

Algebra

June 14, 2017

1 Introduction to Homological Algebra

1.1 Projective, Injective and Flat modules (week 14)

Def 1.

- $M \in \mathbf{Mod}_R$ is **projective** if $\text{Hom}(M, \cdot)$ preserves the *right* exactness.
- $N \in \mathbf{Mod}_R$ is **injective** if $\text{Hom}(\cdot, N)$ preserves the *right* exactness.
- $M \in \mathbf{Mod}_R$ is **flat** if $M \otimes \cdot$ preserves the *left* exactness.

Fact 1.1.1.

- M is projective \iff

$$\begin{array}{ccc} & M & \\ \exists \tilde{f} \swarrow & \downarrow f & \\ M_2 & \longrightarrow & M_3 \longrightarrow 0 \\ 0 \longrightarrow M_1 & \longrightarrow & M_2 \\ & \downarrow g & \swarrow \exists \tilde{g} \\ & N & \end{array}$$
- N is injective \iff
- free \implies projective: If $X = \{x_i \mid i \in \Lambda\}$ and $f : x_i \mapsto a_i$. Since β onto, exists b_i so that $\beta(b_i) = a_i$. we can then set $\tilde{f} : x_i \mapsto b_i$ by the universal property of free module.

$$\begin{array}{ccc} & F(X) & \\ \exists \tilde{f} \swarrow & \downarrow f & \\ M_2 & \xrightarrow{\beta} & M_3 \longrightarrow 0 \end{array}$$

- free \implies flat: Let $F \cong R^{\oplus \Lambda}$ be a free module, and M_1, M_2 be two modules such that $0 \rightarrow M_1 \rightarrow M_2$. Since $R \otimes_R M \cong M$, we have

$$\begin{aligned} 0 \rightarrow M_1 \rightarrow M_2 & \quad \text{exact} \\ \implies 0 \rightarrow R \otimes M_1 \rightarrow R \otimes M_2 & \quad \text{exact} \\ \implies 0 \rightarrow \bigoplus_{i \in \Lambda} R \otimes M_1 \rightarrow \bigoplus_{i \in \Lambda} R \otimes M_2 & \quad \text{exact} \\ \stackrel{(a)}{\implies} 0 \rightarrow R^{\oplus \Lambda} \otimes M_1 \rightarrow R^{\oplus \Lambda} \otimes M_2 & \quad \text{exact} \\ \implies 0 \rightarrow F \otimes M_1 \rightarrow F \otimes M_2 & \quad \text{exact} \end{aligned}$$

Where (a) is by the fact that $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$. Thus F flat.

- If S is a multiplication closed set in R with $1 \in S$, then

$$0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0 \implies 0 \rightarrow M_S \rightarrow N_S \rightarrow L_S \rightarrow 0.$$

We know that $M_S \cong R_S \otimes_R M$. So R_S is a flat R -module. e.g. \mathbb{Q} is a flat \mathbb{Z} -module.

For any $M \in \mathbf{Mod}_R$, a projective module N such that $N \rightarrow M \rightarrow 0$ could be easily found: Simply let $N = F$, a free module on the generating set of M .

Now we shall ask for any module M , does there exist $N \in \mathbf{Mod}_R$ such that N is injective and $0 \rightarrow M \rightarrow N$?

Theorem 1 (Baer's criterion). N is injective $\iff \forall I \subset R$, and a homomorphism f , there exists a homomorphism h such that the following diagram commutes:

$$\begin{array}{ccc} 0 \longrightarrow & I & \longrightarrow R \\ & \downarrow f & \swarrow \exists h \\ & N & \end{array}$$

Proof. “ \Rightarrow ”: See I as an R module, then it is immediate by the definition of injective module.

“ \Leftarrow ”: Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 \\ & & \downarrow g & & \\ & & N & & \end{array}$$

Let $S \triangleq \{(M, \rho) \mid M_1 \subseteq M \subseteq M_2 \text{ and } \rho : M \rightarrow N \text{ extends } g\} \neq \emptyset$ since $(M_1, g) \in S$.

By the routinely proof using Zorn's lemma, exists a maximal element $(M^*, \mu) \in S$.

We claim that $M^* = M_2$. If not, pick $a \in M_2 \setminus M^*$ and let $M' \triangleq M^* + Ra \subsetneq M^*$, $I \triangleq \{r \in R \mid ra \in M^*\}$. Define $f : I \rightarrow N$ with $r \mapsto \mu(ra)$. Then we have an extension $h : R \rightarrow N$ of f .

Now, let $\mu' : M' \rightarrow N = x + ra \mapsto \mu(x) + h(r)$. We shall prove that this map is well-defined: If $x_1 + r_1a = x_2 + r_2a$, then $(r_1 - r_2)a = x_2 - x_1 \in M \implies r_1 - r_2 \in I$. So $h(r_1) - h(r_2) = f(r_1 - r_2) = \mu((r_1 - r_2)a) = \mu(x_2) - \mu(x_1)$, which prove μ' is well defined, and the existence of μ' contradicts the fact that (M^*, μ) is maximal. \square

Def 2. M is **divisible** if $\forall x \in M, r \in R \setminus \{0\}$, there exists $y \in M$ such that $x = ry$, i.e. $rM = M \quad \forall r \in R \setminus \{0\}$.

Prop 1.1.1.

1. Every injective module N over an integral domain is divisible.

Proof. For any $x_0 \in N$ and $r_0 \in R \setminus \{0\}$. Let $I = \langle r_0 \rangle \subset R$. As long as R is an integral domain, $I \cong R$ as an R -module, so the R -module homomorphism $f : I \rightarrow N = rr_0 \mapsto rx_0$ is well-defined. Since N injective, this map extends to $h : R \rightarrow N$. Let $y_0 \triangleq h(1)$, then $r_0y_0 = r_0h(1) = h(r_0) = x_0$. Thus N injective. \square

2. Every divisible module N over an PID is injective.

Proof. For any $I \subseteq R$ and a homomorphism $f : I \rightarrow N$, if $I = 0$ then $h = x \mapsto 0$ is always an extension of f . So assume $I \neq 0$. Since R is a PID, $I = \langle r_0 \rangle$ for some $r_0 \neq 0 \in R$. By the fact that N divisible, exists $y_0 \in N$ such that $r_0y_0 = x_0 \triangleq f(r_0)$.

Now we could define $h : R \rightarrow N$ by $1 \mapsto y_0$. Then $h(r_0) = r_0h(1) = r_0y_0 = x_0$, thus h is an extension of f and N injective. \square

3. If R is a PID, then any quotient N of a injective R -module M is injective.

Proof. By 2., $rM = M$ for any $r \neq 0$, thus $rN = N$ for any $r \neq 0$, and hence N injective. \square

Theorem 2. For any $M \in \mathbf{Mod}_R$, there exists an injective module N containing M .

Proof.

Case 1: $R = \mathbb{Z}$.

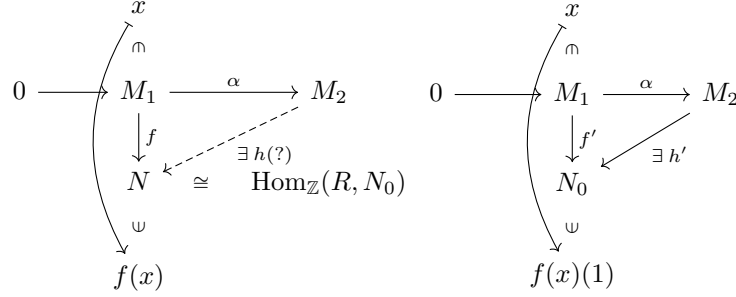
Let $X = \{x_i\}_{i \in \Lambda}$ be a generating set for M and F is free on X . Let f be the natural map from f to M . then $M \cong F/\ker f$.

Define $F' = \bigoplus_{i \in \Lambda} \mathbb{Q}e_i \subset F$, which is obviously a divisible \mathbb{Z} -module. Then $M \subseteq F'/\ker f \triangleq M'$, where M' is injective by proposition 1.1.1.

Case 2: R arbitrary.

We can regard any M as a \mathbb{Z} -module, then there exists an injective module $N_0 \supset M$. Now, we have an R -module $N \triangleq \text{Hom}_{\mathbb{Z}}(R, N_0)$ with multiplication $rf(x) \triangleq x \mapsto f(xr)$.

We claim that N is injective. For any $f :: M_1 \rightarrow N$, and a homomorphism $\alpha :: M_1 \rightarrow M_2$, then α could be take as a \mathbb{Z} -module homomorphism. Define $f' :: M_1 \rightarrow N_0$ by $x \mapsto f(x)(1)$. Since N_0 injective, exists h' , a \mathbb{Z} module homomorphism from M_2 to N_0 .



Now, define

$$\begin{aligned} h : M_2 &\longrightarrow N \\ y &\longmapsto h(y) : R \longrightarrow N_0 \\ 1 &\longmapsto h'(y) \\ r &\longmapsto h'(ry) \end{aligned}$$

We check that h is well-defined.

- $h(y) \in \text{Hom}_{\mathbb{Z}}(R, N_0)$

$$h(y)(r_1 + r_2) = h'((r_1 + r_2)y) = h'(r_1y + r_2y) = h'(r_1y) + h'(r_2y) = h(y)(r_1) + h(y)(r_2)$$

- $h \in \text{Hom}_R(M_2, N)$

$$\begin{aligned} h(r_1y_1 + y_2)(r) &= h'(r(r_1y_1 + y_2)) = h'(rr_1y_1 + ry_2) \\ &= h'(rr_1y_1) + h'(ry_2) \\ &= h(y)(rr_1) + h(y_2)(r) \\ &= (r_1h(y))(r) + h(y_2)(r) \end{aligned}$$

- Show diagram commute $f = h \circ \alpha$ Fix $y \in M_1$, then $\forall r \in R$:

$$\begin{aligned} (h \circ \alpha)(y)(r) &= h(\alpha(y))(r) = h'(r\alpha(y)) \\ &= h'(\alpha(ry)) = f'(ry) \\ &= f(ry)(1) = rf(y)(1) \\ &= f(y)(r) \end{aligned}$$

Thus N_0 injective.

Now notice that, $\text{Hom}_{\mathbb{Z}}(R, \cdot)$ is a left exact functor, so $M \hookrightarrow N_0$ implies $\text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, N_0)$, thus $M \cong \text{Hom}_R(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, N_0) = N$. \square

Prop 1.1.2. TFAE

1. M is projective.
2. Every exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$ split.

3. $\exists M'$ s.t. $M \oplus M' \cong F$: free.

Proof.

