Algebra

June 14, 2017

1 Introduction to Homological Algebra

1.1 Projective, Injective and Flat modules (week 14)

Def 1.

- $M \in \mathbf{Mod}_R$ is **projective** if $\mathrm{Hom}(M, \cdot)$ preserves the *right* exactness.
- $N \in \mathbf{Mod}_R$ is **injective** if $\mathrm{Hom}(\cdot, N)$ preserves the *right* exactness.
- $M \in \mathbf{Mod}_R$ is **flat** if $M \otimes \cdot$ preserves the *left* exactness.

Fact 1.1.1.

- M is projective $\iff$ $\begin{tikzcd} & M \arrow[dl, "\exists \tilde{f}"'] \arrow[d, "f"] & \\ M_2 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$
- N is injective $\iff$ $\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "g"'] & M_2 \arrow[dl, "\exists \tilde{g}"] \\ & N & \end{tikzcd}$
- free $\implies$ projective: If $X = \{x_i \mid i \in \Lambda\}$ and $f : x_i \mapsto a_i$. Since β onto, exists b_i so that $\beta(b_i) = a_i$. we can then set $\tilde{f} : x_i \mapsto b_i$ by the universal property of free module.

$$\begin{tikzcd} & F(X) \arrow[dl, "\exists \tilde{f}"'] \arrow[d, "f"] & \\ M_2 \arrow[r, "\beta"] & M_3 \arrow[r] & 0 \end{tikzcd}$$

- free $\implies$ flat: Let $F \cong R^{\oplus\Lambda}$ be a free module, and M_1, M_2 be two modules such that $0 \to M_1 \to M_2$. Since $R \otimes_R M \cong M$, we have

$$\begin{aligned}
& 0 \to M_1 \to M_2 && \text{exact} \\
\implies & 0 \to R \otimes M_1 \to R \otimes M_2 && \text{exact} \\
\implies & 0 \to \bigoplus_{i\in\Lambda} R \otimes M_1 \to \bigoplus_{i\in\Lambda} R \otimes M_2 && \text{exact} \\
\overset{(a)}{\implies} & 0 \to R^{\oplus\Lambda} \otimes M_1 \to R^{\oplus\Lambda} \otimes M_2 && \text{exact} \\
\implies & 0 \to F \otimes M_1 \to F \otimes M_2 && \text{exact}
\end{aligned}$$

  Where (a) is by the fact that $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$. Thus F flat.

- If S is a multiplication closed set in R with $1 \in S$, then

$$0 \to M \to N \to L \to 0 \implies 0 \to M_S \to N_S \to L_S \to 0.$$

  We know that $M_S \cong R_S \otimes_R M$. So R_S is a flat R-module. e.g. $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

For any $M \in \mathbf{Mod}_R$, a projective module N such that $N \to M \to 0$ could be easily found: Simply let $N = F$, a free module on the generating set of M.

Now we shall ask for any module M, does there exist $N \in \mathbf{Mod}_R$ such that N is injective and $0 \to M \to N$?

Theorem 1 (Baer's criterion). N is injective $\iff$ $\forall\, I \subset R$, and a homomorphism f, there exists a homomorphism h such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists h"] \\ & N & \end{tikzcd}$$

Proof. "$\Rightarrow$": See I as an R module, then it is immediate by the definition of injective module.

"$\Leftarrow$: Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 \\ & & \big\downarrow{\scriptstyle g} & & \\ & & N & & \end{array}$$

Let $S \triangleq \{(M,\rho) \mid M_1 \subseteq M \subseteq M_2 \text{ and } \rho : M \to N \text{ extends } g\} \neq \varnothing$ since $(M_1, g) \in S$.

By the routinely proof using Zorn's lemma, exists a maximal element $(M^*, \mu) \in S$.

We claim that $M^* = M_2$. If not, pick $a \in M_2 \setminus M^*$ and let $M' \triangleq M^* + Ra \supsetneq M^*$, $I \triangleq \{r \in R \mid ra \in M^*\}$. Define $f : I \to N$ with $r \mapsto \mu(ra)$. Then we have an extension $h : R \to N$ of f.

Now, let $\mu' : M' \to N = x + ra \mapsto \mu(x) + h(r)$. We shall prove that this map is well-defined: If $x_1 + r_1 a = x_2 + r_2 a$, then $(r_1 - r_2)a = x_2 - x_1 \in M \implies r_1 - r_2 \in I$. So $h(r_1) - h(r_2) = f(r_1 - r_2) = \mu((r_1 - r_2)a) = \mu(x_2) - \mu(x_1)$, which prove μ' is well defined, and the existence of μ' contradicts the fact that (M^*, μ) is maximal. ☐

Def 2. M is **divisible** if $\forall\, x \in M, r \in R \setminus \{0\}$, there exists $y \in M$ such that $x = ry$, i.e. $rM = M \quad \forall\, r \in R \setminus \{0\}$.

Prop 1.1.1.

1. Every injective module N over an integral domain is divisible.

   Proof. For any $x_0 \in N$ and $r_0 \in R \setminus \{0\}$. Let $I = \langle r_0 \rangle \subset R$. As long as R is an integral domain, $I \cong R$ as an R-module, so the R-module homomorphism $f : I \to N = rr_0 \mapsto rx_0$ is well-defined. Since N injective, this map extends to $h : R \to N$. Let $y_0 \triangleq h(1)$, then $r_0 y_0 = r_0 h(1) = h(r_0) = x_0$. Thus N injective. ☐

2. Every divisible module N over an PID is injective.

   Proof. For any $I \subseteq R$ and a homomorphism $f : I \to N$, if $I = 0$ then $h = x \mapsto 0$ is always an extension of f. So assume $I \neq 0$. Since R is a PID, $I = \langle r_0 \rangle$ for some $r_0 \neq 0 \in R$. By the fact that N divisible, exists $y_0 \in N$ such that $r_0 y_0 = x_0 \triangleq f(r_0)$.

   Now we could define $h : R \to N$ by $1 \mapsto y_0$. Then $h(r_0) = r_0 h(1) = r_0 y_0 = x_0$, thus h is an extension of f and N injective. ☐

3. If R is a PID, then any quotient N of a injective R-module M is injective.

   Proof. By 2., $rM = M$ for any $r \neq 0$, thus $rN = N$ for any $r \neq 0$, and hence N injective. ☐

Theorem 2. For any $M \in \mathbf{Mod}_R$, there exists an injective module N containing M.

Proof.

Case 1: $R = \mathbb{Z}$.

Let $X = \{x_i\}_{i \in \Lambda}$ be a generating set for M and F is free on X. Let f be the natural map from f to M. then $M \cong F/\ker f$.

Define $F' = \bigoplus_{i \in \lambda} \mathbb{Q} e_i \subset F$, which is obviously a divisible $\mathbb{Z}$-module. Then $M \subseteq F'/\ker f \triangleq M'$, where M' is injective by proposition 1.1.1.

Case 2: R arbitrary.

We can regard any M as a $\mathbb{Z}$-module, then there exists an injective module $N_0 \supset M$. Now, we have an R-module $N \triangleq \mathrm{Hom}_{\mathbb{Z}}(R, N_0)$ with multiplication $rf(x) \triangleq x \mapsto f(xr)$.

We claim that N is injective. For any $f :: M_1 \to N$, and a homomorphism $\alpha :: M_1 \to M_2$, then α could be take as a $\mathbb{Z}$-module homomorphism. Define $f' :: M_1 \to N_0$ by $x \mapsto f(x)(1)$. Since N_0 injective, exists h', a $\mathbb{Z}$ module homomorphism from M_2 to N_0.

$$
\begin{tikzcd}
 & x \arrow[dddd, bend right] & & \\
 & \rotatebox{90}{$\in$} & & \\
0 \arrow[r] & M_1 \arrow[rr, "\alpha"] \arrow[d, "f"] & & M_2 \arrow[dll, dashed, "\exists\, h(?)"] \\
 & N & \cong \quad \mathrm{Hom}_{\mathbb{Z}}(R, N_0) & \\
 & \rotatebox{90}{$\ni$} & & \\
 & f(x) & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x \arrow[dddd, bend right] & \\
 & \rotatebox{90}{$\in$} & \\
0 \arrow[r] & M_1 \arrow[r, "\alpha"] \arrow[d, "f'"] & M_2 \arrow[dl, "\exists\, h'"] \\
 & N_0 & \\
 & \rotatebox{90}{$\ni$} & \\
 & f(x)(1) &
\end{tikzcd}
$$

Now, define

$$
\begin{aligned}
h : M_2 &\longrightarrow N \\
y &\longmapsto h(y) : R \longrightarrow N_0 \\
&\qquad\qquad\quad 1 \longmapsto h'(y) \\
&\qquad\qquad\quad r \longmapsto h'(ry)
\end{aligned}
$$

We check that h is well-defined.

- $h(y) \in \mathrm{Hom}_{\mathbb{Z}}(R, N_0)$

  $$h(y)(r_1 + r_2) = h'((r_1 + r_2)y) = h'(r_1 y + r_2 y) = h'(r_1 y) + h'(r_2 y) = h(y)(r_1) + h(y)(r_1)$$

- $h \in \mathrm{Hom}_R(M_2, N)$

  $$
  \begin{aligned}
  h(r_1 y_1 + y_2)(r) &= h'(r(r_1 y_1 + y_2)) = h'(r r_1 y_1 + r y_2) \\
  &= h'(r r_1 y_1) + h'(r y_2) \\
  &= h(y)(r r_1) + h(y_2)(r) \\
  &= (r_1 h(y))(r) + h(y_2)(r)
  \end{aligned}
  $$

- Show diagram commute $f = h \circ \alpha$ Fix $y \in M_1$, then $\forall\, r \in R$:

  $$
  \begin{aligned}
  (h \circ \alpha)(y)(r) &= h(\alpha(y))(r) = h'(r\alpha(y)) \\
  &= h'(\alpha(ry)) = f'(ry) \\
  &= f(ry)(1) = rf(y)(1) \\
  &= f(y)(r)
  \end{aligned}
  $$

Thus N_0 injective.

Now notice that, $\mathrm{Hom}_{\mathbb{Z}}(R, \cdot)$ is a left exact functor, so $M \hookrightarrow N_0$ implies $\mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, N_0)$, thus $M \cong \mathrm{Hom}_R(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, N_0) = N$. $\square$

Prop 1.1.2. TFAE

1. M is projective.
2. Every exact sequence $0 \to M_1 \to M_2 \to M \to 0$ split.

3. $\exists\, M'$ s.t. $M \oplus M' \cong F$: free.

Proof.
(1) $\Rightarrow$ (2) : Since M projective, the map λ with $\beta \circ \lambda = \mathrm{Id}$ exists in the following diagram:

$$\begin{tikzcd}
 & M \arrow[dl, "\exists\,\lambda"'] \arrow[d, "\mathrm{Id}"] & \\
M_2 \arrow[r, "\beta"] & M \arrow[r] & 0
\end{tikzcd}$$

Then λ is a lifting, so $M_2 \cong M_1 \oplus M$ and $0 \to M_1 \to M_2 \to M \to 0$ split.

