Algebra

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0.1 Artinian rings and DVR

0.1.1 Artinian rings

Def 1. R is called an Artinian ring if one of the followings holds:

- any non-empty set of ideals has a minimal element.
- any descending chain of ideals is stationary (DCC).

Goal:

1. $R =\cong R_1 \times \cdots \times R_l$ where R_i is an Artinain local rings.
2. Artinian $\iff$ Noetherian $+ \dim = 0$.

Prop 0.1.1.

- $\sqrt{\mathfrak{m}_i^{n_i} + \mathfrak{m}_j^{n_j}} = \sqrt{\sqrt{\mathfrak{m}_i^{n_i}} + \sqrt{\mathfrak{m}_j^{n_j}}}$

  Proof.
  "$\subseteq$" $\forall\, a \in LHS$, that is, $a^n = b + c$ with $b \in m_i^{n_i} \subseteq \sqrt{m_i^{n_i}}$ and $c \in m_j^{n_j} \subseteq m_j^{n_j}$ then $a \in RHS$.
  "$\supseteq$" $\forall\, a \in RHS$, that is, $a^n = b+c$ with $b^k \in m_i^{n_i}$ and $c^t \in m_j^{n_j}$. Then $(a^n)^{k+t} = (b+c)^{k+t} = b^{k+t} + \cdots + C_t^k b^k c^t + \cdots + c^{k+t}$. Every term either in $m_i^{n_i}$ or $m_j^{m_j}$, then $(a^n)^{k+t} = c + d$ with $c \in m_i^{n_i}$ $d \in m_j^{n_j}$ $\Rightarrow a \in LHS$ $\square$

- If m is prime, $\sqrt{m^n} = m$

  Proof.
  " $\subseteq$ " $a \in LHS \Rightarrow a^k \in m^n$ and m is prime. $\Rightarrow a \in m$.
  " $\subseteq$ " $a \in RHS a \in m \Rightarrow a^n \in LHS$. $\square$

- If m, $m_i, i = 1, \cdots, n$ are prime and $m \supseteq m_1 \cap \cdots \cap m_n$, then $m \supseteq m_i$ for some i.

  Proof.
  Suppose not, then we pick $a_i \in m_i\ m$. $b = a_1 \cdots a_n \in m_i \forall\, i$. $\rightsquigarrow b \in m_1 \cap \cdots \cap m_n \subseteq m$. But, m is prime, exist $a_i \in m$, a contradiction. $\square$

Prop 0.1.2. Let R be an Artinian ring

(1) $I \subseteq R \rightsquigarrow R/I$ is also Artinian.

(2) If R is an integral domain, then R is a field.

  Proof. $\forall \underset{\neq 0}{a} \in R, \langle a \rangle \supseteq \langle a^2 \rangle \supseteq \cdots$ is a descending chain of ideals $\implies \langle a^l \rangle = \langle a^{l+1} \rangle = \cdots$ for some $l \in \mathbb{N} \implies a^l = ba^{l+1} \implies a^l(1 - ab) = 0 \implies ab = 1$ since $a^l \neq 0$. $\square$

(3) $\operatorname{Spec} R = \operatorname{Max} R$. ($\implies \dim R = 0$)

  Proof. $\forall\, p \in \operatorname{Spec} R, R/p$ is an integral domain $\rightsquigarrow R/p$ is a field $\rightsquigarrow p \in \operatorname{Max} R$. $\square$

(4) $|\operatorname{Max} R| < \infty$.