(1) \Rightarrow (2) : Since M projective, the map λ with $\beta \circ \lambda = \text{Id}$ exists in the following diagram:

$$\begin{array}{ccc} & M & \\ \swarrow \exists \lambda & \downarrow \text{Id} & \\ M_2 & \xrightarrow{\beta} & M \longrightarrow 0 \end{array}$$

Then λ is a lifting, so $M_2 \cong M_1 \oplus M$ and $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$ split.

(2) \Rightarrow (3): Let F be a free module on a generating set of M , and $\beta :: F \rightarrow M$ be the natural map, then $0 \rightarrow \ker \beta \rightarrow F \rightarrow M \rightarrow 0$ split, so $F \cong \ker \beta \oplus M$.

(3) \Rightarrow (1): For any $M_2 \rightarrow M_3 \rightarrow 0$, since $M' \oplus M$ free and thus projective, λ' exists in the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \longrightarrow & M' \oplus M & \xleftarrow[\pi]{\mu} & M \longrightarrow 0 \\ & & & & \downarrow \exists \lambda' & & \downarrow f \\ & & & & M_2 & \xrightarrow{\beta} & M_3 \longrightarrow 0 \end{array}$$

Define $\lambda = \lambda' \circ \mu$. Then $\beta \circ \lambda = \beta \circ \lambda' \circ \mu = f \circ \pi \circ \mu = f$. □

Prop 1.1.3. TFAE

1. M is injective.
2. Each exact sequence $0 \rightarrow M \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ split.

Proof. (1) \Rightarrow (2): Similar to the projective case, μ exists in the following diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & M \xrightarrow{\alpha} M_2 \\ & & \downarrow \text{Id} \\ & & M \end{array} \quad \begin{array}{c} \nearrow \exists \mu \\ \nwarrow \end{array}$$

So $M_2 = M \oplus M_3$.

(2) \Rightarrow (1): By theorem 2, there is a module $N \subset M$ so that N is injective.

Consider $0 \longrightarrow M \xrightleftharpoons[\exists \mu]{i} N \longrightarrow \text{coker } i \longrightarrow 0$ split exact and $\mu \circ i = \text{Id}_M$. Since N injective, h' exists in the following diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & M_1 \xrightarrow{\alpha} M_2 \\ & & \downarrow f \\ & & M \\ & \nearrow i \circ f & \downarrow i \\ & & N \end{array} \quad \begin{array}{c} \nearrow \exists h' \\ \nwarrow \mu \end{array}$$

Let $h = \mu \circ h'$, then $h \circ \alpha = \mu \circ h' \circ \alpha = \mu \circ i \circ f = f$. □

Prop 1.1.4. projective \implies flat.

Proof. Observe that $\bigoplus_{i \in \Lambda} M_i$ is flat if and only if M_i is flat for each i , since if $0 \rightarrow N_1 \xrightarrow{\alpha} N_2$ exact, then

$$\begin{array}{ccc} 0 & \longrightarrow & (\bigoplus M_i) \otimes N_1 \xrightarrow{1 \otimes \alpha} (\bigoplus M_i) \otimes N_2 \\ & & \parallel \qquad \qquad \qquad \parallel \\ 0 & \longrightarrow & \bigoplus (M_i \otimes N_1) \xrightarrow{\bigoplus (1 \otimes \alpha)} \bigoplus (M_i \otimes N_2) \\ & & \updownarrow \\ 0 & \longrightarrow & M_i \otimes N_1 \xrightarrow{1 \otimes \alpha} M_i \otimes N_2 \qquad \forall i \in \Lambda \end{array}$$

If M is projective, then by proposition 1.1.2 $\exists M'$ such that $M \oplus M' \cong F$ is free. Since free implies flat, by above, M is flat. \square

Def 3.

- A chain complex C_\bullet of R -modules is a sequence and maps:

$$C_\bullet : \cdots \rightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \rightarrow \cdots \rightarrow C_1 \xrightarrow{d_1} C_0 \rightarrow 0$$

with $d_n \circ d_{n+1} = 0$, $\forall n$. (i.e. $\text{Im } d_{n+1} \subseteq \ker d_n$)

Then define

- $Z_n(C_\bullet) \triangleq \ker d_n$ is the n -cycle.
- $B_n(C_\bullet) \triangleq \text{Im } d_{n+1}$ is the n -boundary.
- $H_n(C_\bullet) \triangleq Z_n(C_\bullet)/B_n(C_\bullet)$ is called the n -th homology.

- A cochain complex C^\bullet of R -modules is a sequence and maps:

$$C^\bullet : 0 \rightarrow C^0 \xrightarrow{d^1} C^1 \rightarrow \cdots \rightarrow C^{n-1} \xrightarrow{d^n} C^n \xrightarrow{d^{n+1}} C^{n+1} \rightarrow \cdots$$

with $d^{n+1} \circ d^n = 0$, $\forall n$. (i.e. $\text{Im } d^n \subseteq \ker d^{n+1}$)

Then define

- $Z^n(C^\bullet) \triangleq \ker d^{n+1}$ is the n -cocycle.
- $B^n(C^\bullet) \triangleq \text{Im } d^n$ is the n -coboundary.
- $H^n(C^\bullet) \triangleq Z^n(C^\bullet)/B^n(C^\bullet)$ is called the n -th cohomology.

- $\varphi : C_\bullet \rightarrow \tilde{C}_\bullet$ is a chain map if the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} \longrightarrow \cdots \\ & & \downarrow \varphi_{n+1} & & \downarrow \varphi_n & & \downarrow \varphi_{n-1} \\ \cdots & \longrightarrow & \tilde{C}_{n+1} & \xrightarrow{\tilde{d}_{n+1}} & \tilde{C}_n & \xrightarrow{\tilde{d}_n} & \tilde{C}_{n-1} \longrightarrow \cdots \end{array}$$

Observe that $\varphi_n(\ker d_n) \subseteq \ker \tilde{d}_n$ and $\varphi_n(\text{Im } d_{n+1}) \subseteq \text{Im } \tilde{d}_{n+1}$. This will induce the following maps:

$$\begin{aligned} \varphi_* : H_n(C_\bullet) &\rightarrow H_n(\tilde{C}_\bullet) \\ x + B_n(C_\bullet) &\mapsto \varphi_n(x) + B_n(\tilde{C}_\bullet) \end{aligned}$$

- $f : C_\bullet \rightarrow \tilde{C}_\bullet$ is null homotopic if $\exists s_n : C_n \rightarrow \tilde{C}_{n+1}$ s.t. $f_n = \tilde{d}_{n+1} \circ s_n + s_{n-1} \circ d_n$, $\forall n$.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} \longrightarrow \cdots \\ & & \downarrow f_{n+1} & \nearrow s_n & \downarrow f_n & \nearrow s_{n-1} & \downarrow f_{n-1} \\ \cdots & \longrightarrow & \tilde{C}_{n+1} & \xrightarrow{\tilde{d}_{n+1}} & \tilde{C}_n & \xrightarrow{\tilde{d}_n} & \tilde{C}_{n-1} \longrightarrow \cdots \end{array}$$

Prop 1.1.5. If f is null homotopic, then $f_* = 0$.

Proof. $f_*(x) = \tilde{d}_{n+1}s_n(x) + s_{n-1}d_n(x) = \tilde{d}_{n+1}s_n(x) \in B_n(\tilde{C}_\bullet) \implies f_*(\bar{x}) = 0$. \square

- Two chain map $f, g : C_\bullet \rightarrow \tilde{C}_\bullet$ are homotopic if $f - g$ is null homotopic. ($f_* = g_*$)
- Let $M \in \mathbf{Mod}_R$. A projection resolution of M is an exact sequence:

$$\cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\alpha} M \rightarrow 0$$

where P_i is projective for all i .

For any M , projection resolution always exists. Let P_0 be a free module on the generators of M . We get $P_0 \xrightarrow{\alpha} M \rightarrow 0$. Similarly, let P_1 be free on $\ker \alpha$, then we could extend the map to $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$. Continue the process we would get a diagram as below, where K_i are the kernels:

$$\begin{array}{ccccccc} \cdots & \rightarrow & P_2 & \xrightarrow{\quad} & P_1 & \xrightarrow{\quad} & P_0 \rightarrow M \rightarrow 0 \\ & & \searrow & & \swarrow & & \swarrow \\ & & & K_1 & & & K_0 \\ & \swarrow & & \searrow & \swarrow & & \searrow \\ 0 & & & & 0 & & 0 \end{array}$$

Theorem 3 (Comparison theorem). Given two chain as following:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 \xrightarrow{\alpha} M \longrightarrow 0 & \text{(projective resolution)} \\ & & \downarrow \exists f_2 & & \downarrow \exists f_1 & & \downarrow \exists f_0 & \downarrow f \\ \cdots & \longrightarrow & \tilde{C}_2 & \xrightarrow{d'_2} & \tilde{C}_1 & \xrightarrow{d'_1} & \tilde{C}_0 \xrightarrow{\alpha'} N \longrightarrow 0 & \text{(exact sequence)} \end{array}$$

Then $\exists f_i : P_i \rightarrow C_i$ s.t. $\{f_i\}$ forms a chain map making the completed diagram commutes. And any two such chain maps are homotopic.

Proof. Using induction on n .

For $n = 0$, the existence of f_0 is guaranteed by the definition of projective module.

$$\begin{array}{ccc} & P_0 & \\ \swarrow \exists f_0 & \downarrow f \circ \alpha & \\ C_0 & \longrightarrow & N \longrightarrow 0 \end{array}$$

For $n > 0$, we claim that $f_{n-1}d_n(P_n) \subseteq \text{Im } d'_n$, since $d'_{n-1}f_{n-1}d_n(x) = f_{n-2}d_{n-1}d_n(x) = 0$ and by the fact that C is exact, $f_{n-1}d_n(x) \in \ker d'_{n-1} = \text{Im } d'_n$. So using the diagram and again by the definition of projective module, f_n exists.

$$\begin{array}{ccc} & P_n & \\ \swarrow \exists f_n & \downarrow f_{n-1} \circ d_n & \\ C_n & \longrightarrow & \text{Im } d'_n \longrightarrow 0 \end{array}$$

Now, for another chain map $\{g_i : P_i \rightarrow C_i\}$, we shall construct suitable $\{s_n\}$ to prove they are homotopic. For $s_{-1} : M \rightarrow C_0$ we could simply pick the zero map. Again, if we could prove that $g_n - f_n - s_{n-1}d_n \in \text{Im } d'_{n+1} = \ker d'_n$, then by the definition of projective module, we would obtain s_n with

$$d_{n+1}s_n = g_n - f_n - s_{n-1}d_n \implies g_n - f_n = d_{n+1}s_n + s_{n-1}d_n$$

immediately. For this we calculate $d'_n(g_n - f_n - s_{n-1}d_n) = g_{n-1}d_n - f_{n-1}d_n - d'_n s_{n-1}d_n$. Notice that $d'_n s_{n-1} = g_{n-1} - f_{n-1} - s_{n-2}d_{n-1}$, and with $d_{n-1}d_n = 0$, we get $d'_n(g_n - f_n - s_{n-1}d_n) = 0$. \square

Def 4. Let $M \in \mathbf{Mod}_R$ and $\cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\alpha} M \rightarrow 0$ be a projective resolution of M . Fix $N \in \mathbf{Mod}_R$. Applying $\text{Hom}_R(\cdot, N)$ will get a complex:

$$0 \rightarrow \text{Hom}_R(M, N) \xrightarrow{\bar{\alpha}} \text{Hom}_R(P_0, N) \xrightarrow{\bar{d}_1} \text{Hom}_R(P_1, N) \rightarrow \cdots$$

Define

- $\text{Ext}_R^0(M, N) = \ker \bar{d}_1 = \text{Im } \bar{\alpha} \cong \text{Hom}_R(M, N)$.
- $\text{Ext}_R^n(M, N) = H^n(\text{Hom}(P_\bullet, N))$, $\forall n \geq 1$.