(2) $\Rightarrow$ (3): Let F be a free module on a generating set of M, and $\beta :: F \to M$ be the natural map, then $0 \to \ker\beta \to F \to M \to 0$ split, so $F \cong \ker\beta \oplus M$.

(3) $\Rightarrow$ (1): For any $M_2 \to M_3 \to 0$, since $M' \oplus M$ free and thus projective, λ' exists in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] & M' \oplus M \arrow[r, shift right, "\pi"'] \arrow[d, "\exists\,\lambda'"] & M \arrow[l, shift right, "\mu"'] \arrow[r] \arrow[d, "f"] & 0 \\
 & & M_2 \arrow[r, "\beta"] & M_3 \arrow[r] & 0
\end{tikzcd}$$

Define $\lambda = \lambda' \circ \mu$. Then $\beta \circ \lambda = \beta \circ \lambda' \circ \mu = f \circ \pi \circ \mu = f$. $\square$

Prop 1.1.3. TFAE

1. M is injective.
2. Each exact sequence $0 \to M \to M_2 \to M_3 \to 0$ split.

Proof. (1) $\Rightarrow$ (2): Similar to the projective case, μ exists in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "\mathrm{Id}"] & M_2 \arrow[dl, dashed, "\exists\,\mu"] \\
 & M &
\end{tikzcd}$$

So $M_2 = M \oplus M_3$.

(2) $\Rightarrow$ (1): By theorem 2, there is a module $N \subset M$ so that N is injective.

Consider $\begin{tikzcd} 0 \arrow[r] & M \arrow[r, shift left, "i"] & N \arrow[l, shift left, "\exists\,\mu"] \arrow[r] & \operatorname{coker} i \arrow[r] & 0 \end{tikzcd}$ split exact and $\mu \circ i = \mathrm{Id}_M$. Since N injective, h' exists in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "\alpha"] \arrow[d, "f"] \arrow[dd, bend right=60, "i \circ f"'] & M_2 \arrow[ddl, "\exists\,h'"] \\
 & M \arrow[d, shift right, "i"'] & \\
 & N \arrow[u, shift right, "\mu"'] &
\end{tikzcd}$$

Let $h = \mu \circ h'$, then $h \circ \alpha = \mu \circ h' \circ \alpha = \mu \circ i \circ f = f$. $\square$

Prop 1.1.4. projective $\Longrightarrow$ flat.

Proof. Observe that $\bigoplus_{i\in\Lambda} M_i$ is flat if and only if M_i is flat for each i, since if $0 \to N_1 \xrightarrow{\alpha} N_2$ exact, then

$$\begin{tikzcd}
0 \arrow[r] & (\bigoplus M_i) \otimes N_1 \arrow[r, "1\otimes\alpha"] \arrow[d, equal] & (\bigoplus M_i) \otimes N_2 \arrow[d, equal] \\
0 \arrow[r] & \bigoplus(M_i \otimes N_1) \arrow[r, "\oplus(1\otimes\alpha)"] \arrow[d, Leftrightarrow] & \bigoplus(M_i \otimes N_2) \\
0 \arrow[r] & M_i \otimes N_1 \arrow[r, "1\otimes\alpha"] & M_i \otimes N_2 & \forall\, i \in \Lambda
\end{tikzcd}$$

If M is projective, then by proposition 1.1.2 $\exists\, M'$ such that $M \oplus M' \cong F$ is free. Since free implies flat, by above, M is flat. $\square$

Def 3.

- A chain complex $C_\bullet$ of R-modules is a sequence and maps:

  $$C_\bullet : \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$$

  with $d_n \circ d_{n+1} = 0, \quad \forall\, n.$ (i.e. $\operatorname{Im} d_{n+1} \subseteq \ker d_n$)

  Then define

  - $Z_n(C_\bullet) \triangleq \ker d_n$ is the n-cycle.
  - $B_n(C_\bullet) \triangleq \operatorname{Im} d_{n+1}$ is the n-boundary.
  - $H_n(C_\bullet) \triangleq Z_n(C_\bullet)/B_n(C_\bullet)$ is called the n-th homology.

- A cochain complex $C^\bullet$ of R-modules is a sequence and maps:

  $$C^\bullet : 0 \to C^0 \xrightarrow{d^1} C^1 \to \cdots \to C^{n-1} \xrightarrow{d^n} C^n \xrightarrow{d^{n+1}} C^{n+1} \to \cdots$$

  with $d^{n+1} \circ d^n = 0, \quad \forall\, n.$ (i.e. $\operatorname{Im} d^n \subseteq \ker d^{n+1}$)

  Then define

  - $Z^n(C^\bullet) \triangleq \ker d^{n+1}$ is the n-cocycle.
  - $B^n(C^\bullet) \triangleq \operatorname{Im} d^n$ is the n-coboundary.
  - $H^n(C^\bullet) \triangleq Z^n(C^\bullet)/B^n(C^\bullet)$ is called the n-th cohomology.

- $\varphi : C_\bullet \to \tilde{C}_\bullet$ is a chain map if the following diagram commutes:

  $$\begin{tikzcd}
  \cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\varphi_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "\varphi_n"] & C_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] & \cdots \\
  \cdots \arrow[r] & \tilde{C}_{n+1} \arrow[r, "\tilde{d}_{n+1}"] & \tilde{C}_n \arrow[r, "\tilde{d}_n"] & \tilde{C}_{n-1} \arrow[r] & \cdots
  \end{tikzcd}$$

  Observe that $\varphi_n(\ker d_n) \subseteq \ker \tilde{d}_n$ and $\varphi_n(\operatorname{Im} d_{n+1}) \subseteq \operatorname{Im} \tilde{d}_{n+1}$. This will induce the following maps:

  $$\begin{aligned} \varphi_* : \quad H_n(C_\bullet) \quad &\to \quad H_n(\tilde{C}_\bullet) \\ x + B_n(C_\bullet) &\mapsto \varphi_n(x) + B_n(\tilde{C}_\bullet) \end{aligned}$$

- $f : C_\bullet \to \tilde{C}_\bullet$ is null homotopic if $\exists\, s_n : C_n \to \tilde{C}_{n+1}$ s.t. $f_n = \tilde{d}_{n+1} \circ s_n + s_{n-1} \circ d_n, \quad \forall\, n.$

  $$\begin{tikzcd}
  \cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dl, red, "s_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, red, "s_{n-1}"] & \cdots \\
  \cdots \arrow[r] & \tilde{C}_{n+1} \arrow[r, "d_{n+1}"] & \tilde{C}_n \arrow[r, "d_n"] & \tilde{C}_{n-1} \arrow[r] & \cdots
  \end{tikzcd}$$

Prop 1.1.5. If f is null homotopic, then $f_* = 0$.

Proof. $f_*(x) = \tilde{d}_{n+1}s_n(x) + s_{n-1}d_n(x) = \tilde{d}_{n+1}s_n(x) \in B_n(\tilde{C}_\bullet) \implies f_*(\bar{x}) = 0$. □

- Two chain map $f, g : C_\bullet \to \tilde{C}_\bullet$ are homotopic if $f - g$ is null homotopic. ($f_* = g_*$)
- Let $M \in \mathbf{Mod}_R$. A projection resolution of M is an exact sequence:

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\alpha} M \to 0$$

where P_i is projective for all i.

For any M, projection resolution always exists. Let P_0 be a free module on the generators of M. We get $P_0 \xrightarrow{\alpha} M \to 0$. Similarly, let P_1 be free on $\ker\alpha$, then we could extend the map to $P_1 \to P_0 \to M \to 0$. Continue the process we would get a diagram as below, where K_i are the kernels:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr] \arrow[dr] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
& & K_1 \arrow[ur] \arrow[dr] & & K_0 \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Theorem 3 (Comparison theorem)**.** Given two chain as following:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, dashed, "\exists f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, dashed, "\exists f_1"] & P_0 \arrow[r, "\alpha"] \arrow[d, dashed, "\exists f_0"] & M \arrow[r] \arrow[d, "f"] & 0 & \text{(projective resolution)} \\
\cdots \arrow[r] & \tilde{C}_2 \arrow[r, "d_2'"] & \tilde{C}_1 \arrow[r, "d_1'"] & \tilde{C}_0 \arrow[r, "\alpha'"] & N \arrow[r] & 0 & \text{(exact sequence)}
\end{tikzcd}$$

Then $\exists\, f_i : P_i \to C_i$ s.t. $\{f_i\}$ forms a chain map making the completed diagram commutes. And any two such chain maps are homotopic.

Proof. Using induction on n.

For $n = 0$, the existence of f_0 is guaranteed by the definition of projective module.

$$\begin{tikzcd}
& P_0 \arrow[dl, dashed, "\exists f_0"'] \arrow[d, "f\circ\alpha"] & \\
C_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

For $n > 0$, we claim that $f_{n-1}d_n(P_n) \subseteq \operatorname{Im} d_n'$, since $d_{n-1}'f_{n-1}d_n(x) = f_{n-2}d_{n-1}d_n(x) = 0$ and by the fact that C is exact, $f_{n-1}d_n(x) \in \ker d_{n-1}' = \operatorname{Im} d_n'$. So using the diagram and again by the definition of projective module, f_n exists.

$$\begin{tikzcd}
& P_n \arrow[dl, dashed, "\exists f_n"'] \arrow[d, "f_{n-1}\circ d_n"] & \\
C_n \arrow[r] & \operatorname{Im} d_n' \arrow[r] & 0
\end{tikzcd}$$

Now, for another chain map $\{g_i : P_i \to C_i\}$, we shall construct suitable $\{s_n\}$ to prove they are homotopic. For $s_{-1} : M \to C_0$ we could simply pick the zero map. Again, if we could prove that $g_n - f_n - s_{n-1}d_n \in \operatorname{Im} d_{n+1}' = \ker d_n'$, then by the definition of projective module, we would obtain s_n with

$$d_{n+1}s_n = g_n - f_n - s_{n-1}d_n \implies g_n - f_n = d_{n+1}s_n + s_{n-1}d_n$$

immediately. For this we calculate $d'_n(g_n - f_n - s_{n-1}d_n) = g_{n-1}d_n - f_{n-1}d_n - d'_n s_{n-1}d_n$. Notice that $d'_n s_{n-1} = g_{n-1} - f_{n-1} - s_{n-2}d_{n-1}$, and with $d_{n-1}d_n = 0$, we get $d'_n(g_n - f_n - s_{n-1}d_n) = 0$. $\square$

Def 4. Let $M \in \mathbf{Mod}_R$ and $\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\alpha} M \to 0$ be a projective resolution of M. Fix $N \in \mathbf{Mod}_R$. Applying $\mathrm{Hom}_R(\cdot, N)$ will get a complex:

$$0 \to \mathrm{Hom}_R(M,N) \xrightarrow{\bar{\alpha}} \mathrm{Hom}_R(P_0,N) \xrightarrow{\bar{d}_1} \mathrm{Hom}_R(P_1,N) \to \cdots$$

Define

- $\mathrm{Ext}^0_R(M,N) = \ker \bar{d}_1 = \mathrm{Im}\,\bar{\alpha} \cong \mathrm{Hom}_R(M,N)$.
- $\mathrm{Ext}^n_R(M,N) = H^n(\mathrm{Hom}(P_\bullet,N)), \quad \forall\, n \geq 1$.

