

Algebra

October 17, 2016

1 Group theory

1.1 Week 1

Def 1. A non-empty set G with a binary function $f : G \times G \rightarrow G, (a, b) \mapsto ab$ is a **group** if it satisfies

1. $(ab)c = a(bc)$.
2. $\exists 1 \in G$ s.t. $1a = a1 = a, \forall a \in G$.
3. $\exists a^{-1} \in G$ s.t. $aa^{-1} = a^{-1}a = 1$.

CONCON

Def 2. Let G be a group. Then G is said to be **abelian** if $\forall a, b \in G, ab = ba$.

Ex 1.1.1. Let G be a semigroup. Then TFAE (the following are equivalent)

1. G is a group.
2. For all $a, b \in G$ and the equations $bx = a, yb = a$, each of them has a solution in G .
3. $\exists e \in G$ s.t. $ae = a \forall a \in G$ and if we fix such e , then $\forall b \in G \exists b' \in G$ s.t. $bb' = e$.

Ex 1.1.2. Let G be a group. Show that

1. $\forall a \in G, a^2 = 1$, then G is abelian.
2. G is abelian $\iff \forall a, b \in G, (ab)^n = a^n b^n$ for three consecutive integer n .

Def 3. Let G be a group and $H \subseteq G, H \neq \emptyset$. Then H is said to be a subgroup of G , denoted by $H \leq G$, if

1. $\forall a, b \in H, ab \in H$.
2. $1 \in H$.
3. $\forall a \in H, a^{-1} \in H$.

useful criterion: $H \leq G \iff \forall a, b \in H, ab^{-1} \in H$.

pf:

\Rightarrow $b \in H \implies b^{-1} \in H$, and $a \in H$, so $ab^{-1} \in H$.

- \Leftarrow
1. $H \neq \emptyset \implies \exists a \in H \implies aa^{-1} = 1 \in H$.
 2. $1, a \in H \implies 1a^{-1} = a^{-1} \in H$.
 3. $a, b^{-1} \in H \implies a(b^{-1})^{-1} = ab \in H$. □

Ex 1.1.1. $(\mathbb{Z}, +, 0) \leq (\mathbb{Q}, +, 0) \leq (\mathbb{R}, +, 0) \leq (\mathbb{C}, +, 0) ; (\mathbb{Q}^\times, \times, 1) \leq (\mathbb{R}^\times, \times, 1) \leq (\mathbb{C}^\times, \times, 1)$

Eg 1.1.2.

- Special linear group $\text{SL}(n, \mathbb{F}) = \{ A \in \text{GL}(n, \mathbb{F}) \mid \det A = 1 \}$
- Orthogonal group $\text{O}(n) = \{ A \in \text{GL}(n, \mathbb{R}) \mid A^t A = I_n \}$
- Unitary group $\text{U}(n) = \{ A \in \text{GL}(n, \mathbb{C}) \mid A^* A = I_n \}$
- Special orthogonal group $\text{SO}(n) = \text{SL}(n, \mathbb{R}) \cap \text{O}(n)$

- Special unitary group $SU(n) = SL(n, \mathbb{C}) \cap U(n)$

Def 4. Let $f : G_1 \rightarrow G_2$. f is called an **isomorphism** if

1. f is 1-1 and onto.
2. $\forall a, b \in G_1, f(ab) = f(a)f(b)$. (**homomorphism**)

, denoted by $G_1 \cong G_2$.

Remark 1. (practice)

1. $f(1) = 1$.
2. $f(a^{-1}) = f(a)^{-1}$.
3. If f is an isomorphism, then $\exists f^{-1}$ is also a homomorphism.

Eg 1.1.3.

- $U(1) = \{ z \in \mathbb{C}^\times \mid \bar{z}z = 1 \}, z = \cos \theta + \sin \theta i$
- $SO(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$

notice that $U(1) \cong SO(2)$. $S^1 = \{ (a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1 \}$, 可被賦予群的結構.

Eg 1.1.4. Let $A \in SU(2) \implies A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \alpha\bar{\alpha} + \beta\bar{\beta} = 1, \alpha, \beta \in \mathbb{C}$.

Quaternion(四元數): $\mathbb{H} = \{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \}$ with $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j (\implies ij = -ji)$.

Let $x = a + bi + cj + dk, \bar{x} = a - bi - cj - dk$, then $N(x) = x\bar{x} = a^2 + b^2 + c^2 + d^2$, For $x \neq 0, N(x) \neq 0, x^{-1} = \frac{1}{N(x)}\bar{x}$

Now, for $x = a + bi + cj + dk = (a + bi) + (c + di)j$. So $SU(2) \cong \{ x \in \mathbb{H}^\times \mid N(x) = 1 \}$. $S^3 = \{ (a, b, c, d) \in \mathbb{R}^4 \mid a^2 + b^2 + c^2 + d^2 = 1 \}$, 可被賦予群的結構.

★ The only spheres with continuous group law are S^1, S^3 .

Ex 1.1.3. Find a way to regard $M_{n \times n}(\mathbb{H})$ as a subset of $M_{2n \times 2n}(\mathbb{C})$, which preserves addition and multiplication, and then there is a way to characterize $GL(n, \mathbb{H})$.

Def 5 (symplectic group). $Sp(n, \mathbb{F}) = \{ A \in GL(2n, \mathbb{F}) \mid A^t J A = J \}$ where $J = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}$.

($A^t J A = J$ preserving non-degenerate skew-symmetric forms)

$Sp(n) = \{ A \in GL(n, \mathbb{H}) \mid A^* A = I_n \}$.

Ex 1.1.4. Show $Sp(n) \cong U(2n) \cap Sp(n, \mathbb{C})$.

Ques: Find the smallest subgroup of $SU(2)$ containing $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

1.2 Week 2

1.2.1 Permutation groups and Dihedral groups

Def 6. A permutation of a set B is a 1-1 and onto function from B to B .

Let $S_B :=$ the set of permutations of B . Then $(S_B, \cdot, \text{Id}_B)$ forms a group.

If $B = \{a_1, \dots, a_n\}$, then $S_B \cong S_{\{1, \dots, n\}}$ and write $S_n = S_{\{1, \dots, n\}}$, called the symmetric group of degree n .

Theorem 1 (Cayley theorem). Any group is isomorphic to a subgroup of some permutation group.

(Hint): Let G be a group. Set $B = G$. Consider $a \in G$ as $\sigma_a : G \rightarrow G, x \mapsto ax$. Then $\sigma_a \in S_G \implies G \leq S_G$.

Fact 1.2.1. S_n is a finite group of order $n!$, i.e. $|S_n| = n!$.

pf: EASY =O

□

Cyclic notation: $\sigma \in S_5$, say $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$. Write $\sigma = (1\ 4)(2\ 3\ 5)$.