Theorem 4 (Indenpedency of the choice of projective resolutions). $\text{Ext}^n(M, N)$ is independent of the choice of the projective resolution used.

Proof. First, consider two projective resolutions of M, \tilde{M} , and map $f : M \rightarrow \tilde{M}$, and two liftings $\{f_i\}, \{g_i\}$. Use $\bar{\cdot}$ to denote the natural transformation from $X \rightarrow Y$ to $\text{Hom}(Y, N) \rightarrow \text{Hom}(X, N)$ by $\bar{f} \triangleq g \mapsto g \circ f$. Then we shall prove that $\bar{f}_\bullet^* = \bar{g}_\bullet^*$, which is to say \bar{f}_\bullet^* is independent of the lifting used.

By comparison theorem (3), $\{f_i\}, \{g_i\}$ are homotopic, and we could write down the diagram below:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\alpha} & M & \longrightarrow & 0 \\ & & f_2 \downarrow g_2 & \swarrow s_1 & f_1 \downarrow g_1 & \swarrow s_0 & f_0 \downarrow g_0 & & \downarrow f & & \\ \cdots & \longrightarrow & \tilde{P}_2 & \xrightarrow{\tilde{d}_2} & \tilde{P}_1 & \xrightarrow{\tilde{d}_1} & \tilde{P}_0 & \xrightarrow{\tilde{\alpha}} & \tilde{M} & \longrightarrow & 0 \end{array}$$

Notice that $\bar{\cdot}$ act linearly, that is, $\bar{f} + \bar{g} = \overline{f+g}$, and $\bar{f}g = \bar{g}\bar{f}$. So we have

$$g_n - f_n = s_{n-1}d_n + \tilde{d}_{n+1}s_n \implies \bar{g}_n - \bar{f}_n = \bar{d}_n\bar{s}_{n-1} + \bar{s}_n\bar{d}_{n+1}$$

and \bar{f}_n, \bar{g}_n are homotopic. Thus by proposition 1.1.5, $\bar{f}_\bullet^* = \bar{g}_\bullet^*$.

Now, let P^\bullet, P'^\bullet be two projective resolution. Consider the diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M \longrightarrow 0 \\ & & \text{Id} \downarrow f_1 & & \text{Id} \downarrow f_0 & & \downarrow \text{Id} \\ \cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow g_1 & & \downarrow g_0 & & \downarrow \text{Id} \\ \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M \longrightarrow 0 \end{array}$$

Then $g_i \circ f_i$ and Id are two liftings, and thus by previous we have $\bar{g}_i^* \circ \bar{f}_i^* = \text{Id}^*$. By symmetry, $\bar{f}_i^* \circ \bar{g}_i^* = \text{Id}^*$, which means that the homology calculated using different resolution are isomorphic. \square

Theorem 5 (Horseshoe Lemma). Given $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ and projective resolutions $P^\bullet \rightarrow L \rightarrow 0, \tilde{P}^\bullet \rightarrow N \rightarrow 0$. Then there is a projective resolution for M such that the following

diagram commutes:

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & P_1 & \longrightarrow & \bar{P}_1 & \longrightarrow & \tilde{P}_1 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_0 & \longrightarrow & \bar{P}_0 & \longrightarrow & \tilde{P}_0 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

Proof. Let $\bar{P}_n \triangleq P_n \oplus \tilde{P}_n$. \bar{P}_n is projective by the fact that sum of projective module are projective. Also $0 \rightarrow P_n \rightarrow P_n \oplus \tilde{P}_n \rightarrow \tilde{P}_n \rightarrow 0$ by injection and projection. It remains to show that the maps in the middle column exists.

By induction on n . Consider the following diagram:

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & P_n & \longrightarrow & P_n \oplus \tilde{P}_n & \longrightarrow & \tilde{P}_n \longrightarrow 0 \\
& & \downarrow \alpha & & \downarrow \bar{\alpha} & \swarrow \exists \sigma & \downarrow \tilde{\alpha} \\
0 & \longrightarrow & P_{n-1} & \xrightarrow{i} & \bar{P}_{n-1} & \xrightarrow{\pi} & \tilde{P}_{n-1} \longrightarrow 0 \\
& & \downarrow d & & \downarrow \bar{d} & & \downarrow \tilde{d} \\
0 & \longrightarrow & P_{n-2} & \longrightarrow & \bar{P}_{n-2} & \longrightarrow & \tilde{P}_{n-2} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

σ exists because \tilde{P}_n is projective. Define

$$\begin{aligned}
\bar{\alpha} : \quad P_n \otimes \tilde{P}_n &\longrightarrow \bar{P}_{n-1} \\
(z, y) &\longmapsto i\alpha(z) + \sigma(y)
\end{aligned}$$

It easy to see that $\bar{\alpha}$ let the diagram commutes.

For any $x \in \ker d$, $\tilde{\pi}(x) = 0$, so $\pi(x) \in \ker \tilde{d} = \text{Im } \tilde{\alpha}$, so exists y satisfy $\pi(x) = \tilde{\alpha}(y)$. Then $\tilde{\alpha}(y) = \pi\sigma(y) = \pi(x) \implies x - \sigma(y) \in \ker \pi = \text{Im } i$. Let z' be the one so that $i(z') = x - \sigma(y)$, tracing the diagram again one would find out $d(z') = 0$, so exists z such that $\alpha(z) = z'$, and then $\bar{\alpha}(z, y) = i\alpha(z) + \sigma(y) = x$, thus $\text{Im } \bar{\alpha} = \ker \bar{d}$. \square

Theorem 6 (Long exact sequence for Ext). If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ exact, then there is a long exact sequence:

$$\begin{aligned}
0 \rightarrow \text{Hom}(N, K) &\rightarrow \text{Hom}(M, K) \rightarrow \text{Hom}(L, K) \\
&\rightarrow \text{Ext}^1(N, K) \rightarrow \text{Ext}^1(M, K) \rightarrow \text{Ext}^1(L, K) \rightarrow \text{Ext}^2(N, K) \rightarrow \dots
\end{aligned}$$

Proof. Taking $\text{Hom}(-, K)$ in the diagram of Horseshoe' lemma (5), we get

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 0 & \longleftarrow & \text{Hom}(P_1, K) & \longleftarrow & \text{Hom}(\bar{P}_1, K) & \longleftarrow & \text{Hom}(\tilde{P}_1, K) \longleftarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 0 & \longleftarrow & \text{Hom}(P_0, K) & \longleftarrow & \text{Hom}(\bar{P}_0, K) & \longleftarrow & \text{Hom}(\tilde{P}_0, K) \longleftarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 0 & \longleftarrow & \text{Hom}(L, K) & \longleftarrow & \text{Hom}(M, K) & \longleftarrow & \text{Hom}(N, K) \longleftarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

Notice that $\text{Hom}(M \oplus N, K) \cong \text{Hom}(M, K) \otimes \text{Hom}(N, K)$, so each row is indeed exact.

By exercise 14.7, the long exact sequence in the statement exists. □

1.2 Ext and Tor (week 15)

Given $M, N \in \mathbf{Mod}_R$, there are two ways to define $\text{Ext}^n(M, N)$:

Def 5 (Ext functor).

- Find any projective resolution $P_\bullet \xrightarrow{\alpha} M \rightarrow 0$, and let $P_M : P_\bullet \rightarrow 0$ (called a *deleted resolution*). We can define $\text{Ext}_{\text{proj}}^n(M, N) = H^n(\text{Hom}(P_M, N))$.
- Find any injective resolution $0 \xrightarrow{\alpha} N \rightarrow E^\bullet$, and let $E_N : 0 \rightarrow E^\bullet$. We can define $\text{Ext}_{\text{inj}}^n(M, N) = H^n(\text{Hom}(M, E_N))$.

Prop 1.2.1. $\text{Ext}_{\text{proj}}^0(M, N) \cong \text{Ext}_{\text{inj}}^0(M, N) \cong \text{Hom}(M, N)$.

Proof.

$$\text{Hom}(P_M, N) : 0 \xrightarrow{\overline{d}_0} \text{Hom}(P_0, N) \xrightarrow{\overline{d}_1} \text{Hom}(P_1, N) \rightarrow \dots$$

$$\text{so } \text{Ext}_{\text{proj}}^0(M, N) = \ker \overline{d}_1 / \text{im } \overline{d}_0 = \ker \overline{d}_1 = \text{im } \alpha = \text{Hom}(M, N). \quad \square$$

Similarly, $\text{Ext}_{\text{inj}}^0(M, N) = \text{Hom}(M, N)$.

Lemma 1.

- If M is projective, then $\text{Ext}_{\text{proj}}^n(M, N) = 0$ for all $n > 0, N \in \mathbf{Mod}_R$.
- If N is injective, then $\text{Ext}_{\text{inj}}^n(M, N) = 0$ for all $n > 0, M \in \mathbf{Mod}_R$.

Proof. If M is projective, then $0 \rightarrow M \xrightarrow{1_M} M \rightarrow 0$ is a projective resolution of M . Its deleted resolution is then $P_M : 0 \rightarrow M \rightarrow 0$. Hence for $n > 0$, $\text{Ext}_{\text{proj}}^n(M, N) = H^n(\text{Hom}(P_M, N)) = 0$.

