Algebra

June 6, 2017

1 Introduction to Homological Algebra

1.1 Projective, Injective and Flat modules (week 14)

Def 1.

- $M \in \mathbf{Mod}_R$ is **projective** if $\mathrm{Hom}(M, \cdot)$ preserves the *right* exactness.
- $N \in \mathbf{Mod}_R$ is **injective** if $\mathrm{Hom}(\cdot, N)$ preserves the *right* exactness.
- $M \in \mathbf{Mod}_R$ is **flat** if $M \otimes \cdot$ preserves the *left* exactness.

Fact 1.1.1.

- M is projective $\iff$ $$\begin{tikzcd} & M \arrow[dl, "\exists \tilde{f}"'] \arrow[d, "f"] & \\ M_2 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$
- N is injective $\iff$ $$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "g"'] & M_2 \arrow[dl, "\exists \tilde{g}"] \\ & N & \end{tikzcd}$$
- free $\implies$ projective: If $X = \{x_i \mid i \in \Lambda\}$ and $f : x_i \mapsto a_i$. Since β onto, exists b_i so that $\beta(b_i) = a_i$. we can then set $\tilde{f} : x_i \mapsto b_i$ by the universal property of free module.
$$\begin{tikzcd} & F(X) \arrow[dl, "\exists \tilde{f}"'] \arrow[d, "f"] & \\ M_2 \arrow[r, "\beta"] & M_3 \arrow[r] & 0 \end{tikzcd}$$
- free $\implies$ flat: Let $F \cong R^{\otimes \Lambda}$ be a free module, and M_1, M_2 be two modules such that $0 \to M_1 \to M_2$. Since $R \otimes_R M \cong M$, we have
$$\begin{aligned} & 0 \to M_1 \to M_2 && \text{exact} \\ \implies & 0 \to R \otimes M_1 \to R \otimes M_2 && \text{exact} \\ \implies & 0 \to \bigoplus_{i \in \Lambda} R \otimes M_1 \to \bigoplus_{i \in \Lambda} R \otimes M_2 && \text{exact} \\ \overset{(a)}{\implies} & 0 \to R^{\otimes \Lambda} \otimes M_1 \to R^{\otimes \Lambda} \otimes M_2 && \text{exact} \\ \implies & 0 \to F \otimes M_1 \to F \otimes M_2 && \text{exact} \end{aligned}$$
Where (a) is by the fact that $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$. Thus F flat.
- If S is a multiplication closed set in R with $1 \in S$, then
$$0 \to M \to N \to L \to 0 \implies 0 \to M_S \to N_S \to L_S \to 0.$$
We know that $M_S \cong R_S \otimes_R M$. So R_S is a flat R-module. e.g. $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

For any $M \in \mathbf{Mod}_R$, a projective module N such that $N \to M \to 0$ could be easily find: Simply let $N = F$, a free module on the set M.

Now we shall ask for any module M, does there exist $N \in \mathbf{Mod}_R$ such that N is injective and $0 \to M \to N$?

Theorem 1 (Boer's criterion)**.** N is injective $\iff$ $\forall\, I \subset R$, and a homomorphism f, there exists a homomorphism h such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists\, h"] \\ & N & \end{tikzcd}$$

Proof. "$\Rightarrow$": See I as an R module, then it is immediate by the definition of injective module.

"$\Leftarrow$: Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "g"] & M_2 \\ & N & \end{tikzcd}$$

Let $S \triangleq \{(M, \rho) \mid M_1 \subseteq M \subseteq M_2 \text{ and } \rho : M \to N \text{ extends } g\} \neq \varnothing$ since $(M_1, g) \in S$.

By the routinely proof using Zorn's lemma, exists a maximal element $(M^*, \mu) \in S$.

We claim that $M^* = M_2$. If not, pick $a \in M_2 \setminus M^*$ and let $M' \triangleq M^* + Ra \supsetneq M^*$, $I \triangleq \{r \in R \mid ra \in M^*\}$. Define $f : I \to N$ with $r \mapsto \mu(ra)$. Then we have an extension $h :: R \to N$ of f.

