

# Algebra

June 5, 2017

# 1 Introduction to Homological Algebra

## 1.1 Projective, Injective and Flat modules (week 14)

Def 1.

- $M \in \mathbf{Mod}_R$  is **projective** if  $\text{Hom}(M, \cdot)$  preserves the *right* exactness.
- $N \in \mathbf{Mod}_R$  is **injective** if  $\text{Hom}(\cdot, N)$  preserves the *right* exactness.
- $M \in \mathbf{Mod}_R$  is **flat** if  $M \otimes \cdot$  preserves the *left* exactness.

Fact 1.1.1.

- $M$  is projective  $\iff$ 

$$\begin{array}{ccc} & M & \\ \exists \tilde{f} \swarrow & \downarrow f & \\ M_2 & \longrightarrow & M_3 \longrightarrow 0 \\ 0 \longrightarrow & M_1 & \longrightarrow M_2 \\ & \downarrow g & \nwarrow \exists \tilde{g} \\ & N & \end{array}$$
- $N$  is injective  $\iff$
- free  $\implies$  projective: If  $X = \{x_i \mid i \in \Lambda\}$  and  $f : x_i \mapsto a_i$ . Since  $\beta$  onto, exists  $b_i$  so that  $\beta(b_i) = a_i$ . we can then set  $\tilde{f} : x_i \mapsto b_i$  by the universal property of free module.

$$\begin{array}{ccc} & F(X) & \\ \exists \tilde{f} \swarrow & \downarrow f & \\ M_2 & \xrightarrow{\beta} & M_3 \longrightarrow 0 \end{array}$$

- free  $\implies$  flat: Let  $F \cong R^{\otimes \Lambda}$  be a free module, and  $M_1, M_2$  be two modules such that  $0 \rightarrow M_1 \rightarrow M_2$ . Since  $R \otimes_R M \cong M$ , we have

$$\begin{aligned} 0 \rightarrow M_1 \rightarrow M_2 & \quad \text{exact} \\ \implies 0 \rightarrow R \otimes M_1 \rightarrow R \otimes M_2 & \quad \text{exact} \\ \implies 0 \rightarrow \bigoplus_{i \in \Lambda} R \otimes M_1 \rightarrow \bigoplus_{i \in \Lambda} R \otimes M_2 & \quad \text{exact} \\ \stackrel{(a)}{\implies} 0 \rightarrow R^{\otimes \Lambda} \otimes M_1 \rightarrow R^{\otimes \Lambda} \otimes M_2 & \quad \text{exact} \\ \implies 0 \rightarrow F \otimes M_1 \rightarrow F \otimes M_2 & \quad \text{exact} \end{aligned}$$

Where (a) is by the fact that  $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$ . Thus  $F$  flat.

- If  $S$  is a multiplication closed set in  $R$  with  $1 \in S$ , then

$$0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0 \implies 0 \rightarrow M_S \rightarrow N_S \rightarrow L_S \rightarrow 0.$$

We know that  $M_S \cong R_S \otimes_R M$ . So  $R_S$  is a flat  $R$ -module. e.g.  $\mathbb{Q}$  is a flat  $\mathbb{Z}$ -module.

For any  $M \in \mathbf{Mod}_R$ , a projective module  $N$  such that  $N \rightarrow M \rightarrow 0$  could be easily find: Simply let  $N = F$ , a free module on the set  $M$ .

Now we shall ask for any module  $M$ , does there exist  $N \in \mathbf{Mod}_R$  such that  $N$  is injective and  $0 \rightarrow M \rightarrow N$ ?

**Theorem 1** (Boer's criterion).  $N$  is injective  $\iff \forall I \subset R$ , and a homomorphism  $f$ , there exists a homomorphism  $h$  such that the following diagram commutes:

$$\begin{array}{ccc} 0 \longrightarrow & I & \longrightarrow R \\ & \downarrow f & \nearrow \exists h \\ & N & \end{array}$$

*Proof.* “ $\Rightarrow$ ”: See  $I$  as an  $R$  module, then it is immediate by the definition of injective module.