The argument applies similarly to injective case. \square

Theorem 7 (Equivalence of Ext_{proj} and Ext_{inj}).

$$\text{Ext}_{\text{proj}}^n(M, N) \cong \text{Ext}_{\text{inj}}^n(M, N).$$

Proof. Let $P_\bullet \rightarrow M \rightarrow 0$ and $0 \rightarrow N \rightarrow E^\bullet$ be projective and injective resolutions, then we have $0 \rightarrow K_0 \rightarrow P_0 \rightarrow M \rightarrow 0$ and $0 \rightarrow N \rightarrow E^0 \rightarrow L^1 \rightarrow 0$ exact.

$$\begin{array}{ccccccc} \cdots & \rightarrow & P_2 & \xrightarrow{\quad} & P_1 & \xrightarrow{\quad} & P_0 \rightarrow M \rightarrow 0 \\ & & \searrow & & \nearrow & & \searrow \\ & & & & K_1 & & K_0 \\ & & \nearrow & & \searrow & & \nearrow \\ 0 & & & & & & 0 \end{array} \quad \begin{array}{ccccccc} 0 \rightarrow N \rightarrow E^0 & \xrightarrow{\quad} & E^1 & \xrightarrow{\quad} & E^2 \rightarrow \cdots \\ & & \searrow & & \nearrow \\ & & & & L^1 & & L^2 \\ & & \nearrow & & \searrow & & \nearrow \\ 0 & & & & & & 0 \end{array}$$

We can construct long exact sequences of homology of $\text{Hom}(\cdot, E_N)$:

$$0 \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(M, E^0) \rightarrow \text{Hom}(M, L^1) \rightarrow \text{Ext}_{\text{inj}}^1(M, N) \rightarrow \text{Ext}_{\text{inj}}^1(M, E^0) = 0$$

$$0 \rightarrow \text{Hom}(P_0, N) \rightarrow \text{Hom}(P_0, E^0) \rightarrow \text{Hom}(P_0, L^1) \rightarrow 0$$

$$0 \rightarrow \text{Hom}(K_0, N) \rightarrow \text{Hom}(K_0, E^0) \rightarrow \text{Hom}(K_0, L^1) \rightarrow \text{Ext}_{\text{inj}}^1(K_0, N) \rightarrow \text{Ext}_{\text{inj}}^1(K_0, E^0) = 0$$

The second sequence is short because P_0 is projective (so $\text{Hom}(P_0, \cdot)$ preserves exactness).

Similarly, for $\text{Hom}(P_M, \cdot)$ we have:

$$0 \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(P_0, N) \rightarrow \text{Hom}(K_0, N) \rightarrow \text{Ext}_{\text{proj}}^1(M, N) \rightarrow \text{Ext}_{\text{proj}}^1(P_0, N) = 0$$

$$\begin{aligned}
0 &\rightarrow \text{Hom}(M, E^0) \rightarrow \text{Hom}(P_0, E^0) \rightarrow \text{Hom}(K_0, E^0) \rightarrow 0 \\
0 &\rightarrow \text{Hom}(M, L^1) \rightarrow \text{Hom}(P_0, L^1) \rightarrow \text{Hom}(K_0, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(M, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(P_0, L^1) = 0
\end{aligned}$$

Combining these sequences together, we got the following 2D diagram:

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{Hom}(M, N) & \longrightarrow & \text{Hom}(M, E^0) & \xrightarrow{\phi} & \text{Hom}(M, L^1) \longrightarrow \text{Ext}_{\text{inj}}^1(M, N) \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{Hom}(P_0, N) & \longrightarrow & \text{Hom}(P_0, E^0) & \xrightarrow{\sigma} & \text{Hom}(P_0, L^1) \longrightarrow 0 \\
& \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\
0 & \longrightarrow & \text{Hom}(K_0, N) & \longrightarrow & \text{Hom}(K_0, E^0) & \xrightarrow{\tau} & \text{Hom}(K_0, L^1) \longrightarrow \text{Ext}_{\text{inj}}^1(K_0, N) \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& \text{Ext}_{\text{proj}}^1(M, N) & & 0 & & \text{Ext}_{\text{proj}}^1(M, L^1) & \\
& \downarrow & & & & \downarrow & \\
& 0 & & & & 0 &
\end{array}$$

By Snake lemma, there is an exact sequence

$$(\ker \alpha \rightarrow) \ker \beta \rightarrow \ker \gamma \rightarrow \text{coker } \alpha \rightarrow \text{coker } \beta (\rightarrow \text{coker } \gamma)$$

and this reads

$$\text{Hom}(M, E^0) \xrightarrow{\phi} \text{Hom}(M, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(M, N) \rightarrow 0$$

Thus $\text{Ext}_{\text{proj}}^1(M, N) \cong \text{coker } \phi \cong \text{Ext}_{\text{inj}}^1(M, N)$.

(From now on, we don't need to distinguish proj/inj for Ext^1 !)

Since σ is onto, $\text{im } \gamma = \text{im}(\gamma \circ \sigma)$. Similarly, $\text{im } \tau = \text{im}(\tau \circ \beta)$.

By the commutativity of the diagram, $\text{im } \gamma = \text{im } \tau$, so

$$\text{Ext}^1(K_0, N) \cong \text{coker } \gamma = \text{Hom}(K_0, L^1) / \text{im } \gamma \cong \text{coker } \tau \cong \text{Ext}^1(M, L^1).$$

Write $K_{-1} := M, L^0 := N$, then $\text{Ext}^1(K_0, L^0) = \text{Ext}^1(K_{-1}, L^1)$ (\star).

Similarly, from the exact sequences

$$0 \rightarrow K_j \rightarrow P_j \rightarrow K_{j-1} \rightarrow 0$$

$$0 \rightarrow L^i \rightarrow E^i \rightarrow L^{i+1} \rightarrow 0$$

, we can obtain $\text{Ext}^1(K_j, L^i) \cong \text{Ext}^1(K_{j-1}, L^{i+1})$ for $i, j \geq 0$.

Now, observe that

$$0 \rightarrow L^{n-1} \rightarrow E^{n-1} \xrightarrow{d_{n-1}} E^n \xrightarrow{d_n} \dots$$

is an injective resolution of L^{n-1} , and $\text{Ext}^1(M, L^{n-1}) \cong \ker \bar{d}_n / \text{im } \bar{d}_{n-1} \cong \text{Ext}_{\text{inj}}^n(M, N)$.

Similarly, for projective resolution we have $\text{Ext}^1(K_{n-2}, N) \cong \text{Ext}_{\text{proj}}^n(M, N)$.

Finally, by (\star),

$$\text{Ext}_{\text{inj}}^n(M, N) \cong \text{Ext}^1(K_{-1}, L^{n-1}) \cong \text{Ext}^1(K_0, L^{n-2}) \cong \dots \cong \text{Ext}^1(K_{n-2}, L^0) \cong \text{Ext}_{\text{proj}}^n(M, N).$$

□

Def 6 (Tor functor). Let $M, N \in \mathbf{Mod}_R$, and $P_\bullet \rightarrow M \rightarrow 0$ be a projective resolution of M , similar to the Ext case, for $n \geq 0$ we can define

$$\mathrm{Tor}_n(M, N) = H_n(P_M \otimes N).$$

Fact 1.2.1. By Horseshoe lemma, short exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ induces a long exact sequence:

$$\cdots \rightarrow \mathrm{Tor}_1(M_1, N) \rightarrow \mathrm{Tor}_1(M_2, N) \rightarrow \mathrm{Tor}_1(M_3, N) \rightarrow M_1 \otimes N \rightarrow M_2 \otimes N \rightarrow M_3 \otimes N \rightarrow 0$$

Prop 1.2.2. If M is flat, then $\mathrm{Tor}_n(M, N) = 0$ for $n > 0, N \in \mathbf{Mod}_R$.

Proof. M is flat $\implies M \otimes \cdot$ is an exact functor. If $Q_\bullet \rightarrow N \rightarrow 0$ is a projective resolution of N , then $\cdots \rightarrow M \otimes Q_1 \rightarrow M \otimes Q_0 \rightarrow M \otimes N \rightarrow 0$ is also exact. By Exercise 15-1, we have

$$\mathrm{Tor}_n(M, N) \cong H_n(M \otimes Q_N) = 0. \quad \square$$

Theorem 8 (Tor for flat resolutions). Let $U_\bullet \rightarrow M \rightarrow 0$ be a flat resolution of M , then for $n \geq 0$,

$$\mathrm{Tor}_n(M, N) \cong H_n(U_M \otimes N).$$

Proof.

$$\begin{array}{ccccccc} \cdots & \rightarrow & U_2 & \xrightarrow{\quad} & U_1 & \xrightarrow{\quad} & U_0 \rightarrow M \rightarrow 0 \\ & & \searrow & & \nearrow & & \nearrow \\ & & & W_1 & & & W_0 \\ & \nearrow & & \searrow & \nearrow & & \searrow \\ 0 & & & & 0 & & 0 \end{array}$$

$$H_\bullet(U_M \otimes N) : \cdots \rightarrow U_2 \otimes N \xrightarrow{d_2 \otimes 1} U_1 \otimes N \xrightarrow{d_1 \otimes 1} U_0 \otimes N \rightarrow 0$$

- $n = 0$:

Since tensor is right exact, $U_1 \otimes N \xrightarrow{d_1 \otimes 1} U_0 \otimes N \xrightarrow{\alpha \otimes 1} M \otimes N \rightarrow 0$ is exact. Hence

$$H_0(U_M \otimes N) = (U_0 \otimes N) / \mathrm{im}(d_1 \otimes 1) = (U_0 \otimes N) / \ker(\alpha \otimes 1) \cong M \otimes N$$

Any projective resolution also has this property, so $\mathrm{Tor}_0(M, N) = H_0(U_M \otimes N)$.

- $n = 1$:

$0 \rightarrow W_0 \rightarrow U_0 \rightarrow M \rightarrow 0$ induces a long exact sequence:

$$\cdots \rightarrow \mathrm{Tor}_1(W_0, N) \rightarrow 0 \rightarrow \mathrm{Tor}_1(M, N) \rightarrow W_0 \otimes N \xrightarrow{i \otimes 1} U_0 \otimes N \rightarrow M \otimes N \rightarrow 0$$

where $\mathrm{Tor}_1(U_0, N) = 0$ because U_0 is flat. We can see that $\mathrm{Tor}_1(M, N) \cong \ker(i \otimes 1)$.

$$\begin{array}{ccccc} U_2 \otimes N & \xrightarrow{d_2 \otimes 1} & U_1 \otimes N & \xrightarrow{d_1 \otimes 1} & U_0 \otimes N \\ & \searrow \beta' \otimes 1 & \nearrow j \otimes 1 & \searrow \alpha' \otimes 1 & \nearrow i \otimes 1 \\ & & W_1 \otimes N & & W_0 \otimes N \\ & & \searrow & & \searrow \\ & & & 0 & 0 \end{array}$$

Since $\alpha' \otimes 1$ is onto, $W_0 \otimes N \cong (U_1 \otimes N) / \ker(\alpha' \otimes 1)$. Also, $x \in \ker(d_1 \otimes 1) \iff (d_1 \otimes 1)(x) = 0 \iff (\alpha' \otimes 1)(x) \in \ker(i \otimes 1)$, so $\ker(i \otimes 1) = (\alpha' \otimes 1)(\ker(d_1 \otimes 1)) \cong \ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1)$. ($\alpha' \otimes 1$ can be considered a quotient map, then $\ker(d_1 \otimes 1)$ descends to $\ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1)$.)

