| = p \implies G/Z_G$  is cyclic  $\implies G$  is abelian. (contradiction)  $\square$

**Prop 1.6.3.** If  $|G| = p^3$  and  $G$  is not abelian, then  $|Z_G| = p$ .

(Abelian:  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p^3\mathbb{Z}$ )

**Prop 1.6.4.** Let  $|G| = p^n$ . Then  $\forall 0 \leq k \leq n, \exists G_k \triangleleft G$  s.t.  $|G_k| = p^k$  and  $G_i \subsetneq G_{i+1}$ .

In general, for a finite group  $G$ ,  $\exists \{1\} = G_r \triangleleft G_{r-1} \triangleleft \dots \triangleleft G_1 \triangleleft G_0 = G$  s.t.  $G_i/G_{i+1}$  is cyclic.

we call  $G$  a solvable group.

*Proof.* By induction on  $n$ ,  $n = 1$  is trivial. For  $n > 1$ , assume that the statement holds for  $n - 1$ . By prop 1.6.1,  $Z_G \neq \{1\}$ .  $\exists a \in Z_G, a \neq 1$ . Let  $\text{ord}(a) = p^l$ , then  $\text{ord}(a^{p^{l-1}}) = p$ .  $\implies$  in any case,  $\exists a \in Z_G$  with  $\text{ord}(a) = p$ .

Now  $|G/\langle a \rangle| = p^{n-1}$ , so by induction hypothesis,  $\forall 0 \leq k \leq n-1, \exists \overline{G_k} \triangleleft G/\langle a \rangle$  s.t.  $|\overline{G_k}| = p^k, \overline{G_i} \leq \overline{G_{i+1}}$ .

By 3rd isom. thm.,  $\exists G_{k+1} \triangleleft G$  s.t.  $\overline{G_k} = G_{k+1}/\langle a \rangle, G_j \leq G_{j+1}$  and  $|G_{k+1}| = p^{k+1}$ .

□

**Prop 1.6.5.** Let a  $p$ -group  $G \curvearrowright X$  with  $|X| < \infty$ . Then  $|X| \equiv |\text{Fix } G| \pmod{p}$ .

**Theorem 13** (Cauchy theorem). Let  $p \mid |G|$ . Then  $\exists a \in G$  s.t.  $\text{ord}(a) = p$ . Consider

$$X = \{ (a_1, \dots, a_p) \mid a_i \in G, a_1 a_2 \dots a_p = 1 \}$$

and the action  $\mathbb{Z}/p\mathbb{Z} \times X \rightarrow X$ :

$$(\overline{k}, (a_1, \dots, a_p)) \mapsto (a_{k+1}, \dots, a_p, a_1, \dots, a_k)$$

(This is well-defined since  $ab = 1 \implies ba = 1$  in a group.) We find that  $(a_1, \dots, a_p) \in \text{Fix } \mathbb{Z}/p\mathbb{Z} \iff a_1 = a_2 = \dots = a_p$ . By prop 1.6.5,  $|\text{Fix } \mathbb{Z}/p\mathbb{Z}| \equiv |X| \pmod{p}$ . And  $|X| = |G|^{p-1} \equiv 0 \pmod{p}$ . Since  $(1, \dots, 1) \in \text{Fix } \mathbb{Z}/p\mathbb{Z}, |\mathbb{Z}/p\mathbb{Z}| \neq 0 \implies |\mathbb{Z}/p\mathbb{Z}| \geq p$ .

So  $\exists (a, \dots, a) \in \text{Fix } \mathbb{Z}/p\mathbb{Z} \implies a^p = 1$ .

Application: Let  $|G| = p^3$  and  $G$  be non-abelian ( $p$  is odd). By prop 1.6.3,  $|G/Z_G| = p^2$ . Since  $G$  is non-abelian, we have  $G/Z_G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ . That is,  $\forall a \in G, a^p \in Z_G$ .

So,

$$\exists \varphi : G \rightarrow Z_G \cong C_p \text{ with } \varphi : a \mapsto a^p$$

Since  $G/Z_G$  is abelian,  $[G, G] \leq Z_G$ . And

$$\begin{cases} |[G, G]| \mid |Z_G| = p \\ G \text{ is non-abelian} \end{cases} \implies [G, G] = Z_G$$

**Def 34.**  $[x, y] = x^{-1}y^{-1}xy \in [G, G], [x, y]^p = 1$ .

So  $a^p b^p = a^p b^p [b, a]^p \dots$  換換換總共需要  $p(p-1)/2$

$$a^p b^p = (ab)^p [b, a]^{\frac{p(p-1)}{2}} = (ab)^p$$

So  $\varphi$  is a group homo.

Now if  $\ker \varphi = G$  ( $\forall a \in G, a^p = 1$ ), i.e.  $\varphi$  is trivial, then  $\varphi$  is useless. Else,  $\exists a \in G$  s.t.  $\text{ord}(a) = p^2$ , then  $H = \langle a \rangle \triangleleft G$ . ( $|G:H| = p$  is the smallest prime dividing  $|G|$ )

Also, in this case,  $\varphi : G \rightarrow Z_G \implies G/\ker \varphi \cong Z_G$ . Let  $E = \ker \varphi, |E| = p^2$ . By the def. of  $\ker \varphi, E \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ .

We find that  $H \cap E = \langle a^p \rangle$ . Pick  $b \in E \setminus H$  and let  $K = \langle b \rangle \implies |K| = p, H \cap K = \{1\}, HK = G$ .

## 1.6.2 Semidirect product

**Fact 1.6.1.**  $K \triangleleft G, H \triangleleft G, K \cap H = \{1\} \implies KH = K \times H$   
 $(\forall k \in K, h \in H, khk^{-1}h^{-1} \in H \cap K = \{1\}, \implies kh = hk)$

**Fact 1.6.2.** Let  $K, H$  be two groups, and  $G = K \rtimes H \implies K \times \{1\} \triangleleft K \rtimes H, \{1\} \times H \triangleleft K \rtimes H$

**Observation 1.**  $K \leq G, H \triangleleft G, K \cap H = \{1\}$  (K 慘 H 好, 簡稱慘好集)

$\implies$  elements in  $KH$  has unique representation ? 好事喔

$KH \iff K \times H$  1-1 corresp,  $(kh) \leftrightarrow (k, h)$

Group operation :  $\forall k_1, k_2 \in K, h_1, h_2 \in H, (k_1 h_1)(k_2 h_2) = k_1 k_2 (k_2^{-1} h_1 k_2) h_2$

Let  $\tau : K \rightarrow \text{Aut}(H), k \mapsto (\tau(k) : h \mapsto khk^{-1})$  (類似  $\in \text{Inn}(H)$  )

**Def 35** (Semi-Direct Product (慘好積)).  $K \times_{\tau} H = \{(k, h) | k \in K, h \in H\}$  with group operation :  $(k_1, h_1)(k_2, h_2) = (k_1 k_2, \tau(k_2^{-1})(h_1)(h_2))$  where  $\tau : K \rightarrow \text{Aut}(H)$  (need not to be inner homomorphism)

Properties:

- Associativity: Good, ex
- The identity =  $(1, 1)$
- Inverse :  $(k, h)^{-1} = (k^{-1}, \tau(k)(h^{-1}))$
- $K \cong K \times \{1\} \leq K \times_{\tau} H : (k_1, 1)(k_2, 1) = (k_1 k_2, \tau(k_2^{-1})(1)1) = (k_1 k_2, 1) \in K \times \{1\}$   
 $H \cong \{1\} \times H \leq K \times_{\tau} H : (1, h+1), (1, h_2) = (1, \tau(1^{-1})(h_1)h_2) = (1, h_1 h_2) \in \{1\} \times K$
- $H \triangleleft K \times_{\tau} H : (k, h)(1, h')(k, h)^{-1} = (k, hh')(k^{-1}, \tau(k)(h^{-1})) = (1, \tau(k)(hh')\tau(k)(h^{-1})) \in H$
- $\tau(k)(h) = khk^{-1} : (k, 1)(1, h)(k^{-1}, 1) = (k, h)(k^{-1}, 1) = (1, \tau(k)(h))$
- If  $\tau$  is trivial  $\implies K \times_{\tau} H \cong K \times H$

**Remark 9.** Some definition swaps the order of  $H$  and  $K$ , i.e.  $(h_1, k_1)(h_2, k_2) = (h_1 \phi(k_1)(h_2), k_1 k_2)$

**Ex 1.6.1.** Show that  $H \rtimes_{\phi} K$  is a group and satisfies the above properties.

**Eg 1.6.1.** Construct a non-abelian group of order 21.

**Fact 1.6.3.**  $\text{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^{\times} \cong C_{p-1}$

Sol :  $\phi_k : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}, \bar{1} \mapsto \bar{k}$

$\phi_{k_2} \circ \phi_{k_1}(1) = \phi_{k_2}(\bar{k}_1) = \phi_{k_2}(1 + \dots + 1) = \bar{k}_2 + \dots + \bar{k}_2 = \overline{k_1 k_2}$

Let  $K = C_3, H = C_7$ , define  $\tau : C_3 \rightarrow \text{Aut}(C_7) \cong C_6, a \mapsto \phi_2$

$\phi_k : b \mapsto b^k$

$G = \langle a, b | a^3 = 1, b^7 = 1, aba^{-1} = b^2 \rangle$

**Eg 1.6.2.**  $p : \text{odd}, |G| = p^3, G$  is non-abelian.

(sol)  $\phi : G \rightarrow Z(G), a \mapsto a^p$  non trivial case  $\exists a \in G$  with  $\text{ord}(a) = p^2$ . Let  $H = \langle a \rangle$  here  $\phi$  is onto and  $E = \ker \phi \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  And  $|H \cap E| = p$   $H \triangleleft G$  because  $[G : H] = p$  Pick  $b \in E \setminus H$  and let  $K = \langle b \rangle \implies |K| = p, K \cap H = \{1\}$  so  $|G| = |KH| = p^3$

**Fact 1.6.4.**  $\text{Aut}(\mathbb{Z}/p^2\mathbb{Z}) \cong (\mathbb{Z}/p^2\mathbb{Z})^{\times}$

Sol :  $\phi_k : \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z}, \bar{1} \mapsto \bar{k}, \gcd(k, p) = 1$

Find a group homo  $\tau : K \implies \text{Aut}(H)$  because  $(1+p)^p \equiv 1 \pmod{p^2}$ ,  $\text{ord}(\overline{1+p}) = p$ . Let  $P = \langle \overline{1+p} \rangle$  is the only subgroup of order  $p$ . (if  $\exists |Q| = p, P \neq Q$  then  $P \cap Q = 1, |PQ| = p^2$  but

$|G| = p(p-1)$ , miserable.) So let  $\tau : b \mapsto (\phi_{1+p} : a \mapsto a^{1+p})$  so  $G = \langle a, b | a^{p^2} = 1, b^p = 1, bab^{-1} = a^{1+p} \rangle$  is a non-abelian group of order  $p^3$ .

**Eg 1.6.3.** Isometry of  $R^n$

**Def 36** (Isometry). An isometry of  $R^n$  is a function  $h : R^n \rightarrow R^n$  that preserves the distance between vectors.

$h = t \circ k$  where  $t$  is translation,  $k$  is an isometry fixing the origin, i.e.  $k \in O(n)$ . Let  $T$  be the group of translations on  $R^n$ ,  $T \cong (R^n, +, 0), t \mapsto t(0)$ .

Let  $\tau : O(n) \rightarrow \text{Aut}(T), A \mapsto L_A : R^n \rightarrow R^n, v \mapsto Av$

$\implies \text{Isom}(R^n) = O(n) \times_{\tau} R^n$

**Eg 1.6.4.** Quaternion  $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$  is not a semi-direct product of any two proper subgroups.

pf: since  $\{\pm 1\}$  is contained in any non-trivial subgroups, can't find  $H \cap K = \{1\}$ .

**Eg 1.6.5.**  $A_4, V_4 = \{1, (12)(34), (14)(23), (13)(24)\} \triangleleft A_4, V_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Let  $H = \langle (123) \rangle \cong C_3$ , define  $\tau : H \rightarrow \text{Aut}(V_4) \cong GL_2(\mathbb{Z}/2\mathbb{Z})$   $(123) \mapsto (\bar{0}\bar{1}; \bar{1}\bar{1})$  so  $A_4 \cong C_3 \times_{\tau} V_4$ .

**Ex 1.6.2.** Construct  $D_n$  as a semi-direct product of  $\mathbb{Z}/n\mathbb{Z}$  and  $\mathbb{Z}/2\mathbb{Z}$ .

**Ex 1.6.3.**

1. Show that  $S_4$  is a semi-direct product of  $V_4$  and  $H = \{\sigma \in S_4 | \sigma(4) = 4\} \cong S_3$ .
2. Show that  $S_n$  is a semi-direct product of  $A_n$  and  $H = \langle (12) \rangle$ .

**Remark 10.**

- $\text{Aut}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \cong GL_2(\mathbb{Z}/p\mathbb{Z})$  (regarded as a vector space over  $\mathbb{Z}/p\mathbb{Z}$ )
- $\text{Aut}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}) \cong \text{Aut}(\mathbb{Z}/p\mathbb{Z}) \times \text{Aut}(\mathbb{Z}/q\mathbb{Z}) \cong C_{p-1} \times C_{q-1}$

## 1.7 Week 7

### 1.7.1 Composition series

Ques: How to simplify a finite group  $G$ ?

Strategy:

- If  $G = \{1\}$ , then done.
- Otherwise, check whether  $G$  has a nontrivial proper normal subgroup.
- If no, then  $G$  is said to be a simple group.
- Otherwise, find a normal subgroup  $G_1$  as large as possible s.t.  $G/G_1$  is simple.
- If  $G_1$  is simple, then done.
- Otherwise, repeat above on  $G_1$  and get  $G_2, \dots, G_n$  s.t.

$$G_n = \{1\} \triangleleft G_1 \triangleleft \dots \triangleleft G_1 \triangleleft G_0 = G \quad \begin{array}{l} G_i/G_{i+1} \text{ is simple} \\ \searrow \\ \text{composition factors} \end{array}$$

Say “it is a composition series” with  $\text{length}(G) = n$ .

Hence simple groups can be regarded as basic building blocks of groups.

The classification of all finite simple groups is given as follows:

1.  $\mathbb{Z}/p\mathbb{Z}$ ,  $p$  is a prime.
2.  $A_n, n \geq 5$ .
3. simple groups of Lie type.
4. 26 sporadic simple groups.

**Eg 1.7.1.**  $G = S_4, G_1 = A_4, G_2 = V_4, G_3 = \langle (1\ 2)(3\ 4) \rangle, G_4 = \{1\} \rightsquigarrow \text{length}(S_4) = 4$ .

factors:  $C_2, C_3, C_2, C_2$ .

**Eg 1.7.2.**  $G = \mathbb{Z}/12\mathbb{Z} = \langle \bar{1} \rangle$ .

- $G_1 = \langle \bar{2} \rangle, G_2 = \langle \bar{4} \rangle, G_3 = \langle \bar{0} \rangle \rightsquigarrow \text{length}(3)$ , factors:  $C_2, C_2, C_3$ .
- $G'_1 = \langle \bar{2} \rangle, G'_2 = \langle \bar{6} \rangle, G'_3 = \langle \bar{0} \rangle \rightsquigarrow \text{length}(3)$ , factors:  $C_2, C_3, C_2$ .
- $G''_1 = \langle \bar{3} \rangle, G''_2 = \langle \bar{6} \rangle, G''_3 = \langle \bar{0} \rangle \rightsquigarrow \text{length}(3)$ , factors:  $C_3, C_2, C_2$ .

**Eg 1.7.3.** Let  $|G| = p^n$ . We know  $\forall 0 \leq k \leq n, \exists G_k \triangleleft G$  with  $|G_k| = p^k$  and  $G_i \leq G_{i+1}$ .

$\text{length}(G) = n$ , factors:  $C_p, \dots, C_p$ . ( $n$  times)

**Theorem 14** (Jordan-Hölder theorem). If  $G$  has a composition series, then any two composition series have the same length and the same factors up to permutation.

**Lemma 1** (Zassenhaus lemma). Let  $H' \triangleleft H \leq G, K' \triangleleft K \leq G$ . Then  $(H \cap K')H' \triangleleft (H \cap K)H', (H' \cap K)K' \triangleleft (H \cap K)K'$  and

$$(H \cap K)H'/(H \cap K')H' \cong (H \cap K)K'/(H' \cap K)K'.$$

**Theorem 15** (Schreier theorem). Any two normal series of  $G$  have equivalent refinements.

refinements: inserting a finite number of subgroups into the normal series.



*Proof.* For two normal series:

$$\begin{aligned}\{1\} &= H_r \triangleleft H_{r-1} \triangleleft \cdots \triangleleft H_1 \triangleleft H_0 = G \\ \{1\} &= K_s \triangleleft K_{s-1} \triangleleft \cdots \triangleleft K_1 \triangleleft K_0 = G\end{aligned}$$

We define

$$\begin{aligned}H_{ij} &= (H_i \cap K_j)H_{i+1} \\ K_{ji} &= (H_i \cap K_j)K_{j+1}.\end{aligned}$$

Then we have

$$\begin{aligned}\{1\} &= H_{(r-1)s} \triangleleft H_{(r-1)(s-1)} \triangleleft \cdots \triangleleft H_{(r-1)0} = H_{r-1} = H_{(r-2)s} \triangleleft \cdots \triangleleft H_{10} = H_1 = H_{0s} \triangleleft \cdots \triangleleft H_{00} = G \\ \{1\} &= K_{(s-1)r} \triangleleft K_{(s-1)(r-1)} \triangleleft \cdots \triangleleft K_{(s-1)0} = K_{s-1} = K_{(s-2)r} \triangleleft \cdots \triangleleft K_{10} = K_1 = K_{0r} \triangleleft \cdots \triangleleft K_{00} = G\end{aligned}$$

Both have size  $= rs$ . By lemma,  $H_{ij}/H_{i(j+1)} \cong K_{ji}/K_{j(i+1)}$ . Note that if  $H_{ij} = H_{i(j+1)}$ , then  $K_{ji} = K_{j(i+1)}$ .  $\square$

*proof of Jordan-Hölder theorem.* Let

$$\begin{cases} \{1\} = G_n \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G & (*) \\ \{1\} = G'_m \triangleleft \cdots \triangleleft G'_1 \triangleleft G'_0 = G & (**) \end{cases}$$

be two composition series.

By Schreier theorem, we get two refined equivalent series  $(*)', (**)'$ . Since  $(*), (**)$  are already composition series,  $(*) = (*)', (**') = (**)'$ . So  $(*), (**)$  are equivalent.  $\square$

*proof of lemma.* First prove  $(H \cap K')H' \triangleleft (H \cap K)H'$ .

- $\forall g \in H \cap K, gK'g^{-1} = K' \rightsquigarrow (gHg^{-1}) \cap (gK'g^{-1}) = H \cap K'$  and  $gH'g^{-1} = H'$ . So

$$g(H \cap K')H'g^{-1} = (H \cap K')H'$$

- $\forall g \in H', ab \in (H \cap K')H',$

To prove

$$(H \cap K)H'/(H \cap K')H' \cong (H \cap K)K'/(H' \cap K)K'.$$

$$\begin{aligned}(H \cap K)H'/(H \cap K')H' &\cong (H \cap K)(H \cap K')H'/(H \cap K')H' \\ &\cong (H \cap K)/(H \cap K) \cap (H \cap K')H' \\ &\cong (H \cap K)/K \cap (H \cap K')H' \\ &\cong (H \cap K)/(H' \cap K)(H \cap K')\end{aligned}$$

$(K \cap (H \cap K')H' = (H' \cap K)(H \cap K')$ , tricky) By symmetry,

$$(H \cap K)K'/(H' \cap K')K' \cong (H \cap K)/(H' \cap K)(H \cap K')$$

$\square$

**Prop 1.7.1.** Let  $|G| < \infty$ . Then  $G$  is solvable  $\iff$  all composition factors are cyclic of prime order.

*Proof.* “ $\Leftarrow$ ”: by def.

“ $\Rightarrow$ ”: If  $G_i/G_{i+1} \cong C_n$  with  $n = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}$ .  $\square$

**Observation.** Let  $K \triangleleft G$ . 把  $K, G/K$  拆成兩個 composition series 的話, 就可以把兩串接起來, 長度就是加起來。

**Ex 1.7.1.** Let  $\{1\} = G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$  be a composition series of  $G$  and  $K \triangleleft G$ . Then after we eliminate equalities,

1.  $\{1\} = (K \cap G_n) \triangleleft (K \cap G_{n-1}) \triangleleft \cdots \triangleleft (K \cap G_1) \triangleleft (K \cap G_0) = K$  is a composition series of  $K$ .
2.  $\{\bar{1}\} = KG_n/K \triangleleft KG_{n-1}/K \triangleleft \cdots \triangleleft KG_1/K \triangleleft KG_0/K = G/K$  is a composition series of  $G/K$ .

**Ex 1.7.2.** Let  $\begin{cases} H \triangleleft G \\ K \triangleleft G \end{cases}$  with  $H \neq K$  s.t.  $G/H, G/K$  are simple. Then  $H/H \cap K, K/K \cap H$  are simple too.

**Ex 1.7.3.** Let  $\{1\} = G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$  be a composition series of length  $n$ . Show by induction on  $n$  that for every composition series of  $G$ :

$$\{1\} = H_m \triangleleft H_{m-1} \triangleleft \cdots \triangleleft H_1 \triangleleft H_0 = G,$$

we have  $m = n$  and

$$\{H_{n-1}/H_n, \dots, H_0/H_1\} = \{G_{n-1}/G_n, \dots, G_0/G_1\}$$

**Ex 1.7.4.** Exhibit all composition series for  $Q_8, D_4, \mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  respectively.

## 1.7.2 Modules over a PID

**Def 37.** Let  $R$  be a ring with 1. A  $R$ -module is an abelian group  $M$  (written additively) on which  $R$  acts linearly.  $R \times M \rightarrow M \quad (r, x) \mapsto rx$

1.  $r(x + y) = rx + ry \quad r \in R, x, y \in M$
2.  $(r_1 + r_2)x = r_1x + r_2x \quad r_1, r_2 \in R, x \in M$
3.  $(r_1r_2)x = r_1(r_2x) \quad r_1, r_2 \in R, x \in M$
4.  $1x = x \quad x \in M$

**Eg 1.7.4.** A  $k$ -vector space is a  $k$ -module.

**Eg 1.7.5.** An abelian group  $G$  can be regarded as a  $\mathbb{Z}$ -module.

$$\mathbb{Z} \times G \rightarrow G \quad \text{by} \quad na = \begin{cases} \underbrace{a + \cdots + a}_{n \text{ times}} & \text{if } n \geq 0 \\ \underbrace{(-a) + \cdots + (-a)}_{n \text{ times}} & \text{if } n < 0 \end{cases}$$

**Eg 1.7.6.** Let  $I$  be an ideal of  $R$ . Then  $I$  can be regarded as an  $R$ -module since  $\forall r \in R, a \in I, \quad ra \in I$ .

**Def 38.** A submodule  $N$  of  $M$  is an additive subgroup of  $M$  s.t.  $\forall r \in R, a \in N, \quad ra \in N$ .

**Prop 1.7.2.** Let  $\phi \neq S \subseteq M$ . The submodule generated by  $S$  is defined to be

$$\langle S \rangle_R = \left\{ \sum_{\text{finite}} r_i x_i \mid x_i \in S, r_i \in R \right\} = \text{the least submodule containing } S$$

$$= \bigcap_{S \subseteq N \subseteq M} N$$

**Def 39.** An  $R$ -module  $M$  is said to be finitely generated if  $\exists x_1, \dots, x_n \in M$  s.t.  $M = \langle x_1, \dots, x_n \rangle_R = Rx_1 + Rx_2 + \dots + Rx_n$

**Eg 1.7.7.**  $R$  is generated by 1 as an  $R$ -module.

**Def 40.** An additive group homo.  $\varphi : M_1 \rightarrow M_2$  is called an  $R$ -module homo. if

$$\varphi(rx) = r\varphi(x) \quad \forall r \in R, x \in M_1$$

**Def 41.** An integral domain  $R$  is called a principal ideal domain (PID) if  $\forall I$  ideal in  $R$ ,  $\exists a \in R$  s.t.  $I = \langle a \rangle_R$ .

**Eg 1.7.8.**  $\mathbb{Z}$  is a PID.

For  $I \subseteq \mathbb{Z}$ ,  $I$  is an additive subgroup, so  $I = m\mathbb{Z} = \langle m \rangle_{\mathbb{Z}}$ .

**Def 42.**  $M$  is said to be a free module of rank  $n$  if  $M \cong R^n = R \oplus \dots \oplus R$  (or  $R \times \dots \times R$ )

**Theorem 16.** If  $R$  is a PID, then any submodule of  $R^n$  is free of rank  $\leq n$ .

*Proof.* By induction on  $n$ . If  $n = 1$ , notice that any submodule is an ideal  $I$  by the closure of submodule. Then since  $R$  is a PID,  $\forall I \subseteq R, \exists a \in R$  s.t.  $I = \langle a \rangle_R = Ra \cong R$  (**as a  $R$ -module**).

Let  $n > 1$  and  $N$  be a submodule of  $R^n$ . Consider

$$\pi_1 : \begin{matrix} R^n & \rightarrow R \\ (r_1, \dots, r_n) & \mapsto r_1 \end{matrix} \quad \text{and} \quad \pi = \pi_1|_N : N \rightarrow R$$

**case 1:**  $\text{Im } \pi = \{0\}$ . In this case,  $N \subseteq \ker \pi_1 \cong R^{n-1}$ . By induction hypothesis,  $N$  is free of rank  $\leq n-1 < n$ .

**case 2:**  $\text{Im } \pi = \langle a \rangle$ , say  $\pi(x) = a$ . Claim:  $N = Rx \oplus \ker \pi, \ker \pi \subseteq \ker \pi_1 \cong R^{n-1}$ .

- $Rx \cap \ker \pi = \{0\}$ :  $rx \in Rx \cap \ker \pi \implies \pi(rx) = 0$ , then  $r\pi(x) = 0$ . But integral domain doesn't have zero divisors, so  $r = 0$  and hence  $rx = 0$ .
- $N \supseteq Rx \oplus \ker \pi$ : Obvious since  $Rx, \ker \pi \subseteq N$ .
- $N \subseteq Rx \oplus \ker \pi$ :  $\forall y \in N, \pi(y) = r_0 a$  for some  $r_0 \in R$ ,  $\pi(y - r_0 x) = 0 \implies y - r_0 x \in \ker \pi$ . So  $N \subseteq Rx \oplus \ker \pi$ .  $\square$

Recall that the elementary matrices are

- $D_i(u) = \text{diag}(1, \dots, 1, u, 1, \dots, 1)$ .  $D_i(u) \in \text{GL}(n, R)$  if  $u$  is a unit.
- $B_{ij}(a) = I_n + ae_{ij}, a \in R, i \neq j$ .  $B_{ij}(a)^{-1} = B_{ij}(-a) \implies B_{ij}(a) \in \text{GL}(n, R)$ .
- $P_{ij} = I_n - e_{ii} - e_{jj} + e_{ij} + e_{ji}$ .

**Fact 1.7.1.** If  $R$  is a PID and  $\langle a, b \rangle_R = \langle d \rangle_R$ , then  $d = \gcd(a, b)$ .

*Proof.*

- $a \in \langle d \rangle_R \implies a = rd$  for some  $r \in R \implies d \mid a$ .  $v \in \langle d \rangle_R \implies d \mid v$ .
- Let  $c \mid a, c \mid b$ , say  $a = k_1c, b = k_2c$ .  $d \in \langle a, b \rangle_R \implies d = x_1a + x_2b$  for some  $x_1, x_2 \in R$ . So  $d = x_1k_1c + x_2k_2c = (x_1k_1 + x_2k_2)c \implies c \mid d$ .  $\square$

**Theorem 17.** Let  $R$  be a PID and  $A \in M_{n \times m}(R)$ . Then  $\exists P \in \text{GL}_n(R)$  and  $Q \in \text{GL}_m(R)$  s.t.

$$PAQ = \begin{pmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_r & \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \quad \text{with } d_i \mid d_{i+1} \quad \forall i = 1, \dots, r-1$$

*Proof.* Define the length  $l(a)$  of  $a \neq 0$  to be  $r$  if  $a = p_1p_2 \dots p_r$  where  $p_1, \dots, p_r$  are prime elements.

prime elements:  $p \mid ab \implies p \mid a$  or  $p \mid b$ .

1. We may assume  $a_{11} \neq 0$  and  $l(a_{11}) \leq l(a_{ij}) \forall a_{ij} \neq 0$ . (換一換就上去了...XD)
2. We may assume  $\begin{cases} a_{11} \mid a_{1k} & \forall k = 2, \dots, m \\ a_{11} \mid a_{k1} & \forall k = 2, \dots, n \end{cases}$ . If  $a_{11} \nmid a_{1k}$ , then we can interchange 2nd and  $k$ th columns to assume  $a = a_{11} \nmid a_{12} = b$ .

Let  $d = \gcd(a, b) \implies \begin{cases} l(d) < l(a) \\ d = ax + by \text{ for some } x, y \in R \end{cases} \implies 1 = \frac{a}{d}x + \frac{b}{d}y$ . Write  $b' = \frac{b}{d}, a' = -\frac{a}{d}$ . Then

$$\begin{pmatrix} -a' & b' \\ y & -x \end{pmatrix} \begin{pmatrix} x & b' \\ y & a' \end{pmatrix} = I_2$$

反正就是移一下減掉, length 會一直變小  $\implies$  這個操作會停.

3. 有這個  $\begin{cases} a_{11} \mid a_{1k} & \forall k = 2, \dots, m \\ a_{11} \mid a_{k1} & \forall k = 2, \dots, n \end{cases}$  就可以全部消掉變成

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & \dots & b_{nm} \end{pmatrix}$$

4. May assume  $a_{11} \mid b_{kl} \quad \forall k, l$ . 不是的話就把該 row 往第一 row 加上去, 重複前面的操作,  $l(a_{11})$  總是變小, 因此會停.
5. 遞迴下去...

最後就弄出想要的矩陣了.  $\square$

## 1.8 Week 8

### 1.8.1 Fundamental theorem of finitely generated abelian groups

**Theorem 18** (Structure theorem of finitely generated module over a PID). Let  $R$  be a PID and  $M$  be a finitely generated  $R$ -module. Then  $M \cong R/d_1R \oplus \cdots \oplus R/d_lR \oplus R^s$ ,  $d_i \in R$  with  $d_i \mid d_{i+1} \quad \forall i = 1, \dots, l-1$  for some  $s \in \mathbb{Z}^{\geq 0}$ .

*Proof.* Let  $M = \langle x_1, \dots, x_n \rangle_R$  and consider

$$\begin{aligned} \varphi : R^n &\rightarrow M \\ e_i &\rightarrow x_i \end{aligned}$$

By 1st isom. thm.,  $R^n / \ker \varphi \cong M$ .

We know  $\ker \varphi \cong R^m$  ( $e'_i \mapsto f_i, e'_i \in R^m$ ) for some  $m \leq n$  and  $\forall x \in \ker \varphi \quad \exists ! x_1, \dots, x_m \in R$  s.t.  $x = \sum_{i=1}^m x_i f_i$ .

Note that  $\ker \varphi \subseteq R^n$ . So we can write  $f_i = \sum_{j=1}^n a_{ji} e_j \quad \forall i = 1, \dots, m$ . Then  $x = \sum x_i \sum a_{ji} e_j = \sum (\sum a_{ji} x_i) e_j$ .

$R$  is a PID  $\implies \exists P \in \text{GL}_n(R), Q \in \text{GL}_m(R)$  s.t.

$$PAQ = \begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_r & \\ & & & 0 \\ & & & & \ddots \end{pmatrix} \quad \text{with } d_i \mid d_{i+1} \quad \forall i = 1, \dots, r-1$$

So consider  $[w_i] = Qe_i$ . Since  $P, Q$  invertible,  $R^n = \bigoplus R w_i$ ,  $\ker \varphi = \bigoplus d_i R w_i$  Hence

$$M \simeq R / \ker \varphi = \bigoplus R w_i / \bigoplus d_i R w_i = \bigoplus R / d_i R$$

□

$$\begin{aligned} R &\rightarrow R w_i / R d'_i w_i \\ 1 &\rightarrow \overline{w_i} \\ r &\rightarrow \overline{r w_i} \end{aligned}$$

**Remark 11.** If  $R$  is commutative, then “ $R^n \cong R^m \implies n = m$ .”

**Theorem 19.** Let  $G$  be a finitely generated abelian group. Then  $G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_l\mathbb{Z} \oplus \mathbb{Z}^s$ ,  $d_i \in \mathbb{Z}$  with  $d_i \mid d_{i+1} \quad \forall i = 1, \dots, l-1$  for some  $s \in \mathbb{Z}^{\geq 0}$ .

Since  $G$  can be regarded as a f.g.  $\mathbb{Z}$ -module and  $\mathbb{Z}$  is a PID, it follows from the main theorem.

$\text{Tor}(G) = \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_l\mathbb{Z} \leq G$  and  $G/\text{Tor}(G) \cong \mathbb{Z}^s$  (free part of  $G$ ).

**Fact 1.8.1.** If  $d = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$ , then  $\mathbb{Z}/d\mathbb{Z} \cong \mathbb{Z}/p_1^{m_1}\mathbb{Z} \oplus \mathbb{Z}/p_2^{m_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_s^{m_s}\mathbb{Z}$ .

**Theorem 20** (Chinese Remainder theorem). Let  $R$  be a commutative ring with 1 and  $I_1, \dots, I_n$  be ideals of  $R$ . Then

$$\begin{aligned} \varphi : R &\rightarrow R/I_1 \times \cdots \times R/I_n \text{ is a ring homo.} \\ r &\mapsto (\overline{r}, \dots, \overline{r}) \end{aligned}$$

and

- (1) if  $I_i, I_j$  are coprime  $\forall i \neq j$ , then  $I_1 I_2 \dots I_n = I_1 \cap I_2 \cap \dots \cap I_n$ .
- (2)  $\varphi$  is surjective  $\iff I_i, I_j$  are coprime  $\forall i \neq j$ .
- (3)  $\varphi$  is injective  $\iff I_1 \cap I_2 \cap \dots \cap I_n = \{0\}$ .

So if  $I_i, I_j$  are coprime  $\forall i \neq j$ , then

$$R/I_1 I_2 \dots I_n \cong R/I_1 \times \dots \times R/I_n.$$

$I_i, I_j$  are coprime  $\iff I_i + I_j = R$ .

*Proof.* we only need to prove (1), (2).

- (1) By induction on  $n$ .  $n = 2$ , need  $I_1 \cap I_2 \subseteq I_1 I_2$ . Indeed,  $I_1 \cap I_2 = (I_1 \cap I_2)R = (I_1 \cap I_2)(I_1 + I_2) \subseteq I_1 I_2$ .

For  $n > 2$ , since  $I_i + I_n = R \quad \forall i = 1, \dots, n-1$ ,  $\exists x_i \in I_i, y_i \in I_n$  s.t.  $x_i + y_i = 1 \quad \forall i = 1, \dots, n-1$ .

So  $x_1 x_2 \dots x_{n-1} = (1 - y_1)(1 - y_2) \dots (1 - y_{n-1}) = 1 - y, y \in I_n \implies I_1 I_2 \dots I_{n-1} + I_n = R$ .

Now,  $I_1 I_2 \dots I_n = (I_1 \dots I_{n-1})I_n = (I_1 \dots I_{n-1}) \cap I_n = I_1 \cap \dots \cap I_n$ .

- (2) " $\Rightarrow$ ": WLOG, we may let  $I_i = I_1, I_j = I_2$ . We have  $x \in R$  s.t.

$$\varphi(x) = (\bar{1}, \bar{0}, \dots, \bar{0}) \quad \text{i.e. } \bar{x} = \bar{1} \text{ in } R/I_1$$

Write  $x \equiv 1 \pmod{I_1}$ . Since  $1 - x \in I_1, x \in I_2$  and  $(1 - x) + x = 1, I_1 + I_2 = R$ .

" $\Leftarrow$ ":  $\forall y \in \text{RHS}, y = (\bar{r}_1, \dots, \bar{r}_n)$ . If we may find that  $x_i \in R$  s.t.  $\varphi(x_i) = (\bar{0}, \dots, \bar{1}, \bar{0}, \dots, \bar{0})$ , then

$$\varphi\left(\sum_{i=1}^n r_i x_i\right) = y$$

It is enough to show, for example,  $\exists x \in R$  s.t.  $\varphi(x) = (\bar{1}, \bar{0}, \dots, \bar{0})$ .

Since  $I_1 + I_i = R \quad \forall i = 2, \dots, n$ ,  $\exists x_i \in I_1, y_i \in I_i$  s.t.  $x_i + y_i = 1 \quad \forall i = 2, \dots, n$ .

So let  $x = y_2 \dots y_n = (1 - x_2) \dots (1 - x_n)$ . We have  $x \in I_2, \dots, I_n$  and  $x \equiv 1 \pmod{I_1}$ .

□

**Eg 1.8.1.**  $|G| = 72$  and  $G$  is abelian:

$$72 = 2 \times 36 = 3 \times 24 = 2 \times 2 \times 18 = 6 \times 12 = 2 \times 6 \times 6$$

Invariant factors

Elementary divisors

**Def 43.** The exponent of  $G$  with  $|G| < \infty$  is

$$\text{Exp}(G) := \min \{m \in \mathbb{N} | g^m = 1 \quad \forall g \in G\}$$

**Ex 1.8.1.**

1. Let  $G$  be abelian with  $|G| = n$ . Show that if  $d \mid n$ , then  $\exists H \leq G$  s.t.  $|H| = d$ .
2. If  $n = 540, d = 90$ , then construct all possible  $G$  and corresponding  $H$ .

**Ex 1.8.2.** Let  $G$  be abelian with  $|G| < \infty$ . Show that  $G$  is cyclic  $\iff \text{Exp}(G) = |G|$ .

**Ex 1.8.3.** Let  $f_i(x) \in \mathbb{Z}[x], i = 1, \dots, k$  with  $\deg f_i = d$  and  $p_1, \dots, p_k$  be distinct primes. Show that  $\exists f(x) \in \mathbb{Z}[x]$  with  $\deg f = d$  s.t.  $\bar{f}(x) = \bar{f}_i(x)$  in  $\mathbb{Z}/p_i\mathbb{Z}[x] \quad \forall i = 1, \dots, k$ .

$$f(x) = a_d x^d + \dots + a_0, \bar{f}(x) = \bar{a}_d x^d + \dots + \bar{a}_0$$

## 1.8.2 Sylow theorems

**Def 44.** Let  $|G| = p^\alpha r$  with  $p \nmid r$ .

1. If  $H \leq G$  with  $|H| = p^\alpha$ , then we call  $H$  a Sylow  $p$ -subgroup of  $G$ .
2.  $\text{Syl}_p(G)$  = the set of all Sylow  $p$ -subgroups of  $G$ .
3.  $n_p = |\text{Syl}_p(G)|$ .

**Lemma 2** (Key lemma). Let  $P \in \text{Syl}_p(G)$  and  $Q$  be a  $p$ -subgroup of  $G$ . Then  $Q \cap N_G(P) = Q \cap P$ .

*Proof.* By Lagrange theorem,  $H = Q \cap N_G(P)$  is also a  $p$ -subgroup of  $N_G(P)$  since  $|H| \mid |Q|$ .

Since  $\begin{cases} P \triangleleft N_G(P) \\ H \leq N_G(P) \end{cases} \implies HP \leq N_G(P)$ , we have

$$|HP| = \frac{|H||P|}{|H \cap P|} = p^{\alpha+k-s}$$

where  $|H \cap P| = p^s, s \leq k$ . Then  $p^{\alpha+k-s} \mid |N_G(P)| \mid |G| = p^\alpha r$ .

So  $k = s \implies H = H \cap P \implies H \leq P \cap Q$ . □

**Theorem 21** (Sylow I).  $\forall 0 \leq k \leq \alpha, \exists H \leq G$  s.t.  $|H| = p^k$ . In particular,  $\text{Syl}_p(G) \neq \emptyset$ .

*Proof.* By induction on  $|G|$ . If  $|G| = 1$ , then  $k = 0, H = \{1\}$ .

Assume  $|G| > 1, k \geq 1, \alpha \geq 1$ .

**case 1:**  $p \mid |Z_G|$ . By Cauchy theorem,  $\exists a \in Z_G$  with  $\text{ord}(a) = p$ . Then  $\langle a \rangle \triangleleft G$  and  $|G/\langle a \rangle| = p^{\alpha-1}r \leq |G|$ . If  $k = 1$ , then  $H = \langle a \rangle$ . Otherwise, we may assume that  $1 \leq k-1 \leq \alpha-1$ . By induction hypothesis,  $\exists H' = G/\langle a \rangle$  s.t.  $|H'| = p^{k-1}$ . By 3rd isom. thm., we can write  $H' = H/\langle a \rangle$  and thus  $|H| = p^k$ .

**case 2:**  $p \nmid |Z_G|$ . By the class equation,  $|G| = |Z_G| + \sum_{i=1}^m \frac{|G|}{|Z_G(a_i)|}, a_i \in Z_G$ .

In this cases,  $\exists a_j$  s.t.  $p \nmid \frac{|G|}{|Z_G(a_j)|} \implies p^\alpha \mid |Z_G(a_j)|$ . And  $Z_G(a_j) \leq G$  since  $a_j \in Z_G$ . By induction hypothesis,  $\exists H \leq Z_G(a_j) \leq G$  s.t.  $|H| = p^k$ . □

**Theorem 22** (Sylow II). Let  $P \in \text{Syl}_p(G)$  and  $Q$  be a  $p$ -subgroup of  $G$ . Then  $\exists a \in G$  s.t.  $Q \leq aPa^{-1}$ . In particular,  $\forall P_1, P_2 \in \text{Syl}_p(G), \exists a \in G$  s.t.  $P_2 = aP_1a^{-1}$ .

*Proof.* Let  $X = \{\text{left cosets of } P\}$  and consider  $\begin{matrix} Q \times X \rightarrow X \\ (a, xP) \mapsto axP \end{matrix}$ .

Observe that  $xP \in \text{Fix } Q \iff axP = xP \quad \forall a \in Q \iff x^{-1}axP = P \quad \forall a \in Q \iff x^{-1}ax \in P \quad \forall a \in Q \iff a \in xPx^{-1} \quad \forall a \in Q$ .

We know  $|\text{Fix } Q| \equiv |X| \pmod{p}$  and  $p \nmid r \implies |\text{Fix } Q| \neq 0 \iff \exists a \in G, Q \leq aPa^{-1}$ .

In particular,  $\begin{cases} P_2 \leq aP_1a^{-1} \\ |P_2| = |aP_1a^{-1}| \end{cases} \implies P_2 = aP_1a^{-1}$ . □

**Theorem 23** (Sylow III).  $n_p \equiv 1 \pmod{p}$  and  $n_p \mid r$ .

*Proof.* • Consider  $\begin{matrix} P \times \text{Syl}_p(G) \rightarrow \text{Syl}_p(G) \\ (a, Q) \mapsto aQa^{-1} \end{matrix}$  where  $P \in \text{Syl}_p(G)$ .

$$P' \in \text{Fix } P \iff aP'a^{-1} = P' \quad \forall a \in P \iff P \leq N_G(P') \cap P = P' \cap P \iff P' = P.$$

$$\text{So } \text{Fix } P = \{P\} \implies n_p \equiv |\text{Fix } P| = 1 \pmod{p}.$$

• Consider  $\begin{matrix} G \times \text{Syl}_p(G) \rightarrow \text{Syl}_p(G) \\ (a, Q) \mapsto aQa^{-1} \end{matrix} \implies$  There is only one orbit  $\text{Syl}_p(G)$ .