Now, let $\mu' :: M' \to N = x + ra \mapsto \mu(x) + h(r)$. We shall prove that this map is well-defined: If $x_1 + r_1 a = x_2 + r_2 a$, then $(r_1 - r_2)a = x_2 - x_1 \in M \implies r_1 - r_2 \in I$. So $h(r_1) - h(r_2) = f(r_1 - r_2) = \mu((r_1 - r_2)a) = \mu(x_2) - \mu(x_1)$, which prove μ' is well defined, and the existence of μ' contradict the fact that (M^*, μ) is maximal. □

Def 2. M is **divisible** if $\forall\, x \in M, r \in R \setminus \{0\}$, there exists $y \in M$ such that $x = ry$, i.e. $rM = M \quad \forall\, r \in R \setminus \{0\}$.

Prop 1.1.1.

1. Every injective module N over an integral domain is divisible.

   Proof. For any x_0 and $r_0 \in R \setminus \{0\}$. Let $I = \langle r_0 \rangle \subset R$. As long as R is an integral domain, $I \cong R$ as an R-module, so the R-module homomorphism $f :: I \to N = rx_0 \mapsto rr_0$ is well-defined. Since N injective, this map extends to $h :: R \to N$. Let $y_0 \triangleq h(1)$, then $r_0 y_0 = r_0 h(1) = h(r_0) = x_0$. Thus N injective. □

2. Every divisible module N over an PID is injective.

   Proof. For any $I \subseteq R$ and a homomorphism $f :: I \to N$, if $I = 0$ then $h = x \mapsto 0$ is always an extension of f. So assume $\forall\, I \neq 0$. Since R is a PID, $I = \langle r_0 \rangle$ for some $r_0 \neq 0 \in R$.' By the fact that N divisible, exists $y_0 \in N$ such that $r_0 y_0 = x_0 \triangleq f(r_0)$.

   Now we could define $h :: R \to N$ by $1 \mapsto y_0$. Then $h(r_0) = r_0 h(1) = r_0 y_0 = x_0$, thus h is an extension of f and N injective. □

3. If R is a PID, then any quotient N of a injective R-module M is injective.

   Proof. By 2., $rM = M$ for any $r \neq 0$, thus $rN = N$ for any $r \neq 0$, and hence N injective. □

Theorem 2. For any $M \in \mathbf{Mod}_R$, exists N injective and contains M.

Proof.

Case 1: $R = \mathbb{Z}$.

Let $X = \{x_i\}_{i \in \Lambda}$ be a generating set for M and F is free on X. Let f be the natural map from f to M. then $M \cong F/\ker f$.

Define $F' = \bigoplus_{i \in \lambda} \mathbb{Q} e_i \subset F$, which is obviously a divisible $\mathbb{Z}$-module. Then $M \subseteq F'/\ker f \triangleq M'$, where M' is injective by proposition 1.1.1.

Case 2: R arbitrary.

We can regard any M as a $\mathbb{Z}$-module, then there exists an injective module $N_0 \supset M$. Now, we have an R-module $N \triangleq \mathrm{Hom}_{\mathbb{Z}}(R, N_0)$ with multiplication $rf(x) \triangleq x \mapsto f(xr)$.

We claim that N is injective. For any $f :: M_1 \to N$, and a homomorphism $\alpha :: M_1 \to M_2$, then α could be take as a $\mathbb{Z}$-module homomorphism. Define $f' :: M_1 \to N_0$ by $x \mapsto f(x)(1)$. Since N_0 injective, exists h', a $\mathbb{Z}$ module homomorphism from M_2 to N_0.

$$\begin{tikzcd}
 & x \arrow[ddddd, bend right, mapsto] & & \\
 & \Cap & & \\
0 \arrow[r] & M_1 \arrow[r, "\alpha"] \arrow[d, "f"] & M_2 \arrow[dl, dashed, "\exists\, h(?)"] & \\
 & N & \cong & Hom_{\mathbb{Z}}(R, N_0) \\
 & \Cup & & \\
 & f(x) & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x \arrow[ddddd, bend right, mapsto] & \\
 & \Cap & \\
0 \arrow[r] & M_1 \arrow[r, "\alpha"] \arrow[d, "f'"] & M_2 \arrow[dl, "\exists\, h'"] \\
 & N_0 & \\
 & \Cup & \\
 & f(x)(1) &
\end{tikzcd}$$

Now, define

$$\begin{aligned}
h : M_2 &\longrightarrow N \\
y &\longmapsto h(y) : R \longrightarrow N_0 \\
&\qquad\qquad\quad 1 \longmapsto h'(y) \\
&\qquad\qquad\quad r \longmapsto h'(ry)
\end{aligned}$$

We check that h is well-defined.

