Localization

Kai-Chi Huang

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1 Localization of rings

Remark 1. In this lecture, all rings are assumed to be commutative rings with 1.

Recall that $\mathbb{Q}$ can be constructed as a "fraction field" of $\mathbb{Z}$. For general rings, fraction field may not exist, but nevertheless we can construct its "ring of fractions".

Roughly speaking, we want to make the smallest ring such that a subset S of R become units.

Def 1. Let R be a ring, $S \subseteq R$ be a multiplicatively closed subset containing 1. We define a ring R_S to be the **localization of R at S**, with a ring homomorphism $\pi : R \to R_S$, if they satisfy the following universal property:

> For any ring T and any ring homomorphism $\psi : R \to T$ with $\psi(1) = 1$ such that $\psi(s)$ is a unit in T for all $s \in S$, there exist a unique ring homomorphism $\Psi : R_S \to T$ such that $\psi = \Psi \circ \pi$.

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[dr, "\psi"'] & R_S \arrow[d, "\exists!\Psi"] \\ & T \end{tikzcd}$$

Theorem 1. The localization R_S exists and is unique up to isomorphism.

Proof. We define an equivalence relation $\sim$ on $R \times S$ with

$(r_1, s_1) \sim (r_2, s_2) \Longleftrightarrow x(s_2r_1 - s_1r_2) = 0$ for some $x \in S$.

- Reflexive : $(r, s) \sim (r, s)$, OK.
- Symmetric : $(r_1, s_1) \sim (r_2, s_2) \Longrightarrow (r_2, s_2) \sim (r_1, s_1)$, OK.
- Transitive : If $(r_1, s_1) \sim (r_2, s_2)$ and $(r_2, s_2) \sim (r_3, s_3)$, then exist $x, y \in S$ such that $x(s_2r_1 - s_1r_2) = 0, y(s_3r_2 - s_2r_3) = 0$, hence $xs_2r_1 = xs_1r_2, ys_3r_2 = ys_2r_3$. Multiply them by ys_3 and xs_1 respectively, we get $xys_3s_2r_1 = xys_3s_1r_2 = xys_2s_1r_3$, that is $(xys_2)(s_3r_1 - s_1r_3) = 0$, so $(r_1, s_3) \sim (r_3, s_1)$.

  Note that the scalar x in $x(s_2r_1 - s_1r_2) = 0$ is required for the transitivity to hold.

Now let $R_S = (R \times S)/\sim$ be the set of equivalence classes. Also, we denote (r, s) by $\frac{r}{s}$.

We can further turn R_S into a ring by allowing addition and multiplication:

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2r_1 + s_1r_2}{s_1s_2}, \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2}$$

After some routine checking, we can confirm that these operations are well-defined.

- If $\frac{r_1}{s_1} = \frac{r_1'}{s_1'}$, then we want to show that $\frac{s_2r_1+s_1r_2}{s_1s_2} = \frac{s_2r_1'+s_1'r_2}{s_1's_2}$.

  Because we have a $x \in S$ such that $x(s_1'r_1 - s_1r_1') = 0$, so $x(s_1's_2(s_2r_1 + s_1r_2) - s_1s_2(s_2r_1' + s_1'r_2)) = s_2^2x(s_1'r_1 - s_1r_1') = 0$.

- If $\frac{r_1}{s_1} = \frac{r_1'}{s_1'}$, then we want to show that $\frac{r_1 r_2}{s_1 s_2} = \frac{r_1' r_2}{s_1' s_2}$.

  Because we have a $x \in S$ such that $x(s_1' r_1 - s_1 r_1') = 0$, so $x(s_1' s_2 r_1 r_2 - s_1 s_2 r_1' r_2) = s_2 x(s_1' r_1 - x s_1 r_1') r_2 = 0$.

In fact, many elementary operations of fractions remains valid in this generalized version, for example, we can do reduction since $\frac{xr}{xs} = \frac{r}{s}$ for all $x \in S$.