We know  $|\text{Syl}_p(G)| = \frac{|G|}{|G_Q|}$  and  $G_Q = N_G(Q)$ . Then  $n_p = \frac{|G|}{|G_Q|} \mid |G|$ . So  $n_p \mid p^\alpha r \implies n_p \mid r$ . □

**Prop 1.8.1.** Let  $|G| = pq$  where  $p, q$  are primes with  $\begin{cases} p < q \\ q \not\equiv 1 \pmod{p} \end{cases}$ . Then  $G \cong C_{pq}$ .

*Proof.*  $n_p = 1 + kp \mid q \implies n_p = 1$  i.e.  $H \in \text{Syl}_p(G) \implies H \triangleleft G$ .

$n_q = 1 + kq \mid p \implies n_q = 1$  i.e.  $K \in \text{Syl}_q(G) \implies K \triangleleft G$ .

Since  $\gcd(p, q) = 1$ ,  $H \cap K = 1$ . Hence  $G = H \times K \cong C_p \times C_q \cong C_{pq}$ . □

**Eg 1.8.2.** Consider  $|G| = 255 = 3 \times 5 \times 17$ .

1. 找兩個 normal subgroup (17, 5 or 3)
2. quot 掉後發現剩下的是 abelian  $\rightsquigarrow [G, G]$  在裡面
3.  $[G, G] = 1$
4. 唱 f.g. xxx thm. 得到  $G \cong \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{17}$ .
5. 中國剩飯定理  $G \cong C_{255}$ .

**Ex 1.8.4.** If  $|G| = 7 \times 11 \times 19$ , then  $G$  is abelian.

**Eg 1.8.3.** No group  $G$  of order  $48 = 2^4 \times 3$  is simple.

1.  $n_2 = 1 + 2k \mid 3 \rightsquigarrow n_2 = 1$  or  $3$ .
2.  $n_2 = 1$  then OK.
3. Assume  $n_2 = 3$ . Let  $P \in \text{Syl}_2(G)$ ,  $X = \{\text{left cosets of } P\}$  ( $|X| = 3$ ).
4. Consider  $\begin{matrix} G \times X \rightarrow X \\ (a, xP) \mapsto axP \end{matrix} \rightsquigarrow \varphi : G \rightarrow S_3$ .
5. 考慮  $\ker \varphi$ .

**Ex 1.8.5.** No group  $G$  of order 36 is simple.

**Ex 1.8.6.** No group  $G$  of order 30 is simple.

**Ex 1.8.7.** Let  $|G| = 385$ . Show that  $\exists P \in \text{Syl}_7(G)$  s.t.  $P \leq Z_G$ .



## 1.9 Week 9

### 1.9.1 Classification

To classify groups of small orders:

- $|G| = 1$ :  $G = \{1\}$
- $|G| = 2$ :  $G \cong C_2$
- $|G| = 3$ :  $G \cong C_3$
- $|G| = 4$ :  $G \cong \mathbb{Z}_4$  or  $\mathbb{Z}_2 \times \mathbb{Z}_2$
- $|G| = 5$ :  $G \cong C_5$
- $|G| = 6$ :  $n_3 = 1, n_2 = 1$  or  $3$ . Let  $H \in \text{Syl}_3(G)$  and  $H \triangleleft G$ . Let  $K \in \text{Syl}_2(G)$ . Also  $H \cap K = \{1\}$  and  $HK = G$  then  $G \cong K \rtimes_\tau H$ 
  - If  $\tau$  is trivial:  $G \cong K \times H \cong C_2 \times C_3 \cong C_6$
  - $\tau : b \mapsto \phi_2 : \langle a \rangle \rightarrow \langle a \rangle$ :  $G \cong K \rtimes_\tau H \cong \langle a, b \mid a^3 = 1, b^2 = 1, bab^{-1} = a^2 = a^{-1} \rangle \cong D_3$
- $|G| = 7$ :  $G \cong C_7$
- $|G| = 8$ :
  - If abelian:  $\mathbb{Z}_8$  or  $\mathbb{Z}_4 \times \mathbb{Z}_2$  or  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
  - If non-abelian:
    - \*  $\nexists a \in G$  with  $\text{ord}(a) = 8$
    - \* Not each  $a \in G$  with  $a^2 = 1$ , otherwise  $G$  is abelian.
    - \*  $\exists a \in G$  with  $\text{ord}(a) = 4$ : Let  $H = \langle a \rangle$  and  $H \triangleleft G$  since  $[G : H] = 2$ . Pick  $b \in G \setminus H$  and  $K = \langle b \rangle$ 
      - $\text{ord}(b) = 2$ :  $H \cap K = \{1\}$  and  $HK = G$  then  $G \cong K \rtimes_\tau H$ ,  $\tau : b \mapsto \phi : a \mapsto a^3$ :  
 $G \cong K \rtimes_\tau H \cong \langle a, b \mid a^4 = 1, b^2 = 1, bab^{-1} = a^3 = a^{-1} \rangle \cong D_4$
      - $\text{ord}(b) = 4$ :  $H \cap K = \langle a^2 = b^2 \rangle$ . Then consider  $bab^{-1} \in H \implies bab^{-1} = 1, a, a^2, a^3$ 
        1.  $1, a$  obviously wrong.
        2.  $bab^{-1} = a^2$ :  $a = a^2aa^{-2} = b^2ab^{-2} = a^4 \implies a^3 = 1$  矛盾
        3. So  $bab^{-1} = a^3 = a^{-1}$ .  
 $G \cong \langle a, b \mid a^4 = 1, b^4 = 1, a^2 = b^2, bab^{-1} = a^3 = a^{-1} \rangle \cong Q_8$
  - $|G| = 9$ :  $G \cong \mathbb{Z}_9$  or  $\mathbb{Z}_3 \times \mathbb{Z}_3$
  - $|G| = 10$ :  $G \cong K \times H \cong C_2 \times C_5 \cong C_{10}$  or  $G \cong D_5$
  - $|G| = 11$ :  $G \cong C_{11}$
  - $|G| = 12$ : Claim: If  $|G| = 12$ , then either  $G$  has a normal Sylow 3-subgroup or  $G \cong A_4$ .

*Proof.* By Sylow 3,  $n_3 = 1 + 3k \mid 4 \implies n_3 = 1$  or  $4$ .

- If  $n_3 = 1$ , then  $G$  has a normal Sylow 3-subgroup.
- Otherwise, let  $P \in \text{Syl}_3(G)$  and  $X = \{\text{left cosets of } P\}$ ,  $|X| = 4$ . Consider  $G \times X \rightarrow X$  defined by  $(a, xP) \mapsto axP$  with  $\phi : G \rightarrow S_4$ . And  $\ker \phi \leq P$ ,  $|P| = 3$  and  $P \not\triangleleft G$  (since  $n_3 = 4$ ), so  $\ker \phi = \{1\}$ .

And since  $n_3 = 4$ , there are 8 elements of order 3 which corresponds to 8 3-cycles in  $A_4$ , thus  $|\text{Im } \phi \cap A_4| \geq 8$ . But  $|\text{Im } \phi \cap A_4| \mid |A_4| = 12 \implies \text{Im } \phi = A_4$

□

Now, for the case where  $\exists H \in \text{Syl}_3(G)$  and  $H \triangleleft G$ . Let  $K \in \text{Syl}_2(G)$ , then  $K \cap H = \{1\}$  and  $KH = G \implies G \cong K \rtimes_\tau H$  for some  $\tau : K \rightarrow \text{Aut}(H) = \{\text{id}, \phi_2\}$

- $\tau$  is trivial:  $\mathbb{Z}_{12}$  or  $\mathbb{Z}_2 \times \mathbb{Z}_6$ .
- $\langle b \rangle = K \cong \mathbb{Z}_4$ :  $\tau(b) = \phi_2 \implies G = \langle a, b \mid a^3 = 1, b^4 = 1, bab^{-1} = a^{-1} \rangle \not\cong D_6, A_4$
- $\langle b \rangle = K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ : Let  $K = \langle b, c \mid b^2 = 1, c^2 = 1, bc = cb \rangle$ , then  $\tau : b \mapsto \phi_2$  and  $c \mapsto \text{id}$  (the other cases are equivalent to this one),  $G = \langle a, b, c \mid a^3 = 1, b^2 = 1, c^2 = 1, bc = cb, bab^{-1} = a^{-1}, cac^{-1} = a \rangle \cong \langle a, b \mid a^3 = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle \times \langle c \rangle \cong D_3 \times C_2 \cong D_6$

**Fact 1.9.1.** For odd  $n$ ,  $D_{2n} \cong D_n \times \mathbb{Z}/2\mathbb{Z}$ .

*Proof.*

$$\begin{aligned} D_{2n} &= \langle a, b \mid a^{2n} = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle \\ H &= \langle a^2, b \mid (a^2)^n = 1, b^2 = 1, b(a^2)b^{-1} = a^{-2} \rangle \cong D_n \\ K &= \langle a^n \rangle \cong C_2 \end{aligned}$$

And  $n$  is odd, so  $H \cap K = \{1\}$  and  $D_{2n} \cong D_n \times C_2$  □

- $|G| = 13$ :  $G \cong C_{13}$
- $|G| = 14$ :  $G \cong C_{14}$  or  $D_7$
- $|G| = 15$ :  $G \cong C_{15}$

**Ex 1.9.1.** Assume that  $K$  is cyclic and  $H$  is an arbitrary group. Let  $\tau_1 : K \rightarrow \text{Aut}(H)$ ,  $\tau_2 : K \rightarrow \text{Aut}(H)$  with  $\tau_1(K) \sim \tau_2(K)$  (conjugate). If  $|K| = \infty$ , then assume that  $\tau_1$  and  $\tau_2$  are injective. Show that  $K \rtimes_{\tau_1} H \cong K \rtimes_{\tau_2} H$ .

**Ex 1.9.2.** Classify  $G$  if  $|G| = p^3$  with  $p$  an odd prime and each nontrivial element of  $G$  has order  $p$ .

**Ex 1.9.3.** Classify groups of order 30.

## 1.9.2 Free groups

A free group generate by a non-empty set  $X$  is that there are no relations satisfied by any of elements in  $X$ .

**Def 45.** A free group on  $X$  is a group  $F$  with an inclusion map  $i : X \rightarrow F$  satisfying the following universal property: For any group  $G$  and any map  $f : X \rightarrow G$ , exists a unique group homo  $\varphi : F \rightarrow G$  that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\quad} & F \\ & \searrow & \downarrow \varphi \\ & & G \end{array}$$

**Theorem 24.**  $F$  exists and is unique up to isomorphism. (Denote it as  $F(X) = F$ ).

*Proof.* For  $X$ , we create a new disjoint set  $X^{-1} = \{x^{-1} : x \in X\}$  and an element  $1 \notin X \cup X^{-1}$ .

Define  $F(X) = \{1\} \cup \left\{ x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} : m \in \mathbb{N}, x_i \in X, \delta_i = \pm 1, x_{i+1}^{\delta_{i+1}} \neq (x_i^{\delta_i})^{-1} \right\}$ , and

$$x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} = y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_n^{\epsilon_n} \iff n = m \text{ and } \delta_i = \epsilon_i \text{ and } x_i = y_i, \forall i$$

For each  $y \in X \cup X^{-1}$ , we define  $\sigma_y : F(X) \rightarrow F(X)$  by

$$\sigma_y(x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m}) = \begin{cases} y x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} & \text{if } x_1^{\delta_1} \neq y^{-1} \\ \begin{cases} x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} & (m \geq 2) \\ 1 & (m = 1) \end{cases} & \text{if } x_1^{\delta_1} = y^{-1} \end{cases}$$

Then  $\sigma_y$  is a permutation of  $F(X)$ , since if  $\sigma_y(x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m}) = \sigma_y(y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_m^{\epsilon_m})$ .

m = n: either  $x_1^{\delta_1} = y_1^{\epsilon_1} = y^{-1}$  or not, then either  $x_2^{\delta_2} x_3^{\delta_3} \cdots x_m^{\delta_m} = y_2^{\epsilon_2} y_3^{\epsilon_3} \cdots y_m^{\epsilon_m}$  or  $y x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} = y y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_m^{\epsilon_m}$ . Both of them leads to  $x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} = y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_m^{\epsilon_m}$ .

m = n+2: Omimi

Also  $\sigma_y$  is onto since omimi. And notice that  $\sigma_{y^{-1}} \circ \sigma_y = id_{F(X)}$

Define  $A = \langle \sigma_x : x \in X \rangle \leq S_{F(X)}$ . and define  $\phi : F(X) \rightarrow A$  by  $\phi(1) = id_{F(X)}$  and  $x_1^{\delta_1} \cdots x_m^{\delta_m} \mapsto \sigma_{x_1}^{\delta_1} \cdots \sigma_{x_m}^{\delta_m}$ . The it is omimi that  $\phi$  is a bijection. So we define  $x :: X \cdot y :: X = \phi^{-1}(\phi(x) \circ \phi(y))$ .

The  $\phi$  in the universal property could be defined as  $\phi(x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m}) = f(x_1)^{\delta_1} \cdots f(x_m)^{\delta_m}$ .  $\square$

**Prop 1.9.1.** Let  $G = \langle a_1, \dots, a_n \rangle$  and  $X = \{x_1, \dots, x_m\}$ . Then  $G \cong F(X)/K$  for some normal subgroup  $K$ .  $K$  is called the subgroup of relations connecting the generators.

Define  $f = x_i :: X_i \rightarrow a_i :: G$ . By universal property,  $\exists \phi = x_i :: F(X) \mapsto a_i :: G$ . Then  $F(x)/\ker \phi \cong G$ .

**Def 46.** Let  $X = \{x_1, x_2, \dots, x_n\}$  and  $R \subset F(X)$ . Let  $N(R)$  be the smallest normal subgroup of  $F(X)$  containing  $R$ , Then  $G = F(X)/N(R)$  is written as  $\langle x_1, \dots, x_n \mid \text{elements of } R \rangle$ , which is called a presentation of  $G$ . If  $|R| < \infty$ , then  $G$  is said to be finitely presented.

**Eg 1.9.1.**

$$D_n = \left\langle \begin{bmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

We find that  $x^n, y^2, xyxy \in \ker \phi$ . Then  $R = \{x^n, y^2, xyxy\} \subseteq \ker \phi \implies N(R) \leq \ker \phi$ . By factor theorem,  $\exists \bar{\phi} :: F(X)/N(R) \rightarrow D_n$ . But notice that

$$|F(x)/N(R)| \leq 2n$$

since  $xyxy = 1 \implies xy = yx^{-1}$ , so every element could be turn into  $x^i y^j$ . Hence  $\bar{\phi}$  is an isomorphism.

**Prop 1.9.2.** Let  $X = \{x_1, x_2, \dots, x_n\}$ . Then  $F(X)/[F(X), F(X)] \cong \mathbb{Z}^n$ .

*Proof.* Define  $f = x_i :: X \mapsto e_i :: \mathbb{Z}^n$ . Then  $\phi = x_i :: F(X) \mapsto e_i :: \mathbb{Z}^n$ . By 1st isomorphism theorem  $F(X)/\ker \phi \cong \mathbb{Z}^n$  which is abelian, so  $[F(X), F(X)] \leq \ker \phi$ . By factor theorem, 一個圖.

Claim that  $\bar{\phi}$  is 1-1.

*Proof.* Since  $F(X)/[F(X), F(X)]$  is abelian,  $\forall a \in F(X)/[F(X), F(X)]$ , we can write  $a = \bar{x}_1^{n_1} \bar{x}_2^{n_2} \cdots \bar{x}_m^{n_m}$ . If  $\bar{\phi}(\bar{a}) = (m_1, \dots, m_n) = 0$  in  $\mathbb{Z}^n$ , then  $m_i = 0, \forall i \implies a = 1$   $\square$

$\square$

## 2 Multilinear algebra

### 2.1 Week 11

#### 2.1.1 Bilinear forms & Groups preserving bilinear forms

**Def 47.** Let  $V$  be a vector space over a field  $F$ .

- A function  $f : V \times V \rightarrow F$  is called a bilinear form if

$$\begin{cases} f(rx_1 + x_2, y) = rf(x_1, y) + f(x_2, y) \\ f(x, ry_1 + y_2) = rf(x, y_1) + f(x, y_2) \end{cases} \quad \forall x_1, x_2, x, y_1, y_2, y \in V, r \in F$$

- $B_F(V, V) = \{ \text{bilinear forms on } V \}$  can be regarded as a vector space over  $F$ .

**Theorem 25.** Let  $\dim V = n$  and  $\beta = \{v_1, \dots, v_n\}$  be a basis for  $V$ . Then  $\exists$  an isomorphism  $\psi_\beta : B_F(V, V) \rightarrow M_{n \times n}(F)$ .

*Proof.* For  $v, w \in V$ , write  $v = \sum_i a_i v_i, w = \sum_j b_j v_j$ , i.e.  $[v]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, [w]_\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ .

For  $f \in B_F(V, V)$ ,  $f(v, w) = \sum_i \sum_j a_i b_j f(v_i, v_j) = \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \begin{pmatrix} f(v_i, v_j) \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ .

Define  $\psi_\beta(f) = A$  with  $A_{ij} = f(v_i, v_j)$ .

- $\psi_\beta$  is a linear transformation.
- $\psi_\beta$  is 1-1.
- $\psi_\beta$  is onto:  $\forall A \in M_{n \times n}(F)$ , we define  $f(v, w) = [v]_\beta^t A [w]_\beta$ . □

**Def 48.** Let  $f \in B_F(V, V)$

- $f$  is said to be symmetric if  $f(v, w) = f(w, v) \quad \forall v, w \in V$ .
- $f$  is said to be skew-symmetric if  $f(v, w) = -f(w, v) \quad \forall v, w \in V$ .
- $f$  is said to be alternating if  $f(v, v) = 0 \quad \forall v \in V$ .

**Remark 12.**

- Alternating  $\implies$  skew-symmetric.
- If  $\text{char } F \neq 2$ , skew-symmetric  $\implies$  alternating.
- If  $\text{char } F = 2$ , symmetric = skew-symmetric.
- $\forall f \in B_F(V, V)$  with  $\text{char } F \neq 2$ ,

$$f_s(u, v) = \frac{1}{2} (f(u, v) + f(v, u))$$

$$f_a(u, v) = \frac{1}{2} (f(u, v) - f(v, u))$$

and  $f(u, v) = f_s(u, v) + f_a(u, v)$ .

So we only need to study “symmetric” & “alternating”.

**Ex 2.1.1.**

1. If  $A$  and  $B$  are congruent ( $B = Q^t A Q$ ) in  $M_{n \times n}(F)$ , then they define the same bilinear form.
2.  $f$  is  $\begin{cases} \text{symmetric} \\ \text{skew-symmetric} \end{cases} \iff \psi_\beta(f)$  is  $\begin{cases} \text{symmetric}(A^t = A) \\ \text{skew-symmetric}(A^t = -A) \end{cases}$

**Observation.** Let  $f \in B_F(V, V)$  and  $v_0 \in V$ .

$$\begin{aligned} L_f(v_0) &= f(v_0, \cdot) \in V' = \text{Hom}(V, F) : \text{the dual space of } V \\ R_f(v_0) &= f(\cdot, v_0) \in V' \end{aligned}$$

The left radical of  $f$  :  $\text{lad}(f) = N(L_f) = \{v \in V \mid f(v, w) = 0 \quad \forall w \in V\}$ .

The right radical of  $f$  :  $\text{rad}(f) = N(R_f) = \{w \in V \mid f(v, w) = 0 \quad \forall v \in V\}$ .

**Ex 2.1.2.**

1.  $\text{rank}(\psi_\beta(f)) = \text{rank}(R_f) = \text{rank}(L_f)$ .
  2. If  $\dim V = n$ , then TFAE ( $\implies f$  : non degenerate)
    - (a)  $\text{rank}(f) = n$ .
    - (b)  $\forall v \in V, v \neq 0, \exists w \in V$  s.t.  $f(v, w) \neq 0$ .
    - (c)  $\text{lad}(f) = \{0\}$ .
    - (d)  $L_f : V \rightarrow V'$  is isom.
- (also, right)

**Theorem 26** (Principal Axis theorem). Let  $\dim V = n$  and  $\text{char } F \neq 2$ . If  $f \in B_F(V, V)$  is symmetric, then  $\exists \beta$  s.t.  $\psi_\beta(f)$  is diagonal.

*Proof.* It is sufficient to find  $\beta = \{v_1, \dots, v_n\}$  s.t.  $f(v_i, v_j) = 0 \quad \forall i \neq j$ .

If  $f = 0$ , then done! Assume  $f \neq 0$ . By induction on  $n$ : If  $n = 1$ , done. Let  $n > 1$ .

Claim 1:  $\exists v_1 \in V$  s.t.  $f(v_1, v_1) \neq 0$ . Assume that  $f(v, v) = 0 \quad \forall v \in V$ .

$$f(v, w) = \frac{1}{2}(f(v+w, v+w) - f(v, v) - f(w, w)) = 0. \quad ^1$$

So  $f = 0$ , which is a contradiction.

Now let  $v_1 \in V$  with  $f(v_1, v_1) \neq 0$ . Let  $W = \langle v_1 \rangle_F$  and  $W^\perp = \{w \in V \mid f(v_1, w) = 0\} \subseteq V$ .

Claim 2:  $V = W \oplus W^\perp$

- $V = W + W^\perp$ : For all  $v \in V$ , let  $a = f(v, v_1)/f(v_1, v_1)$ , then  $v = av_1 + (v - av_1) \triangleq w + w'$  where  $w \in W$  and  $f(w', v_1) = f(v - av_1, v_1) = f(v, v_1) - af(v_1, v_1) = 0$ . So  $w' \in W^\perp$  and thus  $V = W + W^\perp$ .
- $W \cap W^\perp = \{0\}$ : obviously since if  $av_1 \in W$ ,  $f(av_1, v_1) = 0 \iff a = 0 \iff av_1 = 0$ .

Since  $f|_{W^\perp \times W^\perp}$  is a symmetric bilinear form on  $W^\perp$  and  $\dim W^\perp < \dim V$ . By induction hypothesis,  $\exists \{v_2, \dots, v_n\}$  a basis for  $W^\perp$  s.t.  $f(v_i, v_j) = 0 \quad \forall i \neq j$ . Then  $\beta = \{v_1, \dots, v_n\}$ .  $\square$

<sup>1</sup>The argument in class requires  $\text{char } F \geq 4$ , omimi...

**Theorem 27** (Sylvester's theorem). Let  $f \in B_{\mathbb{R}}(V, V)$  be symmetric with  $\dim V = n$ . Then  $\exists \beta$

$$\text{s.t. } \psi_{\beta}(f) = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix}.$$

The triple (# of 1, # of -1, # of 0) is well-defined. (called the signature of  $f$ )

*Proof.* Assume  $V^+ = \langle v_1, \dots, v_p \rangle_F, V^- = \langle v_{p+1}, \dots, v_r \rangle_R, V^{\perp} = \langle v_{r+1}, \dots, v_n \rangle_F$ . ( $V = V^+ \oplus V^- \oplus V^{\perp}$ )

Claim: If  $W$  is a subspace of  $V$  s.t.  $f$  is positive-definite on  $W$ , then  $W, V^-, V^{\perp}$  are independent.

Let  $\langle w_1, w_2, \dots, w_s \rangle$  be a basis of  $W$ . If

$$a_1 w_1 + a_2 w_2 + \dots + a_s w_s = b_{p+1} v_{p+1} + \dots + b_r v_r + c_{r+1} v_{r+1} + \dots + c_n v_n.$$

Let  $w \triangleq a_1 w_1 + \dots + a_s w_s, v \triangleq b_{p+1} v_{p+1} + \dots + b_r v_r + c_{r+1} v_{r+1} + \dots + c_n v_n$ . Since  $w = v$ ,  $f(w, w) = f(v, v)$ . but  $f(w, w) = \sum a_i^2 \geq 0$  and  $f(v, v) = -\sum b_i^2 \leq 0$ . Hence  $a_i = 0, b_i = 0$ . Since  $v_{r+1}, \dots, v_n$  is linearly independent,  $c_i = 0$ . Therefor these vectors are linear independent.

□

**Ex 2.1.3.** Let  $f \in B_F(V, V)$  with  $\text{char } F \neq 2$ . If  $f$  is skew-symmetric, then  $\exists \beta$  s.t.

$$\psi_{\beta}(f) = \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & 1 & \\ & & & & & -1 & 0 & \\ & & & & & & & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix}$$

**Ex 2.1.4.** Study Hermitian form

$\mathsf{T} : V \xrightarrow{\sim} V, f \in B_F(V, V)$ .  $\mathsf{T}$  preserves  $f$  if  $f(\mathsf{T}(v), \mathsf{T}(w)) = f(v, w) \quad \forall v, w \in V$ .

In matrix form, let  $\beta$  be a basis for  $V$ ,  $M = [\mathsf{T}]_{\beta}, A = \psi_{\beta}(f)$ , then  $A = M^t A M$ .

- $f \in B_{\mathbb{R}}(V, V)$  symmetric, non-degenerate:  $\exists \beta$  s.t.  $\psi_{\beta}(f) = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$ .

Then  $\{ \mathsf{T} : V \xrightarrow{\sim} V \text{ preserves } f \} \leftrightarrow \left\{ M \in \text{GL}_n(\mathbb{R}) \mid M^t \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} M = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} \right\} = \text{O}(p, q).$

- $f \in B_{\mathbb{R}}(V, V)$  skew-symmetric, non-degenerate:  $n = 2k$ ,  $\exists \beta$  s.t.  $\psi_{\beta}(f) = J$ .

Then  $\{T : V \xrightarrow{\sim} V \text{ preserves } f\} \leftrightarrow \{M \in \text{GL}_n(\mathbb{R}) | M^t J M = J\}$ , where

$$J = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$$

### 2.1.2 Tensor product

From now on,  $R$  is assumed to be commutative with 1.

**Def 49.** Let  $M_1, \dots, M_n, L$  be  $R$ -modules.

A function  $F : M_1 \times \dots \times M_n \rightarrow L$  is said to be  $n$ -multilinear if  $\forall i$ ,

$$f(x_1, \dots, rx_i + x'_i, \dots, x_n) = rf(x_1, \dots, x_i, \dots, x_n) + f(x_1, \dots, x'_i, \dots, x_n) \quad \forall r \in R, x_i, x'_i \in M_i$$

If  $n = 2$ ,  $f$  is called a bilinear map.

**Def 50.** Let  $M, N$  be  $R$ -modules. A tensor product of  $M$  and  $N$  is an  $R$ -module  $M \otimes_R N$  with a bilinear map  $\rho : M \times N \rightarrow M \otimes_R N$  satisfying the following universal property:

for any  $R$ -module  $W$  and any bilinear map  $f : M \times N \rightarrow W$ ,  $\exists!$   $R$ -module homomorphism  $\varphi : M \otimes_R N \rightarrow W$ ,

$$\begin{array}{ccc} M \times N & \xrightarrow{\rho} & M \otimes_R N \\ & \searrow f & \downarrow \varphi \\ & & W \end{array}$$

**Theorem 28** (Main theorem).  $M \otimes_R N$  exists and is unique up to isom.

*Proof.* Let  $X = M \times N$ . First we construct the free module  $V_1 = \bigoplus_{(x,y) \in X} R \cdot (x, y)$ .

Notice that in  $V_1$ ,

- $(x_1, y_1) + (x_2, y_2) \neq (x_1 + x_2, y_1 + y_2)$ .
- $r(x, y) \neq (rx, ry)$ .
- $r(r_1(x_1, y_1) + \dots + r_n(x_n, y_n)) = rr_1(x_1, y_1) + \dots + rr_n(x_n, y_n)$ .

$$\text{Let } V_0 = \left\langle \begin{array}{l} (x_1 + x_2, y) - (x_1, y) - (x_2, y), \\ (x, y_1 + y_2) - (x, y_1) - (x, y_2), \\ r(x, y) - (rx, y), r(x, y) - (x, ry) \end{array} \middle| x_1, x_2, x \in M, y_1, y_2, y \in N, r \in R \right\rangle_R.$$

Define  $M \otimes_R N = V_1/V_0$  which is an  $R$ -module and  $\rho : M \times N \rightarrow M \otimes_R N$  which is  $R$ -bilinear. (check yourself)

Universal property:  $\forall (x, y) \in M \times N$ ,  $\begin{matrix} R(x, y) \rightarrow W \\ r(x, y) \mapsto rf(x, y) \end{matrix}$ . So, by the universal property of  $\oplus$ ,  $\exists!$   $R$ -module homo.  $\varphi_1 : V_1 \rightarrow W$ :

$$\begin{array}{ccc} M \times N & \xrightarrow{i} & V_1 \\ & \searrow f & \downarrow \varphi_1 \\ & & W \end{array}$$

Claim:  $V_0 \subseteq \ker \varphi_1$ . (check yourself) Then by factor theorem,

$$\begin{array}{ccc} \exists !\varphi : V_1/V_0 & \xrightarrow{\quad} & W \\ & \nwarrow \quad \nearrow & \\ & M \times N & \end{array}$$

□

**Eg 2.1.1.**  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$ .

**Eg 2.1.2.**  $\mathbb{R}[x, y] \cong \mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{R}[y]$ .

*Proof.*  $\mathbb{R}[x] \times \mathbb{R}[y] \rightarrow \mathbb{R}[x, y]$  is bilinear  $\rightsquigarrow \exists !\varphi : \mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{R}[y] \rightarrow \mathbb{R}[x, y]$   
 $(f(x), g(y)) \mapsto f(x)g(y) \quad f(x) \otimes g(y) \mapsto f(x)g(y)$

Conversely,  $\mathbb{R}[x, y] \rightarrow \mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{R}[y]$   
 $h(x, y) = \sum a_{ij}x^i y^j \mapsto \sum a_{ij}x_i \otimes y_j$

□

**Prop 2.1.1.** If  $M = \langle x_1, \dots, x_n \rangle_R$  and  $N = \langle y_1, \dots, y_m \rangle_R$ . Then

$$M \otimes_R N = \langle x_i \otimes y_j \mid i = 1, \dots, n; j = 1, \dots, m \rangle_R.$$

In particular, if  $R$  is a field  $F$ , then  $\dim_F M \otimes_F N = (\dim_F M)(\dim_F N)$ .

*Proof.* Note that  $M \otimes_R N = \langle x \otimes y \mid x \in M, y \in N \rangle$ . Let  $x = \sum_i a_i x_i, y = \sum_j b_j y_j$ . Then  $x \otimes y = \sum_i \sum_j a_i b_j x_i \otimes y_j$ . □

Some canonical isomorphisms:

- $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$ .

*Proof.*  $\forall z \in L, M \times N \rightarrow M \otimes_R (N \otimes_R L)$  is bilinear.  $\exists ! R$ -mod homo.  $\varphi_z : M \otimes_R N \rightarrow$   
 $(x, y) \mapsto x \otimes (y \otimes z)$

$M \otimes_R (N \otimes_R L)$ . Similarly,  $(M \otimes_R N) \times L \rightarrow M \otimes_R (N \otimes_R L)$  is bilinear. (The right is due  
 $(\sum x_i \otimes y_i, z) \mapsto \sum x_i \otimes (y_i \otimes z)$  to  $\varphi_z$  linear, and the left is because  $x \otimes (y \otimes (rz_1 + z_2)) = rx \otimes (y \otimes z_1) + x \otimes (y \otimes z_2)$ .) Hence  
exists unique  $R$ -mod homo.  $\varphi : (M \otimes_R N) \otimes_R L \rightarrow M \otimes_R (N \otimes_R L)$ . By the symmetric  
construction, we have  $\varphi^{-1}$  and  $\varphi^{-1} \circ \varphi = \varphi \circ \varphi^{-1} = 1$ , so the two are isomorphic. □

- $(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$ .

The mapping  $\psi : (M \oplus M') \times N \rightarrow (M \otimes_R N) \oplus (M' \otimes_R N)$  by  $\psi = ((x, x'), y) \mapsto (x \otimes y, x' \otimes y)$   
is bilinear, hence exists  $R$ -mod homomorphism  $\varphi : (M \oplus M') \otimes_R N \rightarrow (M \otimes_R N) \oplus (M' \otimes_R N)$ .

On the other hand, The mapping  $(x, y) : M \times N \mapsto (x, 0) \otimes y : (M \oplus M') \otimes_R N$  is bilinear. So  
exists  $\phi_1 : M \otimes_R N \rightarrow (M \oplus M') \otimes_R N$ , similarly there exists  $\phi_2 : M' \otimes_R N \rightarrow (M \oplus M') \otimes_R N$ .  
Now by the universal property of direct sum, there exists  $\phi : (M \otimes_R N) \oplus (M' \otimes_R N) \rightarrow$   
 $(M \oplus M') \otimes_R N$ . After a careful examine, we have

$$\varphi = (x, x') \otimes y \mapsto (x \otimes y, x' \otimes y), \phi = (x \otimes y, x' \otimes y) \mapsto (x, x') \otimes y$$

Thus  $\phi = \varphi^{-1}$  and hence the two are isomorphic.

**Ex 2.1.5.**

1.  $R \otimes_R M \cong M$ .
2.  $M \otimes_R N \cong N \otimes_R M$ .



**Ex 2.1.6.**  $R/I \otimes_R N \cong N/IN$  where  $IN := \{\sum a_i x_i \mid a_i \in I, x_i \in N\}$ .

**Ex 2.1.7.** Compute  $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q})$ ,  $\dim_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C})$ ,  $\dim_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ ,  $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$

## 2.2 Week 12

### 2.2.1 Tensor product II

By universal property, we get  $\{R\text{-bilinear maps } M \times N \rightarrow L\} \leftrightarrow \text{Hom}_R(M \otimes_R N, L)$ .

Similarly,

$$\begin{aligned}\text{Hom}\left(\bigoplus_{s \in \Lambda} M_s, L\right) &\cong \prod_{s \in \Lambda} \text{Hom}(M_s, L) \\ \text{Hom}\left(N, \prod_{s \in \Lambda} M_s\right) &\cong \prod_{s \in \Lambda} \text{Hom}(N, M_s)\end{aligned}$$

**Fact 2.2.1.**  $f \in \text{Hom}_R(M, M'), g \in \text{Hom}_R(N, N') \rightsquigarrow f \otimes g \in \text{Hom}_R(M \otimes N, M' \otimes N')$  by  $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$ .

*Proof.* Define  $h : M \times N \rightarrow M' \otimes_R N'$   
 $(x, y) \mapsto f(x) \otimes g(y)$  □

Restriction and extension of scalars.

Let  $f : R \rightarrow S$  be a ring homomorphism and  $R, S$  be commutative with 1. Then  $S$  can be regarded as an  $R$ -module.  $\left( \begin{array}{l} R \times S \rightarrow S \\ (r, x) \mapsto f(r)x \end{array} \right)$ .

If  $M$  is a  $S$ -module, then  $M$  is also an  $R$ -module.  $\left( \begin{array}{l} R \times M \rightarrow M \\ (r, a) \mapsto f(r)a \end{array} \right)$ .

If  $N$  is an  $R$ -module, then  $S \otimes_R N$  an  $S$ -module.  $\left( \begin{array}{l} S \times (S \otimes_R N) \rightarrow S \otimes_R N \\ (r, x \otimes a) \mapsto rx \otimes a \end{array} \right)$ .

**Eg 2.2.1** (Important example). Let  $V$  be a real vector space. The complexification of  $V$  is  $V^{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V$  which is a  $\mathbb{C}$ -vector space.

**Ex 2.2.1.** Let  $K \subseteq L$  be an inclusion of fields and let  $E$  be a vector space over  $K$ . Show that  $E^L := L \otimes_K E$  satisfies the following universal property: For any vector space  $U$  over  $L$  and any  $K$ -linear map  $f : E \rightarrow U$ ,  $\exists ! L$ -linear map  $\varphi$ :

$$\begin{array}{ccc} \varphi : 1 \otimes x :: E^L & \xrightarrow{\quad} & f(x) :: U \\ & \nwarrow \quad \nearrow f & \\ & x :: E & \end{array}$$

**Ex 2.2.2.**  $E \rightarrow E^L$  is a covariant functor from the category of vector spaces over  $K$  to the category of vector spaces over  $L$ .

**Eg 2.2.2.**  $\mathbb{Z}^n \cong \mathbb{Z}^m \rightsquigarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^m \rightsquigarrow n = m$ .

**Eg 2.2.3.**  $G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_l\mathbb{Z} \oplus \mathbb{Z}^s, \mathbb{Q} \otimes_{\mathbb{Z}} G = \mathbb{Q}^s$ .

Let  $M, N$  and  $U$  be  $R$ -module. Then

$$\text{Hom}_R(M \otimes_R N, U) \cong \text{Hom}_R(N, \text{Hom}_R(M, U))$$

*Proof.*

- For  $f \in \text{Hom}_R(M \otimes_R N, U)$  and  $a \in N$ , define  $f_a = x \mapsto f(x \otimes a) \in U$ .
  - linear: easy.
  - $\bar{f} : a \mapsto f_a$  is an  $R$ -mod homo.: easy.
  - $\tau : f \mapsto \bar{f}$  is an  $R$ -mod homo.:  $\tau(rf + g)(a)(x) = (rf + g)_a(x) = (rf + g)(x \otimes a) = rf(x \otimes a) + g(x \otimes a) = \dots = r\tau(f)(a)(x) + \tau(g)(a)(x)$
- For  $g \in \text{Hom}_R(N, \text{Hom}_R(M, U))$ , define  $g' = (x, a) \mapsto g(a)(x) \in U$ .
  - $g'$  is  $R$ -bilinear: easy.
  - $\exists ! \tilde{g} : x \otimes a \mapsto g(a)(x)$ .
  - $\sigma : g \mapsto \tilde{g}$  is an  $R$ -mod homo.: easy.
- $\sigma\tau = \text{id}, \tau\sigma = \text{id}$ : easy... □

**Ex 2.2.3.**  $\text{Hom}_R(M, \cdot), M \otimes_R \cdot$  are covariant functors from the category of  $R$ -modules to itself. (is an adjoint pair)

**Fact 2.2.2.**  $\text{Hom}_R(R, M) \cong M$ . By  $f \mapsto f(1)$ .

**Def 51.** An exact sequence  $A \xrightarrow{f_1} B \xrightarrow{f_2} \dots$  is a sequence satisfying  $\text{im } f_k = \ker f_{k+1}$ .

- $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ .
- $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ .

Let  $V, W$  be vector spaces over  $F$ . Then  $V^* \otimes_F W \cong \text{Hom}_F(V, W)$ .

*Proof.* Let  $\alpha = \{e_1, \dots, e_n\}$  and  $\beta = \{f_1, \dots, f_m\}$  be bases for  $V$  and  $W$  respectively. Via  $\alpha, \beta$ ,  $\text{Hom}_F(V, W) \cong \left\langle E_{ij} \left| \begin{matrix} i = 1, \dots, m \\ j = 1, \dots, n \end{matrix} \right. \right\rangle_F$ .  $V^* \otimes W \cong \left\langle e_j^* \otimes f_i \left| \begin{matrix} i = 1, \dots, m \\ j = 1, \dots, n \end{matrix} \right. \right\rangle_F$ . □

## 2.2.2 Tensor algebra

**Def 52.**

- Let  $R$  be a commutative ring with 1. An  $R$ -algebra is a ring  $A$  which is also an  $R$ -module s.t. the multiplication map  $A \times A \rightarrow A$  is  $R$ -bilinear. (  $r(ab) = (ra)b = a(rb)$  )
- Let  $A$  be an  $R$ -algebra. A grading of  $A$  is a collection of  $R$ -submodules  $\{A_n\}_{n=0}^\infty$  ( $n$ -th homogeneous part) s.t.

$$A = \bigoplus_{n=0}^{\infty} A_n \quad \text{and} \quad A_n A_m \subseteq A_{n+m} \quad \forall n, m$$

- A graded  $R$ -algebra is an  $R$ -algebra with a chosen grading.
- $\mathfrak{M}_R$  is the category of  $R$ -modules.
- $\mathfrak{Gr}_R$  is the category of graded  $R$ -algebras. ( $f : A \rightarrow A'$  with  $f(A_n) \subseteq A'_n$ )

**Eg 2.2.4.**  $A = R[x], A_n = \langle x^n \rangle_R$ . If  $I = \langle x+1 \rangle_A$ ,  $I$  is not graded.  $I = \langle x^2 \rangle_A$  is graded.

**Def 53.** An ideal  $I$  is graded in a graded ring  $A$  if and only if  $I = \bigoplus I \cap A_n$ .<sup>2</sup>

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<sup>2</sup>This is not mentioned in class

**Ex 2.2.4.** TFAE

- (1)  $I$  is graded.
- (2)  $\forall a \in I$  write  $a = a_{k_1} + a_{k_2} + \cdots + a_{k_m}, a_{k_i} \in A_{k_i} \implies a_{k_i} \in I$ . ( $a_{k_i}$  is the homogenous component of  $a$ )
- (3)  $A/I$  is a graded ring with  $(A/I)_n = (A_n + I)/I \cong A_n/I \cap A_n$ .

**Ex 2.2.5.**

- (1) If  $I$  is a f.g. graded ideal, then  $I$  has a finite system of generators consisting of homogeneous elements alone.
- (2)  $I, J$  are graded  $\implies I + J, IJ, I \cap J$  are graded.

Observation: Let  $\{M_i\}_{i=1}^\infty$  be a collection of  $R$ -modules.

- $M_1 \otimes_R M_2$  exists.
- $(M_1 \otimes_R M_2) \otimes_R M_3 \cong M_1 \otimes_R (M_2 \otimes_R M_3) \implies M_1 \otimes_R M_2 \otimes_R M_3$  is well-defined. Universal property: for any  $R$ -module  $L$  and a 3-multilinear map  $f : M_1 \times M_2 \times M_3 \rightarrow L$ . (拆括號囉)
- By induction,  $M_1 \otimes \cdots \otimes M_n$  is well-defined and satisfies the universal property. ( $n$ -multilinear map)

Goal: For a given  $R$ -module  $M$ , we intend to construct an graded  $R$ -algebra  $T(M)$  containing  $M$  that is “universal” w.r.t.  $R$ -algebras containing  $M$ .

That is, a tensor algebra is a pair  $(T(M), i)$  where  $T(M)$  is an  $R$ -algebra and  $i :: M \rightarrow T(M)$ , such that for any  $R$ -algebra  $A$  containing  $M$ , which is to say that exist a  $R$ -module homomorphism  $\varphi : M \rightarrow A$ , then  $\exists$  an  $R$ -algebra homomorphism  $\psi :: T(M) \rightarrow A$  such that  $\varphi = \psi \circ i$ .

Construction:

- $\forall k \in \mathbb{N}, T^k(M) := \underbrace{M \otimes \cdots \otimes M}_{k \text{ times}}$ , each  $x_1 \otimes x_2 \otimes \cdots \otimes x_k \in T^k(M)$  is called a  $k$ -tensor.

