

Algebra

June 6, 2017

1 Introduction to Homological Algebra

1.1 Projective, Injective and Flat modules (week 14)

Def 1.

- $M \in \mathbf{Mod}_R$ is **projective** if $\text{Hom}(M, \cdot)$ preserves the *right* exactness.
- $N \in \mathbf{Mod}_R$ is **injective** if $\text{Hom}(\cdot, N)$ preserves the *right* exactness.
- $M \in \mathbf{Mod}_R$ is **flat** if $M \otimes \cdot$ preserves the *left* exactness.

Fact 1.1.1.

- M is projective \iff

$$\begin{array}{ccc} & M & \\ \exists \tilde{f} \swarrow & \downarrow f & \\ M_2 & \longrightarrow & M_3 \longrightarrow 0 \\ 0 \longrightarrow M_1 & \longrightarrow & M_2 \\ & \downarrow g & \nwarrow \exists \tilde{g} \\ & N & \end{array}$$
- N is injective \iff
- free \implies projective: If $X = \{x_i \mid i \in \Lambda\}$ and $f : x_i \mapsto a_i$. Since β onto, exists b_i so that $\beta(b_i) = a_i$. we can then set $\tilde{f} : x_i \mapsto b_i$ by the universal property of free module.

$$\begin{array}{ccc} & F(X) & \\ \exists \tilde{f} \swarrow & \downarrow f & \\ M_2 & \xrightarrow{\beta} & M_3 \longrightarrow 0 \end{array}$$

- free \implies flat: Let $F \cong R^{\otimes \Lambda}$ be a free module, and M_1, M_2 be two modules such that $0 \rightarrow M_1 \rightarrow M_2$. Since $R \otimes_R M \cong M$, we have

$$\begin{aligned} 0 \rightarrow M_1 \rightarrow M_2 & \quad \text{exact} \\ \implies 0 \rightarrow R \otimes M_1 \rightarrow R \otimes M_2 & \quad \text{exact} \\ \implies 0 \rightarrow \bigoplus_{i \in \Lambda} R \otimes M_1 \rightarrow \bigoplus_{i \in \Lambda} R \otimes M_2 & \quad \text{exact} \\ \stackrel{(a)}{\implies} 0 \rightarrow R^{\otimes \Lambda} \otimes M_1 \rightarrow R^{\otimes \Lambda} \otimes M_2 & \quad \text{exact} \\ \implies 0 \rightarrow F \otimes M_1 \rightarrow F \otimes M_2 & \quad \text{exact} \end{aligned}$$

Where (a) is by the fact that $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$. Thus F flat.

- If S is a multiplication closed set in R with $1 \in S$, then

$$0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0 \implies 0 \rightarrow M_S \rightarrow N_S \rightarrow L_S \rightarrow 0.$$

We know that $M_S \cong R_S \otimes_R M$. So R_S is a flat R -module. e.g. \mathbb{Q} is a flat \mathbb{Z} -module.

For any $M \in \mathbf{Mod}_R$, a projective module N such that $N \rightarrow M \rightarrow 0$ could be easily find: Simply let $N = F$, a free module on the set M .

Now we shall ask for any module M , does there exist $N \in \mathbf{Mod}_R$ such that N is injective and $0 \rightarrow M \rightarrow N$?

Theorem 1 (Boer's criterion). N is injective $\iff \forall I \subset R$, and a homomorphism f , there exists a homomorphism h such that the following diagram commutes:

$$\begin{array}{ccc} 0 \longrightarrow & I & \longrightarrow R \\ & \downarrow f & \nwarrow \exists h \\ & N & \end{array}$$

Proof. “ \Rightarrow ”: See I as an R module, then it is immediate by the definition of injective module.

“ \Leftarrow ”: Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 \\ & & \downarrow g & & \\ & & N & & \end{array}$$

Let $S \triangleq \{(M, \rho) \mid M_1 \subseteq M \subseteq M_2 \text{ and } \rho : M \rightarrow N \text{ extends } g\} \neq \emptyset$ since $(M_1, g) \in S$.

By the routinely proof using Zorn's lemma, exists a maximal element $(M^*, \mu) \in S$.

We claim that $M^* = M_2$. If not, pick $a \in M_2 \setminus M^*$ and let $M' \triangleq M^* + Ra \subsetneq M^*$, $I \triangleq \{r \in R \mid ra \in M^*\}$. Define $f : I \rightarrow N$ with $r \mapsto \mu(ra)$. Then we have an extension $h : R \rightarrow N$ of f .

Now, let $\mu' : M' \rightarrow N = x + ra \mapsto \mu(x) + h(r)$. We shall prove that this map is well-defined: If $x_1 + r_1a = x_2 + r_2a$, then $(r_1 - r_2)a = x_2 - x_1 \in M \implies r_1 - r_2 \in I$. So $h(r_1) - h(r_2) = f(r_1 - r_2) = \mu((r_1 - r_2)a) = \mu(x_2) - \mu(x_1)$, which prove μ' is well defined, and the existence of μ' contradict the fact that (M^*, μ) is maximal. \square

Def 2. M is **divisible** if $\forall x \in M, r \in R \setminus \{0\}$, there exists $y \in M$ such that $x = ry$, i.e. $rM = M \quad \forall r \in R \setminus \{0\}$.