- $h(y) \in \mathrm{Hom}_{\mathbb{Z}}(R, N_0)$

  $$h(y)(r_1 + r_2) = h'((r_1 + r_2)y) = h'(r_1 y + r_2 y) = h'(r_1 y) + h'(r_2 y) = h(y)(r_1) + h(y)(r_1)$$

- $h \in \mathrm{Hom}_R(M_2, N)$

  $$\begin{aligned}
  h(r_1 y_1 + y_2)(r) &= h'(r(r_1 y_1 + y_2)) = h'(r r_1 y_1 + r y_2) \\
  &= h'(r r_1 y_1) + h'(r y_2) \\
  &= h(y)(r r_1) + h(y_2)(r) \\
  &= (r_1 h(y))(r) + h(y_2)(r)
  \end{aligned}$$

- Show diagram commute $f = h \circ \alpha$ Fix $y \in M_1$, then $\forall\, r \in R$:

  $$\begin{aligned}
  (h \circ \alpha)(y)(r) &= h(\alpha(y))(r) = h'(r\alpha(y)) \\
  &= h'(\alpha(ry)) = f'(ry) \\
  &= f(ry)(1) = rf(y)(1) \\
  &= f(y)(r)
  \end{aligned}$$

Thus N_0 injective.

Now notice that, $\mathrm{Hom}_{\mathbb{Z}}(R, \cdot)$ is a left exact functor, so $M \hookrightarrow N_0$ implies $\mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, N_0)$, thus $M \cong \mathrm{Hom}_R(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, N_0) = N$. $\square$

Prop 1.1.2. TFAE

1. M is projective.
2. Every exact sequence $0 \to M_1 \to M_2 \to M \to 0$ split.

3. $\exists\, M'$ s.t. $M \oplus M' \cong F$: free.

Proof.
(1) $\Rightarrow$ (2) : Since M projective, the map λ with $\beta \circ \lambda = \mathrm{Id}$ exists in the following diagram:

$$
\begin{tikzcd}
& M \arrow[dl, "\exists\,\lambda"'] \arrow[d, "\mathrm{Id}"] & \\
M_2 \arrow[r, "\beta"] & M \arrow[r] & 0
\end{tikzcd}
$$

Then λ is a lifting, so $M_2 \cong M_1 \oplus M$ and $0 \to M_1 \to M_2 \to M \to 0$ split.

(2) $\Rightarrow$ (3): Let F be a free module on a generating set of M, and $\beta :: F \to M$ be the natural map, then $0 \to \ker\beta \to F \to M \to 0$ split, so $F \cong \ker\beta \oplus M$.

(3) $\Rightarrow$ (1): For any $M_2 \to M_3 \to 0$, since $M' \oplus M$ free and thus projective, λ' exists in the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] & M' \oplus M \arrow[r, shift right, "\pi"'] \arrow[d, "\exists\,\lambda'"] & M \arrow[l, shift right, "\mu"'] \arrow[r] \arrow[d, "f"] & 0 \\
& & M_2 \arrow[r, "\beta"] & M_3 \arrow[r] & 0
\end{tikzcd}
$$

Define $\lambda = \lambda' \circ \mu$. Then $\beta \circ \lambda = \beta \circ \lambda' \circ \mu = f \circ \pi \circ \mu = f$. $\square$

Prop 1.1.3. TFAE

1. M is injective.
2. Each exact sequence $0 \to M \to M_2 \to M_3 \to 0$ split.

Proof. (1) $\Rightarrow$ (2): Similar to the projective case, μ exists in the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "\mathrm{Id}"] & M_2 \arrow[dl, dashed, "\exists\,\mu"] \\
& M &
\end{tikzcd}
$$

So $M_2 = M \oplus M_3$.

(2) $\Rightarrow$ (1): By theorem 2, there is a module $N \subset M$ so that N is injective.