$T^0(M) := R$  and

$$T(M) := \bigoplus_{k=0}^{\infty} T^k(M) = R \oplus T^1(M) \oplus \cdots$$

- define multiplication on  $T(M)$  by:

$$\begin{aligned} T^i(M) \times T^j(M) &\longrightarrow T^{i+j}(M) \\ (x_1 \otimes \cdots \otimes x_i, y_1 \otimes \cdots \otimes y_j) &\longmapsto x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j \end{aligned}$$

- Distribution law: easy.

Proving the universal property: For any  $R$ -algebra  $A$  containing  $M$  and an  $R$ -module homo.  $\varphi : M \rightarrow A$ .  $\forall k \geq 2$ , we define  $f_k : M \times \cdots \times M \rightarrow A$

$$\begin{aligned} f_k : M \times \cdots \times M &\rightarrow A \\ (x_1, \dots, x_k) &\mapsto \varphi(x_1) \cdots \varphi(x_k) \end{aligned}$$

$f_k$  is  $k$ -multilinear  $\rightsquigarrow$

$$\begin{aligned} \exists ! \tilde{f}_k : M \otimes \cdots \otimes M &\rightarrow A \\ x_1 \otimes \cdots \otimes x_k &\mapsto \varphi(x_1) \cdots \varphi(x_k) \end{aligned}$$

By the universal property of  $\bigoplus$ , exists a unique  $R$ -module homo.  $\tilde{\varphi} :: T(M) \rightarrow A$  which make the following diagram commutes.

$$\begin{array}{ccc} \tilde{\varphi} : T(M) & \xrightarrow{\quad} & A \\ & \nwarrow i \quad \nearrow f_k & \\ & T^k(M) & \end{array}$$

$\tilde{\varphi}$  is an  $R$ -algebra homomorphism.

**Def 54.**  $T(M)$  is called the tensor algebra of  $M$ .

**Ex 2.2.6.**  $T$  is a covariant functor from  $\mathfrak{M}_R$  to  $\mathfrak{Gr}_R$ .

**Prop 2.2.1.** Let  $V$  be a vector space over  $F$  with a basis  $\beta = \{v_1, \dots, v_n\}$ . Then

$$\{v_{i_1} \otimes \dots \otimes v_{i_k} \mid \forall j = 1, \dots, k, i_j = 1, \dots, n\}$$

forms a basis for  $T^k(V)$ .  $\dim_F T^k(V) = n^k$ .

$T(V)$  can be regarded as a non-commutative polynomial algebra over  $F$ .

⊙ Symmetrization ( $\text{char } F = 0$ )

$$\begin{aligned} V \times \dots \times V &\longrightarrow T^n(V) \\ (x_1, \dots, x_n) &\longmapsto \frac{1}{n!} \sum_{\tau \in S_n} x_{\tau(1)} \otimes \dots \otimes x_{\tau(n)} \end{aligned}$$

is  $n$ -multilinear.

The symmetrizer operator  $\sigma : T^n(V) \rightarrow T^n(V)$ ,  $\tilde{S}^n(V) := \sigma(T^n(V)) \subseteq T^n(V)$ .

Claim:  $T^n(V) = \tilde{S}^n(V) \oplus C^n(V)$  where

$$C^n(V) = C(V) \cap T^n(V) \quad C(V) = \langle v \otimes w - w \otimes v \mid v, w \in V \rangle$$

## 2.3 Week 13

### 2.3.1 Symmetric and Exterior algebra

**Symmetric algebra** Define

$$\begin{aligned} S : \mathfrak{M}_R &\rightarrow \mathfrak{Gr}_R \\ M &\mapsto T(M)/C(M) \end{aligned} \quad S(M) := T(M)/C(M)$$

where  $C(M)$  is the graded two-sided ideal generated by  $u \otimes v - v \otimes u$  with  $u, v \in M$ .

- $C^k(M) := C(M) \cap T^k(M)$  is the submodule of  $T^k(M)$  generated by all

$$x_1 \otimes \dots \otimes x_k - x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)} \quad \forall x_i \in M, \sigma \in S_k.$$

“ $\subseteq$ ”:  $x_1 \otimes \dots \otimes x_s \otimes (u \otimes v - v \otimes u) \otimes y_1 \otimes \dots \otimes y_t \in C(M) \cap T^k(M)$  with  $s + 2 + t = k$ .

“ $\supseteq$ ”: bubble sort

- $k \geq 2, S^k(M) = T^k(M)/C^k(M) = \langle \bar{x}_1 \otimes \dots \otimes \bar{x}_k \mid x_i \in M \rangle_R$  with  $\bar{x}_1 \otimes \dots \otimes \bar{x}_k = \bar{x}_{\sigma(1)} \otimes \dots \otimes \bar{x}_{\sigma(k)} \quad \forall \sigma \in S_k$

Hence,  $S(M) = \bigoplus_{k=0}^{\infty} S^k(M)$  is a graded commutative  $R$ -algebra.

**Def 55.**  $f : M \times \dots \times M \rightarrow L$  is a symmetric  $k$ -multilinear map if  $f$  is  $k$ -multilinear and

$$f(x_1, \dots, x_k) = f(x_{\sigma(1)}, \dots, x_{\sigma(k)}) \quad \forall \sigma \in S_k$$

- $k \geq 2, S^k(M)$  is universal w.r.t. symmetric  $k$ -multilinear maps on  $M$ : By the universal property of  $T^k(M)$ ,  $\exists!$   $R$ -module homo.  $\tilde{f} : T^k(M) \rightarrow L$ . Now  $C^k(M) \subseteq \ker \tilde{f} \implies \exists!$   $R$ -module homo.  $\bar{f} : S^k(M) \rightarrow L$  by factor thm.
- $S(M)$  satisfies the universal property for maps to a commutative  $R$ -algebra: given a commutative  $R$ -algebra  $A$  and  $f : M \rightarrow A$   $R$ -module homo.,

$$\begin{array}{ccc} M & \xrightarrow{f} & A \\ \downarrow & \nearrow \exists! f' & \uparrow \\ T(M) & \longrightarrow & T(M)/C(M) \end{array}$$

- $S : \mathfrak{M}_R \rightarrow \mathfrak{Gr}_R$  is a covariant functor.

$$- \varphi : M \rightarrow N: R\text{-module homo.} \rightsquigarrow T(\varphi) : T(M) \rightarrow T(N) \rightarrow T(N)/C(N) = S(N)$$

**Ex 2.3.1.** Let  $E$  be a vector space over  $F$  with  $\dim E = n$ .

1. Show that  $S(E) \cong F[x_1, \dots, x_n]$ .
2. Compute  $\dim_F S^k(E)$ .

**Exterior algebra** ( $\text{char } R \neq 2$ )

$$\begin{aligned} \Lambda : \mathfrak{M}_R &\rightarrow \mathfrak{Gr}_R \\ M &\mapsto \Lambda(M) = T(M)/A(M) \end{aligned}$$

where  $A(M)$  is the two sided graded generated by  $v \otimes v \quad \forall v \in M$ .

- $A^k(M) := A(M) \cap T^k(M)$  is the submodule of  $T^k(M)$  generated by all  $x_1 \otimes \dots \otimes x_k$  with  $x_i = x_j$  for some  $i \neq j$ .

(Note:  $(x_1 + x_2) \otimes (x_1 + x_2) = x_1 \otimes x_1 + x_1 \otimes x_2 + x_2 \otimes x_1 + x_2 \otimes x_2 \rightsquigarrow x_1 \otimes x_2 + x_2 \otimes x_1 \in A(M)$ )

- $\Lambda^k(M) \cong T^k(M)/A^k(M) = \langle \overline{x_1 \otimes \dots \otimes x_k} \mid x_i \in M \rangle$  with  $\overline{x_1 \otimes \dots \otimes x_k} = \bar{0}$  if  $x_i = x_j$  for some  $i \neq j$ . We use  $x_1 \wedge \dots \wedge x_k := \overline{x_1 \otimes \dots \otimes x_k}$ .

Note:  $x_1 \wedge x_2 = -x_2 \wedge x_1$ .

**Def 56.**  $f : M \times \dots \times M \rightarrow L$  is an alternating  $k$ -multilinear map if  $f$  is  $k$ -multilinear and  $f(x_1, \dots, x_k) = 0$  when  $x_i = x_j$  for some  $i \neq j$ .

- $k \geq 2$ ,  $\Lambda^k(M)$  is universal w.r.t. alternating  $k$ -multilinear maps on  $M$ :

$$\begin{array}{ccc} M \times \dots \times M & \xrightarrow{\quad} & L \\ \downarrow & \nearrow \exists ! f' & \uparrow \\ T^k(M) & \xrightarrow{\quad} & \Lambda^k(M) \end{array}$$

- $\Lambda(M)$  satisfies the universal property for maps to an  $R$ -algebra  $A$  with  $a^2 = 0 \quad \forall a \in A$ : given an  $R$ -algebra  $A$  and  $f : M \rightarrow A$   $R$ -module homo.,

$$\begin{array}{ccc} M & \xrightarrow{f} & A \\ \downarrow & \nearrow \exists ! f' & \uparrow \\ T(M) & \xrightarrow{\quad} & \Lambda(M) \end{array}$$

- $\Lambda : \mathfrak{M}_R \rightarrow \mathfrak{A}_R$  is a covariant functor.

$$- \varphi : M \rightarrow N: R\text{-module homo.} \rightsquigarrow T(\varphi) : T(M) \rightarrow T(N) \rightarrow T(N)/A(N) = \Lambda(N)$$

**Ex 2.3.2.** Let  $V$  be a vector space over  $F$  with  $\dim V = n$  and  $\varphi : V \rightarrow V$  be a linear transformation.

- (1) Compute  $\Lambda^k(V)$ .
- (2) Determine the map  $\Lambda^k(\varphi) : \Lambda^k(V) \rightarrow \Lambda^k(V)$ .

### Symmetrization and Skew-symmetrization

$$T^k(V) \xrightarrow{\quad} T^k(V)$$

$$\text{Sym} = \sigma : x_1 \otimes \dots \otimes x_k \longmapsto \frac{1}{k!} \sum_{\tau \in S_k} x_{\tau(1)} \otimes \dots \otimes x_{\tau(k)}$$

$$\text{Alt} = \sigma' : x_1 \otimes \dots \otimes x_k \longmapsto \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) x_{\tau(1)} \otimes \dots \otimes x_{\tau(k)}$$

$$\tilde{S}^k(V) = \sigma(T^k(V)) \quad \tilde{\Lambda}^k(V) = \sigma'(T^k(V))$$

- $\sigma^2 = \sigma$  easy  $\rightsquigarrow T^k(V) = \text{Im } \sigma \oplus \ker \sigma = \tilde{S}^k(V) \oplus \ker \sigma$ .
- $\ker \sigma = C^k(V)$ .  $C^k(V) \subseteq \ker \sigma$  is obvious. Assume  $\supsetneq$ , i.e.,  $\exists t \in \ker \sigma$  s.t.  $t \notin C^k(V)$ . Recall  $q : T^k(V) \twoheadrightarrow S^k(V)$ , since  $q$  is the quotient map. Also  $q|_{\tilde{S}^k(V)} \twoheadrightarrow S^k(V)$ , since if  $q(x) = y$ , then it could be easily checked that  $q(\sigma(x)) = y$ , so exists  $t' \in \tilde{S}^k(V)$  satisfies  $q(t') = q(t) \neq 0$ . But then  $q(t - t') = 0 \implies t - t' \in \ker q = C^k(V) \subseteq \ker \sigma$  and because of  $\sigma(t) = 0 \implies \sigma(t') = 0$ . Hence  $t' \in \ker \sigma$ . But then  $t' \in S^k(V) \subseteq \text{Im } \sigma \implies t' \in \text{Im } \sigma \cap \ker \sigma$ , which leads to an ontradiction since  $\sigma$  is a projection.

**Ex 2.3.3.**  $T^k(V) = \tilde{\Lambda}^k(V) \oplus A^k(V)$ .

### 3 Introduction to the linear representation theory of finite groups

#### 3.1 Week 14

##### 3.1.1 Generalities on linear representations

###### Notation

- $G$ : finite group
- $V$ : vector space of finite dim over  $\mathbb{C}$
- $\text{GL}(V)$ : the group of all linear isom.  $V \rightarrow V$

**Def 57.** A group homo.  $\rho : G \rightarrow \text{GL}(V)$  is called a linear representation of  $G$ .  $\dim V$  is called the degree of  $\rho$ . ( $V$  is a representation space)

For a fixed basis  $\beta = \{e_i\}$ ,

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \text{GL}(V) \\ & \searrow R & \downarrow \beta \\ & & \text{GL}_n(\mathbb{C}) \end{array}$$

( $R$  is a matrix representation)

**Eg 3.1.1.** A representation of degree 1 of  $G$  is  $\rho : G \rightarrow \text{GL}(\mathbb{C}) \cong \mathbb{C}^\times$ .

$\text{ord}(g)$  is finite  $\leadsto \rho(g)^m = 1$  for some  $m \in \mathbb{N} \leadsto \rho(g)$  is a root of unity, i.e.  $|\rho(g)| = 1$ .

Note: So,  $\rho : G \rightarrow S^1$ ,  $S^1$  is the unit circle.

1.  $G = \mathbb{Z}/p\mathbb{Z}$ ,  $\rho : \bar{1} \mapsto \zeta_p \in S^1$  with  $\zeta_p^p = 1$ .
2.  $G = S_3$ ,  $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$ .

A permutation representation is  $\rho : \tau \mapsto (\rho(\tau) : e_i \mapsto e_{\tau(i)}) \in \text{GL}(V)$ .

3.  $G = S_3$ ,  $V = \bigoplus_{\sigma \in S_3} \mathbb{C}e_\sigma$ . The regular representation is

$$\rho^{\text{reg}} : \tau \mapsto (\rho^{\text{reg}}(\tau) : e_\sigma \mapsto e_{\tau\sigma}) \in \text{GL}(V).$$

For general  $G$ , with  $V = \bigoplus_{g \in G} \mathbb{C}e_g$ ,

$$\rho^{\text{reg}} : h \mapsto (\rho^{\text{reg}}(h) : e_g \mapsto e_{hg}) \in \text{GL}(V).$$

###### Def 58.

- $\rho : g \mapsto \text{id} \in \text{GL}(V)$ : trivial representation.
- $\rho : G \hookrightarrow \text{GL}(V)$ : faithful representation.
- $\rho, \rho'$  are said to be equivalent if  $\exists$  a linear isom.  $T : V \xrightarrow{\sim} V'$  s.t.

$$\begin{array}{ccc} V & \xrightarrow[\sim]{T} & V' \\ \rho(g) \downarrow & & \downarrow \rho'(g) \\ V & \xrightarrow[\sim]{T} & V' \end{array}$$



**Remark 13.** When we choose two bases  $\beta, \beta'$  for  $V$ ,

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \text{GL}(V) \\ & \searrow R & \downarrow \beta \downarrow \wr \\ & & \text{GL}_n(\mathbb{C}) \end{array} \quad \begin{array}{ccc} G & \xrightarrow{\rho'} & \text{GL}(V) \\ & \searrow R & \downarrow \beta' \downarrow \wr \\ & & \text{GL}_n(\mathbb{C}) \end{array}$$

then  $\rho, \rho'$  are equivalent.

Let  $T : e_i \mapsto e'_i :: V \mapsto V$ . For  $g \in G, R(g) = (a_{ij})$ .

$$T \circ \rho(g) = \rho'(g) \circ T$$

**Def 59.** Let  $\langle \cdot, \cdot \rangle$  be a positive definite Hermitian form on  $V$ .

Then  $T : V \rightarrow V$  is called a unitary operator if  $\langle T(x), T(y) \rangle = \langle x, y \rangle \quad \forall x, y \in V$ .

or  $\forall \beta : \text{orthonormal basis}, [T]_{\beta}^* [T]_{\beta} = [T]_{\beta} [T]_{\beta}^* = I_n$ .

**Theorem 29.**  $\forall \rho : G \rightarrow \text{GL}(V), \exists$  a matrix representation  $R : G \rightarrow U_n$ .

*Proof.* We only need to  $G$ -invariant positive definite Hermitian form on  $V$ . ( $\forall g \in G, \langle \rho(g)x, \rho(g)y \rangle = \langle x, y \rangle \quad \forall x, y \in V$ )

We start with an arbitrary positive definite Hermitian form  $\langle \cdot, \cdot \rangle'$  on  $V$ .

Define a new form  $\langle \cdot, \cdot \rangle$  by

$$\langle x, y \rangle := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle'$$

which is a positive definite Hermitian form, since

$$\begin{aligned} \langle \rho(g)x, \rho(g)y \rangle &\triangleq \frac{1}{|G|} \sum_{h \in G} \langle (\rho(h) \circ \rho(g))(x), (\rho(h) \circ \rho(g))(y) \rangle' \\ &= \frac{1}{|G|} \sum_{gh \triangleq h' \in G} \langle (\rho(h'))(x), (\rho(h'))(y) \rangle' \triangleq \langle x, y \rangle \end{aligned}$$

So with the basis of this hermitian form, every  $\rho(g)$  has a matrix representation  $R(g)$  which is unitary.  $\square$

**Def 60.** Let  $\rho : G \rightarrow \text{GL}(V)$ , For  $W \subset V$  (we use  $\subset$  to denote subspace), if  $\forall x \in W, \rho(g)(x) \in W, \forall g \in G$ , then  $W$  is said to be  $G$ -invariant and

$$\begin{aligned} \rho^W : G &\rightarrow \text{GL}(W) \\ g &\mapsto \rho(g)|_W \end{aligned}$$

is called a subrepresentation of  $\rho$ .

$W$  is  $G$ -invariant  $\rightsquigarrow \rho(g)|_W : W \xrightarrow{\sim} W$ .

**Eg 3.1.2.** Let  $\rho$  be the regular rep. of  $S_3$ .

$W^\circ = \{ \alpha_1 e_1 + \dots + \alpha_6 e_6 \mid \alpha_1 + \dots + \alpha_6 = 0 \}$  is  $G$ -invariant.

$W^1 = \langle e_1 + \dots + e_6 \rangle_{\mathbb{C}}$  is  $G$ -invariant.

**Theorem 30.** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $W \subset V$  be  $G$ -invariant. Then  $\exists W^\circ \subset V$  is still  $G$ -invariant and  $V = W \oplus W^\circ$ .

*Proof.* We can pick an arbitrary  $W'$  with  $V = W \oplus W'$  and  $\pi_1 : V \rightarrow W$  is the projection to  $W$ . Then  $W' = \ker \pi_1$ .

Now we need  $\pi_1$  preserves the  $G$  action ( $G$ -equivariant). Define

$$\pi^\circ = \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} \circ \pi_1 \circ \rho(g) : V \rightarrow W$$

- well-defined:  $\rho(g)(V) \subset V \rightsquigarrow \pi_1 \circ \rho(g)(V) \subset W \rightsquigarrow \rho(g)^{-1} \circ \pi_1 \circ \rho(g)(V) \subseteq W$ .
- surjective:  $\forall y \in W, (\rho(g)^{-1} \circ \pi_1 \circ \rho(g))(y) = (\rho(g)^{-1} \circ \rho(g))(y) = y$  since  $\rho(g)(y) \in W$ . Also,  $\pi^\circ(y) = y, \forall y \in W \implies (\pi^\circ)^2 = \pi^\circ$ . So  $\pi^\circ$  is a projection and hence  $V = \text{Im } \pi^\circ \oplus \ker \pi^\circ$ .
- $G$ -equivariant:  $\forall g' \in G$ ,

$$\begin{aligned} \pi^\circ \circ \rho(g')(x) &= \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} \circ \pi_1 \circ \rho(g)(\rho(g')(x)) \\ &= \rho(g') \frac{1}{|G|} \sum_{gg' \in G} \rho(gg')^{-1} \circ \pi_1 \circ \rho(gg')(x) \\ &= (\rho(g') \circ \pi^\circ)(x) \end{aligned}$$

- $W^\circ := \ker \pi^\circ$  is  $G$ -invariant:  $\forall x \in W^\circ, \pi^\circ(\rho(g)(x)) = \rho(g)(\pi^\circ(x)) = \rho(g)(0) = 0$ . So  $\rho(g)(x) \in W^\circ$ .

$$\begin{array}{ccc} V & \xrightarrow{\pi^\circ} & W \\ \rho(g) \downarrow & & \downarrow \rho(g) \\ V & \xrightarrow{\pi^\circ} & W \end{array}$$

□

**Remark 14.** If  $W \subset V$  is  $G$ -invariant, then  $W^\perp$  is also  $G$ -invariant. (w.r.t. a  $G$ -invariant positive definite Hermitian form)

**Def 61.**  $\rho : G \rightarrow \text{GL}(V)$  is irreducible if  $\rho$  has no proper nontrivial subrepresentations.

**Theorem 31.** Each  $\rho : G \rightarrow \text{GL}(V)$  is a direct sum of irreducible subrepresentations.

*Proof.* By induction on  $\dim V$ . For  $\dim V = 1$ , then  $\rho$  is irreducible.

For  $\dim V > 1$ , if  $\rho$  is irreducible, then done. Otherwise,  $\exists W, W^\circ$  are  $G$ -invariant s.t.  $V = W \oplus W^\circ$  with  $\dim W \geq 1, \dim W^\circ \geq 1$ . By induction hypothesis,  $\rho^W, \rho^{W^\circ}$  are the direct sum of irreducible subrepresentations, and  $\rho = \rho^W \oplus \rho^{W^\circ}$ , done. □

**Remark 15.** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\rho' : G \rightarrow \text{GL}(V')$ .

- $\rho \oplus \rho' : G \rightarrow \text{GL}(V \oplus V')$ . 矩陣是左上右下
- $\rho \otimes \rho' : G \rightarrow \text{GL}(V \otimes V')$ . 矩陣是密密麻麻  $(\sum_{i,j} r_{ip} r'_{jq} (e_i \otimes e'_j))$

### 3.1.2 Character Theory I

Main goal: To determine all equivalence classes of irreducible representations of a finite group  $G$ .

**Def 62.**

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \text{GL}(V) \\ & \searrow R & \downarrow \wr \\ & & \text{GL}_n(\mathbb{C}) \end{array}$$

The character  $\chi_\rho$  if  $\rho$  is the map  $\chi_\rho : G \rightarrow \mathbb{C}$  defined by  $\chi_\rho(g) = \text{Tr}(R(g))$ .

**Remark 16.**

1.  $\chi_\rho$  is independent of the choice of  $\beta = \{e_i\}$  For another basis  $\beta' = \{e'_i\}$ . (Notice that  $\text{Tr}(BA) = \text{Tr}(AB)$ )
2.  $\rho \xrightarrow[\text{equivalent}]{\cong} \rho' \rightsquigarrow \chi_\rho = \chi_{\rho'}$ .

**Def 63.**

- The degree of  $\chi_\rho$  is defined to the degree of  $\rho$  ( $= \dim V$ ).
- $\chi_\rho$  is an irreducible character if  $\rho$  is irreducible.

Basic facts:

1.  $\chi_\rho(1) = n$ .
2.  $\chi_\rho$  is a class function, i.e., it is constant on each conjugacy class.
3.  $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$ : Assume that the eigenvalues of  $R(g)$  are  $\lambda_1, \dots, \lambda_n$ . Then the eigenvalues of  $R(g^{-1})$  are  $\lambda_1^{-1}, \dots, \lambda_n^{-1}$ .

$$0 = \det(\lambda I_n - A) = \det(\lambda I_n (A^{-1} - \lambda^{-1} I_n) A) = \det(\lambda I_n) \det(A^{-1} - \lambda^{-1} I_n) \det(A)$$

So  $\det(A^{-1} - \lambda^{-1} I_n) = 0$ . Then  $g^m = 1 \implies R(g)^m = I_n \implies |\lambda_i| = 1 \implies \lambda_i^{-1} = \overline{\lambda_i}$ .

Thus  $\chi_\rho(g^{-1}) = \text{Tr}(R(g)^{-1}) = \overline{\lambda_1 + \dots + \lambda_n} = \overline{\chi_\rho(g)}$ .

4.  $\chi_{\rho \oplus \rho'} = \chi_\rho + \chi_{\rho'}$ .
5.  $\chi_{\rho \otimes \rho'} = \chi_\rho \chi_{\rho'}$ .

**Def 64.**  $\mathcal{C}(G, \mathbb{C})$  is the vector space of complex functions on  $G$ .

$\chi_\rho \in \mathcal{C}(G) \subset \mathcal{C}(G, \mathbb{C})$  is the vector space of complex class functions of  $G$ .

**Remark 17.** Assume that  $\{C_1, \dots, C_k\}$  is the set of distinct conjugacy classes in  $G$ . Then  $\{f_i(C_j) = \delta_{ij} \mid \forall i = 1, \dots, k\}$  forms a basis for  $\mathcal{C}(G)$  over  $\mathbb{C}$ .

- $\forall f \in \mathcal{C}(G)$ , let  $f(C_i) = a_i$ , then  $f = \sum a_i f_i$ .
- $\sum a_i f_i = 0$ , pick  $x_j \in C_j$ , then  $(\sum a_i f_i)(x_j) = a_j = 0 \quad \forall j = 1, \dots, k$ .

So  $\dim \mathcal{C}(G) = k$ .

**Def 65.**  $\phi, \psi \in \mathcal{C}(G, \mathbb{C})$ , then

$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}$$

is a positive definite Hermitian form on  $\mathcal{C}(G, \mathbb{C})$ .

**Theorem 32** (Main theorem). The set of all irreducible characters of  $G$  forms an orthonormal basis for  $\mathcal{C}(G)$  over  $\mathbb{C}$ . So there are only  $k$  irreducible representations up to equivalent.

**Lemma 3** (Schur's lemma). Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\rho' : G \rightarrow \text{GL}(V')$  be two irr. rep. of  $G$ .

$$\begin{array}{ccc} V & \xrightarrow{\quad \text{T} \quad} & V' \\ \rho(g) \downarrow & & \downarrow \rho'(g) \\ V & \xrightarrow{\quad \text{T} \quad} & V' \end{array} \quad (\text{T} : G\text{-equivariant})$$

Then

1.  $\rho, \rho'$  are not equivalent  $\implies T = 0$ .
2.  $V = V', \rho = \rho' \implies T = \lambda 1_V$  for some  $\lambda \in \mathbb{C}$ .

*Proof.*

1. Assume  $T \neq 0$ . We only need to prove that  $T$  is an isomorphism, and then  $\rho, \rho'$  would be isomorphic by definition. Since  $T$  is  $G$ -equivariant,  $\ker T \leq V$  and  $\text{Im } T \leq V'$  are  $G$ -invariant.  $\rho$  is irreducible  $\implies \ker T = 0$  or  $V$ , but if  $\ker T = V$  then  $T = 0$ , so  $\ker T = 0$ . Similarly,  $\rho'$  is irreducible  $\implies \text{Im } T = 0$  or  $V$ . And by the fact that  $T \neq 0$ ,  $\text{Im } T = V$ . Thus  $T$  is an isomorphism, and consequently  $\rho, \rho'$  are equivalent.
2. Since the vector field is over  $\mathbb{C}$ ,  $T$  has an eigenvalue. Let  $\lambda$  be an eigenvalue of  $T$ , say  $T(v) = \lambda v$  with  $v \neq 0$  in  $V$ . Put  $T' = T - \lambda 1_V$ . Then

$$\rho(g) \circ T' = \rho(g) \circ (T - \lambda 1_V) \stackrel{*}{=} \rho(g) \circ T - \rho(g) \circ \lambda 1_V = T \circ \rho(g) - \lambda 1_V \rho(g) = T' \rho(g)$$

Which  $*$  is due to the linearity of  $\rho(g)$ . Hence  $T'$  is also  $G$ -equivariant.

But  $v \in \ker T'$ , i.e.,  $T'$  is not 1-1. Similar as in 1.,  $\ker T' = \{0\}$  or  $V \implies \ker T' = V \implies T' = 0 \implies T = \lambda 1_V$ .

□

**Coro 3.1.1.** Assume  $\rho, \rho'$  is the same as above. Let  $L : V \rightarrow V'$  be a linear transformation. Define

$$T = \frac{1}{|G|} \sum_{g \in G} \rho'(g)^{-1} L \rho(g).$$

One could easily check that  $T$  is  $G$ -equivariant (i.e.,  $T \circ \rho(g) = \rho'(g) \circ T$ ). Then

1.  $\rho, \rho'$  are not equivalent  $\implies T = 0$ .
2.  $V = V', \rho = \rho' \implies T = \lambda 1_V$ ,  $\lambda = \text{Tr}(T) / \dim V = \text{Tr}(L) / \dim V$ .

**Remark 18.** Let  $\rho \rightarrow_\beta R : G \rightarrow \text{GL}_n(\mathbb{C})$  and  $R(g) = [r_{ij}(g)]$

$\rho' \rightarrow_{\beta'} R' : G \rightarrow \text{GL}_{n'}(\mathbb{C})$  and  $R'(g) = [r'_{ij}(g)]$

and let the matrix representation of  $L$  is  $[L]_\beta^{\beta'} = [x_{\mu\nu}] \in M_{n' \times n}(\mathbb{C})$

Then consider the matrix representation of  $T$ , which is  $[T]_\beta^{\beta'} = [x_{tl}^\circ]$  with

$$x_{tl}^\circ = \frac{1}{|G|} \sum_{\substack{g \in G \\ i=1, \dots, n \\ j=1, \dots, n'}} r'_{tj}(g^{-1}) x_{ji} r_{il}(g)$$

In case 1.,  $x_{tl}^\circ = 0, \forall t, l$ . Since it holds for every  $L$ , which is independent of  $\rho, \rho'$ , fixing  $i, j$  and setting  $x_{ij} = 1$  and 0 otherwise, we get

$$\frac{1}{|G|} \sum_{g \in G} r'_{tj}(g^{-1}) r_{il}(g) = 0, \quad \forall i, j, t, l$$

In case 2.,  $T = \lambda 1_V$ , i.e.  $x_{tl}^\circ = \lambda \delta_{tl}$ .  $\lambda = \frac{\text{Tr}(L)}{n} = \frac{1}{n} \sum_{i=1}^n x_{ii} = \frac{1}{n} \sum_{i,j} \delta_{ji} x_{ji}$

Hence,

$$\frac{1}{|G|} \sum_{g, i, j} r'_{tj}(g^{-1}) x_{ji} r_{il}(g) = \frac{1}{n} \sum_{i,j} \delta_{ji} x_{ji} \delta_{tl}$$

But notice that this equality hold for any  $l$ , which is independent of  $\rho, \rho'$ . So if we fix  $i, j$  and set  $x_{ji} = 1$ , and  $x_{j'i'} = 0$  otherwise, we get

$$\frac{1}{|G|} \sum_{g \in G} r_{tj}(g^{-1}) r_{il}(g) = \frac{1}{n} \delta_{ji} \delta_{tl}$$

**Prop 3.1.1.**

1. If  $\chi_\rho$  is irreducible, then  $\langle \chi_\rho, \chi_\rho \rangle = 1$ .
2. If two irreducible representations  $\rho, \rho'$  are not equivalent, then  $\langle \chi_\rho, \chi_{\rho'} \rangle = 0$ .

*Proof.*

1. Let  $R(g) = [r_{ij}(g)]$  be the matrix representation of  $\rho(g)$ . Then

$$\begin{aligned} \langle \chi_\rho, \chi_\rho \rangle &\triangleq \frac{1}{|G|} \sum_g \chi_\rho(g) \overline{\chi_\rho(g)} = \frac{1}{|G|} \sum_g \chi_\rho(g) \chi_\rho(g^{-1}) \\ &= \frac{1}{|G|} \sum_g \sum_{i,j} r_{ii}(g) r_{jj}(g^{-1}) = \sum_{i,j} \frac{1}{n} \delta_{ij} \delta_{ij} = 1 \end{aligned}$$

2. Let  $R(g) = [r_{ij}(g)], R'(g) = [r'_{ij}(g)]$  be the matrix representation of  $\rho(g), \rho'(g)$ . Then

$$\begin{aligned} \langle \chi_\rho, \chi_{\rho'} \rangle &\triangleq \frac{1}{|G|} \sum_g \chi_\rho(g) \overline{\chi_{\rho'}(g)} = \frac{1}{|G|} \sum_g \chi_\rho(g) \chi_{\rho'}(g^{-1}) \\ &= \frac{1}{|G|} \sum_g \sum_{i,j} r_{ii}(g) r'_{jj}(g^{-1}) = 0 \end{aligned}$$

□

**Remark 19.**  $\langle \chi_\rho, \chi_\rho \rangle = 1 \implies \rho$  is irr.

*Proof.* We write  $\rho = \rho_1^{\oplus m_1} \oplus \dots \oplus \rho_l^{\oplus m_l}$  where  $\rho_1, \dots, \rho_l$  are non-equivalent irr. rep.

$$\chi_\rho = \sum_{i=1}^l m_i \chi_{\rho_i}$$

$$1 = \langle \chi_\rho, \chi_\rho \rangle = \sum_{i=1}^l m_i^2 \implies \exists m_i = 1 \text{ and } m_j = 0 \text{ for } j \neq i$$

So  $\rho \cong \rho_i$ .

□

## 3.2 Week 15

### 3.2.1 Character Theory II

**Prop 3.2.1.** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\rho = \rho^{W_1} \oplus \cdots \oplus \rho^{W_k}$  where  $\rho_i = \rho^{W_i}$  is irr.  $\forall i$ . ( $V \cong W_1 \oplus \cdots \oplus W_k$ )

If  $\tilde{\rho} : G \rightarrow \text{GL}(\tilde{W})$  is an irr. rep. then the number of  $\rho_i$  isomorphic to  $\tilde{\rho}$  is equal to  $\langle \chi_\rho, \chi_{\tilde{\rho}} \rangle$ .

*Proof.* We know  $\chi_\rho = \chi_{\rho_1} + \cdots + \chi_{\rho_k}$ , so

$$\langle \chi_\rho, \chi_{\tilde{\rho}} \rangle = \sum_{i=1}^k \langle \chi_{\rho_i}, \chi_{\tilde{\rho}} \rangle$$

Recall  $\rho_i \cong \tilde{\rho} \implies \langle \chi_{\rho_i}, \chi_{\tilde{\rho}} \rangle = 1$ , otherwise  $\langle \chi_{\rho_i}, \chi_{\tilde{\rho}} \rangle = 0$ . □

**Remark 20.**

1. The number of  $W_i$  isomorphic to  $\tilde{W}$  does not depend on the chosen decomposition. ( $= \langle \chi_\rho, \chi_{\tilde{\rho}} \rangle$ )
2. If  $\chi_\rho = \chi_{\rho'}$ , then  $\rho \cong \rho'$ :  $\langle \chi_\rho, \chi_{\tilde{\rho}} \rangle = \langle \chi_{\rho'}, \chi_{\tilde{\rho}} \rangle$  The type of irr. subrep of  $\rho$  is the same as  $\rho'$ .
3. If  $\chi_1, \dots, \chi_l$  are distinct irr. characters of  $G$ , then since  $x_1, \dots, x_l$  are orthonormal w.r.t.  $\langle \cdot, \cdot \rangle$  in  $\mathcal{C}(G)$ ,  $x_1, \dots, x_l$  are linearly indep. over  $\mathbb{C}$  in  $\mathcal{C}(G)$ .

But  $\dim \mathcal{C}(G) = k = \#$  of conjugacy classes in  $G$ . So  $l \leq k$  i.e. we conclude that there are at most  $k$  mutually non-equivalent irr. rep. of  $G$ , say  $\rho_1, \dots, \rho_l, l \leq k$ .

For any  $\rho : G \rightarrow \text{GL}(V)$ ,  $\rho \cong \rho_1^{\oplus m_1} \oplus \cdots \oplus \rho_l^{\oplus m_l}$  where  $m_i = \langle \chi_\rho, \chi_{\rho_i} \rangle \in \mathbb{Z}^{\geq 0}$ .

**Theorem 33** (Orthogonality relations for  $\chi$ 's). The set of all irr. characters of  $G$  forms an orthonormal **basis**  $\mathcal{C}(G)$  over  $\mathbb{C}$ . In particular, the number of irr. rep. of  $G$  is equal to  $\#$  of conjugacy classes in  $G$ . (up to equivalence)

*Proof.* Let  $\chi_i = \chi_{\rho_i}, i = 1, \dots, l$  be all irr. characters of  $G$  and  $\mathcal{D} = \langle \chi_1, \dots, \chi_l \rangle_{\mathbb{C}} \subseteq \mathcal{C}(G)$ . Then  $\mathcal{C}(G) = \mathcal{D} \oplus \mathcal{D}^\perp$ . Claim:  $\mathcal{D}^\perp = \{0\}$ .

Let  $\varphi \in \mathcal{D}^\perp$ , i.e.  $\langle \varphi, \chi_i \rangle = 0, \forall i = 1, \dots, l$ .

Write  $\rho^{\text{reg}} \cong \rho_1^{\oplus m_1} \oplus \cdots \oplus \rho_l^{\oplus m_l} \implies \chi^{\text{reg}} = m_1 \chi_1 + \cdots + m_l \chi_l$ . By assumption,  $\langle \varphi, \chi_\rho \rangle = 0$ .

For each  $i$ , define  $\mathsf{T}_{\rho_i} \in \text{Hom}_{\mathbb{C}}(V, V)$  by

$$\mathsf{T}_{\rho_i} \triangleq \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \rho_i(g)$$

Then we have

$$\text{Tr}(\mathsf{T}_{\rho_i}) = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \chi_\rho(g) = \overline{\langle \varphi, \chi_\rho \rangle} = 0$$

Also, for all  $h \in G$ .

$$\begin{aligned} \rho_i(h)^{-1} \circ \mathsf{T}_{\rho_i} \circ \rho_i(h) &= \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \rho_i(h)^{-1} \circ \rho_i(g) \circ \rho_i(h) \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(h^{-1}gh)} \rho_i(h^{-1}gh) \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \rho_i(g) = \mathsf{T}_{\rho_i} \end{aligned}$$

Where  $*$  is because  $\varphi$  is a class function. So  $\mathsf{T}_{\rho_i}$  is  $G$ -equivariant. By Schur's lemma,  $\mathsf{T}_{\rho_i} = \lambda_i 1_{W_i}$  where  $\rho_i : G \rightarrow \mathrm{GL}(W_i)$ .

But  $\mathrm{Tr} \mathsf{T}_{\rho_i} = 0 \implies \lambda_i = 0 \implies \mathsf{T}_{\rho_i} = 0$ .

Also, because  $\rho \cong \rho_1^{\oplus m_1} \oplus \dots \oplus \rho_l^{\oplus m_l}$ , if we define

$$\mathsf{T}_{\rho^{\mathrm{reg}}} \triangleq \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \rho^{\mathrm{reg}}(g) \implies \mathsf{T}_{\rho^{\mathrm{reg}}} = \mathsf{T}_{\rho_1^{\oplus m_1}} \oplus \dots \oplus \mathsf{T}_{\rho_k^{\oplus m_k}} = 0$$

Finally, let  $\rho = \rho^{\mathrm{reg}} : G \rightarrow \mathrm{GL}(V)$  with  $V = \bigoplus_{g \in G} \mathbb{C} e_g$ . Then  $\mathsf{T}_{\rho} = 0 \implies \mathsf{T}_{\rho}(e_1) = 0$  and

$$0 = \mathsf{T}_{\rho}(e_1) = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \rho(g)(e_1) = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} e_g$$

Since  $\{e_g\}$  is a basis,  $\overline{\varphi(g)} = 0 \quad \forall g$ . That is,  $\varphi \equiv 0$ . □

**Prop 3.2.2.** Each irr. rep.  $\rho_i : G \rightarrow \mathrm{GL}(W_i)$  is contained in  $\rho^{\mathrm{reg}}$  with multiplicity equal to  $\dim W_i = m_i$ ,  $i = 1, \dots, k$ .

In particular,  $\bigoplus_{g \in G} \mathbb{C} e_g \cong \underbrace{W_1 \oplus \dots \oplus W_1}_{m_1 \text{ times}} \oplus \dots \oplus \underbrace{W_k \oplus \dots \oplus W_k}_{m_k \text{ times}}$ . So  $|G| = m_1^2 + \dots + m_k^2$ .

*Proof.* Let  $\chi^{\mathrm{reg}} := \chi_{\rho^{\mathrm{reg}}}$  and  $\chi_i = \chi_{\rho_i}$ ,  $i = 1, \dots, k$ . Then

$$\langle \chi^{\mathrm{reg}}, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi^{\mathrm{reg}}(g) \chi_i(g^{-1}) = \frac{1}{|G|} |G| \chi_i(1) = m_i$$

□

**Theorem 34** (Divisibility).  $\forall i = 1, \dots, k, \quad \chi_i(1) = m_i \mid |G|$ .

*Proof.* First, we shall proof that for each  $\rho = \rho_i$ ,  $\chi = \chi_i$  and  $j$ , we have

$$\mathsf{T} \triangleq \sum_{g \in C_j} \rho(g) = \frac{|C_j| \chi(g_0)}{m_i} \mathbf{I}_{m_i}, \quad \text{for any } g_0 \in C_j$$

Observe that  $\forall h \in G$ ,

$$\rho(h)^{-1} \circ \mathsf{T} \circ \rho(h) = \sum_{g \in C_j} \rho(h^{-1} g h) = \sum_{g' \in C_j} \rho(g') = \mathsf{T}$$

So  $\mathsf{T}$  is  $G$ -equivariant w.r.t.  $\rho$ .

By Schur's lemma,  $\mathsf{T} = \lambda \mathbf{I}_{m_i}$  for some  $\lambda \in \mathbb{C}$ . And  $\lambda = \mathrm{Tr}(\mathsf{T})/m_i = \sum_{g \in C_j} \chi(g)/m_i = |C_j| \chi(g_0)/m_i$  for any  $g_0 \in C_j$ , thus  $\sum_{g \in C_j} \rho(g) = \frac{|C_j| \chi(g_0)}{m_i} \mathbf{I}_{m_i}$  for any  $g_0 \in C_j$ .

Define  $\lambda_{\mu}(C_i) \triangleq |C_i| \chi_{\mu}(g_i)/m_{\mu}$ . Now, for a  $g \in C_l$ , define  $a_{i,j,l} \triangleq \#\{(g_i, g_j) \in C_i \times C_j \mid g_i g_j = g\}$ , which is indep. of the choice of  $g$ .

We claim that  $\lambda_{\mu}(C_i) \lambda_{\mu}(C_j) = \sum_{l=1}^k a_{i,j,l} \lambda_{\mu}(C_l)$ ,  $\forall i, j, \mu$ . Then

$$\lambda_{\mu}(C_i) \begin{bmatrix} \lambda_{\mu}(C_1) \\ \vdots \\ \lambda_{\mu}(C_k) \end{bmatrix} = A \begin{bmatrix} \lambda_{\mu}(C_1) \\ \vdots \\ \lambda_{\mu}(C_k) \end{bmatrix}, \quad \text{where } A \triangleq \begin{bmatrix} a_{i,1,1} & \dots & a_{i,1,k} \\ \vdots & \ddots & \vdots \\ a_{i,k,1} & \dots & a_{i,k,k} \end{bmatrix}$$

So  $\lambda_{\mu}(C_j)$  is an eigenvalue of  $A$ , i.e.,  $\lambda = \lambda_{\mu}(C_j)$  satisfies  $\det(\lambda I - A) = 0$ . And thus  $\lambda_{\mu}(C_i)$  is an algebraic integer.

