

Algebra Homework

May 23, 2017

1 Week 1

Ex 1.1.

1. Prove that if $[K(\alpha) : K]$ is odd, then $K(\alpha) = K(\alpha^2)$.
2. Given L_1/K and L_2/K with $L_1, L_2 \subseteq L$, show that

$$L_1 \otimes_K L_2 \text{ is a field} \iff [L_1 L_2 : K] = [L_1 : K][L_2 : K]$$

Ex 1.2.

1. Prove that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
2. Determine $[\mathbb{Q}(\sqrt{3 + 2\sqrt{2}}) : \mathbb{Q}]$, $[\mathbb{Q}(\sqrt{3 + 4i} + \sqrt{3 - 4i}) : \mathbb{Q}]$, $[\mathbb{Q}(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) : \mathbb{Q}]$.

Ex 1.3. Let R be a PID and $a \in R$. TFAE:

1. a is an irreducible element.
2. $\langle a \rangle$ is a maximal ideal.
3. $\langle a \rangle$ is a prime ideal.
4. a is a prime element.

Ex 1.4. Let L/K be algebraic and $\tau : L \rightarrow L$ be a monomorphism fixing K . Show that τ is onto. (so τ is isom.)

Ex 1.5.

1. Determine the splitting field L for $x^4 + 2$ over \mathbb{Q} , $[L : \mathbb{Q}]$ and $\text{Aut}(L/\mathbb{Q})$.
2. Determine the splitting field L for $x^6 - 4$ over \mathbb{Q} , $[L : \mathbb{Q}]$ and $\text{Aut}(L/\mathbb{Q})$.

Ex 1.6. Let $L_1, L_2 \subseteq L$ with $[L_1 : K] < \infty$ and $[L_2 : K] < \infty$. Assume L_1 and L_2 are splitting fields over K . Show that

1. $L_1 L_2$ is a splitting fields over K .
2. $L_1 \cap L_2$ is a splitting fields over K .

2 Week 2

Ex 2.1. Show that

$$\mathbb{F}_q \subseteq \mathbb{F}_r \iff r = q^n \text{ for some } n \in \mathbb{N}.$$

Ex 2.2. Let $K = \mathbb{F}_q$ be a finite field.

1. Show that there exists an irreducible polynomial of degree n in $K[x]$ for all $n \in \mathbb{N}$.
2. Let L be a splitting field of $x^n - 1$ over K . Show that $[L : K] = k$ is the least positive integer s.t. $n \mid q^k - 1$.

Ex 2.3. Let K be a finite field. Show that any element in K can be written as the sum of two squares.

3 Week 3

Ex 3.1. Let L/K be a finite extension with $[L : K] = n$. For any field extension M/K , there are at most n monomorphisms from L to M which fix K .

Ex 3.2.

1. If F is a finite field, then F is not algebraically closed.
2. Let F be a finite field and $F(\alpha, \beta)/F$ be an algebraic extension. Show that $\exists c \in F(\alpha, \beta)$ s.t. $F(\alpha, \beta) = F(c)$. i.e. $F(\alpha, \beta)/F$ is a simple extension.

Ex 3.3.

1. Let F be a finite field and G, H be subgroups of $(F^\times, \cdot, 1)$. If $|G| = |H| = n$, then $G = H$.
2. If F is a field such that $(F^\times, \cdot, 1)$ is cyclic, then F is a finite field.

Ex 3.4.

1. For any prime p and any nonzero $a \in \mathbb{F}_p$, prove that $x^p - x + a$ is irreducible and separable.
2. Show that $f(x) = x^3 + px + q \in K[x]$ is separable $\iff 4p^3 + 27q^2 \neq 0$.

Ex 3.5. Let L/K be a separable extension and $f(x) \in K[x]$ be an irreducible polynomial. Assume that $f(x) = f_1(x) \cdots f_n(x)$ for some $f_i(x) \in L[x] \quad \forall i = 1, \dots, n$. Show that if f_i is separable $\forall i$, then f is separable.