“ $\Leftarrow$ ”: Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 \\ & & \downarrow g & & \\ & & N & & \end{array}$$

Let  $S \triangleq \{(M, \rho) \mid M_1 \subseteq M \subseteq M_2 \text{ and } \rho : M \rightarrow N \text{ extends } g\} \neq \emptyset$  since  $(M_1, g) \in S$ .

By the routinely proof using Zorn's lemma, exists a maximal element  $(M^*, \mu) \in S$ .

We claim that  $M^* = M_2$ . If not, pick  $a \in M_2 \setminus M^*$  and let  $M' \triangleq M^* + Ra \subsetneq M^*$ ,  $I \triangleq \{r \in R \mid ra \in M^*\}$ . Define  $f : I \rightarrow N$  with  $r \mapsto \mu(ra)$ . Then we have an extension  $h : R \rightarrow N$  of  $f$ .

Now, let  $\mu' : M' \rightarrow N = x + ra \mapsto \mu(x) + h(r)$ . We shall prove that this map is well-defined: If  $x_1 + r_1a = x_2 + r_2a$ , then  $(r_1 - r_2)a = x_2 - x_1 \in M \implies r_1 - r_2 \in I$ . So  $h(r_1) - h(r_2) = f(r_1 - r_2) = \mu((r_1 - r_2)a) = \mu(x_2) - \mu(x_1)$ , which prove  $\mu'$  is well defined, and the existence of  $\mu'$  contradict the fact that  $(M^*, \mu)$  is maximal.  $\square$

**Def 2.**  $M$  is **divisible** if  $\forall x \in M, r \in R \setminus \{0\}$ , there exists  $y \in M$  such that  $x = ry$ , i.e.  $rM = M \quad \forall r \in R \setminus \{0\}$ .

**Prop 1.1.1.**

1. Every injective module  $N$  over an integral domain is divisible.

*Proof.* For any  $x_0$  and  $r_0 \in R \setminus \{0\}$ . Let  $I = \langle r_0 \rangle \subset R$ . As long as  $R$  is an integral domain,  $I \cong R$  as an  $R$ -module, so the  $R$ -module homomorphism  $f : I \rightarrow N = rx_0 \mapsto rr_0$  is well-defined. Since  $N$  injective, this map extends to  $h : R \rightarrow N$ . Let  $y_0 \triangleq h(1)$ , then  $r_0y_0 = r_0h(1) = h(r_0) = x_0$ . Thus  $N$  injective.  $\square$

2. Every divisible module  $N$  over an PID is injective.

*Proof.* For any  $I \subseteq R$  and a homomorphism  $f : I \rightarrow N$ , if  $I = 0$  then  $h = x \mapsto 0$  is always an extension of  $f$ . So assume  $\forall I \neq 0$ . Since  $R$  is a PID,  $I = \langle r_0 \rangle$  for some  $r_0 \neq 0 \in R$ . By the fact that  $N$  divisible, exists  $y_0 \in N$  such that  $r_0y_0 = x_0 \triangleq f(r_0)$ .

Now we could define  $h : R \rightarrow N$  by  $1 \mapsto y_0$ . Then  $h(r_0) = r_0h(1) = r_0y_0 = x_0$ , thus  $h$  is an extension of  $f$  and  $N$  injective.  $\square$

3. If  $R$  is a PID, then any quotient  $N$  of a injective  $R$ -module  $M$  is injective.

*Proof.* By 2.,  $rM = M$  for any  $r \neq 0$ , thus  $rN = N$  for any  $r \neq 0$ , and hence  $N$  injective.  $\square$