We proof the claim by the following calculating.

$$\begin{aligned}
\lambda_\mu(C_i)\lambda_\mu(C_j)I_{m_\mu} &= (\lambda_\mu(C_i)I_{m_\mu}) (\lambda_\mu(C_j)I_{m_\mu}) = \left( \sum_{g \in C_i} \rho(g) \right) \left( \sum_{g' \in C_j} \rho(g') \right) \\
&= \sum_{\substack{g \in C_i \\ g' \in C_j}} \rho(gg') = \sum_{l=1}^k \sum_{\bar{g} \in C_l} a_{i,j,l} \rho(\bar{g}) \\
&= \sum_{l=1}^k a_{i,j,l} \sum_{\bar{g} \in C_l} \rho(\bar{g}) \\
&= \sum_{l=1}^k a_{i,j,l} \lambda_\mu(C_l) I_{m_\mu}
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{|G|}{m_i} &= \frac{|G|}{m_i} \langle \chi_i, \chi_i \rangle \\
&= \frac{|G|}{m_i} \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_i(g^{-1}) \\
&= \sum_{g \in G} \frac{\chi_i(g)}{m_i} \chi_i(g^{-1}) \\
&= \sum_{j=1}^k \sum_{g \in C_j} \frac{\chi_i(g)}{m_i} \chi_i(g^{-1}) \\
&= \sum_{j=1}^k \frac{|C_j| \chi_i(g_j)}{m_i} \chi_i(g_j^{-1}) \\
&= \sum_{j=1}^k \lambda_i(C_j) \chi_i(g_j^{-1})
\end{aligned}$$

and thus is an algebraic integer.

Also,  $|G|/m_i \in \mathbb{Q}$ , so we conclude that  $|G|/m_i \in \mathbb{Z} \implies m_i \mid |G|$ . □

**Ex 3.2.1.**

1. Show that if  $g \in G$  and  $g \neq 1$ , then  $\sum_{i=1}^k m_i \chi_i(g) = 0$ .
2. Show that each character  $\chi$  of  $G$  with  $\chi(g) = 0 \quad \forall g \neq 1$  is an integral multiple of  $\chi^{\text{reg}}$ .

**Ex 3.2.2.**

1. Let  $|G| < \infty$ . Then  $G$  is abelian  $\iff$  each irr. rep. of  $G$  is of degree 1.
2.  $\{\text{the deg 1 rep. of } G\} = \{\text{the irr. rep. of } G/[G, G]\}$ .

**3.2.2 Applications**

1.  $G = S_3 = D_3$ ,  $6 = 1^2 + 1^2 + 2^2$ .



Classes	1	(1 2)	(1 2 3)
size	1	3	2
$\chi_1$	1	1	1
$\chi_2$	1	-1	1
$\chi_3$	2	0	-1

The permutation representation

deg 4:  $\tilde{\rho} = \rho^W \otimes \rho^W \rightsquigarrow \chi_{\tilde{\rho}} = \chi_3 \cdot \chi_3 = (4, 0, 1)$ .

By inner product with  $\chi_1, \chi_2, \chi_3$ , we can find  $\chi_{\tilde{\rho}} = \chi_1 + \chi_2 + \chi_3 \rightsquigarrow \tilde{\rho} = \rho_1 \oplus \rho_2 \oplus \rho_3$ .

2.  $G = D_4 = \langle x, y \mid x^4 = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$ .  $|G| = 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$ .

Classes	1	$y$	$x$	$x^2$	$xy$
size	1	2	2	1	2
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	1	-1
$\chi_3$	1	1	-1	1	-1
$\chi_4$	1	-1	-1	1	1
$\chi_5$	2	0	0	-2	0

$$\chi^{\text{reg}} = (8, 0, 0, 0, 0) = \chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\chi_5.$$

3.  $G = D_n$ , ( $n$  even)  $[G, G] = H = \langle x^2 \rangle$   
4.  $G = D_n$ , ( $n$  odd)  $[G, G] = H = \langle x \rangle$   
5.  $G = S_4$ .

Classes	1	(1 2)	(1 2 3)	(1 2 3 4)	(1 2)(3 4)
size	1	6	8	6	3
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	-1	1
$\chi_3$	2	0	-1	0	2
$\chi_4$	3	1	0	-1	-1
$\chi_5$	3	-1	0	1	-1

6.  $G = A_4$ ,  $[A_4, A_4] = V_4$ .

Classes	1	(1 2 3)	(1 3 2)	(1 2)(3 4)
size	1	4	4	3
$\chi_1$	1	1	1	1
$\chi_2$	1	$\omega$	$\omega^2$	1
$\chi_3$	1	$\omega^2$	$\omega$	1
$\chi_4$	3	0	0	-1

**Theorem 35** (Product of groups). For  $\rho : G \rightarrow \text{GL}(V)$  and  $\rho' : G' \rightarrow \text{GL}(V')$ , write  $\rho \otimes \rho' : G \times G' \rightarrow \text{GL}(V \otimes V')$ . If  $\{\rho_i\}$  are irreducible representations of  $G$ ,  $\{\rho'_j\}$  are irreducible representations of  $G'$ , then  $\{\rho_i \otimes \rho'_j\}$  are exactly the irreducible representations of  $G \times G'$ .

*Proof.* It is evidence that  $\rho_i \otimes \rho'_j$  is a homomorphism, and hence a representation.

Notice that  $\chi_{\rho \otimes \rho'} = \chi_\rho \odot \chi_{\rho'}$  where  $\chi_\rho \odot \chi_{\rho'}(g, g') = \chi_\rho(g)\chi_{\rho'}(g')$

Now we calculate

$$\begin{aligned}
\langle \chi_{\rho_1} \otimes \chi_{\rho'_1}, \chi_{\rho_2} \otimes \chi_{\rho'_2} \rangle &= \frac{1}{|G||G'|} \sum_{g, g'} \chi_{\rho_1}(g) \chi_{\rho'_1}(g') \chi_{\rho_2}(g) \chi_{\rho'_2}(g') \\
&= \left( \frac{1}{|G|} \sum_g \chi_{\rho_1}(g) \chi_{\rho_2}(g) \right) \left( \frac{1}{|G'|} \sum_{g'} \chi_{\rho'_1}(g') \chi_{\rho'_2}(g') \right) \\
&= \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle \langle \chi_{\rho'_1}, \chi_{\rho'_2} \rangle
\end{aligned}$$

So  $\langle \chi_{\rho} \otimes \chi_{\rho'}, \chi_{\rho} \otimes \chi_{\rho'} \rangle = 1$  hence each  $\chi_{\rho} \otimes \chi_{\rho'}$  is irreducible. And  $\langle \chi_{\rho_1} \otimes \chi_{\rho'_1}, \chi_{\rho_2} \otimes \chi_{\rho'_2} \rangle = 0$  if  $\rho_1 \otimes \rho'_1 \neq \rho_2 \otimes \rho'_2$ , and thus these representations are not isomorphic.

Finally we proof that any irreducible representations of  $G \times G'$  is isomorphic to some  $\rho \otimes \rho'$ .

Let  $\{\rho_1, \dots, \rho_k\}, \{\rho'_1, \dots, \rho'_{k'}\}$  be the sets of irreducible representations of  $G, G'$  respectively. Write  $\chi_i = \chi_{\rho_i}, \chi'_i = \chi_{\rho'_i}$ .

Let  $\mathcal{D} \triangleq \mathcal{C}(G \times G') = \langle \chi_i, \chi'_j \mid i = 1, \dots, k, j = 1, \dots, k' \rangle_{\mathbb{C}} =$ . We claim that  $\mathcal{D}^{\perp} = \{0\}$ .

Let  $f \in \mathcal{D}^{\perp}$ . Then

$$\begin{aligned}
0 &= \frac{1}{|G \times G'|} \sum_{(g, g') \in G \times G'} f(g, g') \overline{\chi_i(g) \chi'_j(g')} \\
&= \frac{1}{|G'|} \sum_{g'} \left( \frac{1}{|G|} \sum_g f(g, g') \overline{\chi_i(g)} \right) \chi'_j(g') \\
&= \left\langle \frac{1}{|G|} \sum_g f(g, \cdot) \overline{\chi_i(g)}, \chi'_j \right\rangle
\end{aligned}$$

Since  $\rho'_j$  are orthonogonal basis of  $\mathcal{C}(G')$ , we have  $\frac{1}{|G|} \sum_g f(g, g') \overline{\chi_i(g)} = 0$  for all  $g'$ . Again,

$$0 = \frac{1}{|G|} \sum_g f(g, g') \overline{\chi_i(g)} = \langle f(\cdot, g'), \chi_i \rangle$$

Hence  $f(g, g') = 0$  for all  $g, g'$ , which implies  $f \equiv 0$ . □

**Ex 3.2.3.** Determine all irr. rep. of  $C_n$ .

**Ex 3.2.4.** Calculate the character table of  $Q_8$ .

**Ex 3.2.5.** Calculate the character table of  $\mathbb{Z}/2\mathbb{Z} \times S_4$  and  $S_3 \times S_4$ .

To calculate  $S_5$ ,  $|S_5| = 120 = 1^2 + 1^2 + 4^2 + 4^2 + 5^2 + 5^2 + 6^2$ .

## 4 Extensions of Groups

### 4.1 Week 16

#### 4.1.1 Extensions of abelian groups

**Def 66.** If a group  $E$  contains a normal subgroup  $N$  and  $E/N \cong G$ , then we call  $E$  an extension of  $N$  by  $G$ , denoted by  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$ .

Ques: When  $N$  and  $G$  are given, how to obtain all extensions of  $N$  by  $G$ .

**Now assume that  $N$  is abelian.**

**Def 67.**  $1 \rightarrow N \rightarrow E \xrightarrow{p} G \rightarrow 1$ .  $l : G \rightarrow E$  is a lifting if  $p \circ l = \text{id}_G$  and  $l(1) = 1$ .

**Remark 21.**  $G \cong E/N = \{xN \mid x \in E\}$ ,  $p \circ l(\bar{x}) = \bar{x}$ ,  $l(\bar{x})$  is a representative of  $xN = \bar{x}$ .

**Prop 4.1.1.**

1.  $\forall \bar{x} \in G, \theta_{\bar{x}} : N \rightarrow N, a \mapsto l(\bar{x})al(\bar{x})^{-1}$ . is independent of the choice of  $l$ .
2.  $\theta : G \rightarrow \text{Aut}(N), \bar{x} \mapsto \theta_{\bar{x}}$  is a group homomorphism.

*Proof.*

1. Suppose  $l' : G \rightarrow E$  is another lifting. Then  $l(\bar{x})N = l'(\bar{x})N$ . So  $l'(\bar{x}) = l(\bar{x})b$  for some  $b \in N$ .  $\forall a \in N$ ,  $l'(\bar{x})al'(\bar{x})^{-1} = l(\bar{x})bab^{-1}l(\bar{x})^{-1} = l(\bar{x})al(\bar{x})^{-1}$  since  $N$  is abelian.
2.  $\theta_{\bar{x}\bar{y}}(a) = l(\bar{x}\bar{y})al(\bar{x}\bar{y})^{-1}$ .

$$\begin{cases} p \circ l(\bar{x}\bar{y}) = \bar{x}\bar{y} \\ p \circ (l(\bar{x})l(\bar{y})) = \bar{x}\bar{y} \end{cases} \rightsquigarrow l(\bar{x}\bar{y}), l(\bar{x})l(\bar{y}) \text{ are liftings of } \bar{x}\bar{y} \quad \square$$

**Def 68.** An extension  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$  splits if  $\exists$  a lifting  $l : G \rightarrow E$  is a group homo.

**Prop 4.1.2.** TFAE

1.  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$  splits.
2.  $\exists$  a subgroup  $K \leq E$  s.t.  $K \cong G$  and  $\begin{cases} K \cap N = \{1\} \\ NK = E \end{cases} \rightsquigarrow E \cong N \rtimes K (\cong N \rtimes G)$ .

*Proof.* (1)  $\Rightarrow$  (2): Let  $K = \text{Im } l$  which is a subgroup since  $l$  is a group homo.

- $l$  is an isomorphism from  $G$  to  $K$ : If  $l(\bar{x}) = l(\bar{y})$ , then  $p \circ l(\bar{x}) = p \circ l(\bar{y}) \rightsquigarrow \bar{x} = \bar{y}$ . So  $l$  is 1-1.
- $E = NK$ :  $\forall x \in E, \bar{x} = p(x) \rightsquigarrow y = l(\bar{x})$  and  $p(x) = p(y) \rightsquigarrow \exists a \in N$  s.t.  $x = ay$ .
- $K \cap N = \{1\}$ :  $a = l(\bar{x}) \in K \cap N \rightsquigarrow 1 = p(a) = p(l(\bar{x})) = \bar{x} \rightsquigarrow a = l(1) = 1$ .

(2)  $\Rightarrow$  (1):

- $p|_K : K \rightarrow G$  is an isom.: onto:  $p(K) = p(NK) = p(E) = G$ , 1-1:  $\ker(p|_K) = N \cap K = \{1\}$ .
- $l = (p|_K)^{-1}$  is a group homo.

Observation: Let  $l : G \rightarrow E$  be a lifting. Then  $E = \bigcup_{\bar{x} \in G} Nl(\bar{x}), \forall x, y \in E$ , write  $x = al(\bar{x}), y = bl(\bar{y}), a, b \in N, \bar{x}, \bar{y} \in G$ .

$$xy = (al(\bar{x})bl(\bar{y})) = al(\bar{x})bl(\bar{x})^{-1}l(\bar{x})l(\bar{y}) = a\theta_{\bar{x}}(b)l(\bar{x})l(\bar{y})$$

Notice that  $l(\bar{x})l(\bar{y})$  and  $l(\bar{x}\bar{y})$  are liftings, so we can write  $l(\bar{x})l(\bar{y}) = f(\bar{x}, \bar{y})l(\bar{x}\bar{y})$  for some  $f(\bar{x}, \bar{y}) \in N$ .  $\square$

**Ex 4.1.1.**  $B^2(G, N) \leq Z^2(G, N)$ .

**Ex 4.1.2.** Show that there are inequivalent extensions of  $N$  by  $G$  with isomorphic middle groups. (Hint:  $N = \mathbb{Z}/p\mathbb{Z}$  with  $p$  is odd,  $E = \mathbb{Z}/p^2\mathbb{Z}$ ,  $a :: N \mapsto x^p :: E$  and please give another morphism  $N \rightarrow E$  by yourself.)

**Def 69.** Given  $1 \rightarrow N \rightarrow E \xrightarrow{p} G \rightarrow 1$  and  $l : G \rightarrow E$ , a factor set is a function  $f : G \times G \rightarrow N$  s.t.  $\forall \bar{x}, \bar{y} \in G, l(\bar{x})l(\bar{y}) = f(\bar{x}, \bar{y})l(\bar{x}\bar{y})$ .

**Prop 4.1.3.** Let  $1 \rightarrow N \rightarrow E \xrightarrow{p} G \rightarrow 1$  and  $l : G \rightarrow E$ . If  $f$  is a factor set, then

- (1)  $f(x, 1) = 1 = f(1, y) \quad \forall x, y \in G$ .
- (2) (cocycle identity)  $\forall x, y, z \in G, f(x, y)f(xy, z) = \theta_x(f(y, z))f(x, yz)$ .  
(i.e.  $f(x, y) + f(xy, z) = xf(y, z) + f(x, yz)$ )

*Proof.*

- (1) Trivial since  $l(x)l(1) = l(1 \cdot x)$ .
- (2) By associativity.  $(l(x)l(y))l(z) = l(x)(l(y)l(z))$ .  
 $(l(x)l(y))l(z) = f(x, y)l(xy)l(z) = f(x, y)f(xy, z)l(xyz)$ , and  
 $l(x)(l(y)l(z)) = l(x)f(y, z)l(yz) = l(x)f(y, z)l^{-1}(x)l(x)l(yz) = \theta_x(f(y, z))f(x, yz)l(xyz)$ .  
Thus  $f(x, y)f(xy, z) = \theta_x(f(y, z))f(x, yz)$ .  $\square$

**Theorem 36.** Let  $\sigma : G \rightarrow \text{Aut}(N), x \mapsto \sigma_x$  be a group homo. and  $f : G \times G \rightarrow N$  satisfies (1),(2) in Prop. 4.1.3. Then  $\exists 1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$  and  $l : G \rightarrow E$  s.t.  $\theta = \sigma$  and  $f$  is the corresponding factor set.

*Proof.* • Define  $E = N \times G$  equipped with the operation

$$(a, x)(b, y) = (a\sigma_x(b)f(x, y), xy)$$

– associativity:

$$\begin{aligned} ((a, x)(b, y))(c, z) &= (a\sigma_x(b)f(x, y), xy)(c, z) \\ &= (a\sigma_x(b)f(x, y)\sigma_{xy}(c)f(xy, z), xyz) \\ &= (a\sigma_x(b)\sigma_{xy}(c)f(x, y)f(xy, z), xyz) \quad (\because N \text{ abelian}) \end{aligned}$$

and

$$\begin{aligned} (a, x)((b, y)(c, z)) &= (a, x)(b\sigma_y(c)f(y, z)) \\ &= (a\sigma_x(b\sigma_y(c)f(y, z))f(x, yz), xyz) \\ &= (a\sigma_x(b)\sigma_{xy}(c)\sigma_x(f(y, z))f(x, yz), xyz) \\ &= (a\sigma_x(b)\sigma_{xy}(c)f(x, y)f(xy, z), xyz) \end{aligned}$$

– identity:  $(1, 1)$ .

– inverse:  $(a, x)^{-1} = (\sigma_{x^{-1}}(a^{-1}f(x, x^{-1})^{-1}), x^{-1})$ .

- $p : E \rightarrow G, (a, x) \mapsto x$  is a group homo by def.
- $i : N \rightarrow E, a \mapsto (a, 1)$  is a group homo.  $(a, 1)(b, 1) = (a\sigma_1(b)f(1, 1), 1) = (ab, 1)$ .
- $\ker p = \text{Im } i$ .
- $\text{Fix } l : G \rightarrow E, a \in N, x \in G$ , say  $l(x) = (b, x)$ .

$$\begin{aligned} l(x)(a, 1)l(x)^{-1} &= (b, x)(a, 1)(b, x)^{-1} = (b\sigma_x(a), x)(\sigma_{x^{-1}}(a^{-1}f(x, x^{-1})^{-1}), x^{-1}) \\ &= (b\sigma_x(a) \cdot (\sigma_x \circ \sigma_{x^{-1}})(b^{-1}f(x, x^{-1})^{-1}) \cdot f(x, x^{-1}), 1) \\ &= (\sigma_x(a), 1) \end{aligned}$$

So  $\theta_x = \sigma_x$ .

- Let  $l : G \rightarrow E, x \mapsto (1, x)$ . Check  $l(x)l(y)l(xy)^{-1} = (f(x, y), 1)$ . Then  $f$  is the corresponding factor set.  $\square$

**Prop 4.1.4.** Let  $1 \rightarrow N \rightarrow E \xrightarrow{p} G \rightarrow 1$  with two liftings  $l_1 : G \rightarrow E, l_2 : G \rightarrow E$  with  $f_1 : G \times G \rightarrow N, f_2 : G \times G \rightarrow N$  respectively.

Then  $\exists h : G \rightarrow N$  with  $h(1) = 1$  and  $\forall x, y \in G, f_2(x, y)f_1(x, y)^{-1} = \theta_x(h(y))h(xy)^{-1}h(x)$ .  
 $(f_2(x, y) - f_1(x, y) = xh(y) - h(xy) + h(x))$

*Proof.* For  $x \in G, \exists h(x) \in N$  s.t.  $l_2(x) = h(x)l_1(x)$ . Since  $l_1(1) = l_2(1) = 1, h(1) = 1$ .

Now,  $l_2(x)l_2(y) = f_2(x, y)l_2(x, y) = f_2(x, y)h(xy)l_1(x, y)$ . and

$$\begin{aligned} l_2(x)l_2(y) &= h(x)l_1(x)h(y)l_1(y) = h(x)l_1(x)h(y)l_1^{-1}(x)l_1(x)l_1(y) \\ &= h(x)\theta_x(h(y))l_1(x)l_1(y) = f_1(x, y)h(x)\theta_x(h(y))l_1(x, y) \end{aligned}$$

So  $f_2(x, y)f_1(x, y)^{-1} = \theta_x(h(y))h(xy)^{-1}h(x)$ .  $\square$

**Remark 22.** A map which has the form  $\tilde{h} : G \times G \rightarrow N, (x, y) \mapsto xh(y) - h(xy) + h(x)$  is called a coboundary map.

**Def 70.**  $Z^2(G, N)$  = the abelian group of all factor sets.

$B^2(G, N)$  = the abelian group of all coboundary maps.

$$H^2(G, N) = Z^2(G, N)/B^2(G, N)$$

**Def 71.** Two extensions  $\begin{cases} 1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1 \\ 1 \rightarrow N \rightarrow E' \rightarrow G \rightarrow 1 \end{cases}$  are equivalent if exists an isomorphism  $\varphi :$

$E \xrightarrow{\sim} E'$  which let the following diagram comutes.

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow 1_N & & \downarrow \varphi & & \downarrow 1_G \\ 1 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & G \longrightarrow 1 \end{array}$$

**Theorem 37.** Two extensions  $\begin{cases} 1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1 \\ 1 \rightarrow N \rightarrow E' \rightarrow G \rightarrow 1 \end{cases}$  are equivalent  $\iff$

Exists mappings  $l : G \rightarrow E, l' : G \rightarrow E'$  with two factor sets  $f, f'$  respectively satisfies  $f - f' \in B^2(G, N)$ .

*Proof.* " $\Rightarrow$ ": Choose  $l : G \rightarrow E$  which has a corresponding factor set  $f : G \times G \rightarrow N$ . Now define  $l' : G \rightarrow E'$  by  $l' = \varphi \circ l$ . Since  $p' \circ l' = p' \circ \varphi \circ l = p \circ l = 1, l'$  is a lifting. Let  $f' : G \times G \rightarrow N$  be its factor set.

Since  $1_N = 1_N \circ \varphi$ ,  $\varphi|_N = 1_N$ . And

$$\begin{aligned} l(x)l(y) &= f(x, y)l(xy) \\ \Rightarrow \varphi(l(x)l(y)) &= \varphi(f(x, y)l(xy)) \\ \Rightarrow l'(x)l'(y) &= \varphi(f(x, y))l'(xy) \\ \Rightarrow f'(x, y) &= \varphi(f(x, y)) \end{aligned}$$

But  $f(x, y) \in N$ ,  $\varphi(f(x, y)) = \varphi|_N(f(x, y)) = f(x, y)$ . So  $f(x, y) = f'(x, y)$ , hence  $f - f' = 0 \in B^2(G, N)$ .

**Ex 4.1.3.**

- (1) Show that  $f' - f \in B^2(G, N)$ .
- (2) “ $\Leftarrow$ ”: Show all details of the following steps:

- $\begin{cases} 1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1 \\ 1 \rightarrow N \rightarrow E(N, G, f, \theta) \rightarrow G \rightarrow 1 \end{cases}$  are equivalent.
- Similarly  $\begin{cases} 1 \rightarrow N \rightarrow E' \rightarrow G \rightarrow 1 \\ 1 \rightarrow N \rightarrow E(N, G, f', \theta') \rightarrow G \rightarrow 1 \end{cases}$  are equivalent.
- $f' - f \rightsquigarrow h : G \rightarrow N$ ,

□

#### 4.1.2 1st and 2nd group cohomology

Let  $N$  be an abelian group and  $G$  be a group with a group homo  $\sigma : G \rightarrow \text{Aut}(N)$  ( $G \curvearrowright N$ )

$e(G, N) = \{\text{equivalence classes of } N \text{ by } G\}$

$$Z^2(G, N) = \{f : G \times G \rightarrow N \mid f(1, v) = f = f(u, 1), f(u, v) + f(uv, w) = uf(v, w) + f(u, vw) \quad u, v, w \in G\}$$

$$B^2(G, N) = \{f : G \times G \rightarrow N \mid \exists h : G \rightarrow N \text{ with } h(1) = 1 \text{ s.t. } f(u, v) = uh(v) - h(uv) + h(u) \quad u, v \in G\}$$

$$H^2(G, N) = Z^2(G, N)/B^2(G, N)$$

Then  $e(G, N) \leftrightarrow H^2(G, N)$ .

**Def 72.**

- $\varphi \in \text{Aut}(E)$  stabilizes  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$  if

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow 1_N & & \downarrow \varphi|_E & & \downarrow 1_G \\ 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \end{array}$$

- $\text{Stab}_E(G, N) = \{\text{stabilizing automorphisms}\} \leq \text{Aut}(E)$

**Def 73.**

- A derivation is a function  $d : G \rightarrow N$  s.t.  $d(uv) = ud(v) + d(u) \quad \forall u, v \in G$ .
- $\text{Der}(G, N) = \{\text{derivations} : G \rightarrow N\}$  is an abelian group with pointwise addition.

**Theorem 38.** Let  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$  with  $\theta = \sigma$ . Then  $\text{Stab}_E(G, N) \cong \text{Der}(G, N)$ . So  $\text{Stab}_E(G, N)$  is abelian.

*Proof.*

- Let  $\varphi \in \text{LHS}$  and fix  $l : G \rightarrow E$ .

$$\begin{array}{ccccccc}
1 & \longrightarrow & N & \longrightarrow & E & \xrightarrow{\quad} & G \longrightarrow 1 \\
& & \downarrow 1_N & & \downarrow \varphi & \swarrow l & \downarrow 1_G \\
1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G \longrightarrow 1
\end{array}
\quad \varphi(al(u)) = \varphi(a)\varphi(l(u)) = ad(u)l(u)$$

- For another  $l' : G \rightarrow E$ , say  $l'(u) = g(u)l(u)$ , where  $g(u) \in N$ , we have

$$d'(u) = \varphi(l'(u))(l'(u))^{-1} = \varphi(g(u)l(u))(g(u)l(u))^{-1} = g(u)\varphi(l(u))l(u)^{-1}g(u)^{-1} = d(u).$$

- $d \in \text{RHS}$ ,

$$\begin{aligned}
d(uv) &= \varphi(l(uv))l(uv)^{-1} \\
&= \varphi(f(u, v)^{-1}l(u)l(v))l(v)^{-1}l(u)^{-1}f(u, v) \\
&= f(u, v)^{-1}d(u)l(u)d(v)l(v)^{-1}l(u)^{-1}f(u, v) \\
&= f(u, v)^{-1}d(u)(l(u)d(v)l(u)^{-1})f(u, v) \\
&= (ud(v))d(u)
\end{aligned}$$

- Conversely,

**Ex 4.1.4.** proof it

- group homo:  $\varphi_2 \circ \varphi_1(al(u)) = \varphi_2(ad_1(u)l(u)) = ad_1(u)\varphi_2(l(u)) = ad_1(u)d_2(u)l(u)$ . That is,  $\varphi_2 \circ \varphi_1 \mapsto d_1d_2$ .  $\square$

**Def 74.**

- $\text{Inn}_E(G, N) = \{\varphi \in \text{Stab}_E(G, N) \mid \varphi : E \rightarrow E, x \mapsto a_0xa_0^{-1} \text{ for some } a_0 \in N\}$ .
- $\text{PDer}(G, N) = \{d \in \text{Der}(G, N) \mid d(u) = ua_1 - a_1 \text{ for some } a_1 \in N\}$ .

**Ex 4.1.5.** Show that  $\text{Inn}_E(G, N) \cong \text{PDer}(G, N)$ .

$$\text{Stab}_E(G, N)/\text{Inn}_E(G, N) \cong \text{Der}(G, N)/\text{PDer}(G, N) = H^1(G, N).$$

**Ex 4.1.6.** Fix  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$ . Show that if  $H^2(G, N) = 0, H^1(G, N) = 0$ , then for  $l : G \rightarrow E$  with  $K = l(G)$ , we get that  $K$  and  $K'$  are conjugate.  
 $l' : G \rightarrow E$  with  $K' = l'(G)$

**Def 75.** Let  $R$  be a commutative ring with 1 and  $G$  be a group. The group ring

$$R[G] = \left\{ \sum_{g \in G} r_g g \mid \text{only finitely many } r_g \text{'s } \neq 0 \text{ in } R \right\}$$

forms an  $R$ -algebra via

$$\begin{aligned}
\sum_{g \in G} r_g g + \sum_{g \in G} r'_g g &= \sum_{g \in G} (r_g + r'_g)g \\
\left( \sum_{g \in G} r_g g \right) \left( \sum_{g' \in G} r'_g g' \right) &= \sum_{g, g' \in G} (r_g r'_g) gg' \\
r \left( \sum_{g \in G} r_g g \right) &= \sum_{g \in G} (rr_g)g
\end{aligned}$$

**Remark 23.**

1.
  - $\{\rho : G \rightarrow \text{GL}(V)\} \leftrightarrow \{V : \mathbb{C}[G]\text{-module}\}.$
  - $\rho : \text{irr} \leftrightarrow V : \text{simple } \mathbb{C}[G]\text{-module (i.e. no nontrivial proper submodule)}$
  - $W \subset V : G\text{-invariant} \leftrightarrow W : \mathbb{C}[G]\text{-submodule}.$
2.  $N : \text{abelian} \rightsquigarrow N : \mathbb{Z}\text{-module and } G \curvearrowright N. \implies N : \mathbb{Z}[G]\text{-module}.$

**Def 76.**  $G \curvearrowright \mathbb{Z}$  trivially. i.e.  $g \cdot n = n \quad \forall g \in G, n \in \mathbb{Z}$ , then  $\mathbb{Z} : \mathbb{Z}[G]\text{-module}.$

- $B_0 = \mathbb{Z}[G][\ ]$ : the free  $\mathbb{Z}[G]$ -module on the symbol  $\[ \]$ .
- $B_1 = \bigoplus_{u \in G} \mathbb{Z}[G][u]$ : the free  $\mathbb{Z}[G]$ -module on the set  $G$ .
- $B_2 = \bigoplus_{u,v \in G} \mathbb{Z}[G][u|v]$ : the free  $\mathbb{Z}[G]$ -module on the set  $G \times G$ .
- $B_3 = \bigoplus_{u,v,w \in G} \mathbb{Z}[G][u|v|w]$ : the free  $\mathbb{Z}[G]$ -module on the set  $G \times G \times G$ .

...

Now apply  $\text{Hom}(\cdot, N)$  to it:

...

**Theorem 39.**  $\text{Ext}_{\mathbb{Z}[G]}^1(\mathbb{Z}, N) := \ker d_2^* / \ker d_1^* \cong \text{Der}(G, N) / \text{PDer}(G, N) = H^1(G, N).$

*Proof.*

- $g \in \ker d_2^* \subseteq \text{Hom}(B_1, N) \implies g \circ d_2 = 0. \dots$
- ...
- Let  $t \in \text{Hom}(B_0, N)$ , say  $t(\[ \]) = a_0 \in N$ .

$$d_1^*(t)([u]) = t \circ d_1([u]) = t(u[\ ] - [\ ]) = ut([\ ]) - t([\ ]) = ua_0 - a_0$$

Then  $d(u) := d_1^*(t)([u]) \implies d \in \text{PDer}(G, N).$

- ...

□

**Remark 24.**  $\text{Ext}_{\mathbb{Z}[G]}^2(\mathbb{Z}, N) \cong H^2(G, N).$



## 5 Fields

### 5.1 Algebraic extensions

**Def 77.**

- $L/K$  is called an **field extension** if  $L$  is a field and  $K$  is a subfield of  $L$ .
- $\alpha \in L$  is **algebraic** over  $K$  if exists  $f(x) \in K[x]$  satisfied  $f(\alpha) = 0$ .
- $L/K$  is called an **algebraic extension** if  $\forall \alpha \in L, \exists f(x) \in K[x]$  such that  $f(\alpha) = 0$ .
- $K(\alpha_1, \alpha_2, \dots, \alpha_n) \triangleq \{P(\alpha_1, \dots, \alpha_n)/Q(\alpha_1, \dots, \alpha_n) : P, Q \in K[x_1, x_2, \dots, x_n] \text{ and } Q \neq 0\}$

**Theorem 40** (Eisenstein criterion).

Let  $f(x) = a_n x^n + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$  with  $\gcd(a_0, a_1, \dots, a_n) = 1$ . Assume that there exists a prime  $p$  s.t.  $p \nmid a_n$  but  $p \mid a_i$  for other  $i \neq n$ , and  $p^2 \nmid a_0$ , then  $f$  is irreducible.

*Proof.* Since  $f$  is primitive, by Gauss lemma, we only need to prove that it is irreducible in  $\mathbb{Q}[x]$ . Consider  $\bar{f}(x)$ , by assumption,  $\bar{f}(x) = \bar{a}_n x^n$ . So if  $f(x) = g(x)h(x)$  with  $\deg g, \deg h \geq 1$ , let  $g(x) = b_r x^r + \dots + b_0, h(x) = c_{n-r} x^{n-r} + \dots + c_0$ , then  $\bar{g}(x) = \bar{b}_r x^r, \bar{h}(x) = \bar{c}_{n-r} x^{n-r}$  for some  $r$ . But then we would find out that  $\bar{b}_0 = \bar{c}_0 = 0$ , and thus  $p^2 \mid a_0$ , which is a contradiction, hence  $f$  is irreducible.  $\square$

**Prop 5.1.1.** Given  $L/K$  and  $\alpha \in L$ , if  $\alpha$  is algebraic over  $K$ , then there exists a unique monic irreducible polynomial  $m_{\alpha, K}(x) \in K[x]$  of minimal degree s.t.  $m_{\alpha, K}(\alpha) = 0$  and for any other  $f(x) \in K[x]$  with  $f(\alpha) = 0$ , we have  $m_{\alpha, K} \mid f$ . We call  $m_{\alpha, K}$  the **minimal polynomial** of  $\alpha$  over  $K$ .

*Proof.* Let  $I$  be the set of all polynomials such that  $f(\alpha) = 0$ , since  $\alpha$  algebraic,  $I \neq \emptyset$ , so pick a monic polynomial  $g(x)$  of minimal degree in  $I$ . For any other  $f(x) \in I$ , write  $f(x) = g(x)q(x) + r(x)$  with  $\deg r < \deg g$ . If  $r(x) \neq 0$ , then  $r(\alpha) = f(\alpha) - q(\alpha)g(\alpha)$ . But then  $r(\alpha) = f(\alpha) - q(\alpha)g(\alpha) = 0$  with  $\deg r < \deg g$ , which contradicts the minimality of  $g$ , thus  $r = 0$ , and hence  $g \mid f$ .

Finally, if  $g(x) = h_1(x)h_2(x)$  with  $\deg h_1, \deg h_2 < \deg g$ , then one of them, say  $h_1(\alpha) = 0$  again contradicts the minimality of  $g$ , hence  $g$  is irreducible.  $\square$

**Prop 5.1.2.** Let  $L/K$  be an extension and  $\alpha \in L$ , the following are equivalent:

- (1)  $\alpha$  is algebraic over  $K$ .
- (2)  $K[\alpha] = K(\alpha)$ .
- (3)  $[K(\alpha) : K] < \infty$ .

*Proof.* (1)  $\Rightarrow$  (2): “ $\subset$ ” trivial.

“ $\supset$ ”: For all  $\beta \in K(\alpha), \beta = g(\alpha)/h(\alpha)$  with  $h(\alpha) \neq 0$ . So  $m_{\alpha, K} \nmid h$ . Since  $m_{\alpha, K}$  is irreducible,  $\gcd(m_{\alpha, K}, h) = 1$ , hence there exists  $a(x), b(x) \in K[x]$  such that  $1 = a(x)h(x) + b(x)m_{\alpha, K}(x)$ . Substitute  $\alpha$  and we get  $1/h(\alpha) = a(\alpha)$ , hence  $\beta = g(\alpha)a(\alpha) \in K[\alpha]$ .

(2)  $\Rightarrow$  (1): Since  $1/\alpha \in K[\alpha]$ , thus  $1/\alpha = f(\alpha)$  for some polynomial  $f$ , hence if we set  $g(x) = xf(x) - 1$ ,  $g(\alpha) = 0$  which implies  $\alpha$  is algebraic.

(1)  $\Rightarrow$  (3): Assume that  $\deg m_{\alpha, K} = n$ , it is easy to see that  $K[\alpha] = \langle 1, \alpha, \dots, \alpha^{n-1} \rangle_K$ . Since (1)  $\Rightarrow$  (2), we have  $[K(\alpha) : K] = [K[\alpha], K] = n$ .

(3)  $\Rightarrow$  (1): Since  $[K(\alpha) : K] = n$ , consider  $1, \alpha, \alpha^2, \dots, \alpha^n$ . Some of these  $n + 1$  elements may be coincident, but nevertheless these elements are linearly dependent. Hence there exists  $a_0, \dots, a_n$  not all zero in  $K$  s.t.  $a_0 + a_1\alpha + \dots + a_n\alpha^n = 0 \Rightarrow \alpha$  is algebraic.  $\square$

**Prop 5.1.3.** Given  $M/L$  and  $L/K$ ,  $[M : K] = [M : L][L : K]$ .

*Proof.* If  $[M : L] = m < \infty$  and  $[L : K] = n < \infty$ , then  $L \cong K^{\oplus n}$ ,  $M \cong L^{\oplus m}$ . So  $M \cong (K^{\oplus n})^{\oplus m} \cong K^{\oplus mn}$ , thus  $[M : K] = mn$ .

Now if  $[M : K] = l < \infty$ , then there exists a basis  $\{z_1, z_2, \dots, z_l\}$  which is a basis for  $M$  over  $K$ . Then  $M = Kz_1 + \dots + Kz_l \subset Lz_1 + \dots + Lz_l \subset M \Rightarrow M = Lz_1 + \dots + Lz_l$ . Hence  $[M : L] < \infty$ . Also, since  $L$  is a  $K$ -linear subspace of  $M$ ,  $[L : K] \leq l \Rightarrow [L : K] < \infty$ . Thus if  $[M : L] = \infty$  or  $[L : K] = \infty$ , then  $[M : K] = \infty$ .  $\square$

**Prop 5.1.4.** Given  $L/K$ , define  $L^{\text{alg}} \triangleq \{\alpha \in L \mid \alpha \text{ is algebraic over } K\}$ , then  $L^{\text{alg}}$  is a subfield of  $L$ .

*Proof.* Notice that if  $\alpha, \beta \in L^{\text{alg}}$ , then  $\beta$  is algebraic over  $K$  implies that  $\beta$  is algebraic over  $K(\alpha)$ . Thus

$$[K(\alpha, \beta) : K] = [K(\alpha)(\beta) : K(\alpha)][K(\alpha) : K] < \infty$$

Also, since  $K(\alpha + \beta), K(\alpha - \beta), K(\alpha\beta), K(\alpha/\beta)$  are all contained in  $K(\alpha, \beta)$ , they are all algebraic over  $K$ , thus these elements are all algebraic, and hence  $L^{\text{alg}}$  is a subfield.  $\square$

**Prop 5.1.5.**  $[L : K] < \infty$  if and only if  $L = K(\alpha_1, \alpha_2, \dots, \alpha_n)$  with each  $\alpha_i$  algebraic over  $K$ . In this case,  $L/K$  is algebraic.

*Proof.* " $\Rightarrow$ ": Let  $[L : K] = n$ , so there is a basis  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  for  $L$  over  $K$ . It is easy to see that  $L = K(\alpha_1, \dots, \alpha_n)$ . Also  $[K(\alpha_i) : K] \leq [L : K] < \infty$ , thus  $\alpha_i$  is algebraic.

" $\Leftarrow$ ": Since  $\alpha_i$  is algebraic over  $K$ ,  $\alpha_i$  is algebraic over  $K(\alpha_1, \dots, \alpha_{i-1})$ . Thus

$$[L : K] = [K(\alpha_1, \dots, \alpha_n) : K(\alpha_1, \dots, \alpha_{n-1})][K(\alpha_1, \dots, \alpha_{n-1}) : K(\alpha_1, \dots, \alpha_{n-2})] \dots [K(\alpha_1) : K] < \infty$$

Moreover,  $\forall \alpha \in L$ ,  $[K(\alpha) : K] \leq [L : K] < \infty$ , so  $\alpha$  is algebraic over  $K$ .  $\square$

**Coro 5.1.1.** Given  $L/K$ , and  $S$  a subset of  $L$ , if  $\forall \alpha \in S$ ,  $\alpha$  is algebraic over  $K$ , then  $K(S)/K$  is algebraic.

*Proof.* If  $\beta \in K(S)$ , by definition we know that there exists  $\alpha_1, \dots, \alpha_n$  such that  $\beta \in K(\alpha_1, \dots, \alpha_n)$ . Thus  $\beta$  is algebraic over  $K$ .  $\square$

**Prop 5.1.6.** If  $M/L$  and  $L/K$  are algebraic, then  $M/K$  is algebraic.

*Proof.* For all  $\alpha \in M$ , since  $\alpha$  is algebraic over  $L$ , there exists  $a_{n-1}, \dots, a_0$  so that  $\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_0 = 0$ , that is,  $\alpha$  is algebraic over  $K(a_0, \dots, a_{n-1})$ .

So  $[K(a_0, \dots, a_{n-1}, \alpha) : K] = [K(a_0, \dots, a_{n-1})(\alpha) : K(a_0, \dots, a_{n-1})][K(a_0, \dots, a_{n-1}) : K] < \infty$ , thus  $\alpha$  is algebraic over  $K$ .  $\square$

**Def 78.** Given  $L/L_1$  and  $L/L_2$ ,  $L_1L_2$  is defined as the smallest subfield of  $L$  containing both  $L_1$  and  $L_2$ .

**Prop 5.1.7.** Let  $[L_1 : K] = m$  and  $[L_2 : K] = n$ .

- (1)  $[L_1 L_2 : K] \leq mn$ .  
(2) If  $\gcd(m, n) = 1$ , then  $[L_1 L_2 : K] = mn$ .

*Proof.* (1): Assume  $L_1 = K(\alpha_1, \dots, \alpha_m)$ ,  $L_2 = K(\beta_1, \dots, \beta_n)$ . We could find that  $L_1 L_2 = K(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n)$ . Notice that  $[K(\beta_1, \dots, \beta_m)(\alpha_i) : K(\beta_1, \dots, \beta_m)] \leq [K(\alpha_i) : K]$ , and thus  $[L_1 L_2 : K] = [K(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) : K(\beta_1, \dots, \beta_n)][K(\beta_1, \dots, \beta_m) : K] \leq [K(\alpha_i, \dots, \alpha_n) : K][K(\beta_1, \dots, \beta_n) : K] = [L_1 : K][L_2 : K]$ .

(2): Notice that  $[L_i : K] \mid [L_1 L_2 : K]$ , so  $mn \mid [L_1 L_2 : K]$ . By (1),  $[L_1 L_2 : K] \leq nm$ , hence  $[L_1 L_2 : K] = nm$ .  $\square$

**Def 79.** Let  $R$  be a commutative ring with 1, and  $I$  be an ideal of  $R$ , then

- $I$  is called a **maximal ideal** if for any ideal  $J$  satisfying  $I \subseteq J$  we have  $J = I$  or  $J = R$ .
- $I$  is called a **prime ideal** if  $I \neq R$  and  $ab \in I \implies a \in I$  or  $b \in I$ .