Theorem 4 (Indenpedency of the choice of projective resolutions)**.** $\mathrm{Ext}^n(M,N)$ is independent of the choice of the projective resolution used.

Proof. First, consider two projective resolutions of $M, \tilde{M}$, and map $f : M \to \tilde{M}$, and two liftings $\{f_i\}, \{g_i\}$. Use $\bar{\cdot}$ to denote the natural transformation from $X \to Y$ to $\mathrm{Hom}(Y,N) \to \mathrm{Hom}(X,N)$ by $\bar{f} \triangleq g \mapsto g \circ f$. Then we shall prove that $\bar{f_\bullet}^* = \bar{g_\bullet}^*$, which is to say $\bar{f_\bullet}^*$ is independent of the lifting used.

By comparison theorem (3), $\{f_i\}, \{g_i\}$ are homotopic, and we could write down the diagram below:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2"', "g_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"', "g_1"] \arrow[dl, "s_1"'] & P_0 \arrow[r, "\alpha"] \arrow[d, "f_0"', "g_0"] \arrow[dl, "s_0"'] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & \tilde{P}_2 \arrow[r, "\tilde{d}_2"] & \tilde{P}_1 \arrow[r, "\tilde{d}_1"] & \tilde{P}_0 \arrow[r, "\tilde{\alpha}"] & \tilde{M} \arrow[r] & 0
\end{tikzcd}$$

Notice that $\bar{\cdot}$ act linearly, that is, $\overline{f+g} = \bar{f} + \bar{g}$, and $\bar{fg} = \bar{g}\bar{f}$. So we have

$$g_n - f_n = s_{n-1}d_n + \tilde{d}_{n+1}s_n \implies \bar{g}_n - \bar{f}_n = \bar{d}_n\bar{s}_{n-1} + \bar{s}_n\bar{d}_{n+1}$$

and $\bar{f}_n, \bar{g}_n$ are homotopic. Thus by proposition 1.1.5, $\bar{f}^*_\bullet = \bar{g}^*_\bullet$.

Now, let $P^\bullet, P'^\bullet$ be two projective resolution. Consider the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] \arrow[dd, bend right, "\mathrm{Id}"'] & P_0 \arrow[r] \arrow[d, "f_0"] \arrow[dd, bend right, "\mathrm{Id}"'] & M \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
\cdots \arrow[r] & P'_1 \arrow[r] \arrow[d, "g_1"] & P'_0 \arrow[r] \arrow[d, "g_0"] & M \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
\cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Then $g_i \circ f_i$ and Id are two liftings, and thus by previous we have $\bar{g}^*_i \circ \bar{f}^*_i = \mathrm{Id}^*$. By symmetry, $\bar{f}^*_i \circ \bar{g}^*_i = \mathrm{Id}^*$, which means that the homology calculated using different resolution are isomorphic. $\square$

Theorem 5 (Horseshoe Lemma)**.** Given $0 \to L \to M \to N \to 0$ and projective resolutions $P^\bullet \to L \to 0$, $\tilde{P}^\bullet \to N \to 0$. Then there is a projective resolution for M such that the following diagram commutes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & \bar{P}_1 \arrow[r] \arrow[d] & \tilde{P}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & \bar{P}_0 \arrow[r] \arrow[d] & \tilde{P}_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Proof. Let $\bar{P}_n \triangleq P_n \oplus \tilde{P}_n$. $\bar{P}_n$ is projective by the fact that sum of projective module are projective. Also $0 \to P_n \to P_n \oplus \tilde{P}_n \to \tilde{P}_n \to 0$ by injection and projection. It remains to show that the maps in the middle column exists.

By induction on n. Consider the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_n \arrow[r] \arrow[d, "\alpha"] & P_n \oplus \tilde{P}_n \arrow[r] \arrow[d, "\bar{\alpha}"] & \tilde{P}_n \arrow[r] \arrow[d, "\tilde{\alpha}"] \arrow[dl, "\exists\,\sigma"] & 0 \\
0 \arrow[r] & P_{n-1} \arrow[r, "i"] \arrow[d, "d"] & \bar{P}_{n-1} \arrow[r, "\pi"] \arrow[d, "\bar{d}"] & \tilde{P}_{n-1} \arrow[r] \arrow[d, "\tilde{d}"] & 0 \\
0 \arrow[r] & P_{n-2} \arrow[r] \arrow[d] & \bar{P}_{n-2} \arrow[r] \arrow[d] & \tilde{P}_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

σ exists because $\tilde{P}_n$ is projective. Define

$$\begin{aligned}
\bar{\alpha}: \qquad P_n \otimes \tilde{P}_n &\longrightarrow \bar{P}_{n-1} \\
(z, y) &\longmapsto i\alpha(z) + \sigma(y)
\end{aligned}$$

It easy to see that $\bar{\alpha}$ let the diagram commutes.

For any $x \in \ker d$, $\tilde{\pi}(x) = 0$, so $\pi(x) \in \ker \tilde{d} = \operatorname{Im} \tilde{\alpha}$, so exists y satisfy $\pi(x) = \tilde{\alpha}(y)$. Then $\tilde{\alpha}(y) = \pi\sigma(y) = \pi(x) \implies x - \sigma(y) \in \ker \pi = \operatorname{Im} i$. Let z' be the one so that $i(z') = x - \sigma(y)$, tracing the diagram again one would find out $d(z') = 0$, so exists z such that $\alpha(z) = z'$, and then $\bar{\alpha}(z, y) = i\alpha(z) + \sigma(y) = x$, thus $\operatorname{Im} \bar{\alpha} = \ker \bar{d}$. ☐

Theorem 6 (Long exact sequence for Ext)**.** If $0 \to L \to M \to N \to 0$ exact, then there is a long exact sequence:

$$\begin{aligned}
0 \to \operatorname{Hom}(N, K) \to \operatorname{Hom}(M, K) \to \operatorname{Hom}(L, K)& \\
\to \operatorname{Ext}^1(N, K) \to \operatorname{Ext}^1(M, K) \to \operatorname{Ext}^1(L, K) \to \operatorname{Ext}^2(N, K) \to \dots&
\end{aligned}$$

Proof. Taking $\mathrm{Hom}(-,K)$ in the diagram of Horseshoe' lemma (5), we get

$$\begin{tikzcd}
& \vdots & \vdots & \vdots & \\
0 & \mathrm{Hom}(P_1,K) \arrow[l] \arrow[u] & \mathrm{Hom}(\bar{P}_1,K) \arrow[l] \arrow[u] & \mathrm{Hom}(\tilde{P}_1,K) \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & \mathrm{Hom}(P_0,K) \arrow[l] \arrow[u] & \mathrm{Hom}(\bar{P}_0,K) \arrow[l] \arrow[u] & \mathrm{Hom}(\tilde{P}_0,K) \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & \mathrm{Hom}(L,K) \arrow[l] \arrow[u] & \mathrm{Hom}(M,K) \arrow[l] \arrow[u] & \mathrm{Hom}(N,K) \arrow[l] \arrow[u] & 0 \arrow[l] \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Notice that $\mathrm{Hom}(M \oplus N, K) \cong \mathrm{Hom}(M,K) \otimes \mathrm{Hom}(N,K)$, so each row is indeed exact. By exercise 14.7, the long exact sequence in the statement exists. $\square$

1.2 Ext and Tor (week 15)

Given $M, N \in \mathbf{Mod}_R$, there are two ways to define $\mathrm{Ext}^n(M, N)$:

Def 5 (Ext functor)**.**

- Find any projective resolution $P_\bullet \xrightarrow{\alpha} M \to 0$, and let $P_M : P_\bullet \to 0$ (called a *deleted resolution*). We can define $\mathrm{Ext}^n_{\mathrm{proj}}(M, N) = H^n(\mathrm{Hom}(P_M, N))$.
- Find any injective resolution $0 \xrightarrow{\alpha} N \to E^\bullet$, and let $E_N : 0 \to E^\bullet$. We can define $\mathrm{Ext}^n_{\mathrm{inj}}(M, N) = H^n(\mathrm{Hom}(M, E_N))$.

Prop 1.2.1. $\mathrm{Ext}^0_{\mathrm{proj}}(M, N) \cong \mathrm{Ext}^0_{\mathrm{inj}}(M, N) \cong \mathrm{Hom}(M, N)$.

Proof.

$$\mathrm{Hom}(P_M, N) : 0 \xrightarrow{\overline{d_0}} \mathrm{Hom}(P_0, N) \xrightarrow{\overline{d_1}} \mathrm{Hom}(P_1, N) \to \cdots$$

so $\mathrm{Ext}^0_{\mathrm{proj}}(M, N) = \ker \overline{d_1} / \operatorname{im} \overline{d_0} = \ker \overline{d_1} = \operatorname{im} \alpha = \mathrm{Hom}(M, N)$. □

Similarly, $\mathrm{Ext}^0_{\mathrm{inj}}(M, N) = \mathrm{Hom}(M, N)$.

Lemma 1.

- If M is projective, then $\mathrm{Ext}^n_{\mathrm{proj}}(M, N) = 0$ for all $n > 0, N \in \mathbf{Mod}_R$.
- If N is injective, then $\mathrm{Ext}^n_{\mathrm{inj}}(M, N) = 0$ for all $n > 0, M \in \mathbf{Mod}_R$.

Proof. If M is projective, then $0 \to M \xrightarrow{\mathbf{1}_M} M \to 0$ is a projective resolution of M. Its deleted resolution is then $P_M : 0 \to M \to 0$. Hence for $n > 0$, $\mathrm{Ext}^n_{\mathrm{proj}}(M, N) = H^n(\mathrm{Hom}(P_M, N)) = 0$.

