

# Algebra

December 29, 2016

# 1 Group theory

## 1.1 Week 1

**Def 1.** A non-empty set  $G$  with a binary function  $f : G \times G \rightarrow G, (a, b) \mapsto ab$  is a **group** if it satisfies

1.  $(ab)c = a(bc)$ .
2.  $\exists 1 \in G$  s.t.  $1a = a1 = a, \forall a \in G$ .
3.  $\exists a^{-1} \in G$  s.t.  $aa^{-1} = a^{-1}a = 1$ .

CONCON

**Def 2.** Let  $G$  be a group. Then  $G$  is said to be **abelian** if  $\forall a, b \in G, ab = ba$ .

**Ex 1.1.1.** Let  $G$  be a semigroup. Then TFAE (the following are equivalent)

1.  $G$  is a group.
2. For all  $a, b \in G$  and the equations  $bx = a, yb = a$ , each of them has a solution in  $G$ .
3.  $\exists e \in G$  s.t.  $ae = a \forall a \in G$  and if we fix such  $e$ , then  $\forall b \in G \exists b' \in G$  s.t.  $bb' = e$ .

**Ex 1.1.2.** Let  $G$  be a group. Show that

1.  $\forall a \in G, a^2 = 1$ , then  $G$  is abelian.
2.  $G$  is abelian  $\iff \forall a, b \in G, (ab)^n = a^n b^n$  for three consecutive integer  $n$ .

**Def 3.** Let  $G$  be a group and  $H \subseteq G, H \neq \emptyset$ . Then  $H$  is said to be a subgroup of  $G$ , denoted by  $H \leq G$ , if

1.  $\forall a, b \in H, ab \in H$ .
2.  $1 \in H$ .
3.  $\forall a \in H, a^{-1} \in H$ .

useful criterion:  $H \leq G \iff \forall a, b \in H, ab^{-1} \in H$ .

*Proof.*

$\Rightarrow$   $b \in H \implies b^{-1} \in H$ , and  $a \in H$ , so  $ab^{-1} \in H$ .

$\Leftarrow$  1.  $H \neq \emptyset \implies \exists a \in H \implies aa^{-1} = 1 \in H$ .

2.  $1, a \in H \implies 1a^{-1} = a^{-1} \in H$ .

3.  $a, b^{-1} \in H \implies a(b^{-1})^{-1} = ab \in H$ . □

**Eg 1.1.1.**  $(\mathbb{Z}, +, 0) \leq (\mathbb{Q}, +, 0) \leq (\mathbb{R}, +, 0) \leq (\mathbb{C}, +, 0) ; (\mathbb{Q}^\times, \times, 1) \leq (\mathbb{R}^\times, \times, 1) \leq (\mathbb{C}^\times, \times, 1)$

**Eg 1.1.2.**

- Special linear group  $\text{SL}(n, \mathbb{F}) = \{ A \in \text{GL}(n, \mathbb{F}) \mid \det A = 1 \}$
- Orthogonal group  $\text{O}(n) = \{ A \in \text{GL}(n, \mathbb{R}) \mid A^t A = I_n \}$

- Unitary group  $U(n) = \{ A \in GL(n, \mathbb{C}) \mid A^* A = I_n \}$
- Special orthogonal group  $SO(n) = SL(n, \mathbb{R}) \cap O(n)$
- Special unitary group  $SU(n) = SL(n, \mathbb{C}) \cap U(n)$

**Def 4.** Let  $f : G_1 \rightarrow G_2$ .  $f$  is called an **isomorphism** if

1.  $f$  is 1-1 and onto.
2.  $\forall a, b \in G_1, f(ab) = f(a)f(b)$ . (**homomorphism**)

, denoted by  $G_1 \cong G_2$ .

**Remark 1.** (practice)

1.  $f(1) = 1$ .
2.  $f(a^{-1}) = f(a)^{-1}$ .
3. If  $f$  is an isomorphism, then  $\exists f^{-1}$  is also a homomorphism.

**Eg 1.1.3.**

- $U(1) = \{ z \in \mathbb{C}^\times \mid \bar{z}z = 1 \}, z = \cos \theta + \sin \theta i$
- $SO(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$

notice that  $U(1) \cong SO(2)$ .  $S^1 = \{ (a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1 \}$ , 可被賦予群的結構.

**Eg 1.1.4.** Let  $A \in SU(2) \implies A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \alpha\bar{\alpha} + \beta\bar{\beta} = 1, \alpha, \beta \in \mathbb{C}$ .

Quaternion(四元數):  $\mathbb{H} = \{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \}$  with  $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$  ( $\implies ij = -ji$ ).

Let  $x = a + bi + cj + dk, \bar{x} = a - bi - cj - dk$ , then  $N(x) = x\bar{x} = a^2 + b^2 + c^2 + d^2$ , For  $x \neq 0, N(x) \neq 0, x^{-1} = \frac{1}{N(x)}\bar{x}$

Now, for  $x = a + bi + cj + dk = (a + bi) + (c + di)j$ . So  $SU(2) \cong \{ x \in \mathbb{H}^\times \mid N(x) = 1 \}$ .  $S^3 = \{ (a, b, c, d) \in \mathbb{R}^4 \mid a^2 + b^2 + c^2 + d^2 = 1 \}$ , 可被賦予群的結構.

★ The only spheres with continuous group law are  $S^1, S^3$ .

**Ex 1.1.3.** Find a way to regard  $M_{n \times n}(\mathbb{H})$  as a subset of  $M_{2n \times 2n}(\mathbb{C})$ , which preserves addition and multiplication, and then there is a way to characterize  $GL(n, \mathbb{H})$ .

**Def 5** (symplectic group).  $Sp(n, \mathbb{F}) = \{ A \in GL(2n, \mathbb{F}) \mid A^t J A = J \}$  where  $J = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}$ .

( $A^t J A = J$  preserving non-degenerate skew-symmetric forms)

$Sp(n) = \{ A \in GL(n, \mathbb{H}) \mid A^* A = I_n \}$ .

**Ex 1.1.4.** Show  $Sp(n) \cong U(2n) \cap Sp(n, \mathbb{C})$ .

Ques: Find the smallest subgroup of  $SU(2)$  containing  $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ .

## 1.2 Week 2

### 1.2.1 Permutation groups and Dihedral groups

**Def 6.** A permutation of a set  $B$  is a 1-1 and onto function from  $B$  to  $B$ .

Let  $S_B :=$  the set of permutations of  $B$ . Then  $(S_B, \cdot, \text{Id}_B)$  forms a group.

If  $B = \{a_1, \dots, a_n\}$ , then  $S_B \cong S_{\{1, \dots, n\}}$  and write  $S_n = S_{\{1, \dots, n\}}$ , called the symmetric group of degree  $n$ .

**Theorem 1** (Cayley theorem). Any group is isomorphic to a subgroup of some permutation group.

(Hint): Let  $G$  be a group. Set  $B = G$ . Consider  $a \in G$  as  $\sigma_a : G \rightarrow G, x \mapsto ax$ . Then  $\sigma_a \in S_G \implies G \leq S_G$ .

**Fact 1.2.1.**  $S_n$  is a finite group of order  $n!$ , i.e.  $|S_n| = n!$ .

*Proof.* EASY =O

□

Cyclic notation:  $\sigma \in S_5$ , say  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$ . Write  $\sigma = (1\ 4)(2\ 3\ 5)$ .

$\Rightarrow$  Any permutation can be written as a product of disjoint cycles.

**Eg 1.2.1.** In  $S_7$ ,  $\sigma_1 = (1\ 2\ 3)(4\ 5\ 6)(7)$ ,  $\sigma_2 = (1\ 3\ 5\ 6)(2\ 4\ 7)$ .

Then  $\sigma_1\sigma_2 = (2\ 5\ 4\ 7\ 3\ 6)$ ,  $\sigma_1^{-1} = (1\ 3\ 2)(4\ 6\ 5)$ .

**Def 7.** A 2 cycle is called a **transposition**.

**Eg 1.2.2.**  $(1\ 2\ 3) = (1\ 3)(1\ 2)$ ,  $(1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2)$ .

Any permutation is a product of 2 cycles.

Useful formula:  $\sigma \in S_n$ ,  $\sigma(j_1 \dots j_m)\sigma^{-1} = (\sigma(j_1) \dots \sigma(j_m))$ .

**Eg 1.2.3.** Let  $\sigma = (1\ 2\ 3)(4\ 5\ 6\ 7)$ ,  $\sigma(2\ 3\ 4)\sigma^{-1} = (3\ 1\ 5)$ .

*Proof.* Note that both sides are functions. For  $i \in \{1, \dots, n\}$ ,

Case 1:  $\exists k$  s.t.  $\sigma(j_k) = i$ , CONCON

Case 2: Otherwise, CONCON

□

**Fact 1.2.2.**  $S_n = \langle (1\ 2), \dots, (1\ n) \rangle$ .

*Proof.*  $(1\ i)^{-1} = (1\ i)$  and  $(i\ j) = (1\ i)(1\ j)(1\ i)^{-1}$ .

□

**Def 8.** Let  $G$  be a group and  $S \subset G$ . The subgroup generated by  $S$  defined to be the smallest subgroup of  $G$  which contains  $S$ , denoted by  $\langle S \rangle$ .

**Ex 1.2.1.**

1.  $S_n = \langle (1\ 2), (2\ 3), \dots, (n-1\ n) \rangle$ ,  $n \geq 2$ .

2.  $S_n = \langle (1\ 2), (1\ 2\ \dots\ n) \rangle, \quad n \geq 2.$

**Def 9.**  $A_n = \{\text{even permutations of } S_n\} \leq S_n, |A_n| = \frac{n!}{2}.$

**Ex 1.2.2.**

1.  $A_n = \langle (1\ 2\ 3), (1\ 2\ 4), \dots, (1\ 2\ n) \rangle, n \geq 3.$
2.  $A_n = \langle (1\ 2\ 3), (2\ 3\ 4), \dots, (n-2\ n-1\ n) \rangle, n \geq 3.$

**Remark 2.**  $\langle S \rangle = \bigcap_{S \subseteq H \leq G} H = \{a_1 a_2 \dots a_k \mid k \in \mathbb{N}, a_i \in S \cup S^{-1}\} \cup \{1\}$

The orthogonal transformations on  $\mathbb{R}^2$ :  $O(2)$ .

Let  $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in O(2).$

略... (這邊討論旋轉和反射的矩陣)

Case 1:  $A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$  is counterclockwise rotation w.r.t.  $\alpha$ .

Case 2:  $A = \begin{pmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{pmatrix}$  is the reflection.  $A^2 = I_2 \implies$  eigenvalues are  $\pm 1$ .

Easy to show that  $L_A(v) = v - 2\langle v, v_2 \rangle v_2$ .

$O(2) = \{\text{rotations}\} \cup \{\text{reflections}\}.$

**Def 10.** The dihedral group  $D_n$  is the group of symmetries of a regular  $n$ -gon.

In general,  $D_n = \langle T, R \mid T^n = 1, R^2 = 1, TR = RT^{-1} \rangle \leq O(2) \leq S_n, |D_n| = 2n.$

**Def 11.** Let  $T$  be a linear transformation from  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ .

- $T$  is called a rotation if  $\exists$  a  $T$ -invariant subspace  $W \subseteq \mathbb{R}^n$  with  $\dim W = 2$  s.t.  $\begin{cases} T|_W \text{ is a rotation} \\ T|_{W^\perp} = \text{id}_{W^\perp} \end{cases}$
- $T$  is called a reflection if  $\exists$  a  $T$ -invariant subspace  $W \subseteq \mathbb{R}^n$  with  $\dim W = 1$  s.t.  $\begin{cases} T|_W = -\text{id}_W \\ T|_{W^\perp} = \text{id}_{W^\perp} \end{cases}$

Main result: the group of orthogonal transformations =  $\langle \text{rotations, reflections} \rangle.$

**Prop 1.2.1.** For  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\exists$  a  $T$ -invariant subspace  $W \subseteq \mathbb{R}^n$  with  $1 \leq \dim W \leq 2$ .

*Proof.* Let  $A = [T]_\alpha \in M_{n \times n}(\mathbb{R}) \subseteq M_{n \times n}(\mathbb{C})$ . Consider  $\widetilde{L}_A: \mathbb{C}^n \rightarrow \mathbb{C}^n, v \mapsto Av$ .

Then  $\exists$  an eigenvalue  $\lambda \in \mathbb{C}$  and an eigenvector  $v \in \mathbb{C}^n$  for  $\widetilde{L}_A$ . Let  $\lambda = \lambda_1 + \lambda_2 i, v = v_1 + v_2 i$ . By definition, we have

$$Av = \widetilde{L}_A(v) = \lambda v = (\lambda_1 + \lambda_2 i)(v_1 + v_2 i) \implies \begin{cases} Av_1 = \lambda_1 v_1 - \lambda_2 v_2 \\ Av_2 = \lambda_2 v_1 + \lambda_1 v_2 \end{cases},$$

so  $W = \langle v_1, v_2 \rangle.$

□

**Ex 1.2.3.**

1. If  $T$  is orthogonal, then  $W^\perp$  is also  $T$ -invariant.

2. Use induction on  $n$  to show the main result.

For  $n = 3, A \in O(3)$ , we have  $A \sim \begin{pmatrix} \cos \alpha & -\sin \alpha & \\ \sin \alpha & \cos \alpha & \\ & & \pm 1 \end{pmatrix}$ .

### 1.2.2 Cyclic groups and internal direct product

**Def 12.** If  $G = \langle a \rangle = \{\dots, a^{-2}, a^{-1}, a, 1, a, a^2, \dots\} = \{a^n \mid n \in \mathbb{Z}\}$ , then  $G$  is a cyclic group generated by  $a$ .

**Eg 1.2.4.**  $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$ .

**Eg 1.2.5.** Let  $A = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} \in \text{SO}(2)$ . Then  $\langle A \rangle = \{I_2, A, A^2, \dots, A^{n-1}\}$  and  $A^n = I_2, A^m = A^r$  where  $m \equiv r \pmod{n}$ .

**Eg 1.2.6.**  $\mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \dots, \overline{(n-1)}\}$  with  $\bar{j} = \{m \in \mathbb{Z} \mid m \equiv j \pmod{n}\}$ .

Define  $\bar{i} + \bar{j} = \begin{cases} \overline{i+j} & \text{if } 0 \leq i+j \leq n \\ \overline{i+j-n} & \text{otherwise} \end{cases} \implies (\mathbb{Z}/n\mathbb{Z}, +, \bar{0})$  forms a group.

**Remark 3.**  $\bar{i} \times \bar{j} = \overline{i \times j}$ .

- 略
- If  $\gcd(j, n) = d, \exists h, k \in \mathbb{Z}$  s.t.  $hj + kn = d$ .

**Def 13.**  $(\mathbb{Z}/n\mathbb{Z})^\times = \{j \in \mathbb{Z}/n\mathbb{Z} \mid \gcd(j, n) = 1\} \implies ((\mathbb{Z}/n\mathbb{Z})^\times, \times, \bar{1})$  forms a group.

**Eg 1.2.7.** 略... 简化剩余系, 原根 (generator)  $(1, 2, 4, p^k, 2p^k, p \text{ is an odd prime})$

**Def 14.**

- The **order** of a finite group  $G$  is the number of elements in  $G$ , denoted by  $|G|$ .
- Let  $a \in G$ , the order of  $a$  is defined to be the least positive integer  $n$  s.t.  $a^n = 1$ , denoted by  $\text{ord}(a) = n$ .
- If  $a^n \neq 1 \quad \forall n \in \mathbb{N}$ , then we call “ $a$  has infinite order”.

**Prop 1.2.2.** Let  $G = \langle a \rangle$  with  $\text{ord}(a) = n$ . Then

1.  $a^m = 1 \iff n \mid m$ .

*Proof.*

$\Leftarrow$ : Let  $m = dn$ , then  $a^m = (a^n)^d = 1$ .

$\Rightarrow$ : Let  $m = qn + r, 0 \leq r < n$ . If  $r \neq 0$ , then  $a^r = a^{m-qn} = (a^m)(a^n)^{-q} = 1$ . But  $r < n$ , which is a contradiction. Hence  $r = 0 \implies n \mid m$ .  $\square$

2.  $\text{ord}(a^r) = n / \gcd(r, n)$ .

*Proof.* Let  $\gcd(r, n) = d, n = dn', r = dr'$  with  $\gcd(n', r') = 1$ . Plan to show “ $\text{ord}(a^r) = n'$ .”

- $(a^r)^{n'} = a^{r'dn'} = (a^n)^{r'} = 1 \implies \text{ord}(a^r) \mid n'$ .
- $1 = (a^r)^{\text{ord}(a^r)} = a^{r \text{ord}(a^r)} \implies n \mid r \text{ord}(a^r) \implies n' \mid r' \text{ord}(a^r) \implies n' \mid \text{ord}(a^r)$ .

□

**Prop 1.2.3.** Any subgroup of a cyclic group is cyclic.

*Proof.* Let  $G = \langle a \rangle$  and  $H \leq G$ . If  $H = \{1\}$ , then  $H = \langle 1 \rangle$ , done!

Otherwise,  $d = \min\{m \in \mathbb{N} \mid a^m \in H\}$ , by well-ordering axiom. Claim  $H = \langle a^d \rangle$ .

⊃:  $a^d \in H$  by the definition of  $d$ .

⊂:  $\forall a^m \in H$ , write  $m = qd + r, 0 \leq r < d$ . If  $r \neq 0$ , then  $a^r = a^{m-qd} = a^m(a^d)^{-q} \in H$ , which is a contradiction. Hence  $r = 0 \implies d \mid m$ .

□

**Ex 1.2.4.**

1.  $\text{ord}(a) = \text{ord}(a^{-1}) = n$ .
2.  $\langle a^r \rangle = \langle a^{\gcd(n,r)} \rangle$ .
3.  $\langle a^{r_1} \rangle = \langle a^{r_2} \rangle \iff \gcd(n, r_1) = \gcd(n, r_2)$ .
4.  $\forall m \mid n, \exists! H \leq \langle a \rangle$  s.t.  $|H| = m$ . Conversely, if  $H \leq \langle a \rangle$ , then  $|H| \mid n$ .

**Prop 1.2.4.** Let  $G = \langle a \rangle$ . Then

1.  $\text{ord}(a) = n \implies G \cong \mathbb{Z}/n\mathbb{Z}$
2.  $\text{ord}(a) = \infty \implies G \cong \mathbb{Z}$

**Ex 1.2.5.** Show Prop 1.2.4.

**Def 15.** Let  $G_1, G_2 \leq G$ .  $G$  is the internal direct product of  $G_1, G_2$  if  $G_1 \times G_2 \rightarrow G, (g_1, g_2) \mapsto g_1 g_2$  is an isom.

**Remark 4.** In this case, we find that

- $G = G_1 G_2 = \{g_1 g_2 \mid g_1 \in G_1, g_2 \in G_2\}$ .
- $G_1 \cap G_2 = \{1\}$ . (consider  $a \neq 1 \in G_1 \cap G_2$ , then  $(1, a) \mapsto a, (a, 1) \mapsto a$ , but the function is 1-1, which is a contradiction.)
- If  $a \in G$  with  $a = g_1 g_2 = g'_1 g'_2$ , then  $(g'_1)^{-1} g_1 = (g'_2) g_2^{-1} \in G_1 \cap G_2 = \{1\} \implies \begin{cases} g_1 = g'_1 \\ g_2 = g'_2 \end{cases}$ .
- For  $g_1 \in G_1, g_2 \in G_2, (g_1, g_2) = (g_1, 1)(1, g_2) = (1, g_2)(g_1, 1) \implies g_1 g_2 = g_2 g_1$ .

**Ex 1.2.6.** TFAE

1.  $G$  is the internal direct product of  $G_1, G_2$ .
2.  $\forall a \in G, \exists! g_1 \in G_1, g_2 \in G_2$  s.t.  $a = g_1 g_2$ ;  $\forall g_1 \in G_1, g_2 \in G_2, g_1 g_2 = g_2 g_1$ .
3.  $G_1 \cap G_2 = \{1\}$ ;  $G = G_1 G_2$ ;  $\forall g_1 \in G_1, g_2 \in G_2, g_1 g_2 = g_2 g_1$ .

**Eg 1.2.8.**

1.  $G = \mathbb{Z}/6\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ ,  $G_1 = \{\bar{0}, \bar{3}\}$ ,  $G_2 = \{\bar{0}, \bar{2}, \bar{4}\}$ . We have  $G \cong G_1 \times G_2$ .
2.  $G = S_3$ ,  $G_1 = \langle (1\ 2) \rangle$ ,  $G_2 = \langle (1\ 2\ 3) \rangle$ . We have  $G_1 \times G_2 \not\cong G$  since  $(1\ 2)(1\ 2\ 3) \neq (1\ 2\ 3)(1\ 2)$ .

**Eg 1.2.9.**  $G = S_3$ ,  $G_1 = \langle (1\ 2) \rangle$ ,  $G_2 = \langle (2\ 3) \rangle$ ,  $G_1 G_2 = \{1, (1\ 2), (2\ 3), (1\ 2\ 3)\} \not\subseteq G$  since  $(1\ 3\ 2) = (1\ 2\ 3)^{-1} \notin G_1 G_2$ .

**Prop 1.2.5.** Let  $H, K \leq G$ . Then  $HK \leq G \iff HK = KH$ .

*Proof.*

$$\Rightarrow: \begin{cases} H \leq HK \\ K \leq HK \end{cases} \implies KH \subseteq HK; \forall hk \in HK, \exists h'k' \in HK \text{ s.t. } (hk)(h'k') = 1 \implies hk = (k')^{-1}(h')^{-1} \in KH \implies HK \subseteq KH.$$

$$\Leftarrow: \text{For } h_1k_1, h_2k_2 \in HK, (h_1k_1)(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1h'k' \in HK.$$

□



## 1.3 Week 3

### 1.3.1 Coset and Quotient Group

Let  $f : G_1 \rightarrow G_2$  be a group homo. Define  $\text{Im } f := f(G_1)$ .

Notice that  $\text{Im } f \leq G_2$ .

*Proof.* Let  $z_1 = f(a_1), z_2 = f(a_2)$ , then  $z_1 z_2^{-1} = f(a_1) f(a_2)^{-1} = f(a_1) f(a_2^{-1}) = f(a_1 a_2^{-1}) \in \text{Im } f$ .  $\square$

**Def 16.**  $\ker f := \{x \in G_1 \mid f(x) = 1\} \leq G_1$ .

**Fact 1.3.1.**

1.  $x \in (\ker f)a \iff f(x) = f(a)$ .
2.  $\ker f = \{1\} \iff f$  is 1-1.

**Def 17.** Let  $H \leq G, \forall a \in G, Ha$  is called a **right coset** of  $H$  in  $G$ .

**Fact 1.3.2.**

1. For 2 right cosets  $Ha, Hb$ , either  $Ha = Hb$  or  $Ha \cap Hb = \emptyset$  must hold.
2.  $\{Ha : a \in G\}$  forms a partition of  $G$ .

**Theorem 2** (Lagrange). Let  $|G| < \infty$  and  $H \leq G, |H| \mid |G|$ .

*Proof.*  $\square$

**Remark 5.**  $r$  is called the **index** of  $H$  in  $G$ , denoted by  $[G : H]$ . (The concept of index can be extended to infinite  $G, H$ .)

**Ex 1.3.1.** no subgroup of  $A_4$  has order 6. (converse of Lagrange thm. is false.)

**Coro 1.3.1.** If  $|G| = p$  is a prime in  $\mathbb{Z}$ , then  $G$  is cyclic.

*Proof.*  $\square$

**Coro 1.3.2.** If  $|G| < \infty, a \in G$ , then  $a^{|G|} = 1$ .

*Proof.*  $\square$

**Remark 6.**

1. Let  $H \leq G, a \in G, aH$  is called a **left coset**.
2.  $\{\text{right cosets of } H\} \leftrightarrow \{\text{right cosets of } H\}$  by  $Ha \mapsto a^{-1}H$ .

Ques: How to make  $\{aH : a \in G\}$  to be a group? For  $aH, bH$ , we must have  $(aH)(bH) = abH$ .

In general,  $(aH)(bH) = abH$  is not well-defined.

**Ex 1.3.1.** Let  $H = \langle (1\ 2) \rangle \leq S_3$ .  $a_1 = (1\ 3), a_2 = (1\ 2\ 3), b_1 = (1\ 3\ 2), b_2 = (2\ 3)$ . 出慘點

If we hope  $a_1b_1H = a_2b_2H$ , then we need  $(a_1b_1)^{-1}a_2b_2 \in H$ .

$$b_1^{-1}a_1^{-1}a_2b_2 = b_1^{-1}b_2b_2^{-1}a_1^{-1}a_2b_2$$

Notice that  $b_1^{-1}b_2, a_1^{-1}a_2 \in H$ , so we need  $b_2^{-1}a_1^{-1}a_2b_2 \in H$ .

**Def 18.** Let  $H \leq G$ .  $H$  is said to be **normal subgroup** of  $G$  if  $\forall g \in G, h \in H, g^{-1}hg \in H$  (or  $g^{-1}Hg \subseteq H$ ), denoted by  $H \triangleleft G$ .

**Def 19.** Let  $H \triangleleft G$ . The set  $\{aH \mid a \in G\}$  forms a group under  $(aH)(bH) = abH, a, b \in G$ . We call it the **quotient group** of  $G$  by  $H$ , denoted by  $G/H$ .

(Note: The identity is  $H = hH$  and  $(aH)^{-1} = a^{-1}H$ .)

**Remark 7.** Define  $q : G \rightarrow G/H, a \mapsto aH$ , called the quotient homomorphism.

**Ex 1.3.2.** Let  $H \leq G$ . Then TFAE

- (a)  $H \triangleleft G$ .
- (b)  $\forall x \in G, xHx^{-1} = H$ .
- (c)  $\forall x \in G, xH = Hx$ .
- (d)  $\forall x, y \in G, (xH)(yH) = (xy)H$ .

Ques: How to find a normal subgroup of  $G$ ?

**Prop 1.3.1.**

- 1. If  $G$  is abelian, then  $\forall H \leq G \rightsquigarrow H \triangleleft G$ . (done by (c))
- 2. If  $H \leq G$  with  $[G : H] = 2$ , then  $H \triangleleft G$ .

**Eg 1.3.2.**  $n \leq 3, [S_n : A_n] = 2 \implies A_n \triangleleft S_n$ .

*Proof.* We can write  $G = H \cup Ha = H \cup aH \implies aH = Ha, \forall a \notin H$ . □

**Def 20.** Define the center of  $G$  to be  $Z_G = \{a \in G \mid ax = xa, \forall x \in G\} \leq G$ .

**Prop 1.3.2.**

- 1.  $Z_G \triangleleft G$ . (by (c) and def.)
- 2. If  $G/Z_G$  is cyclic, then  $G$  is abelian.

*Proof.* Let  $G/Z_G = \langle aZ_G \rangle$ , (let  $\bar{a} := aZ_G$ ) for some  $a \in G$ . For  $x_1, x_2 \in G$ , let  $x_1 = a^{k_1}z_1, x_2 = a^{k_2}z_2$ , then  $x_1x_2 = a^{k_1+k_2}z_1z_2 = x_2x_1$ . ( $z_i$  可以各種交換) □

**Def 21.** The commutator of  $G$  is define to be  $[G, G] = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$ .

**Prop 1.3.3.**  $[G, G] \triangleleft G ; [G, G] = 1 \iff G$  is abelian.

*Proof.*  $\forall x \in G, a \in [G, G], xax^{-1} = xax^{-1}a^{-1}a$  and  $xax^{-1}a^{-1}, a \in [G, G]$ . □

**Ex 1.3.3.**

- 1. If  $H \leq S_n$  and  $\exists \sigma \in H$  is odd, then  $[H : H \cap A_n] = 2$ .