\Rightarrow Any permutation can be written as a product of disjoint cycles.

Eg 1.2.1. In S_7 , $\sigma_1 = (1\ 2\ 3)(4\ 5\ 6)(7)$, $\sigma_2 = (1\ 3\ 5\ 6)(2\ 4\ 7)$.
Then $\sigma_1\sigma_2 = (2\ 5\ 4\ 7\ 3\ 6)$, $\sigma_1^{-1} = (1\ 3\ 2)(4\ 6\ 5)$.

Def 7. A 2 cycle is called a **transposition**.

Eg 1.2.2. $(1\ 2\ 3) = (1\ 3)(1\ 2)$, $(1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2)$.
Any permutation is a product of 2 cycles.

Useful formula: $\sigma \in S_n$, $\sigma(j_1 \dots j_m)\sigma^{-1} = (\sigma(j_1) \dots \sigma(j_m))$.

Eg 1.2.3. Let $\sigma = (1\ 2\ 3)(4\ 5\ 6\ 7)$, $\sigma(2\ 3\ 4)\sigma^{-1} = (3\ 1\ 5)$.

pf: Note that both sides are functions. For $i \in \{1, \dots, n\}$,

Case 1: $\exists k$ s.t. $\sigma(j_k) = i$, CONCON

Case 2: Otherwise, CONCON

□

Fact 1.2.2. $S_n = \langle (1\ 2), \dots, (1\ n) \rangle$.

pf: $(1\ i)^{-1} = (1\ i)$ and $(i\ j) = (1\ i)(1\ j)(1\ i)^{-1}$.

□

Def 8. Let G be a group and $S \subset G$. The subgroup generated by S defined to be the smallest subgroup of G which contains S , denoted by $\langle S \rangle$.

Ex 1.2.1.

$$1. S_n = \langle (1\ 2), (2\ 3), \dots, (n-1\ n) \rangle, \quad n \geq 2.$$

$$2. S_n = \langle (1\ 2), (1\ 2 \dots n) \rangle, \quad n \geq 2.$$

Def 9. $A_n = \{\text{even permutations of } S_n\} \leq S_n, |A_n| = \frac{n!}{2}$.

Ex 1.2.2.

1. $A_n = \langle (1\ 2\ 3), (1\ 2\ 4), \dots, (1\ 2\ n) \rangle, n \geq 3.$
2. $A_n = \langle (1\ 2\ 3), (2\ 3\ 4), \dots, (n-2\ n-1\ n) \rangle, n \geq 3.$

Remark 2. $\langle S \rangle = \bigcap_{S \subseteq H \leq G} H = \{a_1 a_2 \dots a_k \mid k \in \mathbb{N}, a_i \in S \cup S^{-1}\} \cup \{1\}$

The orthogonal transformations on \mathbb{R}^2 : $O(2)$.

Let $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in O(2).$

略... (這邊討論旋轉和反射的矩陣)

Case 1: $A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$ is counterclockwise rotation w.r.t. α .

Case 2: $A = \begin{pmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{pmatrix}$ is the reflection. $A^2 = I_2 \implies$ eigenvalues are ± 1 .

Easy to show that $L_A(v) = v - 2\langle v, v_2 \rangle v_2$.

$O(2) = \{\text{rotations}\} \cup \{\text{reflections}\}.$

Def 10. The dihedral group D_n is the group of symmetries of a regular n -gon.
In general, $D_n = \langle T, R \mid T^n = 1, R^2 = 1, TR = RT^{-1} \rangle \leq O(2) \leq S_n, |D_n| = 2n.$

Def 11. Let T be a linear transformation from $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

- T is called a rotation if \exists a T -invariant subspace $W \subseteq \mathbb{R}^n$ with $\dim W = 2$ s.t. $\begin{cases} T|_W \text{ is a rotation} \\ T|_{W^\perp} = \text{id}_{W^\perp} \end{cases}$
- T is called a reflection if \exists a T -invariant subspace $W \subseteq \mathbb{R}^n$ with $\dim W = 1$ s.t. $\begin{cases} T|_W = -\text{id}_W \\ T|_{W^\perp} = \text{id}_{W^\perp} \end{cases}$

Main result: the group of orthogonal transformations = $\langle \text{rotations, reflections} \rangle.$

Prop 1.2.1. For $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$, \exists a T -invariant subspace $W \subseteq \mathbb{R}^n$ with $1 \leq \dim W \leq 2$.

pf: Let $A = [T]_\alpha \in M_{n \times n}(\mathbb{R}) \subseteq M_{n \times n}(\mathbb{C})$. Consider $\widetilde{L}_A: \mathbb{C}^n \rightarrow \mathbb{C}^n, v \mapsto Av$.

Then \exists an eigenvalue $\lambda \in \mathbb{C}$ and an eigenvector $v \in \mathbb{C}^n$ for \widetilde{L}_A . Let $\lambda = \lambda_1 + \lambda_2 i, v = v_1 + v_2 i$. By definition, we have

$$Av = \widetilde{L}_A(v) = \lambda v = (\lambda_1 + \lambda_2 i)(v_1 + v_2 i) \implies \begin{cases} Av_1 = \lambda_1 v_1 - \lambda_2 v_2 \\ Av_2 = \lambda_2 v_1 + \lambda_1 v_2 \end{cases},$$

so $W = \langle v_1, v_2 \rangle.$

□

Ex 1.2.3.

1. If T is orthogonal, then W^\perp is also T -invariant.
2. Use induction on n to show the main result.

For $n = 3, A \in O(3)$, we have $A \sim \begin{pmatrix} \cos \alpha & -\sin \alpha & \\ \sin \alpha & \cos \alpha & \\ & & \pm 1 \end{pmatrix}.$

1.2.2 Cyclic groups and internal direct product

Def 12. If $G = \langle a \rangle = \{\dots, a^{-2}, a^{-1}, a, 1, a, a^2, \dots\} = \{a^n \mid n \in \mathbb{Z}\}$, then G is a cyclic group generated by a .

Eg 1.2.4. $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$.

Eg 1.2.5. Let $A = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} \in \text{SO}(2)$. Then $\langle A \rangle = \{I_2, A, A^2, \dots, A^{n-1}\}$ and $A^n = I_2$, $A^m = A^r$ where $m \equiv r \pmod{n}$.

Eg 1.2.6. $\mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \dots, \overline{(n-1)}\}$ with $\bar{j} = \{m \in \mathbb{Z} \mid m \equiv j \pmod{n}\}$. Define $\bar{i} + \bar{j} = \begin{cases} \overline{i+j} & \text{if } 0 \leq i+j \leq n \\ \overline{i+j-n} & \text{otherwise} \end{cases} \implies (\mathbb{Z}/n\mathbb{Z}, +, \bar{0})$ forms a group.

