Algebra

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1 Group theory

1.1 Basic

Def 1. A non-empty set G with a binary function $f : G \times G \to G, (a, b) \mapsto ab$ is a *group* if it satisfies

1. $(ab)c = a(bc)$.
2. $\exists 1 \in G$ s.t. $1a = a1 = a, \forall a \in G$.
3. $\exists a^{-1} \in G$ s.t. $aa^{-1} = a^{-1}a = 1$.

CONCON

Def 2. Let G be a group. Then G is said to be *abelian* if $\forall a, b \in G, ab = ba$.

Ex 1.1.1. Let G be a semigroup. Then TFAE (the following are equivalent)

1. G is a group.
2. For all $a, b \in G$ and the equations $bx = a, yb = a$, each of them has a solution in G.
3. $\exists e \in G$ s.t. $ae = a\ \forall a \in G$ and if we fix such e, then $\forall b \in G\ \exists b' \in G$ s.t. $bb' = e$.

Ex 1.1.2. Let G be a group. Show that

1. $\forall a \in G, a^2 = 1$, then G is abelian.
2. G is abelian $\iff \forall a, b \in G, (ab)^n = a^n b^n$ for three consecutive integer n.

Def 3. Let G be a group and $H \subseteq G, H \neq \phi$. Then H is said to be a subgroup of G, denoted by $H \leq G$, if

1. $\forall a, b \in H, ab \in H$.
2. $1 \in H$.
3. $\forall a \in H, a^{-1} \in H$.

<u>useful criterion</u>: $H \leq G \iff \forall a, b \in H, ab^{-1} \in H$.

pf:

$\Rightarrow$ $\quad b \in H \implies b^{-1} \in H$, and $a \in H$, so $ab^{-1} \in H$.

$\Leftarrow$
1. $H \neq \phi \implies \exists a \in H \implies aa^{-1} = 1 \in H$.
2. $1, a \in H \implies 1a^{-1} = a^{-1} \in H$.
3. $a, b^{-1} \in H \implies a(b^{-1})^{-1} = ab \in H$. $\square$

Eg 1.1.1. $(\mathbb{Z}, +, 0) \leq (\mathbb{Q}, +, 0) \leq (\mathbb{R}, +, 0) \leq (\mathbb{C}, +, 0)$; $(\mathbb{Q}^\times, \times, 1) \leq (\mathbb{R}^\times, \times, 1) \leq (\mathbb{C}^\times, \times, 1)$

Eg 1.1.2.

- Special linear group $\mathrm{SL}(n, \mathbb{F}) = \{\, A \in \mathrm{GL}(n, \mathbb{F}) \mid \det A = 1 \,\}$
- Orthogonal group $\mathrm{O}(n) = \{\, A \in \mathrm{GL}(n, \mathbb{R}) \mid A^{\mathrm{t}} A = I_n \,\}$
- Unitary group $\mathrm{U}(n) = \{\, A \in \mathrm{GL}(n, \mathbb{C}) \mid A^* A = I_n \,\}$
- Special orthogonal group $\mathrm{SO}(n) = \mathrm{SL}(n, \mathbb{R}) \cap \mathrm{O}(n)$

- Special unitary group $\mathrm{SU}(n) = \mathrm{SL}(n, \mathbb{C}) \cap \mathrm{U}(n)$

Def 4. Let $f : G_1 \to G_2$. f is called an *isomorphism* if

1. f is 1-1 and onto.
2. $\forall a, b \in G_1, f(ab) = f(a)f(b)$. (*homomorphism*)

, denoted by $G_1 \cong G_2$.

Remark 1. (practice)

1. $f(1) = 1$.
2. $f(a^{-1}) = f(a)^{-1}$.
3. If f is an isomorphism, then $\exists f^{-1}$ is also a homomorphism.

Eg 1.1.3.

- $\mathrm{U}(1) = \{\, z \in \mathbb{C}^{\times} \mid \bar{z}z = 1 \,\}, z = \cos\theta + \sin\theta i$
- $\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$

notice that $\mathrm{U}(1) \cong \mathrm{SO}(2)$. $S^1 = \{\, (a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1 \,\}$, 可被賦予群的結構.

Eg 1.1.4. Let $A \in \mathrm{SU}(2) \implies A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \alpha\bar{\alpha} + \beta\bar{\beta} = 1, \alpha, \beta \in \mathbb{C}$.