Prop 1.1.1.

1. Every injective module N over an integral domain is divisible.

Proof. For any x_0 and $r_0 \in R \setminus \{0\}$. Let $I = \langle r_0 \rangle \subset R$. As long as R is an integral domain, $I \cong R$ as an R -module, so the R -module homomorphism $f : I \rightarrow N = rx_0 \mapsto rr_0$ is well-defined. Since N injective, this map extends to $h : R \rightarrow N$. Let $y_0 \triangleq h(1)$, then $r_0y_0 = r_0h(1) = h(r_0) = x_0$. Thus N injective. \square

2. Every divisible module N over an PID is injective.

Proof. For any $I \subseteq R$ and a homomorphism $f : I \rightarrow N$, if $I = 0$ then $h = x \mapsto 0$ is always an extension of f . So assume $\forall I \neq 0$. Since R is a PID, $I = \langle r_0 \rangle$ for some $r_0 \neq 0 \in R$. By the fact that N divisible, exists $y_0 \in N$ such that $r_0y_0 = x_0 \triangleq f(r_0)$.

Now we could define $h : R \rightarrow N$ by $1 \mapsto y_0$. Then $h(r_0) = r_0h(1) = r_0y_0 = x_0$, thus h is an extension of f and N injective. \square

3. If R is a PID, then any quotient N of a injective R -module M is injective.

Proof. By 2., $rM = M$ for any $r \neq 0$, thus $rN = N$ for any $r \neq 0$, and hence N injective. \square

Theorem 2. For any $M \in \mathbf{Mod}_R$, exists N injective and contains M .

Proof.

Case 1: $R = \mathbb{Z}$.

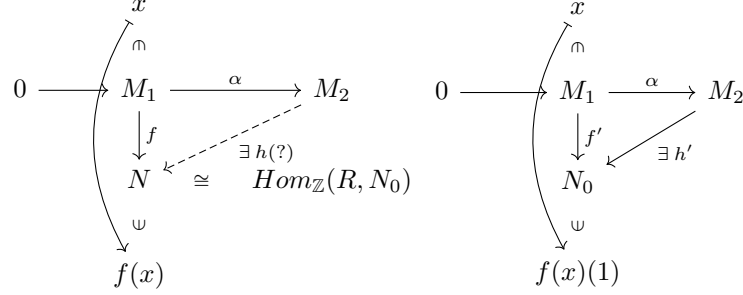
Let $X = \{x_i\}_{i \in \Lambda}$ be a generating set for M and F is free on X . Let f be the natural map from f to M . then $M \cong F/\ker f$.

Define $F' = \bigoplus_{i \in \Lambda} \mathbb{Q}e_i \subset F$, which is obviously a divisible \mathbb{Z} -module. Then $M \subseteq F'/\ker f \triangleq M'$, where M' is injective by proposition 1.1.1.

Case 2: R arbitrary.

We can regard any M as a \mathbb{Z} -module, then there exists an injective module $N_0 \supset M$. Now, we have an R -module $N \triangleq \text{Hom}_{\mathbb{Z}}(R, N_0)$ with multiplication $rf(x) \triangleq x \mapsto f(xr)$.

We claim that N is injective. For any $f :: M_1 \rightarrow N$, and a homomorphism $\alpha :: M_1 \rightarrow M_2$, then α could be take as a \mathbb{Z} -module homomorphism. Define $f' :: M_1 \rightarrow N_0$ by $x \mapsto f(x)(1)$. Since N_0 injective, exists h' , a \mathbb{Z} module homomorphism from M_2 to N_0 .



Now, define

$$\begin{aligned} h : M_2 &\longrightarrow N \\ y &\longmapsto h(y) : R \longrightarrow N_0 \\ 1 &\longmapsto h'(y) \\ r &\longmapsto h'(ry) \end{aligned}$$

We check that h is well-defined.

- $h(y) \in \text{Hom}_{\mathbb{Z}}(R, N_0)$

$$h(y)(r_1 + r_2) = h'((r_1 + r_2)y) = h'(r_1y + r_2y) = h'(r_1y) + h'(r_2y) = h(y)(r_1) + h(y)(r_2)$$

- $h \in \text{Hom}_R(M_2, N)$

$$\begin{aligned} h(r_1y_1 + y_2)(r) &= h'(r(r_1y_1 + y_2)) = h'(rr_1y_1 + ry_2) \\ &= h'(rr_1y_1) + h'(ry_2) \\ &= h(y)(rr_1) + h(y_2)(r) \\ &= (r_1h(y))(r) + h(y_2)(r) \end{aligned}$$

- Show diagram commute $f = h \circ \alpha$ Fix $y \in M_1$, then $\forall r \in R$:

$$\begin{aligned} (h \circ \alpha)(y)(r) &= h(\alpha(y))(r) = h'(r\alpha(y)) \\ &= h'(\alpha(ry)) = f'(ry) \\ &= f(ry)(1) = rf(y)(1) \\ &= f(y)(r) \end{aligned}$$

Thus N_0 injective.