Proof. Consider the set $\left\{ \bigcap_{\text{finite}} \mathfrak{m} \,\middle|\, \mathfrak{m} \in \operatorname{Max} R \right\} \neq \emptyset$. So there exists a minimal element in this set (R is Artinian), say $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k$. Now, for $\mathfrak{m} \in \operatorname{Max} R$, we have $\mathfrak{m} \cap \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k$ since the latter is minimal $\implies \mathfrak{m} \supseteq \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k \rightsquigarrow \mathfrak{m} \supseteq \mathfrak{m}_i$ for some i, by Prop 0.1.1. $\rightsquigarrow m = m_i$, since m_i is max. So $\operatorname{Max} R = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_k\}$. ☐

(5) $\exists\, n_1, \ldots, n_k \in \mathbb{N}$ s.t. $\langle 0 \rangle = \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} = \mathfrak{m}_1^{n_1} \cap \mathfrak{m}_2^{n_2} \cap \cdots \cap \mathfrak{m}_k^{n_k}$.

Proof.

- $\mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} = \mathfrak{m}_1^{n_1} \cap \mathfrak{m}_2^{n_2} \cap \cdots \cap \mathfrak{m}_k^{n_k}$.
  Recall I_i, I_j are coprime for $i \neq j \rightsquigarrow \prod_{i=1}^{n} I_i = \bigcap_{i=1}^{n} I_i$. And, by Prop 0.1.1

$$\sqrt{\mathfrak{m}_i^{n_i} + \mathfrak{m}_j^{n_j}} = \sqrt{\sqrt{\mathfrak{m}_i^{n_i}} + \sqrt{\mathfrak{m}_j^{n_j}}} = \sqrt{\mathfrak{m}_i + \mathfrak{m}_j} = \sqrt{R} = R \rightsquigarrow \mathfrak{m}_i^{n_i} + \mathfrak{m}_j^{n_j} = R.$$

- $\langle 0 \rangle = \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k}$ for suitable $\{n_i\}$ that $\mathfrak{m}_i^{n_i} = \mathfrak{m}_i^{n_i+1}$
  Let $S = J \subseteq R \mid J\mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} \neq 0$. If $\langle 0 \rangle \neq \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k}$, then $\mathfrak{m}_i \in S$. $\rightsquigarrow S \neq \varnothing$. Since R is artinian, exist minimal element $J_0 \in S$. By definition of S, $\exists\, x \in J_0$, $x\mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} \neq 0 \rightsquigarrow \langle x \rangle \in S$ and $\langle x \rangle \subseteq J_0 \Rightarrow \langle x \rangle = J_0$.
  Also, $x\mathfrak{m}_1^{n_1+1} \mathfrak{m}_2^{n_2+1} \cdots \mathfrak{m}_k^{n_k+1} = x\mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} \neq 0 \rightsquigarrow I = x\mathfrak{m}_1 \ldots \mathfrak{m}_k \in S$ and $I \subseteq J_0 = xR \rightsquigarrow I = xR$.

$$(\mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_k) xR = xR \rightsquigarrow (\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_k) xR = xR \rightsquigarrow (\operatorname{Jac} R) xR = xR$$

  By Nakayama's lemma, $xR = 0 \implies x = 0$, which is a contradiction.

☐

(6) The nilradical $\mathfrak{n}_R$ of R is nilpotent.

Proof. By (3), $\mathfrak{n}_R = \operatorname{Jac} R$. Let $n = \max\{n_1, \ldots, n_k\}$ in (5), then $\mathfrak{n}_R^n = (\mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_k)^n = 0$. ☐

<u>Goal 1:</u> $R \cong R_1 \times R_k$ where R_i is Artinian local ring.

Proof. By Chinese Remainder theorem,

$$R \cong R/\langle 0 \rangle = R/\mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} \cong R/\mathfrak{m}_1^{n_1} \times R/\mathfrak{m}_2^{n_2} \times \cdots \times R/\mathfrak{m}_k^{n_k}$$

Let $R_i = R/\mathfrak{m}_i^{n_i}$, then $\bar{\mathfrak{m}} \in \operatorname{Max} R_i$ if $\mathfrak{m} \in \operatorname{Max} R$ and $\mathfrak{m} \supset \mathfrak{m}_i^{n_i} \rightsquigarrow \mathfrak{m} = \mathfrak{m}_i$. So $\operatorname{Max} R_i = \{\, \bar{\mathfrak{m}}_i \,\} \implies R_i$ is a local ring. ☐

Lemma 1. Let V be a K-vector space, TFAE

(1) $\dim_k V < \infty$

(2) V has DCC on subspaces.

