Algebra

June 2, 2017

1 Introduction to Homological Algebra

1.1 Projective, Injective and Flat modules (week 14)

Def 1.

- $M \in \mathbf{Mod}_R$ is **projective** if $\mathrm{Hom}(M, \cdot)$ preserves the *right* exactness.
- $N \in \mathbf{Mod}_R$ is **injective** if $\mathrm{Hom}(\cdot, N)$ preserves the *right* exactness.
- $M \in \mathbf{Mod}_R$ is **flat** if $M \otimes \cdot$ preserves the *left* exactness.

Fact 1.1.1.

- M is projective $\iff$ $$\begin{tikzcd} & M \arrow[dl, "\exists\,\tilde{f}"'] \arrow[d, "f"] & \\ M_2 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$
- N is injective $\iff$ $$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "g"'] & M_2 \arrow[dl, "\exists\,\tilde{g}"] \\ & N & \end{tikzcd}$$
- free $\implies$ projective: If $X = \{\, x_i \mid i \in \Lambda \,\}$ and $f : x_i \mapsto a_i$. Then we can set $\tilde{f} : x_i \mapsto b_i$ for any b_i s.t. $\beta : b_i \mapsto a_i$.
$$\begin{tikzcd} & F(X) \arrow[dl, "\exists\,\tilde{f}"'] \arrow[d, "f"] & \\ M_2 \arrow[r, "\beta"] & M_3 \arrow[r] & 0 \end{tikzcd}$$
- free $\implies$ flat:
$$0 \to M_1 \to M_2 \text{ exact } \Rightarrow 0 \to R \underset{R}{\otimes} M_1 \to R \underset{R}{\otimes} M_2 \text{ exact}$$
since $M_1 \cong R \underset{R}{\otimes} M_1$. Let F is Free on $X = \{x_i\}, i \in \Lambda$, that is $F \cong \underset{x_i \in X}{\oplus} Rx_i \cong R^{\oplus \Lambda}$. And, $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$ (see tensor product section!)
$$\begin{aligned} &0 \to \underset{i\in\Lambda}{\oplus} (R \underset{R}{\otimes} M_1) \to \underset{i\in\Lambda}{\oplus} (R \underset{R}{\otimes} M_2) \text{ exact} \\ \Rightarrow\ &0 \to \big(\underset{i\in\Lambda}{\oplus} R\big) \underset{R}{\otimes} M_1 \to \big(\underset{i\in\Lambda}{\oplus} R\big) \underset{R}{\otimes} M_2 \text{ exact} \end{aligned}$$
Therefore,
$$0 \to F \otimes M_1 \to F \otimes M_2 \text{ exact}$$
- If S is a m.c. set in R with $1 \in S$, then
$$0 \to M \to N \to L \to 0 \implies 0 \to M_S \to N_S \to L_S \to 0.$$
We know that $M_S \cong R_S \otimes_R M$. So R_S is a flat R-module. e.g. $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.
- $\forall\, M \in \mathbf{Mod}_R, \exists\, F$: free on X s.t. $F \to M \to 0$. This is obvious since we can choose X to be the generating set of M.

Ques: $\forall\, M \in \mathbf{Mod}_R$, does there exist $N \in \mathbf{Mod}_R$ is injective s.t. $0 \to M \to N$?

Theorem 1 (Boer's criterion)**.** N is injective $\iff \forall\, I \subset R,$ $$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists\, h"] \\ & N & \end{tikzcd}$$

Proof.

- " $\Rightarrow$ " by the "Fact" of injective.
- " $\Leftarrow$ "
  Consider diagram,
  $$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "g"] & M_2 \\ & N & \end{tikzcd}$$
  Let $S = \{(M, \rho) | M_1 \subseteq M \subseteq M_2 and \rho : M \to N \text{ extend } g\} \neq \emptyset$ By Zorn's lemman, exists a maximal elemant $(M^*, \mu) \in S$.
  <u>Claim</u> : $M^* = M_2$
  If not, pick $a \in M_2 \; M^*$. Let $M' = M^* + Ra \gneq M^*$ and $I = \{r \in R | ra \in M^*\}$ Define $f : I \to N$ with $r \mapsto \mu(ra)$. Then we have extension h,
  $$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "f"] & R \arrow[dl, dashed, "h"] \\ & N & \end{tikzcd}$$
  Now, define $\mu' : M' \to N$ with $x + ra \mapsto \mu(x) + h(r)$
  <u>Well-define</u> :
  $x_1 + r_1 a = x_2 + r_2 a \rightsquigarrow a(r_1 - r_2) = x_2 - x_1 \in M \rightsquigarrow h(r_1) - h(r_2) = h(r_1 - r_2) = f(r_1 - r_2) = \mu(a(r_1 - r_2)) = \mu(x_2 - x_1) = \mu(x_2) - \mu(x_1) \rightsquigarrow \mu(x_1) + h(r_1) = \mu(x_2) + h(r_2)$
  But, μ' is extension of μ. Therefore, $(M', \mu') \gneq (M^*, \mu)$, which is a contradiction to M^* is maximal.

  □

Def 2. M is **divisible** if $\forall\, x \in M, r \in R \setminus \{0\}$, there exists $y \in M$ such that $x = ry$, i.e. $rM = M \quad \forall\, r \in R \setminus \{0\}$.