Now notice that, $\text{Hom}_{\mathbb{Z}}(R, \cdot)$ is a left exact functor, so $M \hookrightarrow N_0$ implies $\text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, N_0)$, thus $M \cong \text{Hom}_R(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, N_0) = N$. \square

Prop 1.1.2. TFAE

1. M is projective.
2. Every exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$ split.

3. $\exists M'$ s.t. $M \oplus M' \cong F$: free.

Proof.

(1) \Rightarrow (2) : Since M projective, the map λ with $\beta \circ \lambda = \text{Id}$ exists in the following diagram:

$$\begin{array}{ccc} & M & \\ \swarrow \exists \lambda & \downarrow \text{Id} & \\ M_2 & \xrightarrow{\beta} & M \longrightarrow 0 \end{array}$$

Then λ is a lifting, so $M_2 \cong M_1 \oplus M$ and $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$ split.

(2) \Rightarrow (3): Let F be a free module on a generating set of M , and $\beta :: F \rightarrow M$ be the natural map, then $0 \rightarrow \ker \beta \rightarrow F \rightarrow M \rightarrow 0$ split, so $F \cong \ker \beta \oplus M$.

(3) \Rightarrow (1): For any $M_2 \rightarrow M_3 \rightarrow 0$, since $M' \oplus M$ free and thus projective, λ' exists in the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \longrightarrow & M' \oplus M & \xleftarrow[\pi]{\mu} & M \longrightarrow 0 \\ & & & & \downarrow \exists \lambda' & & \downarrow f \\ & & & & M_2 & \xrightarrow{\beta} & M_3 \longrightarrow 0 \end{array}$$

Define $\lambda = \lambda' \circ \mu$. Then $\beta \circ \lambda = \beta \circ \lambda' \circ \mu = f \circ \pi \circ \mu = f$. □

Prop 1.1.3. TFAE

1. M is injective.
2. Each exact sequence $0 \rightarrow M \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ split.

Proof. (1) \Rightarrow (2): Similar to the projective case, μ exists in the following diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & M \xrightarrow{\alpha} M_2 \\ & & \downarrow \text{Id} \\ & & M \end{array} \quad \begin{array}{c} \nearrow \exists \mu \\ \nwarrow \end{array}$$

So $M_2 = M \oplus M_3$.

(2) \Rightarrow (1): By theorem 2, there is a module $N \subset M$ so that N is injective.

Consider $0 \longrightarrow M \xrightleftharpoons[\exists \mu]{i} N \longrightarrow \text{coker } i \longrightarrow 0$ split exact and $\mu \circ i = \text{Id}_M$. Since N injective, h' exists in the following diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & M_1 \xrightarrow{\alpha} M_2 \\ & & \downarrow f \\ & & M \\ & \nearrow i \circ f & \downarrow i \\ & & N \end{array} \quad \begin{array}{c} \nearrow \exists h' \\ \nwarrow \mu \end{array}$$

Let $h = \mu \circ h'$, then $h \circ \alpha = \mu \circ h' \circ \alpha = \mu \circ i \circ f = f$ □

Prop 1.1.4. projective \implies flat.

Proof. Observe that $\bigoplus_{i \in \Lambda} M_i$ is flat if and only if M_i is flat for each i , since if $0 \rightarrow N_1 \xrightarrow{\alpha} N_2$ exact, then

$$\begin{array}{ccc}
0 & \longrightarrow & (\bigoplus M_i) \otimes N_1 \xrightarrow{1 \otimes \alpha} (\bigoplus M_i) \otimes N_2 \\
& & \parallel \qquad \qquad \qquad \parallel \\
0 & \longrightarrow & \bigoplus (M_i \otimes N_1) \xrightarrow{\bigoplus (1 \otimes \alpha)} \bigoplus (M_i \otimes N_2) \\
& & \updownarrow \\
0 & \longrightarrow & M_i \otimes N_1 \xrightarrow{1 \otimes \alpha} M_i \otimes N_2 \qquad \qquad \forall i \in \Lambda
\end{array}$$

If M is projective, then by proposition 1.1.2 $\exists M'$ such that $M \oplus M' \cong F$ is free. Since free implies flat, by above, M is flat. \square

1.2 Ext and Tor (week 15)

Given $M, N \in \mathbf{Mod}_R$, there are two ways to define $\text{Ext}^n(M, N)$:

Def 3 (Ext functor).

- Find any projective resolution $P_\bullet \xrightarrow{\alpha} M \rightarrow 0$, and let $P_M : P_\bullet \rightarrow 0$ (called a *deleted resolution*). We can define $\text{Ext}_{\text{proj}}^n(M, N) = H^n(\text{Hom}(P_M, N))$.
- Find any injective resolution $0 \xrightarrow{\alpha} N \rightarrow E^\bullet$, and let $E_N : 0 \rightarrow E^\bullet$. We can define $\text{Ext}_{\text{inj}}^n(M, N) = H^n(\text{Hom}(M, E_N))$.