Ex 3.6.

1. If $\text{char } K = p \neq 0$ and $[L : K] < \infty$ with $p \nmid [L : K]$, then L is separable over K .
2. Let $\text{char } K = p \neq 0$. Show that an algebraic element $\alpha \in L$ is separable over $K \iff K(\alpha) = K(\alpha^{p^n})$ for all $n \geq 1$.

4 Week 4

Ex 4.1.

1. Determine the Galois group of $f(x) = x^5 - 4x + 2$ over \mathbb{Q} .
2. Determine the Galois group of $f(x) = x^3 - 3x + 1$ over \mathbb{Q} .

Ex 4.2. Let $\text{char } K = 0$ and F/K be finite, normal. Let $g(x) \in K[x]$ and L be a splitting field of $g(x)$ over F . Show that L/K is a normal extension.

(Note: $g(x) \in K[x]$ but L is over F)

Ex 4.3.

Def 1.

- A character χ of a group G with values in a field L is a homomorphism $\chi : G \rightarrow L^\times$.
- The characters χ_1, \dots, χ_n of G are said to be linearly independent over L if there is no nontrivial relation

$$a_1\chi_1 + \dots + a_n\chi_n = 0, \quad a_1, \dots, a_n \in L \text{ are not all } 0$$

as a function on G .

1. Show that if χ_1, \dots, χ_n are distinct characters of G with values in L , then they are linearly independent over L .
2. Show that if $\sigma_1, \dots, \sigma_n$ are distinct monomorphisms from K to L , then they are linearly independent over L .
3. Show that distinct automorphisms of K are linearly independent over K .

Ex 4.4.

1. If L/K is Galois, then $\exists f$: irr. in $K[x]$ s.t. L is a splitting field of $f(x)$ over K .
2. TFAE
 - (a) L/K is a Galois extension.
 - (b) K is the fixed field of a subgroup of $\text{Aut}(L)$.
 - (c) K is the fixed field of $\text{Aut}(L/K)$.

Ex 4.5. Find the Galois group of $x^4 - 2$ over \mathbb{Q} . Find all subgroups of this group and find all corresponding intermediate fields between the splitting field of $x^4 - 2$ over \mathbb{Q} and \mathbb{Q} .

Ex 4.6. Find all proper subfields of $\mathbb{Q}\left(\sqrt[3]{5}, \frac{-1+i\sqrt{3}}{2}\right)$ and $\mathbb{Q}(i, \sqrt{7})$ respectively.

5 Week 5

Ex 5.1.

1. Let p be an odd prime with $p \nmid m$. Suppose $a \in \mathbb{Z}$ s.t. $\Phi_m(a) \equiv 0 \pmod{p}$. then $\text{ord}(a) = m$ in $(\mathbb{Z}/p\mathbb{Z})^\times$. (hint: $x^m - 1 = \prod_{d|m} \Phi_d(x)$)
2. Let $a \in \mathbb{Z}$. Show that if p is an odd prime dividing $\Phi_m(a)$, then either $p \mid m$ or $p \equiv 1 \pmod{m}$.

Ex 5.2.

1. Show that $\left[\mathbb{Q} \left(\zeta_n + \frac{1}{\zeta_n} \right) : \mathbb{Q} \right] = \frac{\varphi(n)}{2}$.
2. Find Φ_8, Φ_9 .
3. Show that $x^{16} + 1$ is irreducible in $\mathbb{Q}[x]$ and is reducible in $\mathbb{F}_7[x]$ as a product of 4 quartic polynomials.

Ex 5.3. show that p : odd prime, $(\mathbb{Z}/p^e\mathbb{Z})^\times$ is cyclic of order $p^{e-1}(p-1)$ and $(\mathbb{Z}/2^e\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{e-2}\mathbb{Z}$, $e \geq 2$.