Consider $\begin{tikzcd} 0 \arrow[r] & M \arrow[r, shift left, "i"] & N \arrow[l, shift left, "\exists\,\mu"] \arrow[r] & \operatorname{coker} i \arrow[r] & 0 \end{tikzcd}$ split exact and $\mu \circ i = \mathrm{Id}_M$. Since N injective, h' exists in the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "\alpha"] \arrow[d, "f"] \arrow[dd, bend right=60, "i \circ f"'] & M_2 \arrow[ddl, "\exists\,h'"] \\
& M \arrow[d, shift right, "i"'] & \\
& N \arrow[u, shift right, "\mu"'] &
\end{tikzcd}
$$

Let $h = \mu \circ h'$, then $h \circ \alpha = \mu \circ h' \circ \alpha = \mu \circ i \circ f = f$ $\square$

Prop 1.1.4. projective $\implies$ flat.

Proof. Observe that $\bigoplus_{i\in\Lambda} M_i$ is flat if and only if M_i is flat for each i, since if $0 \to N_1 \xrightarrow{\alpha} N_2$ exact, then

$$\begin{tikzcd}
0 \arrow[r] & (\bigoplus M_i) \otimes N_1 \arrow[r, "1\otimes\alpha"] \arrow[d, equal] & (\bigoplus M_i) \otimes N_2 \arrow[d, equal] & \\
0 \arrow[r] & \bigoplus (M_i \otimes N_1) \arrow[r, "\oplus(1\otimes\alpha)"] \arrow[d, Leftrightarrow] & \bigoplus (M_i \otimes N_2) & \\
0 \arrow[r] & M_i \otimes N_1 \arrow[r, "1\otimes\alpha"] & M_i \otimes N_2 & \forall\, i \in \Lambda
\end{tikzcd}$$

If M is projective, then by proposition 1.1.2 $\exists\, M'$ such that $M \oplus M' \cong F$ is free. Since free implies flat, by above, M is flat. $\square$

1.2 Ext and Tor (week 15)

Given $M, N \in \mathbf{Mod}_R$, there are two ways to define $\mathrm{Ext}^n(M, N)$:

Def 3 (Ext functor)**.**

- Find any projective resolution $P_\bullet \xrightarrow{\alpha} M \to 0$, and let $P_M : P_\bullet \to 0$ (called a *deleted resolution*). We can define $\mathrm{Ext}^n_{\mathrm{proj}}(M, N) = H^n(\mathrm{Hom}(P_M, N))$.
- Find any injective resolution $0 \xrightarrow{\alpha} N \to E^\bullet$, and let $E_N : 0 \to E^\bullet$. We can define $\mathrm{Ext}^n_{\mathrm{inj}}(M, N) = H^n(\mathrm{Hom}(M, E_N))$.

Prop 1.2.1. $\mathrm{Ext}^0_{\mathrm{proj}}(M, N) \cong \mathrm{Ext}^0_{\mathrm{inj}}(M, N) \cong \mathrm{Hom}(M, N)$.

Proof.

$$\mathrm{Hom}(P_M, N) : 0 \xrightarrow{\overline{d_0}} \mathrm{Hom}(P_0, N) \xrightarrow{\overline{d_1}} \mathrm{Hom}(P_1, N) \to \cdots$$

so $\mathrm{Ext}^0_{\mathrm{proj}}(M, N) = \ker \overline{d_1} / \operatorname{im} \overline{d_0} = \ker \overline{d_1} = \operatorname{im} \alpha = \mathrm{Hom}(M, N)$. □

Similarly, $\mathrm{Ext}^0_{\mathrm{inj}}(M, N) = \mathrm{Hom}(M, N)$.

Lemma 1.

- If M is projective, then $\mathrm{Ext}^n_{\mathrm{proj}}(M, N) = 0$ for all $n > 0, N \in \mathbf{Mod}_R$.
- If N is injective, then $\mathrm{Ext}^n_{\mathrm{inj}}(M, N) = 0$ for all $n > 0, M \in \mathbf{Mod}_R$.

Proof. If M is projective, then $0 \to M \xrightarrow{\mathbf{1}_M} M \to 0$ is a projective resolution of M. Its deleted resolution is then $P_M : 0 \to M \to 0$. Hence for $n > 0$, $\mathrm{Ext}^n_{\mathrm{proj}}(M, N) = H^n(\mathrm{Hom}(P_M, N)) = 0$.