**Prop 5.1.8.** Suppose  $R$  is a ring and  $I \subsetneq R$  is an ideal, then

1.  $I$  is maximal  $\iff R/I$  is a field.
2.  $I$  is a prime ideal  $\iff R/I$  is an integral domain.

*Proof.*

1. “ $\implies$ ”: For any  $\bar{r} \in R/I$  with  $\bar{r} \neq 0$ , then  $r \notin I$ . Consider  $\langle r \rangle + I$  which contains  $I$  and is not equal to  $I$  because  $r \notin I$ . Since  $I$  is maximal,  $\langle r \rangle + I = R$ , and thus  $\exists x \in R, y \in I$  such that  $xr + y = 1$ , so  $\bar{x}\bar{r} = \bar{1}$ . Hence every non-zero element has multiply inverse and  $R/I$  is a field.  
“ $\impliedby$ ”: If  $J$  is an ideal such that  $I \subsetneq J$ , pick  $x \in J \setminus I$ , then  $\bar{x} \neq 0$ , so  $\exists r \in J$  such that  $\bar{x}\bar{r} = 1$ . Then  $xr + I = 1 + I \implies \exists y \in I$  s.t.  $xr + y = 1$ . So  $1 \in J$ , and because  $J$  is an ideal,  $J = R$ .
2. By the fact that  $(ab \in I \implies a \in I \text{ or } b \in I) \iff (\bar{a}\bar{b} = 0 \implies \bar{a} = 0 \text{ or } \bar{b} = 0)$  the proof is complete.  $\square$

**Prop 5.1.9.** If  $f(x) \in K[x]$  is irreducible, where  $K$  is a field, then  $\langle f(x) \rangle$  is maximal ideal.

*Proof.* We know that  $K[x]$  is a principle ideal domain, so if  $\langle f(x) \rangle \subseteq J$ , then  $J$  is generated by a element, say  $g(x)$ . Since  $f(x) \in J$ , we could write  $f(x) = g(x)h(x)$ . By the fact that  $f(x)$  is irreducible, either  $g(x)$  is an unit then  $J = R$ , or  $h(x)$  is an unit then  $J = \langle f(x) \rangle$ .  $\square$

**Ex 5.1.1.**  $f(x) = x^2 + 1$  has roots  $\alpha = \pm\sqrt{-1}$ , so  $\mathbb{R}(\sqrt{-1}) \cong \mathbb{R}[x]/\langle x^2 + 1 \rangle$ .

**Theorem 41.** Let  $f(x) \in K[x]$  be monic, irreducible and of degree  $n$ . Then there exists  $L/K$  and  $\alpha \in L$  s.t.  $f(\alpha) = 0$ ,  $L = K(\alpha)$  and  $[L : K] = n$ .

*Proof.* Since  $f(x)$  is irreducible, by prop. 5.1.9  $\langle f(x) \rangle$  is a maximal ideal. Then by prop. 5.1.8  $L = K[x]/\langle f(x) \rangle$  is a field, and  $K$  is a subfield of  $L$  by the inclusion map  $\alpha \mapsto \bar{\alpha}$ . The map is 1-1 since  $\bar{1} \neq 0$  and a field homomorphism is either a 1-1 map or a zero (全洪) map.

Notice that  $L \cong K[\bar{x}]$ , where  $\bar{x}$  is the coset  $x + \langle f(x) \rangle$ . Now let  $\alpha = \bar{x}$ , and it is easy to see that  $f(\alpha) = f(x) + \langle f(x) \rangle = 0$ . Also  $L \cong K[\bar{x}] \cong K(\alpha)$ . Finally,  $m_{\alpha, K} \mid f$  and by the fact that  $f$  is monic and irreducible,  $m_{\alpha, K} = f$  and thus  $[L : K] = \deg m_{\alpha, K} = \deg f = n$ .  $\square$

**Theorem 42.** Let  $f(x) \in K[x]$  be of degree  $n > 0$ . Then there exists  $L/K$  s.t.  $f$  splits over  $L$ , that is,

$$f(x) = \lambda(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) \text{ with } \alpha_1, \alpha_2, \dots, \alpha_n \in L, \lambda \in K$$

In fact,  $L$  can be chosen to be the smallest field over which  $f$  splits and in this case  $[L : K] \leq n!$ .  $L$  is called a *splitting field* for  $f$  over  $K$ .

*Proof.* By induction on  $n$ ,  $n = 1$  is trivial, simply pick  $L = K$ .

For  $n > 1$ , let  $p(x)$  be a monic irreducible factor of  $f(x)$ . By theorem 41, there exists an extension  $K(\alpha_1)$  s.t.  $p(\alpha_1) = 0$ . By division algorithm,  $f(x) = (x - \alpha_1)f_1(x)$  where  $f_1(x) \in K(\alpha_1)[x]$  and  $\deg f_1 = n - 1$ . Using the induction hypothesis, we know that there exists  $L$ , which is an extension of  $K(\alpha_1)$ , s.t.  $f_1$  splits over  $L$ . Hence  $\exists \alpha_2, \alpha_3, \dots, \alpha_n \in L$  s.t.  $f_1(x) = \lambda(x - \alpha_2) \cdots (x - \alpha_n)$ , thus  $f(x) = \lambda(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ . Compare the coefficient of  $x^n$  we know that  $\lambda \in K$ .

More over, observe that  $K(\alpha_1, \dots, \alpha_n)$  is the smallest field containing  $K$  and  $\{\alpha_1, \dots, \alpha_n\}$ . So if we choose  $L = K(\alpha_1, \alpha_2, \dots, \alpha_n)$ , then

$$[L : K] = [K(\alpha_1, \alpha_2, \dots, \alpha_n) : K(\alpha_1, \alpha_2, \dots, \alpha_{n-1})] \cdots [K(\alpha_1) : K] \leq n!$$

Since  $[K(\alpha_1, \alpha_2, \dots, \alpha_k) : K(\alpha_1, \alpha_2, \dots, \alpha_{k-1})] = [K(\alpha_1, \alpha_2, \dots, \alpha_{k-1})(\alpha_k) : K(\alpha_1, \alpha_2, \dots, \alpha_{k-1})]$  and  $\alpha_k$  is a root of  $p(x) \in K(\alpha_1, \alpha_2, \dots, \alpha_{k-1})[x]$  where  $f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_{k-1})p(x)$ .  $\square$

**Eg 5.1.2.** Find a splitting field  $L$  for  $x^8 - 2$  over  $\mathbb{Q}$  and determine  $[L : \mathbb{Q}]$ .

The roots are  $\alpha\zeta^k$  where  $\alpha = \sqrt[8]{2}$  and  $\zeta = e^{2\pi i/8}$ . But  $\zeta = \sqrt{2}(1+i)/2$  where  $\sqrt{2} = \alpha^4$ , so we know that  $L = \mathbb{Q}(\alpha, \zeta) = \mathbb{Q}(\alpha, i)$ . Thus  $[L : \mathbb{Q}] = [\mathbb{Q}(\alpha, i) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}] = 2 \cdot 8 = 16$ .

**Remark 25.**  $\mathbb{Q}[x]/\langle x^8 - 2 \rangle = \mathbb{Q}(\bar{x}) \cong \mathbb{Q}(\sqrt[8]{2}) \cong \mathbb{Q}(\sqrt[8]{2}\zeta)$

**Prop 5.1.10.** Let  $K, L$  be two fields and  $\tau : K \rightarrow L$  be a nontrivial homomorphism. We define  $\bar{\tau} : K[x] \rightarrow \tau(K)[x] \subseteq L[x]$  by

$$a_n x^n + \cdots + a_0 \mapsto \bar{\tau}(f) \triangleq \tau(a_n)x^n + \cdots + \tau(a_0)$$

which is an isomorphism. Also,  $f$  is irreducible implies  $\bar{\tau}(f)$  is irreducible in  $\tau(K)[x]$ .

**Lemma 4.** Let  $K(\alpha)/K$  be algebraic and  $\tau : K \rightarrow L$  be a nontrivial homo, then there exists an extension  $\sigma$  of  $\tau$  from  $K(\alpha)$  to  $L$  if and only if  $\exists \beta \in L$  s.t.  $\bar{\tau}(m_{\alpha, K})(\beta) = 0$ .

In this case  $m_{\beta, \tau(K)} = \bar{\tau}(m_{\alpha, K})$ .

*Proof.* “ $\Rightarrow$ ”: Let  $\beta = \sigma(\alpha)$  and  $m_{\alpha, K} = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ . Then  $\bar{\tau}(m_{\alpha, K})(\beta) = \beta^n + \tau(a_{n-1})\beta^{n-1} + \cdots + \tau(a_0) = \tau(\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0) = 0$

“ $\Leftarrow$ ”: Observe that  $m_{\beta, \tau(K)} = \bar{\tau}(m_{\alpha, K})$  since  $\bar{\tau}(m_{\alpha, K})(\beta) = 0$  and  $\bar{\tau}(m_{\alpha, K})$  is monic and irreducible by prop 5.1.10.  $\sigma$  is then given by the following diagram.

$$\begin{array}{ccccc} K[x] & \xrightarrow[\bar{\tau}]{\sim} & \tau(K)[x] & & \\ \downarrow & & \downarrow & & \\ K(\alpha) & \xleftarrow{\cong} & K[x]/\langle m_{\alpha, K} \rangle & \xrightarrow[\sigma]{\sim} & \tau(K)[x]/\langle m_{\beta, \tau(K)} \rangle \xleftarrow{\cong} \tau(K)(\beta) \subseteq L \end{array}$$

$\square$

**Coro 5.1.2.** Let  $K(\alpha)/K$  be an algebraic extension and  $\tau : K \hookrightarrow L$ . If  $\bar{\tau}(m_{\alpha,K})$  has  $r$  distinct roots in  $L$ , then there are exactly  $r$  extensions of  $\tau$ .

**Theorem 43.** Let  $\tau : K \rightarrow K'$  be an isomorphism of fields. If  $L$  is a splitting field for  $f$  over  $K$  and  $L'$  is a splitting field for  $\bar{\tau}(f)$  over  $K'$ , then  $L \cong L'$

*Proof.* By induction on  $n = \deg f$ . When  $n = 1$ ,  $L = K, L' = K'$ , so  $L \cong L'$ .

Now if  $n > 1$ , assume  $f(\alpha) = 0$  for  $\alpha \in L$ . Then  $\bar{\tau}(m_{\alpha,K}) \mid \bar{\tau}(f)$  and by the fact that  $L'$  is a splitting field for  $\bar{\tau}(f)$ ,  $\exists \beta \in L'$  s.t.  $\bar{\tau}(m_{\alpha,K})(\beta) = 0$ . By lemma 4,  $\exists \tau_o : K(\alpha) \xrightarrow{\sim} K'(\beta)$  with  $\tau_o|_K = \tau$ .

Now, write  $f = (x - \alpha)f_o$ , then  $\bar{\tau}(f) = \bar{\tau}_o(f) = (x - \tau_o(\alpha))\bar{\tau}_o(f_o) = (x - \beta)\bar{\tau}_o(f_o)$ . Then  $L$  and  $L'$  is a splitting field for  $f_o$  over  $K(\alpha)$  and  $\bar{\tau}_o(f_o)$  over  $K(\beta)$  respectively. By induction hypothesis,  $L \cong L'$ .  $\square$

**Coro 5.1.3.** Let  $\tau : K \xrightarrow{\sim} K'$  be an isomorphism of fields, and  $L$  is a splitting field of  $f$  over  $K$ ,  $L'$  is a splitting field of  $\bar{\tau}(f)$  over  $K'$ . Then  $\tau$  could be extend to  $\sigma : L \xrightarrow{\sim} L'$  such that  $\sigma|_K = \tau$ .

## 5.2 Finite field

**Def 80.** A polynomial  $f(x) \in K[x]$  is said to be *separable* if its irreducible factors have no multiple roots in a splitting field  $L$ .

**Def 81.** If  $f(x) = a_n x^n + \cdots + a_1 x + a_0$ , then define  $f'(x) \triangleq n a_n x^{n-1} + \cdots + 2 a_2 x + a_1$ .

**Theorem 44.** Let  $f(x) \in K[x]$  be monic, irreducible of positive degree, then all the roots of  $f(x)$  in a splitting field are simple if and only if  $\gcd(f(x), f'(x)) = 1$ .

*Proof.* “ $\Rightarrow$ ”: We can write  $f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$  where  $\alpha_i$  are distinct roots of  $f$ . Then  $f'(x) = \sum_{i=1}^n f(x)/(x - \alpha_i)$  and we have  $(x - \alpha_i) \nmid f(x)$  for all  $i$ .

“ $\Leftarrow$ ”: Assume  $f(x) = (x - \alpha)^k g(x)$  with  $k \geq 2$ . Then  $f'(x) = k(x - \alpha)^{k-1} g(x) + (x - \alpha)^k g'(x)$  which implies  $(x - \alpha) \mid f(x)$ . So  $(x - \alpha) \mid \gcd(f(x), f'(x))$  and thus  $\gcd(f(x), f'(x)) \neq 1$ .  $\square$

**Remark 26.** The following are equivalent:

1.  $\alpha$  is a multiple root of  $f(x)$ .
2.  $\alpha$  is a common root of  $f(x)$  and  $f'(x)$ .
3.  $m_{\alpha, K} \mid f(x)$  and  $m_{\alpha, K} \mid f'(x)$ .

**Theorem 45.** There is a finite field  $K$  with  $|K| = q \iff q = p^n$  for some prime  $p$  and  $n \in \mathbb{N}$ . In this situation,  $K$  is unique up to isomorphism, denote by  $\mathbb{F}_{p^n}$ .

*Proof.* “ $\Rightarrow$ ”: Let  $p = \text{char } K$  and  $[K : \mathbb{Z}/p\mathbb{Z}] = n$ , then  $|K| = p^n$ .

“ $\Leftarrow$ ”: Let  $K$  be a splitting field for  $f(x) = x^{p^n} - x$  over  $\mathbb{F}_p$ . We claim that the set of all roots of  $f(x)$  forms a field. Since if  $\alpha, \beta$  are two roots of  $f$ , obviously  $\alpha\beta, \alpha\beta^{-1}$  are also roots, and by  $(\alpha \pm \beta)^{p^n} = \alpha^{p^n} \pm \beta^{p^n} = \alpha \pm \beta$  because  $\text{char } K = p$ .  $\alpha \pm \beta$  are also roots, hence the roots form a field. By definition,  $K$  is the smallest field containing  $\mathbb{F}_p$  and roots of  $f(x)$ , so  $K$  is exactly the set of roots of  $f(x)$ .

Also,  $f'(x) = -1$  has no root, so  $f(x)$  has no multiple root which implies  $|K| = p^n$ .

Moreover, if  $K'$  is another finite field with  $|K'| = p^n$ , then for all  $\alpha \in K'$ ,  $\alpha^{p^n} = \alpha$ , so  $\alpha$  is a root of  $f(x)$ , which implies that  $K'$  is a splitting field for  $f(x)$  over  $\mathbb{F}_p$ . By theorem 43,  $K \cong K'$ .  $\square$

**Theorem 46.** Let  $n \in \mathbb{N}$  and  $\mathbb{F}_q$  be a finite field. Then there exists a unique extension  $\mathbb{F}_{q^n}/\mathbb{F}_q$  s.t.  $[\mathbb{F}_{q^n} : \mathbb{F}_q] = n$ , and  $\text{Aut}(\mathbb{F}_{q^n}/\mathbb{F}_q) = \langle \sigma_q \rangle$  with  $\sigma_q = \alpha \mapsto \alpha^q :: \mathbb{F}_{q^n} \mapsto \mathbb{F}_{q^n}$ .  $\sigma_q$  is called the *Frobenius homomorphism*.

*Proof.* By theorem 45,  $q = p^r$  for some prime  $p$  and  $r \in \mathbb{N}$ , so  $q^n = p^{nr}$  which is a power of a prime. Again by theorem 45,  $\mathbb{F}_{q^n}$  is the splitting field for  $x^{p^{nr}} - x$  over  $\mathbb{F}_p$ . Since  $x^q - x \mid x^{q^n} - x$ ,  $\mathbb{F}_q \subseteq \mathbb{F}_{q^n}$  and thus  $[\mathbb{F}_{q^n} : \mathbb{F}_q] = n$ .

Then we proof that  $\sigma_q$  is indeed in  $\text{Aut}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ . We check that

$$\begin{aligned}\sigma_q(\alpha + \beta) &= (\alpha + \beta)^q = \alpha^q + \beta^q = \sigma_q(\alpha) + \sigma_q(\beta) \\ \sigma_q(\alpha\beta) &= (\alpha\beta)^q = \alpha^q \beta^q = \sigma_q(\alpha) \sigma_q(\beta)\end{aligned}$$

Now  $\sigma_q$  is nontrivial since  $\sigma_q$  send 1 to 1, so  $\sigma_q$  is 1-1 and hence an isomorphism since  $\mathbb{F}_q$  is finite. Also, for all  $\alpha \in \mathbb{F}_q$ ,  $\sigma_q(\alpha) = \alpha^q = \alpha$ , hence  $\sigma_q$  fixes  $\mathbb{F}_q$ .

Finally we prove that the order of  $\sigma_q$  is  $n$ . Assume not, so  $\text{ord}(\sigma_q) = m < n$ . Then  $\sigma_q^m = \text{Id} \implies x^{q^m} - x = 0$  for each  $x \in \mathbb{F}_{q^n}$ . But  $x^{q^m} - x = 0$  has at most  $q^m < q^n$  roots, which leads to a contradiction.  $\square$

**Remark 27.** By theorem 10, the multiplication group of  $\mathbb{F}_{q^n}$  is cyclic, so  $\mathbb{F}_{q^n}^\times = \langle \alpha \rangle \subseteq \mathbb{F}_q(\alpha) \setminus \{0\} \subseteq \mathbb{F}_{q^n} \setminus \{0\}$ , hence  $\mathbb{F}_{q^n} = \mathbb{F}_q(\alpha)$ .

**Lemma 5.** Every irreducible polynomial  $f(x)$  in  $\mathbb{F}_{p^n}[x]$  is separable.

*Proof.* Without loss of generality, assume  $f(x)$  is monic.

Since  $\sigma_p$  is an isomorphism,  $\mathbb{F}_{p^n} = \mathbb{F}_{p^n}^p = \{\alpha^p \mid \alpha \in \mathbb{F}_{p^n}\}$ . Now assume  $f(x)$  has a multiple root  $\alpha$ , then  $m_{\alpha, \mathbb{F}_p} = f(x)$  since  $f$  is irreducible. By theorem 44 we also have  $f(x) = m_{\alpha, \mathbb{F}_p} \mid f'(x)$ , but  $\deg f'(x) < \deg f(x)$  so we must have  $f'(x) \equiv 0$ .

Write  $f(x) = a_n x^n + \dots + a_1 x + a_0$ , then  $f'(x) \equiv 0$  implies  $ka_k = 0_{\mathbb{F}_p}$  for each  $k$ , which means that if  $a_k \neq 0 \implies p \mid k$ . So

$$f(x) = a_{mp} x^{mp} + a_{(m-1)p} x^{(m-1)p} + \dots + a_p x^p + a_0 = (a_{mp} x^m + \dots + a_p x + a_0)^p.$$

But this implies  $f(x)$  is reducible, which is a contradiction.  $\square$

**Theorem 47.**  $x^{p^n} - x$  equals the product of all monic irreducible polynomials in  $\mathbb{F}_p[x]$  of degree  $d$  where  $d$  runs through all divisors of  $n$ . i.e.

*Proof.* By lemma, each irreducible polynomial is separable, and if  $f(x), g(x) \in \text{RHS}$ , and  $f(\alpha) = g(\alpha) = 0$ , then  $f = m_{\alpha, \mathbb{F}_p} = g$ . Thus RHS is separable. LHS is separable since  $f' = 1$ , so we could prove the equality by checking that they have same roots.

LHS  $\mid$  RHS:  $\forall \alpha \in \mathbb{F}_{p^n}$ ,  $[\mathbb{F}_p(\alpha) : \mathbb{F}_p] \mid [\mathbb{F}_{p^n} : \mathbb{F}_p] = n$ , thus  $\deg m_{\alpha, \mathbb{F}_p} \mid n$  and hence  $m_{\alpha, \mathbb{F}_p}$  appears in RHS.

RHS  $\mid$  LHS: Assume  $\deg m_{\alpha, \mathbb{F}_p} = d \mid n$ , then  $[\mathbb{F}_p(\alpha) : \mathbb{F}_p] = d$ , so  $\alpha^{p^d} = \alpha$ , and hence  $\alpha = \alpha^{p^d} = \alpha^{p^{2d}} = \dots = \alpha^{p^n}$ .  $\square$

**Def 82.** The Möbius  $\mu$ -function is defined as

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n \text{ has a square factor} \\ (-1)^r, & \text{if } n \text{ is a product of } r \text{ distinct primes} \end{cases}$$

**Theorem 48** (Möbius inversion formula). If  $f(n) = \sum_{d \mid n} g(d)$ , then  $g(n) = \sum_{d \mid n} \mu(d) f\left(\frac{n}{d}\right)$ .

**Remark 28.** Let  $\psi_q(d)$  denote the number of monic irreducible polynomials of degree  $d$  in  $\mathbb{F}_q$ , then  $q^n = \sum_{d \mid n} d \psi_q(d)$ .

Using the convolution notation, we have  $(n \mapsto q^n) = \mathbb{1} * (n \mapsto n \psi_q(n))$ . Where  $\mathbb{1} \triangleq (n \mapsto 1)$ . It could be seen that  $\mathbb{1}^{-1} = \mu$ . Thus  $n \psi_q(n) = \sum_{d \mid n} \mu(d) q^{n/d}$ .

### 5.3 Algebra closure

**Def 83.**

- $L$  is called an **algebraic closure** of  $K$  if  $L/K$  is algebraic and each polynomial  $f(x) \in K[x]$  splits over  $L$ .
- $L$  is said to be **algebraically closed** if for each  $f(x) \in L[x]$ ,  $f(x)$  has a root in  $L$ .

**Prop 5.3.1.** Given  $L/K$ , if  $L$  is algebraically closed, then  $L^{\text{alg}} \triangleq \{ \alpha \in L \mid \alpha \text{ is algebraic over } K \}$  is an algebraic closure of  $K$ .

*Proof.* By prop 5.1.4,  $L^{\text{alg}}$  is a field, and by definition,  $L^{\text{alg}}/K$  is algebraic.

Now we show that for any  $f(x) \in K[x]$ ,  $f(x)$  splits over  $L$ . Using induction,  $\deg f = 1$  is trivial. If  $\deg f > 1$ , then since  $f(x) \in K[x] \subseteq L[x]$ ,  $f$  has a root in  $L$ , say  $\alpha$ . so we could write  $f(x) = (x - \alpha)g(x)$ . Then  $g(x) \in K(\alpha)[x] \subseteq L[x]$ . By induction,  $g(x)$  splits and hence  $f(x)$  splits.

So for any  $f(x) \in K[x]$ ,  $f$  splits over  $L$ . Write  $f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$ , then each  $\alpha_i$  is algebraic over  $K \implies \alpha_i \in L^{\text{alg}}$  and hence  $f(x)$  splits over  $L^{\text{alg}}[x]$ .  $\square$

**Coro 5.3.1.** If  $K$  is algebraically closed, then  $K$  is an algebraic closure of  $K$  itself.

**Prop 5.3.2.** If  $L$  is an algebraic closure of  $K$ , then  $L$  is algebraically closed.

*Proof.* For  $f(x) \in L[x]$ , let  $\alpha$  be a root of  $f(x)$ . Since  $L(\alpha)/L$  and  $L/K$  is algebraic, by prop 5.1.6,  $L(\alpha)/K$  is algebraic. So  $\alpha$  must be in  $L$ , hence  $f(x)$  has a root in  $L$ .  $\square$

**Prop 5.3.3.** The following are equivalent.

1.  $K$  has no nontrivial algebraic extension.
2. For all irreducible polynomial in  $K[x]$  has degree 1.
3. Every polynomial of positive degree in  $K[x]$  has at least one root in  $K$ .
4. Every polynomial of positive degree in  $K[x]$  splits over  $K$ .

In below we would use the Zorn's lemma heavily.

**Lemma 6** (Zorn's lemma). Suppose a partially order set  $P$  has the property that every chain (i.e., a total order subset) has an upper bound in  $P$ , then the set  $P$  contains at least one maximal element.

**Lemma 7.** In a commutative ring  $R$  with 1, any proper ideal  $I \subsetneq R$  is contained in a maximal ideal.

*Proof.* Consider  $S = \{ J \subsetneq R \mid I \subseteq J \} \neq \emptyset$  since  $I \in S$ . Define a partial order on  $S$  by  $J_1 \preceq J_2 \iff J_1 \subseteq J_2$ .

Given a chain  $\{ J_i \mid i \in \Lambda \}$ , let  $J = \bigcup_{i \in \Lambda} J_i$ .  $J$  is an ideal, since if  $x, y \in J$ , then  $x \in J_1, y \in J_2$ . Let  $\tilde{J} = \max(J_1, J_2)$ , then  $x, y \in \tilde{J}$  which implies  $x + y \in \tilde{J}$ , and it is easy to check that for any  $x \in R, y \in J$ ,  $xy \in J$ .

Also,  $J$  is proper since  $1 \notin J$ , or else  $1 \in J_i$  and thus  $J_i = R$  which leads to a contradiction.

By Zorn's lemma, there exists a maximal element in  $S$ , and thus it is a maximal ideal which contains  $I$ .  $\square$



**Theorem 49.** If  $K$  is a field, then there exists an algebraic closure  $L$  of  $K$ .

*Proof.* Let  $S = \{x_f \mid f(x) \in K[x] \text{ with } \deg f \geq 1\}$  be the set of variables indexed by non-constant polynomial in  $K[x]$ . Consider the polynomial ring  $K[S]$  and  $I = \langle f(x_f) : f \in K[x] \text{ with } \deg f \geq 1 \rangle$ , which is an ideal in  $K[S]$ .

We claim that  $I \neq K[S]$ . If not, then  $1 \in I \implies 1 = \sum_{i=1}^n g_i f_i(x_{f_i})$ . Write  $x_i \triangleq x_{f_i}$  for  $i = 1, 2, \dots, n$ . Also, by definition  $g_i$  only involves a finite number of variable in  $S$ , so we could set  $g_i \in K[x_1, x_2, \dots, x_m]$  with  $m \geq n$ . That is,  $1 = \sum_{i=1}^n g_i(x_1, x_2, \dots, x_m) f_i(x_i)$ . Let  $\Sigma$  be a splitting field for  $f(x) = f_1(x) f_2(x) \cdots f_n(x)$  and define  $\alpha_i \in \Sigma$  which satisfies  $f_i(\alpha_i) = 0$  and  $a_i = 0$  for  $n+1 \leq i \leq m$ . Then  $1 = \sum_{i=1}^n g(\alpha_1, \alpha_2, \dots, \alpha_m) f_i(\alpha_i) = 0$ , which leads to a contradiction.

By lemma 7, there exists a maximal ideal  $M$  s.t.  $I \subseteq M$ .

Consider  $K \hookrightarrow F_1 \triangleq K[S]/M$ , and then for all  $f \in K[x]$ ,  $f(\bar{x}_f) = \bar{0}$  in  $F_1$ . By induction,  $\exists F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$  which satisfies  $f(x) \in F_n[x]$  has a root in  $F_{n+1}$ . Let  $F = \bigcup_{i=1}^{\infty} F_i$  which is algebraically closed since if  $f(x) \in F[x]$  then  $f(x) \in F_m[x]$  for some  $m$  and thus  $f(x)$  has a root in  $F_{m+1} \subseteq F$ .

Finally  $L \triangleq \{\alpha \in F \mid \alpha \text{ is algebraic over } K\}$  is an algebraic closure of  $K$ .  $\square$

**Lemma 8.** If  $L_1/K$  is algebraic and  $\tau : K \rightarrow L_2$  is a non-zero homomorphism with  $L_2$  being algebraically closed, then  $\tau$  could be extend to  $\sigma : L_1 \rightarrow L_2$ .

*Proof.* Consider  $S = \{(M, \theta) \mid K \subset M \subset L_1, \theta : M \rightarrow L_2 \text{ with } \theta|_K = \tau\}$ , which is not an empty set since  $(K, \tau) \in S$ .

Define a partial order on  $S$  by  $(M_1, \theta_1) \preceq (M_2, \theta_2) \iff M_1 \subseteq M_2 \wedge \theta_2|_{M_1} = \theta_1$ . Given any chain  $\{(M_i, \theta_i) : i \in \Lambda\}$ , let  $N = \bigcup_{i=1}^{\infty} M_i$  and  $\theta = \alpha \mapsto \theta_i(\alpha)$  if  $\alpha \in M_i$ . It could be check easily that this map is well defined, and  $(N, \theta)$  is a least upper bound in  $S$  for this chain. By Zorn's lemma, there exists a max element  $(M, \sigma)$  in  $S$ .

Now, if  $M \neq L_1$ , then pick  $\alpha \in L_1 \setminus M$ . Since  $L_1/K$  is algebraic, the minimal polynomial  $m_{\alpha, K}$  exists. Since  $L_2$  algebraically closed,  $\sigma(m_{\alpha, K})$  has a root in  $L_2$ , and thus by lemma 4,  $\sigma$  could be extend to  $\sigma' : M(\alpha) \rightarrow L_2$  which contradicts the maximality of  $(M, \sigma)$ . Thus  $M = L_1$ .  $\square$

**Theorem 50.** Any two algebraic closures  $L_1, L_2$  of  $K$  are isomorphic.

*Proof.* Consider the inclusion map  $\text{Id}_K : K \hookrightarrow L_1$ . By Lemma 8,  $\text{Id}_K$  could be extend to  $\sigma : L_2 \rightarrow L_1$  such that  $\sigma|_K = \text{Id}_K$ . Since  $\sigma \neq 0$ ,  $\sigma(L_2) \cong L_2$ . Also,  $L_2$  is algebraically closed implies  $\sigma(L_2)$  is algebraically closed. So for any  $\alpha \in L_1$ ,  $\alpha$  is algebraic over  $K$  and thus over  $\sigma(L_2)$ , which implies  $\alpha \in \sigma(L_2)$ , so  $\sigma$  is onto, hence  $\sigma$  is an isomorphism between  $L_1$  and  $L_2$ .  $\square$

**Eg 5.3.1.** Let  $p$  be a prime.

- Any finite field  $L$  with  $\text{char } L = p$ ,  $L \cong \mathbb{F}_{p^n}$  for some  $n \in \mathbb{N}$ .
- $\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \langle \sigma_p \rangle$  with  $p = \alpha \mapsto \alpha^p \mapsto \alpha^{p^2} \mapsto \dots$ .
- A subfield  $L$  of  $\mathbb{F}_{p^n}$  is isomorphic to  $\mathbb{F}_{p^m}$  with  $m \mid n$  since  $[\mathbb{F}_{p^n} : \mathbb{F}_{p^m}] = d \rightsquigarrow p^{md} = p^n$ .
- $\bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}$  is a field, and it is the algebraic closure of  $\mathbb{F}_p$ .

## 5.4 Separable extension

**Def 84.**

- $\alpha$  is separable over  $K$  if  $m_{\alpha, K}$  is separable over  $K$ .

- $L/K$  is called a **separable extension** if  $\forall \alpha \in L$ ,  $\alpha$  is separable over  $K$ .

**Eg 5.4.1.** Let  $\text{char } K = p$  and  $K^p \subsetneq K$ . Pick  $b \in K \setminus K^p$  and consider  $L$  to be the splitting field of  $x^p - b$  over  $K$ , say  $\alpha \in L$  with  $\alpha^p = b$ . Notice that  $x^p - b = x^p - a^p = (x - a)^p$ , and  $x^p - b$  is irreducible in  $K$ , or else if  $x^p - b = g(x)h(x)$  in  $K[x]$ , then write  $g(x) = (x - \alpha)^k$ ,  $h(x) = (x - \alpha)^{n-k}$ , but then expand  $g(x)$  and we would get  $\alpha^k \in K$ , since  $\alpha^p \in K$  and  $\gcd(k, p) = 1$  implies  $\alpha \in K$  which leads to a contradiction.

By above we know that  $x^p - b$  is inseparable.

**Def 85.**  $K$  is said to be *perfect* if either  $\text{char } K = 0$  or “ $\text{char } K = p$  and  $K = K^p$ ”.

**Eg 5.4.2.** If  $\text{char } K = p$  and  $K/\mathbb{F}_p$  is algebraic, then  $K$  is perfect.

*Proof.* Consider  $\sigma_p : K \rightarrow K$   
 $\alpha \mapsto \alpha^p$ , which is a monomorphism which fixes  $\mathbb{F}_p$ . Since  $K/\mathbb{F}_p$  is algebraic, by the exercise problem,  $\sigma_p$  is an automorphism, so  $K = K^p$ .  $\square$

**Fact 5.4.1.**  $K$  is perfect if and only if for any irreducible polynomial  $f(x) \in K[x]$ ,  $f$  is separable. Also, we can find that an irreducible polynomial  $f(x) \in K[x]$  is not separable over  $K$  if and only if  $\text{char } K = p > 0$  and  $f(x) = g(x^p)$  for some  $g(x) \in K[x]$ , where  $g(x)$  is irreducible and not all coefficients of  $g$  is in  $K^p$ .

Finally, if  $\text{char } K = 0$ , then  $K$  is separable.

**Prop 5.4.1.** Give  $K(\alpha)/K$  with degree  $m_{\alpha, K} = d$  and  $\tau : K \rightarrow L \neq 0$ . If  $\alpha$  is separable over  $K$  and  $\bar{\tau}(m_{\alpha, K})$  splits over  $L$ , then there are exactly  $d$  monomorphisms  $\sigma : K(\alpha) \rightarrow L$  with  $\sigma|_K = \tau$ . Otherwise, if  $\alpha$  is not separable or  $\bar{\tau}(m_{\alpha, K})$  doesn't split over  $L$ , then there are  $r < d$  such monomorphisms.

*Proof.* Observe that  $m_{\alpha, K}$  is separable over  $K$  if and only if  $\bar{\tau}(m_{\alpha, K})$  is separable over  $\tau(K)$ . Extend  $K$  to  $\Sigma$ ,  $\tau(K)$  to  $\Sigma'$ , where  $\Sigma, \Sigma'$  are the splitting field of  $m_{\alpha, K}$  and  $\bar{\tau}(m_{\alpha, K})$ , respectively. Since  $K \cong \tau(K)$ , by theorem 43,  $\Sigma \cong \Sigma'$ . Let  $\tau'$  be the isomorphism which is an extension of  $\tau$ .

If  $m_{\alpha, K} = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_d)$ , then  $\bar{\tau}(m_{\alpha, K}) = (x - \tau'(\alpha_1))(x - \tau'(\alpha_2)) \cdots (x - \tau'(\alpha_n))$ , where  $\tau' : \Sigma \xrightarrow{\sim} \Sigma'$  and  $\alpha_i \neq \alpha_j \iff \tau'(\alpha_i) \neq \tau'(\alpha_j)$ . Thus if  $\alpha$  is separable,  $\bar{\tau}(m_{\alpha, K})$  has  $d$  distinct roots in  $L$ . By corollary 5.1.2, there are exactly  $d$  monomorphisms  $\sigma$  with  $\sigma|_K = \tau$ .

Otherwise, there are  $r$  roots in  $L$  where  $r < d$ , and thus there are  $r < d$  such monomorphisms.  $\square$

**Prop 5.4.2.** Let  $[K' : K] = d$  and  $\tau : K \rightarrow L \neq 0$ . Then  $K'/K$  is separable and  $\forall \alpha \in K'$ ,  $\bar{\tau}(m_{\alpha, K})$  splits over  $L$ , if and only if there are exactly  $d$  monomorphisms  $\sigma : K' \rightarrow L$  with  $\sigma|_K = \tau$ . Otherwise  $\exists r < d$  of such monomorphisms.

*Proof.* By induction on  $d$ , if  $d = 1$  we could simply let  $\sigma = \tau$ .

For  $d > 1$ , consider  $\alpha \in K' \setminus K$ . By prop 5.4.1, there exists exactly  $[K(\alpha) : K]$  monomorphisms  $\tau_1 : K(\alpha) \rightarrow L$ .

Now, for any  $\beta \in K'/K(\alpha)$ ,  $m_{\beta, K(\alpha)} \mid m_{\beta, K}$  and thus  $m_{\beta, K(\alpha)}$  is separable and  $\bar{\tau}_1(m_{\beta, K(\alpha)})$  splits over  $L$  since  $\bar{\tau}(m_{\beta, K})$  splits. These imply that  $K'/K(\alpha)$  is separable and  $\forall \beta \in K'$ ,  $m_{\beta, K(\alpha)}$  splits over  $L$ . Thus,  $K(\alpha)$  satisfies the hypothesis, and by induction, there are exactly  $[K' : K(\alpha)]$  monomorphisms  $\sigma : K' \rightarrow L$  such that  $\sigma|_{K(\alpha)} = \tau_1$ , thus there are  $[K' : K(\alpha)][K(\alpha) : K] = [K' : K]$  such monomorphisms.

Otherwise, we could choose  $\alpha \in K'$  such that  $\bar{\tau}(m_{\alpha,K})$  has fewer than  $[K(\alpha) : K]$  roots in  $L$ , then there are  $r' < [K(\alpha) : K]$  monomorphism  $\tau_1 :: K(\alpha) \rightarrow L$ . By induction, each  $\tau_1$  has  $r''$  extensions  $\sigma :: K' \rightarrow L$  and  $r'' \leq [K' : K(\alpha)]$ . Hence the number of monomorphism equals  $r'r'' < [K' : K]$ .  $\square$

**Lemma 9.** If  $K(\alpha_1, \alpha_2, \dots, \alpha_n)/K$  is algebraic and  $L$  is a splitting field of  $f(x) = \prod_{i=1}^n m_{\alpha_i, K}$  over  $K$ , then for all  $\beta \in K(\alpha_1, \alpha_2, \dots, \alpha_n)$ ,  $m_{\beta, K}$  also splits over  $L$ .

*Proof.* Let  $L = K(R)$  with  $R$  being the set of all roots of  $f(x)$ . Pick any root  $\gamma$  of  $m_{\beta, K}$ . Observe the following diagram:

$$\begin{array}{ccc} K(R) & \xrightarrow[\text{(2) } \sigma]{\sim} & K(R, \gamma) \\ \uparrow & & \uparrow \\ K(\beta) & \xrightarrow[\text{(1) } \tau]{\sim} & K(\gamma) \\ & \nwarrow \quad \nearrow & \\ & K & \end{array}$$

Where (1) holds because these fields are both isomorphic to  $K[x]/\langle m_{\beta, K} \rangle$ .

(2) holds because  $\tau$  obviously fixes  $K$ , and hence  $K(R)$  is a splitting field of  $f$  and  $K(R, \gamma)$  is a splitting field of  $\bar{\tau}(f)$ . By theorem 43,  $K(R)$  and  $K(R, \gamma)$  are isomorphic.

Thus we have  $[K(R) : K] = [K(R, \gamma) : K]$  along with  $[K(R, \gamma) : K] = [K(\gamma, R) : K(R)][K(R) : K]$ . This implies  $[K(\gamma, R) : K(R)] = 1$ , hence  $\gamma \in R$ .  $\square$

**Theorem 51.** Given  $K(\alpha_1, \alpha_2, \dots, \alpha_n)/K$ , if  $\alpha_i$  is separable over  $K_{i-1} \triangleq K(\alpha_1, \dots, \alpha_{i-1})$ , then  $K(\alpha_1, \alpha_2, \dots, \alpha_n)/K$  is separable.

*Proof.* Let  $L$  be a splitting field of  $f(x) = \prod m_{\alpha_i, K}$ .

We claim that there are  $[K_j : K]$  monomorphisms  $\tau_j :: K_j \rightarrow L$  with  $\tau_j|_K = \text{Id}_K$ . Use induction on  $j$ , if  $j = 0$ , then there are only 1 such monomorphism, namely itself  $\text{Id}_K$ .

For  $j > 0$ , observe that  $m_{\alpha_j, K_{i-1}} \mid m_{\alpha_j, K}$ , and since  $\bar{\tau}_{j-1}(m_{\alpha_j, K}) = m_{\alpha_j, K}$  splits over  $L$ ,  $m_{\alpha_j, K_{i-1}}$  also splits over  $L$ . By hypothesis,  $\alpha_j$  is separable over  $K_{j-1}$ , so by prop 5.4.1, there are  $[K_j : K_{j-1}]$  such monomorphisms  $\tau_j :: K_j \rightarrow L$  with  $\tau_j|_{K_{j-1}} = \tau_{j-1}$ . By induction, there are  $[K_{j-1} : K]$  monomorphisms  $\tau_{j-1} :: K_{j-1} \rightarrow L$  with  $\tau_{j-1}|_K = \text{Id}_K$ . Compose these monomorphisms, we know that there exist exactly  $[K_j : K_{j-1}][K_{j-1} : K] = [K_j : K]$  monomorphisms  $\tau_j :: K_j \rightarrow L$  such that  $\tau_j|_K = \text{Id}_K$ .

So there are exactly  $[K_n : K]$  monomorphisms  $\tau :: K(\alpha_1, \dots, \alpha_n) \rightarrow L$  with  $\tau|_K = \text{Id}_K$ . By prop 5.4.2,  $K(\alpha_1, \dots, \alpha_n)$  is separable.  $\square$

**Theorem 52.**  $L/K$  is separable if and only if  $L/M$ ,  $M/K$  are separable.

*Proof.* “ $\Rightarrow$ ”: If  $L/K$  is separable, then  $M/K$  is obviously separable. For any  $\beta \in L$ ,  $m_{\beta, M} \mid m_{\beta, K}$  so  $m_{\beta, M}$  is separable which implies  $L/M$  is separable.

“ $\Leftarrow$ ”: For any  $\alpha \in L$ , write  $m_{\alpha, M} = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ . Then  $m_{\alpha, M}$  is separable implies  $\alpha$  is separable over  $K(a_0, \dots, a_{n-1})$ . Note that  $a_0, \dots, a_{n-1} \in M$  are separable over  $K$ . By theorem 51,  $K(a_0, a_1, \dots, a_{n-1}, \alpha)/K$  is separable, hence each  $\alpha$  is separable over  $K$ , thus  $L/K$  is separable.  $\square$

**Theorem 53** (Primitive element theorem).

- A finite extension is simple if and only if there are only finitely many intermediate fields.
- If  $L/K$  is finite and separable, then  $L/K$  is simple.

## 5.5 Normal extension

**Def 86.**  $L/K$  is called a **normal extension** if  $\forall \alpha \in L$ ,  $m_{\alpha,K}$  splits over  $L$ .

**Theorem 54.**  $L$  is a splitting field of some polynomial  $f(x)$  over  $K$  if and only if  $L/K$  is finite and normal.

*Proof.* “ $\Rightarrow$ ”: Let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be the roots of  $f$ , so  $L = K(\alpha_1, \alpha_2, \dots, \alpha_n)$ , and  $L$  is also a splitting field of  $\prod m_{\alpha_i,K}$  since  $m_{\alpha_i,K} \mid f$ . By lemma 9, for any  $\beta$  in  $L$ ,  $m_{\beta,K}$  splits, thus  $L/K$  is normal and also finite obviously.