Remark 3. $\bar{i} \times \bar{j} = \overline{i \times j}$.

- 略
- If $\gcd(j, n) = d, \exists h, k \in \mathbb{Z}$ s.t. $hj + kn = d$.

Def 13. $(\mathbb{Z}/n\mathbb{Z})^\times = \{j \in \mathbb{Z}/n\mathbb{Z} \mid \gcd(j, n) = 1\} \implies ((\mathbb{Z}/n\mathbb{Z})^\times, \times, \bar{1})$ forms a group.

Eg 1.2.7. 略... 简化剩餘系, 原根 (generator) $(1, 2, 4, p^k, 2p^k, p \text{ is an odd prime})$

Def 14.

- The **order** of a finite group G is the number of elements in G , denoted by $|G|$.
- Let $a \in G$, the order of a is defined to be the least positive integer n s.t. $a^n = 1$, denoted by $\text{ord}(a) = n$.
- If $a^n \neq 1 \quad \forall n \in \mathbb{N}$, then we call “ a has infinite order”.

Prop 1.2.2. Let $G = \langle a \rangle$ with $\text{ord}(a) = n$. Then

1. $a^m = 1 \iff n \mid m$.

pf:

\Leftarrow : Let $m = dn$, then $a^m = (a^n)^d = 1$.

\Rightarrow : Let $m = qn + r, 0 \leq r < n$. If $r \neq 0$, then $a^r = a^{m-qn} = (a^m)(a^n)^{-q} = 1$. But $r < n$, which is a contradiction. Hence $r = 0 \implies n \mid m$. \square

2. $\text{ord}(a^r) = n / \gcd(r, n)$.

pf: Let $\gcd(r, n) = d, n = dn', r = dr'$ with $\gcd(n', r') = 1$. Plan to show “ $\text{ord}(a^r) = n'$.”

- $(a^r)^{n'} = a^{r'n'} = (a^n)^{r'} = 1 \implies \text{ord}(a^r) \mid n'$.
- $1 = (a^r)^{\text{ord}(a^r)} = a^{r \cdot \text{ord}(a^r)} \implies n \mid r \cdot \text{ord}(a^r) \implies n' \mid r' \cdot \text{ord}(a^r) \implies n' \mid \text{ord}(a^r)$.

\square

Prop 1.2.3. Any subgroup of a cyclic group is cyclic.

pf: Let $G = \langle a \rangle$ and $H \leq G$. If $H = \{1\}$, then $H = \langle 1 \rangle$, done!

Otherwise, $d = \min\{m \in \mathbb{N} \mid a^m \in H\}$, by well-ordering axiom. Claim $H = \langle a^d \rangle$.

\supset : $a^d \in H$ by the definition of d .

\subset : $\forall a^m \in H$, write $m = qd + r$, $0 \leq r < d$. If $r \neq 0$, then $a^r = a^{m-qd} = a^m(a^d)^{-q} \in H$, which is a contradiction. Hence $r = 0 \implies d \mid m$.

□

Ex 1.2.4.

1. $\text{ord}(a) = \text{ord}(a^{-1}) = n$.
2. $\langle a^r \rangle = \langle a^{\gcd(n,r)} \rangle$.
3. $\langle a^{r_1} \rangle = \langle a^{r_2} \rangle \iff \gcd(n, r_1) = \gcd(n, r_2)$.
4. $\forall m \mid n, \exists! H \leq \langle a \rangle$ s.t. $|H| = m$. Conversely, if $H \leq \langle a \rangle$, then $|H| \mid n$.

Prop 1.2.4. Let $G = \langle a \rangle$. Then

1. $\text{ord}(a) = n \implies G \cong \mathbb{Z}/n\mathbb{Z}$
2. $\text{ord}(a) = \infty \implies G \cong \mathbb{Z}$

Ex 1.2.5. Show Prop 1.2.4.

Def 15. Let $G_1, G_2 \leq G$. G is the internal direct product of G_1, G_2 if $G_1 \times G_2 \rightarrow G, (g_1, g_2) \mapsto g_1g_2$ is an isom.

Remark 4. In this case, we find that

- $G = G_1G_2 = \{g_1g_2 \mid g_1 \in G_1, g_2 \in G_2\}$.
- $G_1 \cap G_2 = \{1\}$. (consider $a \neq 1 \in G_1 \cap G_2$, then $(1, a) \mapsto a, (a, 1) \mapsto a$, but the function is 1-1, which is a contradiction.)
- If $a \in G$ with $a = g_1g_2 = g'_1g'_2$, then $(g'_1)^{-1}g_1 = (g'_2)g_2^{-1} \in G_1 \cap G_2 = \{1\} \implies \begin{cases} g_1 = g'_1 \\ g_2 = g'_2 \end{cases}$.
- For $g_1 \in G_1, g_2 \in G_2, (g_1, g_2) = (g_1, 1)(1, g_2) = (1, g_2)(g_1, 1) \implies g_1g_2 = g_2g_1$.

Ex 1.2.6. TFAE

1. G is the internal direct product of G_1, G_2 .
2. $\forall a \in G, \exists! g_1 \in G_1, g_2 \in G_2$ s.t. $a = g_1g_2$; $\forall g_1 \in G_1, g_2 \in G_2, g_1g_2 = g_2g_1$.
3. $G_1 \cap G_2 = \{1\}$; $G = G_1G_2$; $\forall g_1 \in G_1, g_2 \in G_2, g_1g_2 = g_2g_1$.

Eg 1.2.8.

1. $G = \mathbb{Z}/6\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}, G_1 = \{\bar{0}, \bar{3}\}, G_2 = \{\bar{0}, \bar{2}, \bar{4}\}$. We have $G \cong G_1 \times G_2$.
2. $G = S_3, G_1 = \langle (1\ 2) \rangle, G_2 = \langle (1\ 2\ 3) \rangle$. We have $G_1 \times G_2 \not\cong G$ since $(1\ 2)(1\ 2\ 3) \neq (1\ 2\ 3)(1\ 2)$.

Eg 1.2.9. $G = S_3, G_1 = \langle (1\ 2) \rangle, G_2 = \langle (2\ 3) \rangle, G_1G_2 = \{1, (1\ 2), (2\ 3), (1\ 2\ 3)\} \not\leq G$ since $(1\ 3\ 2) = (1\ 2\ 3)^{-1} \notin G_1G_2$.