Now, in the diagram $W_1 \otimes N \rightarrow U_1 \otimes N \rightarrow W_0 \otimes N \rightarrow 0$ exact, so $\ker(\alpha' \otimes 1) = \text{im}(j \otimes 1)$. But $\beta' \otimes 1$ is onto, thus $\text{im}(j \otimes 1) = \text{im}(d_2 \otimes 1)$.

Finally,

$$\text{Tor}_1(M, N) \cong \ker(i \otimes 1) \cong \ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1) = \ker(d_1 \otimes 1) / \text{im}(d_2 \otimes 1) = H_1(U_M \otimes N).$$

- $n \geq 2$:

Let's see further in the previous long exact sequences:

$$\cdots \rightarrow \text{Tor}_2(W_0, N) \rightarrow 0 \rightarrow \text{Tor}_2(M, N) \xrightarrow{\sim} \text{Tor}_1(W_0, N) \rightarrow 0 \rightarrow \text{Tor}_1(M, N) \rightarrow \cdots$$

we can see that $\text{Tor}_n(M, N) \cong \text{Tor}_{n-1}(W_0, N)$ for $n \geq 2$.

Now,

$$\cdots U_3 \xrightarrow{d_3} U_2 \xrightarrow{d_2} U_1 \rightarrow W_0 \rightarrow 0$$

is a flat resolution of W_0 , and its homology is $H_{n-1}(U_{W_0} \otimes N) = \ker(d_n \otimes 1) / \text{im}(d_{n-1} \otimes 1) = H_n(U_M \otimes N)$.

By induction, assume it's true for $n - 1$, then

$$H_n(U_M \otimes N) = H_{n-1}(U_{W_0} \otimes N) \cong \text{Tor}_{n-1}(W_0, N) \cong \text{Tor}_n(M, N).$$

□

Eg 1.2.1. $R = \mathbb{Z}, M = \mathbb{Z}/m\mathbb{Z}$ with $m \geq 2$. Then

$$P : 0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{/m\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \rightarrow 0$$

is a projective resolution of $\mathbb{Z}/m\mathbb{Z}$. So for any $N \in \mathbf{Mod}_{\mathbb{Z}}$,

$$H^n(\text{Hom}_{\mathbb{Z}}(P_{\mathbb{Z}/m\mathbb{Z}}, N)) : 0 \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) \xrightarrow{\overline{m}} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) \rightarrow 0,$$

$$\begin{aligned} \text{Ext}_{\mathbb{Z}}^0(\mathbb{Z}/m\mathbb{Z}, N) &= \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, N) \cong {}_mN := \{a \in N \mid ma = 0\} \\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m\mathbb{Z}, N) &\cong N/mN \\ \text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/m\mathbb{Z}, N) &= 0 \quad (\text{for } n \geq 2) \end{aligned}$$

Eg 1.2.2. $\mathbb{Q} = \mathbb{Z}_{(0)}$ is a localization, thus a flat \mathbb{Z} module. Then

$$U : 0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

is a flat resolution of \mathbb{Q}/\mathbb{Z} . For $G \in \mathbf{Mod}_{\mathbb{Z}}$ (i.e. an abelian group),

$$H_n(G \otimes U_{\mathbb{Q}/\mathbb{Z}}) : 0 \rightarrow G \otimes \mathbb{Z} \xrightarrow{1 \otimes i} G \otimes \mathbb{Q} \rightarrow 0$$

$$\begin{aligned} \text{Tor}_0(G, \mathbb{Q}/\mathbb{Z}) &\cong G \otimes \mathbb{Q}/\mathbb{Z} \\ \text{Tor}_1(G, \mathbb{Q}/\mathbb{Z}) &= \ker(1 \otimes i) \cong t(G) := \{a \in G \mid ma = 0 \text{ for some } m \in \mathbb{N}\} \\ \text{Tor}_n(G, \mathbb{Q}/\mathbb{Z}) &= 0 \quad (\text{for } n \geq 2) \end{aligned}$$

Def 7. Let M be a left R -module, then define $M^* := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ as a right R -module by

$$\begin{aligned} fr : M &\rightarrow \mathbb{Q}/\mathbb{Z} \\ x &\mapsto f(rx) \end{aligned}$$

Fact 1.2.2.

1. \mathbb{Q}/\mathbb{Z} is injective.
2. $A = 0 \iff A^* = 0$.
3. $B \hookrightarrow C \iff C^* \twoheadrightarrow B^*$.

Proof.

1. For $m \in \mathbb{Z} \setminus \{0\}$, $m(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$ by $m(\frac{a}{mb} + \mathbb{Z}) \hookrightarrow \frac{a}{b} + \mathbb{Z}$, so \mathbb{Q}/\mathbb{Z} is divisible. But \mathbb{Z} is a PID, so \mathbb{Q}/\mathbb{Z} is injective.
2. $(\Rightarrow) A^* = \text{Hom}_{\mathbb{Z}}(0, \mathbb{Q}/\mathbb{Z}) = 0$.

(\Leftarrow) If $A \neq 0$, then $\exists a \in A, a \neq 0$, so $0 \rightarrow \mathbb{Z}a \xrightarrow{i} A$ is an inclusion.

Since $\mathbb{Z}a$ is a cyclic abelian group, there is a nonzero $g : \mathbb{Z}a \rightarrow \mathbb{Q}/\mathbb{Z}$. (If $\mathbb{Z}a \cong \mathbb{Z}/m\mathbb{Z}$, let $g : a \mapsto \frac{1}{m}$; if $\mathbb{Z}a \cong \mathbb{Z}$, let $g : a \mapsto \frac{1}{2}$.)

But \mathbb{Q}/\mathbb{Z} is injective, so $\exists f : A \rightarrow \mathbb{Q}/\mathbb{Z}$ (i.e. $f \in A^*$), and $f \circ i = g \neq 0$ so $f \neq 0$, thus $A^* \neq 0$.

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z}a & \xrightarrow{i} & A \\ & & \downarrow g & \swarrow \exists f & \\ & & \mathbb{Q}/\mathbb{Z} & & \end{array}$$

3. Since \mathbb{Q}/\mathbb{Z} is injective, $\text{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ is exact. Let $0 \rightarrow \ker f \rightarrow B \xrightarrow{f} C$ exact, applying $\text{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ results in $C^* \xrightarrow{f^*} B^* \rightarrow (\ker f)^* \rightarrow 0$ exact. Thus $\text{coker } f^* = (\ker f)^*$.
By 2., $B \hookrightarrow C \iff \ker f = 0 \iff (\ker f)^* = 0 \iff \text{coker } f^* = 0 \iff C^* \twoheadrightarrow B^*$.

□

Prop 1.2.3. Let M be an R -module, then TFAE

1. M is flat.
2. M^* is injective (as a R -module).
3. $\text{Tor}_1(R/I, M) = 0$ for all ideal $I \subseteq R$.
4. $I \otimes_R M \cong IM$ for all ideal $I \subseteq R$.

Proof.

- 3. \iff 4.

For any ideal $I \subseteq R$, $0 \rightarrow I \xrightarrow{i} R \xrightarrow{q} R/I \rightarrow 0$ is exact. This induces a long exact sequence:

$$\text{Tor}_1(R, M) \rightarrow \text{Tor}_1(R/I, M) \rightarrow I \otimes_R M \xrightarrow{i \otimes 1} R \otimes_R M \xrightarrow{q \otimes 1} R/I \otimes_R M \rightarrow 0$$

- $\text{Tor}_1(R, M) = 0$ since R is a flat R -module.
- $R \otimes_R M \cong M$.
- $R/I \otimes_R M \cong M/IM$ by $(r + I) \otimes a \mapsto (ra + IM)$.

So we have

$$0 \rightarrow \text{Tor}_1(R/I, M) \rightarrow I \otimes_R M \xrightarrow{i'} M \xrightarrow{q'} M/IM \rightarrow 0$$

exact, with $q' : M \rightarrow M/IM$ being exactly the quotient map (one can check that $q \otimes 1 \cong q'$).

Now it's clear that $\text{Tor}_1(R/I, M) = 0 \iff I \otimes_R M \cong \ker(q') \cong IM$.

(The reverse direction requires $I \otimes_R M \cong IM$ being the natural isomorphism $r \otimes b \mapsto rb$, so $i' : IM \rightarrow M$ can then be the natural inclusion.)

- 1. \iff 2.

Let $0 \rightarrow N' \xrightarrow{f} N$, then $\text{Hom}_R(N, M^*) \xrightarrow{\bar{f}} \text{Hom}_R(N', M^*)$.

By the adjoint relation,

$$\text{Hom}_R(N, M^*) = \text{Hom}_R(N, \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(N \otimes_R M, \mathbb{Q}/\mathbb{Z}) = (N \otimes_R M)^*,$$

we have another map $(N \otimes_R M)^* \xrightarrow{(f \otimes 1)^*} (N' \otimes_R M)^*$ isomorphic to the previous one, with its unstarred map $N' \otimes_R M \xrightarrow{f \otimes 1} N \otimes_R M$.

Now, M^* is injective $\iff \bar{f}$ is surjective $\forall N, N' \iff (f \otimes 1)^*$ is surjective $\forall N, N' \iff f \otimes 1$ is injective $\forall N, N' \iff M$ is flat.

- 2. \iff 4.

Similar to the previous section, by Baer's criterion,

$$\begin{aligned} M^* \text{ is injective} &\iff \text{Hom}_R(R, M^*) \twoheadrightarrow \text{Hom}_R(I, M^*), \forall I \subseteq R \\ &\iff (R \otimes_R M)^* \twoheadrightarrow (I \otimes_R M)^*, \forall I \subseteq R \\ &\iff I \otimes_R M \hookrightarrow R \otimes_R M \cong M, \forall I \subseteq R \\ &\iff I \otimes_R M \cong IM, \forall I \subseteq R. \end{aligned}$$

Similarly, this requires the isomorphism of $I \otimes_R M \cong IM$ be natural (the following f).

The map $f : I \otimes_R M \rightarrow IM$
 $r \otimes a \mapsto ra$ is always onto, but may not be 1-1. If it is, $I \otimes_R M \cong IM$.