The argument applies similarly to injective case. □

Theorem 7 (Equivalence of $\mathrm{Ext}_{\mathrm{proj}}$ and $\mathrm{Ext}_{\mathrm{inj}}$)**.**

$$\mathrm{Ext}^n_{\mathrm{proj}}(M, N) \cong \mathrm{Ext}^n_{\mathrm{inj}}(M, N).$$

Proof. Let $P_\bullet \to M \to 0$ and $0 \to N \to E^\bullet$ be projective and injective resolutions, then we have $0 \to K_0 \to P_0 \to M \to 0$ and $0 \to N \to E^0 \to L^1 \to 0$ exact.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr] \arrow[dr] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
& & K_1 \arrow[ur] \arrow[dr] & & K_0 \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & E^0 \arrow[rr] \arrow[dr] & & E^1 \arrow[rr] \arrow[dr] & & E^2 \arrow[r] & \cdots \\
& & & L^1 \arrow[ur] \arrow[dr] & & L^2 \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

We can construct long exact sequences of homology of $\mathrm{Hom}(\cdot, E_N)$:

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, E^0) \to \mathrm{Hom}(M, L^1) \to \mathrm{Ext}^1_{\mathrm{inj}}(M, N) \to \mathrm{Ext}^1_{\mathrm{inj}}(M, E^0) = 0$$

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_0, E^0) \to \mathrm{Hom}(P_0, L^1) \to 0$$

$$0 \to \mathrm{Hom}(K_0, N) \to \mathrm{Hom}(K_0, E^0) \to \mathrm{Hom}(K_0, L^1) \to \mathrm{Ext}^1_{\mathrm{inj}}(K_0, N) \to \mathrm{Ext}^1_{\mathrm{inj}}(K_0, E^0) = 0$$

The second sequence is short because P_0 is projective (so $\mathrm{Hom}(P_0, \cdot)$ preserves exactness).

Similarly, for $\mathrm{Hom}(P_M, \cdot)$ we have:

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(K_0, N) \to \mathrm{Ext}^1_{\mathrm{proj}}(M, N) \to \mathrm{Ext}^1_{\mathrm{proj}}(P_0, N) = 0$$

$$0 \to \mathrm{Hom}(M, E^0) \to \mathrm{Hom}(P_0, E^0) \to \mathrm{Hom}(K_0, E^0) \to 0$$

$$0 \to \mathrm{Hom}(M, L^1) \to \mathrm{Hom}(P_0, L^1) \to \mathrm{Hom}(K_0, L^1) \to \mathrm{Ext}^1_{\mathrm{proj}}(M, L^1) \to \mathrm{Ext}^1_{\mathrm{proj}}(P_0, L^1) = 0$$

Combining these sequences together, we got the following 2D diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Hom}(M,N) \arrow[r] \arrow[d] & \mathrm{Hom}(M,E^0) \arrow[r, "\phi"] \arrow[d] & \mathrm{Hom}(M,L^1) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathrm{inj}}(M,N) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}(P_0,N) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Hom}(P_0,E^0) \arrow[r, "\sigma"] \arrow[d, "\beta"] & \mathrm{Hom}(P_0,L^1) \arrow[r] \arrow[d, "\gamma"] & 0 & \\
0 \arrow[r] & \mathrm{Hom}(K_0,N) \arrow[r] \arrow[d] & \mathrm{Hom}(K_0,E^0) \arrow[r, "\tau"] \arrow[d] & \mathrm{Hom}(K_0,L^1) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathrm{inj}}(K_0,N) \arrow[r] & 0 \\
 & \mathrm{Ext}^1_{\mathrm{proj}}(M,N) \arrow[d] & 0 & \mathrm{Ext}^1_{\mathrm{proj}}(M,L^1) \arrow[d] & & \\
 & 0 & & 0 & &
\end{tikzcd}$$

By Snake lemma, there is an exact sequence

$$(\ker\alpha \to) \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta (\to \operatorname{coker}\gamma)$$

and this reads

$$\mathrm{Hom}(M, E^0) \xrightarrow{\phi} \mathrm{Hom}(M, L^1) \to \mathrm{Ext}^1_{\mathrm{proj}}(M, N) \to 0$$

Thus $\mathrm{Ext}^1_{\mathrm{proj}}(M, N) \cong \operatorname{coker}\phi \cong \mathrm{Ext}^1_{\mathrm{inj}}(M, N)$.
(From now on, we don't need to distinguish proj/inj for Ext^1 !)

Since σ is onto, $\operatorname{im}\gamma = \operatorname{im}(\gamma \circ \sigma)$. Similarly, $\operatorname{im}\tau = \operatorname{im}(\tau \circ \beta)$.

By the commutativity of the diagram, $\operatorname{im}\gamma = \operatorname{im}\tau$, so

$$\mathrm{Ext}^1(K_0, N) \cong \operatorname{coker}\gamma = \mathrm{Hom}(K_0, L^1)/\operatorname{im}\gamma \cong \operatorname{coker}\tau \cong \mathrm{Ext}^1(M, L^1).$$

Write $K_{-1} := M, L^0 := N$, then $\mathrm{Ext}^1(K_0, L^0) = \mathrm{Ext}^1(K_{-1}, L^1)$ $(\star)$.

Similarly, from the exact sequences

$$0 \to K_j \to P_j \to K_{j-1} \to 0$$

$$0 \to L^i \to E^i \to L^{i+1} \to 0$$

, we can obtain $\mathrm{Ext}^1(K_j, L^i) \cong \mathrm{Ext}^1(K_{j-1}, L^{i+1})$ for $i, j \geq 0$.

Now, observe that

$$0 \to L^{n-1} \to E^{n-1} \xrightarrow{d_{n-1}} E^n \xrightarrow{d_n} \cdots$$

is an injective resolution of L^{n-1}, and $\mathrm{Ext}^1(M, L^{n-1}) \cong \ker\overline{d_n}/\operatorname{im}\overline{d_{n-1}} \cong \mathrm{Ext}^n_{\mathrm{inj}}(M, N)$.

Similarly, for projective resolution we have $\mathrm{Ext}^1(K_{n-2}, N) \cong \mathrm{Ext}^n_{\mathrm{proj}}(M, N)$.

Finally, by $(\star)$,

$$\mathrm{Ext}^n_{\mathrm{inj}}(M, N) \cong \mathrm{Ext}^1(K_{-1}, L^{n-1}) \cong \mathrm{Ext}^1(K_0, L^{n-2}) \cong \cdots \cong \mathrm{Ext}^1(K_{n-2}, L^0) \cong \mathrm{Ext}^n_{\mathrm{proj}}(M, N).$$

$\square$

Def 6 (Tor functor)**.** Let $M, N \in \mathbf{Mod}_R$, and $P_\bullet \to M \to 0$ be a projective resolution of M, similar to the Ext case, for $n \geq 0$ we can define

$$\mathrm{Tor}_n(M, N) = H_n(P_M \otimes N).$$

Fact 1.2.1. By Horseshoe lemma, short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ induces a long exact sequence:

$$\cdots \to \mathrm{Tor}_1(M_1, N) \to \mathrm{Tor}_1(M_2, N) \to \mathrm{Tor}_1(M_3, N) \to M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$$

Prop 1.2.2. If M is flat, then $\mathrm{Tor}_n(M, N) = 0$ for $n > 0, N \in \mathbf{Mod}_R$.

Proof. M is flat $\implies$ $M \otimes \cdot$ is an exact functor. If $Q_\bullet \to N \to 0$ is a projective resolution of N, then $\cdots \to M \otimes Q_1 \to M \otimes Q_0 \to M \otimes N \to 0$ is also exact. By Exercise 15-1, we have

$$\mathrm{Tor}_n(M, N) \cong H_n(M \otimes Q_N) = 0. \qquad \square$$

Theorem 8 (Tor for flat resolutions)**.** Let $U_\bullet \to M \to 0$ be a flat resolution of M, then for $n \geq 0$,

$$\mathrm{Tor}_n(M, N) \cong H_n(U_M \otimes N).$$

Proof.

$$\begin{tikzcd}
\cdots \arrow[r] & U_2 \arrow[rr] \arrow[dr] & & U_1 \arrow[rr] \arrow[dr] & & U_0 \arrow[r] & M \arrow[r] & 0 \\
& & W_1 \arrow[ur] \arrow[dr] & & W_0 \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$$H_\bullet(U_M \otimes N) : \cdots \to U_2 \otimes N \xrightarrow{d_2 \otimes \mathbf{1}} U_1 \otimes N \xrightarrow{d_1 \otimes \mathbf{1}} U_0 \otimes N \to 0$$

- $n = 0$:
  Since tensor is right exact, $U_1 \otimes N \xrightarrow{d_1 \otimes \mathbf{1}} U_0 \otimes N \xrightarrow{\alpha \otimes \mathbf{1}} M \otimes N \to 0$ is exact. Hence

  $$H_0(U_M \otimes N) = (U_0 \otimes N)/\operatorname{im}(d_1 \otimes \mathbf{1}) = (U_0 \otimes N)/\ker(\alpha \otimes \mathbf{1}) \cong M \otimes N$$

  Any projective resolution also has this property, so $\mathrm{Tor}_0(M, N) = H_0(U_M \otimes N)$.

- $n = 1$:
  $0 \to W_0 \to U_0 \to M \to 0$ induces a long exact sequence:

  $$\cdots \to \mathrm{Tor}_1(W_0, N) \to 0 \to \mathrm{Tor}_1(M, N) \to W_0 \otimes N \xrightarrow{i \otimes 1} U_0 \otimes N \to M \otimes N \to 0$$

  where $\mathrm{Tor}_1(U_0, N) = 0$ because U_0 is flat. We can see that $\mathrm{Tor}_1(M, N) \cong \ker(i \otimes 1)$.

  $$\begin{tikzcd}
  U_2 \otimes N \arrow[rr, "d_2 \otimes 1"] \arrow[dr, "\beta' \otimes 1"'] & & U_1 \otimes N \arrow[rr, "d_1 \otimes 1"] \arrow[dr, "\alpha' \otimes 1"'] & & U_0 \otimes N \\
  & W_1 \otimes N \arrow[ur, "j \otimes 1"'] \arrow[dr] & & W_0 \otimes N \arrow[ur, "i \otimes 1"'] \arrow[dr] & \\
  & & 0 & & 0
  \end{tikzcd}$$

  Since $\alpha' \otimes 1$ is onto, $W_0 \otimes N \cong (U_1 \otimes N)/\ker(\alpha' \otimes 1)$. Also, $x \in \ker(d_1 \otimes 1) \iff (d_1 \otimes 1)(x) = 0 \iff (\alpha' \otimes 1)(x) \in \ker(i \otimes 1)$, so $\ker(i \otimes 1) = (\alpha' \otimes 1)(\ker(d_1 \otimes 1)) \cong \ker(d_1 \otimes 1)/\ker(\alpha' \otimes 1)$. ($\alpha' \otimes 1$ can be considered a quotient map, then $\ker(d_1 \otimes 1)$ descends to $\ker(d_1 \otimes 1)/\ker(\alpha' \otimes 1)$.)