**Theorem 2.** For any  $M \in \mathbf{Mod}_R$ , exists  $N$  injective and contains  $M$ .

*Proof.*

**Case 1:**  $R = \mathbb{Z}$ .

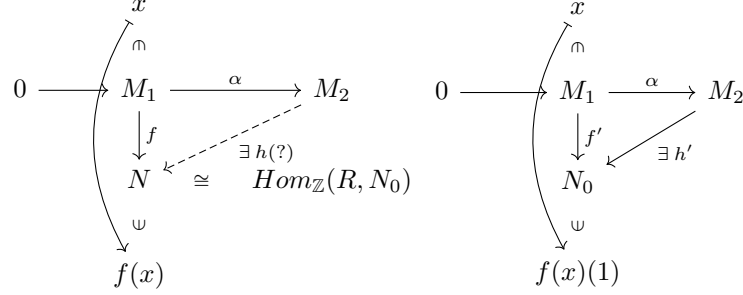
Let  $X = \{x_i\}_{i \in \Lambda}$  be a generating set for  $M$  and  $F$  is free on  $X$ . Let  $f$  be the natural map from  $f$  to  $M$ . then  $M \cong F/\ker f$ .

Define  $F' = \bigoplus_{i \in \Lambda} \mathbb{Q}e_i \subset F$ , which is obviously a divisible  $\mathbb{Z}$ -module. Then  $M \subseteq F'/\ker f \triangleq M'$ , where  $M'$  is injective by proposition 1.1.1.

**Case 2:**  $R$  arbitrary.

We can regard any  $M$  as a  $\mathbb{Z}$ -module, then there exists an injective module  $N_0 \supset M$ . Now, we have an  $R$ -module  $N \triangleq \text{Hom}_{\mathbb{Z}}(R, N_0)$  with multiplication  $rf(x) \triangleq x \mapsto f(xr)$ .

We claim that  $N$  is injective. For any  $f :: M_1 \rightarrow N$ , and a homomorphism  $\alpha :: M_1 \rightarrow M_2$ , then  $\alpha$  could be take as a  $\mathbb{Z}$ -module homomorphism. Define  $f' :: M_1 \rightarrow N_0$  by  $x \mapsto f(x)(1)$ . Since  $N_0$  injective, exists  $h'$ , a  $\mathbb{Z}$  module homomorphism from  $M_2$  to  $N_0$ .



Now, define

$$\begin{aligned} h : M_2 &\longrightarrow N \\ y &\longmapsto h(y) : R \longrightarrow N_0 \\ 1 &\longmapsto h'(y) \\ r &\longmapsto h'(ry) \end{aligned}$$

We check that  $h$  is well-defined.

- $h(y) \in \text{Hom}_{\mathbb{Z}}(R, N_0)$

$$h(y)(r_1 + r_2) = h'((r_1 + r_2)y) = h'(r_1y + r_2y) = h'(r_1y) + h'(r_2y) = h(y)(r_1) + h(y)(r_2)$$

- $h \in \text{Hom}_R(M_2, N)$

$$\begin{aligned} h(r_1y_1 + y_2)(r) &= h'(r(r_1y_1 + y_2)) = h'(rr_1y_1 + ry_2) \\ &= h'(rr_1y_1) + h'(ry_2) \\ &= h(y)(rr_1) + h(y_2)(r) \\ &= (r_1h(y))(r) + h(y_2)(r) \end{aligned}$$

- Show diagram commute  $f = h \circ \alpha$  Fix  $y \in M_1$ , then  $\forall r \in R$ :

$$\begin{aligned} (h \circ \alpha)(y)(r) &= h(\alpha(y))(r) = h'(r\alpha(y)) \\ &= h'(\alpha(ry)) = f'(ry) \\ &= f(ry)(1) = rf(y)(1) \\ &= f(y)(r) \end{aligned}$$

Thus  $N_0$  injective.

Now notice that,  $\text{Hom}_{\mathbb{Z}}(R, \cdot)$  is a left exact functor, so  $M \hookrightarrow N_0$  implies  $\text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, N_0)$ , thus  $M \cong \text{Hom}_R(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, N_0) = N$ .  $\square$

**Prop 1.1.2.** TFAE

1.  $M$  is projective.
2. Every exact sequence  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$  split.

3.  $\exists M'$  s.t.  $M \oplus M' \cong F$ : free.