“ $\Leftarrow$ ”: Since  $L/K$  is a finite extension, we could write  $L = K(\alpha_1, \alpha_2, \dots, \alpha_n)$ . Let  $f = \prod m_{\alpha_i,K}$ , then since  $L/K$  normal, each  $m_{\alpha_i,K}$  splits. It is also easy to see that  $L$  is the smallest field where  $f$  splits, thus  $L$  is a splitting field of  $f$ .  $\square$

**Remark 29.** If  $L/K$  is normal, then for any  $M$  with  $K \subset M \subset L$ , we have  $L/M$  is normal, this is because  $\forall \alpha$ ,  $m_{\alpha,M} \mid m_{\alpha,K}$ , and thus  $m_{\alpha,M}$  splits since  $m_{\alpha,K}$  splits.

But  $M/K$  need not to be normal. For example, Let  $K = \mathbb{Q}$ ,  $L$  be the splitting field of  $x^3 - 2$ , by theorem 54  $L/K$  is normal. Then  $L = \mathbb{Q}(\sqrt[3]{2}, \omega)$  where  $\omega \triangleq e^{2\pi i/3}$ . Let  $M = \mathbb{Q}(\sqrt[3]{2})$  then  $m_{\sqrt[3]{2},K}$  doesn't split in  $M$ , so  $M/K$  is not normal.

**Prop 5.5.1.** Let  $L/K$  be a finite, normal extension and  $L \supset M \supset K$ , then the following are equivalent.

- (a)  $M/K$  is normal.
- (b)  $\forall \sigma \in \text{Aut}(L/K)$ ,  $\sigma(M) \subset M$ .
- (c)  $\forall \sigma \in \text{Aut}(L/K)$ ,  $\sigma(M) = M$ .

*Proof.* (a)  $\Rightarrow$  (b):  $\forall \alpha \in M$ ,  $m_{\alpha,K}(\sigma(\alpha)) = \sigma(m_{\alpha,K}(\alpha)) = 0$ . So  $\sigma(\alpha)$  is a root of  $m_{\alpha,K}$ . Since  $M/K$  normal,  $m_{\alpha,K}$  splits in  $M$  and thus each root of  $m_{\alpha,K}$  is in  $M$ , hence  $\forall m$ ,  $\sigma(m) \in M \Rightarrow \sigma(M) \subset M$ .

(b)  $\Rightarrow$  (c): Since  $L/K$  is algebraic and  $\sigma$  is 1-1, by a homework problem,  $\sigma$  onto.

(c)  $\Rightarrow$  (a): For any  $\alpha \in M$ , let  $\beta \in L$  be a root of  $m_{\alpha,K}$ . By theorem 54, we could assume  $L$  is a splitting field of  $f$  over  $K$ . Consider the following diagram,

$$\begin{array}{ccc}
 L & \xrightarrow[\sigma]{\sim} & L \\
 \uparrow & & \uparrow \\
 K(\alpha) & \xrightarrow[\tau]{\sim} & K(\beta) \\
 & \nwarrow \quad \nearrow & \\
 & K &
 \end{array}$$

Where isomorphism  $\tau$  with  $\tau(\alpha) = \beta$  exists since  $\alpha, \beta$  share the same minimal polynomial, and  $\sigma$  with  $\sigma|_K = \tau$  exists by theorem 43. Since  $\sigma \in \text{Aut}(L/K)$ ,  $\beta = \sigma(\alpha) \in M$ , thus  $M/K$  normal.  $\square$

**Def 87.** Let  $L/K$  is called a *Galois extension* if  $L/K$  is finite, normal and separable. That is,  $L$  is a splitting field of some separable polynomial over  $K$ .

**Theorem 55.** If  $L/K$  is Galois, then  $|\text{Aut}(L/K)| = [L : K]$ . Otherwise,  $|\text{Aut}(L/K)| < [L : K]$ .

*Proof.* Since  $L/K$  is normal, for any  $\alpha$ ,  $m_{\alpha,K}$  splits over  $L$ . Since  $L/K$  is separable,  $m_{\alpha,K}$  has no multiple roots. So there are exactly  $[L : K]$  extensions  $\sigma :: L \rightarrow L$  of  $\text{Id}_K$ .  $\square$

**Def 88.** Given a field  $L$ , define the **fixed field** of  $G$  by  $L^G \triangleq \{\alpha \in L \mid \sigma(\alpha) = \alpha, \forall \sigma \in G\}$ .

**Theorem 56.** If  $G$  is a subgroup of  $\text{Aut}(L)$  with  $|G| < \infty$ , then  $|G| = [L : L^G]$ ,  $G = \text{Aut}(L/L^G)$  and  $L/L^G$  is Galois.

*Proof.* First we prove that  $[L : L^G] \leq |G|$  by contradiction. Assume  $|G| < [L : L^G]$ .

Let  $G = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$  and  $\alpha_1, \alpha_2, \dots, \alpha_{n+1} \in L$  with  $\{\alpha_i\}$  are linearly independent over  $L^G$ .

Consider the equations

$$\begin{cases} \sigma_1(\alpha_1)x_1 + \dots + \sigma_1(\alpha_{n+1})x_{n+1} = 0 \\ \sigma_2(\alpha_1)x_1 + \dots + \sigma_2(\alpha_{n+1})x_{n+1} = 0 \\ \vdots \\ \sigma_n(\alpha_1)x_1 + \dots + \sigma_n(\alpha_{n+1})x_{n+1} = 0 \end{cases}$$

Since the number of variables is more than the number of equations, there is a non-trivial solution. Choose one solution  $(a_1, \dots, a_{n+1})$  having the least amount of nonzero element. By reordering, we could assume the solution is  $(a_1, a_2, \dots, a_m, 0, 0, \dots, 0)$  and it is no harm to assume  $\sigma_1 = 1_G$ . If  $m = 1$ , then  $\sigma_1(\alpha_1)a_1 = \alpha_1 a_1 = 0 \implies a_1 = 0$ , which is a contradiction.

So assume that  $m > 1$ , we have

$$\begin{cases} \sigma_1(\alpha_1)a_1 + \dots + \sigma_1(\alpha_m)a_m = 0 \\ \sigma_2(\alpha_1)a_1 + \dots + \sigma_2(\alpha_m)a_m = 0 \\ \vdots \\ \sigma_n(\alpha_1)a_1 + \dots + \sigma_n(\alpha_m)a_m = 0 \end{cases}$$

By multiplying  $a_m^{-1}$ , we could assume  $a_m = 1$ . The equation about  $\sigma_1$  gives  $\alpha_1 a_1 + \dots + \alpha_m a_m = 0$ , since  $\alpha_i$  is linearly independent, one of  $\{\alpha_i\}$ , say  $\alpha_k$  is not in  $L^G$ , and thus there exists  $t$  such that  $\sigma_t(\alpha_k) \neq \alpha_k$ . Apply  $\sigma_t$  to each equation, we have

$$\sigma_t \sigma_i(\alpha_1) \sigma_t(a_1) + \dots + \sigma_t \sigma_i(\alpha_m) \sigma_t(a_m) = 0, \quad \forall 1 \leq i \leq n$$

But since  $\{\sigma_t \sigma_1, \dots, \sigma_t \sigma_n\} = \{\sigma_1, \dots, \sigma_n\}$ ,  $(\sigma_t(a_1), \sigma_t(a_2), \dots, \sigma_t(a_m), 0, \dots, 0)$  is a solution and thus  $(a_1 - \sigma_t(a_1), \dots, a_m - \sigma_t(a_m), 0, \dots)$  is also a solution of the equations. Since  $\sigma_t(\alpha_k) \neq \alpha_k$ , the solution is not trivial, and because  $a_m = 1$ ,  $a_m - \sigma_t(a_m) = 0$ . Hence this solution has  $m-1$  nonzero element, which contradicts the minimality of the original solution. Thus  $[L : L^G] \leq |\text{Aut}(L/L^G)|$ .

Finally,  $|\text{Aut}(L/L^G)| \leq [L : L^G]$  by theorem 51, thus  $|G| \leq |\text{Aut}(L/L^G)| \leq [L : L^G] \leq |G|$ , hence they are all equal.  $\square$

**Def 89.** Let  $f(x) \in K[x]$  and  $L$  be a splitting field of  $f(x)$  over  $K$ . We use  $\text{Gal}(L/K)$  to denote  $\text{Aut}(L/K)$  and call it the **Galois group** of  $f(x)$ .

**Prop 5.5.2.** Let  $f(x) \in \mathbb{Q}[x]$  be irreducible polynomial of degree  $p$  where  $p$  is a prime. If  $f(x)$  has exactly  $p-2$  roots and 2 complex roots, then the Galois group of  $f(x)$  is  $S_p$ .

*Proof.* Let  $L$  be a splitting field of  $f$  over  $\mathbb{Q}$  and  $R = \{\alpha_1, \alpha_2, \dots, \alpha_p\}$  be the set of all roots of  $f(x)$ . Since  $f(x)$  is irreducible,  $f(x)/a_p = m_{\alpha_i, \mathbb{Q}}, \forall i$ . By lemma 4, for any  $\sigma \in \text{Gal}(L/\mathbb{Q})$ ,  $\sigma$  sends  $\alpha_i$  to another root  $\alpha_j$ . Also,  $\{\alpha_i\}$  generates  $L$  so  $G \triangleq \text{Gal}(L/\mathbb{Q}) \leq S_p$ .

Now, we define an equivalence relation on  $R$  such that  $\alpha_i \sim \alpha_j \iff (\alpha_i \alpha_j) \in G$ , that is,  $\exists \sigma \in G$  such that  $\sigma(\alpha_i) = \alpha_j, \sigma(\alpha_j) = \alpha_i$  and  $\sigma(\alpha_t) = \alpha_t, \forall t \neq i, j$ .

We claim that each equivalence class has the same size. Let  $[\alpha_i], [\alpha_j]$  be two equivalence classes. Since  $\alpha_i, \alpha_j$  share the same minimal polynomial, by lemma 4,  $\exists \sigma, \sigma(\alpha_i) = \alpha_j$ , and  $\sigma$  sends  $[\alpha_i]$

to  $[\alpha_j]$ , since if  $\alpha_k \in [\alpha_i]$ ,  $(\alpha_i \alpha_k) \in G$  and thus  $\sigma(\alpha_i \alpha_k)\sigma^{-1} = (\alpha_j \sigma(\alpha_k)) \in G$ . Since  $\sigma$  is 1-1,  $|\alpha_i| \leq |[\alpha_j]|$ , and by symmetry we have  $|\alpha_i| = |[\alpha_j]|$ .

But then if  $[\alpha_i] = n$ ,  $p = |R| = \sum |[\alpha_j]| = kn$ , so either there are  $p$  equivalence classes with size of 1, which is impossible since the two complex root are equivalent by conjugation, or there are one equivalence class, which means that every 2 cycle is in  $G$ , and thus  $G = S_p$ .  $\square$

## 5.6 Fundamental theorem of Galois theory

**Theorem 57** (Main theorem). Let  $L/K$  be a Galois extension, where  $L$  be a splitting field of a separable polynomial  $f$ , and let  $G = \text{Gal}(L/K)$ . Then:

- (1) There is a 1-1 correspondence from the set of intermediate field to the set of subgroup:

$$\begin{array}{ccc} \{M : K \subseteq M \subseteq L\} & \longleftrightarrow & \{H : H \leq G\} \\ M & \longmapsto & \text{Gal}(L/M) \\ L^H & \longleftarrow & H \end{array}$$

*Proof.* We check these two mappings are the inverse of each other.

By theorem 56,  $\text{Gal}(L/L^H) = H$ .

Now we have  $M \subseteq L^{\text{Gal}(L/M)}$ . Since  $L/M$  is galois,  $[L : M] = |\text{Gal}(L/M)|$ . By theorem 56 again,  $|\text{Gal}(L/M)| = [L : L^{\text{Gal}(L/M)}]$ , thus  $[L : M] = [L : L^{\text{Gal}(L/M)}] \implies M = L^{\text{Gal}(L/M)}$ .  $\square$

- (2) If  $M_1 = L^{H_1}, M_2 = L^{H_2}$ , then  $M_1 \subseteq M_2 \iff H_2 \leq H_1$ .

*Proof.* Obvious.  $\square$

- (3) If  $M = L^H$ , then  $M/K$  is normal if and only if  $H \triangleleft G$ .

*Proof.* For any  $\sigma \in G$ ,

$$\begin{aligned} \tau \in \text{Gal}(L/\sigma(M)) &\iff \tau(\sigma(x)) = \sigma(x), \forall x \in M \\ &\iff \sigma^{-1}\tau\sigma(x) = x, \forall x \in M \\ &\iff \sigma^{-1}\tau\sigma \in \text{Gal}(L/M) \\ &\iff \tau \in \sigma \text{Gal}(L/M)\sigma^{-1} \end{aligned}$$

By prop 5.5.1,  $M/K$  is normal if and only if for all  $\sigma \in G$ ,  $\sigma(M) = M \iff \text{Gal}(L/M) = \text{Gal}(L/\sigma(M))$ . By the discussion above,  $\text{Gal}(L/\sigma(M)) = \sigma \text{Gal}(L/M)\sigma^{-1} = \sigma H \sigma^{-1}$ . Hence  $M/K$  is normal  $\iff H = \sigma H \sigma^{-1}, \forall \sigma \in G \iff H \triangleleft G$ .  $\square$

- (4) If  $H \triangleleft G$ , then  $G/H \cong \text{Gal}(M/K)$ .

*Proof.* Since  $H \triangleleft G$ , by (3) we know that  $M/K$  is Galois. Define  $\varphi = \sigma \mapsto \sigma|_M :: \text{Gal}(L/K) \mapsto \text{Gal}(M/K)$ . The mapping is well defined since  $\sigma(M) = M$  (by prop 5.5.1). Also, this map is onto since by corollary 43, each  $\tau \in \text{Gal}(M/K)$  could be extended to  $\sigma \in \text{Gal}(L/K)$  because  $\bar{\tau}(f) = f$ . Finally, notice that  $\ker \varphi = H$ , thus by the first isomorphism theorem,  $G/H \cong \text{Gal}(M/K)$ .  $\square$

- (5) If  $M_1 = L^{H_1}, M_2 = L^{H_2}$ , then  $M_1 \cap M_2 = L^{\langle H_1, H_2 \rangle}$  and  $M_1 M_2 = L^{H_1 \cap H_2}$ .

**Theorem 58.** Let  $L/K$  be Galois, and  $N/K$  be any extension, then  $LN/N$  is Galois and  $\text{Gal}(LN/N) \cong \text{Gal}(L/L \cap N)$  by the isomorphism  $\varphi : \sigma \mapsto \sigma|_L$ .

*Proof.* Let  $L$  be a splitting field of the separable polynomial  $f(x)$  over  $K$ , say  $L = K(\alpha_1, \dots, \alpha_n)$ . Then  $LN = N(\alpha_1, \dots, \alpha_n)$ , which can be regarded as a splitting field of  $f(x)$  over  $N$ . Thus by theorem 54,  $LN/N$  is Galois.

Now we check that  $\varphi$  is well defined, notice that  $f(\sigma(\alpha_i)) = \sigma(f(\alpha_i)) = 0$  since  $\sigma$  fixes  $K$ , and thus  $f$  sends  $\alpha_i$  to some  $\alpha_j$ . Also,  $\{\alpha_i\}$  generate  $L$  over  $K$ , thus  $\sigma|_L(L) = L$ .

If  $\sigma|_L = \text{Id}_L$ , then  $\sigma(\alpha_i) = \alpha_i, \forall i$ . Since  $\{\alpha_i\}$  generate  $LN$  over  $N$ ,  $\sigma = \text{Id}_{LN}$ . Thus  $\varphi$  is 1-1.

Finally, let  $H = \text{Im } \varphi$ , we claim that  $L^H = L \cap N$ , since

$$\begin{aligned} \alpha \in L^H &\iff \alpha \in L \text{ and } \forall \sigma \in \text{Gal}(LN/N), \sigma|_L(\alpha) = \alpha \\ &\iff \alpha \in L \text{ and } \forall \sigma \in \text{Gal}(LN/N), \sigma(\alpha) = \alpha \\ &\iff \alpha \in L \text{ and } \alpha \in (LN)^{\text{Gal}(LN/N)} \\ &\iff \alpha \in L \text{ and } \alpha \in N \iff \alpha \in L \cap N \end{aligned}$$

□

**Remark 30.** If  $L/K$  is Galois and  $N/K$  is finite, then  $[LN : K] = [L : K][N : K]/[L \cap N : K]$ .

*Proof.*

$$[LN : K]/[N : K] = [LN : N] = \text{Gal}(LN/N) = \text{Gal}(L/L \cap N) = [L : L \cap N] = [L : K]/[L \cap N : K]$$

and the proof is completed. □

## 5.7 Abelian extension

**Def 90.**  $L/K$  is called an abelian extension if  $L/K$  is Galois and  $\text{Gal}(L/K)$  is abelian.

**Eg 5.7.1.** For an extension  $\mathbb{F}_{q^n}/\mathbb{F}_q$  of a finite field,  $\mathbb{F}_{q^n}$  is a splitting field of  $x^{q^n} - x$  over  $\mathbb{F}_p$ , so  $\mathbb{F}_{q^n}/\mathbb{F}_q$  is Galois by theorem 54. By theorem 46, we know that  $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) = \langle \sigma_q \rangle$  is a cyclic group.

**Def 91.**

- The cyclotomic field  $\mathbb{Q}(\zeta_n)$  is the splitting field of  $x^n - 1$  over  $\mathbb{Q}$ .
- $\zeta$  is called an  $n$ th root of unity if  $\zeta^n = 1$ .  $\mathcal{U} = \langle \zeta \rangle$  is the multiplicative group of  $n$ th roots of unity.
- $\zeta_n$  is called a primitive  $n$ th root of unity if  $\zeta^n = 1$  but  $\zeta^m \neq 1, \forall 0 < m < n$ .
- The  $n$ th cyclotomic polynomial is defined as

$$\Phi_n \triangleq \prod_{\gcd(k,n)=1} (x - \zeta_n^k) \implies \deg \Phi_n = \varphi(n)$$

**Prop 5.7.1.**

- $x^n - 1 = \prod_{d|n} \Phi_d$ .

*Proof.* First, Both sides have no multiple root. Then since  $\alpha^n = 1 \iff \text{ord}_\times(\alpha) \mid n$ , we know that two sides has equal roots.  $\square$

- $\Phi_n \in \mathbb{Z}[x]$ .

*Proof.* By induction on  $n$ .  $n = 1$  is trivial. Assume that the statement is true for all  $k < n$ , then since

$$x^n - 1 = \Phi_n \prod_{d|n, d < n} \Phi_d \triangleq \Phi_n \Phi_{<n}$$

But notice that  $\Phi_{<n}$  is monic, so by the long division algorithm, it is easy to see that  $\Phi_n = (x^n - 1)/\Phi_{<n}$  has all coefficients in  $\mathbb{Z}$ .  $\square$

- $\Phi_n$  is irreducible.

*Proof.* Suppose  $\Phi_n = f(x)g(x)$  with  $f$  irreducible, and both  $f, g$  are monic. By Gauss's lemma, we could assume  $f(x), g(x) \in \mathbb{Z}[x]$ . Let  $\zeta_n$  be a primitive  $n$ th root of unity which satisfied  $f(\zeta_n) = 0$  and  $p$  be a prime with  $p \nmid n$ .

Assume that  $g(\zeta_n^p) = 0$ ,  $m_{\zeta_n, \mathbb{Q}} = f \implies f \mid g(x^p)$ , say  $g(x^p) = f(x)h(x)$ . By the long division algorithm, we know that  $h(x) \in \mathbb{Z}[x]$ , since  $f(x) \in \mathbb{Z}[x]$  and monic.

In  $\mathbb{Z}/p\mathbb{Z}[x]$ , we have  $\bar{g}(x)^p = \bar{g}(x^p) = \bar{f}(x)\bar{h}(x)$ , which implies  $\bar{g}, \bar{f}$  has common root, thus  $\bar{\Phi}_n = \bar{f}\bar{g}$  and hence  $x^n - \bar{1}$  has a multiple root. But  $(x^n - \bar{1})' = nx^{n-1} \neq 0$ , and 0 is not a root of  $x^n - \bar{1}$ , which leads to a contradiction.

So we conclude that  $f(\zeta_n^p) = 0$  for any  $p \nmid n$ , which could be extended and show that  $f(\zeta_n^k) = 0$  for any  $\gcd(k, n) = 1$ , thus  $f = \Phi_n$ .  $\square$

- $\mathbb{Q}(\zeta_n)/\mathbb{Q}$  is Galois with  $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \deg \Phi_n = \varphi(n)$ .
- $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ .



*Proof.* Let  $\sigma_k = (\zeta_n \mapsto \zeta_n^k) \in \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ . The isomorphism is given by  $\sigma_k \mapsto \bar{k}$ . Clearly, it is a homomorphism since  $\sigma_k \sigma_h = (\zeta_n \mapsto \zeta_n^{kh}) = \sigma_{kh}$ . Also  $\sigma_k = 1 \iff \bar{k} = 1$ . Finally,  $|\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})| = |\mathbb{F}_n^\times| = \varphi(n)$ , so the map is onto.  $\square$

- Suppose  $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$  with  $p_1, \dots, p_k$  are distinct primes. Define  $L_i \triangleq \mathbb{Q}(\zeta_{p_i^{n_i}})$ . Obviously,  $L_i \subseteq \mathbb{Q}(\zeta_n)$  hence  $L_1 L_2 \cdots L_k \subseteq \mathbb{Q}(\zeta_n)$ , but  $\zeta_n = \zeta_{p_1^{n_1}} \zeta_{p_2^{n_2}} \cdots \zeta_{p_k^{n_k}}$ , so  $L_1 L_2 \cdots L_k \supseteq \mathbb{Q}(\zeta_n)$ . Thus we have  $L_1 L_2 \cdots L_k = \mathbb{Q}(\zeta_n)$ .

**Eg 5.7.2.** Let  $n = p$  be a prime.

- $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) = \mathbb{F}_p^\times = \mathbb{Z}/(p-1)\mathbb{Z}$ .
- For  $H \leq \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ , we shall find  $\mathbb{Q}(\zeta_p)^H$ . Let  $\alpha = \sum_{\tau \in H} \tau(\zeta_p)$ , then it is easy to see that  $\alpha \in \mathbb{Q}(\zeta_p)^H$ . Also, since  $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p-1$ ,  $\zeta_p, \zeta_p^2, \dots, \zeta_p^{p-1}$  is linearly independent, so if some  $\sigma \in G$  satisfy  $\sigma(\alpha) = \alpha$ , then since both  $\sigma(\alpha), \alpha$  are a sum of linearly independent elements,  $\sigma$  must send  $\zeta_p$  to an element  $\tau(\zeta_p)$  for some  $\tau \in H$ , then  $\sigma = \tau \implies \sigma \in H$ . Thus  $\mathbb{Q}(\zeta_p)^H = \mathbb{Q}(\alpha)$ .

**Lemma 10.** If  $L_1/K, L_2/K$  are Galois, then  $L_1 \cap L_2/K, L_1 L_2/K$  are Galois and

$$\text{Gal}(L_1 L_2/K) \cong \{(\sigma, \tau) \mid \sigma|_{L_1 \cap L_2} = \tau|_{L_1 \cap L_2}\} \leq \text{Gal}(L_1/K) \times \text{Gal}(L_2/K)$$

In particular, if  $L_1 \cap L_2 = K$ , then  $\text{Gal}(L_1 L_2/K) \cong \text{Gal}(L_1/K) \times \text{Gal}(L_2/K)$ .

*Proof.* We know that  $L_1 \cap L_2/K$  is finite and separable. Also, for each  $\alpha \in L_1 \cap L_2$ ,  $m_{\alpha, K}$  splits in both  $L_1, L_2$  since they are normal, thus  $m_{\alpha, K}$  splits in  $L_1 \cap L_2$ , hence  $L_1 \cap L_2/K$  is galois.

Similary,  $L_1 L_2$  is finite and separable. Let  $L_1$  be the splitting field of  $f_1$ , and  $L_2$  be the splitting field of  $f_2$ , then  $L_1 L_2$  is the splitting field of the square-free part of  $f_1 f_2$ , hence  $L_1 L_2/K$  normal.

Define  $\varphi = \sigma :: \text{Gal}(L_1 L_2/K) \mapsto (\sigma|_{L_1}, \sigma|_{L_2}) :: \text{Gal}(L_1/K) \times \text{Gal}(L_2/K)$ . Since  $L_1, L_2$  are normal, by proposition 5.5.1,  $\sigma|_{L_i}(L_i) = L_i$  so they are well-defined. Also, it is clear that the map is 1-1.

Now we count the number of the pair  $(\sigma|_{L_1}, \sigma|_{L_2})$ . There are  $[L_1 : K]$  of  $\tau = \sigma|_{L_1}$ , and fixing one, each  $\sigma|_{L_2}$  is an extension of  $\tau|_{L_1 \cap L_2}$ , so there are  $[L_2 : L_1 \cap L_2]$  of such. On the other hand, we have  $|\text{Gal}(L_1 L_2/K)| = [L_1 L_2 : K] = [L_1 L_2 : L_1][L_1 : K] = [L_2 : L_1 \cap L_2][L_1 : K]$ , thus  $\text{Gal}(L_1 L_2/K)$  and  $\{(\sigma|_{L_1}, \sigma|_{L_2})\}$  has the same size, and hence the map is onto.  $\square$

Back to our problem,  $[L_1 L_2 \cdots L_k : \mathbb{Q}] = [\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n) = \varphi(p_1^{n_1}) \cdots \varphi(p_k^{n_k}) = [L_1 : \mathbb{Q}][L_2 : \mathbb{Q}] \cdots [L_k : \mathbb{Q}]$ , thus

$$\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \text{Gal}(\mathbb{Q}(\zeta_{p_1^{n_1}})/\mathbb{Q}) \times \text{Gal}(\mathbb{Q}(\zeta_{p_2^{n_2}})/\mathbb{Q}) \times \cdots \times \text{Gal}(\mathbb{Q}(\zeta_{p_k^{n_k}})/\mathbb{Q})$$

**Theorem 59.** Let  $G$  be a finite abelian group. Then there exists a subfield  $L$  of a cyclotomic field satisfying  $\text{Gal}(L/\mathbb{Q}) \cong G$ .

*Proof.* By the FTFGAG,

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$$

By dirichlet theorem, there are infinitely many prime  $p$  such that  $n \mid p-1$ . Let  $p_i$  be a prime such that  $n_i \mid p_i-1$  and all  $p_i$  are distinct. Then  $G$  is a subgroup of  $\mathbb{Z}/(p_1-1)\mathbb{Z} \times \cdots \times \mathbb{Z}/(p_k-1)\mathbb{Z} \cong \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$  where  $n = p_1 p_2 \cdots p_k$ .  $\square$

### 5.7.1 Kummer extension

In this section, we assume that  $\text{char } K \nmid n$  and  $\zeta$  is a primitive  $n$ th root of unity.

**Def 92.**

- $L/K$  is called a kummer extension of exponent  $n$  if  $\zeta \in K$  and  $L$  is a splitting field of  $(x^n - a_1)(x^n - a_2) \cdots (x^n - a_k)$  over  $K$ .
- Let  $|G| < \infty$ , the exponent  $e(G)$  of  $G$  is the least positive integer  $m$  satisfying  $g^m = 1$  for any  $g \in G$ .

**Theorem 60.** Let  $L$  be a splitting field of  $x^n - a$  over  $K$ , then  $\text{Gal}(L/K(\zeta))$  is cyclic of degree dividing  $n$ . More over  $x^n - a$  is irreducible over  $K(\zeta) \iff [L : K(\zeta)] = n$ .

*Proof.* If  $\alpha$  is a root of  $x^n - a$ , then  $\alpha, \alpha\zeta, \dots, \alpha\zeta^{n-1}$  are roots of  $x^n - a$ , so  $L = K(\alpha, \zeta) = K(\zeta)(\alpha)$ .

Consider  $\varphi : \text{Gal}(L/K(\zeta)) \rightarrow \mathbb{Z}/n\mathbb{Z}$   
 $(\alpha \mapsto \alpha\zeta^k) \mapsto \bar{k}$ . It is easy to see that  $\varphi$  is a homomorphism. Also, if  $\varphi(\sigma) = 0$ ,  $\sigma = (\alpha \mapsto \alpha) = \text{Id}$ . Thus  $\varphi$  is 1-1 and  $\text{Gal}(L/K(\zeta)) \hookrightarrow \mathbb{Z}/n\mathbb{Z}$ .  $\square$

**Def 93.**  $L/K$  is called a cyclic extension if  $L/K$  is Galois and  $\text{Gal}(L/K)$  is cyclic.

**Theorem 61.** If  $L/K$  is a cyclic extension of degree  $n$  and  $\zeta \in K$ , then  $L$  is a splitting field of some irreducible polynomial  $x^n - a$  over  $K$ .

*Proof.* Recall a result proved in HW problem: Distinct automorphisms of  $L$  are linearly independent over  $L$ .

Let  $\text{Gal}(L/K) = \langle \sigma \rangle$  with  $\text{ord}(\sigma) = n$ . Then  $\text{Id}_L + \zeta\sigma + \zeta^2\sigma^2 + \cdots + \zeta^{n-1}\sigma^{n-1} \neq 0$

$$\implies \exists c \in L, \text{ s.t. } \alpha = c + \zeta\sigma(c) + \zeta^2\sigma^2(c) + \cdots + \zeta^{n-1}\sigma^{n-1}(c) \neq 0$$

Observe that  $\sigma(\alpha) = \zeta^{-1}\alpha$ , so  $\alpha \notin K$ . Also  $\sigma(\alpha^n) = \sigma(\alpha)^n = \zeta^{-n}\alpha^n = \alpha^n$ , so  $\alpha^n$  is fixed by  $\text{Gal}(L/K)$ , thus  $a \triangleq \alpha^n \in K$ , and hence  $K(\alpha)$  is a splitting field of  $x^n - a$  over  $K$ .

Now  $\sigma(\alpha) = \zeta^{-1}\alpha \in K(\alpha)$ , so  $\sigma|_{K(\alpha)} \in \text{Gal}(K(\alpha)/K)$ . Also  $\sigma^k(\alpha) = \zeta^{-k}\alpha \implies \text{ord}(\sigma) = n$ . Thus

$$n = [L : K] \geq [K(\alpha) : K] = \text{Gal}(K(\alpha)/K) \geq n \implies L = K(\alpha) \quad \square$$

**Theorem 62.** Let  $L/K$  be a Galois extension such that  $\text{Gal}(L/K)$  is abelian of exponent  $n$  and  $\zeta_n \in K$ , then  $L/K$  is a Kummer extension.

*Proof.* By induction on  $[L : K]$ . If  $[L : K] = 1$  then  $L = K$  and is trivial.

Assume  $[L : K] > 1$ , then by FTFGAG,  $G \triangleq \text{Gal}(L/K) \cong \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \cdots \times \mathbb{Z}/d_s\mathbb{Z}$  with  $d_i \mid d_{i+1}$ . If  $s = 1$  then the theorem degenerates to theorem 61.

So assume  $s > 1$ . Let  $H = \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_{s-1}\mathbb{Z}$ ,  $N = \mathbb{Z}/d_s\mathbb{Z}$  be the corresponding subgroup in  $\text{Gal}(L/K)$ . Set  $M = L^N$ , we have  $[M : K] \leq [L : K]$ . Since any subgroup of abelian group is normal, we have  $\text{Gal}(M/K) \cong \text{Gal}(L/K)/\text{Gal}(L/M) = G/N = H$ .

Denote  $m = d_{s-1}$ ,  $n = d_s$ , we have  $m \mid n$ . Then  $\zeta_n \in K \implies \zeta_m = \zeta_n^{n/m} \in K$ , thus we could pass down the induction, and assume  $M$  is a kummer extension which is a splitting field of  $g = (x^m - b_1)(x^m - b_2) \cdots (x^m - b_{k-1})$  over  $K$  with each  $b_i \in K$ . Let  $\alpha_1, \dots, \alpha_{k-1}$  be all the roots of  $g$ , then  $\alpha_i$  is also a root of  $(x^n - b_1^{n/m})$ . Thus if we define  $a_i \triangleq b_i^{n/m}$ , then  $M$  is also the splitting field of  $(x^n - a_1)(x^n - a_2) \cdots (x^n - a_{k-1})$  over  $K$  since  $\zeta_n \in K$ .

Now, if  $N = \langle \sigma \rangle$ , then  $G \cong H \times N = \{\sigma^k \tau : 0 \leq k < n, \tau \in H\}$ . Since automorphisms are linearly independent, exists  $c \in L$  satisfied

$$0 \neq \alpha = \sum_{\tau \in H} \tau(c) + \zeta \sum_{\tau \in H} \sigma \tau(c) + \cdots + \zeta^{n-1} \sum_{\tau \in H} \sigma^{n-1} \tau(c)$$

Then  $\sigma(\alpha) = \zeta^{-1}\alpha$ , so  $\alpha \notin M$ . Also  $\sigma(\alpha^n) = \alpha^n$  and  $\tau(\alpha^n) = \tau(\alpha)^n = \alpha^n$ , so  $a_k \triangleq \alpha^n \in K$ . Thus  $M(\alpha)$  is a splitting field of  $(x^n - a_k)$  over  $M$ .

Finally,  $n = [L : M] \geq [M(\alpha) : M] = |\text{Gal}(M(\alpha)/M)| \geq n$ , thus  $L = M(\alpha)$ , and hence  $L$  is a splitting field of  $(x^n - a_1)(x^n - a_2) \cdots (x^n - a_k)$ .  $\square$

**Theorem 63.** If  $L/K$  is a kummer extension of exponent  $n$ , then  $\text{Gal}(L/K)$  is abelian of exponent dividing  $n$ .

*Proof.* Let  $L$  be the splitting field of  $(x^n - a_1)(x^n - a_2) \cdots (x^n - a_k)$  with  $\alpha_i = \sqrt[n]{a_i}$ . If  $\sigma \in \text{Gal}(L/K)$ , then  $\sigma$  sends each  $\alpha_i$  to some  $\zeta^{k_{\sigma,i}} \alpha_i$ . So  $\sigma^n = \alpha_i \mapsto \zeta^{k_{\sigma,i}n} \alpha_i = \alpha_i \mapsto \alpha_i = \text{Id}$  and  $\sigma \circ \tau = \alpha_i \mapsto \zeta^{k_{\sigma,i} + k_{\tau,i}} \alpha_i = \tau \circ \sigma$ . by the fact that  $\{\alpha_i\}$  is the generating set of  $L$ . Hence  $\text{Gal}(L/K)$  is abelian and of exponent dividing  $n$ .  $\square$

### 5.7.2 Cubic equations

**Lemma 11.** Let  $\text{char } K \neq 2$  and  $f(x) \in K[x]$  with  $\deg f = n$ . Let  $L = K(\alpha_1, \dots, \alpha_n)$  be a splitting field of  $L$  over  $K$ .

Define  $\delta = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)$ , then  $L^{\text{Gal}(L/K) \cap A_n} = K(\delta)$ . (Here  $\text{Gal}(L/K) \hookrightarrow S_n$ )

*Proof.* Notice that any transposition maps  $\delta$  to  $-\delta$ , so  $\forall \sigma \in \text{Gal}(L/K) \cap A_n$ ,  $\sigma(\delta) = \delta$ , thus  $K(\delta) \subseteq L^{\text{Gal}(L/K) \cap A_n}$ .

Now,  $|\text{Gal}(L/K)/\text{Gal}(L/K) \cap A_n|$  is either 1 or 2. If it is 1, then  $\text{Gal}(L/K) \leq A_n$ , thus  $\delta \in K$  and is trivial. Assume it is 2, then  $\delta$  is not fixed by all permutation, thus  $\delta \notin K$ . But  $D = \delta^2 \in K$  is the discriminant. So we have  $2 = [K(\delta) : K] \leq [L^{\text{Gal}(L/K) \cap A_n} : K] = |\text{Gal}(L^{\text{Gal}(L/K) \cap A_n}/K)| = 2$ , thus  $K(\delta) = L^{\text{Gal}(L/K) \cap A_n}$ .  $\square$

**Prop 5.7.2.** Let  $f(x) = x^3 + px + q$  be irreducible in  $K[x]$  and  $L$  be a splitting field,

- If  $\text{Gal}(L/K) \cong S_3$  then  $\sqrt{D} \notin K$ .
- If  $\text{Gal}(L/K) \cong A_3$  then  $\sqrt{D} \in K$ .

**Def 94.**  $H \leq S_n$  is said to be transitive if for any  $i, j$ , there exists  $\sigma \in H$  such that  $\sigma(i) = j$ .

**Fact 5.7.1.** Let  $f(x)$  be a separable polynomial with degree  $n$ , then

$$f(x) \text{ is irreducible} \iff \text{The Galois group of } f \text{ is transitive in } S_n$$

## 5.8 Solution by radicals

**Def 95.**

1. Given  $L/K$  and  $\alpha \in L$ ,  $\alpha$  is called a radical over  $K$  if  $\alpha^n \in K$  for some  $n \in \mathbb{N}$ .
2.  $L/K$  is called an extension by radicals if there exist  $L = L_n \supset L_{n-1} \supset \cdots \supset L_1 \supset L_0 = K$  s.t.  $\forall i = 1, \dots, n$ ,  $L_i = L_{i-1}(\alpha_i)$  with  $\alpha_i$  a radical over  $L_{i-1}$ .
3.  $f(x) \in K[x]$  is solvable by radicals if there exists  $L/K$ , an extension by radicals, s.t.  $f$  splits over  $L$ .

**Def 96.** (Recall) Let  $G$  be a finite group.  $G$  is solvable if  $\exists \{1\} = G_m \triangleleft G_{m-1} \triangleleft \cdots \triangleleft G_0 = G$  s.t.  $G_{i-1}/G_i$  is cyclic  $\forall i$ .

**Lemma 12.** Given a Galois extension  $L/K$  and  $M = L(\alpha)$  is an extension by a radical, where  $\alpha^n = a \in L$ . Assume that  $\text{char } K \nmid n$ . Then  $\exists N$  s.t.  $N/M$  is an extension by radicals and  $N/K$  is Galois and  $N$  contains  $\zeta_n$ .

*Proof.* We know that  $M(\zeta_n) = L(\zeta_n, \alpha)$  is a splitting field of  $x^n - a$  over  $L$ . If we set

$$f(x) = \prod_{\sigma \in \text{Gal}(L/K)} (x^n - \sigma(a)),$$

then the coefficients of  $f(x)$  are elementary symmetric polynomials in  $\{\sigma(a) \mid \sigma \in \text{Gal}(L/K)\}$ , which are fixed by  $\text{Gal}(L/K)$ , so  $f(x) \in K[x]$ .

Let  $L$  be a splitting field of  $g(x)$  over  $K$ . (since  $L/K$  is Galois) Choose  $N$  as a splitting field of  $f(x)g(x)$  over  $K$ . By def.,  $N/K$  is Galois. Let  $L = K(\beta_1, \dots, \beta_s)$  where  $\beta_1, \dots, \beta_s$  are the roots of  $g(x)$ , then

$$N = K(\beta_1, \dots, \beta_s, \zeta_n, \alpha_\sigma : \sigma \in \text{Gal}(L/K)), \quad \alpha_\sigma^n = \sigma(a) \in L$$

So  $N = M(\zeta_n, \alpha_\sigma : \sigma \in \text{Gal}(L/K) \setminus \{\text{Id}\}) \implies N/M$  is an extension by radicals.  $\square$

**Lemma 13.** Let  $L = L_m \supset L_{m-1} \supset \cdots \supset L_0 = K$  s.t.  $L_i = L_{i-1}(\alpha_i)$  with  $\alpha_i^{n_i} = a_i \in L_{i-1}$ . If  $\text{char } K \nmid n_1 n_2 \cdots n_m$ , then there exists  $N/L$  s.t.  $N/K$  is a Galois extension by radicals and  $\zeta_{n_i} \in N$ ,  $\forall i = 1, \dots, m$ .

*Proof.* By induction on  $m$ . For  $m = 1$ ,  $L_1 \supset L_0 = K$  and  $L_1 = L_0(\alpha_1) = K(\alpha_1)$  where  $\alpha_1^{n_1} \in K$  for some  $n_1 \in \mathbb{N}$ . Set  $N = L(\zeta_{n_1}) = K(\zeta_{n_1}, \alpha_1)$ , done.

For  $m > 1$ , by induction hypothesis,  $\exists N'/L_{m-1}$  s.t.  $N'/K$  is Galois extension by radicals and  $N'$  contains  $\zeta_{n_i}$ ,  $\forall i = 1, \dots, m-1$ . By lemma 12,  $\exists N/N'(\alpha_m)$  is an extension by radicals s.t.  $N/K$  is Galois and  $N$  contains  $\zeta_{n_m}$ .  $\square$

**Prop 5.8.1.** Let  $H \triangleleft G$ . Then  $G$  is solvable  $\iff H, G/H$  are solvable.

*Proof.* “ $\Leftarrow$ ”: Let  $q : G \rightarrow G/H$  be the quotient map,  $Q = G/H$ . The solvable series is given by

$$G = q^{-1}(Q) = q^{-1}(Q_0) \triangleright q^{-1}(Q_1) \triangleright \cdots \triangleright q^{-1}(Q_n) = H = H_0 \triangleright H_1 \triangleright \cdots \triangleright H_m = \{1\}$$

“ $\Rightarrow$ ”:

Claim: Define  $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ ,  $i \in \mathbb{N}$ ;  $G^{(0)} = G$ . Then  $G$  is solvable  $\iff G^{(n)} = \{1\}$  for some  $n$ .

*Proof.* “ $\Leftarrow$ ”: O.K.

“ $\Rightarrow$ ”: Given  $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_m = \{1\}$  with  $G_{i-1}/G_i$  abelian. We have  $G^{(1)} \leq G_1 \rightsquigarrow G^{(2)} \leq [G_1, G_1] \leq G_2 \rightsquigarrow \cdots \rightsquigarrow G^{(n)} \leq G_n = \{1\} \rightsquigarrow G^{(n)} = \{1\}$ .  $\square$

By the claim above:

- $H^{(n)} \leq G^{(n)} = \{1\} \rightsquigarrow H^{(n)} = \{1\} \implies H$  is solvable.
- $q([G, G]) = [q(G), q(G)] = [G/H, G/H] = (G/H)^{(1)} \rightsquigarrow \cdots \rightsquigarrow q(G^{(n)}) = (G/H)^{(n)} \implies G/H$  is solvable.

$\square$

**Theorem 64** (Main Theorem). Under some proper assumption on  $\text{char } K$ , a separable polynomial  $f(x) \in K[x]$  is solvable by radicals  $\iff$  the Galois group of  $f$  is solvable.

**Part A:** Let  $L = L_m \supset \cdots \supset L_0 = K$  s.t.  $L_i = L_{i-1}(\alpha_i)$  with  $\alpha^{n_i} = a_i \in L_{i-1}$  and  $\text{char } K \nmid n_1 \cdots n_m$ . If a separable poly.  $f(x) \in K[x]$  splits over  $L$ , then the Galois group of  $f$  over  $K$  is solvable.

*Proof.* By lemma 13, we can first extend the extension tower and thus assume that  $L/K$  is Galois with each  $\zeta_{n_i}$  in  $L$ . Then each  $L/L_i$  is Galois. If we set  $n = \text{lcm}(n_1, \dots, n_m)$ ,  $L$  also contains  $\zeta = \zeta_n = \zeta_{n_1}^{r_1} \cdots \zeta_{n_m}^{r_m}$ .