Now, in the diagram $W_1 \otimes N \to U_1 \otimes N \to W_0 \otimes N \to 0$ exact, so $\ker(\alpha' \otimes 1) = \operatorname{im}(j \otimes 1)$. But $\beta' \otimes 1$ is onto, thus $\operatorname{im}(j \otimes 1) = \operatorname{im}(d_2 \otimes 1)$.

Finally,

$$\operatorname{Tor}_1(M,N) \cong \ker(i \otimes 1) \cong \ker(d_1 \otimes 1)/\ker(\alpha' \otimes 1) = \ker(d_1 \otimes 1)/\operatorname{im}(d_2 \otimes 1) = H_1(U_M \otimes N).$$

- $n \geq 2$:
  Let's see further in the previous long exact sequences:

  $$\cdots \to \operatorname{Tor}_2(W_0,N) \to 0 \to \operatorname{Tor}_2(M,N) \xrightarrow{\sim} \operatorname{Tor}_1(W_0,N) \to 0 \to \operatorname{Tor}_1(M,N) \to \cdots$$

  we can see that $\operatorname{Tor}_n(M,N) \cong \operatorname{Tor}_{n-1}(W_0,N)$ for $n \geq 2$.

  Now,

  $$\cdots U_3 \xrightarrow{d_3} U_2 \xrightarrow{d_2} U_1 \to W_0 \to 0$$

  is a flat resolution of W_0, and its homology is $H_{n-1}(U_{W_0} \otimes N) = \ker(d_n \otimes 1)/\operatorname{im}(d_{n-1} \otimes 1) = H_n(U_M \otimes N)$.

  By induction, assume it's true for $n-1$, then

  $$H_n(U_M \otimes N) = H_{n-1}(U_{W_0} \otimes N) \cong \operatorname{Tor}_{n-1}(W_0,N) \cong \operatorname{Tor}_n(M,N).$$

$\square$

Eg 1.2.1. $R = \mathbb{Z}, M = \mathbb{Z}/m\mathbb{Z}$ with $m \geq 2$. Then

$$P : 0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{/m\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \to 0$$

is a projective resolution of $\mathbb{Z}/m\mathbb{Z}$. So for any $N \in \mathbf{Mod}_{\mathbb{Z}}$,

$$H^n(\operatorname{Hom}_{\mathbb{Z}}(P_{\mathbb{Z}/m\mathbb{Z}}, N)) : 0 \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},N) \xrightarrow{\overline{m}} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},N) \to 0,$$

$$\begin{array}{rcl} \operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z},N) & = & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z},N) \cong {}_mN := \{a \in N \mid ma = 0\} \\ \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z},N) & \cong & N/mN \\ \operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z},N) & = & 0 \quad (\text{for } n \geq 2) \end{array}$$

Eg 1.2.2. $\mathbb{Q} = \mathbb{Z}_{\langle 0 \rangle}$ is a localization, thus a flat $\mathbb{Z}$ module. Then

$$U : 0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

is a flat resolution of $\mathbb{Q}/\mathbb{Z}$. For $G \in \mathbf{Mod}_{\mathbb{Z}}$ (i.e. an abelian group),

$$H_n(G \otimes U_{\mathbb{Q}/\mathbb{Z}}) : 0 \to G \otimes \mathbb{Z} \xrightarrow{\mathbf{1} \otimes i} G \otimes \mathbb{Q} \to 0$$

$$\begin{array}{rcl} \operatorname{Tor}_0(G,\mathbb{Q}/\mathbb{Z}) & \cong & G \otimes \mathbb{Q}/\mathbb{Z} \\ \operatorname{Tor}_1(G,\mathbb{Q}/\mathbb{Z}) & = & \ker(\mathbf{1} \otimes i) \cong t(G) := \{a \in G \mid ma = 0 \text{ for some } m \in \mathbb{N}\} \\ \operatorname{Tor}_n(G,\mathbb{Q}/\mathbb{Z}) & = & 0 \quad (\text{for } n \geq 2) \end{array}$$

Def 7. Let M be a left R-module, then define $M^* := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ as a right R-module by

$$\begin{aligned} fr : M &\to \mathbb{Q}/\mathbb{Z} \\ x &\mapsto f(rx) \end{aligned}$$

Fact 1.2.2.

1. $\mathbb{Q}/\mathbb{Z}$ is injective.
2. $A = 0 \iff A^* = 0$.
3. $B \hookrightarrow C \iff C^* \twoheadrightarrow B^*$.

Proof.

1. For $m \in \mathbb{Z} \setminus \{0\}$, $m(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$ by $m(\frac{a}{mb} + \mathbb{Z}) \leftarrow\!\shortmid \frac{a}{b} + \mathbb{Z}$, so $\mathbb{Q}/\mathbb{Z}$ is divisible. But $\mathbb{Z}$ is a PID, so $\mathbb{Q}/\mathbb{Z}$ is injective.
2. $(\Rightarrow)$ $A^* = \mathrm{Hom}_{\mathbb{Z}}(0, \mathbb{Q}/\mathbb{Z}) = 0$.

   $(\Leftarrow)$ If $A \neq 0$, then $\exists\, a \in A, a \neq 0$, so $0 \to \mathbb{Z}a \xrightarrow{i} A$ is an inclusion.

   Since $\mathbb{Z}a$ is a cyclic abelian group, there is a nonzero $g : \mathbb{Z}a \to \mathbb{Q}/\mathbb{Z}$. (If $\mathbb{Z}a \cong \mathbb{Z}/m\mathbb{Z}$, let $g : a \mapsto \frac{1}{m}$; if $\mathbb{Z}a \cong \mathbb{Z}$, let $g : a \mapsto \frac{1}{2}$.)

   But $\mathbb{Q}/\mathbb{Z}$ is injective, so $\exists\, f : A \to \mathbb{Q}/\mathbb{Z}$ (i.e. $f \in A^*$), and $f \circ i = g \neq 0$ so $f \neq 0$, thus $A^* \neq 0$.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}a \arrow[r, "i"] \arrow[d, "g"] & A \arrow[dl, "\exists\, f"] \\ & \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

3. Since $\mathbb{Q}/\mathbb{Z}$ is injective, $\mathrm{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ is exact. Let $0 \to \ker f \to B \xrightarrow{f} C$ exact, applying $\mathrm{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ results in $C^* \xrightarrow{f^*} B^* \to (\ker f)^* \to 0$ exact. Thus $\mathrm{coker}\, f^* = (\ker f)^*$.

   By 2., $B \hookrightarrow C \iff \ker f = 0 \iff (\ker f)^* = 0 \iff \mathrm{coker}\, f^* = 0 \iff C^* \twoheadrightarrow B^*$.

□

Prop 1.2.3. Let M be an R-module, then TFAE

1. M is flat.
2. M^* is injective (as a R-module).
3. $\mathrm{Tor}_1(R/I, M) = 0$ for all ideal $I \subseteq R$.
4. $I \otimes_R M \cong IM$ for all ideal $I \subseteq R$.

Proof.

- 3. $\iff$ 4.

  For any ideal $I \subseteq R$, $0 \to I \xrightarrow{i} R \xrightarrow{q} R/I \to 0$ is exact. This induces a long exact sequence:

  $$\mathrm{Tor}_1(R, M) \to \mathrm{Tor}_1(R/I, M) \to I \otimes_R M \xrightarrow{i \otimes \mathbf{1}} R \otimes_R M \xrightarrow{q \otimes \mathbf{1}} R/I \otimes_R M \to 0$$

  - $\mathrm{Tor}_1(R, M) = 0$ since R is a flat R-module.
  - $R \otimes_R M \cong M$.
  - $R/I \otimes_R M \cong M/IM$ by $(r + I) \otimes a \mapsto (ra + IM)$.

  So we have

  $$0 \to \mathrm{Tor}_1(R/I, M) \to I \otimes_R M \xrightarrow{i'} M \xrightarrow{q'} M/IM \to 0$$

  exact, with $q' : M \to M/IM$ being exactly the quotient map (one can check that $q \otimes \mathbf{1} \cong q'$).

  Now it's clear that $\mathrm{Tor}_1(R/I, M) = 0 \iff I \otimes_R M \cong \ker(q') \cong IM$.

  (The reverse direction requires $I \otimes_R M \cong IM$ being the natural isomorphism $r \otimes b \mapsto rb$, so $i' : IM \to M$ can then be the natural inclusion.)

- 1. $\iff$ 2.
  Let $0 \to N' \xrightarrow{f} N$, then $\mathrm{Hom}_R(N, M^*) \xrightarrow{\overline{f}} \mathrm{Hom}_R(N', M^*)$.
  By the adjoint relation,

$$\mathrm{Hom}_R(N, M^*) = \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(N \otimes_R M, \mathbb{Q}/\mathbb{Z}) = (N \otimes_R M)^*,$$

  we have another map $(N \otimes_R M)^* \xrightarrow{(f \otimes \mathbf{1})^*} (N' \otimes_R M)^*$ isomorphic to the previous one, with its unstarred map $N' \otimes_R M \xrightarrow{f \otimes \mathbf{1}} N \otimes_R M$.
  Now, M^* is injective $\iff$ $\overline{f}$ is surjective $\forall\, N, N'$ $\iff$ $(f \otimes \mathbf{1})^*$ is surjective $\forall\, N, N'$ $\iff$ $f \otimes \mathbf{1}$ is injective $\forall\, N, N'$ $\iff$ M is flat.
- 2. $\iff$ 4.
  Similar to the previous section, by Baer's criterion,

$$\begin{aligned} M^* \text{ is injective} \quad &\iff \quad \mathrm{Hom}_R(R, M^*) \twoheadrightarrow \mathrm{Hom}_R(I, M^*), \forall\, I \subseteq R \\ &\iff \quad (R \otimes_R M)^* \twoheadrightarrow (I \otimes_R M)^*, \forall\, I \subseteq R \\ &\iff \quad I \otimes_R M \hookrightarrow R \otimes_R M \cong M, \forall\, I \subseteq R \\ &\iff \quad I \otimes_R M \cong IM, \forall\, I \subseteq R. \end{aligned}$$

  Similarly, this requires the isomorphism of $I \otimes_R M \cong IM$ be natural (the following f).
  The map $\begin{array}{rl} f : I \otimes_R M & \to IM \\ r \otimes a & \mapsto ra \end{array}$ is always onto, but may not be 1-1. If it is, $I \otimes_R M \cong IM$.

□

Prop 1.2.4. For $I, J \subseteq R$ being ideals, then $\mathrm{Tor}_1(R/I, R/J) \cong (I \cap J)/IJ$.

Proof. $0 \to I \xrightarrow{i} R \to R/I \to 0$ induces a long exact sequence

$$0 \to \mathrm{Tor}_1(R/I, R/J) \to I \otimes_R R/J \xrightarrow{i \otimes 1} R \otimes_R R/J \to R/I \otimes_R R/J \to 0,$$

where $\mathrm{Tor}_1(R, R/J) = 0$ since R is flat.