(3) V has ACC on subspaces.

Proof.
<u>Fact</u> : If $V_1 \subseteq V_2$ is finite dim vector space over K, then $V_1 = V_2$ iff $\dim_k V_1 = \dim_k V_2$. Otherwise, $\dim_k V_1 \lneq \dim_k V_2$

(1) $\Leftrightarrow$ (3)
" $\Rightarrow$ " Suppose exists a chain in vector space V with strictly increasing and infinite length,

$$V_1 \subset V_2 \subset \cdots \subseteq V \Rightarrow \dim_k V_1 \lneq \dim_k V_2 \lneq \cdots \leq \dim_k V$$

Then, $\dim_k V$ must be infinite.
" $\Leftarrow$ " If $\dim_k V$ is ininite, let S $= \{b_1, b_2, \dots\}$ be basis of V.

$$\langle b_1 \rangle_K \subset \langle b_1, b_2 \rangle_K \subset \cdots$$

is a infinite accending chain.
Similarily, (1) $\Leftrightarrow$ (2). $\square$

Observation: If R is Northerian and $\dim R = 0$, then $\langle 0 \rangle = \bigcap_{i=1}^{k} q_i$ (primary decomposition) and $\sqrt{q_i} = \mathfrak{m}_i \in \operatorname{Spec} R = \operatorname{Max} R$. Also, $\exists\, n_i\ \mathfrak{m}_1^{n_1}\mathfrak{m}_2^{n_2}\cdots\mathfrak{m}_k^{n_k} = \langle 0 \rangle$
Since $\mathfrak{m}_i$ is finitely generated, $\exists\, n_i$ s.t. $\mathfrak{m}_i^{n_i} \subseteq q_i$. Hence

$$\mathfrak{m}_1^{n_1}\mathfrak{m}_2^{n_2}\cdots\mathfrak{m}_k^{n_k} = \mathfrak{m}_1^{n_1} \cap \mathfrak{m}_2^{n_2} \cap \cdots \cap \mathfrak{m}_k^{n_k} \subseteq q_1 \cap q_2 \cap \cdots \cap q_k = \langle 0 \rangle$$

$$\implies \mathfrak{m}_1^{n_1}\mathfrak{m}_2^{n_2}\cdots\mathfrak{m}_k^{n_k} = \langle 0 \rangle$$

Goal 2: In a ring R, let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be, not necessarily different, maximal ideals in R s.t. $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n = 0$. Then R is Artinian $\iff$ R is Noetherian.

Proof. We have a chain of ideals in R: $\mathfrak{m}_0 = R \supset \mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n = 0$.
Let $M_i = \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_{i-1}/\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_i$ as R-module. Notice that $\mathfrak{m}_i M_i = 0$, we can treat M_i as $R/\mathfrak{m}_i$-module. But $R/\mathfrak{m}_i$ is a field, so M_i can be regarded as a vector space. Hence, by lemma 1

$$M_i \text{ is Artinian} \iff M_i \text{ is Noetherian.}$$

By definition,

$$0 \to \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_i \to \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_{i-1} \to M_i \to 0$$

is exact in $\mathbf{Mod}_R$. By Ex1,

$$\begin{aligned}
\mathfrak{m}_0 = R \text{ Artinian} &\iff \mathfrak{m}_1, M_1 \text{ Artinian} \\
&\iff \mathfrak{m}_1\mathfrak{m}_2, M_1, M_2 \text{ Artinian} \\
&\ \vdots \\
&\iff \underbrace{\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n}_{is\ 0}, M_1, \cdots, M_n \text{ Artinian} \\
&\iff \underbrace{\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n}_{is\ 0}, M_1, \cdots, M_n \text{ Neotherian} \\
&\ \vdots \\
&\iff \mathfrak{m}_1\mathfrak{m}_2, M_1, M_2 \text{ Neotherian} \\
&\iff \mathfrak{m}_1, M_1 \text{ Neotherian} \iff \mathfrak{m}_0 = R \text{ Neotherian}
\end{aligned}$$