Prop 1.2.1. $\text{Ext}_{\text{proj}}^0(M, N) \cong \text{Ext}_{\text{inj}}^0(M, N) \cong \text{Hom}(M, N)$.

Proof.

$$\text{Hom}(P_M, N) : 0 \xrightarrow{\bar{d}_0} \text{Hom}(P_0, N) \xrightarrow{\bar{d}_1} \text{Hom}(P_1, N) \rightarrow \dots$$

$$\text{so } \text{Ext}_{\text{proj}}^0(M, N) = \ker \bar{d}_1 / \text{im } \bar{d}_0 = \ker \bar{d}_1 = \text{im } \alpha = \text{Hom}(M, N). \quad \square$$

Similarly, $\text{Ext}_{\text{inj}}^0(M, N) = \text{Hom}(M, N)$.

Lemma 1.

- If M is projective, then $\text{Ext}_{\text{proj}}^n(M, N) = 0$ for all $n > 0, N \in \mathbf{Mod}_R$.
- If N is injective, then $\text{Ext}_{\text{inj}}^n(M, N) = 0$ for all $n > 0, M \in \mathbf{Mod}_R$.

Proof. If M is projective, then $0 \rightarrow M \xrightarrow{1_M} M \rightarrow 0$ is a projective resolution of M . Its deleted resolution is then $P_M : 0 \rightarrow M \rightarrow 0$. Hence for $n > 0$, $\text{Ext}_{\text{proj}}^n(M, N) = H^n(\text{Hom}(P_M, N)) = 0$.

The argument applies similarly to injective case. \square

Theorem 3 (Equivalence of Ext_{proj} and Ext_{inj}).

$$\text{Ext}_{\text{proj}}^n(M, N) \cong \text{Ext}_{\text{inj}}^n(M, N).$$

Proof. Let $P_\bullet \rightarrow M \rightarrow 0$ and $0 \rightarrow N \rightarrow E^\bullet$ be projective and injective resolutions, then we have $0 \rightarrow K_0 \rightarrow P_0 \rightarrow M \rightarrow 0$ and $0 \rightarrow N \rightarrow E^0 \rightarrow L^1 \rightarrow 0$ exact.

$$\begin{array}{ccccccc} \cdots & \rightarrow & P_2 & \xrightarrow{\quad} & P_1 & \xrightarrow{\quad} & P_0 \rightarrow M \rightarrow 0 \\ & & \searrow & & \nearrow & & \searrow \\ & & & & K_1 & & K_0 \\ & & \nearrow & & \searrow & & \nearrow \\ 0 & & & & & & 0 \end{array} \quad \begin{array}{ccccccc} 0 \rightarrow N \rightarrow E^0 & \xrightarrow{\quad} & E^1 & \xrightarrow{\quad} & E^2 \rightarrow \cdots \\ & & \searrow & & \nearrow \\ & & & & L^1 & & L^2 \\ & & \nearrow & & \searrow & & \nearrow \\ 0 & & & & & & 0 \end{array}$$

We can construct long exact sequences of homology of $\text{Hom}(\cdot, E_N)$:

$$0 \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(M, E^0) \rightarrow \text{Hom}(M, L^1) \rightarrow \text{Ext}_{\text{inj}}^1(M, N) \rightarrow \text{Ext}_{\text{inj}}^1(M, E^0) = 0$$

$$0 \rightarrow \text{Hom}(P_0, N) \rightarrow \text{Hom}(P_0, E^0) \rightarrow \text{Hom}(P_0, L^1) \rightarrow 0$$

$$0 \rightarrow \text{Hom}(K_0, N) \rightarrow \text{Hom}(K_0, E^0) \rightarrow \text{Hom}(K_0, L^1) \rightarrow \text{Ext}_{\text{inj}}^1(K_0, N) \rightarrow \text{Ext}_{\text{inj}}^1(K_0, E^0) = 0$$

The second sequence is short because P_0 is projective (so $\text{Hom}(P_0, \cdot)$ preserves exactness).