Prop 1.2.5. Let $H, K \leq G$. Then $HK \leq G \iff HK = KH$.

pf:

$$\Rightarrow: \begin{cases} H \leq HK \\ K \leq HK \end{cases} \implies KH \subseteq HK ; \forall hk \in HK, \exists h'k' \in HK \text{ s.t. } (hk)(h'k') = 1 \implies hk = (k')^{-1}(h')^{-1} \in KH \implies HK \subseteq KH.$$

$$\Leftarrow: \text{ For } h_1k_1, h_2k_2 \in HK, (h_1k_1)(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1h'k' \in HK.$$

□

1.3 Week 3

1.3.1 Coset and Quotient Group

Let $f : G_1 \rightarrow G_2$ be a group homo. Define $\text{Im } f := f(G_1)$.

Notice that $\text{Im } f \leq G_2$.

pf: Let $z_1 = f(a_1), z_2 = f(a_2)$, then $z_1 z_2^{-1} = f(a_1) f(a_2)^{-1} = f(a_1) f(a_2^{-1}) = f(a_1 a_2^{-1}) \in \text{Im } f$. \square

Def 16. $\ker f := \{x \in G_1 \mid f(x) = 1\} \leq G_1$.

Fact 1.3.1.

1. $x \in (\ker f)a \iff f(x) = f(a)$.
2. $\ker f = \{1\} \iff f$ is 1-1.

Def 17. Let $H \leq G, \forall a \in G, Ha$ is called a **right coset** of H in G .

Fact 1.3.2.

1. For 2 right cosets Ha, Hb , either $Ha = Hb$ or $Ha \cap Hb = \phi$ must hold.
2. $\{Ha : a \in G\}$ forms a partition of G .

Theorem 2 (Lagrange). Let $|G| < \infty$ and $H \leq G, |H| \mid |G|$.

pf:

\square

Remark 5. r is called the **index** of H in G , denoted by $[G : H]$. (The concept of index can be extended to infinite G, H .)

Ex 1.3.1. no subgroup of A_4 has order 6. (converse of Lagrange thm. is false.)

Coro 1.3.1. If $|G| = p$ is a prime in \mathbb{Z} , then G is cyclic.

pf:

\square

Coro 1.3.2. If $|G| < \infty, a \in G$, then $a^{|G|} = 1$.

pf:

\square

Remark 6.

1. Let $H \leq G, a \in G, aH$ is called a **left coset**.
2. $\{\text{right cosets of } H\} \leftrightarrow \{\text{right cosets of } H\}$ by $Ha \mapsto a^{-1}H$.

Ques: How to make $\{aH : a \in G\}$ to be a group? For aH, bH , we must have $(aH)(bH) = abH$. In general, $(aH)(bH) = abH$ is not well-defined.

Ex 1.3.1. Let $H = \langle (1\ 2) \rangle \leq S_3$. $a_1 = (1\ 3), a_2 = (1\ 2\ 3), b_1 = (1\ 3\ 2), b_2 = (2\ 3)$. 出慘點

If we hope $a_1 b_1 H = a_2 b_2 H$, then we need $(a_1 b_1)^{-1} a_2 b_2 \in H$.

$$b_1^{-1} a_1^{-1} a_2 b_2 = b_1^{-1} b_2 b_2^{-1} a_1^{-1} a_2 b_2$$

Notice that $b_1^{-1} b_2, a_1^{-1} a_2 \in H$, so we need $b_2^{-1} a_1^{-1} a_2 b_2 \in H$.

Def 18. Let $H \leq G$. H is said to be **normal subgroup** of G if $\forall g \in G, h \in H, g^{-1}hg \in H$ (or $g^{-1}Hg \subseteq H$), denoted by $H \triangleleft G$.

Def 19. Let $H \triangleleft G$. The set $\{aH \mid a \in G\}$ forms a group under $(aH)(bH) = abH, a, b \in G$. We call it the **quotient group** of G by H , denoted by G/H .
(Note: The identity is $H = hH$ and $(aH)^{-1} = a^{-1}H$.)

Remark 7. Define $q : G \rightarrow G/H, a \mapsto aH$, called the quotient homomorphism.

Ex 1.3.2. Let $H \leq G$. Then TFAE

- (a) $H \triangleleft G$.
- (b) $\forall x \in G, xHx^{-1} = H$.
- (c) $\forall x \in G, xH = Hx$.
- (d) $\forall x, y \in G, (xH)(yH) = (xy)H$.

Ques: How to find a normal subgroup of G ?

Prop 1.3.1.

- 1. If G is abelian, then $\forall H \leq G \rightsquigarrow H \triangleleft G$. (done by (c))
- 2. If $H \leq G$ with $[G : H] = 2$, then $H \triangleleft G$.

Ex 1.3.2. $n \leq 3, [S_n : A_n] = 2 \implies A_n \triangleleft S_n$.

pf: We can write $G = H \cup Ha = H \cup aH \implies aH = Ha, \forall a \notin H$. □

Def 20. Define the center of G to be $Z_G = \{a \in G \mid ax = xa, \forall x \in G\} \leq G$.

Prop 1.3.2.

- 1. $Z_G \triangleleft G$. (by (c) and def.)
- 2. If G/Z_G is cyclic, then G is abelian.

pf: Let $G/Z_G = \langle aZ_G \rangle$, (let $\bar{a} := aZ_G$) for some $a \in G$. For $x_1, x_2 \in G$, let $x_1 = a^{k_1}z_1, x_2 = a^{k_2}z_2$, then $x_1x_2 = a^{k_1+k_2}z_1z_2 = x_2x_1$. (z_i 可以各種交換) □

Def 21. The commutator of G is define to be $[G, G] = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$.

Prop 1.3.3. $[G, G] \triangleleft G ; [G, G] = 1 \iff G$ is abelian.

pf: $\forall x \in G, a \in [G, G], xax^{-1} = xax^{-1}a^{-1}a$ and $xax^{-1}a^{-1}, a \in [G, G]$. □

Ex 1.3.3.

- 1. If $H \leq S_n$ and $\exists \sigma \in H$ is odd, then $[H : H \cap A_n] = 2$.
- 2. For $n \geq 3, [S_n, S_n] = A_n$.

Ex 1.3.4. Let $H \leq G$. Then $H \triangleleft G$ and G/H is abelian $\iff [G, G] \leq H$. (hint: $G/[G, G]$ is "max" among all abelian quotient groups)

1.3.2 Isomorphism theorems & Factor theorem

Theorem 3 (1st isomorphism theorem). Let $f : G_1 \rightarrow G_2$ be a group homo. Then $G_1/\ker f \cong \text{Im } f$.

pf: Define $\varphi : a \ker f \mapsto f(a)$.

- well-defined: $a \ker f = b \ker f \implies a^{-1}b \in \ker f \implies f(a^{-1}b) = 1 \implies f(a)^{-1}f(b) = 1 \implies f(a) = f(b)$.
- group homo: $\varphi((a \ker f)(b \ker f)) = \varphi(ab \ker f) = f(ab) = f(a)f(b) = \varphi(a \ker f)\varphi(b \ker f)$.
- onto: by def. of $\text{Im } f$.
- 1-1: $f(a) = f(b) \implies a \ker f = b \ker f$ (easy).