Consider  $L = L(\zeta) \supset L_{m-1}(\zeta) \supset \cdots \supset L_0(\zeta) = K(\zeta)$  (Note that  $K(\zeta) \supset K$  and  $L/K$  is Galois) and let  $G_i = \text{Gal}(L/L_i(\zeta))$  for each  $i = 0, \dots, m$ .

Define  $L'_i \triangleq L_i(\zeta)$  for all  $i$ . We can find that

- $G_m = \{1\}, G_0 = \text{Gal}(L/K(\zeta))$ .
- Since  $\zeta_n \in L_{i-1}$ ,  $L_i/L_{i-1}$  is normal, so

$$G_{i-1}/G_i = \text{Gal}(L/L'_{i-1})/\text{Gal}(L/L'_i) \cong \text{Gal}(L'_{i-1}/L'_i) = \text{Gal}(L'_i(\alpha_i)/L'_i)$$

is cyclic.

So  $G_0$  is solvable. Moreover,  $K(\zeta)$  is a splitting field of  $x^n - 1$  over  $K$  and  $\text{Gal}(K(\zeta)/K) \leq (\mathbb{Z}/n\mathbb{Z})^\times$ , which is abelian, so it is solvable. Also,  $\text{Gal}(K(\zeta)/K) \cong \text{Gal}(L/K)/G_0$  is solvable.  $\implies \text{Gal}(L/K)$  is solvable. Let  $N$  be a splitting field of  $f$  over  $K \rightsquigarrow L \supset N \rightsquigarrow \text{Gal}(N/K) \cong \text{Gal}(L/K)/\text{Gal}(L/N)$ .

By prop 5.8.1,  $\text{Gal}(N/K)$  is solvable.  $\square$

**Part B:** Let  $f \in K[x]$  be separable and  $L$  be a splitting field of  $f$  over  $K$ . Assume  $\text{char } K \nmid |\text{Gal}(L/K)|$ . If  $\text{Gal}(L/K)$  is solvable, then  $f$  is solvable by radicals.

*Proof.* Let  $n = |\text{Gal}(L/K)|$  and  $\zeta = \zeta_n$ . Let  $N$  be a splitting field of  $f$  over  $K(\zeta)$ , i.e.  $N = LK(\zeta)$ .  $\implies \text{Gal}(N/K(\zeta)) \cong \text{Gal}(L/L \cap K(\zeta)) \leq \text{Gal}(L/K)$ .

So  $\text{Gal}(N/K(\zeta))$  is solvable, say  $\text{Gal}(N/K(\zeta)) = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_m = 1$ ,  $G_{i-1}/G_i$  is cyclic.

If we set  $N_j = N^{G_j}$ , then  $N = N_m \supset N_{m-1} \supset \cdots \supset N_0 = K(\zeta)$  and  $G_j = \text{Gal}(N/N_j)$ ,  $G_{i-1}/G_i \cong \text{Gal}(N_i/N_{i-1})$  is cyclic  $\implies N_i = N_{i-1}(\alpha_i), \alpha_i^{n_i} \in N_{i-1}$ . (kummer extension)

Note that  $n_i = [L_i : L_{i-1}] = |G_{i-1}|/|G_i|$  dividing  $|G_0|$  and  $|G_0| \mid n$ , so  $\zeta_n$  generates  $\zeta_{n_i}$  and  $\text{char } K \nmid n_i$ .

$\implies N/K(\zeta)$  is an extension by radicals  $\rightsquigarrow N/K$  is an extension by radicals.  $\square$

**Remark 31.** In Part A of theorem 64,  $\text{Gal}(K(\zeta)/K) \leq (\mathbb{Z}/n\mathbb{Z})^\times$  may be proper subgroup. We can check the if  $[K(\zeta) : K] \stackrel{?}{=} 4 = \varphi(5)$ .

## 5.9 Ruffini-Abel theorem

**Theorem 65** (Main theorem). Assume  $\text{char } F = 0$ . The general equation of the  $n$ -th degree is not solvable by radicals if  $n \geq 5$ . In fact, let  $f(x) = x^n - t_1x^{n-1} + t_2x^{n-2} - \dots + (-1)^nt_n \in \underbrace{F(t_1, \dots, t_n)}_{=K}[x]$  with  $t_1, \dots, t_n$  variables and  $L$  be a splitting field of  $f$  over  $K$ . Then  $\text{Gal}(L/K) \cong S_n$ .  $S_n$  is not solvable for  $n \geq 5$ .

**Lemma 14.** Let  $L = F(x_1, \dots, x_n)$  and  $s_1, \dots, s_n$  be the elementary symmetric polynomials in  $x_1, \dots, x_n$ .

$$s_k = \sum_{1 \leq j_1 < \dots < j_k \leq n} \prod_{i=1}^k x_{j_i}$$

If  $K = F(s_1, \dots, s_n) \subset L$ , then  $L/K$  is Galois and  $\text{Gal}(L/K) \cong S_n$ .

*Proof.* write  $f(x) = (x - x_1) \dots (x - x_n) = x^n - s_1x^{n-1} + s_2x^{n-2} - \dots + (-1)^ns_n \in K[x]$ . Clearly,  $L$  is a splitting field of  $f$  over  $K \rightsquigarrow L/K$  is Galois and  $\text{Gal}(L/K) \hookrightarrow S_n$ .

Now, for  $\sigma \in S_n$ ,  $\sigma$  can be regarded as an element in  $\text{Gal}(L/K)$ :

$$\begin{aligned} \sigma : L &\rightarrow L \\ x_i &\mapsto x_{\sigma(i)} \end{aligned}$$

Since  $\{\sigma(x_1), \dots, \sigma(x_n)\} = \{x_1, \dots, x_n\} \rightsquigarrow \sigma(s_i) = s_i \quad \forall i \rightsquigarrow \sigma|_K = \text{Id}_K \rightsquigarrow \sigma \in \text{Gal}(L/K)$ .  $\square$

**Coro 5.9.1.**  $L^{S_n} = K = F(s_1, \dots, s_n)$ .

$L^{S_n} = \{f(x_1, \dots, x_n) \in L \mid f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n) \quad \forall \sigma \in S_n\}$  is all symmetric poly.

**Coro 5.9.2.** For any finite group  $G$ , by Cayley thm,  $G \hookrightarrow S_n$  for some  $n$ . so  $\text{Gal}(L/L^G) \cong G$ .

Now we prove the Main theorem:

*Proof.* Let  $L = K(z_1, \dots, z_n)$ . Since  $t_1, \dots, t_n$  are the symmetric poly. w.r.t.  $z_1, \dots, z_n$ ,  $L = F(z_1, \dots, z_n)$ .

Let  $F(s_1, \dots, s_n)$  and  $F(x_1, \dots, x_n)$  be given as in lemma 14.

since  $t_1, \dots, t_n$  are variables,  $\exists \tau : F[t_1, \dots, t_n] \rightarrow F[s_1, \dots, s_n]$  with  $\tau : t_i \mapsto s_i$ . Also, Since  $x_1, \dots, x_n$  are variables,  $\exists \sigma : F[x_1, \dots, x_n] \rightarrow F[z_1, \dots, z_n]$  with  $\sigma : x_i \mapsto z_i$ .

now,  $\sigma \circ \tau(t_i) = \sigma(s_i) = \sigma(\sum x_{j_1} \dots x_{j_i}) = (\sum z_{j_1} \dots z_{j_i}) = t_i \implies \sigma \circ \tau = \text{Id} \implies \tau$  is 1-1 and thus an isom. So there exists an extension  $\tau' : F(t_1, \dots, t_n) \xrightarrow{\sim} F(s_1, \dots, s_n)$ . Note  $\bar{\tau}' : f(x) \mapsto g(x) = x^n - s_1x^{n-1} + \dots + (-1)^ns_n$ .

Let  $F(z_1, \dots, z_n)$  be a splitting field of  $f$  over  $F(t_1, \dots, t_n)$  and  $F(x_1, \dots, x_n)$  be a splitting field of  $g$  over  $F(s_1, \dots, s_n)$  where  $g = \bar{\tau}'(f)$ . There exists  $\sigma' : F(z_1, \dots, z_n) \xrightarrow{\sim} F(x_1, \dots, x_n)$  with  $\sigma'|_{F(t_1, \dots, t_n)} = \tau'$ . So  $\text{Gal}(L/K) \cong S_n$  by lemma 14.  $\square$

**Remark 32.**

- Since  $S_n$  is transitive,  $f$  is irr.
- $\text{char } F = 0 \rightsquigarrow f$  is separable.

## 5.10 Calculation of Galois groups

Let  $f(x)$  be separable in  $K[x]$  and  $L = K(\alpha_1, \dots, \alpha_n)$  be a splitting field of  $f$  over  $K$ . The goal is to find  $\text{Gal}(L/K)$  which is in  $S_n$ .

Define  $\theta \triangleq y_1\alpha_1 + \dots + y_n\alpha_n$ . For each  $\sigma \in S_n$ , define  $\sigma_y(\theta) \triangleq y_{\sigma(1)}\alpha_1 + \dots + y_{\sigma(n)}\alpha_n$  and  $\sigma_\alpha(\theta) = y_1\alpha_{\sigma(1)} + \dots + y_n\alpha_{\sigma(n)}$ . It is easy to see that  $\sigma_y^{-1} = \sigma_\alpha$ .

In  $L(x, y_1, \dots, y_n)$ , we consider  $F(x, \mathbf{y}) = \prod_{\sigma \in S_n} (x - \sigma_y(\theta)) = \prod_{\sigma^{-1} \in S_n} (x - \sigma_\alpha(\theta)) = \prod_{\sigma \in S_n} (x - \sigma_\alpha(\theta))$ .

Since each coefficient of  $F$  is a symmetric polynomial of  $\alpha_1, \dots, \alpha_n$ , by the fundamental theorem of symmetric polynomials, these symmetric polynomials are polynomials of the elementary symmetric polynomials. Thus  $F(x, y) \in K[x, y_1, \dots, y_n]$ .

Decompose  $F$  into irreducible factors in  $K[x, y_1, \dots, y_n]$ , say  $F = F_1 F_2 \dots F_r$ . Notice that for any  $\sigma \in S_n$ ,  $F = \sigma_y F = \sigma_y F_1 \cdot \sigma_y F_2 \dots \sigma_y F_r$ . And each  $F_i$  is map to some  $F_j$ , thus  $\sigma$  induces a permutation of  $F_1, F_2, \dots, F_r$ .

For convenience, assume  $(x - \theta) \mid F_1$ . We have the following lemma:

**Lemma 15.**

$$Q \triangleq \{ \sigma : \sigma_y F_1 = F_1 \} = \{ \sigma : \sigma_y(x - \theta) \mid F_1 \}$$

*Proof.* “ $\subseteq$ ”: Since  $x - \theta \mid F_1$ , so  $\sigma_y(x - \theta) \mid \sigma_y F_1 = F_1$ .

“ $\supseteq$ ”:  $\sigma_y(x - \theta) = x - \sigma_y(\theta) \mid \sigma_y(F_1)$ , so  $\sigma_y(F_1)$  and  $F_1$  has a common factor. Since  $F$  is separable,  $\sigma_y(F_1) = F_1$ .  $\square$

**Prop 5.10.1.**  $\text{Gal}(L/K) = Q$ .

*Proof.* “ $\subseteq$ ”: For each  $\sigma \in \text{Gal}(L/K) \hookrightarrow S_n$ , extend  $\sigma$  to

$$\begin{array}{ccc} \tilde{\sigma} : L(y_1, \dots, y_n) & \rightarrow & L(y_1, \dots, y_n) \\ \alpha \in L & \mapsto & \sigma(\alpha) \\ y_i & \mapsto & y_i \end{array}$$

The automorphism fixes  $K(y_1, \dots, y_n)$ , so  $\tilde{\sigma}(\theta) = \sigma_\alpha(\theta)$  and  $\theta$  share the same minimal polynomial over  $K(y_1, \dots, y_n)$ . By Gauss’s lemma,  $F_1$  is irreducible in  $K[y_1, \dots, y_n][x] \implies F_1$  is irreducible in  $K(y_1, \dots, y_n)[x]$ , thus  $F_1 = m_{\theta, K(y_1, \dots, y_n)} = m_{\sigma_\alpha(\theta), K(y_1, \dots, y_n)}$ , which implies  $(x - \sigma_\alpha(\theta)) \mid F_1$ . So  $\sigma_y^{-1} F_1 = F_1 \implies \sigma^{-1} \in Q \implies \sigma \in Q$ .

“ $\supseteq$ ”: For any  $\sigma \in Q$ ,  $F_1 = m_{\theta, K(y_1, \dots, y_n)} = m_{\sigma_\alpha^{-1}(\theta), K(y_1, \dots, y_n)}$ , so there exists  $\tau \in \text{Aut}(L(\mathbf{y})/K(\mathbf{y}))$  satisfying  $\tau(\theta) = \sigma_\alpha^{-1}(\theta) = \sigma_y(\theta)$ . Since  $L/K$  normal,  $\tau(L) = L$  and thus  $\tau|_L \in \text{Gal}(L/K)$  with  $\tau|_L(\alpha_i) = \alpha_{\sigma^{-1}(i)}$ , which implies that  $\sigma^{-1} \in \text{Gal}(L/K) \implies \sigma \in \text{Gal}(L/K)$ .  $\square$

**Theorem 66.** Let  $f(x)$  be monic, separable, in  $\mathbb{Z}[x]$ . Assume  $p \nmid D = \prod_{i < j} (\alpha_i - \alpha_j)^2$ , then the Galois group of  $\bar{f}(x)$  in  $\mathbb{F}_p[x]$  is a subgroup of the Galois group of  $f(x)$ .

*Proof.* Since  $f$  is separable,  $D \neq 0$ . The discriminant could be calculate by  $D = (-1)^{n(n+1)/2} R(f, f')$  which only depends on the coefficients, so  $\bar{D} \neq 0$  in  $\mathbb{F}_p$  since  $p \nmid D$ . Thus  $f$  separable.

As above, let  $F = F_1 F_2 \dots F_r$  in  $\mathbb{Z}[x, \mathbf{y}]$ . Assume  $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ , then  $\bar{f}(x) = x^n + \bar{a}_{n-1}x^{n-1} + \dots + \bar{a}_0$ . Let  $\alpha_1, \dots, \alpha_n$  and  $\beta_1, \dots, \beta_n$  be their roots, respectively. Define  $\theta_p \triangleq y_1\beta_1 + \dots + y_n\beta_n$ . Since the coefficients of  $F$  are symmetric polynomials of  $\alpha_1, \dots, \alpha_n$ , which only depends on the coefficients of  $f$ , and so is  $F_p(x, y) = \prod_{\sigma \in S_n} (x - \sigma_y(\theta_p))$ , we know that  $F_p(x, y) = \bar{F}(x, y)$ .

Now  $\bar{F} = \bar{F}_1 \bar{F}_2 \dots \bar{F}_r = (G_{1,1} \dots G_{1,q_1})(G_{2,1} \dots G_{2,q_2}) \dots (G_{r,1} \dots G_{r,q_r})$

The Galois group of  $\bar{f}$  is

$$\{\sigma \in S_n : \sigma_y G_{1,j} = G_{1,j}, \forall j\} \subseteq \{\sigma \in S_n : \sigma_y \bar{F}_1 = \bar{F}_1\} = \{\sigma \in S_n : \sigma_y F_1 = F_1\}$$

Where the equality holds because  $\sigma_y \bar{F}_1 = \bar{F}_1 \iff (x - \sigma_y(\theta_p)) \mid \bar{F}_1 \iff (x - \sigma_y(\theta)) \mid F_1 \iff \sigma_y F_1 = F_1$ . Thus the galois group of  $\bar{f}$  is a subgroup of  $f$ .  $\square$

**Fact 5.10.1.**

- Every finite extension of  $\mathbb{F}_p$  is cyclic, so the Galois group of  $\bar{f}(x)$  in  $\mathbb{F}_p[x]$  is cyclic.
- If  $\bar{f}$  is irreducible, then the Galois group of  $\bar{f}$  is transitive on its roots, thus the only possibility is a cycle of length  $n = \deg \bar{f}$  in  $S_n$ .
- If  $\bar{f} = \bar{f}_1 \cdots \bar{f}_r$ , with each  $\bar{f}_i$  irreducible. Let the Galois group be  $\langle \sigma \rangle$ , then  $\sigma$  should be transitive on the roots of each  $\bar{f}_i$ . The only possibility of  $\sigma$  is a permutation composed by cycles of length  $\deg \bar{f}_1, \dots, \deg \bar{f}_r$ . That is,  $\sigma = (\alpha_{1,1} \dots \alpha_{1,m_1}) \cdots (\alpha_{r,1} \dots \alpha_{r,m_r})$  where  $m_i \triangleq \deg \bar{f}_i$ .



## 5.11 Transcendental extensions

**Def 97.** Let  $L/K$  be an extension and  $S \subset L$ .

- $S$  is algebraically dependent over  $K$  if for some  $n \in \mathbb{N}$ , exists  $f(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$  satisfied  $f(a_1, \dots, a_n) = 0$  for some distinct  $a_1, \dots, a_n \in S$ .
- $S$  is algebraically independent over  $K$  if  $S$  is not algebraically dependent.
- $S$  is called a transcendence base for  $L/K$  if  $S$  is algebraically independent and  $L/K(S)$  is algebraic.

**Theorem 67.** Any two transcendence bases for  $L/K$  have the same cardinality.

*Proof.* Pick any transcendence base  $S = \{s_1, \dots, s_n\}$  for  $L/K$ . Let  $T$  be another transcendence base for  $L/K$ . First we deal with the case which  $S$  is finite.

We claim that  $\exists t_1 \in T$  such that  $t_1$  is algebraically independent over  $K(s_2, \dots, s_n)$ .

*Proof.* If not, then all elements of  $T$  is algebraically dependent over  $K(s_2, \dots, s_n)$ . This implies  $K(s_2, \dots, s_n)(T)/K(s_2, \dots, s_n)$  is algebraic. And  $L/K(T)$  is algebraic implies  $L/K(T)(s_2, \dots, s_n)$  is algebraic. Then  $L/K(s_2, \dots, s_n)$  is algebraic, which is a contradiction ( $s_1$  is not).  $\square$

By the claim,  $\{t_1, s_2, \dots, s_n\}$  is algebraic independent. Also, there exists  $f \neq 0$  in  $K[x_1, \dots, x_{n+1}]$  such that  $f(t_1, s_1, \dots, s_n) = 0$  since  $t_1$  is algebraic over  $K(s_1, \dots, s_n)$ . Since  $\{s_1, \dots, s_n\}$  and  $\{t_1, s_2, \dots, s_n\}$  are both algebraically independent,  $t_1, s_1$  must occur in  $f \implies s_1$  is algebraic over  $K(t_1, s_2, \dots, s_n)$ . Then  $K(t_1, s_1, \dots, s_n)/K(t_1, s_2, \dots, s_n)$  is algebraic. Since  $L/K(t_1, s_1, \dots, s_n)$  is algebraic,  $L/K(t_1, s_2, \dots, s_n)$  is algebraic.

Repeating this process, we get find  $t_1, \dots, t_n \in T$  s.t.  $L/K(t_1, \dots, t_n)$  is algebraic. But  $T$  is a transcendence base, so we must have  $T = \{t_1, \dots, t_n\}$ .

Now assume  $S$  is infinite. For another transcendence base  $T$ , we have  $|T| = \infty$ . For  $s \in S$ ,  $s$  is algebraic over  $K(T)$ , and in fact is over  $K(T_s)$  such that  $T_s$  is finite, since algebraic relation involves. Let  $m_{s, K(T)} \in K(T_s)[x]$  for some finite set  $T_s \subset T$ . We claim that  $\bigcup_{s \in S} T_s = T$ .

*Proof.* Let  $U = \bigcup_{s \in S} T_s$ . Clearly  $U \subseteq T$ . And by def,  $K(U)(S)/K(U)$  is algebraic. Also,  $L/K(U)(S)$  is algebraic. So  $L/K(U)$  is algebraic  $\implies T = U$  since  $T$  is a transcendence base.  $\square$

By well ordering principle, we can well-order  $S$  and denote its 1st element by  $s_1$ . Let

$$\begin{cases} T'_{s_1} = T_{s_1} \\ T'_s = T_s \setminus \bigcup_{l < s} T_l \end{cases} \implies \{T'_s\}_{s \in S} \text{ are mutually disjoint}$$

For all  $T'_s$ , choose a fixed ordering of the elements in  $T'_s$ , says  $t_{s,1}, \dots, t_{s,k_s}$ . Define an injection  $\varphi : \bigcup_{s \in S} T'_s \rightarrow S \times \mathbb{N}$  with  $\varphi : t_{s,i} \mapsto (s, i)$ . So  $|T| = \left| \bigcup_{s \in S} T'_s \right| \leq |S \times \mathbb{N}| = |S| |\mathbb{N}| = |S|$  since  $|S| = \infty$ .  $\square$

**Def 98.** Let  $S$  be a transcendence base of  $L/K$ , then we use  $\text{tr deg}_K L$  to denote  $|S|$ .

**Remark 33.** If  $S_1, S_2$  are two transcendence base for  $L/K$ , then it is **not necessarily true** that  $K(S_1) = K(S_2)$ .

**Def 99.**  $L/K$  is called purely transcendental if exists a transcendental base  $S$  such that  $L = K(S)$ .

**Theorem 68** (Lüroth's theorem). If  $L$  is purely transcendental of degree 1 over  $K$ , then any proper intermediate field  $E$  is also purely transcendental of degree 1.

**Lemma 16.** Let  $L = K(t)$  with  $t$  being transcendental over  $K$  and  $u = f(t)/g(t) \in L \setminus K$  with  $\gcd(f(t), g(t)) = 1$ . Assume  $n = \max(\deg f, \deg g)$ , then  $L/K(u)$  is algebraic and  $[L : K(u)] = n$ .

*Proof.* Write

$$f(t) = a_n t^n + \cdots + a_1 t + a_0, \quad g(t) = b_n t^n + \cdots + b_1 t + b_0$$

(note that either  $a_n \neq 0$  or  $b_n \neq 0$ ) Let  $F(x) = f(x) - ug(x) = (a_n - ub_n)x^n + \cdots + (a_1 - ub_1)x + (a_0 - ub_0)$ . Since  $a_n - ub_n \neq 0$ ,  $F(x) \neq 0$  and  $\deg F(x) > 0$ . By def. of  $u$ , we have  $F(t) = 0 \implies t$  is algebraic over  $K(u)$  and  $[K(t) : K(u)] \leq n$ . Now we prove that  $F(x)$  is irreducible over  $K(u)$ . By Gauss's lemma, it suffices to show that  $F(x)$  is irreducible in  $K[u][x] = K[u, x]$ . Assume that  $F(x) = p(u, x)q(u, x)$  with  $\deg_u p = 1$  and  $q \in K[x]$ . Since  $F(x) = f(x) - ug(x)$ , we have  $q \mid f, q \mid g \implies q \mid \gcd(f, g) = 1 \implies q \in K$ . So  $[K(t) : K(u)] = n$ .  $\square$

Now we prove the Lüroth's theorem:

*Proof.* For  $v \in E \setminus K$ , by lemma 16,  $t$  is algebraic over  $K(v) \rightsquigarrow t$  is algebraic over  $E$ .

Let  $m(x) = m_{t,E}$ , then there exists  $\beta(t) \in K(t)$  s.t.  $\beta(t)m(x) = a_n(t)x^n + \cdots + a_1(t)x + a_0(t)$  is primitive in  $K[t][x] = K[t, x]$ . Let  $F(t, x) = \beta(t)m(x)$ .

Since  $t$  is not algebraic over  $K$ , there exists some  $u = \frac{a_i(t)}{a_n(t)} \notin K$ . Write  $u = \frac{f(t)}{g(t)}$  with  $\gcd(f, g) = 1$ . (Note that  $u \in E$ )

By lemma 16,  $[K(t) : K(u)] = r \geq n$ . Now we show that  $r \leq n$ , then  $r = n \implies E = K(u)$ .

Let  $l = f(t)g(x) - g(t)f(x)$ , which is skew-symmetric in  $t$  and  $x$ . Notice that  $g(t)^{-1}l \in E[x]$  and has  $t$  as a zero. So  $m(x) \mid g(t)^{-1}l$  in  $E[x] \implies \beta(t)m(x) \mid \beta(t)g(t)^{-1}l$ . Since  $\beta(t)g(t)^{-1} \in K[t]$ ,  $F(t, x) \mid l$  in  $K(t)[x]$ . Since  $F(t, x)$  is primitive in  $K[t][x]$ ,  $F(t, x) \mid l$  in  $K[t][x]$ .

Say  $l = Fq$  for some  $q(t, x) \in K[t][x]$ . Note that  $\deg_t l \leq r, \deg_t F \geq r \rightsquigarrow \deg_t l = \deg_t F = r, \deg_t q = 0$ . So  $q \in K[x] \rightsquigarrow q$  is primitive in  $K[t][x]$ . By Gauss's lemma,  $F, q$  are primitive, then  $l$  is also primitive in  $K[t][x]$ . Since  $l$  is skew-symmetric in  $t$  and  $x$ ,  $l$  is also primitive in  $K[x][t]$ . But  $q \in K[x]$  and  $q \mid l$ , we have  $q \in K$ . Hence  $n = \deg_x F = \deg_x l = \deg_t l = \deg_t F \geq r$ .  $\square$

## 5.12 Hilbert theorem 90 and Normal basis

Let  $L = K(\alpha)$  with  $f = m_{\alpha, K} = x^n + a_{n-1}x^{n-1} + \cdots + a_0$  being separable. We have known that exists exactly  $n$  monomorphisms  $\sigma_i :: L \rightarrow \overline{K}$  fixing  $K$ , and  $\{\sigma_1(\alpha), \dots, \sigma_n(\alpha)\}$  consists of all roots of  $f$ . So

$$\begin{aligned} x^n + a_{n-1}x^{n-1} + \cdots + a_0 &= (x - \sigma_1(\alpha)) \cdots (x - \sigma_n(\alpha)) \\ \implies -a_{n-1} &= \sigma_1(\alpha) + \cdots + \sigma_n(\alpha) \text{ and } (-1)^n a_0 = \sigma_1(\alpha) \cdots \sigma_n(\alpha) \end{aligned}$$

Consider the  $K$ -linear transformation:

$$\begin{aligned} T_\alpha : K(\alpha) &\rightarrow K(\alpha) \\ v &\mapsto \alpha v \end{aligned}$$

Then

$$[T_\alpha]_\gamma = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}, \quad \text{where } \gamma = \{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$$

And  $\text{Tr}(T_\alpha) = -a_{n-1}$ ,  $\det(T_\alpha) = (-1)^n a_0$ .

**Def 100.** Let  $L/K$  be a Galois extension with  $G = \text{Gal}(L/K)$ . for all  $\alpha \in L$ , define

$$N_{L/K}(\alpha) = \prod_{\sigma \in G} \sigma(\alpha) \quad N_{L/K} :: L^\times \rightarrow K^\times \text{ is multiplicative}$$

$$\text{Tr}_{L/K}(\alpha) = \sum_{\sigma \in G} \sigma(\alpha) \quad \text{Tr}_{L/K} :: L \rightarrow K \text{ is additive}$$

**Theorem 69** (Hilbert theorem 90). Let  $L/K$  is cyclic and  $G = \langle \sigma \rangle$  with  $\text{ord}(\sigma) = n$ , then

1.  $\alpha \in L^\times$  and  $N_{L/K}(\alpha) = 1 \iff \exists \beta \in L^\times, \alpha = \beta/\sigma(\beta)$ .
2.  $\alpha \in L$  and  $\text{Tr}_{L/K}(\alpha) = 0 \iff \exists \beta \in L, \alpha = \beta - \sigma(\beta)$ .

*Proof.*

1. “ $\Leftarrow$ ”:  $N_{L/K}(\alpha) = \prod_{k=0}^{n-1} \sigma^k(\beta/\sigma(\beta)) = 1$ .

“ $\Rightarrow$ ”: Since automorphisms are linearly independent, exists  $c \in L$  such that

$$0 \neq \beta = \text{Id}(c) + \alpha\sigma(c) + \alpha\sigma(\alpha)\sigma^2(c) + \cdots + \alpha\sigma(\alpha)\sigma^2(\alpha) \cdots \sigma^{n-2}(\alpha)\sigma^{n-1}(c)$$

Since  $\alpha\sigma(\alpha\sigma(\alpha)\sigma^2(\alpha) \cdots \sigma^{n-2}(\alpha)) = N_{L/K}(\alpha) = 1$ , it is easy to check that  $\alpha\sigma(\beta) = \beta$ .

2. “ $\Leftarrow$ ”:  $\text{Tr}_{L/K}(\alpha) = \text{Tr}_{L/K}(\beta - \sigma(\beta)) = \sum (\sigma^k(\beta) - \sigma^{k+1}(\beta)) = 0$ .

“ $\Rightarrow$ ”: Choose  $c$  such that  $\beta_1 = c + \sigma(c) + \cdots + \sigma^{n-1}(c) \neq 0$ , so  $\sigma(\beta_1) = \beta_1$ . Let

$$\beta_2 = \alpha\sigma(c) + (\alpha + \sigma(\alpha))\sigma^2(c) + \cdots + (\alpha + \sigma(\alpha) + \cdots + \sigma^{n-2}(\alpha))\sigma^{n-1}(c)$$

Then

$$\beta_2 - \sigma(\beta_2) = \alpha\sigma(c) + \alpha\sigma^2(c) + \cdots + \alpha\sigma^{n-1}(c) + \alpha c = \alpha\beta_1.$$

So let  $\beta \triangleq \beta_2/\beta_1$ , we obtain  $\beta_2/\beta_1 - \sigma(\beta_2/\beta_1) = (\beta_2 - \sigma(\beta_2))/\beta_1 = \alpha$ .  $\square$

**Coro 5.12.1.** Let  $\text{char } K = p$  and  $[L : K] = p$ , then  $L/K$  is Galois and cyclic  $\iff L = K(\alpha)$  where  $\alpha$  is a root of  $x^p - x - a$ .

*Proof.* “ $\Rightarrow$ ”: Let  $\text{Gal}(L/K) = \langle \sigma \rangle$  with  $\text{ord}(\sigma) = p$ . Then  $\text{Tr}_{L/K}(1) = p = 0$ . By theorem 69, exists  $\alpha$  satisfied  $1 = \sigma(\alpha) - \alpha$ . So  $\alpha \notin K$ . Then we have  $1 < [K(\alpha) : K] \mid [L : K] = p$ , so  $[K(\alpha) : K] = p \implies K(\alpha) = L$ .

Notice that  $\sigma^k(\alpha) = \alpha + k$ . Since  $\sigma^k(\alpha)$  iterates through all roots of  $m_{\alpha,K}$  and  $\sigma^k(\alpha) = \alpha + k$ ,  $\alpha, \alpha+1, \dots, \alpha+p-1$  are all the roots of  $m_{\alpha,K}$ . We claim that  $m_{\alpha,K} = x^p - x - a$  where  $a \triangleq \alpha^p - \alpha$ . Since  $\sigma(a) = \sigma(\alpha)^p - \alpha = \alpha^p + p - \alpha = a$ ,  $a$  is fixed by all automorphisms, so  $a \in K$ . Moreover,  $m_{\alpha,K}(\alpha + k) = \alpha^p + k^p - \alpha - k - a = 0$ , thus the proof is completed.

“ $\Leftarrow$ ”: Similarly, we know that all roots of  $x^p - x - a$  are  $\alpha, \alpha+1, \dots, \alpha+p-1$ . Define  $\sigma(\alpha) = \alpha+1$ , then  $\sigma^i(\alpha) = \alpha + i$ , and thus  $\text{ord}(\sigma) = p$ . Hence  $\text{Gal}(L/K) = \langle \sigma \rangle$ .  $\square$

**Coro 5.12.2.** If  $x^2 + dy^2 = 1$  where  $-d$  is not a square, then  $L \triangleq \mathbb{Q}(\sqrt{-d})$  is a splitting field of  $x^2 + d$  over  $\mathbb{Q}$ , so  $N_{L/\mathbb{Q}}(a + b\sqrt{-d}) = a^2 + db^2$ . Since  $[L : \mathbb{Q}] = 2$ , the galois group is obviously cyclic and in fact is  $\langle \sigma \rangle$ , where  $\sigma = (a + b\sqrt{-d}) \mapsto (a - b\sqrt{-d})$ . By theorem 69,

$$x^2 + dy^2 = 1 \iff \exists a + b\sqrt{-d} \text{ s.t. } x + y\sqrt{-d} = \frac{a + b\sqrt{-d}}{a - b\sqrt{-d}} = \frac{(a^2 - db^2) + 2ab\sqrt{-d}}{a^2 + db^2}$$

**Def 101.** Let  $L/K$  be Galois and  $\text{Gal}(L/K) = \{\text{Id} = \sigma_1, \dots, \sigma_n\}$ . A basis for  $L/K$  of the form  $\{\sigma_1(\alpha), \sigma_2(\alpha), \dots, \sigma_n(\alpha)\}$  with  $\alpha \in L$  is called a normal basis for  $L/K$ .

**Lemma 17.**  $\alpha_1, \dots, \alpha_n \in L$  form a basis for  $L/K$  if and only if

$$\begin{vmatrix} \sigma_1(\alpha_1) & \sigma_1(\alpha_2) & \cdots & \sigma_1(\alpha_n) \\ \sigma_2(\alpha_1) & \sigma_2(\alpha_2) & \cdots & \sigma_2(\alpha_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n(\alpha_1) & \sigma_n(\alpha_2) & \cdots & \sigma_n(\alpha_n) \end{vmatrix} \neq 0$$

*Proof.* “ $\Rightarrow$ ”: If not, then the determinant is 0. Then

$$\begin{cases} \sigma_1(\alpha_1)x_1 + \cdots + \sigma_n(\alpha_1)x_n = 0 \\ \sigma_1(\alpha_2)x_1 + \cdots + \sigma_n(\alpha_2)x_n = 0 \\ \vdots \\ \sigma_1(\alpha_n)x_1 + \cdots + \sigma_n(\alpha_n)x_n = 0 \end{cases}$$

has a non-zero solution  $\mathbf{c} = (c_1, \dots, c_n) \in L^n$ . (i.e.,  $\sum c_j \sigma_j(\alpha_i) = 0$  for each  $i$ .) So  $(\sum_j c_j \sigma_j)(\alpha_i) = 0$  for each  $\alpha_i$ , but  $\alpha_i$  is a basis, so  $\sum_j c_j \sigma_j = 0$ , then these automorphisms are linearly dependent, which leads to a contradiction.

“ $\Leftarrow$ ”: If not, then exists  $\mathbf{0} \neq \mathbf{c} = (c_1, \dots, c_n)$  satisfied  $\sum c_i \alpha_i = 0$ . Then  $\sum_i c_i \sigma_j(\alpha_i) = 0$  for each  $j$ . Thus the determinant is 0 which leads to a contradiction.  $\square$

**Lemma 18.** Let  $|K| = \infty$ . Then  $\sigma_1, \dots, \sigma_n$  are algebraically independent over  $L$ .

*Proof.* Let  $f(x_1, \dots, x_n) \in L[x_1, \dots, x_n]$  such that  $f(\sigma_1, \dots, \sigma_n) = 0$ . Let  $\{\alpha_1, \dots, \alpha_n\}$  be a basis for  $L/K$ . Then

$$0 = f(\sigma_1, \dots, \sigma_n) \left( \sum_{i=1}^n r_i \alpha_i \right) = f \left( r_1 \sigma_1 \left( \sum_{i=1}^n \alpha_i \right), \dots, r_n \sigma_n \left( \sum_{i=1}^n \alpha_i \right) \right)$$

So let

$$g(x_1, \dots, x_n) \triangleq f \left( \sum_i \sigma_1(\alpha_i) x_i, \dots, \sum_i \sigma_n(\alpha_i) x_i \right)$$

and write  $g(x_1, \dots, x_n) = \sum_j g_j(x_1, \dots, x_n) \alpha_j$ . Then  $g_j(r_1, \dots, r_n) = 0, \forall \mathbf{r} \in K^n$ . The only polynomial which has infinite zeros (without any relation) is the zero polynomial, thus  $g_j = 0$  for each  $j$ .

Now, by lemma 17,  $\det([\sigma_i(\alpha_j)]) \neq 0$ . So it is possible to solve  $\mathbf{x} = (x_i)$  satisfied  $\mathbf{y} = (y_j) = (\sum_i \sigma_j(\alpha_i) x_i)$ . Thus  $g = 0 \implies f = 0$ .  $\square$

**Theorem 70.** Any Galois extension  $L/K$  has a normal basis.

*Proof.* Case 1:  $L/K$  is cyclic (so all finite field is included).

Let  $\text{Gal}(L/K) = \langle \sigma \rangle$  with  $\text{ord}(\sigma) = n$ .  $\sigma$  could be view as a linear transformation of  $L$  over  $K$ . Thus  $\sigma$  gives  $L$  a  $K[x]$ -module structure by  $(f(x), \alpha) \mapsto f(\sigma)(\alpha)$ .

Since  $K[x]$  is a PID. By the structure theorem, we could write

$$L \cong K[x]/\langle d_1(x) \rangle \oplus \cdots \oplus K[x]/\langle d_s(x) \rangle \quad \text{with } d_i \mid d_{i+1}$$

Since  $\text{Id}, \sigma, \dots, \sigma^{n-1}$  are linearly independent over  $K$ ,  $m_{\sigma, K}$  should have degree at least  $n$ , thus it is clear that  $x^n - 1$  is the minimal polynomial of  $\sigma$ , thus  $d_s(x) = x^n - 1$ . But the characteristic polynomial of  $\sigma$  has degree at most  $n$ , thus  $d_1(x) \cdots d_s(x) = x^n - 1$ . So  $L \cong K[x]/\langle x^n - 1 \rangle$ . Let  $\alpha \in L$  such that  $\text{Ann}(\alpha) = \langle x^n - 1 \rangle$ , then  $L = K[x]\alpha$ . Hence  $L = \langle \alpha, \sigma(\alpha), \dots, \sigma^{n-1}(\alpha) \rangle$ .

Case 2:  $|K| = \infty$ . Let  $\text{Gal}(L/K) = \{\sigma_1, \dots, \sigma_n\}$ . Define  $y_{i,j} = x_k$  so that  $\sigma_i \sigma_j = \sigma_k$ . Consider

$$f(x_1, \dots, x_n) = \begin{vmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n} \end{vmatrix}$$

This determinant is a non-zero polynomial in  $x_1, x_2, \dots, x_n$ . Since if we fix  $\sigma_1$ , for each  $\sigma_i$ , exists unique  $j$  so that  $\sigma_i \sigma_j = \sigma_1$ . So the determinant has a  $x_1^n$  term and is not zero. Then  $f(\sigma_1, \dots, \sigma_n) \neq 0$  by lemma 18. Thus there exists  $\alpha \in L$  s.t.  $\det([\sigma_i \sigma_j(\alpha)]) = f(\sigma_1, \dots, \sigma_n)(\alpha) \neq 0$ . So by lemma 17,  $\{\sigma_i(\alpha)\}$  is a basis.  $\square$

## 6 Commutative Algebra

### 6.1 ED, PID and UFD

We shall consider  $R$  to be a integral domain below.

**Def 102.** A function  $N : R \rightarrow \mathbb{N}$  with  $N(0) = 0$  is called a norm on  $R$ .

**Def 103.**  $R$  is called a Euclidean domain if exists a norm  $N$  on  $R$  satisfy

$$\forall a, b \in R, \exists q, r \in R \text{ s.t. } a = qb + r \text{ with } r = 0 \text{ or } N(r) < N(b)$$

**Eg 6.1.1.**

- $\mathbb{Z}$  is a ED with  $N(n) = |n|$ .
- $K[x]$  is a ED with  $N(f) = \deg f, \forall f \in K[x]$ .

**Def 104.**  $A_d$  is defined to be the ring of integers in the quadratic field  $\mathbb{Q}(\sqrt{d})$  with  $d \neq 1$  and  $d$  is square-free. That is,

$$A_d \triangleq \{ \alpha \in \mathbb{Q}(\sqrt{d}) \mid \alpha \text{ is integral over } \mathbb{Z} \}$$

**Theorem 71.**

- If  $d \equiv 1 \pmod{4}$ , then

$$A_d = \left\{ a + b \frac{1 + \sqrt{d}}{2} : a, b \in \mathbb{Z} \right\}$$

- Else,  $d \equiv 2, 3 \pmod{4}$ , then

$$A_d = \{ a + b\sqrt{d} : a, b \in \mathbb{Z} \}$$

**Theorem 72.**  $A_d$  is a ED if  $d = 2, 3, 5, -1, -2, -3, -7, -11$ . Hence  $A_d$  is also PID and UFD.

**Eg 6.1.2.**  $A_{-5}$  is not a ED.

*Proof.* Consider  $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ . Notice that  $1 + \sqrt{-5}$  is irreducible, since if  $1 + \sqrt{-5} = \alpha\beta$ , then  $6 = N(1 + \sqrt{-5}) = N(\alpha)N(\beta)$ . But there is  $a^2 + 5b^2 = 2$  or  $3$  has no integer solution. Also  $1 + \sqrt{-5} \nmid 2, 3$ . Since if  $(1 + \sqrt{-5})\alpha = 2$ , then  $N(1 + \sqrt{-5})N(\alpha) = N(2)$ , but  $N(1 + \sqrt{-5}) = 6$ .  $\square$

#### 6.1.1 $A_{-1}$ and $A_{-3}$

First,  $\alpha$  is a unit  $\iff N(\alpha) = 1$ . so we have:

- $A_{-1}$ :  $\pm 1, \pm i$ .
- $A_{-3}$ :  $\pm 1, \pm \omega, \pm \omega^2$ .

If  $\alpha$  is a prime in  $A_{-1}$  or  $A_{-3}$ , then  $N(\alpha) = p$  or  $p^2$  for some prime integer  $p$ .

Let  $N(\alpha) = \alpha\bar{\alpha} = p_1 \cdots p_n$  in  $\mathbb{Z}$

**Def 105.** If  $p$  is add and  $a \not\equiv 0 \pmod{p}$ , then

- If  $x^2 \equiv a \pmod{p}$  is solvable, then define  $\left(\frac{a}{p}\right) = 1$ .
- Else  $x^2 \equiv a \pmod{p}$  is not solvable and define  $\left(\frac{a}{p}\right) = -1$ .

**Prop 6.1.1.**

- $a \equiv b \pmod{p} \implies \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$ .

## 6.2 Primary decomposition

**Def 106.**

- The radical of an ideal  $I$  is defined by  $\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$ .
- $I$  is radical if  $\sqrt{I} = I$ .

**Def 107.** The **nilradical** is defined as  $\sqrt{\langle 0 \rangle} \triangleq \{a \in R \mid a^n = 0 \text{ for some } n \in \mathbb{N}\}$ . Elements in it are called nilpotent.

**Prop 6.2.1.**  $\sqrt{\langle 0 \rangle} = \bigcap_{P \in \text{Spec } R} P$ , where  $\text{Spec } R$  is the set of prime ideals in  $R$ .