2. For  $n \geq 3$ ,  $[S_n, S_n] = A_n$ .

**Ex 1.3.4.** Let  $H \leq G$ . Then  $H \triangleleft G$  and  $G/H$  is abelian  $\iff [G, G] \leq H$ . (hint:  $G/[G, G]$  is "max" among all abelian quotient groups)

### 1.3.2 Isomorphism theorems & Factor theorem

**Theorem 3** (1st isomorphism theorem). Let  $f : G_1 \rightarrow G_2$  be a group homo. Then  $G_1/\ker f \cong \text{Im } f$ .

*Proof.* Define  $\varphi : a \ker f \mapsto f(a)$ .

- well-defined:  $a \ker f = b \ker f \implies a^{-1}b \in \ker f \implies f(a^{-1}b) = 1 \implies f(a)^{-1}f(b) = 1 \implies f(a) = f(b)$ .
- group homo:  $\varphi((a \ker f)(b \ker f)) = \varphi(ab \ker f) = f(ab) = f(a)f(b) = \varphi(a \ker f)\varphi(b \ker f)$ .
- onto: by def. of  $\text{Im } f$ .
- 1-1:  $f(a) = f(b) \implies a \ker f = b \ker f$  (easy).

□

**Theorem 4** (Factor theorem). Let  $f : G_1 \rightarrow G_2$  be a group homo. and  $H \triangleleft G_1, H \leq \ker f$ . Then  $\exists$  a group homo.  $\varphi : G/H \rightarrow G_2$  s.t.

$$\begin{array}{ccc} G_1 & \xrightarrow{q} & G/H \\ & \searrow f & \downarrow \varphi \\ & & G_2 \end{array}$$

**Eg 1.3.3.** Let  $G = \langle a \rangle$  with  $\text{ord}(a) = n$ . Then  $G \cong \mathbb{Z}/n\mathbb{Z}$ . (1st isom. thm.)

**Eg 1.3.4.**  $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}, 4\mathbb{Z} \leq 2\mathbb{Z}$ , so by factor thm.,  $\mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ .

**Eg 1.3.5.**  $\det : \text{GL}(n, \mathbb{F}) \rightarrow \mathbb{F}^\times \implies \text{GL}(n, \mathbb{F})/\text{SL}(n, \mathbb{F}) \cong \mathbb{F}^\times$

**Eg 1.3.6.**  $\text{sgn} : S_n \rightarrow \{\pm 1\} \implies S_n/A_n \cong \{\pm 1\}$

**Theorem 5** (2nd isomorphism theorem). Let  $H \leq G, K \triangleleft G$ . Then  $HK/K \cong H/H \cap K$ .

*Proof.* First,  $\begin{cases} H \leq G \\ K \triangleleft G \end{cases} \implies HK = KH \implies HK \leq G; K \triangleleft G \implies K \triangleleft HK$ .

Define  $\varphi : H \rightarrow HK/K, h \mapsto hK$ . which is a group homo.

- onto:  $\forall (hk)K, hK = hK, \text{ so } \varphi(h) = hK = hkK$ .
- Find  $\ker \varphi$ :  $a \in \ker \varphi \iff \begin{cases} a \in H \\ aK = K \end{cases} \iff a \in H \cap K, \text{ so } \ker \varphi = H \cap K$ .

Then by 1st isom. thm.

□

**Eg 1.3.7.**  $G = \text{GL}(2, \mathbb{C}), H = \text{SL}(2, \mathbb{C}), K = \mathbb{C}^\times I_2 = Z_G \triangleleft G$ .

By 2nd isom. thm.,  $G/K \cong H/\{\pm I_2\}$ . ( $G = HK, \{\pm I_2\} = H \cap K$ )

projective linear group:  $\text{PGL}(2, \mathbb{C}) = G/K$ .

projective special linear group:  $\text{PSL}(2, \mathbb{C}) = H/H \cap K$ .

齊次座標...OTL

**Ex 1.3.5.**

1. Let  $H_1 \triangleleft G_1, H_2 \triangleleft G_2$ . Then  $(H_1 \times H_2) \triangleleft (G_1 \times G_2)$  and  $G_1 \times G_2 / H_1 \times H_2 \cong G_1 / H_1 \times G_2 / H_2$ .
2. Let  $H \triangleleft G, K \triangleleft G$  s.t.  $G = HK$ . Then  $G/H \cap K \cong G/H \times G/K$ .

**Ex 1.3.6.** Let  $H \triangleleft G$  with  $[G : H] = p$ , which is a prime in  $\mathbb{Z}$ . Then  $\forall K \leq G$ , either (1)  $K \leq H$  or (2)  $G = HK$  and  $[K : K \cap H] = p$ .

**Theorem 6** (3rd isomorphism theorem). Let  $K \triangleleft G$ .

1. There is a 1-1 correspondence between  $\{H \leq G \mid K \leq H\}$  and  $\{\text{subgroups of } G/K\}$ . ( $H \triangleleft G$  ... normal)

*Proof.* Define  $\varphi : H \mapsto H/K$ . ( $H/K \leq G/K$ )

- 1-1: Assume  $H_1/K = H_2/K$ . For  $a \in H_1$ ,  $aK \in H_1/K = H_2/K$ . so  $\exists b \in H_2$  s.t.  $aK = bK \implies b^{-1}a \in K \leq H_2 \implies a \in bH_2 = H_2$ . So  $H_1 \leq H_2$ . By symmetry,  $H_2 \leq H_1$ , and thus  $H_1 = H_2$ .
- onto: Given a subgroup  $Q$  of  $G/K$ , consider  $H = q^{-1}(Q)$  where  $q : G \rightarrow G/K$ .
  - $H \leq G$ :  $\forall a, b \in H, q(a), q(b) \in Q \implies q(a)q(b)^{-1} \in Q \implies q(ab^{-1}) \in Q \implies ab^{-1} \in H \implies H \leq G$ .
  - $K \leq H$ :  $\forall a \in K, q(a) = aK = K \in Q \implies a \in H \implies K \leq H$ .
  - $Q = H/K$ :  $\forall aK \in Q, aK = q(a) \implies a \in H \implies aK \in H/K \implies Q \subseteq H/K$ .  
And  $\forall aK \in H/K (a \in H), q(a) \in Q \implies H/K \subseteq Q$ . So  $Q = H/K$ .
- $H \triangleleft G, K \leq H \iff \forall g \in G, gHg^{-1} = H, K \leq H \iff \forall \bar{g} \in G/K, \bar{g}(H/K)\bar{g}^{-1} = H/K \iff H/K \triangleleft G/K$ .  $\square$

2. If  $H \triangleleft G$  with  $K \leq H$ , then  $(G/K)/(H/K) \cong G/H$ .

*Proof.* Define  $\varphi : G \rightarrow (G/K)/(H/K)$  with  $\varphi : a \mapsto aK(H/K)$ .

- onto: ... easy.
- Find  $\ker \varphi$ :  $a \in \ker \varphi \iff aK(H/K) = H/K \iff aK \in H/K \iff a \in H$ .

By 1st isom. thm.,  $(G/K)/(H/K) \cong G/H$ .  $\square$

**Eg 1.3.8.**  $m\mathbb{Z} + n\mathbb{Z}/m\mathbb{Z} \cong n\mathbb{Z}/m\mathbb{Z} \cap n\mathbb{Z}$ . ( $m\mathbb{Z} + n\mathbb{Z} = \gcd(m, n)\mathbb{Z}, m\mathbb{Z} \cap n\mathbb{Z} = \text{lcm}(m, n)\mathbb{Z}$ )

Ques:  $G/K \cong G'/K'$  and  $K \cong K' \not\Rightarrow G \cong G'$ .

**Eg 1.3.9.**  $Q_8$  and  $D_4$  交給陳力

Extension problem: given two groups  $A, B$ , how to find  $G$  and  $K \triangleleft G$ , s.t.  $K \cong A, G/K \cong B$ ?  
( $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ , short exact sequence)

(e.g.  $G = A \times B, K = A \times \{1\}$ )

## 1.4 Week 4

### 1.4.1 Universal property and direct sum & product

In general, let  $f_1 : G_1 \rightarrow G, f_2 : G_2 \rightarrow G$  are group homo.  $f_1 \times f_2 : G_1 \times G_2 \rightarrow G, (a, b) \mapsto f_1(a)f_2(b)$ . But we have  $(a, b) = (a, 1)(1, b) = (1, b)(a, 1)$ , so  $f_1(a)f_2(b) = f_2(b)f_1(a) \implies$  need  $G$  to be abelian.

So we intend to define the direct sum in the category of abelian group.

Notation: For abelian groups, we use “+” to denote the group operation and “0” to denote the identity.

**Def 22.** Given a non-empty family of abelian groups  $\{G_s \mid s \in \Lambda\}$ , a (external) direct sum of  $\{G_s \mid s \in \Lambda\}$  is an abelian group  $\bigoplus_{s \in \Lambda} G_s$  with the embedding mappings  $i_{s_0} : G_{s_0} \rightarrow \bigoplus_{s \in \Lambda} G_s, \forall s_0 \in \Lambda$  satisfying the universal property:

for any abelian group  $H$  and group homo.  $\varphi_s : G_s \rightarrow H, \forall s \in \Lambda, \exists!$  group homo.  $\varphi : \bigoplus_{s \in \Lambda} G_s \rightarrow H$  s.t. 又一個 $\complement$ 圖

**Theorem 7.**  $\bigoplus_{s \in \Lambda} G_s$  exists and is unique up to isomorphisms.

*Proof.* Existence:  $\bigoplus_{s \in \Lambda} G_s = \{(g_s)_{s \in \Lambda} \mid g_s \in G_s, \text{ almost all of the } g_s \text{' are } 0\}$  and

$$i_{s_0} : G_{s_0} \rightarrow \bigoplus_{s \in \Lambda} G_s, a_{s_0} \mapsto (g_s)_{s \in \Lambda} \text{ with } g_{s_0} = a_{s_0}, g_s = 0, \forall s \neq s_0.$$

group operation:  $(g_s)_{s \in \Lambda} + (g'_s)_{s \in \Lambda} := (g_s + g'_s)_{s \in \Lambda} \in \bigoplus_{s \in \Lambda} G_s$ . 這邊也一個 $\complement$ 圖

Uniqueness: Assume  $\exists$  another  $G$  satisfies the universal property, 一個大 $\complement$ 圖  $(G, \bigoplus_{s \in \Lambda} G_s)$  互相有唯一一個映射可以 keep  $i_{s_0}, \varphi \circ \psi = \text{id}_G, \psi \circ \varphi = \text{id}_{\bigoplus_{s \in \Lambda} G_s}$   $\square$

**Def 23.** Given a non-empty family of groups  $\{G_s \mid s \in \Lambda\}$ , a direct product of  $\{G_s \mid s \in \Lambda\}$  is a group  $\prod_{s \in \Lambda} G_s$  with projections  $p_{s_0} : \prod_{s \in \Lambda} G_s \rightarrow G_{s_0}, \forall s_0 \in \Lambda$  satisfying the following universal property:

for any group  $H$  with group homo.  $\varphi_s : H \rightarrow G_s, \forall s \in \Lambda, \exists! \varphi : H \rightarrow \prod_{s \in \Lambda} G_s$  s.t. 又一個 $\complement$ 圖

**Theorem 8.**  $\prod_{s \in \Lambda} G_s$  exists and is unique up to isomorphisms.

*Proof.* Existence:  $\prod_{s \in \Lambda} G_s = \{(g_s)_{s \in \Lambda} \mid g_s \in G_s\}$  and

$$p_{s_0} : \prod_{s \in \Lambda} G_s \rightarrow G_{s_0}, (g_s)_{s \in \Lambda} \mapsto g_{s_0}, \forall s_0 \in \Lambda$$

- group operation:  $(g_s)_{s \in \Lambda} \cdot (g'_s)_{s \in \Lambda} := (g_s g'_s)_{s \in \Lambda} \in \prod_{s \in \Lambda} G_s$ .
- Define  $\varphi$ : 這邊也一個 $\complement$ 圖 which is uniquely defined.

Uniqueness: Assume  $\exists$  another  $G$  satisfies the universal property, 一個大 $\complement$ 圖  $(G, \prod_{s \in \Lambda} G_s)$  互相有唯一一個映射可以 keep  $i_{s_0}, \varphi \circ \psi = \text{id}_G, \psi \circ \varphi = \text{id}_{\prod_{s \in \Lambda} G_s}$   $\square$

**Ex 1.4.1.** Google the definition of the **direct limit** and show the existence and uniqueness.

**Ex 1.4.2.** Google the definition of the **inverse limit** and show the existence and uniqueness.

Motivation:  $\zeta_m$  is called an  $m$ -th root of unity if  $\zeta_m^m = 1$ .

$$\varinjlim_n \mathbb{Z}/2^n \mathbb{Z} \cong \{2^n\text{-th roots of unity} : n \in \mathbb{N}\}$$

$$\varprojlim_n \mathbb{Z}/2^n\mathbb{Z} = \left( \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2^n\mathbb{Z} / \langle i_k(a) - i_j(f_{kj}(a)) \mid k \leq j, a \in \mathbb{Z}/2^k\mathbb{Z} \rangle \right)$$

where  $f_{kj} : \mathbb{Z}/2^k\mathbb{Z} \rightarrow \mathbb{Z}/2^j\mathbb{Z}$ .

Inverse limit:

$$\varprojlim \mathbb{Z}/2^n\mathbb{Z} = \left\{ (n_1, n_2, \dots) \in \prod_n \mathbb{Z}/2^n\mathbb{Z} \mid \forall i < j, n_i \equiv n_j \pmod{2^{i+1}} \right\}$$

#### 1.4.2 Rings and fields

**Def 24.** A **ring** is a non-empty set  $R$  with two operations  $R \times R \rightarrow R$

$$(a, b) \mapsto a + b \quad \text{and} \quad (a, b) \mapsto ab$$

satisfying

1.  $(R, +, 0)$  is an abelian group.
2.  $(R, \cdot)$  is a semigroup. (if it is a monoid, then it is called “a ring with 1.”)
3. (Distributive laws)  $\forall a, b, c \in R, \begin{cases} a(b + c) = ab + ac \\ (b + c)a = ba + ca \end{cases}$

**Eg 1.4.1.**  $\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/n\mathbb{Z}, M_{n \times n}(\mathbb{F})$

**Eg 1.4.2.** Let  $G$  be an abelian group. Define (endomorphism, automorphism)

$$\text{End}(G) := \{ \text{group homo. } G \rightarrow G \} \quad \text{Aut}(G) := \{ \text{group isom. } G \rightarrow G \}$$

A natural ring structure on  $\text{End}(G)$  is:

$$\forall a \in G, \begin{cases} (f + g)(a) := f(a) + g(a) \\ (f \cdot g)(a) := f(g(a)) \end{cases}$$

**Eg 1.4.3.**  $\mathbb{Z}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Z} \} \subset \mathbb{R}$ .

**Def 25.** Let  $R$  be a ring with 1.

- (a)  $\forall a \in R, a \neq 0$ ,  $a$  is called a unit if  $\exists a^{-1} \in R$ .
- (b)  $(R^\times = \{ \text{units in } R \}, \cdot, 1)$  forms a group.
- (c)  $R$  is called a division ring if  $R \setminus \{0\} = R^\times$ .
- (d)  $R$  is said to be commutative if  $ab = ba, \forall a, b \in R$ .
- (e)  $R$  is a field if  $R$  is a commutative division ring.
- (f)  $a \neq 0$  is called a left zero divisor if  $\exists b \in R, b \neq 0$  s.t.  $ab = 0$ .
- (g)  $a$  is called a zero divisor if  $a$  is either a left or right zero divisor.
- (h)  $R$  is called an integral domain if  $R$  is a commutative ring without zero divisors.

Fact:

1. fields  $\implies$  integral domains.
2. finite + integral domain  $\implies$  fields.

*Proof.* Let  $R = \{0, a_1, \dots, a_n\}$ , for  $a \in R, a \neq 0, aa_i = aa_j \implies a(a_i - a_j) = 0 \implies i = j$ .  
So  $\{0, aa_1, \dots, aa_n\} = R \implies \exists a_i$  s.t.  $aa_i = 1$ .  $\square$

**Prop 1.4.1.** TFAE

1.  $\mathbb{Z}/n\mathbb{Z}$  is an integral domain.
2.  $\mathbb{Z}/n\mathbb{Z}$  is a field.
3.  $n = p$  is a prime.

easy to prove.

**Def 26.**

- $f : R_1 \rightarrow R_2$  is called a ring homomorphism if  $\forall a, b \in R, \begin{cases} f(a+b) = f(a) + f(b) \\ f(ab) = f(a)f(b) \end{cases}$ .
- $\text{Im } f$  is a subring of  $R_2$ .
- $\ker f = \{x \in R_1 \mid f(x) = 0\}$  is an additive group of  $R_1$  and  $\forall r \in R_1, x \in \ker f, f(rx) = f(r)f(x) = f(r)0 = 0 \implies rx \in \ker f, xr \in \ker f$ .
- $R_1/\ker f$  is an additive group and  $R_1/\ker f \cong \text{Im } f$  (additive isomorphism).

**Def 27.** Let  $I$  be an additive subgroup of  $R$ .  $I$  is called an ideal if  $\forall r \in R, x \in I, rx \in I, xr \in I$ .  
( $R/I, +, \cdot$ ) forms a quotient ring under

$$\forall r_1, r_2 \in R, (r_1 + I)(r_2 + I) = r_1 r_2 + I$$

well-defined: easy to show.

**Ex 1.4.3.** State and show the isomorphism theorems and the factor theorem.

**Prop 1.4.2.** If  $R$  is a ring with 1, then  $\exists!$  ring homo.  $\varphi : \mathbb{Z} \rightarrow R$  s.t.  $\varphi(1) = 1$ .

*Proof.* Let  $\varphi : \mathbb{Z} \rightarrow R$  is a ring homo. s.t.  $\varphi(1) = 1$ . Then  $\forall n \in \mathbb{Z}, \varphi(n) = \varphi(1) + \dots + \varphi(1) = n1$ .  
Now  $\forall n, m \in \mathbb{Z}, \varphi(n)\varphi(m) = (n1)(m1) = n(m1) = (nm)1$  by the distributive law. So  $\varphi$  is well-defined and unique.  $\square$

**Def 28.** In Prop 1.4.2,  $\ker \varphi = m\mathbb{Z}$  for some  $m > 0$ . We call  $m$  the characteristic of  $R$ , denoted by  $\text{char } R = m$ .

**Prop 1.4.3.**

1. If  $R$  is an integral domain, then  $\text{char } R = 0$  or  $p$ , where  $p$  is a prime. (try to prove this)
2. In the case of  $\text{char } R = p, \forall a, b \in R, (a+b)^p = a^p + b^p$ .

*Proof.*

$$(a+b)^p = a^p + \binom{p}{1}a^{p-1}b + \dots + b^p = a^p + b^p$$

because  $p \mid \binom{p}{i} \implies \binom{p}{i}a^{p-i}b^i = 0$ .  $\square$

**Ex 1.4.4.** Let  $F$  be a field. Show that

1. if  $\text{char } F = 0$ , then  $\mathbb{Q} \hookrightarrow$  subfield of  $F$ .
2. if  $\text{char } F = p$ , then  $\mathbb{Z}/p\mathbb{Z} \hookrightarrow$  subfield of  $F$ .

**Theorem 9.** If  $F$  is a finite field, then  $|F| = p^n$  for some  $n \in \mathbb{N}$  and  $p$  is a prime.

*Proof.* By Ex. 1.4.4,  $\text{char } F = p$ ,  $p$  is a prime and  $\mathbb{Z}/p\mathbb{Z} \hookrightarrow F$ .

We have  $\mathbb{Z}/p\mathbb{Z} \times F \rightarrow F, (r, v) \mapsto rv$ .  $F$  can be regarded as a vector space over  $\mathbb{Z}/p\mathbb{Z}$ .

Let  $\dim_{\mathbb{Z}/p\mathbb{Z}} F = n$ , then  $F \cong (\mathbb{Z}/p\mathbb{Z})^n \implies |F| = p^n$ . □

**Theorem 10.** Let  $F$  be a field. Then any finite subgroup  $G$  of  $(F^\times, \cdot, 1)$  is cyclic.

*Proof.* Let  $|G| = n$ . Define  $h$  to be the max order of an element in  $G$ , say  $a^h = 1$ .

If  $h = n$ , then  $|\langle a \rangle| = h = n = |G|$  and  $\langle a \rangle \subseteq G$ , so  $G = \langle a \rangle$ .

Otherwise,  $h < n$ . We know that  $x^h - 1$  has at most  $h$  roots. So  $\exists b \in G$  is not a root of  $x^h - 1$ . Let  $\text{ord}(b) = h'$ , so  $h' \mid n$  and  $h' \nmid h$ . So  $\exists$  a prime  $p$  s.t.  $p^r \mid h'$  but  $p^r \nmid h$ .

Write  $h = mp^s$ ,  $s < r$  and  $\gcd(m, p) = 1 \implies \text{ord}(a^{p^s}) = m$ .

Write  $h' = qp^r \implies \text{ord}(b^q) = p^r$ .

Since  $\gcd(m, p^r) = 1$ ,  $\text{ord}(a^{p^s} b^q) = mp^r > mp^s = h$ , which is a contradiction. □

**Ex 1.4.5.**

1. Let  $a, b \in G$  with  $ab = ba$  and  $\text{ord}(a) = m, \text{ord}(b) = n$ . If  $\gcd(m, n) = 1$ , then  $\text{ord}(ab) = mn$ .  
In general, is the order of  $ab$  equal to  $\text{lcm}(m, n)$ ?
2. Let  $G$  be a finite group and  $H, K \leq G$ . Then  $|HK| = \frac{|H||K|}{|H \cap K|}$ .



## 1.5 Week 5

### 1.5.1 Group actions I

**Def 29.** A group  $G$  is said to act on a nonempty set  $X$  if  $\exists$  a map  $G \times X \rightarrow X$  with  $(g, x) \mapsto gx$  s.t.

1.  $1x = x$
2.  $(g_1g_2)x = g_1(g_2x) \quad \forall g_1, g_2 \in G$

**Prop 1.5.1.**  $\{\text{actions of } G\} \leftrightarrow \{\text{group homo. } G \rightarrow S_X\}$

*Proof.* Given an action  $(g, x) \mapsto gx$ , consider  $\varphi : G \rightarrow S_X$  s.t.  $\varphi : g \mapsto (\tau_g : x \mapsto gx)$ .

- 1-1:  $gx = gy \implies g^{-1}(gx) = y \implies x = y$ .
- onto:  $\forall y \in X$ , let  $x = g^{-1}y$ , then  $y = gx$ .
- group homo.:  $\varphi(gg') = (\tau_{gg'} : x \mapsto gg'x) = \tau_g \circ \tau_{g'} = \varphi(g)\varphi(g')$ .

Conversely, given a group homo.  $\varphi : G \rightarrow S_X$ , consider  $(g, x) \mapsto \varphi(g)(x)$ .

- $1x = \varphi(1)(x) = \text{Id}(x) = x$ .
- $g_1g_2x = \varphi(g_1g_2)(x) = \varphi(g_1) \circ \varphi(g_2)(x) = g_1(g_2x)$ . □

**Def 30.** A representation of  $G$  on a vector space  $V$  is a group action of  $G$  on  $V$  linearly. i.e.  $\exists$  group homo.  $\varphi : G \rightarrow \text{GL}(V)$ .

**Eg 1.5.1.**

$$\mathbb{Z}/m\mathbb{Z} \rightarrow \text{SO}(2), \quad \bar{k} \mapsto \begin{pmatrix} \cos \frac{2k\pi}{m} & -\sin \frac{2k\pi}{m} \\ \sin \frac{2k\pi}{m} & \cos \frac{2k\pi}{m} \end{pmatrix}$$

**Eg 1.5.2.**

$$S_n \rightarrow \text{GL}(n, \mathbb{R}), \quad \sigma \mapsto (\tau_\sigma : e_i \mapsto e_{\sigma(i)})$$

**Remark 8.**

1. An action  $G \times X \rightarrow X$  is said to be faithful if the corresponding group homo.  $\varphi : G \hookrightarrow S_X$ , denoted by  $G \curvearrowright X$ .
2. In general,  $\ker \varphi = \{g \in G \mid gx = x \quad \forall x \in X\} = \bigcap_{x \in X} \{g \mid gx = x\}$ .  
Define  $G_x = \{g \mid gx = x\} \leq G$  is the isotropy subgroup of  $G$  at  $x$ . (the stabilizer of  $G$  at  $x$ )
3.  $\varphi : G \rightarrow S_X \implies G/\ker \varphi \hookrightarrow S_X$ . So  $G/\ker \varphi \times X \rightarrow X$  is faithful.
4. Let  $\mathcal{C}(X) = \{f : X \rightarrow \mathbb{C}\}$ . If  $G \curvearrowright X$ , then  $G \curvearrowright \mathcal{C}(X)$  by  $G \times \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  with  $(g, f) \mapsto gf(x) = f(g^{-1}x)$ .  
The reason:  $(g_1g_2)f(x) = f((g_1g_2)^{-1}x) = f(g_2^{-1}g_1^{-1}x) = g_1(g_2f)(x)$ .

**Def 31.** Let  $G \curvearrowright X$  and  $x \in X$ .

- The **orbit** of  $x$  is defined to be  $Gx = \{gx \mid g \in G\}$ .
- $G \curvearrowright X$  is said to be transitive if  $\exists$  only one orbit. i.e.  $\forall x, y \in X, \exists g \in G$  s.t.  $y = gx$ .

The set of orbits forms a partition:  $x \sim y \iff \exists g \in G \text{ s.t. } y = gx$ .

**Prop 1.5.2.** Let  $G \curvearrowright X$  and  $x \in X$ . Then  $|Gx| = [G : G_x]$ .

In particular,  $|G| < \infty \implies |G| = |Gx||G_x| \quad \forall x \in X$ .

*Proof.* Define  $\psi : Gx \rightarrow \{\text{left coset of } G_x\}$  as  $\psi : gx \mapsto gG_x$ .

- well-defined and 1-1:  $g_1x = g_2x \iff g_2^{-1}g_1x = x \iff g_2^{-1}g_1 \in G_x \iff g_2^{-1}g_1G_x = G_x \iff g_1G_x = g_2G_x$ .
- onto:  $\forall g \in G, \psi(gx) = gG_x$ . □

### 1.5.2 Action by left multiplication

- The action  $G \times G \rightarrow G, (g, x) \mapsto gx$  is associated with  $\varphi : G \hookrightarrow S_G$ . It is faithful (Cayley theorem) and transitive.
- Let  $H \leq G$  and  $X := \{\text{left coset of } H\}$ . The group action  $(g, xH) \mapsto gxH \rightsquigarrow \varphi : G \rightarrow S_X$ .

$$\ker \varphi = \bigcap_{x \in G} \underbrace{xHx^{-1}}_{\text{a conjugate of } H} \leq H$$

which is the largest normal subgroup in  $G$  contained in  $H$ .

*Proof.* If  $\begin{cases} N \triangleleft G \\ N \leq H \end{cases}, \forall x \in G, xNx^{-1} \leq xHx^{-1} \implies N = N(xx^{-1}) = xNx^{-1} \leq xHx^{-1}$ . □

**Prop 1.5.3.** Let  $H \leq G$  with  $[G : H] = p$  being the smallest prime dividing  $|G|$ . Then  $H \triangleleft G$ .