Hints:

1. Check $(1+p)^{p^{e-1}} \equiv 1 \pmod{p^e}$ but $(1+p)^{p^{e-2}} \not\equiv 1 \pmod{p^e}$. And for $e \geq 3$, $(1+2^2)^{2^{e-2}} \equiv 1 \pmod{2^e}$ but $(1+2^2)^{2^{e-3}} \not\equiv 1 \pmod{2^e}$.
2. If each Sylow p -subgroup of G is normal, then G is isomorphic to the product of all sylow p -subgroups.

Ex 5.4.

- (a) Let $\mathbb{C}(t)$ be the field of rational functions over \mathbb{C} and L be a splitting field of $x^n - t$ over $\mathbb{C}(t)$. Find $\text{Gal}(L/\mathbb{C}(t))$.
- (b) Let $\mathbb{F}_p(t)$ be the field of rational functions over \mathbb{F}_p and L be a splitting field of $x^3 - 2t$ over $\mathbb{F}_p(t)$. Find $\text{Gal}(L/\mathbb{F}_p(t))$.

Ex 5.5. Let $\text{char } K \neq 2, 3$ and $f(x) = x^4 + px^2 + qx + r$ be irr. and separable with roots $\alpha_1, \dots, \alpha_4$. Let $L = K(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $G_f = \text{Gal}(L/K) \leq S_4$. Set $\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$, $\beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$, $\beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3$.

- (a) Show that $L^{G_f \cap V} = K(\beta_1, \beta_2, \beta_3)$ and $\text{Gal}(K(\beta_1, \beta_2, \beta_3)/K) \cong G_f/G_f \cap V$ where $V = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \leq S_4$.
- (b) Show that there exists i s.t. $\beta_i \in K \iff G_f \leq D_4$.
- (c) Let $h(x) = (x - \beta_1)(x - \beta_2)(x - \beta_3) \in K[x]$ with discriminant $D(h)$, Show that
 - (1) If $h(x)$ is irr. and $D(h) \notin K^2$, then $G_f \cong S_4$.
 - (2) If $h(x)$ is irr. and $D(h) \in K^2$, then $G_f \cong A_4$.
 - (3) If $h(x)$ splits completely in $K[x]$, then $G_f \cong V$.
 - (4) Let $h(x)$ has one root in K . Then
 - (i) If $f(x)$ is irr. over $K(\beta_1, \beta_2, \beta_3)$, then $G_f \cong D_4$.
 - (ii) If $f(x)$ is reducible over $K(\beta_1, \beta_2, \beta_3)$, then $G_f \cong C_4$.

6 Week 6

Ex 6.1. Is $f(x) = 2x^5 - 10x + 5 \in \mathbb{Q}[x]$ solvable by radicals? Justify your answer!

Ex 6.2. Show that if $|G| = p^2q$ where p, q are distinct primes, then G is solvable.

Ex 6.3. Solve $x^4 + ax + b = 0$ in terms of radicals.

Ex 6.4. power sum: $p_k = \sum_{i=1}^n x_i^k$. show that newton identities: $s_0 = 1$,

$$ks_k = \sum_{i=1}^k (-1)^{i-1} s_{k-i} p_i, \quad p_k = \sum_{i=1}^{k-1} (-1)^{i+k-1} s_{k-i} p_i + (-1)^{k-1} k s_k$$

(s_k are the elementary symmetric polynomials in x_1, \dots, x_n .)

Ex 6.5. For any prime $p \geq 5$. Let $k, m, n_1, \dots, n_{k-2} \in \mathbb{Z}$ s.t.

$$\begin{cases} k \text{ is odd and } > 3, \\ m \text{ is even and } > 0, \\ n_1, \dots, n_{k-2} \text{ are even and } n_1 < n_2 < \dots < n_{k-2}. \end{cases}$$

Consider $g(x) = (x^2 + m)(x - n_1) \dots (x - n_{k-2})$ and $f(x) = g(x) - 2 \in \mathbb{Z}[x]$.