*Proof.* “ $\subset$ ”: Notice that  $a^n = 0 \in P$  for any prime ideal  $P$ . By the definition of prime ideal, either  $a \in P$  or  $a^{n-1} \in P$ . No matter which, eventually we would get  $a \in P$ .

“ $\supset$ ”: Let  $\mathcal{S} \triangleq \{I : \text{ideal in } R \mid a^n \notin I, \forall n \in \mathbb{N}\}$ . By the routine argument of Zorn’s lemma, exists maximal element  $Q$  in  $\mathcal{S}$ . We claim that  $\mathcal{S}$  is a prime ideal.

For each  $x, y \notin Q$ , we have  $Q + Rx \supsetneq Q$  and  $Q + Ry \supsetneq Q$ . By the maximality of  $Q$ , these two ideals are not in  $\mathcal{S}$ . So exists  $n, m$  such that  $a^n \in Q + Rx$ ,  $a^m \in Q + Ry$  which implies  $a^{n+m} \in Q + Rxy$ , so  $Q + Rxy \notin \mathcal{S}$ , thus  $xy \notin Q$ , hence  $Q$  is prime.  $\square$

**Coro 6.2.1.**

$$\sqrt{I} = \bigcap_{\substack{P \supset I \\ P \in \text{Spec } R}} P$$

*Proof.* Notice that  $\text{Spec } R/I = \{P \in \text{Spec } R \mid R \subset I\}$ . By the proposition above,

$$\sqrt{\langle \bar{0} \rangle} = \bigcap_{\bar{P} \in \text{Spec } R/I} \bar{P} \implies \sqrt{I} = \bigcap_{\substack{P \supset I \\ P \in \text{Spec } R}} P$$

$\square$

**Def 108.** An ideal  $q$  of  $R$  is called primary if  $q \neq R$  and “ $xy \in q$  and  $x \notin q$ ” implies  $y^n \in q$  for some  $n \in \mathbb{N}$ .

**Prop 6.2.2.**

- prime  $\implies$  primary.
- $\sqrt{\text{primary}} \implies$  prime. Also, if  $q$  is primary, then  $p = \sqrt{q}$  is the smallest prime ideal containing  $q$ , we say  $q$  is  $p$ -primary.

*Proof.* The first one is obvious.

If  $q$  is primary and  $\sqrt{q} = p$ . For any  $xy \in p$  and  $x \notin p$ , there exists  $n$  so that  $x^n y^n \in q$ , and for this  $n$ ,  $x^n \notin q$ . Thus  $(y^n)^m \in q$  for some  $m$ , hence  $y \in p$ . We conclude that  $p$  is a prime ideal.

Finally, by corollary 6.2.1,

$$p = \sqrt{q} = \bigcap_{\substack{P \supset q \\ P \in \text{Spec } R}} P \subset P, \quad \forall P \text{ prime},$$

thus  $p$  is indeed the smallest.  $\square$

**Eg 6.2.1.** The primary ideals in  $\mathbb{Z}$  are  $\langle 0 \rangle$  and  $\langle p^m \rangle$  where  $p$  is a prime.

*Proof.* If  $q = \langle a \rangle$  is primary, then  $\sqrt{q} = \langle p \rangle$  is prime, and  $p^n \in \langle a \rangle$ . So  $ab = p^n$  which implies  $a = p^m$  for some  $m$ .  $\square$

**Def 109.** An ideal  $I$  is said to be **irreducible** if  $I = q_1 \cap q_2 \implies I = q_1 \vee I = q_2$ .

**Def 110.** Define  $(I : x) = \{a \in R \mid ax \in I\}$ .

**Theorem 73.** In a Noetherian ring  $R$ , every irreducible ideal  $I$  is primary.

*Proof.* Let  $xy \in I$  and  $x \notin I$ . Consider  $(I : y) \subseteq (I : y^2) \subseteq \dots$ . Since  $R$  is Noetherian, exists  $n$  such that  $(I : y^n) = (I : y^m)$  for any  $m \geq n$ .

We claim that  $I = (I + Ry^n) \cap (I + Rx)$ .

- “ $\subseteq$ ”: Obvious.
- “ $\supseteq$ ”: For any  $b \in (I + Ry^n) \cap (I + Rx)$ , write  $b = a_1 + r_1y^n = a_2 + r_2x$ . Then  $r_1y^{n+1} = a_2y - a_1y + r_2xy \in I$  since  $a_1, a_2, xy \in I$ . So  $r_1 \in (I : y^{n+1}) = (I : y^n) \implies r_1y^n \in I$ . Thus  $b = a_1 + r_1y^n \in I$ .

Now by the fact that  $I$  is irreducible and  $I \neq I + Rx$  since  $x \notin I$ , thus  $I = I + Ry^n \implies y^n \in I$ .  $\square$

**Theorem 74.** In a Noetherian ring  $R$ , every ideal is a finite intersection of irreducible ideals.

*Proof.* If not, let  $\mathcal{I} \triangleq \{I : \text{ideal in } R \mid I \text{ is not a finite intersection of irreducible ideals}\}$  and  $\mathcal{I}$  is not an empty set. Since  $R$  is Noetherian, the set has a maximal element  $I_0$ . Then  $I_0$  is not irreducible (or else it is an intersection of itself, which is irreducible). Write  $I_0 = I_1 \cap I_2$ , with  $I_1, I_2 \neq I_0$ . Then  $I_1, I_2 \notin \mathcal{I}$ , so these two ideals could be written as a finite intersection of irreducible ideals, implying that  $I_0$  could also be written as a finite intersection of irreducible ideals, which is a contradiction.  $\square$

**Prop 6.2.3.** Let  $q$  be a  $p$ -primary ideal and  $x \in R$ .

1. If  $x \in q$ , then  $(q : x) = R$ .

*Proof.* In this case  $1 \in (q : x)$ , thus  $(q : x) = R$ .  $\square$

2. If  $x \notin q$ , then  $(q : x)$  is  $p$ -primary.

*Proof.* For any  $y \in (q : x)$ ,  $xy \in q$  but  $x \notin q$ , thus  $y^n \in q \implies y \in p$ . Hence

$$q \subset (q : x) \subset p \implies p = \sqrt{q} \subset \sqrt{(q : x)} \subset \sqrt{p} = p$$

and thus  $(q : x)$  is  $p$ -primary.

For any  $y, z$  with  $yz \in (q : x)$  but  $y \notin (q : x)$ , which is equivalent to  $xyz \in q$  but  $xy \notin q$ . Since  $q$  primary,  $z^n \in q \subset (q : x)$ .  $\square$

3. If  $x \notin p$ , then  $(q : x) = q$ .



*Proof.*

$$\begin{cases} y \in (q : x) \\ x \notin p \end{cases} \implies \begin{cases} xy \in (q : x) \\ x^n \notin q, \forall n \in \mathbb{N} \end{cases} \implies y \in q$$

□

**Prop 6.2.4.** If each  $q_i$  are  $p$ -primary, then  $q \triangleq \bigcap_{i=1}^n q_i$  is  $p$ -primary.

*Proof.* We check that  $\sqrt{q} = \bigcap_{i=1}^n \sqrt{q_i} = \bigcap_{i=1}^n p = p$ .

Also, if  $xy \in q$  with  $x \notin q$ , then  $x \notin q_k$  for some  $k$ . But  $xy \in q_k$ , thus  $y^n \in q_k$ . Since  $\sqrt{q} = q_k$ ,  $(y^n)^{m'} = y^m \in p \subset q$ , thus  $q$  is  $p$ -primary. □

**Def 111.** A **primary decomposition** of  $I = q_1 \cap \dots \cap q_n$  is **minimal** if  $\sqrt{q_1}, \dots, \sqrt{q_n}$  are distinct and  $q_i \not\supseteq \bigcap_{j \neq i} q_j$ .

A minimal primary decomposition of an ideal always exists in Noetherian ring since by theorem 74, the ideal could be written as a finite intersection of irreducible ideals, and then by theorem 73, these ideals are primary. Now If  $\sqrt{q_i} = \sqrt{q_j}$  happen in these ideal, we could remove these two ideals and add  $q' = \sqrt{q_i} \cap \sqrt{q_j}$ . By proposition 6.2.4,  $q'$  is also primary. And if  $q_i \subseteq \bigcap_{j \neq i} q_j$ , we could simply remove  $q_i$ .

**Theorem 75** (Uniqueness of primary decomposition). Let  $I = \bigcap_{i=1}^n q_i$  be a minimal decomposition of  $I$ . If  $p_i = \sqrt{q_i}, \forall i$ , then we have

$$\{p_i\} = \left\{ \sqrt{(I : x)} \mid x \in R \wedge \sqrt{(I : x)} \in \text{Spec } R \right\}$$

which is independent of the decomposition.

*Proof.* “ $\supset$ ”: Let  $x \in R \setminus I$ , then  $(I : x) = (\bigcap_{i=1}^n q_i : x) = \bigcap_{i=1}^n (q_i : x)$ . By proposition 6.2.3, we have  $\sqrt{(I : x)} = \bigcap \sqrt{(q_i : x)} = \bigcap_{x \notin q_i} p_i$ .

Now, we have the following observation. “If  $p \in \text{Spec } R$  with  $p = \bigcap_{i=1}^n J_i$ , then  $p = J_j$  for some  $j$ .” If not, then  $J_i \not\subset p$  for all  $i$ , so we could pick  $x_i \in J_i \setminus p$ . But then  $x_1 x_2 \dots x_n \in \bigcap J_i \in p$  since  $J_i$  are ideals, which leads to a contradiction since  $p$  is prime.

So if  $\sqrt{(I : x)}$  is a prime, then it is equal to some  $p_i$ .

“ $\subset$ ”: By assumption,  $q_i \not\supseteq \bigcap_{j \neq i} q_j$  for each  $i$ , thus we could pick  $x \in \bigcap_{j \neq i} q_j \setminus q_i$ , then  $\sqrt{(I : x)} = \bigcap_j \sqrt{(q_j : x)} = \sqrt{(q_i : x)} = p_i$ . □

**Def 112.** If  $\{p_i\}$  is the unique prime ideals from the minimal primary decomposition of  $I$ .

- $\{p_i\}$  is said to be associated with  $I$  or to belong to  $I$ .
- The minimal elements in  $\{p_i\}$  are called isolated primes.
- The other are called embedded primes.

**Eg 6.2.2.** Let  $R = k[x, y]$  and  $I = \langle x^2, xy \rangle$ . If  $P_1 = \langle x \rangle, P_2 = \langle x, y \rangle$ , then  $I = P_1 \cap P_2^2$ .  $P_1$  is isolated, while  $P_2$  is embedded.

### 6.3 The equivalence of algebra and geometry

In the following,  $k$  will be an algebraically closed field.

**Def 113.** The category of affine algebraic sets  $\mathcal{G}$ , which its objects and morphisms are defined as following.

**objects:** The objects are affine algebraic sets in  $k^n$ .

An **affine algebraic set** is the common zero set of  $\{F_i\}_{i \in \Lambda} \subset k[x_1, \dots, x_n]$  in  $k^n$ . We denote it by  $V = \mathcal{V}(\{F_i\}_{i \in \Lambda}) \subset k^n$ . (In fact,  $I = \langle F_i : i \in \Lambda \rangle$  is Noetherian, so  $I = \langle F_1, \dots, F_n \rangle$  and  $V = \mathcal{V}(I)$ .)

**morphisms:** The morphisms are the polynomial map from  $k^n$  to  $k^m$ .

A **polynomial map** is a mapping as following:

$$\begin{aligned} k^n &\longrightarrow k^m \\ \alpha &\longmapsto (F_1(\alpha), \dots, F_m(\alpha)) \end{aligned}$$

where each  $F_i$  is a polynomial in  $K[x_1, \dots, x_n]$ .

Given two affine algebraic sets  $V \subset k^n$  and  $W \subset k^m$ , if a map  $F : V \rightarrow W$  is the restriction of a polynomial map from  $k^n$  to  $k^m$ , then  $F$  is a morphism from  $V$  to  $W$ .

Moreover, if  $F : V \rightarrow W$  and  $G : W \rightarrow V$  satisfy  $F \circ G = \text{Id}$  and  $G \circ F = \text{Id}$ , then we say  $V \cong W$ .

**Def 114.** The category of finitely generated reduced  $k$ -algebra  $\mathcal{A}$ , which its objects and morphisms are defined as following.

**objects:** The objects are the reduced finitely generated  $k$ -algebra  $R$ .

A finitely generated  $k$ -algebra  $R$  is reduced if  $R$  has no non-zero nilpotent elements.

**morphisms:** The morphisms are the  $k$ -algebra homomorphisms.

**Eg 6.3.1.** It is easy to see that  $\mathcal{V}(0) = k^n$  and  $\mathcal{V}(1) = \emptyset$ .

#### 6.3.1 One-one correspondence between affine algebraic sets and radical ideals

**Def 115.** Define  $\mathcal{I}(V) = \{f \in k[x_1, \dots, x_n] \mid f(\alpha) = 0, \forall \alpha \in V\}$ .

The one-one correspondence is given by

$$\begin{aligned} \{\text{affine algebraic sets in } \mathbb{A}_k^n\} &\longleftrightarrow \{\text{radical ideals in } k[x_1, \dots, x_n]\} \\ V &\longmapsto \mathcal{I}(V) \\ \mathcal{V}(I) &\longleftarrow I \end{aligned}$$

**Prop 6.3.1.**

- $\sqrt{\mathcal{I}(V)} = \mathcal{I}(V)$ .

*Proof.* For all  $f^n \in \mathcal{I}(V)$ ,  $f^n(\alpha) = 0, \forall \alpha \in V \implies f(\alpha) = 0, \forall \alpha \in V$ . Thus  $f \in \mathcal{I}(V)$ .  $\square$

- If  $V$  is an affine set, then  $\mathcal{V}(\mathcal{I}(V)) = V$ .

*Proof.* “ $\supset$ ”:  $\forall \alpha \in V, f \in \mathcal{I}(V), f(\alpha) = 0 \implies \alpha \in \mathcal{V}(\mathcal{I}(V))$ .

“ $\subset$ ”: Since  $V$  is an affine set,  $V = \mathcal{V}(I)$ , then  $I \subset \mathcal{I}(V)$ , so  $\mathcal{V}(\mathcal{I}(V)) \subset \mathcal{V}(I) = V$ .  $\square$

**Lemma 19.** Given  $T/S/R$ , a tower of rings. If  $R$  is Noetherian,  $T/S$  is a module finite and  $T/R$  is a ring finite, then  $S/R$  is a ring finite.

*Proof.* Let  $T = R[a_1, \dots, a_n] = S\omega_1 + \dots + S\omega_m$ . Then  $a_i = \sum_j r_{i,k} \omega_k$  for some  $r_{i,k}$  and  $\omega_{i,j} = \sum t_{i,j,k} w_k$  for some  $t_{i,j,k}$ .

Let  $S' = R[\{r_{i,k}\}, \{t_{i,j,k}\}] \subseteq S$ , which is Noetherian by the Hilbert basis theorem ( $R$  Noetherian  $\implies R[x]$  Noetherian). Thus  $T = S'\omega_1 + \dots + S'\omega_m$  is a Noetherian  $S'$ -module by the fact that finitely generated module over a Noetherian ring is a Noetherian module.

Since  $S \subset T$ ,  $S$  is a finitely generated  $S'$  submodule, so  $S = S'v_1 + \dots + S'v_r = R[\{r_{i,k}\}, \{t_{i,j,k}\}, \{v_i\}]$ .  $\square$

**Lemma 20.** If  $S = k(z_1, \dots, z_p)$ ,  $p > 0$  with each  $z_i$  transcendental, then  $S/k$  is not ring finite.

*Proof.* If not, say  $S = k[f_1, \dots, f_n]$  with  $f_i = g_i/h_i$ ,  $g_i, h_i \in k[z_1, \dots, z_p]$ . Then for any irreducible polynomial  $p$  such that  $p \nmid h_i$  for each  $h_i$  (This polynomial exists since for each  $h_i$  there are only finite degree 1 factors). Then  $1/p \notin k[f_1, \dots, f_n]$  by checking the divisibility of the denominator under addition and multiplication, which leads to a contradiction.  $\square$

**Lemma 21.** If  $A/k$  is an extension of fields and ring finite, then  $A/k$  is algebraic.

*Proof.* If  $A/k$  is transcendental and let  $\{z_1, \dots, z_t\}$  be a transcendental base. Then  $A/k(z_1, \dots, z_t)$  is algebraic, thus a module finite. By lemma 19,  $k(z_1, \dots, z_t)$  is ring finite, which contradict with lemma 20.  $\square$

**Theorem 76** (Weak form of Hilbert Nullstellensatz).

$$I \subsetneq k[x_1, \dots, x_n] \implies v(I) \neq \emptyset$$

*Proof.* Since  $I$  proper, by lemma 7, exists a maximal ideal  $M$  such that  $I \subseteq M$ . Consider  $K \triangleq k[x_1, \dots, x_n]/M = k[\bar{x}_1, \dots, \bar{x}_n]$ . By proposition 5.1.8,  $K$  is a field, and by lemma 21,  $K/k$  is algebraic. Since  $k$  is already algebraically closed,  $K = k$  and hence each  $\bar{x}_i \in k$ . Let  $\alpha \triangleq (\bar{x}_1, \dots, \bar{x}_n) \in A_k^n$ , then for any  $f \in M$ ,  $f(\alpha) = f(\bar{x}_1, \dots, \bar{x}_n) = \bar{f} = 0$ , thus  $\alpha \in \mathcal{V}(M) \subseteq \mathcal{V}(I)$ .  $\square$

**Theorem 77** (Strong form of Hilbert Nullstellensatz).  $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$

*Proof.* “ $\supset$ ”: If  $f^n \in I$ , then  $f(\alpha) = 0, \forall \alpha \in \mathcal{V}(I) \implies f^n(\alpha) = 0, \forall \alpha \in \mathcal{V}(I)$ , thus  $f^n \in \mathcal{I}(\mathcal{V}(I))$ .

“ $\subset$ ”: If  $\mathcal{I}(\mathcal{V}(I)) = 0$ , then  $I \subseteq \sqrt{I} \subseteq \mathcal{I}(\mathcal{V}(I)) = 0$ , thus  $I = 0$ .

Otherwise, exists  $0 \neq f \in \mathcal{I}(\mathcal{V}(I))$ , Let  $J = \langle I, ft - 1 \rangle \subset k[x_1, \dots, x_n, t]$ . If  $(a_1, \dots, a_n, t_0)$  is a zero of  $J$ , then  $ft - 1 \in J \implies -1 = f(a_1, \dots, a_n)t_0 - 1 = 0$ , which is a contradiction, so by theorem 76,  $J = k[x_1, \dots, x_n, t]$ .

Write  $1 = \sum h_i f_i + s(ft - 1)$ , where each  $f_i \in I$  and  $h_i, s \in k[x_1, \dots, x_n, t]$ . This is a equation of variables, so if we set  $t = 1/f$ , the equation still holds. Now each  $h_i$  would be the form  $\sum p_i/f^{k_i}$ , so we could multiply each side by a suitable  $f^\rho$  and get  $f^\rho = \sum c_i f_i$  with each  $c_i \in k[x_1, \dots, x_n]$ . This implies  $f^\rho \in I$ , thus  $f \in \sqrt{I}$ .  $\square$

**Def 116.** Let  $V \in \mathcal{G}$ , the coordinate ring of  $V$  is  $k[V] \triangleq k[x_1, \dots, x_n]/\mathcal{I}(V)$

### 6.3.2 Equivalence of $\mathcal{G}$ and $\mathcal{A}$

We define a functor  $F$  from  $\mathcal{G}$  to  $\mathcal{A}$  by

$$\begin{aligned} F : \quad \mathcal{G} &\longrightarrow \mathcal{A} \\ V &\longmapsto k[V] \end{aligned}$$

And For a polynomial map  $f : V \rightarrow W$ , define

$$\begin{aligned} F(f) = f^* : \quad k[W] &\longrightarrow k[V] \\ g &\longmapsto g \circ f \end{aligned}$$

Conversely, define a functor  $G$  by

$$\begin{aligned} G : \quad \mathcal{A} &\longrightarrow \mathcal{G} \\ k[x_1, \dots, x_n]/I &\longmapsto \mathcal{V}(I) \end{aligned}$$

Then if

$$\begin{aligned} \varphi : \quad k[\dots]/I &\longrightarrow k[\dots]/J \\ \bar{x}_i &\longmapsto \bar{f}_i \end{aligned}$$

Define

$$\begin{aligned} G(\varphi) = \psi : \quad \mathcal{V}(J) &\longrightarrow \mathcal{V}(I) \\ \alpha = (a_1, \dots, a_m) &\longmapsto (f_1(\alpha), \dots, f_n(\alpha)) \end{aligned}$$

## 6.4 Gröbner basis

### 6.4.1 Division algorithm in $K[X_1, \dots, X_n]$

**Eg 6.4.1.**  $I = \langle xy - 1, y^2 - 1 \rangle \subseteq K[x, y]$ ,  $f_1 = xy - 1$  and  $f_2 = y^2 - 1$ ,  $G = \{f_1, f_2\}$ . Does  $f = x^2y + xy^2 + y^2 \in I$ ?

- Choose a lexicographic monomial ordering:  $x > y$
- The multidegree  $\partial(f) = (2, 1)$ ,  $\partial(f_1) = (1, 1)$ ,  $\partial(f_2) = (0, 2)$
- The leading term  $\text{LT}(f) = x^2y$ ,  $\text{LT}(f_1) = xy$ ,  $\text{LT}(f_2) = y^2$
- $\text{LT}(f) = x \text{LT}(f_1) \Rightarrow f = x f_1 + xy^2 + y^2 + x \Rightarrow f = \underset{h_1}{(x+y)}f_1 + \underset{h_2}{(1)}f_2 + \underset{\bar{f}^G}{(x+y+1)}$  or  

$$f = \underset{h_1}{x}f_1 + \underset{h_2}{(x+1)}f_2 + \underset{\bar{f}^G}{(2x+1)}.$$

Note: Divisor  $h_1$ ,  $h_2$  and remainder  $\bar{f}^G$  are not unique!!

**Def 117.** Fix a monomial ordering and let  $I$  be an ideal of  $K[X_1, \dots, X_n]$ . The ideal of leading terms in  $I$  is defined to be  $\text{LT}(I) = \langle \text{LT}(f) \mid f \in I \rangle$ .

**Remark 34.** Let  $I = \langle f_1, \dots, f_n \rangle$ . In general,  $\langle \text{LT}(f_1), \dots, \text{LT}(f_n) \rangle \subsetneq \text{LT}(I)$ .

**Eg 6.4.2.** Let  $f_1 = xy^2 + y$ ,  $f_2 = x^2y$ . And,  $xf_1 - yf_2 = xy \in \langle f_1, f_2 \rangle$  but  $xy \notin \langle xy^2, x^2y \rangle$ .

**Def 118.**  $G = \{g_1, \dots, g_m\}$  is called a Gröbner basis of  $I$  if  $I = \langle g_1, \dots, g_m \rangle$  and  $\text{LT}(I) = \langle \text{LT}(g_1), \dots, \text{LT}(g_m) \rangle$ .

**Prop 6.4.1.** Let  $g_1, \dots, g_m \in I$ , then  $\text{LT}(I) = \langle \text{LT}(g_1), \dots, \text{LT}(g_m) \rangle \implies I = \langle g_1, \dots, g_m \rangle$ .

*Proof.*  $\forall f \in I$ , do the division process. Then  $f = \sum_{i=1}^m h_i g_i + r$ , either  $r = 0$  or  $\star = \text{no term of } r \text{ is divisible by any of } \text{LT}(g_1), \dots, \text{LT}(g_m)$ . Assume  $r \neq 0$ , then  $r = f - \sum_{i=1}^m h_i g_i \in I \Rightarrow \text{LT}(r) \in \text{LT}(I) = \langle \text{LT}(g_1), \dots, \text{LT}(g_m) \rangle$ , which is a contradiction. Hence,  $r = 0$  (i.e.  $f \in \langle g_1, \dots, g_m \rangle$ ).  $\square$

**Theorem 78.** Each ideal  $I$  has a Gröbner basis.

*Proof.* By Hilbert basis thm,  $\text{LT}(I) = \langle f_1, \dots, f_m \rangle$  for some  $f_i$ 's. Write  $f_i = \sum_{j=1}^{m_i} h_{ij} \text{LT}(g_{ij})$  with  $h_{ij} \in K[X_1, \dots, X_n]$ ,  $g_{ij} \in I$ . Then  $\text{LT}(I) = \langle \text{LT}(g_{ij}) \mid i = 1, \dots, m, j = 1, \dots, m_i \rangle$ . By prop 6.4.1, This is Gröbner basis.  $\square$

**Theorem 79.** Let  $G = \{g_1, \dots, g_m\}$  be a Gröbner basis of  $I$ , then

- $\forall f \in K[X_1, \dots, X_n]$ ,  $f = f_I + r$  where  $f_I \in I, r = \star$  are unique.

*Proof.* By division algorithm,  $f = f_I + \underset{\star}{r} = f'_I + \underset{\star}{r}'$ , then  $\underset{\star}{r} - \underset{\star}{r}' = f_I - f'_I$ . But if  $\underset{\star}{r} - \underset{\star}{r}' \neq 0$ , then  $\text{LT}(\underset{\star}{r} - \underset{\star}{r}') \in \text{LT}(I) = \langle \text{LT}(g_1), \dots, \text{LT}(g_m) \rangle$ , which is a contradiction. Hence,  $\underset{\star}{r} - \underset{\star}{r}' = 0 \Rightarrow f_I = f'_I$ .  $\square$

- $f \in I \iff r = 0$ .

*Proof.* Suppose  $f \in I$ , then  $f = f_I + \underset{\star}{r}$ , and if  $\underset{\star}{r} \neq 0$ ,  $\underset{\star}{r} = f - f_I \in I$ , which is a contradiction. Hence,  $\underset{\star}{r} = 0$ . Conversely, if  $\underset{\star}{r} = 0$ ,  $f = f_I \in I$ .  $\square$

### 6.4.2 Buchberger's algorithm

**Def 119.** Let  $f, g \in K[x_1, \dots, x_n]$  and  $M$  be the monic least common multiple of  $\text{LT}(f)$  and  $\text{LT}(g)$ .  $S(f, g) = \frac{M}{\text{LT}(f)}f - \frac{M}{\text{LT}(g)}g$  is called an S-polynomial of  $f, g$ .

Let  $I = \langle g_1, \dots, g_m \rangle$  and  $G = \{g_1, \dots, g_m\}$ . A Gröner basis of  $I$  can be constructed by the following algorithm:

1. Initially let  $G_0 \leftarrow G$ .
2. Repeatly construct  $G_{i+1} \leftarrow G_i \cup (\{S(f, g) \bmod G_i \mid f, g \in G_i\} \setminus \{0\})$ , until once  $G_{i+1} = G_i$ , then  $G_i$  is a Gröbner basis of  $I$ .

**Lemma 22.** Let  $f_1, \dots, f_m \in K[x_1, \dots, x_n]$  with  $a_1, \dots, a_m \in K$  satisfying  $\partial(f_1) = \partial(f_2) = \dots = \partial(f_m) = \alpha$  and  $h = \sum_{i=1}^m a_i f_i$  with  $\partial(h) < \alpha$ . Then  $h = \sum_{i=2}^m b_i S(f_{i-1}, f_i)$  for some  $b_i \in K$ .

*Proof.* Write  $f_i = c_i f'_i$  with  $c_i \in K$  and  $f'_i$  being monic of multidegree  $\alpha$ . Note:  $S(f_i, f_j) = f'_i - f'_j$  since all multidegree are equal. Then,

$$\begin{aligned} h &= \sum_{i=1}^m (a_i c_i f'_i) \\ &= a_1 c_1 (f'_1 - f'_2) + (a_1 c_1 + a_2 c_2)(f'_2 - f'_3) + \dots + (a_1 c_1 + \dots + a_{m-1} c_{m-1})(f'_{m-1} - f'_m) \\ &\quad + (a_1 c_1 + \dots + a_m c_m) f'_m \\ &= \sum_{i=2}^m b_i S(f_{i-1}, f_i) + b_{m+1} f'_m \text{ with } b_i = \sum_{j=1}^{i-1} a_j c_j. \end{aligned}$$

Also, in this equality,  $f'_m$  is the only term that has multidegree  $\alpha$  (other terms have multidegree less than  $\alpha$ ). So  $b_{m+1} = 0$  must hold. Then, we have  $h = \sum_{i=2}^m b_i S(f_{i-1}, f_i)$ .  $\square$

**Theorem 80** (Buchberger's criterion). Assume  $I = \langle g_1, \dots, g_m \rangle$ , then  $G = \{g_1, \dots, g_m\}$  is a Gröbner basis of  $I \iff S(g_i, g_j) \equiv 0 \pmod{G}$  for each  $i, j$ .

*Proof.*

- Suppose  $G$  is a Gröbner basis of  $I$ .  $S(g_i, g_j) \in I \Rightarrow S(g_i, g_j) = 0$  by thm 79.
- Converely, suppose  $S(g_i, g_j) \equiv 0 \pmod{G} \forall i, j$ . For  $f \in I$ ,  $f \underset{\text{not division}}{=} \sum_{i=1}^m h_i g_i$  for some  $h_i \in K[x_1, \dots, x_n]$ . Define  $\alpha = \max\{\partial(h_1 g_1), \dots, \partial(h_m g_m)\}$ . We have  $\partial(f) \leq \alpha$  and we can select an expression  $f = \sum_{i=1}^m h_i g_i$  for f s.t  $\alpha$  is minimal.
- Claim:  $\partial(f) = \alpha$
- (pf) Rewrite,

$$\begin{aligned} f &= \sum_{i=1}^m h_i g_i \\ &= \sum_{\partial(h_i g_i) = \alpha} h_i g_i + \sum_{\partial(h_i g_i) < \alpha} h_i g_i \quad (\text{the first term} \neq 0 \text{ since } \alpha \text{ is minimal.}) \\ &= \sum_{\partial(h_i g_i) = \alpha} \text{LT}(h_i) g_i + \sum_{\partial(h_i g_i) = \alpha} (h_i - \text{LT}(h_i) g_i) + \sum_{\partial(h_i g_i) < \alpha} h_i g_i \end{aligned}$$

Let  $\text{LT}(h_i) = a_i h_i^0$  with  $h_i^0$  being a monic monomial. Comparing the multidegree on both side,  $\partial\left(\sum_{\partial(h_i g_i) = \alpha} a_i h_i^0 g_i\right) < \alpha$  By lemma 22,  $\sum_{\partial(h_i g_i) = \alpha} (a_i h_i^0 g_i) = c_{12} S(h_{i_1}^0 g_{i_1}, h_{i_2}^0 g_{i_2}) + \dots$  (finite)

where  $\partial(h_{i_1}g_{i_1}) = \partial(h_{i_2}g_{i_2}) = \cdots = \alpha$ . By def, if we set  $\beta_{st} = M_{st}$  = the monic lcm of  $\text{LT}(g_{i_s}), \text{LT}(g_{i_t})$ , then

$$\begin{aligned} S(h_{i_s}^0 g_{i_s}, h_{i_t}^0 g_{i_t}) &= \frac{X^\alpha}{\text{LT}(h_{i_s}^0 g_{i_s})} h_{i_s}^0 g_{i_s} - \frac{X^\alpha}{\text{LT}(h_{i_t}^0 g_{i_t})} h_{i_t}^0 g_{i_t} \\ &= X^{\alpha-\beta_{st}} \left( \frac{X^{\beta_{st}}}{h_{i_s}^0 \text{LT}(g_{i_s})} h_{i_s}^0 g_{i_s} - \frac{X^{\beta_{st}}}{h_{i_t}^0 \text{LT}(g_{i_t})} h_{i_t}^0 g_{i_t} \right) \\ &= X^{\alpha-\beta_{st}} S(g_{i_s}, g_{i_t}) \\ &= X^{\alpha-\beta_{st}} \sum_{j=1}^m l_j g_j \text{ (by division)} \end{aligned}$$

- Then,  $\partial(l_j g_j) < \beta_{st} \implies \partial(f) \geq \alpha$ . Therefore,  $\partial(f) = \alpha \implies \text{LT}(f) = \sum_{\partial(h_i g_i) = \alpha} \text{LT}(h_i) \text{LT}(g_i) \implies \text{LT}(f) \in \langle \text{LT}(g_1), \dots, \text{LT}(g_m) \rangle$ .

□

**Theorem 81.** The Buchberger's algorithm will terminate

*Proof.* .

- $\langle \text{LT}(G_i) \rangle \subsetneq \langle \text{LT}(G_{i+1}) \rangle$  if  $G_i \neq G_{i+1}$

$$G_i \neq G_{i+1} \implies \exists f, g \in G_i \text{ s.t. } S(f, g) \not\equiv 0 \pmod{G} \implies \text{LT}(S(f, g)) \notin \langle \text{LT}(G_i) \rangle$$

- $\langle \text{LT}(G_0) \rangle \subsetneq \langle \text{LT}(G_1) \rangle \subsetneq \cdots$  is not possible since  $K[x_1, \dots, x_n]$  is a Noetherian ring. (Noetherian ACC condition).

□

## 6.5 Applications of Gröbner basis

**Def 120.** Let  $I \subseteq K[x_1, \dots, x_n]$  and  $x_1 > x_2 > \cdots > x_n$ .  $I_i \triangleq I \cap K[x_{i+1}, \dots, x_n]$  is called the  $i$ -th elimination ideal of  $I$ .

**Theorem 82** (Elimination theorem). Let  $G = \{g_1, \dots, g_m\}$  be a Gröbner basis of  $I \neq 0$  with ordering  $x_1 > \cdots > x_n$ . Then  $G_i \triangleq G \cap K[x_{i+1}, \dots, x_n]$  is a Gröbner basis of  $I_i$  (i.e.,  $\langle \text{LT}(G_i) \rangle = \text{LT}(I_i)$ ).

**Eg 6.5.1.** Find  $V = \mathcal{V}(x + y - z, x^2 + y^2 - z^3, x^3 + y^3 - z^5)$ .

We compute a Gröbner basis of  $I = \langle f_1, \dots, f_3 \rangle$  with respect to the ordering  $x > y > z$ . The Gröbner basis is  $\{x + y - z, 2y^2 - 2yz - z^3 + z^2, 2z^5 - 3z^4 + z^3\}$ .

**Eg 6.5.2.**

$$\begin{aligned} f: \mathbb{A}^1 &\longrightarrow \mathbb{A}^3 \\ t &\longmapsto (t^4, t^3, t^2) \end{aligned}$$

We compute a Gröbner basis of  $I = \langle t^4 - x, t^3 - y, t^2 - z \rangle$  with respect to  $t > x > y > z$ . The Gröbner basis is  $\{-t^2 + z, ty - z^2, tz - y, x - z^2, y^2 - z^3\}$ .

**Eg 6.5.3.**

$$\begin{aligned} f : V = \mathcal{V}(x^3 - x^2z - y^2z) &\longrightarrow \mathbb{A}^3 \\ (x, y, z) &\longmapsto (x^2z - y^2z, 2xyz, -z^3) \end{aligned}$$

The ideal is  $\langle x^3 - x^2z - y^2z, u - x^2z + y^2z, v - 2xyz, w + z^3 \rangle$  has a Gröbner basis  $\langle \dots, u^2 + v^2 - w^2 \rangle$ .

**Theorem 83.** Let  $I, J$  be two ideals of  $K[x_1, \dots, x_n]$ , then  $I \cap J = (t\tilde{I} + (1-t)\tilde{J}) \cap K[x_1, \dots, x_n]$ .

**Eg 6.5.4.**  $I = \langle y^2, x - yz \rangle$ ,  $J = \langle x, z \rangle$ . We shall find  $I \cap J$ .

$tI + (1-t)J = \langle tx - tyz, ty^2, (1-t)x, (1-t)z \rangle$  has a Gröbner basis  $\{f_1, f_2, f_3, f_4, xy, x - yz\}$ , so  $I \cap J = \langle xy, x - yz \rangle$ .

**Theorem 84.** Let  $L = \langle f_1, \dots, f_s \rangle \subsetneq K[x_1, \dots, x_n]$ , then  $f \in \sqrt{I} \iff \langle f_1, \dots, f_s, 1 - tf \rangle = K[x_1, \dots, x_n, t]$ .

**Eg 6.5.5.** Let  $I = \langle xy^2 + 2y^2, x^4 - 2x^2 + 1 \rangle$ , and we are to determine  $f = y - x^2 + 1$  is in  $\sqrt{I}$  or not.

**Prop 6.5.1.** An affine algebraic set  $V$  in  $\mathbb{A}_k^n$  has a unique minimal decomposition.  $V = V_1 \cap V_2 \cap \dots \cap V_m$  with  $V_i$  irreducible and  $V_i \not\subset V_j$ .

**Theorem 85 (Decomposition).** Assume  $\sqrt{I} = I$  and  $I \subset J$ , then  $\mathcal{V}(I : J) = \overline{\mathcal{V}(I) \setminus \mathcal{V}(J)}$ . and  $\mathcal{V}(I) = \mathcal{V}(J) \cup \mathcal{V}(I : J)$ .

**Eg 6.5.6.** Let  $I = \langle xz - y^2, x^3 - yz \rangle$  and  $V = \mathcal{V}(I)$ .

Notice that  $\langle xz - y^2, x^3 - yz \rangle \subseteq \langle x, y \rangle = J$ , so  $(I : J) = (I : \langle x \rangle) \cap (I : \langle y \rangle)$ .

First we calculate  $(I : x)$ . Notice that we know how to calculate  $I \cap \langle x \rangle$  now. After a calculation,  $I \cap \langle x \rangle = \{x^2z - xy^2, x^4 - xyz, x^3y - xz^2\}$ , so  $(I : x) = \langle f_1/x, f_2/x, f_3/x \rangle = I + \langle x^2y - z^3 \rangle$ . Similarly one could find that  $(I : y) = (I : x)$ , thus  $(I : J) = (I : x)$ .

Hence  $V = \mathcal{V}(x, y) \cap \mathcal{V}(xz - y^2, x^3 - yz, x^2y - z^2)$ .

**Prop 6.5.2.** In general, if  $W \subseteq \mathbb{A}_k^n$  is an affine algebraic set defined by  $x_i = f_i(t_1, \dots, t_m)$ , then  $W$  is irreducible.

**Prop 6.5.3.** If  $f : V \rightarrow W$  with  $\overline{f(V)} = \mathcal{V}(\ker f^*)$ , where  $f^* : k[W] \rightarrow k[V]$ .



## 6.6 Local Rings

From now on,  $R$  is a commutative ring with 1.

**Def 121.**  $R$  is called a local ring if it has a unique maximal ideal.

**Prop 6.6.1.**

- (1)  $R$  is a local ring.
- (2) The set of non-units forms an ideal.
- (3)  $\exists M \in \text{Max } R$  s.t.  $1 + m$  is a unit  $\forall m \in M$ .

*Proof.*

- (1)  $\Rightarrow$  (2): Let  $M$  be the unique maximal ideal of  $R$ . Then  $R \setminus M = \{\text{units in } R\}$  since  $M$  is unique  $\rightsquigarrow M = \{\text{non-units}\}$ .
- (2)  $\Rightarrow$  (3): This ideal must be a maximal ideal  $M$  since it can't be extended. Now,  $1 \notin M \rightsquigarrow 1 + m \notin M$ . So  $1 + m$  is a unit.
- (3)  $\Rightarrow$  (1): If there exists another maximal ideal  $N$ , then  $M + N = R$ . Say  $m \in M, n \in N$  s.t.  $m + n = 1$ , then  $n = 1 - m$  is a unit  $\implies N = R$ , which is a contradiction.

□

**Eg 6.6.1.**  $k[[x]]$  is a local ring with the unique maximal ideal  $\langle x \rangle$ .

**Prop 6.6.2.** The following sets are correspondent ( $k$  is algebraically closed):

- (1)  $\mathbb{A}_k^n$
- (2)  $\text{Max } k[x_1, \dots, x_n]$
- (3)  $\text{Hom}_k(k[x_1, \dots, x_n], k)$

*Proof.*

□

**Remark 35.** Inspired by the correspondence,

**Eg 6.6.2.** Let  $P \in \text{Spec } R$ . If  $S = R \setminus P$ , then  $S$  is multiplicatively closed set with  $1 \in S$  and  $R_P \triangleq R_S$  is a local ring.

**Def 122.** A property of an  $R$ -module  $M$  is said to be a local property if

$$M \text{ has this property} \iff M_P \text{ (as an } R_P\text{-module) has this property } \forall P \in \text{Spec } R$$

**Prop 6.6.3.** TFAE

- (1)  $M = 0$
- (2)  $M_P = 0 \quad \forall P \in \text{Spec } R$
- (3)  $M_Q = 0 \quad \forall Q \in \text{Max } R$

**Coro 6.6.1.** Let  $N \subseteq M$ , TFAE (consider  $M/N$ )

- (1)  $N = M$
- (2)  $N_P = M_P \quad \forall P \in \text{Spec } R$

$$(3) N_Q = M_Q \quad \forall Q \in \text{Max } R$$

**Prop 6.6.4.** TFAE

- (1)  $0 \rightarrow M \xrightarrow{\phi} N$  exact
- (2)  $0 \rightarrow M_P \xrightarrow{\phi_P} N_P$  exact  $\forall P \in \text{Spec } R$
- (3)  $0 \rightarrow M_Q \xrightarrow{\phi_Q} N_Q$  exact  $\forall Q \in \text{Max } R$

**Def 123.**

- Let  $R \subseteq S$ .  $\bar{R} = \{x \in S \mid x \text{ is integral over } R\}$  is called the integral closure of  $R$  in  $S$ .
- $R$  is integrally closed in  $S$  if  $R = \bar{R}$ .
- An integral domain  $R$  is called normal if  $R$  is integrally closed in its field of fractions.

**Theorem 86.** UFD is normal.

*Proof.* Let  $R$  be a UFD and  $K$  be its field of fractions. If  $a \in K$  is integral over  $R$  and  $a^n + r_1 a^{n-1} + \cdots + r_n = 0$ . Write  $a = u/s$  with  $\gcd(u, s) = 1$ . Then  $u^n + r_1 s u^{n-1} + \cdots + r_n s^n = 0$ . Now if  $s$  is a non-unit, says  $p \mid s$  with  $p$  is a prime. Then  $p \mid u$  obviously  $\leadsto p \mid \gcd(u, s) = 1$ , which is a contradiction. So  $s$  is a unit  $\implies a \in R$ .  $\square$

**Prop 6.6.5.**

- Let  $S/R$  is an integral extension and  $T \subset R$  be a m.c. set with  $1 \in T$ . Then  $S_T$  is also integral over  $R_T$ .
- Let  $S/R$  be an arbitrary extension and  $T \subset R$  be m.c. with  $1 \in T$ . Then  $(\bar{R})_T = \overline{(R_T)}$  in  $S_T$ .

*Proof.*  $\square$

**Prop 6.6.6.** “Being normal” is a local property. TFAE

- (1)  $R$  is normal
- (2)  $R_P$  is normal  $\forall P \in \text{Spec } R$
- (3)  $R_Q$  is normal  $\forall Q \in \text{Max } R$

**Prop 6.6.7.** TFAE

- (1)  $M$  is a flat  $R$ -module
- (2)  $M_P$  is a flat  $R$ -module  $\forall P \in \text{Spec } R$
- (3)  $M_Q$  is a flat  $R$ -module  $\forall Q \in \text{Max } R$

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