In this ring, we have $1 = \frac{1}{1}$ and $0 = \frac{0}{1}$. Define the ring homomorphism $\pi : R \to R_S$ by $\pi(r) = \frac{r}{1}$. It is easy to check that π is a well defined ring homomorphism. More, $\pi(s) = \frac{s}{1}$ is a unit for all $s \in S$ since $\frac{s}{1} \cdot \frac{1}{s} = \frac{s}{s} = \frac{1}{1}$.

Now let us consider the universal property.

Let $\psi : R \to T$ be a ring homomorphism, and $\psi(s)$ is a unit for all $s \in S$. If the Ψ in the universal property exists, it must have $\Psi(\frac{r}{1}) = \psi(r)$, so $\Psi(\frac{r}{s}) = \Psi(\frac{r}{1} \cdot (\frac{s}{1})^{-1}) = \Psi(\frac{r}{1}) \cdot \Psi(\frac{s}{1})^{-1} = \psi(r)\psi(s)^{-1}$. (so if it exists, it must be unique.)

To check this Ψ is well-defined, consider a pair $\frac{r}{s} = \frac{r'}{s'}$, we have $x(s'r - r's) = 0$ for some $x \in S$. Hence $\psi(x)(\psi(s')\psi(r) - \psi(r')\psi(s)) = 0$, and $\psi(x)\psi(r)\psi(s)^{-1} = \psi(x)\psi(r')\psi(s')^{-1}$. Because $\psi(x)$ is a unit in T, we have $\psi(r)\psi(s)^{-1} = \psi(r')\psi(s')^{-1}$, i.e. $\Psi(\frac{r}{s}) = \Psi(\frac{r'}{s'})$.

It's easy to check that Ψ is a ring homomorphism, so R_S satisfies the universal property. By the routine argument of universal property, R_S is unique up to isomorphism. ☐

Notice that in general, π may not be injective. So let us consider its kernel:

Proposition 1. $\ker \pi = \{r \in R \mid \exists s \in S \text{ such that } sr = 0\}$

Proof. $r \in \ker \pi \iff \pi(r) = 0 \iff \frac{r}{1} = \frac{0}{1} \iff \exists s \in S$ such that $s(1 \cdot r - 1 \cdot 0) = 0$ $\iff \exists s \in S$ such that $sr = 0$. ☐

Corollary 1. $\pi : R \to R_S$ is an injection if and only if S contains no zero divisors of R.

Corollary 2. If R is an integral domain, let $S = R \setminus \{0\}$, then R_S is a field, and π is an injection (so R is a subring of R_S). This R_S is called the **fraction field** of R.

Example 1. $\mathbb{Z}_S = \mathbb{Q}$. This is the classical construction of rational number from integers.

Example 2. Let K be a field, then $K[x]_S = K(x)$, the field of rational functions in x.

Example 3. Let $a \in R$, then $S = \{a^n \mid n \geq 0\}$ is multiplicatively closed and contains 1. R_S is then a ring with denominators of powers of a. Such R_S is often denoted by R_a.

For example, $\mathbb{Z}_2 = \mathbb{Z}[\frac{1}{2}] \subseteq \mathbb{Q}$

Example 4. Let $P \subsetneq$ be a prime ideal in R, then $S = R \setminus P$ is multiplicatively closed (as $x, y \notin P \Rightarrow xy \notin P$) and contains 1. Such R_S is called *localiztion at prime* P, and often denoted by R_P.

For example, $\mathbb{Z}_{\langle p \rangle} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, p \nmid b\} \subseteq \mathbb{Q}$.

After constructing the localization ring R_S, we can now consider the relation between ideals in R and ideals in R_S.

Def 2.

1. Let I be an ideal of R, then its *extension* to R_S is defined as $I^e := R_S \pi(I)$.
2. Let J be an ideal of R_S, then its *contraction* to R is defined as $J^c := \pi^{-1}(J)$.

Proposition 2.

1. For any ideal J of R_S, $(J^c)^e = J$.
2. For any ideal I of R, $(I^e)^c = \{r \in R \mid sr \in I \text{ for some } s \in S\}$.
   In particular, $I^e = R_S \iff I \cap S \neq \emptyset$.
3. If R is Noetherian, then R_S is also Noetherian.
4. There is a 1-1 correspondence:

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{prime ideals } P \text{ of } R \\ \text{with } P \cap S = \emptyset \end{array} \right\} & \longleftrightarrow & \{\text{prime ideals of } R_S\} \\ I & \longmapsto & I^e \\ J^c & \longleftarrow\!\shortmid & J \end{array}$$

Proof.