1. Show that f is irr. in $\mathbb{Z}[x]$
2. Show that f has exactly two non-real roots for $m \gg 0$. If $k = p$, then $G_f \cong S_p$.

7 Week 7

Ex 7.1.

1. Let $\alpha_1, \dots, \alpha_n$ be roots of $f(x)$ and then

$$\delta = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{vmatrix}$$

Show that

$$D = \begin{vmatrix} n & p_1 & \dots & p_{n-1} \\ p_1 & p_2 & \dots & p_n \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_n & \dots & p_{2n-2} \end{vmatrix}, \quad p_i = \sum_{k=1}^n \alpha_k^i$$

2. If $f(x) = x^n + px + q$, then $D = y_{n+1}n^n q^{n-1} + y_n(n-1)^{n-1}p^n$, where

$$y_n = \begin{cases} 1, & n \equiv 1, 2 \pmod{4} \\ -1, & n \equiv 0, 3 \pmod{4} \end{cases}$$

Ex 7.2. A transitive subgroup G of S_n containing a transposition and an $(n-1)$ -cycle is S_n .

Ex 7.3.

1. If $f(x) = x^5 - x - 1$, then $G_f \cong S_5$.
2. If $f(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$, then $G_f \cong \mathbb{Z}/5\mathbb{Z}$.

8 Week 8

Ex 8.1. Use Zorn's lemma to show the existence of a transcendence base S of any extension L/K . (S may equal to \emptyset)

Ex 8.2. Given $L/M, M/K$, show that $\text{tr deg}_K L = \text{tr deg}_M L + \text{tr deg}_K M$.

Ex 8.3. Show that for any extension L/K , $\text{Tr} : L \rightarrow K$ is surjective.

Ex 8.4. If L/K with $|L| < \infty$, show that $N_{L/K} : L^\times \rightarrow K^\times$ is also surjective.

9 Week 9

Ex 9.1.

1. $\text{ED} \implies \text{PID}$.
2. $\text{ED} \implies \text{GCD domain}$.

Ex 9.2. A_{-19} is a PID but not a ED.

10 Week 10

Ex 10.1.

1. If \sqrt{I} is a maximal ideal, then I is primary.
2. A power of a maximal ideal is m -primary.

Ex 10.2.

Def 2. The Jacobson radical of R is the intersection of all maximal ideals of R and is denoted by $\text{Jac } R$.

Show that $x \in \text{Jac } R \iff 1 - rx$ is a unit $\forall r \in R$.

Ex 10.3.

1. Show that if M is a finitely generated R -module and $IM = M$ with $I \subseteq \text{Jac } R$, then $M = 0$.
2. Show that if M is a finitely generated R -module, N is a submodule of M and $I \subseteq \text{Jac } R$, then

$$M = IM + N \implies M = N.$$

Ex 10.4.

1. $V_1 \subseteq V_2 \iff \mathcal{I}(V_1) \supseteq \mathcal{I}(V_2)$.
2. $\mathcal{I}(V_1 \cup V_2) = \mathcal{I}(V_1) \cap \mathcal{I}(V_2)$.
3. $V_1 \cup V_2 = \mathcal{V}(\mathcal{I}(V_1)\mathcal{I}(V_2))$.
4. $\bigcap_{\lambda \in \Lambda} V_\lambda = \mathcal{V}\left(\sum_{\lambda \in \Lambda} \mathcal{I}(V_\lambda)\right)$.
5. V is irr. $\iff \mathcal{I}(V) \in \text{Spec } k[x_1, \dots, x_n]$.

Ex 10.5.

1. $f(\alpha) \in W \quad \forall \alpha \in V$.
2. $\varphi = f^*$.
3. For different representations of R , $R \cong k[x_1, \dots, x_n]/I_1 \xrightarrow[\Psi^{-1}]{\Psi} k[z_1, \dots, z_l]/I_2$ and $\begin{cases} V_1 = \mathcal{V}(I_1) \\ V_2 = \mathcal{V}(I_2) \end{cases}$.
Show that Ψ (and Ψ^{-1}) will give an isom. of V_1 and V_2 .