*Proof.* Let  $X = \{a_1H, \dots, a_pH\}$  (all left cosets of  $H$ ) and  $\varphi : G \rightarrow S_p$  be the associated group homo. for the group action  $(g, a_iH) \mapsto ga_iH$ .

By the 1st isom. thm.,  $G/\ker \varphi \hookrightarrow S_p$ .

By Lagrange thm.  $|G/\ker \varphi| \mid |S_p| = p!$  and  $|G/\ker \varphi| \mid |G| \implies |G/\ker \varphi| \mid p$ .

So  $|G/\ker \varphi| = 1$  or  $p$ .

If  $|G/\ker \varphi| = 1 \implies G = \ker \varphi \leq H \subsetneq G$ , which is a contradiction.

So  $|G/\ker \varphi| = p \implies [G : \ker \varphi] = p \implies [G : H][H : \ker \varphi] = p \implies [H : \ker \varphi] = 1 \implies H = \ker \varphi \triangleleft G$ . □

### 1.5.3 Action by conjugation

- The action  $G \times G \rightarrow G, (g, x) \mapsto gxg^{-1}$  is associated with the group homo.  $\varphi : G \rightarrow S_G, g \mapsto (\tau_g : x \mapsto gxg^{-1})$ .

$$\text{Inn}(G) := \{\tau_g \mid g \in G\}$$

**Fact 1.5.1.**  $\tau_g$  is an automorphism. (isom.  $G \rightarrow G$ )

So  $\varphi : G \twoheadrightarrow \text{Inn}(G) \leq \text{Aut}(G) \leq S_G$ .

$\ker \varphi = \{g \in G \mid gxg^{-1} = x \quad \forall x \in G\} = Z_G$ .

By the 1st isom. thm.,  $G/\ker \varphi \cong \text{Inn}(G)$ .

- The conjugacy class:  $Gx = \{gxg^{-1} \mid g \in G\} = \text{Cl}(x)$ .
- The centralizer of  $x$  in  $G$ :  $G_x = \{g \in G \mid gxg^{-1} = x\} = Z_G(x)$ .

$$|\text{Cl}(x)| = [G : Z_G(x)], \text{ if } |G| < \infty, |G| = |\text{Cl}(x)||Z_G(x)|$$

- For  $H \triangleleft G$ , define  $G \times H \rightarrow H$   $(g, h) \mapsto ghg^{-1}$  with the group homo.  $\varphi : G \rightarrow \text{Aut}(H)$ .

$$\ker \varphi = \{g \in G \mid gxg^{-1} = x \quad \forall x \in H\} = Z_G(H) \implies G/Z_G(H) \leq \text{Aut}(H)$$

- The normalizer of  $H$  in  $G$ :  $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$

**Theorem 11** (Normalizer-Centralizer theorem). If  $H \leq G$  then  $N_G(H)/Z_G(H) \hookrightarrow \text{Aut}(H)$ .

*Proof.* Define  $\varphi = g :: N_G(H) \mapsto (h \mapsto ghg^{-1}) :: \text{Aut}(H)$ . Then  $\ker \varphi = Z_G(H)$ , so  $N_G(H)/Z_G(H) \cong \text{Im } \varphi \leq \text{Aut}(H)$ .  $\square$

## 1.6 Week 6

### 1.6.1 Group actions II

**Def 32.** Let  $G \curvearrowright X$  and  $|X| < \infty$ . Write  $\text{Fix } G := \{x \in X \mid gx = x \ \forall g \in G\}$ .

- $x \in \text{Fix } G, Gx = \{x\}$ .
- $x \notin \text{Fix } G, |Gx| = [G : G_x]$ .

Let  $\{G_{x_1}, \dots, G_{x_n}\}$  be the set of distinct orbits. After rearrangement, assume  $x_1, \dots, x_r \in \text{Fix } G, x_{r+1}, \dots, x_n \notin \text{Fix } G$ . Then

$$|X| = |\text{Fix } G| + \sum_{i=r+1}^n [G : G_{x_i}]$$

**Theorem 12** (class equation). Let  $|G| < \infty$ . Then either  $G = Z_G$  or  $\exists a_1, \dots, a_m \in G \setminus Z_G$  s.t.

$$|G| = |Z_G| + \sum_{i=1}^n [G : G_{a_i}]$$

*Proof.* Consider the action  $(g, x) \mapsto gxg^{-1}$ , then

$$\text{Fix } G = \{x \in G \mid gxg^{-1} = x \ \forall g \in G\} = Z_G$$

It follows from the above argument. □

**Def 33.**  $G$  is called a  $p$ -group if  $|G| = p^n$ , where  $p$  is a prime,  $n \in \mathbb{N}$ .

**Prop 1.6.1.** If  $G$  is a  $p$ -group, then  $Z_G \neq \{1\}$ .

*Proof.* Let  $|G| = p^n$ . If  $G = Z_G$ , then done. Otherwise, by the class equation (use action by conjugation),  $|G| = |Z_G| + \sum_{i=1}^n [G : G_{a_i}]$ ,  $a_i \notin Z_G$ .

$$G_{a_i} = Z_G(a_i), \text{ so } a_i \notin Z_G \implies Z_G(a_i) \subsetneq G \implies p \mid [G : Z_G(a_i)] = \frac{|G|}{|Z_G(a_i)|}.$$

$$\text{So } |Z_G| = |G| - \sum_{i=1}^n [G : Z_G(a_i)] \implies p \mid |Z_G| \implies Z_G \neq \{1\}. \quad \square$$

**Prop 1.6.2.** If  $|G| = p^2$ , then  $G$  is abelian. ( $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  and  $\mathbb{Z}/p^2\mathbb{Z}$ )

*Proof.* Assume that  $G$  is not abelian. By prop 1.6.1,  $|Z_G| = p \implies |G/Z_G| = p \implies G/Z_G$  is cyclic  $\implies G$  is abelian. (contradiction) □

**Prop 1.6.3.** If  $|G| = p^3$  and  $G$  is not abelian, then  $|Z_G| = p$ .

(Abelian:  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p^3\mathbb{Z}$ )

**Prop 1.6.4.** Let  $|G| = p^n$ . Then  $\forall 0 \leq k \leq n, \exists G_k \triangleleft G$  s.t.  $|G_k| = p^k$  and  $G_i \triangleleft G_{i+1}$ .

In general, for a finite group  $G$ ,  $\exists \{1\} = G_r \triangleleft G_{r-1} \triangleleft \dots \triangleleft G_1 \triangleleft G_0 = G$  s.t.  $G_i/G_{i+1}$  is cyclic.

we call  $G$  a solvable group.

*Proof.* By induction on  $n$ ,  $n = 1$  is trivial. For  $n > 1$ , assume that the statement holds for  $n - 1$ . By prop 1.6.1,  $Z_G \neq \{1\}$ .  $\exists a \in Z_G, a \neq 1$ . Let  $\text{ord}(a) = p^l$ , then  $\text{ord}(a^{p^{l-1}}) = p$ .  $\implies$  in any case,  $\exists a \in Z_G$  with  $\text{ord}(a) = p$ .

Now  $|G/\langle a \rangle| = p^{n-1}$ , so by induction hypothesis,  $\forall 0 \leq k \leq n-1, \exists \overline{G_k} \triangleleft G/\langle a \rangle$  s.t.  $|\overline{G_k}| = p^k, \overline{G_i} \leq \overline{G_{i+1}}$ .

By 3rd isom. thm.,  $\exists G_{k+1} \triangleleft G$  s.t.  $\overline{G_k} = G_{k+1}/\langle a \rangle, G_j \leq G_{j+1}$  and  $|G_{k+1}| = p^{k+1}$ .

□

**Prop 1.6.5.** Let a  $p$ -group  $G \curvearrowright X$  with  $|X| < \infty$ . Then  $|X| \equiv |\text{Fix } G| \pmod{p}$ .

**Theorem 13** (Cauchy theorem). Let  $p \mid |G|$ . Then  $\exists a \in G$  s.t.  $\text{ord}(a) = p$ . Consider

$$X = \{ (a_1, \dots, a_p) \mid a_i \in G, a_1 a_2 \dots a_p = 1 \}$$

and the action  $\mathbb{Z}/p\mathbb{Z} \times X \rightarrow X$ :

$$(\overline{k}, (a_1, \dots, a_p)) \mapsto (a_{k+1}, \dots, a_p, a_1, \dots, a_k)$$

(This is well-defined since  $ab = 1 \implies ba = 1$  in a group.) We find that  $(a_1, \dots, a_p) \in \text{Fix } \mathbb{Z}/p\mathbb{Z} \iff a_1 = a_2 \dots a_p$ . By prop 1.6.5,  $|\text{Fix } \mathbb{Z}/p\mathbb{Z}| \equiv |X| \pmod{p}$ . And  $|X| = |G|^{p-1} \equiv 0 \pmod{p}$ . Since  $(1, \dots, 1) \in \text{Fix } \mathbb{Z}/p\mathbb{Z}, |\mathbb{Z}/p\mathbb{Z}| \neq 0 \implies |\mathbb{Z}/p\mathbb{Z}| \geq p$ .

So  $\exists (a, \dots, a) \in \text{Fix } \mathbb{Z}/p\mathbb{Z} \implies a^p = 1$ .

Application: Let  $|G| = p^3$  and  $G$  be non-abelian ( $p$  is odd). By prop 1.6.3,  $|G/Z_G| = p^2$ . Since  $G$  is non-abelian, we have  $G/Z_G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ . That is,  $\forall a \in G, a^p \in Z_G$ .

So,

$$\exists \varphi : G \rightarrow Z_G \cong C_p \text{ with } \varphi : a \mapsto a^p$$

Since  $G/Z_G$  is abelian,  $[G, G] \leq Z_G$ . And

$$\begin{cases} |[G, G]| \mid |Z_G| = p \\ G \text{ is non-abelian} \end{cases} \implies [G, G] = Z_G$$

**Def 34.**  $[x, y] = x^{-1}y^{-1}xy \in [G, G], [x, y]^p = 1$ .

So  $a^p b^p = a^p b^p [b, a]^p \dots$  換換換總共需要  $p(p-1)/2$

$$a^p b^p = (ab)^p [b, a]^{\frac{p(p-1)}{2}} = (ab)^p$$

So  $\varphi$  is a group homo.

Now if  $\ker \varphi = G$  ( $\forall a \in G, a^p = 1$ ), i.e.  $\varphi$  is trivial, then  $\varphi$  is useless. Else,  $\exists a \in G$  s.t.  $\text{ord}(a) = p^2$ , then  $H = \langle a \rangle \triangleleft G$ . ( $[G : H] = p$  is the smallest prime dividing  $|G|$ )

Also, in this case,  $\varphi : G \rightarrow Z_G \implies G/\ker \varphi \cong Z_G$ . Let  $E = \ker \varphi, |E| = p^2$ . By the def. of  $\ker \varphi, E \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ .

We find that  $H \cap E = \langle a^p \rangle$ . Pick  $b \in E \setminus H$  and let  $K = \langle b \rangle \implies |K| = p, H \cap K = \{1\}, HK = G$ .

## 1.6.2 Semidirect product

**Fact 1.6.1.**  $K \triangleleft G, H \triangleleft G, K \cap H = \{1\} \implies KH = K \times H$   
 $(\forall k \in K, h \in H, khk^{-1}h^{-1} \in H \cap K = \{1\}, \implies kh = hk)$

**Fact 1.6.2.** Let  $K, H$  be two groups, and  $G = K \times H \implies K \times \{1\} \triangleleft K \times H, \{1\} \times H \triangleleft K \times H$

**Observation 1.**  $K \leq G, H \triangleleft G, K \cap H = \{1\}$  (K 慘 H 好, 簡稱慘好集)

$\implies$  elements in  $KH$  has unique representation? 好事喔

$KH \iff K \times H$  1-1 corresp,  $(kh) \leftrightarrow (k, h)$

Group operation :  $\forall k_1, k_2 \in K, h_1, h_2 \in H, (k_1 h_1)(k_2 h_2) = k_1 k_2 (k_2^{-1} h_1 k_2) h_2$   
Let  $\tau : K \rightarrow \text{Aut}(H), k \mapsto (\tau(k) : h \mapsto k h k^{-1})$  (類似  $\in \text{Inn}(H)$ )

**Def 35** (Semi-Direct Product (慘好積)).  $K \times_{\tau} H = \{(k, h) | k \in K, h \in H\}$  with group operation :  $(k_1, h_1)(k_2, h_2) = (k_1 k_2, \tau(k_2^{-1})(h_1)(h_2))$  where  $\tau : K \rightarrow \text{Aut}(H)$  (need not to be inner homomorphism)

Properties:

- Associativity: Good, ex
- The identity =  $(1, 1)$
- Inverse :  $(k, h)^{-1} = (k^{-1}, \tau(k)(h^{-1}))$
- $K \cong K \times \{1\} \leq K \times_{\tau} H : (k_1, 1)(k_2, 1) = (k_1 k_2, \tau(k_2^{-1})(1)1) = (k_1 k_2, 1) \in K \times \{1\}$   
 $H \cong \{1\} \times H \leq K \times_{\tau} H : (1, h_1)(1, h_2) = (1, \tau(1^{-1})(h_1)h_2) = (1, h_1 h_2) \in \{1\} \times H$
- $H \triangleleft K \times_t H : (k, h)(1, h')(k, h)^{-1} = (k, h h')(k^{-1}, \tau(k)(h^{-1})) = (1, \tau(k)(h h')\tau(k)(h^{-1})) \in H$
- $\tau(k)(h) = k h k^{-1} : (k, 1)(1, h)(k^{-1}, 1) = (k, h)(k^{-1}, 1) = (1, \tau(k)(h))$
- If  $\tau$  is trivial  $\implies K \times_t H \cong K \times H$

**Remark 9.** Some definition swaps the order of  $H$  and  $K$ , i.e.  $(h_1, k_1)(h_2, k_2) = (h_1 \phi(k_1)(h_2), k_1 k_2)$

**Ex 1.6.1.** Show that  $H \rtimes_{\phi} K$  is a group and satisfies the above properties.

**Eg 1.6.1.** Construct a non-abelian group of order 21.

**Fact 1.6.3.**  $\text{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^{\times} \cong C_{p-1}$

Sol :  $\phi_k : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}, \bar{1} \mapsto \bar{k}$

$$\phi_{k_2} \circ \phi_{k_1}(1) = \phi_{k_2}(\bar{k}_1) = \phi_{k_2}(1 + \dots + 1) = \bar{k}_2 + \dots + \bar{k}_2 = \overline{k_1 k_2}$$

Let  $K = C_3, H = C_7$ , define  $\tau : C_3 \rightarrow \text{Aut}(C_7) \cong C_6, a \mapsto \phi_2$

$$\phi_k : b \mapsto b^k$$

$$G = \langle a, b | a^3 = 1, b^7 = 1, aba^{-1} = b^2 \rangle$$

**Eg 1.6.2.**  $p : \text{odd}, |G| = p^3, G$  is non-abelian.

(sol)  $\phi : G \rightarrow Z(G), a \mapsto a^p$  non trivial case  $\exists a \in G$  with  $\text{ord}(a) = p^2$ . Let  $H = \langle a \rangle$  here  $\phi$  is onto and  $E = \ker \phi \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  And  $|H \cap E| = p$   $H \triangleleft G$  because  $[G : H] = p$  Pick  $b \in E \setminus H$  and let  $K = \langle b \rangle \implies |K| = p, K \cap H = \{1\}$  so  $|G| = |KH| = p^3$

**Fact 1.6.4.**  $\text{Aut}(\mathbb{Z}/p^2\mathbb{Z}) \cong (\mathbb{Z}/p^2\mathbb{Z})^{\times}$

Sol :  $\phi_k : \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z}, \bar{1} \mapsto \bar{k}, \gcd(k, p) = 1$

Find a group homo  $\tau : K \implies \text{Aut}(H)$  because  $(1+p)^p \equiv 1 \pmod{p^2}$ ,  $\text{ord}(\overline{1+p}) = p$ . Let  $P = \langle \overline{1+p} \rangle$  is the only subgroup of order  $p$ . (if  $\exists |Q| = p, P \neq Q$  then  $P \cap Q = 1, |PQ| = p^2$  but  $|G| = p(p-1)$ , miserable.) So let  $\tau : b \mapsto (\phi_{1+p} : a \mapsto a^{1+p})$  so  $G = \langle a, b | a^{p^2} = 1, b^p = 1, bab^{-1} = a^{1+p} \rangle$  is a non-abelian group of order  $p^3$ .

**Eg 1.6.3.** Isometry of  $R^n$

**Def 36** (Isometry). An isometry of  $R^n$  is a function  $h : R^n \rightarrow R^n$  that preserves the distance between vectors.

$h = t \circ k$  where  $t$  is translation,  $k$  is an isometry fixing the origin, i.e.  $k \in O(n)$ . Let  $T$  be the group of translations on  $R^n$ ,  $T \cong (R^n, +, 0), t \mapsto t(0)$ .

Let  $\tau : O(n) \rightarrow \text{Aut}(T), A \mapsto L_A : R^n \rightarrow R^n, v \mapsto Av$

$$\implies \text{Isom}(R^n) = O(n) \times_{\tau} R^n$$

**Eg 1.6.4.** Quaternion  $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$  is not a semi-direct product of any two proper subgroups.

pf: since  $\{\pm 1\}$  is contained in any non-trivial subgroups, can't find  $H \cap K = \{1\}$ .

**Eg 1.6.5.**  $A_4, V_4 = \{1, (12)(34), (14)(23), (13)(24)\} \triangleleft A_4, V_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Let  $H = \langle (123) \rangle \cong C_3$ , define  $\tau : H \rightarrow \text{Aut}(V_4) \cong GL_2(\mathbb{Z}/2\mathbb{Z})$   $(123) \mapsto (\bar{0}\bar{1}; \bar{1}\bar{1})$  so  $A_4 \cong C_3 \times_{\tau} V_4$ .

**Ex 1.6.2.** Construct  $D_n$  as a semi-direct product of  $\mathbb{Z}/n\mathbb{Z}$  and  $\mathbb{Z}/2\mathbb{Z}$ .

**Ex 1.6.3.**

1. Show that  $S_4$  is a semi-direct product of  $V_4$  and  $H = \{\sigma \in S_4 | \sigma(4) = 4\} \sim S_3$ .
2. Show that  $S_n$  is a semi-direct product of  $A_n$  and  $H = \langle (12) \rangle$ .

**Remark 10.**

- $\text{Aut}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \cong GL_2(\mathbb{Z}/p\mathbb{Z})$  (regarded as a vector space over  $\mathbb{Z}/p\mathbb{Z}$ )
- $\text{Aut}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}) \cong \text{Aut}(\mathbb{Z}/p\mathbb{Z}) \times \text{Aut}(\mathbb{Z}/q\mathbb{Z}) \cong C_{p-1} \times C_{q-1}$

## 1.7 Week 7

### 1.7.1 Composition series

Ques: How to simplify a finite group  $G$ ?

Strategy:

- If  $G = \{1\}$ , then done.
- Otherwise, check whether  $G$  has a nontrivial proper normal subgroup.
- If no, then  $G$  is said to be a simple group.
- Otherwise, find a normal subgroup  $G_1$  as large as possible s.t.  $G/G_1$  is simple.
- If  $G_1$  is simple, then done.
- Otherwise, repeat above on  $G_1$  and get  $G_2, \dots, G_n$  s.t.

$$G_n = \{1\} \triangleleft G_1 \triangleleft \dots \triangleleft G_1 \triangleleft G_0 = G \quad G_i/G_{i+1} \text{ is simple} \quad \searrow \text{composition factors}$$

Say “it is a composition series” with  $\text{length}(G) = n$ .

Hence simple groups can be regarded as basic building blocks of groups.

The classification of all finite simple groups is given as follows:

1.  $\mathbb{Z}/p\mathbb{Z}$ ,  $p$  is a prime.
2.  $A_n$ ,  $n \geq 5$ .
3. simple groups of Lie type.
4. 26 sporadic simple groups.

**Eg 1.7.1.**  $G = S_4, G_1 = A_4, G_2 = V_4, G_3 = \langle (1\ 2)(3\ 4) \rangle, G_4 = \{1\} \rightsquigarrow \text{length}(S_4) = 4$ .  
factors:  $C_2, C_3, C_2, C_2$ .

**Eg 1.7.2.**  $G = \mathbb{Z}/12\mathbb{Z} = \langle \bar{1} \rangle$ .

- $G_1 = \langle \bar{2} \rangle, G_2 = \langle \bar{4} \rangle, G_3 = \langle \bar{0} \rangle \rightsquigarrow \text{length}(3)$ , factors:  $C_2, C_2, C_3$ .
- $G'_1 = \langle \bar{2} \rangle, G'_2 = \langle \bar{6} \rangle, G'_3 = \langle \bar{0} \rangle \rightsquigarrow \text{length}(3)$ , factors:  $C_2, C_3, C_2$ .
- $G''_1 = \langle \bar{3} \rangle, G''_2 = \langle \bar{6} \rangle, G''_3 = \langle \bar{0} \rangle \rightsquigarrow \text{length}(3)$ , factors:  $C_3, C_2, C_2$ .

**Eg 1.7.3.** Let  $|G| = p^n$ . We know  $\forall 0 \leq k \leq n, \exists G_k \triangleleft G$  with  $|G_k| = p^k$  and  $G_i \not\leq G_{i+1}$ .  
 $\text{length}(G) = n$ , factors:  $C_p, \dots, C_p$ . ( $n$  times)

**Theorem 14** (Jordan-Hölder theorem). If  $G$  has a composition series, then any two composition series have the same length and the same factors up to permutation.

**Lemma 1** (Zassenhaus lemma). Let  $H' \triangleleft H \leq G, K' \triangleleft K \leq G$ . Then  $(H \cap K')H' \triangleleft (H \cap K)H', (H' \cap K)K' \triangleleft (H \cap K)K'$  and

$$(H \cap K)H' / (H \cap K')H' \cong (H \cap K)K' / (H' \cap K)K'.$$

**Theorem 15** (Schreier theorem). Any two normal series of  $G$  have equivalent refinements.  
refinements: inserting a finite number of subgroups into the normal series.



*Proof.* For two normal series:

$$\begin{aligned}\{1\} &= H_r \triangleleft H_{r-1} \triangleleft \cdots \triangleleft H_1 \triangleleft H_0 = G \\ \{1\} &= K_s \triangleleft K_{s-1} \triangleleft \cdots \triangleleft K_1 \triangleleft K_0 = G\end{aligned}$$

We define

$$\begin{aligned}H_{ij} &= (H_i \cap K_j)H_{i+1} \\ K_{ji} &= (H_i \cap K_j)K_{j+1}.\end{aligned}$$

Then we have

$$\begin{aligned}\{1\} &= H_{(r-1)s} \triangleleft H_{(r-1)(s-1)} \triangleleft \cdots \triangleleft H_{(r-1)0} = H_{r-1} = H_{(r-2)s} \triangleleft \cdots \triangleleft H_{10} = H_1 = H_{0s} \triangleleft \cdots \triangleleft H_{00} = G \\ \{1\} &= K_{(s-1)r} \triangleleft K_{(s-1)(r-1)} \triangleleft \cdots \triangleleft K_{(s-1)0} = K_{s-1} = K_{(s-2)r} \triangleleft \cdots \triangleleft K_{10} = K_1 = K_{0r} \triangleleft \cdots \triangleleft K_{00} = G\end{aligned}$$

Both have size  $= rs$ . By lemma,  $H_{ij}/H_{i(j+1)} \cong K_{ji}/K_{j(i+1)}$ . Note that if  $H_{ij} = H_{i(j+1)}$ , then  $K_{ji} = K_{j(i+1)}$ .  $\square$

*proof of Jordan-Hölder theorem.* Let

$$\begin{cases} \{1\} = G_n \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G & (*) \\ \{1\} = G'_m \triangleleft \cdots \triangleleft G'_1 \triangleleft G'_0 = G & (**) \end{cases}$$

be two composition series.

By Schreier theorem, we get two refined equivalent series  $(*)', (**)'$ . Since  $(*), (**)$  are already composition series,  $(*) = (*)', (**') = (**)'$ . So  $(*), (**)$  are equivalent.  $\square$

*proof of lemma.* First prove  $(H \cap K')H' \triangleleft (H \cap K)H'$ .

- $\forall g \in H \cap K, gK'g^{-1} = K' \rightsquigarrow (gHg^{-1}) \cap (gK'g^{-1}) = H \cap K'$  and  $gH'g^{-1} = H'$ . So

$$g(H \cap K')H'g^{-1} = (H \cap K')H'$$

- $\forall g \in H', ab \in (H \cap K')H',$

To prove

$$(H \cap K)H'/(H \cap K')H' \cong (H \cap K)K'/(H' \cap K)K'.$$

$$\begin{aligned}(H \cap K)H'/(H \cap K')H' &\cong (H \cap K)(H \cap K')H'/(H \cap K')H' \\ &\cong (H \cap K)/(H \cap K) \cap (H \cap K')H' \\ &\cong (H \cap K)/K \cap (H \cap K')H' \\ &\cong (H \cap K)/(H' \cap K)(H \cap K')\end{aligned}$$

$(K \cap (H \cap K')H' = (H' \cap K)(H \cap K')$ , tricky) By symmetry,

$$(H \cap K)K'/(H' \cap K')K' \cong (H \cap K)/(H' \cap K)(H \cap K')$$

$\square$

**Prop 1.7.1.** Let  $|G| < \infty$ . Then  $G$  is solvable  $\iff$  all composition factors are cyclic of prime order.

*Proof.* “ $\Leftarrow$ ”: by def.

“ $\Rightarrow$ ”: If  $G_i/G_{i+1} \cong C_n$  with  $n = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}$ .  $\square$

**Observation.** Let  $K \triangleleft G$ . 把  $K, G/K$  拆成兩個 composition series 的話, 就可以把兩串接起來, 長度就是加起來。

**Ex 1.7.1.** Let  $\{1\} = G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$  be a composition series of  $G$  and  $K \triangleleft G$ . Then after we eliminate equalities,

1.  $\{1\} = (K \cap G_n) \triangleleft (K \cap G_{n-1}) \triangleleft \cdots \triangleleft (K \cap G_1) \triangleleft (K \cap G_0) = K$  is a composition series of  $K$ .
2.  $\{\bar{1}\} = KG_n/K \triangleleft KG_{n-1}/K \triangleleft \cdots \triangleleft KG_1/K \triangleleft KG_0/K = G/K$  is a composition series of  $G/K$ .

**Ex 1.7.2.** Let  $\begin{cases} H \triangleleft G \\ K \triangleleft G \end{cases}$  with  $H \neq K$  s.t.  $G/H, G/K$  are simple. Then  $H/H \cap K, K/K \cap H$  are simple too.

**Ex 1.7.3.** Let  $\{1\} = G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$  be a composition series of length  $n$ . Show by induction on  $n$  that for every composition series of  $G$ :

$$\{1\} = H_m \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_1 \triangleleft H_0 = G,$$

we have  $m = n$  and

$$\{H_{n-1}/H_n, \dots, H_0/H_1\} = \{G_{n-1}/G_n, \dots, G_0/G_1\}$$

**Ex 1.7.4.** Exhibit all composition series for  $Q_8, D_4, \mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  respectively.