1. "$\subseteq$": Since $\pi(\pi^{-1}(J)) \subseteq J$, so $(J^c)^e = R_S(\pi(\pi^{-1}(J))) \subseteq R_S J = J$.
   "$\supseteq$": For $x = \frac{r}{s} \in J$, $\frac{r}{1} = \frac{s}{1}\frac{r}{s} \in J$, so $r \in \pi^{-1}(J) = J^c$. Then, $x = \frac{r}{s} = \frac{1}{s}\frac{r}{1} = \frac{1}{s}\pi(r) \in R_S\pi(J^c) = (J^c)^e$.
2. "$\supseteq$": If $r \in R$ and exists some $s \in S$ such that $sr \in I$, then $\pi(r) = \frac{r}{1} = \frac{1}{s}\frac{sr}{1} \in I^e$, hence $r \in (I^e)^c$.
   "$\subseteq$": For $r \in (I^e)^c$, $\pi(r) = \frac{r}{1} \in I^e = R_S\pi(I)$. So we have $\frac{r}{1} = \sum_{i=1}^n \frac{a_i}{s_i} r_i$ for some $a_i \in R, s_i \in S, r_i \in I$. But $\sum_{i=1}^n \frac{a_i}{s_i} r_i = \frac{\sum_{i=1}^n \left(\prod_{j\neq i} s_j\right) a_i r_i}{\prod_{i=1}^n s_i} = \frac{a}{s}$ with $a \in I, s \in S$. So $\frac{r}{1} = \frac{a}{s}$, that means there exists $x \in S$ that $x(sr - a) = 0$, then $xsr = xa \in I$. But $xs \in S$, so $r \in \text{RHS}$.
   Also, $I^e = R_S \iff \frac{1}{1} \in I^e \iff 1 \in (I^e)^c \iff s \cdot 1 = s \in I$ for some $s \in S \iff I \cap S \neq \emptyset$.
3. If R is Noetherian, consider an ascending chain of ideals of R_S : $J_1 \subseteq J_2 \subseteq \cdots$. Contracting this chain will give an ascending chain of ideals of R : $J_1^c \subseteq J_2^c \subseteq \cdots$. Because R is Noetherian, this chain must stop at some J_n^c (i.e. $J_n^c = J_{n+1}^c = \cdots$). If we extend this chain back to R_S, we'll get $(J_1^c)^e \subseteq (J_2^c)^e \subseteq \cdots$, and this chain stop at $(J_n^c)^e$. By 1., we have $(J_i^c)^e = J_i$, so this chain is identical to the original chain, so the original chain stops at J_n. Now we can conclude that R_S is Noetherian.
4. "$\supseteq$": Let $Q \subsetneq R_S$ be a prime ideal of R_S, then $Q^c \cap S = \emptyset$. (otherwise, by 2., $Q = R_S$.) For $x, y \in R$ such that $xy \in Q^c$, we have $\pi(xy) = \frac{xy}{1} \in Q$. But $\frac{xy}{1} = \frac{x}{1}\frac{y}{1}$ and Q is prime, so $\frac{x}{1} \in Q$ or $\frac{y}{1} \in Q$. This implies $x = \pi^{-1}(\frac{x}{1}) \in Q^c$ or $y \in Q^c$, hence Q^c is a prime ideal.
   "$\subseteq$": Let P be a prime ideal of R with $P \cap D = \emptyset$. By 2., $P^e \subsetneq R_S$. For $\frac{x}{s}, \frac{y}{t} \in R_S$, if $\frac{xy}{st} \in P^e$, then $\frac{xy}{1} = \frac{st}{1}\frac{xy}{st} \in P^e$, so $xy \in (P^e)^c$. Again by 2., there exists $z \in S$ such that $zxy \in P$. But P is prime, and $z \notin P$, so $x \in P$ or $y \in P$. From this we can get $\frac{x}{1} \in P^e$ or $\frac{y}{1} \in P^e$, so $\frac{x}{s} \in P^e$ or $\frac{y}{t} \in P^e$. This shows P^e is a prime ideal of R_S.
   More, if $r \in (P^e)^c$, there is a $s \in S$ such that $sr \in P$, but $s \notin P$, so $r \in P$. This is just saying $(P^e)^c = P$.
   Combine with the fact that $(Q^c)^e = Q$, it's now clear that $\cdot^e$ and $\cdot^c$ are inverses of each other, hence form a bijection between these two sets of prime ideals.