□

Prop 1.2.4. For $I, J \subseteq R$ being ideals, then $\text{Tor}_1(R/I, R/J) \cong (I \cap J)/IJ$.

Proof. $0 \rightarrow I \xrightarrow{i} R \rightarrow R/I \rightarrow 0$ induces a long exact sequence

$$0 \rightarrow \text{Tor}_1(R/I, R/J) \rightarrow I \otimes_R R/J \xrightarrow{i \otimes 1} R \otimes_R R/J \rightarrow R/I \otimes_R R/J \rightarrow 0,$$

where $\text{Tor}_1(R, R/J) = 0$ since R is flat.

Also $I \otimes_R R/J \cong I/IJ, R \otimes_R R/J \cong R/J$, so we have $I/IJ \xrightarrow{i'} R/J$ with $\text{Tor}_1(R/I, R/J) \cong \ker(i \otimes 1) \cong \ker i'$.

But $i' : I/IJ \rightarrow R/J$
 $x + IJ \mapsto x + J$, so $\bar{x} \in \ker i' \iff x \in I \text{ and } x \in J \iff x \in I \cap J$, hence $\ker i' \cong (I \cap J)/IJ$.

□

1.3 Koszul complex (week 16)

In this section, we assume that R is commutative with 1.

Def 8. Let $L \in \mathbf{Mod}_R$, with $f : L \rightarrow R$ an R -linear map, define

$$\begin{aligned} d_f : \Lambda^n L &\rightarrow \Lambda^{n-1} L \\ x_1 \wedge \cdots \wedge x_n &\mapsto \sum_{i=1}^n (-1)^{i+1} f(x_i) x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \end{aligned}$$

where $\Lambda^n L$ is the n -th exterior power of L , and \hat{x}_i means omitting x_i .

Then we can define a chain complex called **Koszul complex**:

$$K_\bullet(f) : \cdots \rightarrow \Lambda^n L \xrightarrow{d_f} \Lambda^{n-1} L \rightarrow \cdots \rightarrow \Lambda^2 L \xrightarrow{d_f} L \xrightarrow{f} R$$

Also, d_f can be considered as a graded R -homomorphism of degree -1 :

$$\begin{aligned} d_f : \Lambda L &\rightarrow \Lambda L \\ x \wedge y &\mapsto d_f(x) \wedge y + (-1)^{\deg x} \cdot x \wedge d_f(y) \end{aligned}$$

where ΛL is the exterior algebra of L , and x, y are any homogeneous elements of ΛL .

Def 9. Let $(C_\bullet, d), (C'_\bullet, d')$ be chain complexes of R -modules, define their *tensor product* to be a chain complex $C_\bullet \otimes C'_\bullet$ with

$$(C_\bullet \otimes C'_\bullet)_n = \bigoplus_{i=0}^n (C_i \otimes_R C'_{n-i})$$

with the boundary maps being

$$\begin{aligned} d \otimes d' : (C_\bullet \otimes C'_\bullet)_n &\rightarrow (C_\bullet \otimes C'_\bullet)_{n-1} \\ \sum_{i=0}^n x_i \otimes y_{n-i} &\mapsto \sum_{i=0}^n (d(x_i) \otimes y_{n-i} + (-1)^i \cdot x_i \otimes d'(y_{n-i})) \end{aligned}$$

One can verify that

$$\begin{aligned} (d \otimes d') \circ (d \otimes d')(x \otimes y) &= (d \otimes d')(d(x) \otimes y + (-1)^{\deg x} \cdot x \otimes d'(y)) \\ &= d \circ d(x) \otimes y + (-1)^{\deg x-1} \cdot d(x) \otimes d'(y) \\ &\quad + (-1)^{\deg x} \cdot d(x) \otimes d'(y) + x \otimes d' \circ d'(y) \\ &= 0 \end{aligned}$$

Prop 1.3.1. Let $L_1, L_2 \in \mathbf{Mod}_R, f_1 \in \text{Hom}_R(L_1, R), f_2 \in \text{Hom}_R(L_2, R)$. Define

$$\begin{aligned} f = f_1 + f_2 : L_1 \oplus L_2 &\rightarrow R \\ (x, y) &\mapsto f_1(x) + f_2(y) \end{aligned}$$

then

$$\begin{aligned} K_\bullet(f_1) \otimes K_\bullet(f_2) &\cong K_\bullet(f) \\ \bigoplus_{i=0}^n (\Lambda^i L_1 \otimes_R \Lambda^{n-i} L_2) &\cong \Lambda^n(L_1 \oplus L_2) \end{aligned}$$

with $d_{f_1} \otimes d_{f_2} = d_f$.

Proof. Exercise 16-1(2). □

Def 10. Let $L = \bigoplus_{i=1}^n Re_i$ be a free R -module, and $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i \in R$, define

$$K_{\bullet}(\mathbf{x}) := K_{\bullet}(f), \text{ with } \begin{array}{l} f : L \rightarrow R \\ e_i \mapsto x_i \end{array}.$$

Coro 1.3.1. $K_{\bullet}(\mathbf{x}) \cong K_{\bullet}(x_1) \otimes \dots \otimes K_{\bullet}(x_n)$ with $K_{\bullet}(x_i) : 0 \rightarrow R \xrightarrow{x_i} R$.

Prop 1.3.2. Let $x \in R$ and (C_{\bullet}, ∂) be a chain complex of R -modules, then there exist ρ, π s.t.

$$0 \rightarrow C_{\bullet} \xrightarrow{\rho} C_{\bullet} \otimes K_{\bullet}(x) \xrightarrow{\pi} C_{\bullet}(-1) \rightarrow 0$$

is exact, where $(C_{\bullet}(-1))_n = C_{n-1}$.

Proof. Since $K_{\bullet}(x) : 0 \rightarrow R \xrightarrow{x} R$, so

$$(C_{\bullet} \otimes K_{\bullet}(x))_n = (C_i \otimes_R R) \oplus (C_{i-1} \otimes_R R),$$

and the boundary map is

$$\begin{array}{ccc} d : (C_i \otimes_R R) \oplus (C_{i-1} \otimes_R R) & \rightarrow & (C_{i-1} \otimes_R R) \oplus (C_{i-2} \otimes_R R) \\ (z_1 \otimes r_1, z_2 \otimes r_2) & \mapsto & (\partial z_1 \otimes r_1 + (-1)^{i-1} z_2 \otimes x r_2, \partial z_2 \otimes r_2) \end{array}.$$

Under the isomorphism $C_i \otimes_R R \cong C_i$, the boundary map become

$$\begin{array}{ccc} d : C_i \oplus C_{i-1} & \rightarrow & C_{i-1} \oplus C_{i-2} \\ \begin{pmatrix} r_1 z_1 \\ r_2 z_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \partial & (-1)^{i-1} x \\ 0 & \partial \end{pmatrix} \begin{pmatrix} r_1 z_1 \\ r_2 z_2 \end{pmatrix} \end{array}$$

Let

$$\begin{array}{ccc} \rho_i : C_i \rightarrow C_i \oplus C_{i-1} & \text{and} & \pi_i : C_i \oplus C_{i-1} \rightarrow C_{i-1} \\ z_1 \mapsto (z_1, 0) & & (z_1, z_2) \mapsto z_2 \end{array}$$

then

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_i & \xrightarrow{\rho_i} & C_i \oplus C_{i-1} & \xrightarrow{\pi_i} & C_{i-1} \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow d & & \downarrow \partial \\ 0 & \longrightarrow & C_{i-1} & \xrightarrow{\rho_{i-1}} & C_{i-1} \oplus C_{i-2} & \xrightarrow{\pi_{i-1}} & C_{i-2} \longrightarrow 0 \end{array}$$

commutes and exact:

- $d \circ \rho(z_1) = d(z_1, 0) = (\partial z_1, 0)$
- $\rho \circ \partial(z_1) = \rho(\partial z_1) = (\partial z_1, 0)$
- $\partial \circ \pi(z_1, z_2) = \partial(z_2) = \partial z_2$
- $\pi \circ d(z_1, z_2) = \pi(\partial z_1 + (-1)^{i-1} x z_2, \partial z_2) = \partial z_2$

□

Coro 1.3.2. This induces a long exact sequence

$$\dots \rightarrow H_i(C_{\bullet}) \xrightarrow{\rho_*} H_i(C_{\bullet} \otimes K_{\bullet}(x)) \xrightarrow{\pi_*} H_i(C_{\bullet}(-1)) \xrightarrow{\pm x} H_{i-1}(C_{\bullet}) \rightarrow \dots$$

Proof. We only need to show the connection homomorphism is indeed $\pm x$.

Given $z \in C_{i-1}$ with $\partial z = 0$,

$$z \xrightarrow{\pi^{-1}} (0, z) \xrightarrow{d} ((-1)^{i-1} x z, 0) \xrightarrow{\rho^{-1}} (-1)^{i-1} x z.$$

□

Def 11. We call x to be C_\bullet -regular, if x is not a zero divisor of C_i and $C_i/xC_i \neq 0$, for all $i \geq 0$.

Prop 1.3.3. If x is C_\bullet -regular, then $H_i(C_\bullet \otimes K_\bullet(x)) \cong H_i(C_\bullet/xC_\bullet)$ for all $i \geq 0$.

Proof. Let

$$\begin{aligned} \phi_i : C_i \oplus C_{i-1} &\rightarrow C_i/xC_i \\ (z_1, z_2) &\mapsto \overline{z_1}, \end{aligned}$$

then

$$\begin{array}{ccc} C_i \oplus C_{i-1} & \xrightarrow{\phi_i} & C_i/xC_i \\ \downarrow d_i & & \downarrow \bar{\partial}_i \\ C_{i-1} \oplus C_{i-2} & \xrightarrow{\phi_{i-1}} & C_{i-1}/xC_{i-1} \end{array}$$

commutes.

- $\bar{\partial} \circ \phi_i(z_1, z_2) = \bar{\partial}(z_1) = \overline{\partial z_1}$.
- $\phi_{i-1} \circ d(z_1, z_2) = \phi_{i-1}(\partial z_1 + (-1)^{i-1}xz_2, \partial z_2) = \overline{\partial z_1}$, since $xz_2 \in xC_{i-1}$.

Now we need to show the induced maps

$$\begin{aligned} \phi_{*i} : \ker d_i / \text{im } d_{i+1} &\rightarrow \ker \bar{\partial}_i / \text{im } \bar{\partial}_{i+1} \\ \overline{(z_1, z_2)} &\mapsto \overline{z_1} = \overline{z_1} + \text{im } \bar{\partial}_{i+1} \end{aligned}$$

are isomorphisms.