Also $I \otimes_R R/J \cong I/IJ, R \otimes_R R/J \cong R/J$, so we have $I/IJ \xrightarrow{i'} R/J$ with $\mathrm{Tor}_1(R/I, R/J) \cong \ker(i \otimes 1) \cong \ker i'$.

But $\begin{array}{rl} i' : \; I/IJ & \to R/J \\ x + IJ & \mapsto x + J \end{array}$, so $\overline{x} \in \ker i' \iff x \in I$ and $x \in J \iff x \in I \cap J$, hence $\ker i' \cong (I \cap J)/IJ$. □

1.3 Koszul complex (week 16)

In this section, we assume that R is commutative with 1.

Def 8. Let $L \in \mathbf{Mod}_R$, with $f : L \to R$ an R-linear map, define

$$\begin{aligned} \mathrm{d}_f : \quad \Lambda^n L \quad &\to \quad \Lambda^{n-1} L \\ x_1 \wedge \cdots \wedge x_n &\mapsto \sum_{i=1}^{n} (-1)^{i+1} f(x_i) x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \end{aligned}$$

where $\Lambda^n L$ is the n-th exterior power of L, and $\hat{x}_i$ means omitting x_i.

Then we can define a chain complex called **Koszul complex**:

$$K_\bullet(f) : \cdots \to \Lambda^n L \xrightarrow{\mathrm{d}_f} \Lambda^{n-1} L \to \cdots \to \Lambda^2 L \xrightarrow{\mathrm{d}_f} L \xrightarrow{f} R$$

Also, d_f can be considered as a graded R-homomorphism of degree -1:

$$\begin{aligned} \mathrm{d}_f : \ \Lambda L \ &\to \quad \Lambda L \\ x \wedge y &\mapsto \mathrm{d}_f(x) \wedge y + (-1)^{\deg x} \cdot x \wedge \mathrm{d}_f(y) \end{aligned}$$

where ΛL is the exterior algebra of L, and x, y are any homogeneous elements of ΛL.

Def 9. Let $(C_\bullet, d), (C'_\bullet, d')$ be chain complexes of R-modules, define their *tensor product* to be a chain complex $C_\bullet \otimes C'_\bullet$ with

$$(C_\bullet \otimes C'_\bullet)_n = \bigoplus_{i=0}^{n} \left(C_i \otimes_R C'_{n-i}\right)$$

with the boundary maps being

$$\begin{aligned} d \otimes d' : \ (C_\bullet \otimes C'_\bullet)_n \ &\to \quad (C_\bullet \otimes C'_\bullet)_{n-1} \\ \sum_{i=0}^{n} x_i \otimes y_{n-i} &\mapsto \sum_{i=0}^{n} \left(d(x_i) \otimes y_{n-i} + (-1)^i \cdot x_i \otimes d'(y_{n-i})\right) \end{aligned}$$

One can verify that

$$\begin{aligned} (d \otimes d') \circ (d \otimes d')(x \otimes y) &= (d \otimes d')(d(x) \otimes y + (-1)^{\deg x} \cdot x \otimes d'(y)) \\ &= d \circ d(x) \otimes y + (-1)^{\deg x - 1} \cdot d(x) \otimes d'(y) \\ &\quad + (-1)^{\deg x} \cdot d(x) \otimes d'(y) + x \otimes d' \circ d'(y) \\ &= 0 \end{aligned}$$

Prop 1.3.1. Let $L_1, L_2 \in \mathbf{Mod}_R, f_1 \in \mathrm{Hom}_R(L_1, R), f_2 \in \mathrm{Hom}_R(L_2, R)$. Define

$$\begin{aligned} f = f_1 + f_2 : L_1 \oplus L_2 &\to \quad R \\ (x, y) \ &\mapsto f_1(x) + f_2(y) \end{aligned},$$

then

$$\begin{aligned} K_\bullet(f_1) \otimes K_\bullet(f_2) \quad &\cong \quad K_\bullet(f) \\ \bigoplus_{i=0}^{n} \left(\Lambda^i L_1 \otimes_R \Lambda^{n-i} L_2\right) \quad &\cong \quad \Lambda^n(L_1 \oplus L_2) \end{aligned}$$

with $\mathrm{d}_{f_1} \otimes \mathrm{d}_{f_2} = \mathrm{d}_f$.

Proof. Exercise 16-1(2). $\square$

Def 10. Let $L = \bigoplus_{i=1}^{n} Re_i$ be a free R-module, and $\mathbf{x} = (x_1, \cdots, x_n)$ with $x_i \in R$, define

$$K_\bullet(\mathbf{x}) := K_\bullet(f), \text{with } \begin{array}{c} f : L \to R \\ e_i \mapsto x_i \end{array}.$$

Coro 1.3.1. $K_\bullet(\mathbf{x}) \cong K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_n)$ with $K_\bullet(x_i) : 0 \to R \xrightarrow{x_i} R$.

Prop 1.3.2. Let $x \in R$ and $(C_\bullet, \partial)$ be a chain complex of R-modules, then there exist ρ, π s.t.

$$0 \to C_\bullet \xrightarrow{\rho} C_\bullet \otimes K_\bullet(x) \xrightarrow{\pi} C_\bullet(-1) \to 0$$

is exact, where $(C_\bullet(-1))_n = C_{n-1}$.

Proof. Since $K_\bullet(x) : 0 \to R \xrightarrow{x} R$, so

$$(C_\bullet \otimes K_\bullet(x))_n = (C_i \otimes_R R) \oplus (C_{i-1} \otimes_R R),$$

and the boundary map is

$$\begin{array}{rcl} \mathrm{d} : (C_i \otimes_R R) \oplus (C_{i-1} \otimes_R R) \to & & (C_{i-1} \otimes_R R) \oplus (C_{i-2} \otimes_R R) \\ (z_1 \otimes r_1, z_2 \otimes r_2) & \mapsto & \big(\partial z_1 \otimes r_1 + (-1)^{i-1} z_2 \otimes x r_2, \partial z_2 \otimes r_2\big) \end{array}.$$

Under the isomorphism $C_i \otimes_r R \cong C_i$, the boundary map become

$$\begin{array}{rcl} \mathrm{d} : C_i \oplus C_{i-1} \to & & C_{i-1} \oplus C_{i-2} \\ \begin{pmatrix} r_1 z_1 \\ r_2 z_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \partial & (-1)^{i-1} x \\ 0 & \partial \end{pmatrix} \begin{pmatrix} r_1 z_1 \\ r_2 z_2 \end{pmatrix} \end{array}$$

Let

$$\begin{array}{rcl} \rho_i : C_i & \to & C_i \oplus C_{i-1} \\ z_1 & \mapsto & (z_1, 0) \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_i : C_i \oplus C_{i-1} & \to & C_{i-1} \\ (z_1, z_2) & \mapsto & z_2 \end{array}$$

then

$$\begin{tikzcd} 0 \arrow[r] & C_i \arrow[r, "\rho_i"] \arrow[d, "\partial"] & C_i \oplus C_{i-1} \arrow[r, "\pi_i"] \arrow[d, "\mathrm{d}"] & C_{i-1} \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & C_{i-1} \arrow[r, "\rho_{i-1}"] & C_{i-1} \oplus C_{i-2} \arrow[r, "\pi_{i-1}"] & C_{i-2} \arrow[r] & 0 \end{tikzcd}$$

commutes and exact:

- $\mathrm{d} \circ \rho(z_1) = \mathrm{d}(z_1, 0) = (\partial z_1, 0)$
- $\rho \circ \partial(z_1) = \rho(\partial z_1) = (\partial z_1, 0)$
- $\partial \circ \pi(z_1, z_2) = \partial(z_2) = \partial z_2$
- $\pi \circ \mathrm{d}(z_1, z_2) = \pi(\partial z_1 + (-1)^{i-1} x z_2, \partial z_2) = \partial z_2$

□

Coro 1.3.2. This induces a long exact sequence

$$\cdots \to H_i(C_\bullet) \xrightarrow{\rho_*} H_i(C_\bullet \otimes K_\bullet(x)) \xrightarrow{\pi_*} H_i(C_\bullet(-1)) \xrightarrow{\pm x} H_{i-1}(C_\bullet) \to \cdots.$$

Proof. We only need to show the connection homomorphism is indeed $\pm x$.

Given $z \in C_{i-1}$ with $\partial z = 0$,

$$z \xmapsto{\pi^{-1}} (0, z) \overset{d}{\mapsto} \big((-1)^{i-1} x z, 0\big) \xmapsto{\rho^{-1}} (-1)^{i-1} x z.$$

□

Def 11. We call x to be $C_\bullet$-regular, if x is not a zero divisor of C_i and $C_i/xC_i \neq 0$, for all $i \geq 0$.

Prop 1.3.3. If x is $C_\bullet$-regular, then $H_i(C_\bullet \otimes K_\bullet(x)) \cong H_i(C_\bullet/xC_\bullet)$ for all $i \geq 0$.

Proof. Let

$$\begin{array}{rcl} \phi_i : C_i \oplus C_{i-1} & \to & C_i/xC_i \\ (z_1, z_2) & \mapsto & \overline{z_1} \end{array},$$

then

$$\begin{array}{ccc} C_i \oplus C_{i-1} & \xrightarrow{\phi_i} & C_i/xC_i \\ \downarrow{\scriptstyle \mathrm{d}_i} & & \downarrow{\scriptstyle \overline{\partial}_i} \\ C_{i-1} \oplus C_{i-2} & \xrightarrow{\phi_{i-1}} & C_{i-1}/xC_{i-1} \end{array}$$

commutes.

- $\overline{\partial} \circ \phi_i(z_1, z_2) = \overline{\partial}(z_1) = \overline{\partial z_1}$.
- $\phi_{i-1} \circ \mathrm{d}(z_1, z_2) = \phi_{i-1}(\partial z_1 + (-1)^{i-1} x z_2, \partial z_2) = \overline{\partial z_1}$, since $xz_2 \in xC_{i-1}$.

Now we need to show the induced maps

$$\begin{array}{rcl} \phi_{*i} : \ker \mathrm{d}_i / \operatorname{im} \mathrm{d}_{i+1} & \to & \ker \overline{\partial}_i / \operatorname{im} \overline{\partial}_{i+1} \\ \overline{(z_1, z_2)} & \mapsto & \overline{\overline{z_1}} = \overline{z_1} + \operatorname{im} \overline{\partial}_{i+1} \end{array}$$

are isomorphisms.

- Onto:
  For $\overline{z} \in \ker \overline{\partial}_i$ with $\partial z = xz' \in xC_{i-1}$, $z' \in C_{i-1}$. Then $\phi_i(z, (-1)^i z') = \overline{z}$, and $\mathrm{d}(z, (-1)^i z') = (\partial z - xz', (-1)^i \partial z') = (0, 0)$, so $(z, (-1)^i z') \in \ker \mathrm{d}_i$. (Since $x \partial z' = \partial(xz') = \partial^2 z = 0$, and x is not a zero divisor of C_i, so $\partial z' = 0$.)