$\square$

Corollary 3. Localization at prime ideal P results in a local ring R_P with the unique maximal ideal P^e.

Proof. For maximal ideal $Q \subsetneq R_P$, then its contraction must have $Q^c \cap S = \{0\}$. Since $S = R \setminus P$, so $Q^c \subseteq P$, $Q = (Q^c)^e \subseteq P^e$. But Q is maximal, so $Q = P^e$. $\square$

2 Localization of modules

The concept of localization can also be applied on R-modules. Its construction is almost the same as the ring version:

Def 3. Let M be an R-module, and S be a multiplicatively closed subset of R containing 1.

We define a R_S-module M_S to be the **localization of M at S**, with an R-module homomorphism $\pi : M \to M_S$, if they satisfy the following universal property:

> For any R-module N such that the multiplication map by $s : \begin{array}{ccc} N & \to & N \\ x & \mapsto & sx \end{array}$ is bijective for every $s \in S$, and any R-module homomorphism $\psi : M \to N$, there exist a unique R-module homomorphism $\Psi : M_S \to N$ such that $\psi = \Psi \circ \pi$.

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "\psi"'] & M_S \arrow[d, "\exists!\Psi"] \\ & N \end{tikzcd}$$

Theorem 2. The localization M_S exists and is unique up to isomorphism.

Proof. The proof is essentially the same as the ring version, by defing an equivalence relation $\sim$ on $M \times S$ with

$$(a_1, s_1) \sim (a_2, s_2) \iff x(s_2a_1 - s_1a_2) = 0 \text{ for some } x \in S.$$

and let $M_S = (M \times S)/\sim$.

The only difference is that we turn M_S into an R_S-module this time:

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2a_1 + s_1a_2}{s_1s_2}, \quad \frac{r}{s} \cdot \frac{a}{b} = \frac{ra}{sb}$$

Also, we define $\pi : M \to M_S$ by $\pi(a) = \frac{a}{1}$.

Notice that we've already upgraded M_S into an R_S-module, since the R_S-scalar multiplication is well-defined. $\square$

Corollary 4. $\ker \pi = \{a \in M \mid \exists s \in S \text{ such that } sa = 0\}$

Remark 2. The ring R can be regarded as a self-module. Let I be an ideal of R, then I can also be regarded as a submodule of R.

In this perspective, the extension of ideal is just the localization of module:

$$I^e = R_S\pi(I) = I_S$$

In fact, M_S is just the *extension of R_S scalars* from the R-module M.

Proposition 3. $M_S \cong R_S \otimes_R M$ as R_S-modules.

Proof. First, R_S can be regarded as a R-module by restricting scalar products to R. Since the map

$$\phi : \begin{array}{ccc} R_S \times M & \longrightarrow & M_S \\ (\frac{r}{s}, a) & \longmapsto & \frac{ra}{s} \end{array}$$

is R-bilinear, it induces an R-module homomorphism

$$\psi : \begin{array}{ccc} R_S \otimes_R M & \longrightarrow & M_S \\ \frac{r}{s} \otimes a & \longmapsto & \frac{ra}{s} \end{array}$$

Because all elements in M_S can be written as the form $\frac{a}{s}$, and $\psi(\frac{1}{s} \otimes a) = \frac{a}{s}$, so ψ is onto.

If $\psi(\frac{r}{s} \otimes a) = \frac{ra}{s} = \frac{0}{1}$, by definition exists $u \in S$ such that $u(1 \cdot ra - s \cdot 0) = ura = 0$, so $\frac{r}{s} \otimes a = \frac{1}{us} \otimes ura = 0$, this means ψ is 1-1.