Similarly, for $\text{Hom}(P_M, \cdot)$ we have:

$$0 \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(P_0, N) \rightarrow \text{Hom}(K_0, N) \rightarrow \text{Ext}_{\text{proj}}^1(M, N) \rightarrow \text{Ext}_{\text{proj}}^1(P_0, N) = 0$$

$$\begin{aligned}
0 &\rightarrow \text{Hom}(M, E^0) \rightarrow \text{Hom}(P_0, E^0) \rightarrow \text{Hom}(K_0, E^0) \rightarrow 0 \\
0 &\rightarrow \text{Hom}(M, L^1) \rightarrow \text{Hom}(P_0, L^1) \rightarrow \text{Hom}(K_0, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(M, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(P_0, L^1) = 0
\end{aligned}$$

Combining these sequences together, we got the following 2D diagram:

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{Hom}(M, N) & \longrightarrow & \text{Hom}(M, E^0) & \xrightarrow{\phi} & \text{Hom}(M, L^1) \longrightarrow \text{Ext}_{\text{inj}}^1(M, N) \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{Hom}(P_0, N) & \longrightarrow & \text{Hom}(P_0, E^0) & \xrightarrow{\sigma} & \text{Hom}(P_0, L^1) \longrightarrow 0 \\
& \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\
0 & \longrightarrow & \text{Hom}(K_0, N) & \longrightarrow & \text{Hom}(K_0, E^0) & \xrightarrow{\tau} & \text{Hom}(K_0, L^1) \longrightarrow \text{Ext}_{\text{inj}}^1(K_0, N) \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& \text{Ext}_{\text{proj}}^1(M, N) & & 0 & & \text{Ext}_{\text{proj}}^1(M, L^1) & \\
& \downarrow & & & & \downarrow & \\
& 0 & & & & 0 &
\end{array}$$

By Snake lemma, there is an exact sequence

$$(\ker \alpha \rightarrow) \ker \beta \rightarrow \ker \gamma \rightarrow \text{coker } \alpha \rightarrow \text{coker } \beta (\rightarrow \text{coker } \gamma)$$

and this reads

$$\text{Hom}(M, E^0) \xrightarrow{\phi} \text{Hom}(M, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(M, N) \rightarrow 0$$

Thus $\text{Ext}_{\text{proj}}^1(M, N) \cong \text{coker } \phi \cong \text{Ext}_{\text{inj}}^1(M, N)$.

(From now on, we don't need to distinguish proj/inj for Ext^1 !)

Since σ is onto, $\text{im } \gamma = \text{im}(\gamma \circ \sigma)$. Similarly, $\text{im } \tau = \text{im}(\tau \circ \beta)$.

By the commutativity of the diagram, $\text{im } \gamma = \text{im } \tau$, so

$$\text{Ext}^1(K_0, N) \cong \text{coker } \gamma = \text{Hom}(K_0, L^1) / \text{im } \gamma \cong \text{coker } \tau \cong \text{Ext}^1(M, L^1).$$

Write $K_{-1} := M, L^0 := N$, then $\text{Ext}^1(K_0, L^0) = \text{Ext}^1(K_{-1}, L^1)$ (\star).

Similarly, from the exact sequences

$$0 \rightarrow K_j \rightarrow P_j \rightarrow K_{j-1} \rightarrow 0$$

$$0 \rightarrow L^i \rightarrow E^i \rightarrow L^{i+1} \rightarrow 0$$

, we can obtain $\text{Ext}^1(K_j, L^i) \cong \text{Ext}^1(K_{j-1}, L^{i+1})$ for $i, j \geq 0$.

Now, observe that

$$0 \rightarrow L^{n-1} \rightarrow E^{n-1} \xrightarrow{d_{n-1}} E^n \xrightarrow{d_n} \dots$$

is an injective resolution of L^{n-1} , and $\text{Ext}^1(M, L^{n-1}) \cong \ker \overline{d_n} / \text{im } \overline{d_{n-1}} \cong \text{Ext}_{\text{inj}}^n(M, N)$.

Similarly, for projective resolution we have $\text{Ext}^1(K_{n-2}, N) \cong \text{Ext}_{\text{proj}}^n(M, N)$.

Finally, by (\star),

$$\text{Ext}_{\text{inj}}^n(M, N) \cong \text{Ext}^1(K_{-1}, L^{n-1}) \cong \text{Ext}^1(K_0, L^{n-2}) \cong \dots \cong \text{Ext}^1(K_{n-2}, L^0) \cong \text{Ext}_{\text{proj}}^n(M, N).$$

□

Def 4 (Tor functor). Let $M, N \in \mathbf{Mod}_R$, and $P_\bullet \rightarrow M \rightarrow 0$ be a projective resolution of M , similar to the Ext case, for $n \geq 0$ we can define

$$\mathrm{Tor}_n(M, N) = H_n(P_M \otimes N).$$

Fact 1.2.1. By Horseshoe lemma, short exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ induces a long exact sequence:

$$\cdots \rightarrow \mathrm{Tor}_1(M_1, N) \rightarrow \mathrm{Tor}_1(M_2, N) \rightarrow \mathrm{Tor}_1(M_3, N) \rightarrow M_1 \otimes N \rightarrow M_2 \otimes N \rightarrow M_3 \otimes N \rightarrow 0$$

Prop 1.2.2. If M is flat, then $\mathrm{Tor}_n(M, N) = 0$ for $n > 0, N \in \mathbf{Mod}_R$.