## 1.7.2 Modules over a PID

**Def 37.** Let  $R$  be a ring with 1. A  $R$ -module is an abelian group  $M$  (written additively) on which  $R$  acts linearly.  $R \times M \rightarrow M \quad (r, x) \mapsto rx$

1.  $r(x + y) = rx + ry \quad r \in R, x, y \in M$
2.  $(r_1 + r_2)x = r_1x + r_2x \quad r_1, r_2 \in R, x \in M$
3.  $(r_1r_2)x = r_1(r_2x) \quad r_1, r_2 \in R, x \in M$
4.  $1x = x \quad x \in M$

**Eg 1.7.4.** A  $k$ -vector space is a  $k$ -module.

**Eg 1.7.5.** An abelian group  $G$  can be regarded as a  $\mathbb{Z}$ -module.

$$\begin{aligned} \mathbb{Z} \times G &\rightarrow G \\ (n, a) &\mapsto na \end{aligned} \quad \text{by} \quad na = \begin{cases} \underbrace{a + \cdots + a}_{n \text{ times}} & \text{if } n \geq 0 \\ \underbrace{(-a) + \cdots + (-a)}_{n \text{ times}} & \text{if } n < 0 \end{cases}$$

**Eg 1.7.6.** Let  $I$  be an ideal of  $R$ . Then  $I$  can be regarded as an  $R$ -module since  $\forall r \in R, a \in I, \quad ra \in I$ .

**Def 38.** A submodule  $N$  of  $M$  is an additive subgroup of  $M$  s.t.  $\forall r \in R, a \in N, \quad ra \in N$ .

**Prop 1.7.2.** Let  $\phi \neq S \subseteq M$ . The submodule generated by  $S$  is defined to be

$$\begin{aligned} \langle S \rangle_R &= \left\{ \sum_{\text{finite}} r_i x_i \mid x_i \in S, r_i \in R \right\} = \text{the least submodule containing } S \\ &= \bigcap_{S \subset N \subset M} N \end{aligned}$$

**Def 39.** An  $R$ -module  $M$  is said to be finitely generated if  $\exists x_1, \dots, x_n \in M$  s.t.  $M = \langle x_1, \dots, x_n \rangle_R = Rx_1 + Rx_2 + \dots + Rx_n$

**Eg 1.7.7.**  $R$  is generated by 1 as an  $R$ -module.

**Def 40.** An additive group homo.  $\varphi : M_1 \rightarrow M_2$  is called an  $R$ -module homo. if

$$\varphi(rx) = r\varphi(x) \quad \forall r \in R, x \in M_1$$

**Def 41.** An integral domain  $R$  is called a principal ideal domain (PID) if  $\forall I$  ideal in  $R$ ,  $\exists a \in R$  s.t.  $I = \langle a \rangle_R$ .

**Eg 1.7.8.**  $\mathbb{Z}$  is a PID.

For  $I \subseteq \mathbb{Z}$ ,  $I$  is an additive subgroup, so  $I = m\mathbb{Z} = \langle m \rangle_{\mathbb{Z}}$ .

**Def 42.**  $M$  is said to be a free module of rank  $n$  if  $M \cong R^n = R \oplus \dots \oplus R$  (or  $R \times \dots \times R$ )

**Theorem 16.** If  $R$  is a PID, then any submodule of  $R^n$  is free of rank  $\leq n$ .

*Proof.* By induction on  $n$ . If  $n = 1$ , notice that any submodule is an ideal  $I$  by the closure of submodule. Then since  $R$  is a PID,  $\forall I \subseteq R, \exists a \in R$  s.t.  $I = \langle a \rangle_R = Ra \cong R$  (**as a  $R$ -module**).

Let  $n > 1$  and  $N$  be a submodule of  $R^n$ . Consider

$$\pi_1 : \begin{matrix} R^n & \rightarrow & R \\ (r_1, \dots, r_n) & \mapsto & r_1 \end{matrix} \quad \text{and} \quad \pi = \pi_1|_N : N \rightarrow R$$

**case 1:**  $\text{Im } \pi = \{0\}$ . In this case,  $N \subseteq \ker \pi_1 \cong R^{n-1}$ . By induction hypothesis,  $N$  is free of rank  $\leq n-1 < n$ .

**case 2:**  $\text{Im } \pi = \langle a \rangle$ , say  $\pi(x) = a$ . Claim:  $N = Rx \oplus \ker \pi, \ker \pi \subseteq \ker \pi_1 \cong R^{n-1}$ .

- $Rx \cap \ker \pi = \{0\}$ :  $rx \in Rx \cap \ker \pi \implies \pi(rx) = 0$ , then  $r\pi(x) = 0$ . But integral domain doesn't have zero divisors, so  $r = 0$  and hence  $rx = 0$ .
- $N \supseteq Rx \oplus \ker \pi$ : Obvious since  $Rx, \ker \pi \subseteq N$ .
- $N \subseteq Rx \oplus \ker \pi$ :  $\forall y \in N, \pi(y) = r_0 a$  for some  $r_0 \in R$ ,  $\pi(y - r_0 x) = 0 \implies y - r_0 x \in \ker \pi$ . So  $N \subseteq Rx \oplus \ker \pi$ .  $\square$

Recall that the elementary matrices are

- $D_i(u) = \text{diag}(1, \dots, 1, u, 1, \dots, 1)$ .  $D_i(u) \in \text{GL}(n, R)$  if  $u$  is a unit.
- $B_{ij}(a) = I_n + ae_{ij}, a \in R, i \neq j$ .  $B_{ij}(a)^{-1} = B_{ij}(-a) \implies B_{ij}(a) \in \text{GL}(n, R)$ .
- $P_{ij} = I_n - e_{ii} - e_{jj} + e_{ij} + e_{ji}$ .

**Fact 1.7.1.** If  $R$  is a PID and  $\langle a, b \rangle_R = \langle d \rangle_R$ , then  $d = \gcd(a, b)$ .

*Proof.*

- $a \in \langle d \rangle_R \implies a = rd$  for some  $r \in R \implies d \mid a$ .  $v \in \langle d \rangle_R \implies d \mid v$ .
- Let  $c \mid a, c \mid b$ , say  $a = k_1 c, b = k_2 c$ .  $d \in \langle a, b \rangle_R \implies d = x_1 a + x_2 b$  for some  $x_1, x_2 \in R$ . So  $d = x_1 k_1 c + x_2 k_2 c = (x_1 k_1 + x_2 k_2) c \implies c \mid d$ .  $\square$

**Theorem 17.** Let  $R$  be a PID and  $A \in M_{n \times m}(R)$ . Then  $\exists P \in \text{GL}_n(R)$  and  $Q \in \text{GL}_m(R)$  s.t.

$$PAQ = \begin{pmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_r & \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \quad \text{with } d_i \mid d_{i+1} \quad \forall i = 1, \dots, r-1$$

*Proof.* Define the length  $l(a)$  of  $a \neq 0$  to be  $r$  if  $a = p_1 p_2 \dots p_r$  where  $p_1, \dots, p_r$  are prime elements.  
prime elements:  $p \mid ab \implies p \mid a$  or  $p \mid b$ .

1. We may assume  $a_{11} \neq 0$  and  $l(a_{11}) \leq l(a_{ij}) \forall a_{ij} \neq 0$ . (換一換就上去了...XD)
2. We may assume  $\begin{cases} a_{11} \mid a_{1k} & \forall k = 2, \dots, m \\ a_{11} \mid a_{k1} & \forall k = 2, \dots, n \end{cases}$ . If  $a_{11} \nmid a_{1k}$ , then we can interchange 2nd and  $k$ th columns to assume  $a = a_{11} \nmid a_{12} = b$ .

Let  $d = \gcd(a, b) \implies \begin{cases} l(d) < l(a) \\ d = ax + by \text{ for some } x, y \in R \end{cases} \implies 1 = \frac{a}{d}x + \frac{b}{d}y$ . Write  $b' = \frac{b}{d}, a' = -\frac{a}{d}$ . Then

$$\begin{pmatrix} -a' & b' \\ y & -x \end{pmatrix} \begin{pmatrix} x & b' \\ y & a' \end{pmatrix} = I_2$$

反正就是移一下減掉, length 會一直變小  $\implies$  這個操作會停.

3. 有這個  $\begin{cases} a_{11} \mid a_{1k} & \forall k = 2, \dots, m \\ a_{11} \mid a_{k1} & \forall k = 2, \dots, n \end{cases}$  就可以全部消掉變成

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & \dots & b_{nm} \end{pmatrix}$$

4. May assume  $a_{11} \mid b_{kl} \quad \forall k, l$ . 不是的話就把該 row 往第一 row 加上去, 重複前面的操作,  $l(a_{11})$  總是變小, 因此會停.
5. 遞迴下去...

最後就弄出想要的矩陣了.

□

## 1.8 Week 8

### 1.8.1 Fundamental theorem of finitely generated abelian groups

**Theorem 18** (Structure theorem of finitely generated module over a PID). Let  $R$  be a PID and  $M$  be a finitely generated  $R$ -module. Then  $M \cong R/d_1R \oplus \cdots \oplus R/d_lR \oplus R^s$ ,  $d_i \in R$  with  $d_i \mid d_{i+1} \quad \forall i = 1, \dots, l-1$  for some  $s \in \mathbb{Z}^{\geq 0}$ .

*Proof.* Let  $M = \langle x_1, \dots, x_n \rangle_R$  and consider

$$\begin{aligned} \varphi : R^n &\rightarrow M \\ e_i &\rightarrow x_i \end{aligned}$$

By 1st isom. thm.,  $R^n / \ker \varphi \cong M$ .

We know  $\ker \varphi \cong R^m$  ( $e'_i \mapsto f_i, e'_i \in R^m$ ) for some  $m \leq n$  and  $\forall x \in \ker \varphi \quad \exists! x_1, \dots, x_m \in R$  s.t.  $x = \sum_{i=1}^m x_i f_i$ .

Note that  $\ker \varphi \subseteq R^n$ . So we can write  $f_i = \sum_{j=1}^n a_{ji} e_j \quad \forall i = 1, \dots, m$ . Then  $x = \sum x_i \sum a_{ji} e_j = \sum (\sum a_{ji} x_i) e_j$ .

$R$  is a PID  $\implies \exists P \in \text{GL}_n(R), Q \in \text{GL}_m(R)$  s.t.

$$PAQ = \begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_r & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \quad \text{with } d_i \mid d_{i+1} \quad \forall i = 1, \dots, r-1$$

So consider  $[w_i] = Qe_i$ . Since  $P, Q$  invertible,  $R^n = \bigoplus R w_i$ ,  $\ker \varphi = \bigoplus d_i R w_i$  Hence

$$M \simeq R / \ker \varphi = \bigoplus R w_i / \bigoplus d_i R w_i = \bigoplus R / d_i R$$

□

$$\begin{aligned} R &\rightarrow R w_i / R d'_i w_i \\ 1 &\rightarrow \overline{w_i} \\ r &\rightarrow \overline{r w_i} \end{aligned}$$

**Remark 11.** If  $R$  is commutative, then " $R^n \cong R^m \implies n = m$ ."

**Theorem 19.** Let  $G$  be a finitely generated abelian group. Then  $G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_l\mathbb{Z} \oplus \mathbb{Z}^s$ ,  $d_i \in \mathbb{Z}$  with  $d_i \mid d_{i+1} \quad \forall i = 1, \dots, l-1$  for some  $s \in \mathbb{Z}^{\geq 0}$ .

Since  $G$  can be regarded as a f.g.  $\mathbb{Z}$ -module and  $\mathbb{Z}$  is a PID, it follows from the main theorem.

$\text{Tor}(G) = \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_l\mathbb{Z} \leq G$  and  $G/\text{Tor}(G) \cong \mathbb{Z}^s$  (free part of  $G$ ).

**Fact 1.8.1.** If  $d = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$ , then  $\mathbb{Z}/d\mathbb{Z} \cong \mathbb{Z}/p_1^{m_1}\mathbb{Z} \oplus \mathbb{Z}/p_2^{m_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_s^{m_s}\mathbb{Z}$ .

**Theorem 20** (Chinese Remainder theorem). Let  $R$  be a commutative ring with 1 and  $I_1, \dots, I_n$  be ideals of  $R$ . Then

$$\begin{aligned} \varphi : R &\rightarrow R/I_1 \times \cdots \times R/I_n \text{ is a ring homo.} \\ r &\mapsto (\overline{r}, \dots, \overline{r}) \end{aligned}$$

and

- (1) if  $I_i, I_j$  are coprime  $\forall i \neq j$ , then  $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n$ .
- (2)  $\varphi$  is surjective  $\iff I_i, I_j$  are coprime  $\forall i \neq j$ .

(3)  $\varphi$  is injective  $\iff I_1 \cap I_2 \cap \dots \cap I_n = \{0\}$ .

So if  $I_i, I_j$  are coprime  $\forall i \neq j$ , then

$$R/I_1 I_2 \dots I_n \cong R/I_1 \times \dots \times R/I_n.$$

$I_i, I_j$  are coprime  $\iff I_i + I_j = R$ .

*Proof.* we only need to prove (1), (2).

(1) By induction on  $n$ .  $n = 2$ , need  $I_1 \cap I_2 \subseteq I_1 I_2$ . Indeed,  $I_1 \cap I_2 = (I_1 \cap I_2)R = (I_1 \cap I_2)(I_1 + I_2) \subseteq I_1 I_2$ .

For  $n > 2$ , since  $I_i + I_n = R \quad \forall i = 1, \dots, n-1$ ,  $\exists x_i \in I_i, y_i \in I_n$  s.t.  $x_i + y_i = 1 \quad \forall i = 1, \dots, n-1$ .

So  $x_1 x_2 \dots x_{n-1} = (1 - y_1)(1 - y_2) \dots (1 - y_{n-1}) = 1 - y, y \in I_n \implies I_1 I_2 \dots I_{n-1} + I_n = R$ .

Now,  $I_1 I_2 \dots I_n = (I_1 \dots I_{n-1})I_n = (I_1 \dots I_{n-1}) \cap I_n = I_1 \cap \dots \cap I_n$ .

(2) “ $\Rightarrow$ ”: WLOG, we may let  $I_i = I_1, I_j = I_2$ . We have  $x \in R$  s.t.

$$\varphi(x) = (\bar{1}, \bar{0}, \dots, \bar{0}) \quad \text{i.e. } \bar{x} = \bar{1} \text{ in } R/I_1$$

Write  $x \equiv 1 \pmod{I_1}$ . Since  $1 - x \in I_1, x \in I_2$  and  $(1 - x) + x = 1, I_1 + I_2 = R$ .

“ $\Leftarrow$ ”:  $\forall y \in \text{RHS}, y = (\bar{r}_1, \dots, \bar{r}_n)$ . If we may find that  $x_i \in R$  s.t.  $\varphi(x_i) = (\bar{0}, \dots, \bar{1}, \bar{0}, \dots, \bar{0})$ , then

$$\varphi\left(\sum_{i=1}^n r_i x_i\right) = y$$

It is enough to show, for example,  $\exists x \in R$  s.t.  $\varphi(x) = (\bar{1}, \bar{0}, \dots, \bar{0})$ .

Since  $I_1 + I_i = R \quad \forall i = 2, \dots, n$ ,  $\exists x_i \in I_1, y_i \in I_i$  s.t.  $x_i + y_i = 1 \quad \forall i = 2, \dots, n$ .

So let  $x = y_2 \dots y_n = (1 - x_2) \dots (1 - x_n)$ . We have  $x \in I_2, \dots, I_n$  and  $x \equiv 1 \pmod{I_1}$ .

□

**Eg 1.8.1.**  $|G| = 72$  and  $G$  is abelian:

$$72 = 2 \times 36 = 3 \times 24 = 2 \times 2 \times 18 = 6 \times 12 = 2 \times 6 \times 6$$

Invariant factors

Elementary divisors

**Def 43.** The exponent of  $G$  with  $|G| < \infty$  is

$$\text{Exp}(G) := \min \{m \in \mathbb{N} | g^m = 1 \quad \forall g \in G\}$$

**Ex 1.8.1.**

1. Let  $G$  be abelian with  $|G| = n$ . Show that if  $d \mid n$ , then  $\exists H \leq G$  s.t.  $|H| = d$ .
2. If  $n = 540, d = 90$ , then construct all possible  $G$  and corresponding  $H$ .

**Ex 1.8.2.** Let  $G$  be abelian with  $|G| < \infty$ . Show that  $G$  is cyclic  $\iff \text{Exp}(G) = |G|$ .

**Ex 1.8.3.** Let  $f_i(x) \in \mathbb{Z}[x], i = 1, \dots, k$  with  $\deg f_i = d$  and  $p_1, \dots, p_k$  be distinct primes. Show that  $\exists f(x) \in \mathbb{Z}[x]$  with  $\deg f = d$  s.t.  $\bar{f}(x) = \bar{f}_i(x)$  in  $\mathbb{Z}/p_i \mathbb{Z}[x] \quad \forall i = 1, \dots, k$ .

$$f(x) = a_d x^d + \dots + a_0, \bar{f}(x) = \overline{a_d} x^d + \dots + \overline{a_0}$$

### 1.8.2 Sylow theorems

**Def 44.** Let  $|G| = p^\alpha r$  with  $p \nmid r$ .

1. If  $H \leq G$  with  $|H| = p^\alpha$ , then we call  $H$  a Sylow  $p$ -subgroup of  $G$ .
2.  $\text{Syl}_p(G)$  = the set of all Sylow  $p$ -subgroups of  $G$ .
3.  $n_p = |\text{Syl}_p(G)|$ .

**Lemma 2** (Key lemma). Let  $P \in \text{Syl}_p(G)$  and  $Q$  be a  $p$ -subgroup of  $G$ . Then  $Q \cap N_G(P) = Q \cap P$ .

*Proof.* By Lagrange theorem,  $H = Q \cap N_G(P)$  is also a  $p$ -subgroup of  $N_G(P)$  since  $|H| \mid |Q|$ .

Since  $\begin{cases} P \triangleleft N_G(P) \\ H \leq N_G(P) \end{cases} \implies HP \leq N_G(P)$ , we have

$$|HP| = \frac{|H||P|}{|H \cap P|} = p^{\alpha+k-s}$$

where  $|H \cap P| = p^s, s \leq k$ . Then  $p^{\alpha+k-s} \mid |N_G(P)| \mid |G| = p^\alpha r$ .

So  $k = s \implies H = H \cap P \implies H \leq P \cap Q$ . □

**Theorem 21** (Sylow I).  $\forall 0 \leq k \leq \alpha, \exists H \leq G$  s.t.  $|H| = p^k$ . In particular,  $\text{Syl}_p(G) \neq \emptyset$ .

*Proof.* By induction on  $|G|$ . If  $|G| = 1$ , then  $k = 0, H = \{1\}$ .

Assume  $|G| > 1, k \geq 1, \alpha \geq 1$ .

**case 1:**  $p \mid |Z_G|$ . By Cauchy theorem,  $\exists a \in Z_G$  with  $\text{ord}(a) = p$ . Then  $\langle a \rangle \triangleleft G$  and  $|G/\langle a \rangle| = p^{\alpha-1}r \leq |G|$ . If  $k = 1$ , then  $H = \langle a \rangle$ . Otherwise, we may assume that  $1 \leq k-1 \leq \alpha-1$ . By induction hypothesis,  $\exists H' = G/\langle a \rangle$  s.t.  $|H'| = p^{k-1}$ . By 3rd isom. thm., we can write  $H' = H/\langle a \rangle$  and thus  $|H| = p^k$ .

**case 2:**  $p \nmid |Z_G|$ . By the class equation,  $|G| = |Z_G| + \sum_{i=1}^m \frac{|G|}{|Z_G(a_i)|}, a_i \in Z_G$ .

In this cases,  $\exists a_j$  s.t.  $p \nmid \frac{|G|}{|Z_G(a_j)|} \implies p^\alpha \mid |Z_G(a_j)|$ . And  $Z_G(a_j) \leq G$  since  $a_j \in Z_G$ . By induction hypothesis,  $\exists H \leq Z_G(a_j) \leq G$  s.t.  $|H| = p^k$ . □

**Theorem 22** (Sylow II). Let  $P \in \text{Syl}_p(G)$  and  $Q$  be a  $p$ -subgroup of  $G$ . Then  $\exists a \in G$  s.t.  $Q \leq aPa^{-1}$ . In particular,  $\forall P_1, P_2 \in \text{Syl}_p(G), \exists a \in G$  s.t.  $P_2 = aP_1a^{-1}$ .

*Proof.* Let  $X = \{\text{left cosets of } P\}$  and consider  $\begin{matrix} Q \times X \rightarrow X \\ (a, xP) \mapsto axP \end{matrix}$ .

Observe that  $xP \in \text{Fix } Q \iff axP = xP \quad \forall a \in Q \iff x^{-1}axP = P \quad \forall a \in Q \iff x^{-1}ax \in P \quad \forall a \in Q \iff a \in xPx^{-1} \quad \forall a \in Q$ .

We know  $|\text{Fix } Q| \equiv |X| \pmod{p}$  and  $p \nmid r \implies |\text{Fix } Q| \neq 0 \iff \exists a \in G, Q \leq aPa^{-1}$ .

In particular,  $\begin{cases} P_2 \leq aP_1a^{-1} \\ |P_2| = |aP_1a^{-1}| \end{cases} \implies P_2 = aP_1a^{-1}$ . □

**Theorem 23** (Sylow III).  $n_p \equiv 1 \pmod{p}$  and  $n_p \mid r$ .

*Proof.* • Consider  $\begin{matrix} P \times \text{Syl}_p(G) \rightarrow \text{Syl}_p(G) \\ (a, Q) \mapsto aQa^{-1} \end{matrix}$  where  $P \in \text{Syl}_p(G)$ .

$P' \in \text{Fix } P \iff aP'a^{-1} = P' \quad \forall a \in P \iff P \leq N_G(P') \cap P = P' \cap P \iff P' = P$ .

So  $\text{Fix } P = \{P\} \implies n_p \equiv |\text{Fix } P| = 1 \pmod{p}$ .

- Consider  $\begin{pmatrix} G \times \text{Syl}_p(G) \rightarrow \text{Syl}_p(G) \\ a, Q \mapsto aQa^{-1} \end{pmatrix} \implies$  There is only one orbit  $\text{Syl}_p(G)$ .

We know  $|\text{Syl}_p(G)| = \frac{|G|}{|G_Q|}$  and  $G_Q = N_G(Q)$ . Then  $n_p = \frac{|G|}{|G_Q|} \mid |G|$ . So  $n_p \mid p^\alpha r \implies n_p \mid r$ .

□

**Prop 1.8.1.** Let  $|G| = pq$  where  $p, q$  are primes with  $\begin{cases} p < q \\ q \not\equiv 1 \pmod{p} \end{cases}$ . Then  $G \cong C_{pq}$ .

*Proof.*  $n_p = 1 + kp \mid q \implies n_p = 1$  i.e.  $H \in \text{Syl}_p(G) \implies H \triangleleft G$ .

$n_q = 1 + kq \mid p \implies n_q = 1$  i.e.  $K \in \text{Syl}_q(G) \implies K \triangleleft G$ .

Since  $\gcd(p, q) = 1$ ,  $H \cap K = 1$ . Hence  $G = H \times K \cong C_p \times C_q \cong C_{pq}$ .

□

**Eg 1.8.2.** Consider  $|G| = 255 = 3 \times 5 \times 17$ .

1. 找兩個 normal subgroup (17, 5 or 3)
2. quot 掉後發現剩下的是 abelian  $\rightsquigarrow [G, G]$  在裡面
3.  $[G, G] = 1$
4. 唱 f.g. xxx thm. 得到  $G \cong \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{17}$ .
5. 中國剩飯定理  $G \cong C_{255}$ .

**Ex 1.8.4.** If  $|G| = 7 \times 11 \times 19$ , then  $G$  is abelian.

**Eg 1.8.3.** No group  $G$  of order  $48 = 2^4 \times 3$  is simple.

1.  $n_2 = 1 + 2k \mid 3 \rightsquigarrow n_2 = 1$  or 3.
2.  $n_2 = 1$  then OK.
3. Assume  $n_2 = 3$ . Let  $P \in \text{Syl}_2(G)$ ,  $X = \{\text{left cosets of } P\}$  ( $|X| = 3$ ).
4. Consider  $\begin{pmatrix} G \times X \rightarrow X \\ a, xP \mapsto axP \end{pmatrix} \rightsquigarrow \varphi : G \rightarrow S_3$ .
5. 考慮  $\ker \varphi$ .

**Ex 1.8.5.** No group  $G$  of order 36 is simple.

**Ex 1.8.6.** No group  $G$  of order 30 is simple.

**Ex 1.8.7.** Let  $|G| = 385$ . Show that  $\exists P \in \text{Syl}_7(G)$  s.t.  $P \leq Z_G$ .



## 1.9 Week 9

### 1.9.1 Classification

To classify groups of small orders:

- $|G| = 1$ :  $G = \{1\}$
- $|G| = 2$ :  $G \cong C_2$
- $|G| = 3$ :  $G \cong C_3$
- $|G| = 4$ :  $G \cong \mathbb{Z}_4$  or  $\mathbb{Z}_2 \times \mathbb{Z}_2$
- $|G| = 5$ :  $G \cong C_5$
- $|G| = 6$ :  $n_3 = 1, n_2 = 1$  or  $3$ . Let  $H \in \text{Syl}_3(G)$  and  $H \triangleleft G$ . Let  $K \in \text{Syl}_2(G)$ . Also  $H \cap K = \{1\}$  and  $HK = G$  then  $G \cong K \rtimes_\tau H$ 
  - If  $\tau$  is trivial:  $G \cong K \times H \cong C_2 \times C_3 \cong C_6$
  - $\tau : b \mapsto \phi_2 : \langle a \rangle \rightarrow \langle a \rangle$ :  $G \cong K \rtimes_\tau H \cong \langle a, b \mid a^3 = 1, b^2 = 1, bab^{-1} = a^2 = a^{-1} \rangle \cong D_3$
- $|G| = 7$ :  $G \cong C_7$
- $|G| = 8$ :
  - If abelian:  $\mathbb{Z}_8$  or  $\mathbb{Z}_4 \times \mathbb{Z}_2$  or  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
  - If non-abelian:
    - \*  $\nexists a \in G$  with  $\text{ord}(a) = 8$
    - \* Not each  $a \in G$  with  $a^2 = 1$ , otherwise  $G$  is abelian.
    - \*  $\exists a \in G$  with  $\text{ord}(a) = 4$ : Let  $H = \langle a \rangle$  and  $H \triangleleft G$  since  $[G : H] = 2$ . Pick  $b \in G \setminus H$  and  $K = \langle b \rangle$ 
      - $\text{ord}(b) = 2$ :  $H \cap K = \{1\}$  and  $HK = G$  then  $G \cong K \rtimes_\tau H$ ,  $\tau : b \mapsto \phi : a \mapsto a^3$ :  $G \cong K \rtimes_\tau H \cong \langle a, b \mid a^4 = 1, b^2 = 1, bab^{-1} = a^3 = a^{-1} \rangle \cong D_4$
      - $\text{ord}(b) = 4$ :  $H \cap K = \langle a^2 = b^2 \rangle$ . Then consider  $bab^{-1} \in H \implies bab^{-1} = 1, a, a^2, a^3$ 
        1.  $1, a$  obviously wrong.
        2.  $bab^{-1} = a^2$ :  $a = a^2aa^{-2} = b^2ab^{-2} = a^4 \implies a^3 = 1$  矛盾
        3. So  $bab^{-1} = a^3 = a^{-1}$ .  
 $G \cong \langle a, b \mid a^4 = 1, b^4 = 1, a^2 = b^2, bab^{-1} = a^3 = a^{-1} \rangle \cong Q_8$
- $|G| = 9$ :  $G \cong \mathbb{Z}_9$  or  $\mathbb{Z}_3 \times \mathbb{Z}_3$
- $|G| = 10$ :  $G \cong K \times H \cong C_2 \times C_5 \cong C_{10}$  or  $G \cong D_5$
- $|G| = 11$ :  $G \cong C_{11}$
- $|G| = 12$ : Claim: If  $|G| = 12$ , then either  $G$  has a normal Sylow 3-subgroup or  $G \cong A_4$ .