□

Theorem 4 (Factor theorem). Let $f : G_1 \rightarrow G_2$ be a group homo. and $H \triangleleft G_1, H \leq \ker f$. Then \exists a group homo. $\varphi : G/H \rightarrow G_2$ s.t. 一個正圖

Eg 1.3.3. Let $G = \langle a \rangle$ with $\text{ord}(a) = n$. Then $G \cong \mathbb{Z}/n\mathbb{Z}$. (1st isom. thm.)

Eg 1.3.4. $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}, 4\mathbb{Z} \leq 2\mathbb{Z}$, so by factor thm., $\mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$.

Eg 1.3.5. $\det : \text{GL}(n, \mathbb{F}) \rightarrow \mathbb{F}^\times \implies \text{GL}(n, \mathbb{F})/\text{SL}(n, \mathbb{F}) \cong \mathbb{F}^\times$

Eg 1.3.6. $\text{sgn} : S_n \rightarrow \{\pm 1\} \implies S_n/A_n \cong \{\pm 1\}$

Theorem 5 (2nd isomorphism theorem). Let $H \leq G, K \triangleleft G$. Then $HK/K \cong H/H \cap K$.

pf: First, $\begin{cases} H \leq G \\ K \triangleleft G \end{cases} \implies HK = KH \implies HK \leq G ; K \triangleleft G \implies K \triangleleft HK$.

Define $\varphi : H \rightarrow HK/K, h \mapsto hK$. which is a group homo.

- onto: $\forall (hk)K, hK = hK$, so $\varphi(h) = hK = hkK$.
- Find $\ker \varphi$: $a \in \ker \varphi \iff \begin{cases} a \in H \\ aK = K \end{cases} \iff a \in H \cap K$, so $\ker \varphi = H \cap K$.

Then by 1st isom. thm.

□

Eg 1.3.7. $G = \text{GL}(2, \mathbb{C}), H = \text{SL}(2, \mathbb{C}), K = \mathbb{C}^\times I_2 = Z_G \triangleleft G$.

By 2nd isom. thm., $G/K \cong H/\{\pm I_2\}$. ($G = HK, \{\pm I_2\} = H \cap K$)

projective linear group: $\text{PGL}(2, \mathbb{C}) = G/K$.

projective special linear group: $\text{PSL}(2, \mathbb{C}) = H/H \cap K$.

齊次座標...OTL

Ex 1.3.5.

1. Let $H_1 \triangleleft G_1, H_2 \triangleleft G_2$. Then $(H_1 \times H_2) \triangleleft (G_1 \times G_2)$ and $G_1 \times G_2 / H_1 \times H_2 \cong G_1 / H_1 \times G_2 / H_2$.
2. Let $H \triangleleft G, K \triangleleft G$ s.t. $G = HK$. Then $G / H \cap K \cong G / H \times G / K$.

Ex 1.3.6. Let $H \triangleleft G$ with $[G : H] = p$, which is a prime in \mathbb{Z} . Then $\forall K \leq G$, either (1) $K \leq H$ or (2) $G = HK$ and $[K : K \cap H] = p$.

Theorem 6 (3rd isomorphism theorem). Let $K \triangleleft G$.

1. There is a 1-1 correspondence between $\{H \leq G \mid K \leq H\}$ and $\{\text{subgroups of } G/K\}$. ($H \triangleleft G$... normal)

pf: Define $\varphi : H \mapsto H/K$. ($H/K \leq G/K$)

- 1-1: Assume $H_1/K = H_2/K$. For $a \in H_1$, $aK \in H_1/K = H_2/K$. so $\exists b \in H_2$ s.t. $aK = bK \implies b^{-1}a \in K \leq H_2 \implies a \in bH_2 = H_2$. So $H_1 \leq H_2$. By symmetry, $H_2 \leq H_1$, and thus $H_1 = H_2$.
- onto: Given a subgroup Q of G/K , consider $H = q^{-1}(Q)$ where $q : G \rightarrow G/K$.
 - $H \leq G$: $\forall a, b \in H, q(a), q(b) \in Q \implies q(a)q(b)^{-1} \in Q \implies q(ab^{-1}) \in Q \implies ab^{-1} \in H \implies H \leq G$.
 - $K \leq H$: $\forall a \in K, q(a) = aK = K \in Q \implies a \in H \implies K \leq H$.
 - $Q = H/K$: $\forall aK \in Q, aK = q(a) \implies a \in H \implies aK \in H/K \implies Q \subseteq H/K$.
And $\forall aK \in H/K (a \in H), q(a) \in Q \implies H/K \subseteq Q$. So $Q = H/K$.
- $H \triangleleft G, K \leq H \iff \forall g \in G, gHg^{-1} = H, K \leq H \iff \forall \bar{g} \in G/K, \bar{g}(H/K)\bar{g}^{-1} = H/K \iff H/K \triangleleft G/K$. \square

2. If $H \triangleleft G$ with $K \leq H$, then $(G/K)/(H/K) \cong G/H$.

pf: Define $\varphi : G \rightarrow (G/K)/(H/K)$ with $\varphi : a \mapsto aK(H/K)$.

- onto: ... easy.
- Find $\ker \varphi$: $a \in \ker \varphi \iff aK(H/K) = H/K \iff aK \in H/K \iff a \in H$.

By 1st isom. thm., $(G/K)/(H/K) \cong G/H$. \square

Eg 1.3.8. $m\mathbb{Z} + n\mathbb{Z}/m\mathbb{Z} \cong n\mathbb{Z}/m\mathbb{Z} \cap n\mathbb{Z}$. ($m\mathbb{Z} + n\mathbb{Z} = \gcd(m, n)\mathbb{Z}, m\mathbb{Z} \cap n\mathbb{Z} = \text{lcm}(m, n)\mathbb{Z}$)

Ques: $G/K \cong G'/K'$ and $K \cong K' \not\Rightarrow G \cong G'$.

Eg 1.3.9. Q_8 and D_4 交給陳力

Extension problem: given two groups A, B , how to find G and $K \triangleleft G$, s.t. $K \cong A, G/K \cong B$?
($1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$, short exact sequence)
(e.g. $G = A \times B, K = A \times \{1\}$)

1.4 Week 4

1.4.1 Universal property and direct sum & product

In general, let $f_1 : G_1 \rightarrow G, f_2 : G_2 \rightarrow G$ are group homo. $f_1 \times f_2 : G_1 \times G_2 \rightarrow G, (a, b) \mapsto f_1(a)f_2(b)$. But we have $(a, b) = (a, 1)(1, b) = (1, b)(a, 1)$, so $f_1(a)f_2(b) = f_2(b)f_1(a) \implies$ need G to be abelian.