Proof. Since $0 \rightarrow M \xrightarrow{1_M} M \rightarrow 0$ is a flat resolution of M . □

Theorem 4 (Tor for flat resolutions). Let $U_\bullet \rightarrow M \rightarrow 0$ be a flat resolution of M , then for $n \geq 0$,

$$\mathrm{Tor}_n(M, N) \cong H_n(U_M \otimes N)$$

Proof.

$$\begin{array}{ccccccc} \cdots & \rightarrow & U_2 & \xrightarrow{\quad} & U_1 & \xrightarrow{\quad} & U_0 \rightarrow M \rightarrow 0 \\ & & \searrow & & \swarrow & & \swarrow \\ & & & W_1 & & & W_0 \\ & \nearrow & & \searrow & \nearrow & & \searrow \\ 0 & & & & 0 & & 0 \end{array}$$

$$H_\bullet(U_M \otimes N) : \cdots \rightarrow U_2 \otimes N \xrightarrow{d_2 \otimes 1} U_1 \otimes N \xrightarrow{d_1 \otimes 1} U_0 \otimes N \rightarrow 0$$

- $n = 0$:

Since tensor is right exact, $U_1 \otimes N \xrightarrow{d_1 \otimes 1} U_0 \otimes N \xrightarrow{\alpha \otimes 1} M \otimes N \rightarrow 0$ is exact. Hence

$$H_0(U_M \otimes N) = (U_0 \otimes N) / \mathrm{im}(d_1 \otimes 1) = (U_0 \otimes N) / \ker(\alpha \otimes 1) \cong M \otimes N$$

Any projective resolution also has this property, so $\mathrm{Tor}_0(M, N) = H_0(U_M \otimes N)$.

- $n = 1$:

$0 \rightarrow W_0 \rightarrow U_0 \rightarrow M \rightarrow 0$ induces a long exact sequence:

$$\cdots \rightarrow \mathrm{Tor}_1(W_0, N) \rightarrow 0 \rightarrow \mathrm{Tor}_1(M, N) \rightarrow W_0 \otimes N \xrightarrow{i \otimes 1} U_0 \otimes N \rightarrow M \otimes N \rightarrow 0$$

where $\mathrm{Tor}_1(U_0, N) = 0$ because U_0 is flat. We can see that $\mathrm{Tor}_1(M, N) \cong \ker(i \otimes 1)$.

$$\begin{array}{ccccc} U_2 \otimes N & \xrightarrow{d_2 \otimes 1} & U_1 \otimes N & \xrightarrow{d_1 \otimes 1} & U_0 \otimes N \\ & \searrow \beta' \otimes 1 & \nearrow j \otimes 1 & \searrow \alpha' \otimes 1 & \nearrow i \otimes 1 \\ & & W_1 \otimes N & & W_0 \otimes N \\ & & & \searrow & \searrow \\ & & & & 0 \end{array}$$

Since $\alpha' \otimes 1$ is onto, $W_0 \otimes N \cong (U_1 \otimes N) / \ker(\alpha' \otimes 1)$. Also, $x \in \ker(d_1 \otimes 1) \iff (d_1 \otimes 1)(x) = 0 \iff (\alpha' \otimes 1)(x) \in \ker(i \otimes 1)$, so $\ker(i \otimes 1) = (\alpha' \otimes 1)(\ker(d_1 \otimes 1)) \cong \ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1)$. ($\alpha' \otimes 1$ can be considered a quotient map, then $\ker(d_1 \otimes 1)$ descends to $\ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1)$.)

Now, in the diagram $W_1 \otimes N \rightarrow U_1 \otimes N \rightarrow W_0 \otimes N \rightarrow 0$ exact, so $\ker(\alpha' \otimes 1) = \mathrm{im}(j \otimes 1)$. But $\beta' \otimes 1$ is onto, thus $\mathrm{im}(j \otimes 1) = \mathrm{im}(d_2 \otimes 1)$.

Finally,

$$\mathrm{Tor}_1(M, N) \cong \ker(i \otimes 1) \cong \ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1) = \ker(d_1 \otimes 1) / \mathrm{im}(d_2 \otimes 1) = H_1(U_M \otimes N).$$

- $n \geq 2$:

Let's see further in the previous long exact sequences:

$$\cdots \rightarrow \mathrm{Tor}_2(W_0, N) \rightarrow 0 \rightarrow \mathrm{Tor}_2(M, N) \xrightarrow{\sim} \mathrm{Tor}_1(W_0, N) \rightarrow 0 \rightarrow \mathrm{Tor}_1(M, N) \rightarrow \cdots$$

we can see that $\mathrm{Tor}_n(M, N) \cong \mathrm{Tor}_{n-1}(W_0, N)$ for $n \geq 2$.

Now,

$$\cdots U_3 \xrightarrow{d_3} U_2 \xrightarrow{d_2} U_1 \rightarrow W_0 \rightarrow 0$$

is a flat resolution of W_0 , and its homology is $H_{n-1}(U_{W_0} \otimes N) = \ker(d_n \otimes 1) / \mathrm{im}(d_{n-1} \otimes 1) = H_n(U_M \otimes N)$.