Quaternion(四元數): $\mathbb{H} = \{\, a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \,\}$ with $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j (\implies ij = -ji)$.
Let $x = a + bi + cj + dk, \bar{x} = a - bi - cj - dk$, then $N(x) = x\bar{x} = a^2 + b^2 + c^2 + d^2$, For $x \neq 0, N(x) \neq 0, x^{-1} = \frac{1}{N(x)}\bar{x}$
Now, for $x = a + bi + cj + dk = (a + bi) + (c + di)j$. So $\mathrm{SU}(2) \cong \{\, x \in \mathbb{H}^{\times} \mid N(x) = 1 \,\}$. $S^3 = \{\, (a, b, c, d) \in \mathbb{R}^4 \mid a^2 + b^2 + c^2 + d^2 = 1 \,\}$, 可被賦予群的結構.
★ The only spheres with continuous group law are S^1, S^3.

Ex 1.1.3. Find a way to regard $M_{n\times n}(\mathbb{H})$ as a subset of $M_{2n\times 2n}(\mathbb{C})$, which preserves addition and multiplication, and then there is a way to characterize $\mathrm{GL}(n, \mathbb{H})$.

Def 5 (symplectic group)**.** $\mathrm{Sp}(n, \mathbb{F}) = \{\, A \in \mathrm{GL}(2n, \mathbb{F}) \mid A^{\mathrm{t}}JA = J \,\}$ where $J = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}$.
($A^{\mathrm{t}}JA = J$ preserving non-degenerate skew-symmetric forms)
$\mathrm{Sp}(n) = \{\, A \in \mathrm{GL}(n, \mathbb{H}) \mid A^*A = I_n \,\}$.

Ex 1.1.4. Show $\mathrm{Sp}(n) \cong \mathrm{U}(2n) \cap \mathrm{Sp}(n, \mathbb{C})$.

Ques: Find the smallest subgroup of $\mathrm{SU}(2)$ containing $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

1.2 Permutation groups and Dihedral groups

Def 6. A permutation of a set B is a 1-1 and onto function from B to B.
Let $S_B :=$ the set of permutations of B. Then $(S_B, \cdot, \mathrm{Id}_B)$ forms a group.
If $B = \{a_1, \ldots, a_n\}$, then $S_B \cong S_{\{1,\ldots,n\}}$ and write $S_n = S_{\{1,\ldots,n\}}$, called the symmetric group of degree n.

Theorem 1 (Cayley theorem)**.** Any group is isomorphic to a subgroup of some permutation group.
(Hint): Let G be a group. Set $B = G$. Consider $a \in G$ as $\sigma_a : G \to G, x \mapsto ax$. Then $\sigma_a \in S_G \implies G \leq S_G$.

Fact 1. S_n is a finite group of order $n!$, i.e. $|S_n| = n!$.

pf: EASY =O $\square$

<u>Cyclic notation</u>: $\sigma \in S_5$, say $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$. Write $\sigma = (1\ 4)(2\ 3\ 5)$.
$\Rightarrow$ Any permutation can be written as a product of disjoint cycles.

Eg 1.2.1. In S_7, $\sigma_1 = (1\ 2\ 3)(4\ 5\ 6)(7), \sigma_2 = (1\ 3\ 5\ 6)(2\ 4\ 7)$.
Then $\sigma_1\sigma_2 = (2\ 5\ 4\ 7\ 3\ 6), \sigma_1^{-1} = (1\ 3\ 2)(4\ 6\ 5)$.

Def 7. A 2 cycle is called a *transposition*.

Eg 1.2.2. $(1\ 2\ 3) = (1\ 3)(1\ 2), (1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2)$.
Any permutation is a product of 2 cycles.

<u>Useful formula</u>: $\sigma \in S_n$, $\sigma(j_1\ \ldots\ j_m)\sigma^{-1} = (\sigma(j_1)\ \ldots\ \sigma(j_m))$.

Eg 1.2.3. Let $\sigma = (1\ 2\ 3)(4\ 5\ 6\ 7)$, $\sigma(2\ 3\ 4)\sigma^{-1} = (3\ 1\ 5)$.

pf: Note that both sides are functions. For $i \in \{1, \ldots, n\}$,

<u>Case 1</u>: $\exists k$ s.t. $\sigma(j_k) = i$, CONCON

<u>Case 2</u>: Otherwise, CONCON

$\square$

Fact 2. $S_n = \langle (1\ 2), \ldots, (1\ n) \rangle$.

pf: $(1\ i)^{-1} = (1\ i)$ and $(i\ j) = (1\ i)(1\ j)(1\ i)^{-1}$. $\square$

Def 8. Let G be a group and $S \subset G$. The subgroup generated by S defined to be the smallest subgroup of G which contains S, denoted by $\langle S \rangle$.