  Now, $\phi_{*i}\left(\overline{(z, (-1)^i z')}\right) = \overline{\overline{z}}$, so ϕ_{*i} is onto.
- 1-1:
  Let $(z, z') \in \ker \mathrm{d}_i$ with $\phi_i(z, z') = \overline{z} \in \operatorname{im} \overline{\partial}_{i+1}$, i.e. $\overline{z} = \partial \overline{z''}$ with $z'' \in C_{i+1}$. This means $z - \partial z'' = xz'''$ with $z''' \in C_i$, so $\partial(z - \partial z'') = \partial z = x \partial z'''$.

  On the other hand, $\mathrm{d}(z, z') = (\partial z + (-1)^{i-1} xz', \partial z') = (0, 0)$, so $\partial z = (-1)^i xz', \partial z' = 0$.

  So $\mathrm{d}(z'', (-1)^i z''') = (\partial z'' + (-1)^{2i} xz'''), (-1)^i \partial z''') = (z, z')$, i.e. $(z, z') \in \operatorname{im} d_{i+1}$. ($\partial z = x \partial z''' = (-1)^i xz'$, since x is not a zero divisor, so $\partial z''' = (-1)^i z'$.)

  Hence, $\phi_{*i}\left(\overline{(z_1, z_2)}\right) = \overline{0}$ implies $\overline{(z_1, z_2)} = \overline{0}$, so ϕ_{*i} is 1-1.

$\square$

Def 12. Let $M \in \mathbf{Mod}_R$. A sequence $\{a_1, \cdots, a_m\}, m \geq 0$ is said to be M-regular if

- $M/\langle a_1, \cdots, a_m \rangle M \neq 0$.
- a_{i+1} is not a zero divisor of $M/\langle a_1, \cdots, a_i \rangle M$ for $0 \leq i \leq m-1$.

Theorem 9. If $\mathbf{x} = (x_1, \cdots, x_n)$ is an R-regular sequence, then $K_\bullet(\mathbf{x}) \to R/\langle x_1, \cdots, x_n \rangle \to 0$ is a free resolution of $R/\langle x_1, \cdots, x_n \rangle$.

Proof. Since its modules are $K_i(\mathbf{x}) = \Lambda^i R^n \cong R^{\binom{n}{i}}$, i.e. free R-modules, so we only need to show the exactness.

By induction on n,

- $n = 1$: $K_\bullet(x_1) : 0 \to R \xrightarrow{x_1} R \to R/\langle x_1 \rangle \to 0$ exact.

- $n > 1$: Assume that $\mathbf{x}' = (x_1, \cdots, x_{n-1})$ and $K_\bullet(\mathbf{x}') \to R/\langle x_1, \cdots, x_{n-1}\rangle \to 0$ exact, i.e. $H_i(K_\bullet(\mathbf{x}')) = 0$ for $i > 0$.

  Since we have $K_\bullet(\mathbf{x}) \cong K_\bullet(\mathbf{x}') \otimes K_\bullet(x_n)$ and a long exact sequence
  $$\cdots \to H_i(K_\bullet(\mathbf{x}')) \to H_i(K_\bullet(\mathbf{x})) \to H_i(K_\bullet(\mathbf{x}')(-1)) \xrightarrow{\pm x_n} H_{i-1}(K_\bullet(\mathbf{x})) \to \cdots$$
  where $H_i(K_\bullet(\mathbf{x}')(-1)) = H_{i-1}(K_\bullet(\mathbf{x}'))$.

  For $i > 1$, the sequence becomes
  $$\cdots \to 0 \to H_i(K_\bullet(\mathbf{x})) \to 0 \xrightarrow{\pm x_n} \cdots,$$
  so $H_i(K_\bullet(\mathbf{x})) = 0$.

  For $i = 1$, we have $H_0(K_\bullet(\mathbf{x})) \cong R/\langle x_1, \cdots, x_{n-1}\rangle$, so
  $$0 \to H_1(K_\bullet(\mathbf{x})) \to R/\langle x_1, \cdots, x_{n-1}\rangle \xrightarrow{\pm x_n} R/\langle x_1, \cdots, x_{n-1}\rangle$$
  But x_n is not a zero divisor of $R/\langle x_1, \cdots, x_{n-1}\rangle$, so the map $\pm x_n$ is 1-1, then $H_1(K_\bullet(\mathbf{x})) \cong \ker(\pm x_n) = 0$.

□

Eg 1.3.1. Let $\mathbf{x} = (x_1, x_2)$, then
$$K_\bullet(\mathbf{x}) : 0 \to R \xrightarrow{\alpha} R^2 \xrightarrow{\beta} R \xrightarrow{q} R/\langle x_1, x_2\rangle \to 0$$
with $\alpha : r \mapsto (-x_2 r, x_1 r)$ and $\beta : (r_1, r_2) \mapsto x_1 r_1 + x_2 r_2$.

Coro 1.3.3. Let $I = \langle x_1, \cdots, x_n\rangle \subset R$ be an ideal with $\{x_1, \cdots, x_n\}$ be R-regular, then R/I has *projective dimension* $\mathrm{pd}(R/I) = n$, i.e. the shortest projective resolution of R/I has length n.

Proof. $K_\bullet(\mathbf{x})$ is already a projective resolution of length N, so we only need to show that there's no shorter ones.

The left side of $K_\bullet(\mathbf{x})$ reads
$$0 \to \Lambda^n R^n \xrightarrow{d_n} \Lambda^{n-1} R^n \to \cdots$$

But
$$\Lambda^n R^n = R(e_1 \wedge \cdots \wedge e_n) \cong R, \quad \Lambda^{n-1} R^n = \bigoplus_{i=1}^n R(e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e^n) \cong R^n$$
so
$$\begin{aligned} d_n : R &\to R^n \\ r &\mapsto (x_1 r, -x_2 r, \cdots, (-1)^{n-1} x_n r) \end{aligned}$$

Taking tensor with R/I, we get
$$0 \to R \otimes_R R/I \xrightarrow{d_n \otimes \mathbf{1}} R^n \otimes_R R/I \to \cdots$$
but $R \otimes_R R/I \cong R/I, R^n \otimes_R R/I \cong (R/I)^n$, so
$$\begin{aligned} d_n \otimes \mathbf{1} : R/I &\to (R/I)^n \\ \overline{r} &\mapsto (\overline{x_1 r}, \overline{-x_2 r}, \cdots, \overline{(-1)^{n-1} x_n r}) \end{aligned}$$

Now,
$$\mathrm{Tor}_n(R/I, R/I) = H_n(K_\bullet(\mathbf{x}) \otimes R/I) = \ker(d_n \otimes \mathbf{1}) = \mathrm{Ann}_{R/I}\, I = \{\overline{r} \in R/I \mid rI = I\} = R/I \neq 0.$$

($R/I \neq 0$ is because $\{x_1, \cdots, x_n\}$ is R-regular.) Thus, any projective resolution can't have length shorter than n since that will imply $\mathrm{Tor}_n(R/I, R/I) = 0$. □

Remark 1. Let $I = \langle x_1, \cdots, x_n \rangle$ generated by R-regular sequence $\{x_1, \cdots, x_n\}$, then

- $\mathrm{Tor}_n(R/I, M) \cong \mathrm{Ann}_M I$.
- $\mathrm{Ext}^n(R/I, M) \cong M/IM$.

1.4 Derived category

Def 13.

- $\mathcal{C}$ is a pre-additive category if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group $\forall\, X, Y \in \mathcal{C}$ s.t.

$$X \xrightarrow{u} Y \overset{f}{\underset{g}{\rightrightarrows}} Z \xrightarrow{v} T$$

  with $(f+g)u = fu + gu$ and $v(f+g) = vf + vg$.
- addivitve category: a pre-additive category $\mathcal{C}$ s.t.
  - There exists a zero object 0 s.t. $\forall\, X$, $\mathrm{Hom}_{\mathcal{C}}(0, X) = \{0\} = \mathrm{Hom}_{\mathcal{C}}(X, 0)$.
  - Finite sum and finite products exist.

Def 14.

- $f \in \mathrm{Hom}(B, C)$ is called a monomorphism if $\forall\, X \xrightarrow{g} B \xrightarrow{f} C$ with $f \circ g = 0 \implies g = 0$.
- $f \in \mathrm{Hom}(B, C)$ is called a epimorphism if $\forall\, B \xrightarrow{f} C \xrightarrow{h} D$ with $h \circ f = 0 \implies h = 0$.
- a kernel of $f \in \mathrm{Hom}(B, C)$ is a morphism $i : A \to B$ s.t. $f \circ i = 0$ and $\forall\, g : X \to B$ with $f \circ g = 0$, we have

$$\begin{tikzcd} A \arrow[r, "i"] & B \arrow[r, "f"] & C \\ & X \arrow[ul, dashed, "\exists!"] \arrow[u, "g"] \arrow[ur, "0"'] & \end{tikzcd}$$

- a cokernel of $f \in \mathrm{Hom}(B, C)$ is a morphism $p : C \to D$ s.t. $p \circ f = 0$ and $\forall\, h : C \to Y$ with $h \circ f = 0$, we have

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "0"'] & C \arrow[r, "p"] \arrow[d, "h"] & D \arrow[dl, dashed, "\exists!"] \\ & Y & \end{tikzcd}$$

Remark 2.

- If i is a kernel of f, then i is a monomorphism.
- If p is a cokernel of f, then p is a epimorphism.

Remark 3. An epimonrphism may not be a cokernel. Consider $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$ which is an epimorphism in the category of f.g. free $\mathbb{Z}$-modules. If $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$ is the cokernel of $G \xrightarrow{f} \mathbb{Z}$, then

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "0"'] & \mathbb{Z} \arrow[r, "\times 3"] \arrow[d, "\times 2"] & \mathbb{Z} \arrow[dl, dashed, "\exists!\, \tilde{f}"] \\ & \mathbb{Z} & \end{tikzcd}$$

This implies $\tilde{f} : 1 \mapsto \frac{2}{3}$, which is impossible.

Def 15. $\mathcal{A}$ is an **abelian category** if it is an additive category s.t.

- kernels and cokernels always exist in $\mathcal{A}$.
- every monomorphism is a kernel and every epimorphism is a cokernel.

Fact 1.4.1. If $\mathcal{A}$ is an abelian category, then:

- every morphism is expressible as the composite of an epimorphism and a monomorphism. Given $f : B \to C$, we have

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, two heads] & & C \\ & \operatorname{Im} f \arrow[ur, hook] & \end{tikzcd}$$

  where $\operatorname{Im} f$ is unique up to isomorphism.