Hence ψ is an isomorphism between $R_S \otimes_R M$ and M_S.

If we upgrade $R_S \otimes_R M$ to R_S-module by multiplying R_S scalars to the left side ($\frac{b}{c}(\frac{r}{s} \otimes a) = \frac{br}{cs} \otimes a$), then ψ can also be upgraded to an R_S-module isomorphism. □

Proposition 4. Let M, N be R-modules, and $\varphi : M \to N$ be an R-module homomorphism. Then there is an induced R_S-module homomorphism φ_S such that

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\pi_M"] & N \arrow[d, "\pi_N"] \\ M_S \arrow[r, "\varphi_S"'] & N_S \end{tikzcd}$$

commutes.

Proof. First we use the tensor product form $M_S \cong R_S \otimes_R M$, then the localization map becomes $\pi'_M : a \mapsto \frac{1}{1} \otimes a$.

Now, just let

$$\varphi_S = \mathbf{1}_{R_S} \otimes \varphi : \begin{array}{ccc} R_S \otimes_R M & \longrightarrow & R_S \otimes_R N \\ \frac{r}{s} \otimes a & \longmapsto & \frac{r}{s} \otimes \varphi(a) \end{array}$$

, then it's automatically well-defined (tensor product of functions), and $\varphi'_S(\pi'_M(a)) = \varphi_M(\frac{1}{1} \otimes a) = \frac{1}{1} \otimes \varphi(a) = \pi'_N(\varphi(a))$.

Again by multiplying R_S scalar to the left side, φ_S can be upgraded to an R_S-module homomorphism.

Now we can recover the desired homomorphism:

$$\varphi_S : \begin{array}{ccc} M_S & \longrightarrow & N_S \\ \frac{a}{b} & \longmapsto & \frac{\varphi(a)}{b} \end{array}$$

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\pi_M"] & N \arrow[d, "\pi_N"] \\ M_S \arrow[r, "\varphi_S"'] \arrow[d, "\wr"] & N_S \arrow[d, "\wr"] \\ R_S \otimes_R M \arrow[r, "\varphi'_S"', "=\mathbf{1}_{R_S} \otimes \varphi"''] & R_S \otimes_R N \end{tikzcd}$$

□

Now we show that localization at S behaves like a functor.

Proposition 5. Localization at S is a covariant functor from the category of R-modules to the category of R_S-modules.

Proof.

- $(\mathbf{1}_M)_S(\frac{a}{b}) = \frac{\mathbf{1}_M(a)}{b} = \frac{a}{b} = \mathbf{1}_{M_S}(\frac{a}{b})$.
- $(g \circ f)_S(\frac{a}{b}) = \frac{(g \circ f)(a)}{b} = \frac{g(f(a))}{b} = (g_S \circ f_S)(\frac{a}{b})$

□

As a functor, $\cdot_S$ commutes with many algebraic operations, one of the most important properties is the exactness:

Proposition 6. Localization at S is exact. That is, for any short exact sequence of R-modules

$$\mathbf{C} : 0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$$

, the induced sequence of R_S-modules

$$\mathbf{C_S} : 0 \to L_S \xrightarrow{\psi_S} M_S \xrightarrow{\varphi_S} N_S \to 0$$

is also exact.

Proof.

- Exactness at L_S:

  For $\frac{a}{b} \in L_S$, if $\psi_S(\frac{a}{b}) = 0$, then $\frac{\psi(a)}{b} = 0$ since ψ is 1-1. By definition $\exists x \in S$ such that $x\psi(a) = \psi(xa) = 0$. Now $\frac{a}{b} = \frac{xa}{xb} = 0$, so ψ_S is 1-1.

- Exactness at N_S:

  For every $\frac{a}{b} \in N_S$, we have some $x \in M$ such that $\varphi(x) = a$ since φ is onto. So $\varphi_S(\frac{x}{b}) = \frac{\varphi(x)}{b} = \frac{a}{b}$, φ_S is onto.