Note: Goal 2 is accomplish by recongnizing that,

- R is Artinian $\implies \mathfrak{m}_1^{n_1}\cdots\mathfrak{m}_k^{n_k} = \langle 0 \rangle$ by prop 0.1.2 (4).
- R is Neother + dim 0 $\implies \mathfrak{m}_1^{n_1\cdots\mathfrak{m}_k^{n_k} = \langle 0 \rangle}$ by Observation.

$\square$

0.1.2 DVR (Discrete Valuation Ring)

Def 2.

(1) Let K be a field. A discrete valuation of K is $\nu : K^\times \twoheadrightarrow \mathbb{Z}$ ($\nu(0) = \infty$) s.t.

- $\nu(xy) = \nu(x) + \nu(y)$.
- $\nu(x \pm y) = \min\{\nu(x), \nu(y)\}$.

(2) The valuation ring of ν is $R = \{ x \in K \mid \nu(x) \geq 0 \}$, called a DVR.

- $\mathfrak{m} = \{ x \in R \mid \nu(x) > 0 \}$ is the unique maximal ideal in R since $\nu(x) = 0 \iff x$ is a unit.
- Let $t \in R$ with $\nu(t) = 1$, then $\mathfrak{m} = \langle t \rangle$.
- Let $I \subseteq \mathfrak{m}$ and define $m = \min\{ l \in \mathbb{N} \mid x = t^l u \quad \forall x \in I \}$. Then $I = \langle t^m \rangle$.

Prop 0.1.3. R is a DVR $\iff$ R is 1-dimensional normal, Noetherian local domain.

Proof. □

0.1.3 Dedekind domains

Def 3. A Dedekind domain is a Noetherian normal domain of $\dim 1$.

Def 4. Let R be an integral domain and $K = \mathrm{Frac}(R)$. A nonzero R-submodule I of K is called a fractional ideal of R if $\exists\, 0 \neq a \in R$ s.t. $aI \subset R$.

Eg 0.1.1. If $I = \langle f_1, \ldots, f_n \rangle_R$ with $f_i = \frac{a_i}{b_i} \in K$, then $a = b_1 b_2 \cdots b_n$ and $aI \subset R \implies I$ is fractional.

In general, if R is a Noetherian, then every fractional ideal I of R is finitely generated.

Def 5. A fractional ideal I of R is invertible if $\exists\, J$: a fractional ideal of R s.t. $IJ = R$.

Prop 0.1.4.

1. If I is invertible, then $J = I^{-1}$ is unique and equals $J = (R : I) \triangleq \{ a \in K \mid aI \subset R \}$.

   Proof. □

2. If I is invertible, then$I$ is a finitely generated R-module.

   Proof. □

3. Let R be a local domain but not a field, $K = \mathrm{Frac}(R)$. Then R is a DVR $\iff$ every nonzero fractional ideal I of R is invertible.

   Proof. □

Theorem 1. Let R be an integral domain and $K = \mathrm{Frac}(R)$. TFAE

(a) R is a Dedekind domain.

(b) R is Noetherian and R_P is a DVR for all $P \in \mathrm{Spec}\, R$.

(c) Every nonzero fractional ideal of R is invertible.

(d) Every nonzero proper ideal of R can be written (uniquely) as a product of powers of prime ideals.

Proof.

(a)⇔(b):

- R is normal $\iff$ R_P is normal for all $P \in \operatorname{Spec} R$.
- $\dim R_P = 1 \quad \forall\, P \in \operatorname{Spec} R \iff h(P) = 1 \quad \forall\, 0 \neq P \in \operatorname{Spec} R \iff \dim R = 1$.

(b)⇔(c):

(a)⇔(b):

(a)(b)(c)⇒(d):

(d)⇒(c):

□