The argument applies similarly to injective case. □

Theorem 3 (Equivalence of $\mathrm{Ext}_{\mathrm{proj}}$ and $\mathrm{Ext}_{\mathrm{inj}}$)**.**

$$\mathrm{Ext}^n_{\mathrm{proj}}(M, N) \cong \mathrm{Ext}^n_{\mathrm{inj}}(M, N).$$

Proof. Let $P_\bullet \to M \to 0$ and $0 \to N \to E^\bullet$ be projective and injective resolutions, then we have $0 \to K_0 \to P_0 \to M \to 0$ and $0 \to N \to E^0 \to L^1 \to 0$ exact.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr] \arrow[dr] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
& & K_1 \arrow[ur] \arrow[dr] & & K_0 \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & E^0 \arrow[rr] \arrow[dr] & & E^1 \arrow[rr] \arrow[dr] & & E^2 \arrow[r] & \cdots \\
& & & L^1 \arrow[ur] \arrow[dr] & & L^2 \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

We can construct long exact sequences of homology of $\mathrm{Hom}(\cdot, E_N)$:

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, E^0) \to \mathrm{Hom}(M, L^1) \to \mathrm{Ext}^1_{\mathrm{inj}}(M, N) \to \mathrm{Ext}^1_{\mathrm{inj}}(M, E^0) = 0$$

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_0, E^0) \to \mathrm{Hom}(P_0, L^1) \to 0$$

$$0 \to \mathrm{Hom}(K_0, N) \to \mathrm{Hom}(K_0, E^0) \to \mathrm{Hom}(K_0, L^1) \to \mathrm{Ext}^1_{\mathrm{inj}}(K_0, N) \to \mathrm{Ext}^1_{\mathrm{inj}}(K_0, E^0) = 0$$

The second sequence is short because P_0 is projective (so $\mathrm{Hom}(P_0, \cdot)$ preserves exactness).

Similarly, for $\mathrm{Hom}(P_M, \cdot)$ we have:

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(K_0, N) \to \mathrm{Ext}^1_{\mathrm{proj}}(M, N) \to \mathrm{Ext}^1_{\mathrm{proj}}(P_0, N) = 0$$

$$0 \to \mathrm{Hom}(M, E^0) \to \mathrm{Hom}(P_0, E^0) \to \mathrm{Hom}(K_0, E^0) \to 0$$

$$0 \to \mathrm{Hom}(M, L^1) \to \mathrm{Hom}(P_0, L^1) \to \mathrm{Hom}(K_0, L^1) \to \mathrm{Ext}^1_{\mathrm{proj}}(M, L^1) \to \mathrm{Ext}^1_{\mathrm{proj}}(P_0, L^1) = 0$$

Combining these sequences together, we got the following 2D diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Hom}(M,N) \arrow[r] \arrow[d] & \mathrm{Hom}(M,E^0) \arrow[r, "\phi"] \arrow[d] & \mathrm{Hom}(M,L^1) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathrm{inj}}(M,N) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}(P_0,N) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Hom}(P_0,E^0) \arrow[r, "\sigma"] \arrow[d, "\beta"] & \mathrm{Hom}(P_0,L^1) \arrow[r] \arrow[d, "\gamma"] & 0 & \\
0 \arrow[r] & \mathrm{Hom}(K_0,N) \arrow[r] \arrow[d] & \mathrm{Hom}(K_0,E^0) \arrow[r, "\tau"] \arrow[d] & \mathrm{Hom}(K_0,L^1) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathrm{inj}}(K_0,N) \arrow[r] & 0 \\
 & \mathrm{Ext}^1_{\mathrm{proj}}(M,N) \arrow[d] & 0 & \mathrm{Ext}^1_{\mathrm{proj}}(M,L^1) \arrow[d] & & \\
 & 0 & & 0 & &
\end{tikzcd}$$

By Snake lemma, there is an exact sequence

$$(\ker \alpha \to) \ker \beta \to \ker \gamma \to \operatorname{coker} \alpha \to \operatorname{coker} \beta (\to \operatorname{coker} \gamma)$$

and this reads

$$\mathrm{Hom}(M, E^0) \xrightarrow{\phi} \mathrm{Hom}(M, L^1) \to \mathrm{Ext}^1_{\mathrm{proj}}(M, N) \to 0$$

Thus $\mathrm{Ext}^1_{\mathrm{proj}}(M, N) \cong \operatorname{coker} \phi \cong \mathrm{Ext}^1_{\mathrm{inj}}(M, N)$.
(From now on, we don't need to distinguish proj/inj for Ext^1 !)