Prop 1.1.1.

1. Every injective module N over an integral domain is divisible.

   Proof. For x_0, $r_0 \in R \; \{0\}$,
   $$\begin{tikzcd} & r_0 \arrow[dd, phantom, "\Cap"] \arrow[dddd, bend right, mapsto] & \\ & & \\ 0 \arrow[r] & \langle r_0 \rangle \arrow[r, hook] \arrow[d, "f"] & R \arrow[dl, "h"] \\ & N \arrow[d, phantom, "\Cup"] & \\ & x_0 & \end{tikzcd}$$
   Let $y_0 := h(1)$. Then, $r_0 y_0 = r_0 h(1) = h(r_0) = x_0$. (why the ID is required ?) □

2. Every divisible module N over an PID is injective.

   Proof. If $I = 0$, let h(1) = arbitrary is always let diagram commute. Now, let $\forall\, I \neq 0\; I = \langle r_0 \rangle$

for some $r_0 \neq 0 \in R$

$$
\begin{tikzcd}
 & r_0 \arrow[dddd, mapsto, bend right] & \\
 & \rotatebox[origin=c]{90}{$\in$} & \\
0 \arrow[r] & \langle r_0 \rangle \arrow[d, "f"] \arrow[r, hook] & R \arrow[dl, dashed, "h"] \\
 & N & \\
 & \rotatebox[origin=c]{-90}{$\in$} & \\
 & x_0 := f(r_0) &
\end{tikzcd}
$$

Then, $\exists\, y_0 \rightsquigarrow r_0 y_0 = x_0$. Define $h(1) = y_0$. The diagram commute. $\square$

Theorem 2. $\forall\, M \in \mathbf{Mod}_R$, $\exists\, N$ is injective s.t. $M \hookrightarrow N$.

Proof.

- Case 1 $R = \mathbb{Z}$

- Case 2 R is arbitrary
  We can regard M_1 as $\mathbb{Z}$ module, therefore $\exists\, N_0$ a $\mathbb{Z}$ module with inclusion map. $M_1 \hookrightarrow N_0$.
  Now, we have R module $N := Hom_{\mathbb{Z}}(R, N_0)$
  Claim: N is injective

$$
\begin{tikzcd}
 & x \arrow[dddd, mapsto, bend right] & & \\
 & \rotatebox[origin=c]{90}{$\in$} & & \\
0 \arrow[r] & M_1 \arrow[d, "f"] \arrow[rr] & & M_2 \arrow[dll, dashed, "\exists\, h?"] \\
 & N \arrow[r, phantom, "\cong"] & Hom_{\mathbb{Z}}(R, N_0) & \\
 & \rotatebox[origin=c]{-90}{$\in$} & & \\
 & f(x) & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x \arrow[dddd, mapsto, bend right] & \\
 & \rotatebox[origin=c]{90}{$\in$} & \\
0 \arrow[r] & M_1 \arrow[d, "f'"] \arrow[r] & M_2 \arrow[dl, "\exists\, h'"] \\
 & N_0 & \\
 & \rotatebox[origin=c]{-90}{$\in$} & \\
 & f(x)(1) &
\end{tikzcd}
\quad \text{in } Mod_R
$$

$\boldsymbol{Mod}_R$ $\qquad\qquad$ $\boldsymbol{Mod}_{\mathbb{Z}}$

  Now, define

$$
\begin{aligned}
h : M_2 &\longrightarrow N \\
y &\longmapsto h(y) : R \longrightarrow N_0 \\
&\qquad\qquad 1 \longmapsto h'(y) \\
&\qquad\qquad r \longmapsto h'(ry)
\end{aligned}
$$

  well-define

  1. Show $h(y) \in Hom_{\mathbb{Z}}(R, N_0)$

$$h(y)(r_1 + kr_2) = h'((r_1 + kr_2)y) = h'(r_1 + kr_2 y) = h'(r_1) + kh'(r_2) = h(y)(r_1) + kh(y)(r_1)$$

  2. Show $h \in Hom_R(M_2, N)$

$$
\begin{aligned}
h(y_1 + r_2 y_2)(r)\ \forall\, r \in R &= h'(r(y_1 + r_2 y_2))\ \forall\, r \in R = h'(ry_1 + rr_2 y_2)\ \forall\, r \in R \\
&= h'(ry_1) + h'(rr_2 y_2)\ \forall\, r \in R = h(y)(r) + h(r_2 y_2)(r)\ \ \forall\, r \in R \\
&= h(y)(r) + (r_2 h(y_2))(r)\ \forall\, r \in R
\end{aligned}
$$

3. Show diagram commute

$\square$

Prop 1.1.2. TFAE

1. M is projective.
2. $\forall \quad 0 \to M_1 \to M_2 \to M \to 0$ is split exact.
3. $\exists\, M'$ s.t. $M \oplus M' \cong F$: free.

Prop 1.1.3. TFAE

1. M is projective.
2. $\forall \quad 0 \to M_1 \to M_2 \to M \to 0$ is split exact.
3. $\exists\, M'$ s.t. $M \oplus M' \cong F$: free.

Prop 1.1.4. TFAE

1. M is injective.
2. $\forall \quad 0 \to M \to M_2 \to M_3 \to 0$ is split exact.

Prop 1.1.5. projective $\implies$ flat.