*Proof.* By Sylow 3,  $n_3 = 1 + 3k \mid 4 \implies n_3 = 1$  or  $4$ .

- If  $n_3 = 1$ , then  $G$  has a normal Sylow 3-subgroup.
- Otherwise, let  $P \in \text{Syl}_3(G)$  and  $X = \{\text{left cosets of } P\}$ ,  $|X| = 4$ . Consider  $G \times X \rightarrow X$  defined by  $(a, xP) \mapsto axP$  with  $\phi : G \rightarrow S_4$ . And  $\ker \phi \leq P$ ,  $|P| = 3$  and  $P \not\triangleleft G$  (since  $n_3 = 4$ ), so  $\ker \phi = \{1\}$ .

And since  $n_3 = 4$ , there are 8 elements of order 3 which corresponds to 8 3-cycles in  $A_4$ , thus  $|\text{Im } \phi \cap A_4| \geq 8$ . But  $|\text{Im } \phi \cap A_4| \mid |A_4| = 12 \implies \text{Im } \phi = A_4$

□

Now, for the case where  $\exists H \in \text{Syl}_3(G)$  and  $H \triangleleft G$ . Let  $K \in \text{Syl}_2(G)$ , then  $K \cap H = \{1\}$  and  $KH = G \implies G \cong K \rtimes_\tau H$  for some  $\tau : K \rightarrow \text{Aut}(H) = \{\text{id}, \phi_2\}$

- $\tau$  is trivial:  $\mathbb{Z}_{12}$  or  $\mathbb{Z}_2 \times \mathbb{Z}_6$ .
- $\langle b \rangle = K \cong \mathbb{Z}_4$ :  $\tau(b) = \phi_2 \implies G = \langle a, b \mid a^3 = 1, b^4 = 1, bab^{-1} = a^{-1} \rangle \not\cong D_6, A_4$
- $\langle b \rangle = K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ : Let  $K = \langle b, c \mid b^2 = 1, c^2 = 1, bc = cb \rangle$ , then  $\tau : b \mapsto \phi_2$  and  $c \mapsto \text{id}$  (the other cases are equivalent to this one),  $G = \langle a, b, c \mid a^3 = 1, b^2 = 1, c^2 = 1, bc = cb, bab^{-1} = a^{-1}, cac^{-1} = a \rangle \cong \langle a, b \mid a^3 = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle \times \langle c \rangle \cong D_3 \times C_2 \cong D_6$

**Fact 1.9.1.** For odd  $n$ ,  $D_{2n} \cong D_n \times \mathbb{Z}/2\mathbb{Z}$ .

*Proof.*

$$\begin{aligned} D_{2n} &= \langle a, b \mid a^{2n} = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle \\ H &= \langle a^2, b \mid (a^2)^n = 1, b^2 = 1, b(a^2)b^{-1} = a^{-2} \rangle \cong D_n \\ K &= \langle a^n \rangle \cong C_2 \end{aligned}$$

And  $n$  is odd, so  $H \cap K = \{1\}$  and  $D_{2n} \cong D_n \times C_2$  □

- $|G| = 13$ :  $G \cong C_{13}$
- $|G| = 14$ :  $G \cong C_{14}$  or  $D_7$
- $|G| = 15$ :  $G \cong C_{15}$

**Ex 1.9.1.** Assume that  $K$  is cyclic and  $H$  is an arbitrary group. Let  $\tau_1 : K \rightarrow \text{Aut}(H)$ ,  $\tau_2 : K \rightarrow \text{Aut}(H)$  with  $\tau_1(K) \sim \tau_2(K)$  (conjugate). If  $|K| = \infty$ , then assume that  $\tau_1$  and  $\tau_2$  are injective. Show that  $K \rtimes_{\tau_1} H \cong K \rtimes_{\tau_2} H$ .

**Ex 1.9.2.** Classify  $G$  if  $|G| = p^3$  with  $p$  an odd prime and each nontrivial element of  $G$  has order  $p$ .

**Ex 1.9.3.** Classify groups of order 30.

## 1.9.2 Free groups

A free group generate by a non-empty set  $X$  is that there are no relations satisfied by any of elements in  $X$ .

**Def 45.** A free group on  $X$  is a group  $F$  with an inclusion map  $i : X \rightarrow F$  satisfying the following universal property: For any group  $G$  and any map  $f : X \rightarrow G$ , exists a unique group homo  $\varphi : F \rightarrow G$  that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\quad} & F \\ & \searrow & \downarrow \varphi \\ & & G \end{array}$$

**Theorem 24.**  $F$  exists and is unique up to isomorphism. (Denote it as  $F(X) = F$ ).

*Proof.* For  $X$ , we create a new disjoint set  $X^{-1} = \{x^{-1} : x \in X\}$  and an element  $1 \notin X \cup X^{-1}$ .

Define  $F(X) = \{1\} \cup \left\{ x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} : m \in \mathbb{N}, x_i \in X, \delta_i = \pm 1, x_{i+1}^{\delta_{i+1}} \neq (x_i^{\delta_i})^{-1} \right\}$ , and

$$x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} = y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_n^{\epsilon_n} \iff n = m \text{ and } \delta_i = \epsilon_i \text{ and } x_i = y_i, \forall i$$

For each  $y \in X \cup X^{-1}$ , we define  $\sigma_y : F(X) \rightarrow F(X)$  by

$$\sigma_y(x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m}) = \begin{cases} y x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} & \text{if } x_1^{\delta_1} \neq y^{-1} \\ \begin{cases} x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} & (m \geq 2) \\ 1 & (m = 1) \end{cases} & \text{if } x_1^{\delta_1} = y^{-1} \end{cases}$$

Then  $\sigma_y$  is a permutation of  $F(X)$ , since if  $\sigma_y(x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m}) = \sigma_y(y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_m^{\epsilon_m})$ .

$m = n$ : either  $x_1^{\delta_1} = y_1^{\epsilon_1} = y^{-1}$  or not, then either  $x_2^{\delta_2} x_3^{\delta_3} \cdots x_m^{\delta_m} = y_2^{\epsilon_2} y_3^{\epsilon_3} \cdots y_m^{\epsilon_m}$  or  $y x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} = y y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_m^{\epsilon_m}$ . Both of them leads to  $x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m} = y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_m^{\epsilon_m}$ .

$m = n+2$ : Omimi

Also  $\sigma_y$  is onto since omimi. And notice that  $\sigma_{y^{-1}} \circ \sigma_y = id_{F(X)}$

Define  $A = \langle \sigma_x : x \in X \rangle \leq S_{F(X)}$ . and define  $\phi : F(X) \rightarrow A$  by  $\phi(1) = id_{F(X)}$  and

$x_1^{\delta_1} \cdots x_m^{\delta_m} \mapsto \sigma_{x_1}^{\delta_1} \cdots \sigma_{x_m}^{\delta_m}$ . The it is omimi that  $\phi$  is a bijection. So we define  $x :: X \cdot y :: X = \phi^{-1}(\phi(x) \circ \phi(y))$ .

The  $\phi$  in the universal property could be defined as  $\phi(x_1^{\delta_1} x_2^{\delta_2} \cdots x_m^{\delta_m}) = f(x_1)^{\delta_1} \cdots f(x_m)^{\delta_m}$ .  $\square$

**Prop 1.9.1.** Let  $G = \langle a_1, \cdots, a_n \rangle$  and  $X = \{x_1, \cdots, x_m\}$ . Then  $G \cong F(X)/K$  for some normal subgroup  $K$ .  $K$  is called the subgroup of relations connecting the generators.

Define  $f = x_i :: X_i \rightarrow a_i :: G$ . By universal property,  $\exists \phi = x_i :: F(X) \mapsto a_i :: G$ . Then  $F(x)/\ker \phi \cong G$ .

**Def 46.** Let  $X = \{x_1, x_2, \cdots, x_n\}$  and  $R \subset F(X)$ . Let  $N(R)$  be the smallest normal subgroup of  $F(X)$  containing  $R$ , Then  $G = F(X)/N(R)$  is written as  $\langle x_1, \cdots, x_n \mid \text{elements of } R \rangle$ , which is called a presentation of  $G$ . If  $|R| < \infty$ , then  $G$  is said to be finitely presented.

**Eg 1.9.1.**

$$D_n = \left\langle \begin{bmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

We find that  $x^n, y^2, xyxy \in \ker \phi$ . Then  $R = \{x^n, y^2, xyxy\} \subseteq \ker \phi \implies N(R) \leq \ker \phi$ . By factor theorem,  $\exists \bar{\phi} : F(X)/N(R) \rightarrow D_n$ . But notice that

$$|F(X)/N(R)| \leq 2n$$

since  $xyxy = 1 \implies xy = yx^{-1}$ , so every element could be turn into  $x^i y^j$ . Hence  $\bar{\phi}$  is an isomorphism.

**Prop 1.9.2.** Let  $X = \{x_1, x_2, \cdots, x_n\}$ . Then  $F(X)/[F(X), F(X)] \cong \mathbb{Z}^n$ .

*Proof.* Define  $f = x_i :: X \mapsto e_i :: \mathbb{Z}^n$ . Then  $\phi = x_i :: F(X) \mapsto e_i :: \mathbb{Z}^n$ . By 1st isomorphism theorem  $F(X)/\ker \phi \cong \mathbb{Z}^n$  which is abelian, so  $[F(X), F(X)] \leq \ker \phi$ . By factor theorem, 一個圖.

Claim that  $\bar{\phi}$  is 1-1.

*Proof.* Since  $F(X)/[F(X), F(X)]$  is abelian,  $\forall a \in F(X)/[F(X), F(X)]$ , we can write  $a = \bar{x}_1^{n_1} \bar{x}_2^{n_2} \cdots \bar{x}_m^{n_m}$ . If  $\bar{\phi}(\bar{a}) = (m_1, \cdots, m_n) = 0$  in  $\mathbb{Z}^n$ , then  $m_i = 0, \forall i \implies a = 1$   $\square$

$\square$

## 2 Multilinear algebra

### 2.1 Week 11

#### 2.1.1 Bilinear forms & Groups preserving bilinear forms

**Def 47.** Let  $V$  be a vector space over a field  $F$ .

- A function  $f : V \times V \rightarrow F$  is called a bilinear form if

$$\begin{cases} f(rx_1 + x_2, y) = rf(x_1, y) + f(x_2, y) \\ f(x, ry_1 + y_2) = rf(x, y_1) + f(x, y_2) \end{cases} \quad \forall x_1, x_2, x, y_1, y_2, y \in V, r \in F$$

- $B_F(V, V) = \{ \text{bilinear forms on } V \}$  can be regarded as a vector space over  $F$ .

**Theorem 25.** Let  $\dim V = n$  and  $\beta = \{v_1, \dots, v_n\}$  be a basis for  $V$ . Then  $\exists$  an isomorphism  $\psi_\beta : B_F(V, V) \rightarrow M_{n \times n}(F)$ .

*Proof.* For  $v, w \in V$ , write  $v = \sum_i a_i v_i, w = \sum_j b_j v_j$ , i.e.  $[v]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, [w]_\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ .

For  $f \in B_F(V, V)$ ,  $f(v, w) = \sum_i \sum_j a_i b_j f(v_i, v_j) = (a_1 \dots a_n) \begin{pmatrix} f(v_i, v_j) \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ .

Define  $\psi_\beta(f) = A$  with  $A_{ij} = f(v_i, v_j)$ .

- $\psi_\beta$  is a linear transformation.
- $\psi_\beta$  is 1-1.
- $\psi_\beta$  is onto:  $\forall A \in M_{n \times n}(F)$ , we define  $f(v, w) = [v]_\beta^t A [w]_\beta$ . □

**Def 48.** Let  $f \in B_F(V, V)$

- $f$  is said to be symmetric if  $f(v, w) = f(w, v) \quad \forall v, w \in V$ .
- $f$  is said to be skew-symmetric if  $f(v, w) = -f(w, v) \quad \forall v, w \in V$ .
- $f$  is said to be alternating if  $f(v, v) = 0 \quad \forall v \in V$ .

**Remark 12.**

- Alternating  $\implies$  skew-symmetric.
- If  $\text{char } F \neq 2$ , skew-symmetric  $\implies$  alternating.
- If  $\text{char } F = 2$ , symmetric = skew-symmetric.
- $\forall f \in B_F(V, V)$  with  $\text{char } F \neq 2$ ,

$$\begin{aligned} f_s(u, v) &= \frac{1}{2} (f(u, v) + f(v, u)) \\ f_a(u, v) &= \frac{1}{2} (f(u, v) - f(v, u)) \end{aligned}$$

and  $f(u, v) = f_s(u, v) + f_a(u, v)$ .

So we only need to study “symmetric” & “alternating”.

**Ex 2.1.1.**

1. If  $A$  and  $B$  are congruent ( $B = Q^t A Q$ ) in  $M_{n \times n}(F)$ , then they define the same bilinear form.
2.  $f$  is  $\begin{cases} \text{symmetric} \\ \text{skew-symmetric} \end{cases} \iff \psi_\beta(f)$  is  $\begin{cases} \text{symmetric}(A^t = A) \\ \text{skew-symmetric}(A^t = -A) \end{cases}$

**Observation.** Let  $f \in B_F(V, V)$  and  $v_0 \in V$ .

$$\begin{aligned} L_f(v_0) &= f(v_0, \cdot) \in V' = \text{Hom}(V, F) : \text{the dual space of } V \\ R_f(v_0) &= f(\cdot, v_0) \in V' \end{aligned}$$

The left radical of  $f$  :  $\text{lrad}(f) = N(L_f) = \{v \in V \mid f(v, w) = 0 \quad \forall w \in V\}$ .

The right radical of  $f$  :  $\text{rrad}(f) = N(R_f) = \{w \in V \mid f(v, w) = 0 \quad \forall v \in V\}$ .

**Ex 2.1.2.**

1.  $\text{rank}(\psi_\beta(f)) = \text{rank}(R_f) = \text{rank}(L_f)$ .
  2. If  $\dim V = n$ , then TFAE ( $\implies f$  : non degenerate)
    - (a)  $\text{rank}(f) = n$ .
    - (b)  $\forall v \in V, v \neq 0, \exists w \in V$  s.t.  $f(v, w) \neq 0$ .
    - (c)  $\text{lrad}(f) = \{0\}$ .
    - (d)  $L_f : V \rightarrow V'$  is isom.
- (also, right)

**Theorem 26** (Principal Axis theorem). Let  $\dim V = n$  and  $\text{char } F \neq 2$ . If  $f \in B_F(V, V)$  is symmetric, then  $\exists \beta$  s.t.  $\psi_\beta(f)$  is diagonal.

*Proof.* It is sufficient to find  $\beta = \{v_1, \dots, v_n\}$  s.t.  $f(v_i, v_j) = 0 \quad \forall i \neq j$ .

If  $f = 0$ , then done! Assume  $f \neq 0$ . By induction on  $n$ : If  $n = 1$ , done. Let  $n > 1$ .

Claim 1:  $\exists v_1 \in V$  s.t.  $f(v_1, v_1) \neq 0$ . Assume that  $f(v, v) = 0 \quad \forall v \in V$ .

$$f(v, w) = \frac{1}{4}f(v+w, v+w) - \frac{1}{4}f(v-w, v-w) = 0$$

So  $f = 0$ , which is a contradiction.

Now let  $v_1 \in V$  with  $f(v_1, v_1) \neq 0$ . Let  $W = \langle v_1 \rangle_F$  and  $W^\perp = \{w \in V \mid f(v_1, w) = 0\} \subseteq V$ .

Claim 2:  $V = W \oplus W^\perp$

- $V = W + W^\perp$ : For all  $v \in V$ , let  $a = f(v, v_1)/f(v_1, v_1)$ , then  $v = av_1 + (v - av_1) \triangleq w + w'$  where  $w \in W$  and  $f(w', v_1) = f(v - av_1, v_1) = f(v, v_1) - af(v_1, v_1) = 0$ . So  $w' \in W^\perp$  and thus  $V = W + W^\perp$ .
- $W \cap W^\perp = \{0\}$ : obviously since if  $av_1 \in W$ ,  $f(av_1, v_1) = 0 \iff a = 0 \iff av_1 = 0$ .

Since  $f|_{W^\perp \times W^\perp}$  is a symmetric bilinear form on  $W^\perp$  and  $\dim W^\perp < \dim V$ . By induction hypothesis,  $\exists \{v_2, \dots, v_n\}$  a basis for  $W^\perp$  s.t.  $f(v_i, v_j) = 0 \quad \forall i \neq j$ . Then  $\beta = \{v_1, \dots, v_n\}$ .  $\square$

**Theorem 27** (Sylvester's theorem). Let  $f \in B_{\mathbb{R}}(V, V)$  be symmetric with  $\dim V = n$ . Then  $\exists \beta$

$$\text{s.t. } \psi_{\beta}(f) = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix}.$$

The triple (# of 1, # of -1, # of 0) is well-defined. (called the signature of  $f$ )

*Proof.* Assume  $V^+ = \langle v_1, \dots, v_p \rangle_F$ ,  $V^- = \langle v_{p+1}, \dots, v_r \rangle_R$ ,  $V^\perp = \langle v_{r+1}, \dots, v_n \rangle_F$ . ( $V = V^+ \oplus V^- \oplus V^\perp$ )

Claim: If  $W$  is a subspace of  $V$  s.t.  $f$  is positive-definite on  $W$ , then  $W, V^-, V^\perp$  are independent.

Let  $\langle w_1, w_2, \dots, w_s \rangle$  be a basis of  $W$ . If

$$a_1 w_1 + a_2 w_2 + \dots + a_s w_s = b_{p+1} v_{p+1} + \dots + b_r v_r + c_{r+1} v_{r+1} + \dots + c_n v_n.$$

Let  $w \triangleq a_1 w_1 + \dots + a_s w_s$ ,  $v \triangleq b_{p+1} v_{p+1} + \dots + b_r v_r + c_{r+1} v_{r+1} + \dots + c_n v_n$ . Since  $w = v$ ,  $f(w, w) = f(v, v)$ . but  $f(w, w) = \sum a_i^2 \geq 0$  and  $f(v, v) = -\sum b_i^2 \leq 0$ . Hence  $a_i = 0, b_i = 0$ . Since  $v_{r+1}, \dots, v_n$  is linear independent,  $c_i = 0$ . Therefor these vectors are linear independent.

□

**Ex 2.1.3.** Let  $f \in B_F(V, V)$  with  $\text{char } F \neq 2$ . If  $f$  is skew-symmetric, then  $\exists \beta$  s.t.

$$\psi_{\beta}(f) = \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & 1 & \\ & & & & & -1 & 0 & \\ & & & & & & & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix}$$

**Ex 2.1.4.** Study Hermitian form

$T : V \xrightarrow{\sim} V, f \in B_F(V, V)$ .  $T$  preserves  $f$  if  $f(T(v), T(w)) = f(v, w) \quad \forall v, w \in V$ .

In matrix form, let  $\beta$  be a basis for  $V$ ,  $M = [T]_{\beta}$ ,  $A = \psi_{\beta}(f)$ , then  $A = M^t A M$ .

- $f \in B_{\mathbb{R}}(V, V)$  symmetric, non-degenerate:  $\exists \beta$  s.t.  $\psi_{\beta}(f) = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$ .

Then  $\{ T : V \xrightarrow{\sim} V \text{ preserves } f \} \leftrightarrow \left\{ M \in \text{GL}_n(\mathbb{R}) \mid M^t \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} M = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} \right\} = O(p, q)$ .

- $f \in B_{\mathbb{R}}(V, V)$  skew-symmetric, non-degenerate:  $n = 2k$ ,  $\exists \beta$  s.t.  $\psi_{\beta}(f) = J$ .

Then  $\{ T : V \xrightarrow{\sim} V \text{ preserves } f \} \leftrightarrow \{ M \in \text{GL}_n(\mathbb{R}) \mid M^t J M = J \}$ , where

$$J = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$$

### 2.1.2 Tensor product

From now on,  $R$  is assumed to be commutative with 1.

**Def 49.** Let  $M_1, \dots, M_n, L$  be  $R$ -modules.

A function  $F : M_1 \times \dots \times M_n \rightarrow L$  is said to be  $n$ -multilinear if  $\forall i$ ,

$$f(x_1, \dots, rx_i + x'_i, \dots, x_n) = rf(x_1, \dots, x_i, \dots, x_n) + f(x_1, \dots, x'_i, \dots, x_n) \quad \forall r \in R, x_i, x'_i \in M_i$$

If  $n = 2$ ,  $f$  is called a bilinear map.

**Def 50.** Let  $M, N$  be  $R$ -modules. A tensor product of  $M$  and  $N$  is an  $R$ -module  $M \otimes_R N$  with a bilinear map  $\rho : M \times N \rightarrow M \otimes_R N$  satisfying the following universal property:

for any  $R$ -module  $W$  and any bilinear map  $f : M \times N \rightarrow W$ ,  $\exists!$   $R$ -module homomorphism  $\varphi : M \otimes_R N \rightarrow W$ ,

$$\begin{array}{ccc} M \times N & \xrightarrow{\rho} & M \otimes_R N \\ & \searrow f & \downarrow \varphi \\ & & W \end{array}$$

**Theorem 28** (Main theorem).  $M \otimes_R N$  exists and is unique up to isom.

*Proof.* Let  $X = M \times N$ . First we construct the free module  $V_1 = \bigoplus_{(x,y) \in X} R \cdot (x, y)$ .

Notice that in  $V_1$ ,

- $(x_1, y_1) + (x_2, y_2) \neq (x_1 + x_2, y_1 + y_2)$ .
- $r(x, y) \neq (rx, ry)$ .
- $r(r_1(x_1, y_1) + \dots + r_n(x_n, y_n)) = rr_1(x_1, y_1) + \dots + rr_n(x_n, y_n)$ .

$$\text{Let } V_0 = \left\langle \begin{array}{l} (x_1 + x_2, y) - (x_1, y) - (x_2, y), \\ (x, y_1 + y_2) - (x, y_1) - (x, y_2), \\ r(x, y) - (rx, y), r(x, y) - (x, ry) \end{array} \middle| x_1, x_2, x \in M, y_1, y_2, y \in N, r \in R \right\rangle_R.$$

Define  $M \otimes_R N = V_1/V_0$  which is an  $R$ -module and  $\rho : M \times N \rightarrow M \otimes_R N$   $(x, y) \mapsto (x, y) + V_0 = x \otimes y$  which is  $R$ -bilinear. (check yourself)

Universal property:  $\forall (x, y) \in M \times N$ ,  $\begin{array}{l} R(x, y) \rightarrow W \\ r(x, y) \mapsto rf(x, y) \end{array}$ . So, by the universal property of  $\oplus$ ,  $\exists!$   $R$ -module homo.  $\varphi_1 : V_1 \rightarrow W$ :

$$\begin{array}{ccc} M \times N & \xrightarrow{i} & V_1 \\ & \searrow f & \downarrow \varphi_1 \\ & & W \end{array}$$

Claim:  $V_0 \subseteq \ker \varphi_1$ . (check yourself) Then by factor theorem,

$$\begin{array}{ccc} \exists! \varphi : V_1/V_0 & \longrightarrow & W \\ & \nwarrow \quad \nearrow & \\ & M \times N & \end{array}$$

□

**Eg 2.1.1.**  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$ .

**Eg 2.1.2.**  $\mathbb{R}[x, y] \cong \mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{R}[y]$ .

*Proof.*  $\mathbb{R}[x] \times \mathbb{R}[y] \rightarrow \mathbb{R}[x, y]$  is bilinear  $\rightsquigarrow \exists! \varphi : \mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{R}[y] \rightarrow \mathbb{R}[x, y]$   
 $(f(x), g(y)) \mapsto f(x)g(y) \mapsto f(x) \otimes g(y) \mapsto f(x)g(y)$ .

Conversely,  $\mathbb{R}[x, y] \rightarrow \mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{R}[y]$   
 $h(x, y) = \sum a_{ij} x^i y^j \mapsto \sum a_{ij} x_i \otimes y_j$ . □

**Prop 2.1.1.** If  $M = \langle x_1, \dots, x_n \rangle_R$  and  $N = \langle y_1, \dots, y_m \rangle_R$ . Then

$$M \otimes_R N = \langle x_i \otimes y_j \mid i = 1, \dots, n; j = 1, \dots, m \rangle_R.$$

In particular, if  $R$  is a field  $F$ , then  $\dim_F M \otimes_F N = (\dim_F M)(\dim_F N)$ .

*Proof.* Note that  $M \otimes_R N = \langle x \otimes y \mid x \in M, y \in N \rangle$ . Let  $x = \sum_i a_i x_i, y = \sum_j b_j y_j$ . Then  $x \otimes y = \sum_i \sum_j a_i b_j x_i \otimes y_j$ . □

Some canonical isomorphisms:

- $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$ .

*Proof.*  $\forall z \in L, M \times N \rightarrow M \otimes_R (N \otimes_R L)$  is bilinear.  $\exists! R$ -mod homo.  $\varphi_z : M \otimes_R N \rightarrow$   
 $(x, y) \mapsto x \otimes (y \otimes z)$

$M \otimes_R (N \otimes_R L)$ . Similarly,  $(M \otimes_R N) \times L \rightarrow M \otimes_R (N \otimes_R L)$  is bilinear. (The right is due  
 $(\sum x_i \otimes y_i, z) \mapsto \sum x_i \otimes (y_i \otimes z)$  to  $\varphi_z$  linear, and the left is because  $x \otimes (y \otimes (rz_1 + z_2)) = rx \otimes (y \otimes z_1) + x \otimes (y \otimes z_2)$ .) Hence  
exists unique  $R$ -mod homo.  $\varphi : (M \otimes_R N) \otimes_R L \rightarrow M \otimes_R (N \otimes_R L)$ . By the symmetric  
construction, we have  $\varphi^{-1}$  and  $\varphi^{-1} \circ \varphi = \varphi \circ \varphi^{-1} = 1$ , so the two are isomorphic. □

- $(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$ .

The mapping  $\psi : (M \oplus M') \times N \rightarrow (M \otimes_R N) \oplus (M' \otimes_R N)$  by  $\psi = ((x, x'), y) \mapsto (x \otimes y, x' \otimes y)$   
is bilinear, hence exists  $R$ -mod homomorphism  $\varphi : (M \oplus M') \otimes_R N \rightarrow (M \otimes_R N) \oplus (M' \otimes_R N)$ .

On the other hand, The mapping  $(x, y) : M \times N \mapsto (x, 0) \otimes y : (M \oplus M') \otimes_R N$  is bilinear. So  
exists  $\phi_1 : M \otimes_R N \rightarrow (M \oplus M') \otimes_R N$ , similarly there exists  $\phi_2 : M' \otimes_R N \rightarrow (M \oplus M') \otimes_R N$ .  
Now by the universal property of direct sum, there exists  $\phi : (M \otimes_R N) \oplus (M' \otimes_R N) \rightarrow$   
 $(M \oplus M') \otimes_R N$ . After a careful examine, we have

$$\varphi = (x, x') \otimes y \mapsto (x \otimes y, x' \otimes y), \phi = (x \otimes y, x' \otimes y) \mapsto (x, x') \otimes y$$

Thus  $\phi = \varphi^{-1}$  and hence the two are isomorphic.

**Ex 2.1.5.**

1.  $R \otimes_R M \cong M$ .
2.  $M \otimes_R N \cong N \otimes_R M$ .