- **Onto:**

For $\bar{z} \in \ker \bar{\partial}_i$ with $\partial z = xz' \in xC_{i-1}$, $z' \in C_{i-1}$. Then $\phi_i(z, (-1)^i z') = \bar{z}$, and $d(z, (-1)^i z') = (\partial z - xz', (-1)^i \partial z') = (0, 0)$, so $(z, (-1)^i z') \in \ker d_i$. (Since $x\partial z' = \partial(xz') = \partial^2 z = 0$, and x is not a zero divisor of C_i , so $\partial z' = 0$.)

Now, $\phi_{*i}(\overline{(z, (-1)^i z')}) = \bar{z}$, so ϕ_{*i} is onto.

- **1-1:**

Let $(z, z') \in \ker d_i$ with $\phi_i(z, z') = \bar{z} \in \text{im } \bar{\partial}_{i+1}$, i.e. $\bar{z} = \partial z''$ with $z'' \in C_{i+1}$. This means $z - \partial z'' = xz'''$ with $z''' \in C_i$, so $\partial(z - \partial z'') = \partial z = x\partial z'''$.

On the other hand, $d(z, z') = (\partial z + (-1)^{i-1}xz', \partial z') = (0, 0)$, so $\partial z = (-1)^i xz'$, $\partial z' = 0$.

So $d(z'', (-1)^i z''') = (\partial z'' + (-1)^{2i}xz''', (-1)^i \partial z''') = (z, z')$, i.e. $(z, z') \in \text{im } d_{i+1}$. ($\partial z = x\partial z''' = (-1)^i xz'$, since x is not a zero divisor, so $\partial z''' = (-1)^i z'$.)

Hence, $\phi_{*i}(\overline{(z_1, z_2)}) = \bar{0}$ implies $\overline{(z_1, z_2)} = \bar{0}$, so ϕ_{*i} is 1-1.

□

Def 12. Let $M \in \mathbf{Mod}_R$. A sequence $\{a_1, \dots, a_m\}$, $m \geq 0$ is said to be M -regular if

- $M/\langle a_1, \dots, a_m \rangle M \neq 0$.
- a_{i+1} is not a zero divisor of $M/\langle a_1, \dots, a_i \rangle M$ for $0 \leq i \leq m-1$.

Theorem 9. If $\mathbf{x} = (x_1, \dots, x_n)$ is an R -regular sequence, then $K_\bullet(\mathbf{x}) \rightarrow R/\langle x_1, \dots, x_n \rangle \rightarrow 0$ is a free resolution of $R/\langle x_1, \dots, x_n \rangle$.

Proof. Since its modules are $K_i(\mathbf{x}) = \Lambda^i R^n \cong R^{\binom{n}{i}}$, i.e. free R -modules, so we only need to show the exactness.

By induction on n ,

- $n = 1$: $K_\bullet(x_1) : 0 \rightarrow R \xrightarrow{x_1} R \rightarrow R/\langle x_1 \rangle \rightarrow 0$ exact.

- $n > 1$: Assume that $\mathbf{x}' = (x_1, \dots, x_{n-1})$ and $K_\bullet(\mathbf{x}') \rightarrow R/\langle x_1, \dots, x_{n-1} \rangle \rightarrow 0$ exact, i.e. $H_i(K_\bullet(\mathbf{x}')) = 0$ for $i > 0$.

Since we have $K_\bullet(\mathbf{x}) \cong K_\bullet(\mathbf{x}') \otimes K_\bullet(x_n)$ and a long exact sequence

$$\cdots \rightarrow H_i(K_\bullet(\mathbf{x}')) \rightarrow H_i(K_\bullet(\mathbf{x})) \rightarrow H_i(K_\bullet(\mathbf{x}')(-1)) \xrightarrow{\pm x_n} H_{i-1}(K_\bullet(\mathbf{x})) \rightarrow \cdots$$

where $H_i(K_\bullet(\mathbf{x}')(-1)) = H_{i-1}(K_\bullet(\mathbf{x}'))$.

For $i > 1$, the sequence becomes

$$\cdots \rightarrow 0 \rightarrow H_i(K_\bullet(\mathbf{x})) \rightarrow 0 \xrightarrow{\pm x_n} \cdots,$$

so $H_i(K_\bullet(\mathbf{x})) = 0$.

For $i = 1$, we have $H_0(K_\bullet(\mathbf{x})) \cong R/\langle x_1, \dots, x_{n-1} \rangle$, so

$$0 \rightarrow H_1(K_\bullet(\mathbf{x})) \rightarrow R/\langle x_1, \dots, x_{n-1} \rangle \xrightarrow{\pm x_n} R/\langle x_1, \dots, x_{n-1} \rangle$$

But x_n is not a zero divisor of $R/\langle x_1, \dots, x_{n-1} \rangle$, so the map $\pm x_n$ is 1-1, then $H_1(K_\bullet(\mathbf{x})) \cong \ker(\pm x_n) = 0$.

□

Eg 1.3.1. Let $\mathbf{x} = (x_1, x_2)$, then

$$K_\bullet(\mathbf{x}) : 0 \rightarrow R \xrightarrow{\alpha} R^2 \xrightarrow{\beta} R \xrightarrow{q} R/\langle x_1, x_2 \rangle \rightarrow 0$$

with $\alpha : r \mapsto (-x_2r, x_1r)$ and $\beta : (r_1, r_2) \mapsto x_1r_1 + x_2r_2$.

Coro 1.3.3. Let $I = \langle x_1, \dots, x_n \rangle \subset R$ be an ideal with $\{x_1, \dots, x_n\}$ be R -regular, then R/I has *projective dimension* $\text{pd}(R/I) = n$, i.e. the shortest projective resolution of R/I has length n .

Proof. $K_\bullet(\mathbf{x})$ is already a projective resolution of length N , so we only need to show that there's no shorter ones.

The left side of $K_\bullet(\mathbf{x})$ reads

$$0 \rightarrow \Lambda^n R^n \xrightarrow{d_n} \Lambda^{n-1} R^n \rightarrow \cdots$$

But

$$\Lambda^n R^n = R(e_1 \wedge \cdots \wedge e_n) \cong R, \quad \Lambda^{n-1} R^n = \bigoplus_{i=1}^n R(e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n) \cong R^n$$

so

$$d_n : R \rightarrow R^n \\ r \mapsto (x_1r, -x_2r, \dots, (-1)^{n-1}x_nr)$$

Taking tensor with R/I , we get

$$0 \rightarrow R \otimes_R R/I \xrightarrow{d_n \otimes 1} R^n \otimes_R R/I \rightarrow \cdots$$

but $R \otimes_R R/I \cong R/I$, $R^n \otimes_R R/I \cong (R/I)^n$, so

$$d_n \otimes 1 : R/I \rightarrow \frac{(R/I)^n}{\overline{(x_1\bar{r}, -x_2\bar{r}, \dots, (-1)^{n-1}x_n\bar{r})}}$$

Now,

$$\text{Tor}_n(R/I, R/I) = H_n(K_\bullet(\mathbf{x}) \otimes R/I) = \ker(d_n \otimes 1) = \text{Ann}_{R/I} I = \{\bar{r} \in R/I \mid rI = I\} = R/I \neq 0.$$

($R/I \neq 0$ is because $\{x_1, \dots, x_n\}$ is R -regular.) Thus, any projective resolution can't have length shorter than n since that will imply $\text{Tor}_n(R/I, R/I) = 0$. □

Remark 1. Let $I = \langle x_1, \dots, x_n \rangle$ generated by R -regular sequence $\{x_1, \dots, x_n\}$, then

- $\text{Tor}_n(R/I, M) \cong \text{Ann}_M I$.
- $\text{Ext}^n(R/I, M) \cong M/IM$.

1.4 Derived category

Def 13.

- \mathcal{C} is a pre-additive category if $\text{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group $\forall X, Y \in \mathcal{C}$ s.t.

$$X \xrightarrow{u} Y \xrightarrow[f]{g} Z \xrightarrow{v} T$$

with $(f + g)u = fu + gu$ and $v(f + g) = vf + vg$.

- additive category: a pre-additive category \mathcal{C} s.t.
 - There exists a zero object 0 s.t. $\forall X, \text{Hom}_{\mathcal{C}}(0, X) = \{0\} = \text{Hom}_{\mathcal{C}}(X, 0)$.
 - Finite sum and finite products exist.

Def 14.

- $f \in \text{Hom}(B, C)$ is called a monomorphism if $\forall X \xrightarrow{g} B \xrightarrow{f} C$ with $f \circ g = 0 \implies g = 0$.
- $f \in \text{Hom}(B, C)$ is called an epimorphism if $\forall B \xrightarrow{f} C \xrightarrow{h} D$ with $h \circ f = 0 \implies h = 0$.
- a kernel of $f \in \text{Hom}(B, C)$ is a morphism $i : A \rightarrow B$ s.t. $f \circ i = 0$ and $\forall g : X \rightarrow B$ with $f \circ g = 0$, we have

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{f} & C \\ & \nwarrow \exists! & \uparrow g & \searrow 0 & \\ & & X & & \end{array}$$

- a cokernel of $f \in \text{Hom}(B, C)$ is a morphism $p : C \rightarrow D$ s.t. $p \circ f = 0$ and $\forall h : C \rightarrow Y$ with $h \circ f = 0$, we have

$$\begin{array}{ccccc} B & \xrightarrow{f} & C & \xrightarrow{p} & D \\ & \searrow 0 & \downarrow h & \swarrow \exists! & \\ & & Y & & \end{array}$$

Remark 2.

- If i is a kernel of f , then i is a monomorphism.
- If p is a cokernel of f , then p is an epimorphism.

Remark 3. An epimorphism may not be a cokernel. Consider $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$ which is an epimorphism in the category of f.g. free \mathbb{Z} -modules. If $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$ is the cokernel of $G \xrightarrow{f} \mathbb{Z}$, then

$$\begin{array}{ccccc} G & \xrightarrow{f} & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} \\ & \searrow 0 & \downarrow \times 2 & \swarrow \exists! \tilde{f} & \\ & & \mathbb{Z} & & \end{array}$$

This implies $\tilde{f} : 1 \mapsto \frac{2}{3}$, which is impossible.

Def 15. \mathcal{A} is an **abelian category** if it is an additive category s.t.

- kernels and cokernels always exist in \mathcal{A} .
- every monomorphism is a kernel and every epimorphism is a cokernel.

Fact 1.4.1. If \mathcal{A} is an abelian category, then:

- every morphism is expressible as the composite of an epimorphism and a monomorphism. Given $f : B \rightarrow C$, we have

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ & \searrow & \nearrow \\ & \text{Im } f & \end{array}$$

where $\text{Im } f$ is unique up to isomorphism.