Ex 10.6.

1. Show that if $V = \mathcal{V}(y - x^2) \subseteq \mathbb{A}_{\mathbb{C}}^2$, then

$$\begin{array}{l} f : \mathbb{A}_{\mathbb{C}}^1 \rightarrow V \\ t \mapsto (t, t^2) \end{array} \quad \text{is an isom.}$$

2. Show that if $V = \mathcal{V}(y^2 - x^3) \subseteq \mathbb{A}_{\mathbb{C}}^2$, then

$$\begin{array}{l} f : \mathbb{A}_{\mathbb{C}}^1 \rightarrow V \\ t \mapsto (t^2, t^3) \end{array} \quad \text{is bijective but not an isom.}$$

11 Week 11

Ex 11.1.

- (a) Show that $\{y - x^2, z - x^3\}$ is not a Gröbner basis for $x > y > z$.
- (b) Find a Gröbner basis for $\langle y - x^2, z - x^3 \rangle$.

Ex 11.2.

- (a) Find a Gröbner basis of $I = \langle -x^3 + y, x^2y - y^2 \rangle$.
- (b) Check $x^6 - x^5y \stackrel{?}{\in} I$.

Def 3.

- A Gröbner basis $G = \{g_1, \dots, g_m\}$ of I is said to be minimal if $\text{LT}(g_i)$ is monic for all i and $\forall j, \text{LT}(g_i) \notin \langle \text{LT}(G \setminus \{g_j\}) \rangle$.
- A minimal Gröbner basis $G = \{g_1, \dots, g_m\}$ is said to be reduced if for all j , no term in g_j is divisible by any of $\text{LT}(g_1), \dots, \text{LT}(g_{j-1}), \text{LT}(g_{j+1}), \dots, \text{LT}(g_m)$.

Ex 11.3.

1. Show that for a given monomial ordering, every non-zero ideal I in $k[x_1, \dots, x_n]$ has a unique reduced Gröbner basis.
2. Show that $I = \langle x^2y + xy^2 - 2y, x^2 + xy - x + y^2 - 2y, xy^2 - x - y + y^3 \rangle$ and $J = \langle x - y^2, xy - y, x^2 - y \rangle$ are equal.

Ex 11.4. Let $I = \langle x^2 + xy^5 + y^4, xy^6 - xy^3 + y^5 - y^2, xy^5 - xy^2 \rangle, x > y$. Find the reduced Gröbner basis of I .

Ex 11.5.

1. $\left(I : \left(\sum_{i=1}^r J_i \right) \right) = \bigcap_{i=1}^r (I : J_i)$. Here, $(I : J) \triangleq \{x \in R \mid xJ \subseteq I\}$.
2. $\left(\bigcap_{i=1}^r I_i : J \right) = \bigcap_{i=1}^r (I_i : J)$.
3. $((I_1 : I_2) : I_3) = (I_1 : I_2 I_3)$.

12 Week 12

Ex 12.1. The following sets are correspondent:

- (1) V (an affine algebraic set)
- (2) $\text{Max } k[V]$
- (3) $\text{Hom}_k(k[V], k)$

Def 4. Let S/R be an extension of rings and $I \subseteq R$ be an ideal. $a \in S$ is said to be integral over I if $\exists f(x) = x^n + r_1x^{n-1} + \cdots + r_n$ with $n > 0$ and $r_i \in I$ s.t. $f(a) = 0$.

Ex 12.2. Let $a \in S$. Show that TFAE

- (1) a is integral over I .
- (2) $R[a]$ is a finitely generated R -module and $a \in \sqrt{IR[a]}$. Here $IR[a]$ is an ideal in $R[a]$ generated by I .
- (3) There exists a subring S' of S (with $R[a] \subset S'$) s.t. S' is a finitely generated R -module and $a \in \sqrt{IS'}$.