**Ex 2.1.6.**  $R/I \otimes_R N \cong N/IN$  where  $IN := \{\sum a_i x_i \mid a_i \in I, x_i \in N\}$ .

**Ex 2.1.7.** Compute  $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}), \dim_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}), \dim_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}), \dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$



## 2.2 Week 12

### 2.2.1 Tensor product II

By universal property, we get  $\{R\text{-bilinear maps } M \times N \rightarrow L\} \leftrightarrow \text{Hom}_R(M \otimes_R N, L)$ .

Similarly,

$$\begin{aligned}\text{Hom}\left(\bigoplus_{s \in \Lambda} M_s, L\right) &\cong \prod_{s \in \Lambda} \text{Hom}(M_s, L) \\ \text{Hom}\left(N, \prod_{s \in \Lambda} M_s\right) &\cong \prod_{s \in \Lambda} \text{Hom}(N, M_s)\end{aligned}$$

**Fact 2.2.1.**  $f \in \text{Hom}_R(M, M'), g \in \text{Hom}_R(N, N') \rightsquigarrow f \otimes g \in \text{Hom}_R(M \otimes N, M' \otimes N')$  by  $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$ .

*Proof.* Define  $h : M \times N \rightarrow M' \otimes_R N'$   
 $(x, y) \mapsto f(x) \otimes g(y)$  □

Restriction and extension of scalars.

Let  $f : R \rightarrow S$  be a ring homomorphism and  $R, S$  be commutative with 1. Then  $S$  can be regarded as an  $R$ -module.  $\left( \begin{array}{c} R \times S \rightarrow S \\ (r, x) \mapsto f(r)x \end{array} \right)$ .

If  $M$  is a  $S$ -module, then  $M$  is also an  $R$ -module.  $\left( \begin{array}{c} R \times M \rightarrow M \\ (r, a) \mapsto f(r)a \end{array} \right)$ .

If  $N$  is an  $R$ -module, then  $S \otimes_R N$  an  $S$ -module.  $\left( \begin{array}{c} S \times (S \otimes_R N) \rightarrow S \otimes_R N \\ (r, x \otimes a) \mapsto rx \otimes a \end{array} \right)$ .

**Eg 2.2.1** (Important example). Let  $V$  be a real vector space. The complexification of  $V$  is  $V^{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V$  which is a  $\mathbb{C}$ -vector space.

**Ex 2.2.1.** Let  $K \subseteq L$  be an inclusion of fields and let  $E$  be a vector space over  $K$ . Show that  $E^L := L \otimes_K E$  satisfies the following universal property: For any vector space  $U$  over  $L$  and any  $K$ -linear map  $f : E \rightarrow U$ ,  $\exists!$   $L$ -linear map  $\varphi$ :

$$\begin{array}{ccc} \varphi : 1 \otimes x :: E^L & \xrightarrow{\quad} & f(x) :: U \\ & \nwarrow \quad \nearrow f & \\ & x :: E & \end{array}$$

**Ex 2.2.2.**  $E \rightarrow E^L$  is a covariant functor from the category of vector spaces over  $K$  to the category of vector spaces over  $L$ .

**Eg 2.2.2.**  $\mathbb{Z}^n \cong \mathbb{Z}^m \rightsquigarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^m \rightsquigarrow n = m$ .

**Eg 2.2.3.**  $G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_l\mathbb{Z} \oplus \mathbb{Z}^s, \mathbb{Q} \otimes_{\mathbb{Z}} G = \mathbb{Q}^s$ .

Let  $M, N$  and  $U$  be  $R$ -module. Then

$$\text{Hom}_R(M \otimes_R N, U) \cong \text{Hom}_R(N, \text{Hom}_R(M, U))$$

*Proof.*

- For  $f \in \text{Hom}_R(M \otimes_R N, U)$  and  $a \in N$ , define  $f_a = x :: M \mapsto f(x \otimes a) :: U$ .
  - linear: easy.
  - $\bar{f} : a \mapsto f_a$  is an  $R$ -mod homo.: easy.
  - $\tau : f \mapsto \bar{f}$  is an  $R$ -mod homo.:  $\tau(rf + g)(a)(x) = (rf + g)_a(x) = (rf + g)(x \otimes a) = rf(x \otimes a) + g(x \otimes a) = \dots = r\tau(f)(a)(x) + \tau(g)(a)(x)$
- For  $g \in \text{Hom}_R(N, \text{Hom}_R(M, U))$ , define  $g' = (x, a) :: M \times N \mapsto g(a)(x) :: U$ .
  - $g'$  is  $R$ -bilinear: easy.
  - $\exists! \tilde{g} : x \otimes a \mapsto g(a)(x)$ .
  - $\sigma : g \mapsto \tilde{g}$  is an  $R$ -mod homo.: easy.
- $\sigma\tau = \text{id}, \tau\sigma = \text{id}$ : easy... □

**Ex 2.2.3.**  $\text{Hom}_R(M, \cdot), M \otimes_R \cdot$  are covariant functors from the category of  $R$ -modules to itself. (is an adjoint pair)

**Fact 2.2.2.**  $\text{Hom}_R(R, M) \cong M$ . By  $f \mapsto f(1)$ .

**Def 51.** An exact sequence  $A \xrightarrow{f_1} B \xrightarrow{f_2} \dots$  is a sequence satisfied  $\text{im } f_k = \ker f_{k+1}$ .

- $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ .
- $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ .

Let  $V, W$  be vector spaces over  $F$ . Then  $V^* \otimes_F W \cong \text{Hom}_F(V, W)$ .

*Proof.* Let  $\alpha = \{e_1, \dots, e_n\}$  and  $\beta = \{f_1, \dots, f_m\}$  be bases for  $V$  and  $W$  respectively. Via  $\alpha, \beta$ ,  $\text{Hom}_F(V, W) \cong \left\langle E_{ij} \left| \begin{matrix} i = 1, \dots, m \\ j = 1, \dots, n \end{matrix} \right. \right\rangle_F$ .  $V^* \otimes W \cong \left\langle e_j^* \otimes f_i \left| \begin{matrix} i = 1, \dots, m \\ j = 1, \dots, n \end{matrix} \right. \right\rangle_F$ . □

## 2.2.2 Tensor algebra

**Def 52.**

- Let  $R$  be a commutative ring with 1. An  $R$ -algebra is a ring  $A$  which is also an  $R$ -module s.t. the multiplication map  $A \times A \rightarrow A$  is  $R$ -bilinear. (  $r(ab) = (ra)b = a(rb)$  )
- Let  $A$  be an  $R$ -algebra. A grading of  $A$  is a collection of  $R$ -submodules  $\{A_n\}_{n=0}^\infty$  ( $n$ -th homogeneous part) s.t.

$$A = \bigoplus_{n=0}^{\infty} A_n \quad \text{and} \quad A_n A_m \subseteq A_{n+m} \quad \forall n, m$$

- A graded  $R$ -algebra is an  $R$ -algebra with a chosen grading.
- $\mathfrak{M}_R$  is the category of  $R$ -modules.
- $\mathfrak{Gr}_R$  is the category of graded  $R$ -algebras. ( $f : A \rightarrow A'$  with  $f(A_n) \subseteq A'_n$ )

**Ex 2.2.4.**  $A = R[x], A_n = \langle x^n \rangle_R$ . If  $I = \langle x + 1 \rangle_A$ ,  $I$  is not graded.  $I = \langle x^2 \rangle_A$  is graded.

**Def 53.** An ideal  $I$  is graded in a graded ring  $A$  if and only if  $I = \bigoplus I \cap A_n$ .<sup>1</sup>

---

<sup>1</sup>This is not mentioned in class

**Ex 2.2.4.** TFAE

- (1)  $I$  is graded.
- (2)  $\forall a \in I$  write  $a = a_{k_1} + a_{k_2} + \cdots + a_{k_m}, a_{k_i} \in A_{k_i} \implies a_{k_i} \in I$ . ( $a_{k_i}$  is the homogeneous component of  $a$ )
- (3)  $A/I$  is a graded ring with  $(A/I)_n = (A_n + I)/I \cong A_n/I \cap A_n$ .

**Ex 2.2.5.**

- (1) If  $I$  is a f.g. graded ideal, then  $I$  has a finite system of generators consisting of homogeneous elements alone.
- (2)  $I, J$  are graded  $\implies I + J, IJ, I \cap J$  are graded.

Observation: Let  $\{M_i\}_{i=1}^\infty$  be a collection of  $R$ -modules.

- $M_1 \otimes_R M_2$  exists.
- $(M_1 \otimes_R M_2) \otimes_R M_3 \cong M_1 \otimes_R (M_2 \otimes_R M_3) \implies M_1 \otimes_R M_2 \otimes_R M_3$  is well-defined. Universal property: for any  $R$ -module  $L$  and a 3-multilinear map  $f : M_1 \times M_2 \times M_3 \rightarrow L$ . (拆括號囉)
- By induction,  $M_1 \otimes \cdots \otimes M_n$  is well-defined and satisfies the universal property. ( $n$ -multilinear map)

Goal: For a given  $R$ -module  $M$ , we intend to construct an graded  $R$ -algebra  $T(M)$  containing  $M$  that is “universal” w.r.t.  $R$ -algebras containing  $M$ .

That is, a tensor algebra is a pair  $(T(M), i)$  where  $T(M)$  is an  $R$ -algebra and  $i :: M \rightarrow T(M)$ , such that for any  $R$ -algebra  $A$  containing  $M$ , which is to say that exist a  $R$ -module homomorphism  $\varphi : M \rightarrow A$ , then exists an  $R$ -algebra homomorphism  $\psi :: T(M) \rightarrow A$  such that  $\varphi = \psi \circ i$ .

Construction:

- $\forall k \in \mathbb{N}, T^k(M) := \underbrace{M \otimes \cdots \otimes M}_{k \text{ times}}$ , each  $x_1 \otimes x_2 \otimes \cdots \otimes x_k \in T^k(M)$  is called a  $k$ -tensor.

$T^0(M) := R$  and

$$T(M) := \bigoplus_{k=0}^{\infty} T^k(M) = R \oplus T^1(M) \oplus \cdots$$

- define multiplication on  $T(M)$  by:

$$\begin{aligned} T^i(M) \times T^j(M) &\longrightarrow T^{i+j}(M) \\ (x_1 \otimes \cdots \otimes x_i, y_1 \otimes \cdots \otimes y_j) &\longmapsto x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j \end{aligned}$$

- Distribution law: easy.

Proving the universal property: For any  $R$ -algebra  $A$  containing  $M$  and an  $R$ -module homo.  $\varphi : M \rightarrow A$ .  $\forall k \geq 2$ , we define  $f_k : M \times \cdots \times M \rightarrow A$

$$\begin{aligned} f_k : M \times \cdots \times M &\rightarrow A \\ (x_1, \dots, x_k) &\mapsto \varphi(x_1) \cdots \varphi(x_k) \end{aligned}$$

$f_k$  is  $k$ -multilinear  $\rightsquigarrow$

$$\begin{aligned} \exists! \tilde{f}_k : M \otimes \cdots \otimes M &\rightarrow A \\ x_1 \otimes \cdots \otimes x_k &\mapsto \varphi(x_1) \cdots \varphi(x_k) \end{aligned}$$

By the universal property of  $\bigoplus$ , exists a unique  $R$ -module homo.  $\tilde{\varphi} :: T(M) \rightarrow A$  which make the following diagram commutes.

$$\begin{array}{ccc} \tilde{\varphi} : T(M) & \xrightarrow{\quad} & A \\ & \nwarrow i \quad \nearrow f_k & \\ & T^k(M) & \end{array}$$

$\tilde{\varphi}$  is an  $R$ -algebra homomorphism.

**Def 54.**  $T(M)$  is called the tensor algebra of  $M$ .

**Ex 2.2.6.**  $T$  is a covariant functor from  $\mathfrak{M}_R$  to  $\mathfrak{Gr}_R$ .

**Prop 2.2.1.** Let  $V$  be a vector space over  $F$  with a basis  $\beta = \{v_1, \dots, v_n\}$ . Then

$$\{v_{i_1} \otimes \dots \otimes v_{i_k} \mid \forall j = 1, \dots, k, i_j = 1, \dots, n\}$$

forms a basis for  $T^k(V)$ .  $\dim_F T^k(V) = n^k$ .

$T(V)$  can be regarded as a non-commutative polynomial algebra over  $F$ .

⊙ Symmetrization ( $\text{char } F = 0$ )

$$\begin{aligned} V \times \dots \times V &\longrightarrow T^n(V) \\ (x_1, \dots, x_n) &\longmapsto \frac{1}{n!} \sum_{\tau \in S_n} x_{\tau(1)} \otimes \dots \otimes x_{\tau(n)} \end{aligned}$$

is  $n$ -multilinear.

The symmetrizer operator  $\sigma : T^n(V) \rightarrow T^n(V)$ ,  $\tilde{S}^n(V) := \sigma(T^n(V)) \subseteq T^n(V)$ .

Claim:  $T^n(V) = \tilde{S}^n(V) \oplus C^n(V)$  where

$$C^n(V) = C(V) \cap T^n(V) \quad C(V) = \langle v \otimes w - w \otimes v \mid v, w \in V \rangle$$

## 2.3 Week 13

### 2.3.1 Symmetric and Exterior algebra

**Symmetric algebra** Define

$$\begin{aligned} S : \mathfrak{M}_R &\rightarrow \mathfrak{Gr}_R \\ M &\mapsto T(M)/C(M) \end{aligned} \quad S(M) := T(M)/C(M)$$

where  $C(M)$  is the graded two-sided ideal generated by  $u \otimes v - v \otimes u$  with  $u, v \in M$ .

- $C^k(M) := C(M) \cap T^k(M)$  is the submodule of  $T^k(M)$  generated by all

$$x_1 \otimes \dots \otimes x_k - x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)} \quad \forall x_i \in M, \sigma \in S_k.$$

“ $\subseteq$ ”:  $x_1 \otimes \dots \otimes x_s \otimes (u \otimes v - v \otimes u) \otimes y_1 \otimes \dots \otimes y_t \in C(M) \cap T^k(M)$  with  $s + 2 + t = k$ .

“ $\supseteq$ ”: bubble sort

- $k \geq 2, S^k(M) = T^k(M)/C^k(M) = \langle \bar{x}_1 \otimes \dots \otimes \bar{x}_k \mid x_i \in M \rangle_R$  with  $\bar{x}_1 \otimes \dots \otimes \bar{x}_k = \bar{x}_{\sigma(1)} \otimes \dots \otimes \bar{x}_{\sigma(k)} \quad \forall \sigma \in S_k$

Hence,  $S(M) = \bigoplus_{k=0}^{\infty} S^k(M)$  is a graded commutative  $R$ -algebra.

**Def 55.**  $f : M \times \dots \times M \rightarrow L$  is a symmetric  $k$ -multilinear map if  $f$  is  $k$ -multilinear and

$$f(x_1, \dots, x_k) = f(x_{\sigma(1)}, \dots, x_{\sigma(k)}) \quad \forall \sigma \in S_k$$

- $k \geq 2, S^k(M)$  is universal w.r.t. symmetric  $k$ -multilinear maps on  $M$ : By the universal property of  $T^k(M)$ ,  $\exists!$   $R$ -module homo.  $\tilde{f} : T^k(M) \rightarrow L$ . Now  $C^k(M) \subseteq \ker \tilde{f} \implies \exists!$   $R$ -module homo.  $\bar{f} : S^k(M) \rightarrow L$  by factor thm.
- $S(M)$  satisfies the universal property for maps to a commutative  $R$ -algebra: given a commutative  $R$ -algebra  $A$  and  $f : M \rightarrow A$   $R$ -module homo.,

$$\begin{array}{ccc} M & \xrightarrow{f} & A \\ \downarrow & \nearrow \exists! f' & \uparrow \\ T(M) & \longrightarrow & T(M)/C(M) \end{array}$$

- $S : \mathfrak{M}_R \rightarrow \mathfrak{Gr}_R$  is a covariant functor.

$$- \varphi : M \rightarrow N: R\text{-module homo.} \rightsquigarrow T(\varphi) : T(M) \rightarrow T(N) \rightarrow T(N)/C(N) = S(N)$$

**Ex 2.3.1.** Let  $E$  be a vector space over  $F$  with  $\dim E = n$ .

1. Show that  $S(E) \cong F[x_1, \dots, x_n]$ .
2. Compute  $\dim_F S^k(E)$ .

**Exterior algebra** ( $\text{char } R \neq 2$ )

$$\begin{aligned} \Lambda : \mathfrak{M}_R &\rightarrow \mathfrak{Gr}_R \\ M &\mapsto \Lambda(M) = T(M)/A(M) \end{aligned}$$

where  $A(M)$  is the two sided graded generated by  $v \otimes v \quad \forall v \in M$ .

- $A^k(M) := A(M) \cap T^k(M)$  is the submodule of  $T^k(M)$  generated by all  $x_1 \otimes \dots \otimes x_k$  with  $x_i = x_j$  for some  $i \neq j$ .  
(Note:  $(x_1 + x_2) \otimes (x_1 + x_2) = x_1 \otimes x_1 + x_1 \otimes x_2 + x_2 \otimes x_1 + x_2 \otimes x_2 \rightsquigarrow x_1 \otimes x_2 + x_2 \otimes x_1 \in A(M)$ )

- $\Lambda^k(M) \cong T^k(M)/A^k(M) = \langle \overline{x_1 \otimes \dots \otimes x_k} \mid x_i \in M \rangle$  with  $\overline{x_1 \otimes \dots \otimes x_k} = \bar{0}$  if  $x_i = x_j$  for some  $i \neq j$ . We use  $x_1 \wedge \dots \wedge x_k := \overline{x_1 \otimes \dots \otimes x_k}$ .

Note:  $x_1 \wedge x_2 = -x_2 \wedge x_1$ .

**Def 56.**  $f : M \times \dots \times M \rightarrow L$  is an alternating  $k$ -multilinear map if  $f$  is  $k$ -multilinear and  $f(x_1, \dots, x_k) = 0$  when  $x_i = x_j$  for some  $i \neq j$ .

- $k \geq 2$ ,  $\Lambda^k(M)$  is universal w.r.t. alternating  $k$ -multilinear maps on  $M$ :

$$\begin{array}{ccc} M \times \dots \times M & \xrightarrow{\quad} & L \\ \downarrow & \nearrow \exists! f' & \uparrow \\ T^k(M) & \xrightarrow{\quad} & \Lambda^k(M) \end{array}$$

- $\Lambda(M)$  satisfies the universal property for maps to an  $R$ -algebra  $A$  with  $a^2 = 0 \quad \forall a \in A$ : given an  $R$ -algebra  $A$  and  $f : M \rightarrow A$   $R$ -module homo.,

$$\begin{array}{ccc} M & \xrightarrow{f} & A \\ \downarrow & \nearrow \exists! f' & \uparrow \\ T(M) & \xrightarrow{\quad} & \Lambda(M) \end{array}$$

- $\Lambda : \mathfrak{M}_R \rightarrow \mathfrak{Gr}_R$  is a covariant functor.

$$- \varphi : M \rightarrow N: R\text{-module homo.} \rightsquigarrow T(\varphi) : T(M) \rightarrow T(N) \rightarrow T(N)/A(N) = \Lambda(N)$$

**Ex 2.3.2.** Let  $V$  be a vector space over  $F$  with  $\dim V = n$  and  $\varphi : V \rightarrow V$  be a linear transformation.

- (1) Compute  $\Lambda^k(V)$ .
- (2) Determine the map  $\Lambda^k(\varphi) : \Lambda^k(V) \rightarrow \Lambda^k(V)$ .

### Symmetrization and Skew-symmetrization

$$T^k(V) \xrightarrow{\quad} T^k(V)$$

$$\text{Sym} = \sigma : x_1 \otimes \dots \otimes x_k \longmapsto \frac{1}{k!} \sum_{\tau \in S_k} x_{\tau(1)} \otimes \dots \otimes x_{\tau(k)}$$

$$\text{Alt} = \sigma' : x_1 \otimes \dots \otimes x_k \longmapsto \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) x_{\tau(1)} \otimes \dots \otimes x_{\tau(k)}$$

$$\tilde{S}^k(V) = \sigma(T^k(V)) \quad \tilde{\Lambda}^k(V) = \sigma'(T^k(V))$$

- $\sigma^2 = \sigma$  easy  $\rightsquigarrow T^k(V) = \text{Im } \sigma \oplus \ker \sigma = \tilde{S}^k(V) \oplus \ker \sigma$ .
- $\ker \sigma = C^k(V)$ .  $C^k(V) \subseteq \ker \sigma$  is obvious. Assume  $\supsetneq$ , i.e.,  $\exists t \in \ker \sigma$  s.t.  $t \notin C^k(V)$ . Recall  $q : T^k(V) \twoheadrightarrow S^k(V)$ , since  $q$  is the quotient map. Also  $q|_{\tilde{S}^k(V)} \twoheadrightarrow S^k(V)$ , since if  $q(x) = y$ , then it could be easily checked that  $q(\sigma(x)) = y$ , so exists  $t' \in \tilde{S}^k(V)$  satisfies  $q(t') = q(t) \neq 0$ . But then  $q(t - t') = 0 \implies t - t' \in \ker q = C^k(V) \subseteq \ker \sigma$  and because of  $\sigma(t) = 0 \implies \sigma(t') = 0$ . Hence  $t' \in \ker \sigma$ . But then  $t' \in S^k(V) \subseteq \text{Im } \sigma \implies t' \in \text{Im } \sigma \cap \ker \sigma$ , which leads to an contradiction since  $\sigma$  is a projection.

**Ex 2.3.3.**  $T^k(V) = \tilde{\Lambda}^k(V) \oplus A^k(V)$ .

### 3 Introduction to the linear representation theory of finite groups

#### 3.1 Week 14

##### 3.1.1 Generatlities on linear representations

###### Notation

- $G$ : finite group
- $V$ : vector space of finite dim over  $\mathbb{C}$
- $\text{GL}(V)$ : the group of all linear isom.  $V \rightarrow V$

**Def 57.** A group homo.  $\rho : G \rightarrow \text{GL}(V)$  is called a linear representation of  $G$ .  $\dim V$  is called the degree of  $\rho$ . ( $V$  is a representation space)

For a fixed basis  $\beta = \{e_i\}$ ,

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \text{GL}(V) \\ & \searrow R & \downarrow \beta \\ & & \text{GL}_n(\mathbb{C}) \end{array}$$

( $R$  is a matrix representation)

**Eg 3.1.1.** A representation of degree 1 of  $G$  is  $\rho : G \rightarrow \text{GL}(\mathbb{C}) \cong \mathbb{C}^*$ .

$\text{ord}(g)$  is finite  $\rightsquigarrow \rho(g)^m = 1$  for some  $m \in \mathbb{N} \rightsquigarrow \rho(g)$  is a root of unity, i.e.  $|\rho(g)| = 1$ .

Note: So,  $\rho : G \rightarrow S^1$ ,  $S^1$  is the unit circle.

1.  $G = \mathbb{Z}/p\mathbb{Z}$ ,  $\rho : \bar{1} :: G \mapsto \zeta_p :: S^1$  with  $\zeta_p^p = 1$ .

2.  $G = S_3$ ,  $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$ .

A permutation representation is  $\rho : \tau :: S_3 \mapsto (\rho(\tau) : e_i \mapsto e_{\tau(i)}) :: \text{GL}(V)$ .

3.  $G = S_3$ ,  $V = \bigoplus_{\sigma \in S_3} \mathbb{C}e_\sigma$ . The regular representation is

$$\rho^{\text{reg}} : \tau :: G \mapsto (\rho^{\text{reg}}(\tau) : e_\sigma \mapsto e_{\tau\sigma}) :: \text{GL}(V).$$

For general  $G$ , with  $V = \bigoplus_{g \in G} \mathbb{C}e_g$ ,

$$\rho^{\text{reg}} : h :: G \mapsto (\rho^{\text{reg}}(h) : e_g \mapsto e_{hg}) :: \text{GL}(V).$$

###### Def 58.

- $\rho : g :: G \mapsto \text{id} :: \text{GL}(V)$ : trivial representation.
- $\rho : G \hookrightarrow \text{GL}(V)$ : faithful representation.
- $\rho, \rho'$  are said to be equivalent if  $\exists$  a linear isom.  $T : V \xrightarrow{\sim} V'$  s.t.

$$\begin{array}{ccc} V & \xrightarrow[\sim]{T} & V' \\ \rho(g) \downarrow & & \downarrow \rho'(g) \\ V & \xrightarrow[\sim]{T} & V' \end{array}$$

**Remark 13.** When we choose two bases  $\beta, \beta'$  for  $V$ ,

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \text{GL}(V) \\ & \searrow R & \downarrow \beta \wr \\ & & \text{GL}_n(\mathbb{C}) \end{array} \quad \begin{array}{ccc} G & \xrightarrow{\rho'} & \text{GL}(V) \\ & \searrow R & \downarrow \beta' \wr \\ & & \text{GL}_n(\mathbb{C}) \end{array}$$

then  $\rho, \rho'$  are equivalent.

Let  $T : e_i :: V \mapsto e'_i :: V$ . For  $g \in G, R(g) = (a_{ij})$ .

$$T \circ \rho(g) = \rho'(g) \circ T$$

**Def 59.** Let  $\langle \cdot, \cdot \rangle$  be a positive definite Hermitian form on  $V$ .

Then  $T : V \rightarrow V$  is called a unitary operator if  $\langle T(x), T(y) \rangle = \langle x, y \rangle \quad \forall x, y \in V$ .

or  $\forall \beta$  : orthonormal basis,  $[T]_{\beta}^* [T]_{\beta} = [T]_{\beta} [T]_{\beta}^* = I_n$ .

**Theorem 29.**  $\forall \rho : G \rightarrow \text{GL}(V), \exists$  a matrix representation  $R : G \rightarrow U_n$ .

*Proof.* We only need to  $G$ -invariant positive definite Hermitian form on  $V$ . ( $\forall g \in G, \langle \rho(g)x, \rho(g)y \rangle = \langle x, y \rangle \quad \forall x, y \in V$ )

We start with an arbitrary positive definite Hermitian form  $\langle \cdot, \cdot \rangle'$  on  $V$ .

Define a new form  $\langle \cdot, \cdot \rangle$  by

$$\langle x, y \rangle := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle'$$

which is a positive definite Hermitian form, since

$$\begin{aligned} \langle \rho(g)x, \rho(g)y \rangle &\triangleq \frac{1}{|G|} \sum_{h \in G} \langle (\rho(h) \circ \rho(g))(x), (\rho(h) \circ \rho(g))(y) \rangle' \\ &= \frac{1}{|G|} \sum_{gh \triangleq h' \in G} \langle (\rho(h'))(x), (\rho(h'))(y) \rangle' \triangleq \langle x, y \rangle \end{aligned}$$

So with the basis of this hermitian form, every  $\rho(g)$  has a matrix representation  $R(g)$  which is unitary.  $\square$

**Def 60.** Let  $\rho : G \rightarrow \text{GL}(V)$ , For  $W \subset V$  (we use  $\subset$  to denote subspace), if  $\forall x \in W, \rho(g)(x) \in W, \forall g \in G$ , then  $W$  is said to be  $G$ -invariant and

$$\begin{aligned} \rho^W : G &\rightarrow \text{GL}(W) \\ g &\mapsto \rho(g)|_W \end{aligned}$$

is called a subrepresentation of  $\rho$ .

$W$  is  $G$ -invariant  $\rightsquigarrow \rho(g)|_W : W \xrightarrow{\sim} W$ .

**Eg 3.1.2.** Let  $\rho$  be the regular rep. of  $S_3$ .

$W^\circ = \{ \alpha_1 e_1 + \cdots + \alpha_6 e_6 \mid \alpha_1 + \cdots + \alpha_6 = 0 \}$  is  $G$ -invariant.