Since σ is onto, $\operatorname{im} \gamma = \operatorname{im}(\gamma \circ \sigma)$. Similarly, $\operatorname{im} \tau = \operatorname{im}(\tau \circ \beta)$.

By the commutativity of the diagram, $\operatorname{im} \gamma = \operatorname{im} \tau$, so

$$\mathrm{Ext}^1(K_0, N) \cong \operatorname{coker} \gamma = \mathrm{Hom}(K_0, L^1)/\operatorname{im} \gamma \cong \operatorname{coker} \tau \cong \mathrm{Ext}^1(M, L^1).$$

Write $K_{-1} := M, L^0 := N$, then $\mathrm{Ext}^1(K_0, L^0) = \mathrm{Ext}^1(K_{-1}, L^1)$ $(\star)$.

Similarly, from the exact sequences

$$0 \to K_j \to P_j \to K_{j-1} \to 0$$

$$0 \to L^i \to E^i \to L^{i+1} \to 0$$

, we can obtain $\mathrm{Ext}^1(K_j, L^i) \cong \mathrm{Ext}^1(K_{j-1}, L^{i+1})$ for $i, j \geq 0$.

Now, observe that

$$0 \to L^{n-1} \to E^{n-1} \xrightarrow{d_{n-1}} E^n \xrightarrow{d_n} \cdots$$

is an injective resolution of L^{n-1}, and $\mathrm{Ext}^1(M, L^{n-1}) \cong \ker \overline{d_n} / \operatorname{im} \overline{d_{n-1}} \cong \mathrm{Ext}^n_{\mathrm{inj}}(M, N)$.

Similarly, for projective resolution we have $\mathrm{Ext}^1(K_{n-2}, N) \cong \mathrm{Ext}^n_{\mathrm{proj}}(M, N)$.

Finally, by $(\star)$,

$$\mathrm{Ext}^n_{\mathrm{inj}}(M, N) \cong \mathrm{Ext}^1(K_{-1}, L^{n-1}) \cong \mathrm{Ext}^1(K_0, L^{n-2}) \cong \cdots \cong \mathrm{Ext}^1(K_{n-2}, L^0) \cong \mathrm{Ext}^n_{\mathrm{proj}}(M, N).$$

$\square$

Def 4 (Tor functor)**.** Let $M, N \in \mathbf{Mod}_R$, and $P_\bullet \to M \to 0$ be a projective resolution of M, similar to the Ext case, for $n \geq 0$ we can define

$$\mathrm{Tor}_n(M, N) = H_n(P_M \otimes N).$$

Fact 1.2.1. By Horseshoe lemma, short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ induces a long exact sequence:

$$\cdots \to \mathrm{Tor}_1(M_1, N) \to \mathrm{Tor}_1(M_2, N) \to \mathrm{Tor}_1(M_3, N) \to M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$$

Prop 1.2.2. If M is flat, then $\mathrm{Tor}_n(M, N) = 0$ for $n > 0, N \in \mathbf{Mod}_R$.

Proof. Since $0 \to M \xrightarrow{\mathbf{1}_M} M \to 0$ is a flat resolution of M. □

Theorem 4 (Tor for flat resolutions)**.** Let $U_\bullet \to M \to 0$ be a flat resolution of M, then for $n \geq 0$,

$$\mathrm{Tor}_n(M, N) \cong H_n(U_M \otimes N)$$

.

Proof.

$$\begin{tikzcd}
\cdots \arrow[r] & U_2 \arrow[rr] \arrow[dr] & & U_1 \arrow[rr] \arrow[dr] & & U_0 \arrow[r] & M \arrow[r] & 0 \\
& & W_1 \arrow[ur] \arrow[dr] & & W_0 \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$$H_\bullet(U_M \otimes N) : \cdots \to U_2 \otimes N \xrightarrow{d_2 \otimes \mathbf{1}} U_1 \otimes N \xrightarrow{d_1 \otimes \mathbf{1}} U_0 \otimes N \to 0$$

- $n = 0$:
  Since tensor is right exact, $U_1 \otimes N \xrightarrow{d_1 \otimes \mathbf{1}} U_0 \otimes N \xrightarrow{\alpha \otimes \mathbf{1}} M \otimes N \to 0$ is exact. Hence

  $$H_0(U_M \otimes N) = (U_0 \otimes N)/\operatorname{im}(d_1 \otimes \mathbf{1}) = (U_0 \otimes N)/\ker(\alpha \otimes \mathbf{1}) \cong M \otimes N$$

  Any projective resolution also has this property, so $\mathrm{Tor}_0(M, N) = H_0(U_M \otimes N)$.