So we intend to define the direct sum in the category of abelian group.

Notation: For abelian groups, we use “+” to denote the group operation and “0” to denote the identity.

Def 22. Given a non-empty family of abelian groups $\{G_s \mid s \in \Lambda\}$, a (external) direct sum of $\{G_s \mid s \in \Lambda\}$ is an abelian group $\bigoplus_{s \in \Lambda} G_s$ with the embedding mappings $i_{s_0} : G_{s_0} \rightarrow \bigoplus_{s \in \Lambda} G_s, \forall s_0 \in \Lambda$ satisfying the universal property:

for any abelian group H and group homo. $\varphi_s : G_s \rightarrow H \forall s \in \Lambda, \exists!$ group homo. $\varphi : \bigoplus_{s \in \Lambda} G_s \rightarrow H$ s.t. 又一個 τ 圖

Theorem 7. $\bigoplus_{s \in \Lambda} G_s$ exists and is unique up to isomorphisms.

pf: Existence: $\bigoplus_{s \in \Lambda} G_s = \{(g_s)_{s \in \Lambda} \mid g_s \in G_s, \text{ almost all of the } g_s' \text{ are } 0\}$ and

$$i_{s_0} : G_{s_0} \rightarrow \bigoplus_{s \in \Lambda} G_s, a_{s_0} \mapsto (g_{s_0})_{s \in \Lambda} \text{ with } g_{s_0} = a_{s_0}, g_s = 0, \forall s \neq s_0.$$

group operation: $(g_s)_{s \in \Lambda} + (g'_s)_{s \in \Lambda} := (g_s + g'_s)_{s \in \Lambda} \in \bigoplus_{s \in \Lambda} G_s$. 這邊也一個 τ 圖

Uniqueness: Assume \exists another G satisfies the universal property, 一個大 τ 圖 $(G, \bigoplus_{s \in \Lambda} G_s)$ 互相有唯一一個映射可以 keep $i_{s_0}, \varphi \circ \psi = \text{id}_G, \psi \circ \varphi = \text{id}_{\bigoplus_{s \in \Lambda} G_s}$ \square

Def 23. Given a non-empty family of groups $\{G_s \mid s \in \Lambda\}$, a direct product of $\{G_s \mid s \in \Lambda\}$ is a group $\prod_{s \in \Lambda} G_s$ with projections $p_{s_0} : \prod_{s \in \Lambda} G_s \rightarrow G_{s_0}, \forall s_0 \in \Lambda$ satisfying the following universal property:

for any group H with group homo. $\varphi_s : H \rightarrow G_s, \forall s \in \Lambda, \exists! \varphi : H \rightarrow \prod_{s \in \Lambda} G_s$ s.t. 又一個 τ 圖

Theorem 8. $\prod_{s \in \Lambda} G_s$ exists and is unique up to isomorphisms.

pf: Existence: $\prod_{s \in \Lambda} G_s = \{(g_s)_{s \in \Lambda} \mid g_s \in G_s\}$ and

$$p_{s_0} : \prod_{s \in \Lambda} G_s \rightarrow G_{s_0}, (g_s)_{s \in \Lambda} \mapsto g_{s_0}, \forall s_0 \in \Lambda$$

- group operation: $(g_s)_{s \in \Lambda} \cdot (g'_s)_{s \in \Lambda} := (g_s g'_s)_{s \in \Lambda} \in \prod_{s \in \Lambda} G_s$.
- Define φ : 這邊也一個 τ 圖 which is uniquely defined.

Uniqueness: Assume \exists another G satisfies the universal property, 一個大 τ 圖 $(G, \prod_{s \in \Lambda} G_s)$ 互相有唯一一個映射可以 keep $i_{s_0}, \varphi \circ \psi = \text{id}_G, \psi \circ \varphi = \text{id}_{\prod_{s \in \Lambda} G_s}$ \square

Ex 1.4.1. Google the definition of the **direct limit** and show the existence and uniqueness.

Ex 1.4.2. Google the definition of the **inverse limit** and show the existence and uniqueness.

Motivation: ζ_m is called an m -th root of unity if $\zeta_m^m = 1$.

$$\varinjlim_n \mathbb{Z}/2^n \mathbb{Z} \cong \{2^n\text{-th roots of unity} : n \in \mathbb{N}\}$$

$$\varinjlim_n \mathbb{Z}/2^n \mathbb{Z} = \left(\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2^n \mathbb{Z} \right) / \langle i_k(a) - i_j(f_{kj}(a)) \mid k \leq j, a \in \mathbb{Z}/2^k \mathbb{Z} \rangle$$

where $f_{kj} : \mathbb{Z}/2^k\mathbb{Z} \rightarrow \mathbb{Z}/2^j\mathbb{Z}$.
Inverse limit:

$$\varprojlim \mathbb{Z}/2^n\mathbb{Z} = \left\{ (n_1, n_2, \dots) \in \prod_n \mathbb{Z}/2^n\mathbb{Z} \mid \forall i < j, n_i \equiv n_j \pmod{2^{i+1}} \right\}$$

1.4.2 Rings and fields

Def 24. A **ring** is a non-empty set R with two operations $R \times R \rightarrow R$

$$(a, b) \mapsto a + b \quad \text{and} \quad (a, b) \mapsto ab$$

satisfying

1. $(R, +, 0)$ is an abelian group.
2. (R, \cdot) is a semigroup. (if it is a monoid, then it is called “a ring with 1.”)
3. (Distributive laws) $\forall a, b, c \in R, \begin{cases} a(b + c) = ab + ac \\ (b + c)a = ba + ca \end{cases}$

Eg 1.4.1. $\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/n\mathbb{Z}, M_{n \times n}(\mathbb{F})$

Eg 1.4.2. Let G be an abelian group. Define (endomorphism, automorphism)

$$\text{End}(G) := \{ \text{group homo. } G \rightarrow G \} \quad \text{Aut}(G) := \{ \text{group isom. } G \rightarrow G \}$$

A natural ring structure on $\text{End}(G)$ is:

$$\forall a \in G, \begin{cases} (f + g)(a) := f(a) + g(a) \\ (f \cdot g)(a) := f(g(a)) \end{cases}$$

Eg 1.4.3. $\mathbb{Z}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Z} \} \subset \mathbb{R}$.

Def 25. Let R be a ring with 1.

- (a) $\forall a \in R, a \neq 0$, a is called a unit if $\exists a^{-1} \in R$.
- (b) $(R^\times = \{\text{units in } R\}, \cdot, 1)$ forms a group.
- (c) R is called a division ring if $R \setminus \{0\} = R^\times$.
- (d) R is said to be commutative if $ab = ba, \forall a, b \in R$.
- (e) R is a field if R is a commutative division ring.
- (f) $a \neq 0$ is called a left zero divisor if $\exists b \in R, b \neq 0$ s.t. $ab = 0$.
- (g) a is called a zero divisor if a is either a left or right zero divisor.
- (h) R is called an integral domain if R is a commutative ring without zero divisors.