*Proof.*

(1)  $\Rightarrow$  (2) : Since  $M$  projective, the map  $\lambda$  with  $\beta \circ \lambda = \text{Id}$  exists in the following diagram:

$$\begin{array}{ccc} & M & \\ \swarrow \exists \lambda & \downarrow \text{Id} & \\ M_2 & \xrightarrow{\beta} & M \longrightarrow 0 \end{array}$$

Then  $\lambda$  is a lifting, so  $M_2 \cong M_1 \oplus M$  and  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$  split.

(2)  $\Rightarrow$  (3): Let  $F$  be a free module on a generating set of  $M$ , and  $\beta :: F \rightarrow M$  be the natural map, then  $0 \rightarrow \ker \beta \rightarrow F \rightarrow M \rightarrow 0$  split, so  $F \cong \ker \beta \oplus M$ .

(3)  $\Rightarrow$  (1): For any  $M_2 \rightarrow M_3 \rightarrow 0$ , since  $M' \oplus M$  free and thus projective,  $\lambda'$  exists in the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \longrightarrow & M' \oplus M & \xleftarrow[\pi]{\mu} & M \longrightarrow 0 \\ & & & & \downarrow \exists \lambda' & & \downarrow f \\ & & & & M_2 & \xrightarrow{\beta} & M_3 \longrightarrow 0 \end{array}$$

Define  $\lambda = \lambda' \circ \mu$ . Then  $\beta \circ \lambda = \beta \circ \lambda' \circ \mu = f \circ \pi \circ \mu = f$ . □

**Prop 1.1.3.** TFAE

1.  $M$  is injective.
2. Each exact sequence  $0 \rightarrow M \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  split.

*Proof.* (1)  $\Rightarrow$  (2): Similar to the projective case,  $\mu$  exists in the following diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & M \xrightarrow{\alpha} M_2 \\ & & \downarrow \text{Id} \\ & & M \end{array} \quad \begin{array}{c} \nearrow \exists \mu \\ \nwarrow \end{array}$$

So  $M_2 = M \oplus M_3$ .

(2)  $\Rightarrow$  (1): By theorem 2, there is a module  $N \subset M$  so that  $N$  is injective.

Consider  $0 \longrightarrow M \xrightleftharpoons[\exists \mu]{i} N \longrightarrow \text{coker } i \longrightarrow 0$  split exact and  $\mu \circ i = \text{Id}_M$ . Since  $N$  injective,  $h'$  exists in the following diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & M_1 \xrightarrow{\alpha} M_2 \\ & & \downarrow f \\ & & M \\ & \nearrow i \circ f & \downarrow i \\ & & N \end{array} \quad \begin{array}{c} \nwarrow \exists h' \\ \nearrow \mu \end{array}$$

Let  $h = \mu \circ h'$ , then  $h \circ \alpha = \mu \circ h' \circ \alpha = \mu \circ i \circ f = f$  □

**Prop 1.1.4.** projective  $\implies$  flat.

*Proof.* Observe that  $\bigoplus_{i \in \Lambda} M_i$  is flat if and only if  $M_i$  is flat for each  $i$ , since if  $0 \rightarrow N_1 \xrightarrow{\alpha} N_2$  exact, then

$$\begin{array}{ccc}
0 & \longrightarrow & (\bigoplus M_i) \otimes N_1 \xrightarrow{1 \otimes \alpha} (\bigoplus M_i) \otimes N_2 \\
& & \parallel \\
0 & \longrightarrow & \bigoplus (M_i \otimes N_1) \xrightarrow{\bigoplus (1 \otimes \alpha)} \bigoplus (M_i \otimes N_2) \\
& & \updownarrow \\
0 & \longrightarrow & M_i \otimes N_1 \xrightarrow{1 \otimes \alpha} M_i \otimes N_2 \qquad \forall i \in \Lambda
\end{array}$$

If  $M$  is projective, then by proposition 1.1.2  $\exists M'$  such that  $M \oplus M' \cong F$  is free. Since free implies flat, by above,  $M$  is flat.  $\square$

