Algebra

June 2, 2017

1 Introduction to Homological Algebra

Projective, Injective and Flat modules (week 14)

Def 1.

- $M \in \mathbf{Mod}_R$ is **projective** if $\mathrm{Hom}(M,\cdot)$ preserves the *right* exactness.
- $N \in \mathbf{Mod}_R$ is **injective** if $\mathrm{Hom}(\cdot, N)$ preserves the *right* exactness.
- $M \in \mathbf{Mod}_R$ is flat if $M \otimes \cdot$ preserves the *left* exactness.

Fact 1.1.1.

- $\begin{array}{cccc} \bullet & M \text{ is projective} & \Longleftrightarrow & \stackrel{\exists \, \tilde{f}}{\swarrow} & \stackrel{M}{\downarrow_f} \\ & & M_2 \longrightarrow M_3 \longrightarrow 0 \\ \bullet & N \text{ is injective} & \Longleftrightarrow & 0 \longrightarrow M_1 \longrightarrow M_2 \\ \bullet & & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & N \end{array}$
- free \implies projective: If $X = \{x_i \mid i \in \Lambda\}$ and $f: x_i \mapsto a_i$. Then we can set $\tilde{f}: x_i \mapsto b_i$ for any b_i s.t. $\beta: b_i \mapsto a_i$.

$$F(X)$$

$$\downarrow^{f} \qquad \downarrow^{f}$$

$$M_{2} \xrightarrow{\beta} M_{3} \longrightarrow 0$$

• free \Longrightarrow flat:

$$0 \to M_1 \to M_2 \text{ exact } \Rightarrow 0 \to R \underset{R}{\otimes} M_1 \to R \underset{R}{\otimes} M_2 \text{ exact}$$

since $M_1 \cong R \underset{R}{\otimes} M_1$. Let F is Free on $X = \{x_i\}, i \in \Lambda$, that is $F \cong \underset{x_i \in X}{\oplus} Rx_i \cong R^{\oplus \Lambda}$. And, $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$ (see tensor product section!)

$$0 \to \bigoplus_{i \in \Lambda} (R \otimes M_1) \to \bigoplus_{i \in \Lambda} (R \otimes M_2) \text{ exact}$$

$$\Rightarrow 0 \to \left(\bigoplus_{i \in \Lambda} R\right) \otimes M_1 \to \left(\bigoplus_{i \in \Lambda} R\right) \otimes M_2 \text{ exact}$$

Therefore,

$$0 \to F \otimes M_1 \to F \otimes M_2$$
 exact

• If S is a m.c. set in R with $1 \in S$, then

$$0 \to M \to N \to L \to 0 \implies 0 \to M_S \to N_S \to L_S \to 0.$$

We know that $M_S \cong R_S \otimes_R M$. So R_S is a flat R-module. e.g. \mathbb{Q} is a flat \mathbb{Z} -module.

• $\forall M \in \mathbf{Mod}_R, \exists F$: free on X s.t. $F \to M \to 0$. This is obvious since we can choose X to be the generating set of M.

Ques: $\forall M \in \mathbf{Mod}_R$, does there exist $N \in \mathbf{Mod}_R$ is injective s.t. $0 \to M \to N$?

Theorem 1 (Boer's criterion).
$$N$$
 is injective $\iff \forall I \subset R, 0 \xrightarrow{f} I \xrightarrow{f} R$

Proof.

- " \Rightarrow " by the "Fact" of injective.
- " = "

Consider diagram,

$$0 \longrightarrow M_1 \longrightarrow M_2$$

$$\downarrow^g$$

$$N$$

Let $S = \{(M, \rho) | M_1 \subseteq M \subseteq M_2 \text{ and } \rho : M \to N \text{ extend } g\} \neq \emptyset$ By Zorn's lemman, exists a maximal elemant $(M^*, \mu) \in S$.