- Exactness at M_S ($\operatorname{im}\psi_S = \ker\varphi_S$):

  ”$\supseteq$”: For $\frac{a}{b} \in \ker\varphi_S \subseteq M_S$, $\varphi_S(\frac{a}{b}) = \frac{\varphi(a)}{b} = 0$, so $\exists x \in S$ such that $x\varphi(a) = \varphi(xa) = 0$. This imply $xa \in \ker\varphi = \operatorname{im}\psi$, so $\exists c \in L$ such that $\psi(c) = xa$. Now $\psi_S(\frac{c}{xb}) = \frac{\psi(c)}{xb} = \frac{xa}{xb} = \frac{a}{b}$, then $\frac{a}{b} \in \operatorname{im}\psi_S$.

  ”$\subseteq$”: For $\frac{a}{b} \in \operatorname{im}\psi_S \subseteq M_S$, $\exists \frac{c}{d} \in L_S$ such that $\psi_S(\frac{c}{d}) = \frac{\psi(c)}{d} = \frac{a}{b}$. So $\varphi_S(\frac{a}{b}) = \varphi_S(\frac{\psi(c)}{d}) = \frac{\varphi(\psi(c))}{d} = \frac{0}{d} = 0$, thus $\frac{a}{b} \in \ker\varphi_S$.

$\square$

Proposition 7. Let I, J be ideals of R, and M, N, L be R-modules. For 1. $\sim$ 3., assume N, L are submodules of M.

1. $(N + L)_S = N_S + L_S$
2. $(N \cap L)_S = N_S \cap L_S$
3. N_S is a submodule of M_S, and $M_S/N_S \cong (M/N)_S$
4. $(I + J)_S = I_S + J_S$
5. $(I \cap J)_S = I_S \cap J_S$
6. $R_S/I_S \cong (R/I)_{\bar{S}}$
7. $(\sqrt{I})_S = \sqrt{I_S}$
8. $(L \oplus N)_S \cong L_S \oplus N_S$
9. $(L \otimes_R N)_S \cong L_S \otimes_{R_S} N_S$

Proof.

1. ”$\subseteq$”: For $x = \frac{a+b}{c} \in (N + L)_S$ with $a \in N, b \in L, c \in S$, we have $x = \frac{a}{c} + \frac{b}{c} \in N_S + L_S$.

   ”$\supseteq$”: For $x = \frac{a}{b} + \frac{c}{d} \in N_S + L_S$, $x = \frac{ad+cb}{bd} \in (N + L)_S$, since $ad \in N, cb \in L, bd \in S$.

2. ”$\subseteq$”: For $x = \frac{a}{b} \in (N \cap L)_S$ with $a \in N \cap L$, then $\frac{a}{b} \in N_S$ and $\frac{a}{b} \in L_S$.

   ”$\supseteq$”: For $x = \frac{a}{b} \in N_S \cap L_S$, we can write $\frac{a}{b} = \frac{c}{d}$ for some $c \in N$. So $\exists u \in S$ such that $uda = ubc \in N$. Similarly, $\exists u', d' \in S$ such that $u'd'a \in L$. Now $udu'd'a \in N \cap L$, so $x = \frac{udu'd'a}{udu'd'b} \in (N \cap L)_S$.

3. By 1st isomorphism theorem, we have the following exact sequence:

$$0 \to N \to M \to M/N \to 0$$

Since the localization is still exact:

$$0 \to N_S \to M_S \to (M/N)_S \to 0$$

The map $N_S \to M_S$ is injective means N_S can be regarded as a submodule of M_S. Again by 1st isomorphism theorem, we have $M_S/N_S \cong (M/N)_S$.