  Proof. Consider the following diagram:

$$\begin{tikzcd} \ker f \arrow[r, hook, "i"] & B \arrow[r, "f"] \arrow[d, two heads, "p'"] \arrow[dr, dashed, "\exists\,!\mu"] & C \arrow[r, two heads, "p"] & \operatorname{coker} f \\ & \operatorname{coker} i \arrow[r, dashed, "\exists\,!\sigma"'] \arrow[ur, dashed, "\exists\,!\gamma"'] & \ker p \arrow[u, hook, "i'"'] & \end{tikzcd}$$

  Where μ, γ exists because i, p are kernel and cokernel. Now, $i'\mu i = fi = 0$, and since i' is a monomorphism, $\mu i = 0$. Moreover, since p is the cokernel of i, there exists a unique σ letting the diagram commute.

  By exercise, σ is both a monomorphism and epimorphism. In an abelian catagory, this implies that σ is actually an isomorphism (i.e., σ^{-1} exists). $\square$

- $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if f is monomorphism, g is epimorphism and $\operatorname{Im} f = \ker g$.

Theorem 10 (Freyd-Mitchell theorem)**.** A small abelian category is equivalent to a full subcategory of a category of R-modules.

Def 16.

- $I \in \operatorname{Obj} \mathcal{A}$ is injective if the functor $\operatorname{Hom}(-, I)$ is exact.
- An abelian category is said to be **enough injectives** if for any $A \in \operatorname{Obj} \mathcal{A}$, there exists an injective object I such that $A \hookrightarrow I$.

Def 17. Given a functor $F : \mathcal{A} \to \mathcal{B}$ satisfy:

1. F is additive, which is to say F is a group homomorphism $\operatorname{Hom}(A, A') \to \operatorname{Hom}(FA, FA')$.
2. F is left exact. If $0 \to A' \to A \to A'' \to 0$, then $0 \to FA' \to FA \to FA''$.

Then the derived functor $R^iF : \mathcal{A} \to \mathcal{B}$ is defined as

$$R^iF(A) = \begin{cases} F(A), & \text{if } i = 0 \\ H^i(F(I^\bullet)), & \text{else} \end{cases}$$

Our goal is to construct the derived category $D^+(\mathcal{A})$ and $D^+(\mathcal{B})$ letting RF be a exact functor.

Def 18. Let $\mathcal{A}$ be an abelian category.

- $\operatorname{Kom}(\mathcal{A})$ is the category of complexes over $\mathcal{A}$.

- $K(\mathcal{A})$ is the homotopy category of $\mathcal{A}$, defined by $\mathrm{Obj}(K(\mathcal{A})) = \mathrm{Obj}(\mathrm{Kom}(\mathcal{A}))$ and

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim,$$

  where $\sim$ indicates homotopy equivalences.

Remark 4.

- $\mathrm{Hom}_{K(\mathcal{A})}(I_A^\bullet, I_B^\bullet) \cong \mathrm{Hom}_{\mathcal{A}}(A, B)$ by comparison theorem (3).
- It could be shown that $K(\mathcal{A})$ is additive but may not be abelian.

Def 19. $f \in \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet)$ is called a quasi-isomorphism if $H^n(f)$ is an isomorphism between $H^n(A^\bullet)$ and $H^n(B^\bullet)$ for each n.

Eg 1.4.1.

- A quasi-isomorphism is often not invertible. For example:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

- Given $0 \to A \to I^\bullet$,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

  are two quasi-isomorphic complexes.

Def 20. Let $\mathcal{B}$ be a category. A class of morphism $S \subset \mathrm{Mor}(\mathcal{B})$ is said to be **localizing** if

1. S is closed under composition with $\mathrm{Id}_X \in S$ for each object X in $\mathcal{B}$.
2. Extension condition holds: For each $f \in \mathrm{Mor}\,\mathcal{B}$, $s \in S$, exists $g \in \mathrm{Mor}\,\mathcal{B}$, $t \in S$ such that $ft = sg$. The dual version should hold as well.
3. For any $f, g \in \mathrm{Hom}(X, Y)$,

$$\exists\, s \in S \text{ s.t. } sf = sg \iff \exists\, t \in S \text{ s.t. } ft = gt.$$

Theorem 11. If S is localizing, then exists a category $\mathcal{B}[S^{-1}]$ with a functor $Q : \mathcal{B} \to \mathcal{B}[S^{-1}]$ such that

1. $Q(s)$ is an isomorphism for each $s \in S$.
2. Given another functor $F : B \to B'$ satisfy condition 1, there exists a unique functor $G : \mathcal{B}[S^{-1}] \to B'$ such that $F = G \circ Q$.

Proof. Define a roof to be a pair (s, t) with

$$\begin{tikzcd}
& X' \arrow[dl, "S \ni s"'] \arrow[dr, "t"] & \\
X & & Y
\end{tikzcd}$$

Also, define $(s, f) \sim (t, g)$ if there exists Z such that

$$\begin{tikzcd}
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & Z \arrow[u, "p"] \arrow[d, "q"] & Y \\
 & X'' \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}$$

with $sp = tq \in S$ and $fp = gq$.

First we check that "$\sim$" is indeed an equivalence relation. $(s, f) \sim (s, f)$ and $(s, f) \sim (t, g) \implies (t, g) \sim (s, f)$ are trivial. If $(s, f) \sim (t, g)$ and $(t, g) \sim (u, h)$, then we have the following diagram:

$$\begin{tikzcd}
 & X' \arrow[dl, "s"'] \arrow[ddr, "f"] & & & \\
 & Z \arrow[u, "p"] \arrow[d, "q"] & & & \\
A & X'' \arrow[l, "t"] \arrow[r, "g"'] & B & R \arrow[ull, "\exists\, z"'] \arrow[dll, "\exists\, a"] & T \arrow[l, "\exists\, w"] \\
 & W \arrow[u, "r"] \arrow[d, "v"] & & & \\
 & X''' \arrow[uul, "u"] \arrow[uur, "h"'] & & &
\end{tikzcd}$$

Using definition 2. on $tr \in S$ and sp, there are morphism z, a with $z \in S$ and $spz = tra$. Moreover, $tqz = spz = tra$, if we let $b = qz, c = ra$, then by 3., morphism $w \in S$ exists with $bw = cw$. Define $x = pzw, y = vaw$, we have $sx = spzw = tqzw = tbw = tcw = traw = uvaw = uy$ and $sx \in S$ since $sx = spzw$ and sp, z, w are all in S. Similarly, $fx = hy$, thus $(s, f) \sim (u, h)$. Hence we've just proved that $\sim$ is an equivalence relation.

Now we could construct the localized category as following: The objects are $\mathrm{Obj}(\mathcal{B}[S^{-1}]) = \mathcal{B}$ and $\mathrm{Mor}(\mathcal{B}[S^{-1}]) = \{$ equivalence classes under $\sim \}$. $[(t, g)] \circ [(s, f)] = [(su, gh)]$ could be defined as in the following diagram:

$$\begin{tikzcd}
 & & \exists\, U \quad \text{by 2.} \arrow[dl, "S \ni u"'] \arrow[dr, "h"] & & \\
 & X \arrow[dl, "s"'] \arrow[dr, "f"] & & Y \arrow[dl, "t"'] \arrow[dr, "g"] & \\
A & & B & & C
\end{tikzcd}$$

$\square$

Finally, define functor Q by $Q(X) = X, \forall\, X \in \mathrm{Obj}(\mathcal{B})$ and $Q(f) = [(\mathrm{Id}_X, f)]$. For the universal property, if F is another functor making every morphism in S be invertible, then the functor G exists uniquely by $G([(s, f)]) = F(f)F(s)^{-1}$.

Def 21. The mapping cone of a chain map f between two chain $X^\bullet \xrightarrow{f} Y^\bullet$ is defined as a chain with $\mathrm{cone}(f)^n = X^{n+1} \oplus Y^n$, and the chain map is defined as

$$\begin{array}{rccc} d_{\mathrm{cone}(f)}: & \mathrm{cone}(f)^n = X^{n+1} \oplus Y^n & \longrightarrow & \mathrm{cone}(f)^{n+1} X^{n+2} \oplus Y^{n+1} \\ & (x_{n+1}, y_n) & \xmapsto[\begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}]{} & \big(-d_X(x^{n+1}), f(x^{n+1}) + d_Y(y_n)\big) \end{array}$$

It is easy to see that $d^2_{\mathrm{cone}(f)} = 0$.

Prop 1.4.1. Suppose that $f : X^\bullet \to Y^\bullet$ is a chain map, then there is a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & \mathrm{cone}(f) & \longrightarrow & X[+1] & \longrightarrow & 0 \\ & & d & \longmapsto & (0,d) & & & & \\ & & & & (c,d) & \longmapsto & -c & & \end{array}$$

Proof. It is easy to see that the rows are exact. Tracing the diagram shows that the diagram commutes. □

Coro 1.4.1. There exists a long exact sequence of homology:

$$\cdots \to H^m(Y^\bullet) \to H^m(\mathrm{cone}(f)) \to H^{m+1}(X^\bullet) \xrightarrow{\delta} H^{m+1}(Y^\bullet) \to H^{m+1}(\mathrm{cone}(f)) \to \cdots$$

Where the connecting homomorphism $\delta = f^*$.

Proof. Tracing the diagram below as in the snake lemma,

$$\begin{tikzcd} & X^m \oplus Y^{m-1} \arrow[r] \arrow[d] & X^m \arrow[d] \\ Y^m \arrow[r] & X^{m+1} \oplus Y^m \arrow[r] & X^{m+1} \end{tikzcd}$$

Suppose $\bar{x} \in H^m(X^\bullet)$, then $d_X(x) = 0$, so $d(-x, 0) = (dx, -f(x))$ with $dx = 0$, which implies $-f(x) :: Y^m \mapsto d(-x, 0) :: X^{m+1} \oplus Y^m$, so $\delta = -f^*$ (出問題)... □

Coro 1.4.2. $\mathrm{cone}(f)$ exact $\iff$ f quasi-isomorphic.

Proof. Directly by the exact sequence

$$H^{m-1}(\mathrm{cone}(f)) \to H^m(X^\bullet) \to H^m(Y^\bullet) \to H^m(\mathrm{cone}(f))$$

□

Theorem 12. Let $\mathcal{A}$ be an abelian category and $K(\mathcal{A})$ be the homotopy category. Then the class of quasi-isomorphisms are localizing.

Proof. We check that:

1. It is closed under composition: If f, g are quasi-isomorphic, then $(fg)^* = f^*g^*$ is a isomorphism since both f^*, g^* are, thus fg is quasi-isomorphic.

2. The diagram could be completed:

$$\begin{tikzcd}
\exists\, W^\bullet \arrow[r, dashed] \arrow[d, dashed] & Z^\bullet \arrow[d, "g\colon \text{q-iso}"] \\
X^\bullet \arrow[r, "f"] & Y^\bullet
\end{tikzcd}$$

$\square$