## 1.2 Ext and Tor (week 15)

Given  $M, N \in \mathbf{Mod}_R$ , there are two ways to define  $\text{Ext}^n(M, N)$ :

**Def 3** (Ext functor).

- Find any projective resolution  $P_\bullet \xrightarrow{\alpha} M \rightarrow 0$ , and let  $P_M : P_\bullet \rightarrow 0$  (called a *deleted resolution*). We can define  $\text{Ext}_{\text{proj}}^n(M, N) = H^n(\text{Hom}(P_M, N))$ .
- Find any injective resolution  $0 \xrightarrow{\alpha} N \rightarrow E^\bullet$ , and let  $E_N : 0 \rightarrow E^\bullet$ . We can define  $\text{Ext}_{\text{inj}}^n(M, N) = H^n(\text{Hom}(M, E_N))$ .

**Prop 1.2.1.**  $\text{Ext}_{\text{proj}}^0(M, N) \cong \text{Ext}_{\text{inj}}^0(M, N) \cong \text{Hom}(M, N)$ .

*Proof.*

$$\text{Hom}(P_M, N) : 0 \xrightarrow{\overline{d}_0} \text{Hom}(P_0, N) \xrightarrow{\overline{d}_1} \text{Hom}(P_1, N) \rightarrow \dots$$

$$\text{so } \text{Ext}_{\text{proj}}^0(M, N) = \ker \overline{d}_1 / \text{im } \overline{d}_0 = \ker \overline{d}_1 = \text{im } \alpha = \text{Hom}(M, N). \quad \square$$

Similarly,  $\text{Ext}_{\text{inj}}^0(M, N) = \text{Hom}(M, N)$ .

**Lemma 1.**

- If  $M$  is projective, then  $\text{Ext}_{\text{proj}}^n(M, N) = 0$  for all  $n > 0, N \in \mathbf{Mod}_R$ .
- If  $N$  is injective, then  $\text{Ext}_{\text{inj}}^n(M, N) = 0$  for all  $n > 0, M \in \mathbf{Mod}_R$ .

*Proof.* If  $M$  is projective, then  $0 \rightarrow M \xrightarrow{1_M} M \rightarrow 0$  is a projective resolution of  $M$ . Its deleted resolution is then  $P_M : 0 \rightarrow M \rightarrow 0$ . Hence for  $n > 0$ ,  $\text{Ext}_{\text{proj}}^n(M, N) = H^n(\text{Hom}(P_M, N)) = 0$ .

The argument applies similarly to injective case.  $\square$

**Theorem 3** (Equivalence of  $\text{Ext}_{\text{proj}}$  and  $\text{Ext}_{\text{inj}}$ ).

$$\text{Ext}_{\text{proj}}^n(M, N) \cong \text{Ext}_{\text{inj}}^n(M, N)$$

.

*Proof.* Let  $P_\bullet \rightarrow M \rightarrow 0$  and  $0 \rightarrow N \rightarrow E^\bullet$  be projective and injective resolutions, then we have  $0 \rightarrow K_0 \rightarrow P_0 \rightarrow M \rightarrow 0$  and  $0 \rightarrow N \rightarrow E^0 \rightarrow L^1 \rightarrow 0$  exact.

$$\begin{array}{ccccccc} \cdots & \rightarrow & P_2 & \xrightarrow{\quad} & P_1 & \xrightarrow{\quad} & P_0 \rightarrow M \rightarrow 0 \\ & & \searrow & & \nearrow & \searrow & \nearrow \\ & & & K_1 & & & K_0 \\ & \nearrow & & \searrow & \nearrow & & \searrow \\ 0 & & & & 0 & & 0 \end{array} \quad \begin{array}{ccccccc} 0 \rightarrow N \rightarrow E^0 & \xrightarrow{\quad} & E^1 & \xrightarrow{\quad} & E^2 \rightarrow \cdots \\ & \searrow & \nearrow & \searrow & \nearrow \\ & & L^1 & & L^2 \\ & \nearrow & \searrow & \nearrow & \searrow \\ 0 & & & 0 & & 0 \end{array}$$