Proof. Consider the following diagram:

$$\begin{array}{ccccccc} \ker f & \xleftarrow{i} & B & \xrightarrow{f} & C & \xrightarrow{p} & \text{coker } f \\ & & \downarrow p' & \swarrow \exists! \mu & \searrow \exists! \nu & & \uparrow i' \\ & & \text{coker } i & \xrightarrow[\exists! \sigma]{} & \ker p & & \end{array}$$

where μ, ν exist because i', p' are kernel and cokernel. Now, $i'\mu i = fi = 0$, and since i' is a monomorphism, $\mu i = 0$. Moreover, since p' is the cokernel of i , there exists a unique σ letting the diagram commute.

By exercise, σ is both a monomorphism and epimorphism. In an abelian category, this implies that σ is actually an isomorphism (i.e., σ^{-1} exists). \square

- $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is exact if f is monomorphism, g is epimorphism and $\text{Im } f = \ker g$.

Theorem 10 (Freyd-Mitchell theorem). A small abelian category is equivalent to a full subcategory of a category of R -modules.

Def 16.

- $I \in \text{Obj } \mathcal{A}$ is injective if the functor $\text{Hom}(-, I)$ is exact.
- An abelian category is said to be **enough injectives** if for any $A \in \text{Obj } \mathcal{A}$, there exists an injective object I such that $A \hookrightarrow I$.

Def 17. Given a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ satisfy:

1. F is additive, which is to say F is a group homomorphism $\text{Hom}(A, A') \rightarrow \text{Hom}(FA, FA')$.
2. F is left exact. If $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$, then $0 \rightarrow FA' \rightarrow FA \rightarrow FA''$.

Then the derived functor $R^i F : \mathcal{A} \rightarrow \mathcal{B}$ is defined as

$$R^i F(A) = \begin{cases} F(A), & \text{if } i = 0 \\ H^i(F(I^\bullet)), & \text{else} \end{cases}$$

Our goal is to construct the derived category $D^+(\mathcal{A})$ and $D^+(\mathcal{B})$ letting RF be a exact functor.

Def 18. Let \mathcal{A} be an abelian category.

- $\text{Kom}(\mathcal{A})$ is the category of complexes over \mathcal{A} .

- $K(\mathcal{A})$ is the homotopy category of \mathcal{A} , defined by $\text{Obj}(K(\mathcal{A})) = \text{Obj}(\text{Kom}(\mathcal{A}))$ and

$$\text{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \text{Hom}_{\text{Kom}(\mathcal{A})}(A^\bullet, B^\bullet) / \sim,$$

where \sim indicates homotopy equivalences.

Remark 4.

- $\text{Hom}_{K(\mathcal{A})}(I_A^\bullet, I_B^\bullet) \cong \text{Hom}_{\mathcal{A}}(A, B)$ by comparison theorem (3).
- It could be shown that $K(\mathcal{A})$ is additive but may not be abelian.

Def 19. $f \in \text{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet)$ is called a quasi-isomorphism if $H^n(f)$ is an isomorphism between $H^n(A^\bullet)$ and $H^n(B^\bullet)$ for each n .

Eg 1.4.1. • A quasi-isomorphism is often not invertible. For example:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & 0 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \longrightarrow \dots \end{array}$$

- Given $0 \rightarrow A \rightarrow I^\bullet$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 \longrightarrow \dots \end{array}$$

are two quasi-isomorphic complexes.

Def 20. Let \mathcal{B} be a category. A class of morphism $S \subset \text{Mor}(\mathcal{B})$ is said to be **localizing** if

1. S is closed under composition with $\text{Id}_X \in S$ for each object X in \mathcal{B} .
2. Extension condition holds: For each $f \in \text{Mor } \mathcal{B}$, $s \in S$, exists $g \in \text{Mor } \mathcal{B}$, $t \in S$ such that $ft = sg$. The dual version should hold as well.
3. For any $f, g \in \text{Hom}(X, Y)$,

$$\exists s \in S \text{ s.t. } sf = sg \iff \exists t \in S \text{ s.t. } ft = gt.$$

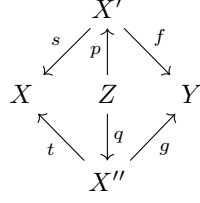
Theorem 11. If S is localizing, then exists a category $\mathcal{B}[S^{-1}]$ with a functor $Q : \mathcal{B} \rightarrow \mathcal{B}[S^{-1}]$ such that

1. $Q(s)$ is an isomorphism for each $s \in S$.
2. Given another functor $F : \mathcal{B} \rightarrow \mathcal{B}'$ satisfy condition 1, there exists a unique functor $G : \mathcal{B}[S^{-1}] \rightarrow \mathcal{B}'$ such that $F = G \circ Q$.

Proof. Define a roof to be a pair (s, t) with

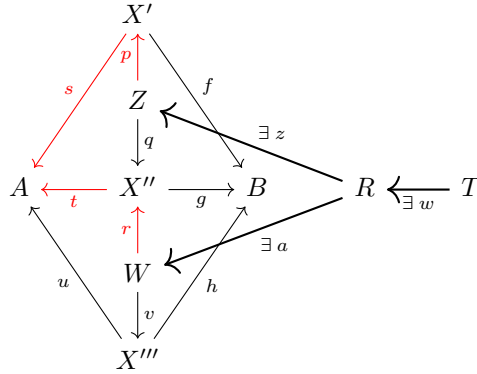
$$\begin{array}{ccc} & X' & \\ s \ni s \swarrow & & \searrow t \\ X & & Y \end{array}$$

Also, define $(s, f) \sim (t, g)$ if there exists Z such that



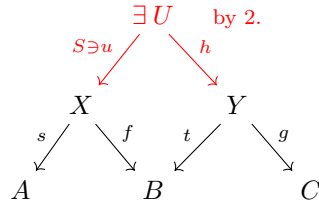
with $sp = tq \in S$ and $fp = gq$.

First we check that “ \sim ” is indeed an equivalence relation. $(s, f) \sim (s, f)$ and $(s, f) \sim (t, g) \implies (t, g) \sim (s, f)$ are trivial. If $(s, f) \sim (t, g)$ and $(t, g) \sim (u, h)$, then we have the following diagram:



Using definition 2. on $tr \in S$ and sp , there are morphism z, a with $z \in S$ and $spz = tra$. Moreover, $tqz = spz = tra$, if we let $b = qz, c = ra$, then by 3., morphism $w \in S$ exists with $bw = cw$. Define $x = pzw, y = vaw$, we have $sx = spzw = tqzw = tbw = tcw = traw = uvaw = uy$ and $sx \in S$ since $sx = spzw$ and sp, z, w are all in S . Similarly, $fx = hy$, thus $(s, f) \sim (u, h)$. Hence we've proved that \sim is an equivalence relation.

Now we could construct the localized category as following: The objects are $\text{Obj}(\mathcal{B}[S^{-1}]) = \mathcal{B}$ and $\text{Mor}(\mathcal{B}[S^{-1}]) = \{ \text{equivalence classes under } \sim \}$. $[(t, g)] \circ [(s, f)] = [(su, gh)]$ could be defined as in the following diagram:



□

Finally, define functor Q by $Q(X) = X, \forall X \in \text{Obj}(\mathcal{B})$ and $Q(f) = [(\text{Id}_X, f)]$. For the universal property, if F is another functor making every morphism in S be invertible, then the functor G exists uniquely by $G([(s, f)]) = F(f)F(s)^{-1}$.

Def 21. The mapping cone of a chain map f between two chain $X^\bullet \xrightarrow{f} Y^\bullet$ is defined as a chain with $\text{cone}(f)^n = X^{n+1} \oplus Y^n$, and the chain map is defined as

$$d_{\text{cone}(f)} : \quad \text{cone}(f)^n = X^{n+1} \oplus Y^n \longrightarrow \text{cone}(f)^{n+1} X^{n+2} \oplus Y^{n+1}$$

$$(x_{n+1}, y_n) \longmapsto \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix} \begin{pmatrix} x_{n+1} \\ y_n \end{pmatrix} = (-d_X(x_{n+1}), f(x_{n+1}) + d_Y(y_n))$$

It is easy to see that $d_{\text{cone}(f)}^2 = 0$.

Prop 1.4.1. Suppose that $f : X^\bullet \rightarrow Y^\bullet$ is a chain map, then there is a short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \longrightarrow & \text{cone}(f) & \longrightarrow & X[+1] \longrightarrow 0 \\ & & d & \longmapsto & (0, d) & & \\ & & (c, d) & \longmapsto & -c & & \end{array}$$

Proof. It is easy to see that the rows are exact. Tracing the diagram shows that the diagram commutes. \square

Coro 1.4.1. There exists a long exact sequence of homology:

$$\cdots \rightarrow H^m(Y^\bullet) \rightarrow H^m(\text{cone}(f)) \rightarrow H^{m+1}(X^\bullet) \xrightarrow{\delta} H^{m+1}(Y^\bullet) \rightarrow H^{m+1}(\text{cone}(f)) \rightarrow \cdots$$

Where the connecting homomorphism $\delta = f^*$.

Proof. Tracing the diagram below as in the snake lemma,

$$\begin{array}{ccccc} X^m \oplus Y^{m-1} & \longrightarrow & X^m & & \\ \downarrow & & \downarrow & & \\ Y^m & \longrightarrow & X^{m+1} \oplus Y^m & \longrightarrow & X^{m+1} \end{array}$$

Suppose $\bar{x} \in H^m(X^\bullet)$, then $d_X(x) = 0$, so $d(-x, 0) = (dx, -f(x))$ with $dx = 0$, which implies $-f(x) :: Y^m \mapsto d(-x, 0) :: X^{m+1} \oplus Y^m$, so $\delta = -f^*$ (Chu Wen Ti)... \square

Coro 1.4.2. $\text{cone}(f)$ exact $\iff f$ quasi-isomorphic.

Proof. Directly by the exact sequence

$$H^{m-1}(\text{cone}(f)) \rightarrow H^m(X^\bullet) \rightarrow H^m(Y^\bullet) \rightarrow H^m(\text{cone}(f))$$

\square

Theorem 12. Let \mathcal{A} be an abelian category and $K(\mathcal{A})$ be the homotopy category. Then the class of quasi-isomorphisms are localizing.

Proof. We check that:

1. It is closed under composition: If f, g are quasi-isomorphic, then $(fg)^* = f^*g^*$ is a isomorphism since both f^*, g^* are, thus fg is quasi-isomorphic.

2. The diagram could be completed:

$$\begin{array}{ccc} \exists W^\bullet & \dashrightarrow & Z^\bullet \\ \downarrow & & \downarrow g: \text{q-iso} \\ X^\bullet & \xrightarrow{f} & Y^\bullet \end{array}$$

□