Fact:

1. fields \implies integral domains.
2. finite + integral domain \implies fields.

pf: Let $R = \{0, a_1, \dots, a_n\}$, for $a \in R, a \neq 0$, $aa_i = aa_j \implies a(a_i - a_j) = 0 \implies i = j$. So $\{0, aa_1, \dots, aa_n\} = R \implies \exists a_i$ s.t. $aa_i = 1$. \square

Prop 1.4.1. TFAE

1. $\mathbb{Z}/n\mathbb{Z}$ is an integral domain.
2. $\mathbb{Z}/n\mathbb{Z}$ is a field.
3. $n = p$ is a prime.

easy to prove.

Def 26.

- $f : R_1 \rightarrow R_2$ is called a ring homomorphism if $\forall a, b \in R, \begin{cases} f(a+b) = f(a) + f(b) \\ f(ab) = f(a)f(b) \end{cases}$.
- $\text{Im } f$ is a subring of R_2 .
- $\ker f = \{x \in R_1 \mid f(x) = 0\}$ is an additive group of R_1 and $\forall r \in R_1, x \in \ker f, f(rx) = f(r)f(x) = f(r)0 = 0 \implies rx \in \ker f, xr \in \ker f$.
- $R_1/\ker f$ is an additive group and $R_1/\ker f \cong \text{Im } f$ (additive isomorphism).

Def 27. Let I be an additive subgroup of R . I is called an ideal if $\forall r \in R, x \in I, rx \in I, xr \in I$. $(R/I, +, \cdot)$ forms a quotient ring under

$$\forall r_1, r_2 \in R, (r_1 + I)(r_2 + I) = r_1 r_2 + I$$

well-defined: easy to show.

Ex 1.4.3. State and show the isomorphism theorems and the factor theorem.

Prop 1.4.2. If R is a ring with 1, then $\exists!$ ring homo. $\varphi : \mathbb{Z} \rightarrow R$ s.t. $\varphi(1) = 1$.

pf: Let $\varphi : \mathbb{Z} \rightarrow R$ is a ring homo. s.t. $\varphi(1) = 1$. Then $\forall n \in \mathbb{Z}, \varphi(n) = \varphi(1) + \dots + \varphi(1) = n1$. Now $\forall n, m \in \mathbb{Z}, \varphi(n)\varphi(m) = (n1)(m1) = n(m1) = (nm)1$ by the distributive law. So φ is well-defined and unique. \square

Def 28. In Prop 1.4.2, $\ker \varphi = m\mathbb{Z}$ for some $m > 0$. We call m the characteristic of R , denoted by $\text{char } R = m$.

Prop 1.4.3.

1. If R is an integral domain, then $\text{char } R = 0$ or p , where p is a prime. (try to prove this)
2. In the case of $\text{char } R = p, \forall a, b \in R, (a+b)^p = a^p + b^p$.

pf:

$$(a+b)^p = a^p + \binom{p}{1}a^{p-1}b + \dots + b^p = a^p + b^p$$

$$\text{because } p \mid \binom{p}{i} \implies \binom{p}{i}a^{p-i}b^i = 0.$$

\square

Ex 1.4.4. Let F be a field. Show that

1. if $\text{char } F = 0$, then $\mathbb{Q} \hookrightarrow$ subfield of F .
2. if $\text{char } F = p$, then $\mathbb{Z}/p\mathbb{Z} \hookrightarrow$ subfield of F .

Theorem 9. If F is a finite field, then $|F| = p^n$ for some $n \in \mathbb{N}$ and p is a prime.

pf: By Ex. 1.4.4, $\text{char } F = p$, p is a prime and $\mathbb{Z}/p\mathbb{Z} \hookrightarrow F$.

We have $\mathbb{Z}/p\mathbb{Z} \times F \rightarrow F, (r, v) \mapsto rv$. F can be regarded as a vector space over $\mathbb{Z}/p\mathbb{Z}$.

Let $\dim_{\mathbb{Z}/p\mathbb{Z}} F = n$, then $F \cong (\mathbb{Z}/p\mathbb{Z})^n \implies |F| = p^n$. □

Theorem 10. Let F be a field. Then any finite subgroup G of $(F^\times, \cdot, 1)$ is cyclic.

pf: Let $|G| = n$. Define h to be the max order of an element in G , say $a^h = 1$.

If $h = n$, then $|\langle a \rangle| = h = n = |G|$ and $\langle a \rangle \subseteq G$, so $G = \langle a \rangle$.

Otherwise, $h < n$. We know that $x^h - 1$ has at most h roots. So $\exists b \in G$ is not a root of $x^h - 1$.

Let $\text{ord}(b) = h'$, so $h' \mid n$ and $h' \nmid h$. So \exists a prime p s.t. $p^r \mid h'$ but $p^r \nmid h$.

Write $h = mp^s$, $s < r$ and $\gcd(m, p) = 1 \implies \text{ord}(a^{p^s}) = m$.

Write $h' = qp^r \implies \text{ord}(b^q) = p^r$.

Since $\gcd(m, p^r) = 1$, $\text{ord}(a^{p^s} b^q) = mp^r > mp^s = h$, which is a contradiction. □

Ex 1.4.5.

1. Let $a, b \in G$ with $ab = ba$ and $\text{ord}(a) = m, \text{ord}(b) = n$. If $\gcd(m, n) = 1$, then $\text{ord}(ab) = mn$.
In general, is the order of ab equal to $\text{lcm}(m, n)$?

2. Let G be a finite group and $H, K \leq G$. Then $|HK| = \frac{|H||K|}{|H \cap K|}$.

1.5 Week 5

1.5.1 Group actions I

Def 29. A group G is said to act on a nonempty set X if \exists a map $G \times X \rightarrow X$ with $(g, x) \mapsto gx$ s.t.

1. $1x = x$
2. $(g_1g_2)x = g_1(g_2x) \quad \forall g_1, g_2 \in G$

Prop 1.5.1. $\{\text{actions of } G\} \leftrightarrow \{\text{group homo. } G \rightarrow S_X\}$

pf: Given an action $(g, x) \mapsto gx$, consider $\varphi : G \rightarrow S_X$ s.t. $\varphi : g \mapsto (\tau_g : x \mapsto gx)$.

- 1-1: $gx = gy \implies g^{-1}(gx) = y \implies x = y$.
- onto: $\forall y \in X$, let $x = g^{-1}y$, then $y = gx$.
- group homo.: $\varphi(gg') = (\tau_{gg'} : x \mapsto gg'x) = \tau_g \circ \tau_{g'} = \varphi(g)\varphi(g')$.