By induction, assume it's true for $n-1$, then

$$H_n(U_M \otimes N) = H_{n-1}(U_{W_0} \otimes N) \cong \mathrm{Tor}_{n-1}(W_0, N) \cong \mathrm{Tor}_n(M, N).$$

□

Eg 1.2.1. $R = \mathbb{Z}, M = \mathbb{Z}/m\mathbb{Z}$ with $m \geq 2$. Then

$$P : 0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{/m\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \rightarrow 0$$

is a projective resolution of $\mathbb{Z}/m\mathbb{Z}$. So for any $N \in \mathbf{Mod}_{\mathbb{Z}}$,

$$H^n(\mathrm{Hom}_{\mathbb{Z}}(P_{\mathbb{Z}/m\mathbb{Z}}, N)) : 0 \rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) \xrightarrow{\bar{m}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) \rightarrow 0,$$

$$\begin{aligned} \mathrm{Ext}_{\mathbb{Z}}^0(\mathbb{Z}/m\mathbb{Z}, N) &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, N) \cong {}_mN := \{a \in N \mid ma = 0\} \\ \mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m\mathbb{Z}, N) &\cong N/mN \\ \mathrm{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/m\mathbb{Z}, N) &= 0 \quad (\text{for } n \geq 2) \end{aligned}$$

Eg 1.2.2. $\mathbb{Q} = \mathbb{Z}_{\langle 0 \rangle}$ is a localization, thus a flat \mathbb{Z} module. Then

$$U : 0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

is a flat resolution of \mathbb{Q}/\mathbb{Z} . For $G \in \mathbf{Mod}_{\mathbb{Z}}$ (i.e. an abelian group),

$$H_n(G \otimes U_{\mathbb{Q}/\mathbb{Z}}) : 0 \rightarrow G \otimes \mathbb{Z} \xrightarrow{1 \otimes i} G \otimes \mathbb{Q} \rightarrow 0$$

$$\begin{aligned} \mathrm{Tor}_0(G, \mathbb{Q}/\mathbb{Z}) &\cong G \otimes \mathbb{Q}/\mathbb{Z} \\ \mathrm{Tor}_1(G, \mathbb{Q}/\mathbb{Z}) &= \ker(1 \otimes i) \cong t(G) := \{a \in G \mid ma = 0 \text{ for some } m \in \mathbb{N}\} \\ \mathrm{Tor}_n(G, \mathbb{Q}/\mathbb{Z}) &= 0 \quad (\text{for } n \geq 2) \end{aligned}$$

Def 5. Let M be a left R -module, then define

$$M^* := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$$

as a right R -module by

$$\begin{aligned} fr : M &\rightarrow \mathbb{Q}/\mathbb{Z} \\ x &\mapsto f(rx) \end{aligned}$$

Fact 1.2.2.

1. \mathbb{Q}/\mathbb{Z} is injective.
2. $A = 0 \iff A^* = 0$.
3. $B \hookrightarrow C \iff C^* \twoheadrightarrow B^*$.

Proof.

1. For $m \in \mathbb{Z} \setminus \{0\}$, $m(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$ by $m(\frac{a}{mb} + \mathbb{Z}) \hookrightarrow \frac{a}{b} + \mathbb{Z}$, so \mathbb{Q}/\mathbb{Z} is divisible. But \mathbb{Z} is a PID, so \mathbb{Q}/\mathbb{Z} is injective.
2. $(\Rightarrow) A^* = \text{Hom}_{\mathbb{Z}}(0, \mathbb{Q}/\mathbb{Z}) = 0$.

(\Leftarrow) If $A \neq 0$, then $\exists a \in A, a \neq 0$, so $0 \rightarrow \mathbb{Z}a \xrightarrow{i} A$ is an inclusion.

Since $\mathbb{Z}a$ is a cyclic abelian group, there is a nonzero $g : \mathbb{Z}a \rightarrow \mathbb{Q}/\mathbb{Z}$. (If $\mathbb{Z}a \cong \mathbb{Z}/m\mathbb{Z}$, let $g : a \mapsto \frac{1}{m}$; if $\mathbb{Z}a \cong \mathbb{Z}$, let $g : a \mapsto \frac{1}{2}$.)

But \mathbb{Q}/\mathbb{Z} is injective, so $\exists f : A \rightarrow \mathbb{Q}/\mathbb{Z}$ (i.e. $f \in A^*$), and $f \circ i = g \neq 0$ so $f \neq 0$, thus $A^* \neq 0$.

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z}a & \xrightarrow{i} & A \\ & & \downarrow g & \swarrow \exists f & \\ & & \mathbb{Q}/\mathbb{Z} & & \end{array}$$

3. Since \mathbb{Q}/\mathbb{Z} is injective, $\text{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ is exact. Let $0 \rightarrow \ker f \rightarrow B \xrightarrow{f} C$ exact, applying $\text{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ results in $C^* \xrightarrow{f^*} B^* \rightarrow (\ker f)^* \rightarrow 0$ exact. Thus $\text{coker } f^* = (\ker f)^*$.
By 2., $B \hookrightarrow C \iff \ker f = 0 \iff (\ker f)^* = 0 \iff \text{coker } f^* = 0 \iff C^* \twoheadrightarrow B^*$.