Ex 12.3. Show that $\sqrt{I\bar{R}} = \{a \in S \mid a \text{ is integral over } I\}$ where \bar{R} is the integral closure of R in S .

Ex 12.4. If $\phi : R_1 \rightarrow R_2$ is a ring homomorphism and $p \in \text{Spec } R_1$, then

$$p = \phi^{-1}(q) \text{ for some } q \in \text{Spec } R_2 \iff p = \phi^{-1}(R_2\phi(p))$$

Ex 12.5. In a UFD R , $\forall p \in \text{Spec } R$ with $h(p) = 1$, $p = \langle \alpha \rangle$ for some prime element $\alpha \in R$.

13 Week 13

Ex 13.1. Let $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ be exact in \mathbf{Mod}_R . Show that N, L are Artinian (Noetherian) $\iff M$ is Artinian (Noetherian).

Ex 13.2. Let R be an Artinian local ring with maximal ideal \mathfrak{m} , then TFAE:

1. R is a PID
2. \mathfrak{m} is principal
3. $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leq 1$

Ex 13.3. (Krull's Intersection Theorem) Let R be a Noetherian local ring with maximal ideal \mathfrak{m} , show that

$$\bigcap_{n=0}^{\infty} \mathfrak{m}^n = \langle 0 \rangle.$$

Def 5. Let R be an integral domain and $I, J \subseteq R$. We say $I \mid J$ if $\exists I' \subseteq R$ s.t. $II' = J$ (so $J \subseteq I$)

Ex 13.4. Let R be a Dedekind domain, $I, J \subseteq R$ and
$$\begin{cases} I = P_1^{e_1} P_2^{e_2} \cdots P_n^{e_n} \\ J = P_1^{f_1} P_2^{f_2} \cdots P_n^{f_n} \end{cases}, e_i, f_j \geq 0.$$
 Show that

1. $I \subseteq J \iff J \mid I \iff f_i \leq e_i$.
2. $I + J = \langle I, J \rangle = P_1^{d_1} P_2^{d_2} \cdots P_n^{d_n}$ where $d_i = \min\{f_i, e_i\}$.

Ex 13.5. Let R be a Dedekind domain, $I \subseteq R$ and $I = P_1^{e_1} P_2^{e_2} \cdots P_n^{e_n}$. Show that

1. $\exists J \subseteq R$ with $I + J = R$ s.t. $IJ = \langle a \rangle$ for some $a \in R$.
2. $I \subseteq R \implies R/I$ is a principal ring.
3. $I \subseteq R$, let $a \in I$ with $a \neq 0$. $\implies I = \langle a, b \rangle$ for some $b \in I$.

14 Week 14

Ex 14.1. If $0 \rightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \rightarrow 0$ is exact in \mathbf{Mod}_R , then for $M, N \in \mathbf{Mod}_R$,

- $0 \rightarrow \mathrm{Hom}_R(M, M_1) \xrightarrow{\alpha'} \mathrm{Hom}_R(M, M_2) \xrightarrow{\beta'} \mathrm{Hom}_R(M, M_3)$ is exact,
- $0 \rightarrow \mathrm{Hom}_R(M_1, N) \xrightarrow{\beta''} \mathrm{Hom}_R(M_2, N) \xrightarrow{\alpha''} \mathrm{Hom}_R(M_3, N)$ is exact,
- $M \otimes_R M_1 \xrightarrow{1_M \otimes \alpha} M \otimes_R M_2 \xrightarrow{1_M \otimes \beta} M \otimes_R M_3 \rightarrow 0$ is exact.

Ex 14.2. Give examples to explain why $\mathrm{Hom}_R(M, \cdot)$, $\mathrm{Hom}_R(\cdot, N)$, $M \otimes_R \cdot$ are not exact functors.

Ex 14.3. Show that $h(y) \in \mathrm{Hom}_{\mathbb{Z}}(R, N_0)$, $h \in \mathrm{Hom}_R(M_2, N)$ and $h \circ \alpha = f$.