Ex 1.2.1.

1. $S_n = \langle (1\ 2), (2\ 3), \ldots, (n-1\ n) \rangle$.
2. $S_n = \langle (1\ 2), (1\ 2\ \ldots\ n) \rangle$.

Def 9. $A_n = \{\text{even permutations of } S_n\} \leq S_n, |A_n| = \frac{n!}{2}$.

Ex 1.2.2.

1. $A_n = \langle (1\ 2\ 3), (1\ 2\ 4), \ldots, (1\ 2\ n) \rangle, n \geq 3$.

2. $A_n = \langle (1\ 2\ 3), (2\ 3\ 4), \ldots, (n-2\ n-1\ n) \rangle, n \geq 3$.

Remark 2. $\langle S \rangle = \bigcap_{S \subseteq H \leq G} H = \{a_1 a_2 \ldots a_k \mid k \in \mathbb{N}, a_i \in S \cup S^{-1}\} \cup \{1\}$

The orthogonal transformations on $\mathbb{R}^2$: $\mathrm{O}(2)$.
Let $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in \mathrm{O}(2)$.
略... (這邊討論旋轉和反射的矩陣)

<u>Case 1</u>: $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ is counterclockwise roration w.r.t. α.

<u>Case 2</u>: $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ is the reflection. $A^2 = I_2 \implies$ eigenvalues are ± 1.

Easy to show that $\mathrm{L}_A(v) = v - 2\langle v, v_2 \rangle v_2$.

$\mathrm{O}(2) = \{\text{rotations}\} \cup \{\text{reflections}\}$.

Def 10. The dihedral group D_n is the group of symmetries of a regular n-gon.
In general, $D_n = \langle \mathrm{T}, \mathrm{R} \mid \mathrm{T}^n = 1, \mathrm{R}^2 = 1, \mathrm{TR} = \mathrm{RT}^{-1} \rangle \leq O(2) \leq S_n, |D_n| = 2n$.

Def 11. Let T be a linear transformation from $\mathbb{R}^n \to \mathbb{R}^n$.

- T is called a rotation if $\exists$ a T-invariant subspace $W \subseteq \mathbb{R}^n$ with $\dim W = 2$ s.t. $\begin{cases} \mathrm{T}\big|_W \text{ is a rotation} \\ \mathrm{T}\big|_{W^\perp} = \mathrm{id}_{W^\perp} \end{cases}$

- T is called a reflection if $\exists$ a T-invariant subspace $W \subseteq \mathbb{R}^n$ with $\dim W = 2$ s.t. $\begin{cases} \mathrm{T}\big|_W = -\mathrm{id}_W \\ \mathrm{T}\big|_{W^\perp} = \mathrm{id}_{W^\perp} \end{cases}$

<u>Main result</u>: the group of orthogonal transformations $= \langle \text{rotations}, \text{reflections} \rangle$.
<u>Prop</u>: For $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^n$, $\exists$ a T-invariant subspace $W \subseteq \mathbb{R}^n$ with $1 \leq \dim W \leq 2$.

pf: Let $A = [\mathrm{T}]_\alpha \in M_{n\times n}(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{C})$. Consider $\widetilde{\mathrm{L}_A} : \mathbb{C}^n \to \mathbb{C}^n, v \mapsto Av$.
Then $\exists$ an eigenvalue $\lambda \in \mathbb{C}$ and an eigenvector $v \in \mathbb{C}^n$ for $\widetilde{\mathrm{L}_A}$. Let $\lambda = \lambda_1 + \lambda_2 i, v = v_1 + v_2 i$. By definition, we have

$$Av = \widetilde{\mathrm{L}_A}(v) = \lambda v = (\lambda_1 + \lambda_2 i)(v_1 + v_2 i) \implies \begin{cases} Av_1 = \lambda_1 v_1 - \lambda_2 v_2 \\ Av_1 = \lambda_2 v_1 + \lambda_1 v_2 \end{cases},$$

so $W = \langle v_1, v_2 \rangle$. $\square$

Ex 1.2.3.

1. If T is orthogonal, then $W^\perp$ is also T-invariant.

2. Use induction on n to show the main result.

For $n = 3, A \in \mathrm{O}(3)$, we have $A \sim \begin{pmatrix} \cos\alpha & -\sin\alpha & \\ \sin\alpha & \cos\alpha & \\ & & \pm 1 \end{pmatrix}$.