We can construct long exact sequences of homology of  $\text{Hom}(\cdot, E_N)$ :

$$0 \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(M, E^0) \rightarrow \text{Hom}(M, L^1) \rightarrow \text{Ext}_{\text{inj}}^1(M, N) \rightarrow \text{Ext}_{\text{inj}}^1(M, E^0) = 0$$

$$0 \rightarrow \text{Hom}(P_0, N) \rightarrow \text{Hom}(P_0, E^0) \rightarrow \text{Hom}(P_0, L^1) \rightarrow 0$$

$$0 \rightarrow \text{Hom}(K_0, N) \rightarrow \text{Hom}(K_0, E^0) \rightarrow \text{Hom}(K_0, L^1) \rightarrow \text{Ext}_{\text{inj}}^1(K_0, N) \rightarrow \text{Ext}_{\text{inj}}^1(K_0, E^0) = 0$$

The second sequence is short because  $P_0$  is projective (so  $\text{Hom}(P_0, \cdot)$  preserves exactness).

Similarly, for  $\text{Hom}(P_M, \cdot)$  we have:

$$0 \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(P_0, N) \rightarrow \text{Hom}(K_0, N) \rightarrow \text{Ext}_{\text{proj}}^1(M, N) \rightarrow \text{Ext}_{\text{proj}}^1(P_0, N) = 0$$

$$0 \rightarrow \text{Hom}(M, E^0) \rightarrow \text{Hom}(P_0, E^0) \rightarrow \text{Hom}(K_0, E^0) \rightarrow 0$$

$$0 \rightarrow \text{Hom}(M, L^1) \rightarrow \text{Hom}(P_0, L^1) \rightarrow \text{Hom}(K_0, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(M, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(P_0, L^1) = 0$$

Combining these sequences together, we got the following 2D diagram:

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{Hom}(M, N) & \longrightarrow & \text{Hom}(M, E^0) & \xrightarrow{\phi} & \text{Hom}(M, L^1) \longrightarrow \text{Ext}_{\text{inj}}^1(M, N) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Hom}(P_0, N) & \longrightarrow & \text{Hom}(P_0, E^0) & \xrightarrow{\sigma} & \text{Hom}(P_0, L^1) \longrightarrow 0 \\
& & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
0 & \longrightarrow & \text{Hom}(K_0, N) & \longrightarrow & \text{Hom}(K_0, E^0) & \xrightarrow{\tau} & \text{Hom}(K_0, L^1) \longrightarrow \text{Ext}_{\text{inj}}^1(K_0, N) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \text{Ext}_{\text{proj}}^1(M, N) & & 0 & & \text{Ext}_{\text{proj}}^1(M, L^1) \\
& & \downarrow & & & & \downarrow \\
& & 0 & & & & 0
\end{array}$$

By Snake lemma, there is an exact sequence

$$(\ker \alpha \rightarrow) \ker \beta \rightarrow \ker \gamma \rightarrow \text{coker } \alpha \rightarrow \text{coker } \beta (\rightarrow \text{coker } \gamma)$$

and this reads

$$\text{Hom}(M, E^0) \xrightarrow{\phi} \text{Hom}(M, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(M, N) \rightarrow 0$$

Thus  $\text{Ext}_{\text{proj}}^1(M, N) \cong \text{coker } \phi \cong \text{Ext}_{\text{inj}}^1(M, N)$ .

(From now on, we don't need to distinguish proj/inj for  $\text{Ext}^1$  !)

Since  $\sigma$  is onto,  $\text{im } \gamma = \text{im}(\gamma \circ \sigma)$ . Similarly,  $\text{im } \tau = \text{im}(\tau \circ \beta)$ .

By the commutativity of the diagram,  $\text{im } \gamma = \text{im } \tau$ , so

$$\text{Ext}^1(K_0, N) \cong \text{coker } \gamma = \text{Hom}(K_0, L^1) / \text{im } \gamma \cong \text{coker } \tau \cong \text{Ext}^1(M, L^1)$$

Write  $K_{-1} := M, L^0 := N$ , then  $\text{Ext}^1(K_0, L^0) = \text{Ext}^1(K_{-1}, L^1)$  ( $\star$ ).