□

Prop 1.2.3. Let M be an R -module, then TFAE

1. M is flat.
2. M^* is injective (as a R -module).
3. $\text{Tor}_1(R/I, M) = 0$ for all ideal $I \subseteq R$.
4. $I \otimes_R M \cong IM$ for all ideal $I \subseteq R$.

Proof.

- 3. \iff 4.

For any ideal $I \subseteq R$, $0 \rightarrow I \xrightarrow{i} R \xrightarrow{q} R/I \rightarrow 0$ is exact. This induces a long exact sequence:

$$\text{Tor}_1(R, M) \rightarrow \text{Tor}_1(R/I, M) \rightarrow I \otimes_R M \xrightarrow{i \otimes \mathbf{1}} R \otimes_R M \xrightarrow{q \otimes \mathbf{1}} R/I \otimes_R M \rightarrow 0$$

- $\text{Tor}_1(R, M) = 0$ since R is a flat R -module.
- $R \otimes_R M \cong M$.
- $R/I \otimes_R M \cong M/IM$ by $(r + I) \otimes a \mapsto (ra + IM)$.

So we have

$$0 \rightarrow \text{Tor}_1(R/I, M) \rightarrow I \otimes_R M \xrightarrow{i'} M \xrightarrow{q'} M/IM \rightarrow 0$$

exact, with $q' : M \rightarrow M/IM$ being exactly the quotient map (one can check that $q \otimes \mathbf{1} \cong q'$).

Now it's clear that $\text{Tor}_1(R/I, M) = 0 \iff I \otimes_R M \cong \ker(q') \cong IM$.

(The reverse direction requires $I \otimes_R M \cong IM$ being the natural isomorphism $r \otimes b \mapsto rb$, so $i' : IM \rightarrow M$ can then be the natural inclusion.)

- 1. \iff 2.

Let $0 \rightarrow N' \xrightarrow{f} N$, then $\text{Hom}_R(N, M^*) \xrightarrow{\bar{f}} \text{Hom}_R(N', M^*)$.

By the adjoint relation,

$$\text{Hom}_R(N, M^*) = \text{Hom}_R(N, \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(N \otimes_R M, \mathbb{Q}/\mathbb{Z}) = (N \otimes_R M)^*,$$

we have another map $(N \otimes_R M)^* \xrightarrow{(f \otimes 1)^*} (N' \otimes_R M)^*$ isomorphic to the previous one, with its unstarred map $N' \otimes_R M \xrightarrow{f \otimes 1} N \otimes_R M$.

Now, M^* is injective $\iff \bar{f}$ is surjective $\forall N, N' \iff (f \otimes 1)^*$ is surjective $\forall N, N' \iff f \otimes 1$ is injective $\forall N, N' \iff M$ is flat.

- 2. \iff 4.

Similar to the previous section, by Baer's criterion,

$$\begin{aligned} M^* \text{ is injective} &\iff \text{Hom}_R(R, M^*) \twoheadrightarrow \text{Hom}_R(I, M^*), \forall I \subseteq R \\ &\iff (R \otimes_R M)^* \twoheadrightarrow (I \otimes_R M)^*, \forall I \subseteq R \\ &\iff I \otimes_R M \hookrightarrow R \otimes_R M \cong M, \forall I \subseteq R \\ &\iff I \otimes_R M \cong IM, \forall I \subseteq R. \end{aligned}$$

Similarly, this requires the isomorphism of $I \otimes_R M \cong IM$ be natural (the following f).

The map $f : I \otimes_R M \rightarrow IM$
 $r \otimes a \mapsto ra$ is always onto, but may not be 1-1. If it is, $I \otimes_R M \cong IM$.

□

Prop 1.2.4. For $I, J \subseteq R$ being ideals, then $\text{Tor}_1(R/I, R/J) \cong (I \cap J)/IJ$.

Proof. $0 \rightarrow I \xrightarrow{i} R \rightarrow R/I \rightarrow 0$ induces a long exact sequence

$$0 \rightarrow \text{Tor}_1(R/I, R/J) \rightarrow I \otimes_R R/J \xrightarrow{i \otimes 1} R \otimes_R R/J \rightarrow R/I \otimes_R R/J \rightarrow 0,$$

where $\text{Tor}_1(R, R/J) = 0$ since R is flat.

Also $I \otimes_R R/J \cong I/IJ, R \otimes_R R/J \cong R/J$, so we have $I/IJ \xrightarrow{i'} R/J$ with $\text{Tor}_1(R/I, R/J) \cong \ker(i \otimes 1) \cong \ker i'$.

But $i' : I/IJ \rightarrow R/J$
 $x + IJ \mapsto x + J$, so $\bar{x} \in \ker i' \iff x \in I \text{ and } x \in J \iff x \in I \cap J$, hence $\ker i' \cong (I \cap J)/IJ$.

□