4. If wee see I, J as submodules of self-module R, this directly follows 1.

5. Also directly by 2.

6. Directly by 3., we have the isomorphism of R-modules:

$$\begin{array}{ccc} R_S/I_S & \cong & (R/I)_S \\ \overline{\frac{a}{b}} & \longmapsto & \frac{\bar{a}}{b} \\ \frac{a}{b} + I_S & \longmapsto & \frac{a+I}{b} \end{array}$$

But in RHS, the fractions is equivalent if the denominator differs by an element $x \in I$:

$$\frac{a+I}{b} = \frac{a+I}{b+x}$$

then we can replace the denominator with $\bar{b}$ without violating the well-definedness, so this is also an isomorphism:

$$\begin{array}{ccc} R_S/I_S & \cong & (R/I)_{\bar{S}} \\ \overline{\frac{a}{b}} & \longmapsto & \frac{\bar{a}}{\bar{b}} \\ \frac{a}{b} + I_S & \longmapsto & \frac{a+I}{b+I} \end{array}$$

By checking the multiplication,

$$\begin{array}{ccc} (\frac{a}{b} + I_S) \cdot (\frac{c}{d} + I_S) & \longmapsto & \frac{a+I}{b+I} \cdot \frac{c+I}{d+I} \\ \frac{ac}{bd} + I_S & \longmapsto & \frac{ac+I}{bd+I} \end{array}$$

we can upgrade the isomorphism to a ring one.

7. "$\subseteq$": For $x \in (\sqrt{I})_S$, we can write $x = \frac{a}{b}$ with $a \in \sqrt{I}, b \in S$. Assume $a^n \in I$, then $x^n = \frac{a^n}{b^n} \in I_S$, so $x \in \sqrt{I_S}$.

   "$\supseteq$": For $x = \frac{a}{b} \in \sqrt{I_S}$, if $(\frac{a}{b})^n = \frac{a^n}{b^n} = \frac{c}{d} \in I_S$, then $\exists y \in S$ such that $yda^n = yb^nc \in I$, so $(yda)^n = y^n d^n a^n \in I$, i.e. $yda \in \sqrt{I}$. Now $x = \frac{yda}{ydb} \in (\sqrt{I})_S$.

8. We say an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\psi"] & M \arrow[r, "\varphi"] & N \arrow[r] \arrow[l, bend left, "l"] & 0 \end{tikzcd}$$

splits, if there exists an R-module homomorphism $l : N \to M$ such that $\varphi \circ l = \mathbf{1}_N$. we call this l a *lifting*.

Recall that in group extension, $0 \to N \to E \to G \to 0$ splits if and only if $E \cong N \rtimes G$ is a semi-direct product.

Similarly, because the additive groups of R-modules are abelian, this exact sequence splits if and only if $M \cong L \oplus N$ and ψ, φ are natural inclusion and projection, respectively.

So let $M = L \oplus N$, then

$$0 \to L \xrightarrow{\psi} L \oplus N \xrightarrow{\varphi} N \to 0$$

is a splitting exact sequence.

Hence,

$$0 \to L_S \xrightarrow{\psi_S} (L \oplus N)_S \xrightarrow{\varphi_S} N_S \to 0$$

is also exact and splits, which means $(L \oplus N)_S \cong L_S \oplus N_S$.

Alternative proof:

Without loss of generality, assume $N, L \subseteq N \oplus L$, i.e. $N \oplus L$ is the internal direct sum of N and L. (If not, take their embedding $N \cong N' \subseteq N \oplus L$ and $L \cong L' \subseteq N \oplus L$.) Then $N \oplus L = N + L$ with $N \cap L = \{0\}$.

By 1., $(N \oplus L)_S = (N + L)_S = N_S + L_S$. Also, by 2, if $\frac{a}{b} \in N_S \cap L_S = (N \cap L)_S$, then $\frac{a}{b} = 0$. So $N_S \cap L_S = \{0\}$, which means $N_S + L_S = N_S \oplus L_S$.

9.
$$\begin{aligned} L_S \otimes_{R_S} N_S &\cong L_S \otimes_{R_S} (R_S \otimes_R N) \\ &\cong (L_S \otimes_{R_S} R_S) \otimes_R N \quad (\star) \\ &\cong L_S \otimes_R N \\ &\cong (R_S \otimes_R L) \otimes_R N \\ &\cong R_S \otimes_R (L \otimes_R N) \\ &\cong (L \otimes_R N)_S \end{aligned}$$

where $(\star)$ is because the associativity of tensor products holds even when the two tensor products are over different rings, as long as the middle module is both R- and R_S-module.

$\square$