Similarly, from the exact sequences

$$0 \rightarrow K_j \rightarrow P_j \rightarrow K_{j-1} \rightarrow 0$$

$$0 \rightarrow L^i \rightarrow E^i \rightarrow L^{i+1} \rightarrow 0$$

, we can obtain  $\text{Ext}^1(K_j, L^i) \cong \text{Ext}^1(K_{j-1}, L^{i+1})$  for  $i, j \geq 0$ .

Now, observe that

$$0 \rightarrow L^{n-1} \rightarrow E^{n-1} \xrightarrow{d_{n-1}} E^n \xrightarrow{d_n} \dots$$

is an injective resolution of  $L^{n-1}$ , and  $\text{Ext}^1(M, L^{n-1}) \cong \ker \overline{d_n} / \text{im } \overline{d_{n-1}} \cong \text{Ext}_{\text{inj}}^n(M, N)$ .

Similarly, for projective resolution we have  $\text{Ext}^1(K_{n-2}, N) \cong \text{Ext}_{\text{proj}}^n(M, N)$ .



Finally, by  $(\star)$ ,

$$\text{Ext}_{\text{inj}}^n(M, N) \cong \text{Ext}^1(K_{-1}, L^{n-1}) \cong \text{Ext}^1(K_0, L^{n-2}) \cong \cdots \cong \text{Ext}^1(K_{n-2}, L^0) \cong \text{Ext}_{\text{proj}}^n(M, N)$$

□

**Def 4** (Tor functor). Let  $M, N \in \mathbf{Mod}_R$ , and  $P_\bullet \rightarrow M \rightarrow 0$  be a projective resolution of  $M$ , similar to the Ext case, for  $n \geq 0$  we can define

$$\text{Tor}_n(M, N) = H_n(P_M \otimes N)$$

**Fact 1.2.1.** By Horseshoe lemma, short exact sequence  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  induces a long exact sequence:

$$\cdots \rightarrow \text{Tor}_1(M_1, N) \rightarrow \text{Tor}_1(M_2, N) \rightarrow \text{Tor}_1(M_3, N) \rightarrow M_1 \otimes N \rightarrow M_2 \otimes N \rightarrow M_3 \otimes N \rightarrow 0$$

**Prop 1.2.2.** If  $M$  is flat, then  $\text{Tor}_n(M, N) = 0$  for  $n > 0, N \in \mathbf{Mod}_R$ .

*Proof.* Since  $0 \rightarrow M \xrightarrow{1_M} M \rightarrow 0$  is a flat resolution of  $M$ .

□

**Theorem 4** (Tor for flat resolutions). Let  $U_\bullet \rightarrow M \rightarrow 0$  be a flat resolution of  $M$ , then for  $n \geq 0$ ,

$$\text{Tor}_n(M, N) \cong H_n(U_M \otimes N)$$

*Proof.*

$$\begin{array}{ccccccc} \cdots & \rightarrow & U_2 & \xrightarrow{\quad} & U_1 & \xrightarrow{\quad} & U_0 \rightarrow M \rightarrow 0 \\ & & \searrow & & \swarrow & & \nearrow \\ & & & W_1 & & & W_0 \\ & \nearrow & & \searrow & \nearrow & & \searrow \\ 0 & & & & 0 & & 0 \end{array}$$

$$H_\bullet(U_M \otimes N) : \cdots \rightarrow U_2 \otimes N \xrightarrow{d_2 \otimes 1} U_1 \otimes N \xrightarrow{d_1 \otimes 1} U_0 \otimes N \rightarrow 0$$

- $n = 0$ :

Since tensor is right exact,  $U_1 \otimes N \xrightarrow{d_1 \otimes 1} U_0 \otimes N \xrightarrow{\alpha \otimes 1} M \otimes N \rightarrow 0$  is exact. Hence

$$H_0(U_M \otimes N) = (U_0 \otimes N) / \text{im}(d_1 \otimes 1) = (U_0 \otimes N) / \ker(\alpha \otimes 1) \cong M \otimes N$$

Any projective resolution also has this property, so  $\text{Tor}_0(M, N) = H_0(U_M \otimes N)$ .