- $n = 1$:
  $0 \to W_0 \to U_0 \to M \to 0$ induces a long exact sequence:

  $$\cdots \to \mathrm{Tor}_1(W_0, N) \to 0 \to \mathrm{Tor}_1(M, N) \to W_0 \otimes N \xrightarrow{i \otimes 1} U_0 \otimes N \to M \otimes N \to 0$$

  where $\mathrm{Tor}_1(U_0, N) = 0$ because U_0 is flat. We can see that $\mathrm{Tor}_1(M, N) \cong \ker(i \otimes 1)$.

  $$\begin{tikzcd}
  U_2 \otimes N \arrow[rr, "d_2 \otimes 1"] \arrow[dr, "\beta' \otimes 1"'] & & U_1 \otimes N \arrow[rr, "d_1 \otimes 1"] \arrow[dr, "\alpha' \otimes 1"'] & & U_0 \otimes N \\
  & W_1 \otimes N \arrow[ur, "j \otimes 1"'] \arrow[dr] & & W_0 \otimes N \arrow[ur, "i \otimes 1"'] \arrow[dr] & \\
  & & 0 & & 0
  \end{tikzcd}$$

  Since $\alpha' \otimes 1$ is onto, $W_0 \otimes N \cong (U_1 \otimes N)/\ker(\alpha' \otimes 1)$. Also, $x \in \ker(d_1 \otimes 1) \iff (d_1 \otimes 1)(x) = 0 \iff (\alpha' \otimes 1)(x) \in \ker(i \otimes 1)$, so $\ker(i \otimes 1) = (\alpha' \otimes 1)(\ker(d_1 \otimes 1)) \cong \ker(d_1 \otimes 1)/\ker(\alpha' \otimes 1)$. ($\alpha' \otimes 1$ can be considered a quotient map, then $\ker(d_1 \otimes 1)$ descends to $\ker(d_1 \otimes 1)/\ker(\alpha' \otimes 1)$.)

  Now, in the diagram $W_1 \otimes N \to U_1 \otimes N \to W_0 \otimes N \to 0$ exact, so $\ker(\alpha' \otimes 1) = \operatorname{im}(j \otimes 1)$. But $\beta' \otimes 1$ is onto, thus $\operatorname{im}(j \otimes 1) = \operatorname{im}(d_2 \otimes 1)$.

Finally,

$$\mathrm{Tor}_1(M, N) \cong \ker(i \otimes 1) \cong \ker(d_1 \otimes 1)/\ker(\alpha' \otimes 1) = \ker(d_1 \otimes 1)/\operatorname{im}(d_2 \otimes 1) = H_1(U_M \otimes N).$$

- $n \geq 2$:
  Let's see further in the previous long exact sequences:

  $$\cdots \to \mathrm{Tor}_2(W_0, N) \to 0 \to \mathrm{Tor}_2(M, N) \xrightarrow{\sim} \mathrm{Tor}_1(W_0, N) \to 0 \to \mathrm{Tor}_1(M, N) \to \cdots$$

  we can see that $\mathrm{Tor}_n(M, N) \cong \mathrm{Tor}_{n-1}(W_0, N)$ for $n \geq 2$.

  Now,

  $$\cdots U_3 \xrightarrow{d_3} U_2 \xrightarrow{d_2} U_1 \to W_0 \to 0$$

  is a flat resolution of W_0, and its homology is $H_{n-1}(U_{W_0} \otimes N) = \ker(d_n \otimes 1)/\operatorname{im}(d_{n-1} \otimes 1) = H_n(U_M \otimes N)$.

  By induction, assume it's true for $n-1$, then

  $$H_n(U_M \otimes N) = H_{n-1}(U_{W_0} \otimes N) \cong \mathrm{Tor}_{n-1}(W_0, N) \cong \mathrm{Tor}_n(M, N).$$

$\square$