$W^1 = \langle e_1 + \cdots + e_6 \rangle_{\mathbb{C}}$  is  $G$ -invariant.

**Theorem 30.** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $W \subset V$  be  $G$ -invariant. Then  $\exists W^\circ \subset V$  is still  $G$ -invariant and  $V = W \oplus W^\circ$ .



*Proof.* We can pick an arbitrary  $W'$  with  $V = W \oplus W'$  and  $\pi_1 : V \rightarrow W$  is the projection to  $W$ . Then  $W' = \ker \pi_1$ .

Now we need  $\pi_1$  preserves the  $G$  action ( $G$ -equivariant). Define

$$\pi^\circ = \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} \circ \pi_1 \circ \rho(g) : V \rightarrow W$$

- well-defined:  $\rho(g)(V) \subset V \rightsquigarrow \pi_1 \circ \rho(g)(V) \subset W \rightsquigarrow \rho(g)^{-1} \circ \pi_1 \circ \rho(g)(V) \subseteq W$ .
- surjective:  $\forall y \in W, (\rho(g)^{-1} \circ \pi_1 \circ \rho(g))(y) = (\rho(g)^{-1} \circ \rho(g))(y) = y$  since  $\rho(g)(y) \in W$ . Also,  $\pi^\circ(y) = y, \forall y \in W \implies (\pi^\circ)^2 = \pi^\circ$ . So  $\pi^\circ$  is a projection and hence  $V = \text{Im } \pi^\circ \oplus \ker \pi^\circ$ .
- $G$ -equivariant:  $\forall g' \in G$ ,

$$\begin{aligned} \pi^\circ \circ \rho(g')(x) &= \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} \circ \pi_1 \circ \rho(g)(\rho(g')(x)) \\ &= \rho(g') \frac{1}{|G|} \sum_{gg' \in G} \rho(gg')^{-1} \circ \pi_1 \circ \rho(gg')(x) \\ &= (\rho(g') \circ \pi^\circ)(x) \end{aligned}$$

- $W^\circ := \ker \pi^\circ$  is  $G$ -invariant:  $\forall x \in W^\circ, \pi^\circ(\rho(g)(x)) = \rho(g)(\pi^\circ(x)) = \rho(g)(0) = 0$ . So  $\rho(g)(x) \in W^\circ$ .

$$\begin{array}{ccc} V & \xrightarrow{\pi^\circ} & W \\ \rho(g) \downarrow & & \downarrow \rho(g) \\ V & \xrightarrow{\pi^\circ} & W \end{array}$$

□

**Remark 14.** If  $W \subset V$  is  $G$ -invariant, then  $W^\perp$  is also  $G$ -invariant. (w.r.t. a  $G$ -invariant positive definite Hermitian form)

**Def 61.**  $\rho : G \rightarrow \text{GL}(V)$  is irreducible if  $\rho$  has no proper nontrivial subrepresentations.

**Theorem 31.** Each  $\rho : G \rightarrow \text{GL}(V)$  is a direct sum of irreducible subrepresentations.

*Proof.* By induction on  $\dim V$ . For  $\dim V = 1$ , then  $\rho$  is irreducible.

For  $\dim V > 1$ , if  $\rho$  is irreducible, then done. Otherwise,  $\exists W, W^\circ$  are  $G$ -invariant s.t.  $V = W \oplus W^\circ$  with  $\dim W \geq 1, \dim W^\circ \geq 1$ . By induction hypothesis,  $\rho^W, \rho^{W^\circ}$  are the direct sum of irreducible subrepresentations, and  $\rho = \rho^W \oplus \rho^{W^\circ}$ , done. □

**Remark 15.** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\rho' : G \rightarrow \text{GL}(V')$ .

- $\rho \oplus \rho' : G \rightarrow \text{GL}(V \oplus V')$ . 矩陣是左上右下
- $\rho \otimes \rho' : G \rightarrow \text{GL}(V \otimes V')$ . 矩陣是密密麻麻  $(\sum_{i,j} r_{ip} r'_{jq} (e_i \otimes e'_j))$

### 3.1.2 Character Theory I

Main goal: To determine all equivalence classes of irreducible representations of a finite group  $G$ .

**Def 62.**

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \text{GL}(V) \\ & \searrow R & \downarrow \wr \\ & & \text{GL}_n(\mathbb{C}) \end{array}$$

The character  $\chi_\rho$  if  $\rho$  is the map  $\chi_\rho : G \rightarrow \mathbb{C}$  defined by  $\chi_\rho(g) = \text{Tr}(R(g))$ .

**Remark 16.**

1.  $\chi_\rho$  is independent of the choice of  $\beta = \{e_i\}$  For another basis  $\beta' = \{e'_i\}$ . (Notice that  $\text{Tr}(BA) = \text{Tr}(AB)$ )
2.  $\rho \xrightarrow[\text{equivalent}]{\cong} \rho' \rightsquigarrow \chi_\rho = \chi_{\rho'}$ .

**Def 63.**

- The degree of  $\chi_\rho$  is defined to the degree of  $\rho$  ( $= \dim V$ ).
- $\chi_\rho$  is an irreducible character if  $\rho$  is irreducible.

Basic facts:

1.  $\chi_\rho(1) = n$ .
2.  $\chi_\rho$  is a class function, i.e., it is constant on each conjugacy class.
3.  $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$ : Assume that the eigenvalues of  $R(g)$  are  $\lambda_1, \dots, \lambda_n$ . Then the eigenvalues of  $R(g^{-1})$  are  $\lambda_1^{-1}, \dots, \lambda_n^{-1}$ .  

$$0 = \det(\lambda I_n - A) = \det(\lambda I_n(A^{-1} - \lambda^{-1}I_n)A) = \det(\lambda I_n) \det(A^{-1} - \lambda^{-1}I_n) \det(A)$$

So  $\det(A^{-1} - \lambda^{-1}I_n) = 0$ . Then  $g^m = 1 \implies R(g)^m = I_n \implies |\lambda_i| = 1 \implies \lambda_i^{-1} = \overline{\lambda_i}$ .  
Thus  $\chi_\rho(g^{-1}) = \text{Tr}(R(g)^{-1}) = \overline{\lambda_1 + \dots + \lambda_n} = \overline{\chi_\rho(g)}$ .
4.  $\chi_{\rho \oplus \rho'} = \chi_\rho + \chi_{\rho'}$ .
5.  $\chi_{\rho \otimes \rho'} = \chi_\rho \chi_{\rho'}$ .

**Def 64.**  $\mathcal{C}(G, \mathbb{C})$  is the vector space of complex functions on  $G$ .

$\chi_\rho \in \mathcal{C}(G) \subset \mathcal{C}(G, \mathbb{C})$  is the vector space of complex class functions of  $G$ .

**Remark 17.** Assume that  $\{C_1, \dots, C_k\}$  is the set of distinct conjugacy classes in  $G$ . Then  $\{f_i(C_j) = \delta_{ij} \mid \forall i = 1, \dots, k\}$  forms a basis for  $\mathcal{C}(G)$  over  $\mathbb{C}$ .

- $\forall f \in \mathcal{C}(G)$ , let  $f(C_i) = a_i$ , then  $f = \sum a_i f_i$ .
- $\sum a_i f_i = 0$ , pick  $x_j \in C_j$ , then  $(\sum a_i f_i)(x_j) = a_j = 0 \quad \forall j = 1, \dots, k$ .

So  $\dim \mathcal{C}(G) = k$ .

**Def 65.**  $\phi, \psi \in \mathcal{C}(G, \mathbb{C})$ , then

$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}$$

is a positive definite Hermitian form on  $\mathcal{C}(G, \mathbb{C})$ .

**Theorem 32** (Main theorem). The set of all irreducible characters of  $G$  forms an orthonormal basis for  $\mathcal{C}(G)$  over  $\mathbb{C}$ . So there are only  $k$  irreducible representations up to equivalent.

**Lemma 3** (Schur's lemma). Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\rho' : G \rightarrow \text{GL}(V')$  be two irr. rep. of  $G$ .

$$\begin{array}{ccc} V & \xrightarrow{\quad \text{T} \quad} & V' \\ \rho(g) \downarrow & & \downarrow \rho'(g) \\ V & \xrightarrow{\quad \text{T} \quad} & V' \end{array} \quad (\text{T} : G\text{-equivariant})$$

Then

1.  $\rho, \rho'$  are not equivalent  $\implies \mathbf{T} = 0$ .
2.  $V = V', \rho = \rho' \implies \mathbf{T} = \lambda 1_V$  for some  $\lambda \in \mathbb{C}$ .

*Proof.*

1. Assume  $\mathbf{T} \neq 0$ . We only need to prove that  $\mathbf{T}$  is an isomorphism, and then  $\rho, \rho'$  would be isomorphic by definition. Since  $\mathbf{T}$  is  $G$ -equivariant,  $\ker \mathbf{T} \leq V$  and  $\text{Im } \mathbf{T} \leq V'$  are  $G$ -invariant.  $\rho$  is irreducible  $\implies \ker \mathbf{T} = 0$  or  $V$ , but if  $\ker \mathbf{T} = V$  then  $\mathbf{T} = 0$ , so  $\ker \mathbf{T} = 0$ . Similarly,  $\rho'$  is irreducible  $\implies \text{Im } \mathbf{T} = 0$  or  $V$ . And by the fact that  $\mathbf{T} \neq 0$ ,  $\text{Im } \mathbf{T} = V$ . Thus  $\mathbf{T}$  is an isom, and consequently  $\rho, \rho'$  are equivalent.
2. Since the vector field is over  $\mathbb{C}$ ,  $\mathbf{T}$  has an eigenvalue. Let  $\lambda$  be an eigenvalue of  $\mathbf{T}$ , say  $\mathbf{T}(v) = \lambda v$  with  $v \neq 0$  in  $V$ . Put  $\mathbf{T}' = \mathbf{T} - \lambda 1_V$ . Then

$$\rho(g) \circ \mathbf{T}' = \rho(g) \circ (\mathbf{T} - \lambda 1_V) \stackrel{*}{=} \rho(g) \circ \mathbf{T} - \rho(g) \circ \lambda 1_V = \mathbf{T} \circ \rho(g) - \lambda 1_V \rho(g) = \mathbf{T}' \rho(g)$$

Which  $*$  is due to the linearity of  $\rho(g)$ . Hence  $\mathbf{T}'$  is also  $G$ -equivariant.

But  $v \in \ker \mathbf{T}'$ , i.e.,  $\mathbf{T}'$  is not 1-1. By 1.,  $\mathbf{T}' = 0 \implies \mathbf{T} = \lambda 1_V$ .

□

**Coro 3.1.1.** Assume  $\rho, \rho'$  is the same as above. Let  $\mathbf{L} : V \rightarrow V'$  be a linear transformation. Define

$$\mathbf{T} = \frac{1}{|G|} \sum_{g \in G} \rho'(g)^{-1} \mathbf{L} \rho(g).$$

One could easily check that  $\mathbf{T}$  is  $G$ -equivariant (i.e.,  $\mathbf{T} \circ \rho(g) = \rho'(g) \circ \mathbf{T}$ ). Then

1.  $\rho, \rho'$  are not equivalent  $\implies \mathbf{T} = 0$ .
2.  $V = V', \rho = \rho' \implies \mathbf{T} = \lambda 1_V$ ,  $\lambda = \text{Tr}(\mathbf{T}) / \dim V = \text{Tr}(\mathbf{L}) / \dim V$ .

**Remark 18.** Let  $\rho \rightarrow_{\beta} R : G \rightarrow \text{GL}_n(\mathbb{C})$  and  $R(g) = [r_{ij}(g)]$   
 $\rho' \rightarrow_{\beta'} R' : G \rightarrow \text{GL}_{n'}(\mathbb{C})$  and  $R'(g) = [r'_{ij}(g)]$

and let the matrix representation of  $\mathbf{L}$  is  $[\mathbf{L}]_{\beta}^{\beta'} = [x_{\mu\nu}] \in M_{n' \times n}(\mathbb{C})$

Then consider the matrix representation of  $\mathbf{T}$ , which is  $[\mathbf{T}]_{\beta}^{\beta'} = [x_{tl}^{\circ}]$  with

$$x_{tl}^{\circ} = \frac{1}{|G|} \sum_{\substack{g \in G \\ i=1, \dots, n \\ j=1, \dots, n'}} r'_{tj}(g^{-1}) x_{ji} r_{il}(g)$$

In case 1.,  $x_{tl}^{\circ} = 0, \forall t, l$ . Since it holds for every  $\mathbf{L}$ , which is independent of  $\rho, \rho'$ , fixing  $i, j$  and setting  $x_{ij} = 1$  and 0 otherwise, we get

$$\frac{1}{|G|} \sum_{g \in G} r'_{tj}(g^{-1}) r_{il}(g) = 0, \quad \forall i, j, t, l$$

In case 2.,  $\mathbf{T} = \lambda 1_V$ , i.e.  $x_{tl}^{\circ} = \lambda \delta_{tl}$ .  $\lambda = \frac{\text{Tr}(\mathbf{L})}{n} = \frac{1}{n} \sum_{i=1}^n x_{ii} = \frac{1}{n} \sum_{i,j} \delta_{ji} x_{ji}$

Hence,

$$\frac{1}{|G|} \sum_{g, i, j} r'_{tj}(g^{-1}) x_{ji} r_{il}(g) = \frac{1}{n} \sum_{i,j} \delta_{ji} x_{ji} \delta_{tl}$$

But notice that this equality hold for any  $L$ , which is independent of  $\rho, \rho'$ . So if we fix  $i, j$  and set  $x_{ji} = 1$ , and  $x_{j'i'} = 0$  otherwise, we get

$$\frac{1}{|G|} \sum_{g \in G} r_{tj}(g^{-1}) r_{il}(g) = \frac{1}{n} \delta_{ji} \delta_{tl}$$

**Prop 3.1.1.**

1. If  $\chi_\rho$  is irreducible, then  $\langle \chi_\rho, \chi_\rho \rangle = 1$ .
2. If two irreducible representations  $\rho, \rho'$  are not equivalent, then  $\langle \chi_\rho, \chi_{\rho'} \rangle = 0$ .

*Proof.*

1. Let  $R(g) = [r_{ij}(g)]$  be the matrix representation of  $\rho(g)$ . Then

$$\begin{aligned} \langle \chi_\rho, \chi_\rho \rangle &\triangleq \frac{1}{|G|} \sum_g \chi_\rho(g) \overline{\chi_\rho(g)} = \frac{1}{|G|} \sum_g \chi_\rho(g) \chi_\rho(g^{-1}) \\ &= \frac{1}{|G|} \sum_g \sum_{i,j} r_{ii}(g) r_{jj}(g^{-1}) = \sum_{i,j} \frac{1}{n} \delta_{ij} \delta_{ij} = 1 \end{aligned}$$

2. Let  $R(g) = [r_{ij}(g)]$ ,  $R'(g) = [r'_{ij}(g)]$  be the matrix representation of  $\rho(g), \rho'(g)$ . Then

$$\begin{aligned} \langle \chi_\rho, \chi_{\rho'} \rangle &\triangleq \frac{1}{|G|} \sum_g \chi_\rho(g) \overline{\chi_{\rho'}(g)} = \frac{1}{|G|} \sum_g \chi_\rho(g) \chi_{\rho'}(g^{-1}) \\ &= \frac{1}{|G|} \sum_g \sum_{i,j} r_{ii}(g) r'_{jj}(g^{-1}) = 0 \end{aligned}$$

□

**Remark 19.**  $\langle \chi_\rho, \chi_\rho \rangle = 1 \implies \rho$  is irr.

*Proof.* We write  $\rho = \rho_1^{\oplus m_1} \oplus \dots \oplus \rho_l^{\oplus m_l}$  where  $\rho_1, \dots, \rho_l$  are non-equivalent irr. rep.

$$\chi_\rho = \sum_{i=1}^l m_i \chi_{\rho_i}$$

$$1 = \langle \chi_\rho, \chi_\rho \rangle = \sum_{i=1}^l m_i^2 \implies \exists m_i = 1 \text{ and } m_j = 0 \text{ for } j \neq i$$

So  $\rho \cong \rho_i$ .

□

## 3.2 Week 15

### 3.2.1 Character Theory II

**Prop 3.2.1.** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\rho = \rho^{W_1} \oplus \cdots \oplus \rho^{W_k}$  where  $\rho_i = \rho^{W_i}$  is irr.  $\forall i$ . ( $V \cong W_1 \oplus \cdots \oplus W_k$ )

If  $\tilde{\rho} : G \rightarrow \text{GL}(\tilde{W})$  is an irr. rep. then the number of  $\rho_i$  isomorphic to  $\tilde{\rho}$  is equal to  $\langle \chi_\rho, \chi_{\tilde{\rho}} \rangle$ .

*Proof.* We know  $\chi_\rho = \chi_{\rho_1} + \cdots + \chi_{\rho_k}$ , so

$$\langle \chi_\rho, \chi_{\tilde{\rho}} \rangle = \sum_{i=1}^k \langle \chi_{\rho_i}, \chi_{\tilde{\rho}} \rangle$$

Recall  $\rho_i \cong \tilde{\rho} \implies \langle \chi_{\rho_i}, \chi_{\tilde{\rho}} \rangle = 1$ , otherwise  $\langle \chi_{\rho_i}, \chi_{\tilde{\rho}} \rangle = 0$ . □

**Remark 20.**

1. The number of  $W_i$  isomorphic to  $\tilde{W}$  does not depend on the chosen decomposition. ( $= \langle \chi_\rho, \chi_{\tilde{\rho}} \rangle$ )
2. If  $\chi_\rho = \chi_{\rho'}$ , then  $\rho \cong \rho'$ :  $\langle \chi_\rho, \chi_{\tilde{\rho}} \rangle = \langle \chi_{\rho'}, \chi_{\tilde{\rho}} \rangle$  The type of irr. subrep of  $\rho$  is the same as  $\rho'$ .
3. If  $\chi_1, \dots, \chi_l$  are distinct irr. characters of  $G$ , then since  $x_1, \dots, x_l$  are orthonormal w.r.t.  $\langle \cdot, \cdot \rangle$  in  $\mathcal{C}(G)$ ,  $x_1, \dots, x_l$  are linearly indep. over  $\mathbb{C}$  in  $\mathcal{C}(G)$ .

But  $\dim \mathcal{C}(G) = k = \#$  of conjugacy classes in  $G$ . So  $l \leq k$  i.e. we conclude that there are at most  $k$  mutually non-equivalent irr. rep. of  $G$ , say  $\rho_1, \dots, \rho_l, l \leq k$ .

For any  $\rho : G \rightarrow \text{GL}(V)$ ,  $\rho \cong \rho_1^{\oplus m_1} \oplus \cdots \oplus \rho_l^{\oplus m_l}$  where  $m_i = \langle \chi_\rho, \chi_{\rho_i} \rangle \in \mathbb{Z}^{\geq 0}$ .

**Theorem 33** (Orthogonality relations for  $\chi$ 's). The set of all irr. characters of  $G$  forms an orthonormal **basis**  $\mathcal{C}(G)$  over  $\mathbb{C}$ . In particular, the number of irr. rep. of  $G$  is equal to  $\#$  of conjugacy classes in  $G$ . (up to equivalence)

*Proof.* Let  $\chi_i = \chi_{\rho_i}, i = 1, \dots, l$  be all irr. characters of  $G$  and  $\mathcal{D} = \langle \chi_1, \dots, \chi_l \rangle_{\mathbb{C}} \subseteq \mathcal{C}(G)$ . Then  $\mathcal{C}(G) = \mathcal{D} \oplus \mathcal{D}^\perp$ . Claim:  $\mathcal{D}^\perp = \{0\}$ .

Let  $\varphi \in \mathcal{D}^\perp$ , i.e.  $\langle \varphi, \chi_i \rangle = 0, \forall i = 1, \dots, l$ .

Write  $\rho^{\text{reg}} \cong \rho_1^{\oplus m_1} \oplus \cdots \oplus \rho_l^{\oplus m_l} \implies \chi^{\text{reg}} = m_1 \chi_1 + \cdots + m_l \chi_l$ . By assumption,  $\langle \varphi, \chi_\rho \rangle = 0$ .

For each  $i$ , define  $\mathsf{T}_{\rho_i} \in \text{Hom}_{\mathbb{C}}(V, V)$  by

$$\mathsf{T}_{\rho_i} \triangleq \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \rho_i(g)$$

Then we have

$$\text{Tr}(\mathsf{T}_{\rho_i}) = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \chi_\rho(g) = \overline{\langle \varphi, \chi_\rho \rangle} = 0$$

Also, for all  $h \in G$ .

$$\begin{aligned} \rho(h)^{-1} \circ \mathsf{T}_{\rho_i} \circ \rho(h) &= \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \rho(h)^{-1} \circ \rho(g) \circ \rho(h) \\ &\stackrel{*}{=} \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(h^{-1}gh)} \rho(h^{-1}gh) \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \rho(g) = \mathsf{T}_{\rho_i} \end{aligned}$$

Where  $*$  is because  $\varphi$  is a class function. So  $\mathsf{T}_{\rho_i}$  is  $G$ -equivariant. By Schur's lemma,  $\mathsf{T}_{\rho_i} = \lambda_i 1_{W_i}$  where  $\rho_i : G \rightarrow \mathrm{GL}(W_i)$ .

But  $\mathrm{Tr} \mathsf{T}_{\rho_i} = 0 \implies \lambda_i = 0 \implies \mathsf{T}_{\rho_i} = 0$ .

Also, because  $\rho \cong \rho_1^{\oplus m_1} \oplus \dots \oplus \rho_l^{\oplus m_l}$ , if we define

$$\mathsf{T}_{\rho^{\mathrm{reg}}} \triangleq \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \rho^{\mathrm{reg}}(g) \implies \mathsf{T}_{\rho^{\mathrm{reg}}} = \mathsf{T}_{\rho_1}^{\oplus m_1} \oplus \dots \oplus \mathsf{T}_{\rho_k}^{\oplus m_k} = 0$$

Finally, let  $\rho = \rho^{\mathrm{reg}} : G \rightarrow \mathrm{GL}(V)$  with  $V = \bigoplus_{g \in G} \mathbb{C} e_g$ . Then  $\mathsf{T}_{\rho} = 0 \implies \mathsf{T}_{\rho}(e_1) = 0$  and

$$0 = \mathsf{T}_{\rho}(e_1) = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \rho(g)(e_1) = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} e_g$$

Since  $\{e_g\}$  is a basis,  $\overline{\varphi(g)} = 0 \quad \forall g$ . That is,  $\varphi \equiv 0$ .  $\square$

**Prop 3.2.2.** Each irr. rep.  $\rho_i : G \rightarrow \mathrm{GL}(W_i)$  is contained in  $\rho^{\mathrm{reg}}$  with multiplicity equal to  $\dim W_i = m_i$ ,  $i = 1, \dots, k$ .

In particular,  $\bigoplus_{g \in G} \mathbb{C} e_g \cong \underbrace{W_1 \oplus \dots \oplus W_1}_{m_1 \text{ times}} \oplus \dots \oplus \underbrace{W_1 \oplus \dots \oplus W_k}_{m_k \text{ times}}$ . So  $|G| = m_1^2 + \dots + m_k^2$ .

*Proof.* Let  $\chi^{\mathrm{reg}} := \chi_{\rho^{\mathrm{reg}}}$  and  $\chi_i = \chi_{\rho_i}$ ,  $i = 1, \dots, k$ . Then

$$\langle \chi^{\mathrm{reg}}, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi^{\mathrm{reg}}(g) \chi_i(g^{-1}) = \frac{1}{|G|} |G| \chi_i(1) = m_i$$

$\square$

**Theorem 34** (Divisibility).  $\forall i = 1, \dots, k, \quad \chi_i(1) = m_i \mid |G|$ .

*Proof.* First, we shall proof that for each  $\rho = \rho_i$ ,  $\chi = \chi_i$  and  $j$ , we have

$$\mathsf{T} \triangleq \sum_{g \in C_j} \rho(g) = \frac{|C_j| \chi(g_0)}{m_i} 1_{m_i}, \quad \text{for any } g_0 \in C_j$$

Observe that  $\forall h \in G$ ,

$$\rho(h)^{-1} \circ \mathsf{T} \circ \rho(h) = \sum_{g \in C_j} \rho(h^{-1} g h) = \sum_{g' \in C_j} \rho(g') = \mathsf{T}$$

So  $\mathsf{T}$  is  $G$ -equivariant w.r.t.  $\rho$ .

By Schur's lemma,  $\mathsf{T} = \lambda 1_{m_i}$  for some  $\lambda \in \mathbb{C}$ . And  $\lambda = \mathrm{Tr}(\mathsf{T})/m_i = \sum_{g \in C_j} \chi(g)/m_i = |C_j| \chi(g_0)/m_i$  for any  $g_0 \in C_j$ , thus  $\sum_{g \in C_j} \rho(g) = \frac{|C_j| \chi(g_0)}{m_j} 1_{m_j}$  for any  $g_0 \in C_j$ .

Define  $\lambda_{\mu}(C_i) \triangleq |C_i| \chi_{\mu}(g_i)/m_{\mu}$ . Now, for a  $g \in C_l$ , define  $a_{i,j,l} \triangleq \#\{(g_i, g_j) \in C_i \times C_j \mid g_i g_j = g\}$ , which is indep. of the choice of  $g$ .

We claim that  $\lambda_{\mu}(C_i) \lambda_{\mu}(C_j) = \sum_{l=1}^k a_{i,j,l} \lambda_{\mu}(C_l)$ ,  $\forall i, j, \mu$ . Then

$$\lambda_{\mu}(C_i) \begin{bmatrix} \lambda_{\mu}(C_1) \\ \vdots \\ \lambda_{\mu}(C_k) \end{bmatrix} = A \begin{bmatrix} \lambda_{\mu}(C_1) \\ \vdots \\ \lambda_{\mu}(C_k) \end{bmatrix}, \quad \text{where } A \triangleq \begin{bmatrix} a_{i,1,1} & \dots & a_{i,1,k} \\ \vdots & \ddots & \vdots \\ a_{i,k,1} & \dots & a_{i,k,k} \end{bmatrix}$$

So  $\lambda_{\mu}(C_j)$  is an eigenvalue of  $A$ , i.e.,  $\lambda = \lambda_{\mu}(C_j)$  satisfies  $\det(\lambda I - A) = 0$ . And thus  $\lambda_{\mu}(C_i)$  is an algebraic integer.

We proof the claim by the following calculating.

$$\begin{aligned}
\lambda_\mu(C_i)\lambda_\mu(C_j)I_{m_\mu} &= (\lambda_\mu(C_i)I_{m_\mu}) (\lambda_\mu(C_j)I_{m_\mu}) = \left( \sum_{g \in C_i} \rho(g) \right) \left( \sum_{g' \in C_j} \rho(g') \right) \\
&= \sum_{\substack{g \in C_i \\ g' \in C_j}} \rho(gg') = \sum_{l=1}^k \sum_{\bar{g} \in C_l} a_{i,j,l} \rho(\bar{g}) \\
&= \sum_{l=1}^k a_{i,j,l} \sum_{\bar{g} \in C_l} \rho(\bar{g}) \\
&= \sum_{l=1}^k a_{i,j,l} \lambda_\mu(C_l) I_{m_\mu}
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{|G|}{m_i} &= \frac{|G|}{m_i} \langle \chi_i, \chi_i \rangle \\
&= \frac{|G|}{m_i} \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_i(g^{-1}) \\
&= \sum_{g \in G} \frac{\chi_i(g)}{m_i} \chi_i(g^{-1}) \\
&= \sum_{j=1}^k \sum_{g \in C_j} \frac{\chi_i(g)}{m_i} \chi_i(g^{-1}) \\
&= \sum_{j=1}^k \frac{|C_j| \chi_i(g_j)}{m_i} \chi_i(g_j^{-1}) \\
&= \sum_{j=1}^k \lambda_i(C_j) \chi_i(g_j^{-1})
\end{aligned}$$

and thus is an algebraic integer.

