Algebra

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1 Group theory

Def 1. A non-empty set G with a binary function $f : G \times G \to G, (a, b) \mapsto ab$ is a *group* if it satisfies

1. $(ab)c = a(bc)$.
2. $\exists 1 \in G$ s.t. $1a = a1 = a, \forall a \in G$.
3. $\exists a^{-1} \in G$ s.t. $aa^{-1} = a^{-1}a = 1$.

CONCON

Def 2. Let G be a group. Then G is said to be *abelian* if $\forall a, b \in G, ab = ba$.

Ex 1. Let G be a semigroup. Then TFAE (the following are equivalent)

1. G is a group.
2. For all $a, b \in G$ and the equations $bx = a, yb = a$, each of them has a solution in G.
3. $\exists e \in G$ s.t. $ae = a \ \forall a \in G$ and if we fix such e, then $\forall b \in G \ \exists b' \in G$ s.t. $bb' = e$.

Ex 2. Let G be a group. Show that

1. $\forall a \in G, a^2 = 1$, then G is abelian.
2. G is abelian $\iff \forall a, b \in G, (ab)^n = a^n b^n$ for three consecutive integer n.

Def 3. Let G be a group and $H \subseteq G, H \neq \phi$. Then H is said to be a subgroup of G, denoted by $H \leq G$, if

1. $\forall a, b \in H, ab \in H$.
2. $1 \in H$.
3. $\forall a \in H, a^{-1} \in H$.

<u>useful criterion</u>: $H \leq G \iff \forall a, b \in H, ab^{-1} \in H$.

pf:

$\Rightarrow$ $\quad b \in H \implies b^{-1} \in H$, and $a \in H$, so $ab^{-1} \in H$.

$\Leftarrow$
1. $H \neq \phi \implies \exists a \in H \implies aa^{-1} = 1 \in H$.
2. $1, a \in H \implies 1a^{-1} = a^{-1} \in H$.
3. $a, b^{-1} \in H \implies a(b^{-1})^{-1} = ab \in H$. $\square$

Eg 1. $(\mathbb{Z}, +, 0) \leq (\mathbb{Q}, +, 0) \leq (\mathbb{R}, +, 0) \leq (\mathbb{C}, +, 0)$; $(\mathbb{Q}^\times, \times, 1) \leq (\mathbb{R}^\times, \times, 1) \leq (\mathbb{C}^\times, \times, 1)$

Eg 2.

- Special linear group $\mathrm{SL}(n, \mathbb{F}) = \{ A \in \mathrm{GL}(n, \mathbb{F}) \mid \det A = 1 \}$
- Orthogonal group $\mathrm{O}(n) = \{ A \in \mathrm{GL}(n, \mathbb{R}) \mid A^{\mathrm{t}} A = I_n \}$
- Unitary group $\mathrm{U}(n) = \{ A \in \mathrm{GL}(n, \mathbb{C}) \mid A^* A = I_n \}$
- Special orthogonal group $\mathrm{SO}(n) = \mathrm{SL}(n, \mathbb{R}) \cap \mathrm{O}(n)$
- Special unitary group $\mathrm{SU}(n) = \mathrm{SL}(n, \mathbb{C}) \cap \mathrm{U}(n)$

Def 4. Let $f : G_1 \to G_2$. f is called an *isomorphism* if

1. f is 1-1 and onto.
2. $\forall a, b \in G_1, f(ab) = f(a)f(b)$. (*homomorphism*)

, denoted by $G_1 \cong G_2$.

Remark 1. (practice)

1. $f(1) = 1$.
2. $f(a^{-1}) = f(a)^{-1}$.
3. If f is an isomorphism, then $\exists f^{-1}$ is also a homomorphism.

Eg 3.

- $\mathrm{U}(1) = \{\, z \in \mathbb{C}^\times \mid \bar{z}z = 1 \,\}, z = \cos\theta + \sin\theta i$
- $\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$

notice that $\mathrm{U}(1) \cong \mathrm{SO}(2)$. $S^1 = \{\, (a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1 \,\}$,

Eg 4. Let $A \in \mathrm{SU}(2) \implies A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \alpha\bar{\alpha} + \beta\bar{\beta} = 1, \alpha, \beta \in \mathbb{C}$.

<u>Quaternion</u>($\mathbb{H} = \{\, a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \,\} i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j(\implies ij = -ji)$
$x = a + bi + cj + dk, \bar{x} = a - bi - cj - dk N(x) = x\bar{x} = a^2 + b^2 + c^2 + d^2 x \neq 0, N(x) \neq 0, x^{-1} = \frac{1}{N(x)}\bar{x}$
$x = a + bi + cj + dk = (a + bi) + (c + di)j \mathrm{SU}(2) \cong \{\, x \in \mathbb{H}^\times \mid N(x) = 1 \,\} S^3 = \{\, (a, b, c, d) \in \mathbb{R}^4 \mid a^2 + b^2 + c^2 + d^2 = 1 \,\}$
$\bigstar S^1, S^3$

Ex 3. Find a way to regard $M_{n\times n}(\mathbb{H})$ as a subset of $M_{2n\times 2n}(\mathbb{C})$, which preserves addition and multiplication, and then there is a way to characterize $\mathrm{GL}(n, \mathbb{H})$.

Def 5 (symplectic group)**.** $\mathrm{Sp}(n, \mathbb{F}) = \{\, A \in \mathrm{GL}(2n, \mathbb{F}) \mid A^{\mathrm{t}} J A = J \,\}$ where $J = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}$.
($A^{\mathrm{t}} J A = J$ preserving non-degenerate skew-symmetric forms)
$\mathrm{Sp}(n) = \{\, A \in \mathrm{GL}(n, \mathbb{H}) \mid A^* A = I_n \,\}$.

Ex 4. Show $\mathrm{Sp}(n) \cong \mathrm{U}(2n) \cap \mathrm{Sp}(n, \mathbb{C})$.

$\mathrm{SU}(2) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$