- $n = 1$ :

$0 \rightarrow W_0 \rightarrow U_0 \rightarrow M \rightarrow 0$  induces a long exact sequence:

$$\cdots \rightarrow \text{Tor}_1(W_0, N) \rightarrow 0 \rightarrow \text{Tor}_1(M, N) \rightarrow W_0 \otimes N \xrightarrow{i \otimes 1} U_0 \otimes N \rightarrow M \otimes N \rightarrow 0$$

where  $\text{Tor}_1(U_0, N) = 0$  because  $U_0$  is flat. We can see that  $\text{Tor}_1(M, N) \cong \ker(i \otimes 1)$ .

$$\begin{array}{ccccc} U_2 \otimes N & \xrightarrow{d_2 \otimes 1} & U_1 \otimes N & \xrightarrow{d_1 \otimes 1} & U_0 \otimes N \\ & \searrow \beta' \otimes 1 & \nearrow j \otimes 1 & \searrow \alpha' \otimes 1 & \nearrow i \otimes 1 \\ & & W_1 \otimes N & & W_0 \otimes N \\ & & \searrow & & \searrow \\ & & & 0 & 0 \end{array}$$

Since  $\alpha' \otimes 1$  is onto,  $W_0 \otimes N \cong (U_1 \otimes N) / \ker(\alpha' \otimes 1)$ . Also,  $x \in \ker(d_1 \otimes 1) \iff (d_1 \otimes 1)(x) = 0 \iff (\alpha' \otimes 1)(x) \in \ker(i \otimes 1)$ , so  $\ker(i \otimes 1) = (\alpha' \otimes 1)(\ker(d_1 \otimes 1)) \cong \ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1)$ . ( $\alpha' \otimes 1$  can be considered a quotient map, then  $\ker(d_1 \otimes 1)$  descends to  $\ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1)$ .)

Now, in the diagram  $W_1 \otimes N \rightarrow U_1 \otimes N \rightarrow W_0 \otimes N \rightarrow 0$  exact, so  $\ker(\alpha' \otimes 1) = \text{im}(j \otimes 1)$ . But  $\beta' \otimes 1$  is onto, thus  $\text{im}(j \otimes 1) = \text{im}(d_2 \otimes 1)$ .

Finally,

$$\text{Tor}_1(M, N) \cong \ker(i \otimes 1) \cong \ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1) = \ker(d_1 \otimes 1) / \text{im}(d_2 \otimes 1) = H_1(U_M \otimes N)$$

.

- $n \geq 2$ :

Let's see further in the previous long exact sequences:

$$\cdots \rightarrow \text{Tor}_2(W_0, N) \rightarrow 0 \rightarrow \text{Tor}_2(M, N) \xrightarrow{\sim} \text{Tor}_1(W_0, N) \rightarrow 0 \rightarrow \text{Tor}_1(M, N) \rightarrow \cdots$$

we can see that  $\text{Tor}_n(M, N) \cong \text{Tor}_{n-1}(W_0, N)$  for  $n \geq 2$ .

Now,

$$\cdots U_3 \xrightarrow{d_3} U_2 \xrightarrow{d_2} U_1 \rightarrow W_0 \rightarrow 0$$

is a flat resolution of  $W_0$ , and its homology is  $H_{n-1}(U_{W_0} \otimes N) = \ker(d_n \otimes 1) / \text{im}(d_{n-1} \otimes 1) = H_n(U_M \otimes N)$ .

By induction, assume it's true for  $n-1$ , then

$$H_n(U_M \otimes N) = H_{n-1}(U_{W_0} \otimes N) \cong \text{Tor}_{n-1}(W_0, N) \cong \text{Tor}_n(M, N)$$

.

□