Also,  $|G|/m_i \in \mathbb{Q}$ , so we conclude that  $|G|/m_i \in \mathbb{Z} \implies m_i \mid |G|$ . □

**Ex 3.2.1.**

1. Show that if  $g \in G$  and  $g \neq 1$ , then  $\sum_{i=1}^k m_i \chi_i(g) = 0$ .
2. Show that each character  $\chi$  of  $G$  with  $\chi(g) = 0 \quad \forall g \neq 1$  is an integral multiple of  $\chi^{\text{reg}}$ .

**Ex 3.2.2.**

1. Let  $|G| < \infty$ . Then  $G$  is abelian  $\iff$  each irr. rep. of  $G$  is of degree 1.
2.  $\{\text{the deg 1 rep. of } G\} = \{\text{the irr. rep. of } G/[G, G]\}$ .

**3.2.2 Applications**

1.  $G = S_3 = D_3$ ,  $6 = 1^2 + 1^2 + 2^2$ .

Classes	1	(1 2)	(1 2 3)
size	1	3	2
$\chi_1$	1	1	1
$\chi_2$	1	-1	1
$\chi_3$	2	0	-1

The permutation representation

deg 4:  $\tilde{\rho} = \rho^W \otimes \rho^W \rightsquigarrow \chi_{\tilde{\rho}} = \chi_3 \cdot \chi_3 = (4, 0, 1)$ .

By inner product with  $\chi_1, \chi_2, \chi_3$ , we can find  $\chi_{\tilde{\rho}} = \chi_1 + \chi_2 + \chi_3 \rightsquigarrow \tilde{\rho} = \rho_1 \oplus \rho_2 \oplus \rho_3$ .

2.  $G = D_4 = \langle x, y \mid x^4 = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$ .  $|G| = 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$ .

Classes	1	$y$	$x$	$x^2$	$xy$
size	1	2	2	1	2
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	1	-1
$\chi_3$	1	1	-1	1	-1
$\chi_4$	1	-1	-1	1	1
$\chi_5$	2	0	0	-2	0

$\chi^{\text{reg}} = (8, 0, 0, 0, 0) = \chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\chi_5$ .

3.  $G = D_n$ , ( $n$  even)  $[G, G] = H = \langle x^2 \rangle$

4.  $G = D_n$ , ( $n$  odd)  $[G, G] = H = \langle x \rangle$

5.  $G = S_4$ .

Classes	1	(1 2)	(1 2 3)	(1 2 3 4)	(1 2)(3 4)
size	1	6	8	6	3
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	-1	1
$\chi_3$	2	0	-1	0	2
$\chi_4$	3	1	0	-1	-1
$\chi_5$	3	-1	0	1	-1

6.  $G = A_4$ ,  $[A_4, A_4] = V_4$ .

Classes	1	(1 2 3)	(1 3 2)	(1 2)(3 4)
size	1	4	4	3
$\chi_1$	1	1	1	1
$\chi_2$	1	$\omega$	$\omega^2$	1
$\chi_3$	1	$\omega^2$	$\omega$	1
$\chi_4$	3	0	0	-1

**Theorem 35** (Product of groups). For  $\rho : G \rightarrow \text{GL}(V)$  and  $\rho' : G' \rightarrow \text{GL}(V')$ , write  $\rho \otimes \rho' : G \times G' \rightarrow \text{GL}(V \otimes V')$ . If  $\{\rho_i\}$  are irreducible representations of  $G$ ,  $\{\rho'_j\}$  are irreducible representations of  $G'$ , then  $\{\rho_i \otimes \rho'_j\}$  are exactly the irreducible representations of  $G \times G'$ .

*Proof.* It is evidence that  $\rho_i \otimes \rho'_j$  is a homomorphism, and hence a representation.

Notice that  $\chi_{\rho \otimes \rho'} = \chi_\rho \odot \chi_{\rho'}$  where  $\chi_\rho \odot \chi_{\rho'}(g, g') = \chi_\rho(g)\chi_{\rho'}(g')$

Now we calculate

$$\begin{aligned}
\langle \chi_{\rho_1} \otimes \chi_{\rho'_1}, \chi_{\rho_2} \otimes \chi_{\rho'_2} \rangle &= \frac{1}{|G||G'|} \sum_{g, g'} \chi_{\rho_1}(g) \chi_{\rho'_1}(g') \chi_{\rho_2}(g) \chi_{\rho'_2}(g') \\
&= \left( \frac{1}{|G|} \sum_g \chi_{\rho_1}(g) \chi_{\rho_2}(g) \right) \left( \frac{1}{|G'|} \sum_{g'} \chi_{\rho'_1}(g') \chi_{\rho'_2}(g') \right) \\
&= \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle \langle \chi_{\rho'_1}, \chi_{\rho'_2} \rangle
\end{aligned}$$



So  $\langle \chi_\rho \otimes \chi_{\rho'}, \chi_\rho \otimes \chi_{\rho'} \rangle = 1$  hence each  $\chi_\rho \otimes \chi_{\rho'}$  is irreducible. And  $\langle \chi_{\rho_1} \otimes \chi_{\rho'_1}, \chi_{\rho_2} \otimes \chi_{\rho'_2} \rangle = 0$  if  $\rho_1 \otimes \rho'_1 \neq \rho_2 \otimes \rho'_2$ , and thus these representations are not isomorphic.

Finally we proof that any irreducible representations of  $G \times G'$  is isomorphic to some  $\rho \otimes \rho'$ .

Let  $\{\rho_1, \dots, \rho_k\}, \{\rho'_1, \dots, \rho'_{k'}\}$  be the sets of irreducible representations of  $G, G'$  respectively. Write  $\chi_i = \chi_{\rho_i}, \chi'_i = \chi_{\rho'_i}$ .

Let  $\mathcal{D} \triangleq \mathcal{C}(G \times G') = \langle \chi_i, \chi'_j \mid i = 1, \dots, k, j = 1, \dots, k' \rangle_{\mathbb{C}}$ . We claim that  $\mathcal{D}^\perp = \{0\}$ .

Let  $f \in \mathcal{D}^\perp$ . Then

$$\begin{aligned} 0 &= \frac{1}{|G \times G'|} \sum_{(g, g') \in G \times G'} f(g, g') \overline{\chi_i(g) \chi'_j(g')} \\ &= \frac{1}{|G'|} \sum_{g'} \left( \frac{1}{|G|} \sum_g f(g, g') \overline{\chi_i(g)} \right) \chi'_j(g') \\ &= \left\langle \frac{1}{|G|} \sum_g f(g, \cdot) \overline{\chi_i(g)}, \chi'_j(g') \right\rangle \end{aligned}$$

Since  $\rho'_j$  are orthonormal basis of  $\mathcal{C}(G')$ , we have  $\frac{1}{|G|} \sum_g f(g, g') \overline{\chi_i(g)} = 0$  for all  $g'$ . Again,

$$0 = \frac{1}{|G|} \sum_g f(g, g') \overline{\chi_i(g)} = \langle f(\cdot, g'), \chi_i(g) \rangle$$

Hence  $f(g, g') = 0$  for all  $g, g'$ , which implies  $f \equiv 0$ . □

**Ex 3.2.3.** Determine all irr. rep. of  $C_n$ .

**Ex 3.2.4.** Calculate the character table of  $Q_8$ .

**Ex 3.2.5.** Calculate the character table of  $\mathbb{Z}/2\mathbb{Z} \times S_4$  and  $S_3 \times S_4$ .

To calculate  $S_5$ ,  $|S_5| = 120 = 1^2 + 1^2 + 4^2 + 4^2 + 5^2 + 5^2 + 6^2$ .

## 4 Extensions of Groups

### 4.1 Week 16

#### 4.1.1 Extensions of abelian groups

**Def 66.** If a group  $E$  contains a normal subgroup  $N$  and  $E/N \cong G$ , then we call  $E$  an extension of  $N$  by  $G$ , denoted by  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$ .

Ques: When  $N$  and  $G$  are given, how to obtain all extensions of  $N$  by  $G$ .

**Now assume that  $N$  is abelian.**

**Def 67.**  $1 \rightarrow N \rightarrow E \xrightarrow{p} G \rightarrow 1$ .  $l : G \rightarrow E$  is a lifting if  $p \circ l = \text{id}_G$  and  $l(1) = 1$ .

**Remark 21.**  $G \cong E/N = \{xN \mid x \in E\}$ ,  $p \circ l(\bar{x}) = \bar{x}$ ,  $l(\bar{x})$  is a representative of  $xN = \bar{x}$ .

**Prop 4.1.1.**

1.  $\forall \bar{x} \in G, \theta_{\bar{x}} : N \rightarrow N, a \mapsto l(\bar{x})al(\bar{x})^{-1}$ . is independent of the choice of  $l$ .
2.  $\theta : G \rightarrow \text{Aut}(N), \bar{x} \mapsto \theta_{\bar{x}}$  is a group homomorphism.

*Proof.*

1. Suppose  $l' : G \rightarrow E$  is another lifting. Then  $l(\bar{x})N = l'(\bar{x})N$ . So  $l'(\bar{x}) = l(\bar{x})b$  for some  $b \in N$ .  $\forall a \in N, l'(\bar{x})al'(\bar{x})^{-1} = l(\bar{x})bab^{-1}l(\bar{x})^{-1} = l(\bar{x})al(\bar{x})^{-1}$  since  $N$  is abelian.
2.  $\theta_{\bar{x}\bar{y}}(a) = l(\bar{x}\bar{y})al(\bar{x}\bar{y})^{-1}$ .

$$\begin{cases} p \circ l(\bar{x}\bar{y}) = \bar{x}\bar{y} \\ p \circ (l(\bar{x})l(\bar{y})) = \bar{x}\bar{y} \end{cases} \rightsquigarrow l(\bar{x}\bar{y}), l(\bar{x})l(\bar{y}) \text{ are liftings of } \bar{x}\bar{y} \quad \square$$

**Def 68.** An extension  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$  splits if  $\exists$  a lifting  $l : G \rightarrow E$  is a group homo.

**Prop 4.1.2.** TFAE

1.  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$  splits.
2.  $\exists$  a subgroup  $K \leq E$  s.t.  $K \cong G$  and  $\begin{cases} K \cap N = \{1\} \\ NK = E \end{cases} \rightsquigarrow E \cong N \rtimes K (\cong N \rtimes G)$ .

*Proof.* (1)  $\Rightarrow$  (2): Let  $K = \text{Im } l$  which is a subgroup since  $l$  is a group homo.

- $l$  is an isomorphism from  $G$  to  $K$ : If  $l(\bar{x}) = l(\bar{y})$ , then  $p \circ l(\bar{x}) = p \circ l(\bar{y}) \rightsquigarrow \bar{x} = \bar{y}$ . So  $l$  is 1-1.
- $E = NK$ :  $\forall x \in E, \bar{x} = p(x) \rightsquigarrow y = l(\bar{x})$  and  $p(x) = p(y) \rightsquigarrow \exists a \in N$  s.t.  $x = ay$ .
- $K \cap N = \{1\}$ :  $a = l(\bar{x}) \in K \cap N \rightsquigarrow 1 = p(a) = p(l(\bar{x})) = \bar{x} \rightsquigarrow a = l(1) = 1$ .

(2)  $\Rightarrow$  (1):

- $p|_K : K \rightarrow G$  is an isom.: onto:  $p(K) = p(NK) = p(E) = G$ , 1-1:  $\ker(p|_K) = N \cap K = \{1\}$ .

- $l = (p|_K)^{-1}$  is a group homo.

Observation: Let  $l : G \rightarrow E$  be a lifting. Then  $E = \bigcup_{\bar{x} \in G} Nl(\bar{x})$ ,  $\forall x, y \in E$ , write  $x = al(\bar{x})$ ,  $y = bl(\bar{y})$ ,  $a, b \in N$ ,  $\bar{x}, \bar{y} \in G$ .

$$xy = (al(\bar{x})bl(\bar{y})) = al(\bar{x})bl(\bar{x})^{-1}l(\bar{x})l(\bar{y}) = a\theta_{\bar{x}}(b)l(\bar{x})l(\bar{y})$$

Notice that  $l(\bar{x})l(\bar{y})$  and  $l(\bar{x}\bar{y})$  are liftings, so we can write  $l(\bar{x})l(\bar{y}) = f(\bar{x}, \bar{y})l(\bar{x}\bar{y})$  for some  $f(\bar{x}, \bar{y}) \in N$ .  $\square$

**Ex 4.1.1.**  $B^2(G, N) \leq Z^2(G, N)$ .

**Ex 4.1.2.** Show that there are inequivalent extensions of  $N$  by  $G$  with isomorphic middle groups. (Hint:  $N = \mathbb{Z}/p\mathbb{Z}$  with  $p$  is odd,  $E = \mathbb{Z}/p^2\mathbb{Z}$ ,  $a :: N \mapsto x^p :: E$  and please give another morphism  $N \rightarrow E$  by yourself.)

**Def 69.** Given  $1 \rightarrow N \rightarrow E \xrightarrow{p} G \rightarrow 1$  and  $l : G \rightarrow E$ , a factor set is a function  $f : G \times G \rightarrow N$  s.t.  $\forall \bar{x}, \bar{y} \in G$ ,  $l(\bar{x})l(\bar{y}) = f(\bar{x}, \bar{y})l(\bar{x}\bar{y})$ .

**Prop 4.1.3.** Let  $1 \rightarrow N \rightarrow E \xrightarrow{p} G \rightarrow 1$  and  $l : G \rightarrow E$ . If  $f$  is a factor set, then

- (1)  $f(x, 1) = 1 = f(1, y) \quad \forall x, y \in G$ .
- (2) (cocycle identity)  $\forall x, y, z \in G$ ,  $f(x, y)f(xy, z) = \theta_x(f(y, z))f(x, yz)$ .  
(i.e.  $f(x, y) + f(xy, z) = xf(y, z) + f(x, yz)$ )

*Proof.*

- (1) Trivial since  $l(x)l(1) = l(1 \cdot x)$ .
- (2) By associativity.  $(l(x)l(y))l(z) = l(x)(l(y)l(z))$ .  
 $(l(x)l(y))l(z) = f(x, y)l(xy)l(z) = f(x, y)f(xy, z)l(xyz)$ , and  
 $l(x)(l(y)l(z)) = l(x)f(y, z)l(yz) = l(x)f(y, z)l^{-1}(x)l(x)l(yz) = \theta_x(f(y, z))f(x, yz)l(xyz)$ .  
Thus  $f(x, y)f(xy, z) = \theta_x(f(y, z))f(x, yz)$ .  $\square$

**Theorem 36.** Let  $\sigma : G \rightarrow \text{Aut}(N)$ ,  $x \mapsto \sigma_x$  be a group homo. and  $f : G \times G \rightarrow N$  satisfies (1),(2) in Prop. 4.1.3. Then  $\exists 1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$  and  $l : G \rightarrow E$  s.t.  $\theta = \sigma$  and  $f$  is the corresponding factor set.

*Proof.* • Define  $E = N \times G$  equipped with the operation

$$(a, x)(b, y) = (a\sigma_x(b)f(x, y), xy)$$

– associativity:

$$\begin{aligned} ((a, x)(b, y))(c, z) &= (a\sigma_x(b)f(x, y), xy)(c, z) \\ &= (a\sigma_x(b)f(x, y)\sigma_{xy}(c)f(xy, z), xyz) \\ &= (a\sigma_x(b)\sigma_{xy}(c)f(x, y)f(xy, z), xyz) \quad (\because N \text{ abelian}) \end{aligned}$$

and

$$\begin{aligned} (a, x)((b, y)(c, z)) &= (a, x)(b\sigma_y(c)f(y, z)) \\ &= (a\sigma_x(b\sigma_y(c)f(y, z))f(x, yz), xyz) \\ &= (a\sigma_x(b)\sigma_{xy}(c)\sigma_x(f(y, z))f(x, yz), xyz) \\ &= (a\sigma_x(b)\sigma_{xy}(c)f(x, y)f(xy, z), xyz) \end{aligned}$$

- identity:  $(1, 1)$ .
- inverse:  $(a, x)^{-1} = (\sigma_{x^{-1}}(a^{-1}f(x, x^{-1})^{-1}), x^{-1})$ .
- $p : E \rightarrow G, (a, x) \mapsto x$  is a group homo by def.
- $i : N \rightarrow E, a \mapsto (a, 1)$  is a group homo.  $(a, 1)(b, 1) = (a\sigma_1(b)f(1, 1), 1) = (ab, 1)$ .
- $\ker p = \text{Im } i$ .
- $\text{Fix } l : G \rightarrow E, a \in N, x \in G$ , say  $l(x) = (b, x)$ .

$$\begin{aligned} l(x)(a, 1)l(x)^{-1} &= (b, x)(a, 1)(b, x)^{-1} = (b\sigma_x(a), x)(\sigma_{x^{-1}}(a^{-1}f(x, x^{-1})^{-1}), x^{-1}) \\ &= (b\sigma_x(a) \cdot (\sigma_x \circ \sigma_{x^{-1}})(b^{-1}f(x, x^{-1})^{-1}) \cdot f(x, x^{-1}), 1) \\ &= (\sigma_x(a), 1) \end{aligned}$$

So  $\theta_x = \sigma_x$ .

- Let  $l : G \rightarrow E, x \mapsto (1, x)$ . Check  $l(x)l(y)l(xy)^{-1} = (f(x, y), 1)$ . Then  $f$  is the corresponding factor set.  $\square$

**Prop 4.1.4.** Let  $1 \rightarrow N \rightarrow E \xrightarrow{p} G \rightarrow 1$  with two liftings  $l_1 : G \rightarrow E, l_2 : G \rightarrow E$  with  $f_1 : G \times G \rightarrow N, f_2 : G \times G \rightarrow N$  respectively.

Then  $\exists h : G \rightarrow N$  with  $h(1) = 1$  and  $\forall x, y \in G, f_2(x, y)f_1(x, y)^{-1} = \theta_x(h(y))h(xy)^{-1}h(x)$ .  
 $(f_2(x, y) - f_1(x, y) = xh(y) - h(xy) + h(x))$

*Proof.* For  $x \in G, \exists h(x) \in N$  s.t.  $l_2(x) = h(x)l_1(x)$ . Since  $l_1(1) = l_2(1) = 1, h(1) = 1$ .

Now,  $l_2(x)l_2(y) = f_2(x, y)l_2(x, y) = f_2(x, y)h(xy)l_1(x, y)$ . and

$$\begin{aligned} l_2(x)l_2(y) &= h(x)l_1(x)h(y)l_1(y) = h(x)l_1(x)h(y)l_1^{-1}(x)l_1(x)l_1(y) \\ &= h(x)\theta_x(h(y))l_1(x)l_1(y) = f_1(x, y)h(x)\theta_x(h(y))l_1(x, y) \end{aligned}$$

So  $f_2(x, y)f_1(x, y)^{-1} = \theta_x(h(y))h(xy)^{-1}h(x)$ .  $\square$

**Remark 22.** A map which has the form  $\tilde{h} : G \times G \rightarrow N, (x, y) \mapsto xh(y) - h(xy) + h(x)$  is called a coboundary map.

**Def 70.**  $Z^2(G, N)$  = the abelian group of all factor sets.

$B^2(G, N)$  = the abelian group of all coboundary maps.

$H^2(G, N) = Z^2(G, N)/B^2(G, N)$

**Def 71.** Two extensions  $\begin{cases} 1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1 \\ 1 \rightarrow N \rightarrow E' \rightarrow G \rightarrow 1 \end{cases}$  are equivalent if exists an isomorphism  $\varphi : E \xrightarrow{\sim} E'$  which let the following diagram comutes.

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow 1_N & & \downarrow \varphi & & \downarrow 1_G \\ 1 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & G \longrightarrow 1 \end{array}$$

**Theorem 37.** Two extensions  $\begin{cases} 1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1 \\ 1 \rightarrow N \rightarrow E' \rightarrow G \rightarrow 1 \end{cases}$  are equivalent  $\iff$

Exists mappings  $l : G \rightarrow E, l' : G \rightarrow E'$  with two factor sets  $f, f'$  respectively satisfies  $f - f' \in B^2(G, N)$ .

*Proof.* “ $\Rightarrow$ ”: Choose  $l : G \rightarrow E$  which has a corresponding factor set  $f : G \times G \rightarrow N$ . Now define  $l' : G \rightarrow E'$  by  $l' = \varphi \circ l$ . Since  $p' \circ l' = p' \circ \varphi \circ l = p \circ l = 1$ ,  $l'$  is a lifting. Let  $f' : G \times G \rightarrow N$  be its factor set.

Since  $1_N = 1_N \circ \varphi$ ,  $\varphi|_N = 1_N$ . And

$$\begin{aligned} l(x)l(y) &= f(x, y)l(x, y) \\ \Rightarrow \varphi(l(x)l(y)) &= \varphi(f(x, y)l(x, y)) \\ \Rightarrow l'(x)l'(y) &= \varphi(f(x, y))l'(x, y) \\ \Rightarrow f'(x, y) &= \varphi(f(x, y)) \end{aligned}$$

But  $f(x, y) \in N$ ,  $\varphi(f(x, y)) = \varphi|_N(f(x, y)) = f(x, y)$ . So  $f(x, y) = f'(x, y)$ , hence  $f - f' = 0 \in B^2(G, N)$ .

**Ex 4.1.3.**

- (1) Show that  $f' - f \in B^2(G, N)$ .
- (2) “ $\Leftarrow$ ”: Show all details of the following steps:

- $\begin{cases} 1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1 \\ 1 \rightarrow N \rightarrow E(N, G, f, \theta) \rightarrow G \rightarrow 1 \end{cases}$  are equivalent.
- Similarly  $\begin{cases} 1 \rightarrow N \rightarrow E' \rightarrow G \rightarrow 1 \\ 1 \rightarrow N \rightarrow E(N, G, f', \theta') \rightarrow G \rightarrow 1 \end{cases}$  are equivalent.
- $f' - f \rightsquigarrow h : G \rightarrow N$ ,

□

#### 4.1.2 1st and 2nd group cohomology

Let  $N$  be an abelian group and  $G$  be a group with a group homo  $\sigma : G \rightarrow \text{Aut}(N)$  ( $G \curvearrowright N$ )

$e(G, N) = \{\text{equivalence classes of } N \text{ by } G\}$

$Z^2(G, N) = \{f : G \times G \rightarrow N \mid f(1, v) = f = f(u, 1), f(u, v) + f(uv, w) = uf(v, w) + f(u, vw) \quad u, v, w \in G\}$

$B^2(G, N) = \{f : G \times G \rightarrow N \mid \exists h : G \rightarrow N \text{ with } h(1) = 1 \text{ s.t. } f(u, v) = uh(v) - h(uv) + h(u) \quad u, v \in G\}$

$H^2(G, N) = Z^2(G, N)/B^2(G, N)$

Then  $e(G, N) \leftrightarrow H^2(G, N)$ .

**Def 72.**

- $\varphi \in \text{Aut}(E)$  stabilizes  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$  if

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow 1_N & & \downarrow \varphi|_N & & \downarrow 1_G \\ 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \end{array}$$

- $\text{Stab}_E(G, N) = \{\text{stabilizing automorphisms}\} \leq \text{Aut}(E)$

**Def 73.**

- A derivation is a function  $d : G \rightarrow N$  s.t.  $d(uv) = ud(v) + d(u) \quad \forall u, v \in G$ .
- $\text{Der}(G, N) = \{\text{derivations} : G \rightarrow N\}$  is an abelian group with pointwise addition.

**Theorem 38.** Let  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$  with  $\theta = \sigma$ . Then  $\text{Stab}_E(G, N) \cong \text{Der}(G, N)$ . So  $\text{Stab}_E(G, N)$  is abelian.

*Proof.*

- Let  $\varphi \in \text{LHS}$  and fix  $l : G \rightarrow E$ .

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \longrightarrow & E & \xrightarrow{\quad} & G \longrightarrow 1 \\ & & \downarrow 1_N & & \downarrow \varphi & \swarrow l & \downarrow 1_G \\ 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \end{array} \quad \varphi(al(u)) = \varphi(a)\varphi(l(u)) = ad(u)l(u)$$

- For another  $l' : G \rightarrow E$ , say  $l'(u) = g(u)l(u)$ , where  $g(u) \in N$ , we have

$$d'(u) = \varphi(l'(u))(l'(u))^{-1} = \varphi(g(u)l(u))(g(u)l(u))^{-1} = g(u)\varphi(l(u))l(u)^{-1}g(u)^{-1} = d(u).$$

- $d \in \text{RHS}$ ,

$$\begin{aligned} d(uv) &= \varphi(l(uv))l(uv)^{-1} \\ &= \varphi(f(u, v)^{-1}l(u)l(v))l(v)^{-1}l(u)^{-1}f(u, v) \\ &= f(u, v)^{-1}d(u)l(u)d(v)l(v)l(v)^{-1}l(u)^{-1}f(u, v)^{-1} \\ &= f(u, v)^{-1}d(u)(l(u)d(v)l(u)^{-1})f(u, v)^{-1} \\ &= (ud(v))d(u) \end{aligned}$$

- Conversely,

**Ex 4.1.4.** proof it

- group homo:  $\varphi_2 \circ \varphi_1(al(u)) = \varphi_2(ad_1(u)l(u)) = ad_1(u)\varphi_2(l(u)) = ad_1(u)d_2(u)l(u)$ . That is,  $\varphi_2 \circ \varphi_1 \mapsto d_1d_2$ .  $\square$

**Def 74.**

- $\text{Inn}_E(G, N) = \{\varphi \in \text{Stab}_E(G, N) \mid \varphi E \rightarrow E, x \mapsto a_0 x a_0^{-1} \text{ for some } a_0 \in N\}$ .
- $\text{PDer}(G, N) = \{d \in \text{Der}(G, N) \mid d(u) = ua_1 - a_1 \text{ for some } a_1 \in N\}$ .

**Ex 4.1.5.** Show that  $\text{Inn}_E(G, N) \cong \text{PDer}(G, N)$ .

$$\text{Stab}_E(G, N)/\text{Inn}_E(G, N) \cong \text{Der}(G, N)/\text{PDer}(G, N) = H^1(G, N).$$

**Ex 4.1.6.** Fix  $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$ . Show that if  $H^2(G, N) = 0, H^1(G, N) = 0$ , then for  $l : G \rightarrow E$  with  $K = l(G)$  and  $l' : G \rightarrow E$  with  $K' = l'(G)$ , we get that  $K$  and  $K'$  are conjugate.

**Def 75.** Let  $R$  be a commutative ring with 1 and  $G$  be a group. The group ring

$$R[G] = \left\{ \sum_{g \in G} r_g g \mid \text{only finitely many } r_g \text{'s } \neq 0 \text{ in } R \right\}$$

forms an  $R$ -algebra via

$$\begin{aligned}\sum_{g \in G} r_g g + \sum_{g \in G} r'_g g &= \sum_{g \in G} (r_g + r'_g) g \\ \left( \sum_{g \in G} r_g g \right) \left( \sum_{g' \in G} r'_g g' \right) &= \sum_{g, g' \in G} (r_g r'_g) g g' \\ r \left( \sum_{g \in G} r_g g \right) &= \sum_{g \in G} (r r_g) g\end{aligned}$$