 $\underline{\text{Claim}}: M^* = M_2$

If not, pick $a \in M_2$ M^* . Let $M' = M^* + Ra \supseteq M^*$ and $I = \{r \in R | ra \in M^*\}$ Define $f: I \to N$ with $r \mapsto \mu(ra)$. Then we have extension h,

$$0 \longrightarrow I \underset{N}{\longleftrightarrow} R$$

Now, define $\mu': M' \to N$ with $x + ra \mapsto \mu(x) + h(r)$

Well-define:

 $x_1 + r_1 a = x_2 + r_2 a \leadsto a(r_1 - r_2) = x_2 - x_1 \in M \leadsto h(r_1) - h(r_2) = h(r_1 - r_2) = f(r_1 - r_2) = \mu(a(r_1 - r_2)) = \mu(x_2 - x_1) = \mu(x_2) - \mu(x_1) \leadsto \mu(x_1) + h(r_1) = \mu(x_2) + h(r_2)$

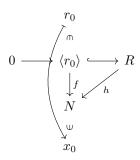
But, μ' is extension of μ . Therefore, $(M', \mu') \geq (M^*, \mu)$, which is a contradiction to M^* is maximal.

Def 2. M is **divisible** if $\forall x \in M, r \in R \setminus \{0\}$, there exists $y \in M$ such that x = ry, i.e. $rM = M \quad \forall r \in R \setminus \{0\}$.

Prop 1.1.1.

1. Every injective module N over an integral domain is divisible.

Proof. For $x_0, r_0 \in R \{0\},\$

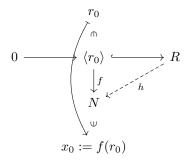


Let $y_0 := h(1)$. Then, $r_0 y_0 = r_0 h(1) = h(r_0) = x_0$. (why the ID is required?)

2. Every divisible module N over an PID is injective.

Proof. If I = 0, let h(1) = arbitrary is always let diagram commute. Now, let $\forall I \neq 0$ $I = \langle r_0 \rangle$

for some $r_0 \neq 0 \in R$



Then, $\exists y_0 \leadsto r_0 y_0 = x_0$. Define $h(1) = y_0$. The diagram commute.

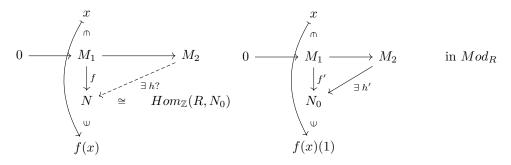
Theorem 2. $\forall M \in \mathbf{Mod}_R, \exists N \text{ is injective s.t. } M \hookrightarrow N.$

Proof.

- Case 1 $R = \mathbb{Z}$
- Case 2 R is arbitrary

We can regard M_1 as \mathbb{Z} module, therefore $\exists N_0$ a \mathbb{Z} module with inclusion map. $M_1 \hookrightarrow N_0$. Now, we have R module $N := Hom_{\mathbb{Z}}(R, N_0)$

 $\underline{\text{Claim}}$: N is injective



 $oldsymbol{Mod}_R oldsymbol{Mod}_{\mathbb{Z}}$

Now, define

$$h: M_2 \longrightarrow N$$

$$y \longmapsto h(y): R \longrightarrow N_0$$

$$1 \longmapsto h'(y)$$

$$r \longmapsto h'(ry)$$

well-define

1. Show $h(y) \in Hom_{\mathbb{Z}}(R, N_0)$

$$h(y)(r_1+kr_2) = h'((r_1+kr_2)y) = h'(r_1+kr_2y) = h'(r_1)+kh'(r_2) = h(y)(r_1)+kh(y)(r_1)$$

2. Show $h \in Hom_R(M_2, N)$

$$h(y_1 + r_2 y_2)(r) \ \forall \ r \in R = h'(r(y_1 + r_2 y_2)) \ \forall \ r \in R = h'(ry_1 + rr_2 y_2) \ \forall \ r \in R$$
$$= h'(ry_1) + h'(rr_2 y_2) \ \forall \ r \in R = h(y)(r) + h(r_2 y_2)(r) \ \ \forall \ r \in R$$
$$= h(y)(r) + (r_2 h(y_2))(r) \ \forall \ r \in R$$

3. Show diagram commute

Prop 1.1.2. TFAE

- 1. M is projective.
- 2. $\forall 0 \to M_1 \to M_2 \to M \to 0$ is split exact.

3. $\exists M'$ s.t. $M \oplus M' \cong F$: free.

Prop 1.1.3. TFAE

- 1. M is projective.
- 2. $\forall 0 \to M_1 \to M_2 \to M \to 0$ is split exact.
- 3. $\exists M'$ s.t. $M \oplus M' \cong F$: free.

Prop 1.1.4. TFAE

- 1. M is injective.
- 2. $\forall 0 \to M \to M_2 \to M_3 \to 0$ is split exact.

Prop 1.1.5. projective \implies flat.