Conversely, given a group homo. $\varphi : G \rightarrow S_X$, consider $(g, x) \mapsto \varphi(g)(x)$.

- $1x = \varphi(1)(x) = \text{Id}(x) = x$.
- $g_1g_2x = \varphi(g_1g_2)(x) = \varphi(g_1) \circ \varphi(g_2)(x) = g_1(g_2x)$. □

Def 30. A representation of G on a vector space V is a group action of G on V linearly. i.e. \exists group homo. $\varphi : G \rightarrow \text{GL}(V)$.

Eg 1.5.1.

$$\mathbb{Z}/m\mathbb{Z} \rightarrow \text{SO}(2), \quad \bar{k} \mapsto \begin{pmatrix} \cos \frac{2k\pi}{m} & -\sin \frac{2k\pi}{m} \\ \sin \frac{2k\pi}{m} & \cos \frac{2k\pi}{m} \end{pmatrix}$$

Eg 1.5.2.

$$S_n \rightarrow \text{GL}(n, \mathbb{R}), \quad \sigma \mapsto (\tau_\sigma : e_i \mapsto e_{\sigma(i)})$$

Remark 8.

1. An action $G \times X \rightarrow X$ is said to be faithful if the corresponding group homo. $\varphi : G \hookrightarrow S_X$, denoted by $G \curvearrowright X$.
2. In general, $\ker \varphi = \{g \in G \mid gx = x \quad \forall x \in X\} = \bigcap_{x \in X} \{g \mid gx = x\}$.
Define $G_x = \{g \mid gx = x\} \leq G$ is the isotropy subgroup of G at x . (the stabilizer of G at x)
3. $\varphi : G \rightarrow S_X \implies G/\ker \varphi \hookrightarrow S_X$. So $G/\ker \varphi \times X \rightarrow X$ is faithful.
4. Let $\mathcal{C}(X) = \{f : X \rightarrow \mathbb{C}\}$. If $G \curvearrowright X$, then $G \curvearrowright \mathcal{C}(X)$ by $G \times \mathcal{C}(X) \rightarrow \mathcal{C}(X)$ with $(g, f) \mapsto gf(x) = f(g^{-1}x)$.
The reason: $(g_1g_2)f(x) = f((g_1g_2)^{-1}x) = f(g_2^{-1}g_1^{-1}x) = g_1(g_2f)(x)$.

Def 31. Let $G \curvearrowright X$ and $x \in X$.

- The **orbit** of x is defined to be $Gx = \{gx \mid g \in G\}$.
- $G \curvearrowright X$ is said to be transitive if \exists only one orbit. i.e. $\forall x, y \in X, \exists g \in G$ s.t. $y = gx$.

The set of orbits forms a partition: $x \sim y \iff \exists g \in G$ s.t. $y = gx$.

Prop 1.5.2. Let $G \curvearrowright X$ and $x \in X$. Then $|Gx| = [G : G_x]$.
In particular, $|G| < \infty \implies |G| = |Gx||G_x| \quad \forall x \in X$.

pf. Define $\psi : Gx \rightarrow \{\text{left coset of } G_x\}$ as $\psi : gx \mapsto gG_x$.

- well-defined and 1-1: $g_1x = g_2x \iff g_2^{-1}g_1x = x \iff g_2^{-1}g_1 \in G_x \iff g_2^{-1}g_1G_x = G_x \iff g_1G_x = g_2G_x$.
- onto: $\forall g \in G, \psi(gx) = gG_x$. □

1.5.2 Action by left multiplication

- The action $G \times G \rightarrow G, (g, x) \mapsto gx$ is associated with $\varphi : G \hookrightarrow S_G$. It is faithful (Cayley theorem) and transitive.
- Let $H \leq G$ and $X := \{\text{left coset of } H\}$.

1.6 Week 6

1.6.1 Group actions II

Def 32. Let $G \curvearrowright X$ and $|X| < \infty$. Write $\text{Fix } G := \{x \in X \mid gx = x \ \forall g \in G\}$.

- $x \in \text{Fix } G, Gx = \{x\}$.
- $x \notin \text{Fix } G, |Gx| = [G : G_x]$.

Let $\{G_{x_1}, \dots, G_{x_n}\}$ be the set of distinct orbits. After rearrangement, assume $x_1, \dots, x_r \in \text{Fix } G, x_{r+1}, \dots, x_n \notin \text{Fix } G$. Then

$$|X| = |\text{Fix } G| + \sum_{i=r+1}^n [G : G_{x_i}]$$

Theorem 11 (class equation). Let $|G| < \infty$. Then either $G = Z_G$ or $\exists a_1, \dots, a_m \in G \setminus Z_G$ s.t.

$$|G| = |Z_G| + \sum_{i=1}^n [G : G_{a_i}]$$

pf: Consider the action $(g, x) \mapsto gxg^{-1}$, then

$$\text{Fix } G = \{x \in G \mid gxg^{-1} = x \ \forall g \in G\} = Z_G$$

It follows from the above argument. □

Def 33. G is called a p -group if $|G| = p^n$, where p is a prime, $n \in \mathbb{N}$.

Prop 1.6.1. If G is a p -group, then $Z_G \neq \{1\}$.

pf: Let $|G| = p^n$. If $G = Z_G$, then done. Otherwise, by the class equation (use action by conjugation), $|G| = |Z_G| + \sum_{i=1}^n [G : G_{a_i}]$, $a_i \notin Z_G$.

$$G_{a_i} = Z_G(a_i), \text{ so } a_i \notin Z_G \implies Z_G(a_i) \subsetneq G \implies p \mid [G : Z_G(a_i)] = \frac{|G|}{|Z_G(a_i)|}.$$

$$\text{So } |Z_G| = |G| - \sum_{i=1}^n [G : Z_G(a_i)] \implies p \mid |Z_G| \implies Z_G \neq \{1\}. \quad \square$$

Prop 1.6.2. If $|G| = p^2$, then G is abelian. ($\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p^2\mathbb{Z}$)

pf: Assume that G is not abelian. By prop 1.6.1, $|Z_G| = p \implies |G/Z_G| = p \implies G/Z_G$ is cyclic $\implies G$ is abelian. (contradiction) □

Prop 1.6.3. If $|G| = p^3$ and G is not abelian, then $|Z_G| = p$.

(Abelian: $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p^3\mathbb{Z}$)

Prop 1.6.4. Let $|G| = p^n$. Then $\forall 0 \leq k \leq n, \exists G_k \triangleleft G$ s.t. $|G_k| = p^k$ and $G_i \leq G_{i+1}$.

In general, for a finite group G , $\exists \{1\} = G_r \triangleleft G_{r-1} \triangleleft \dots \triangleleft G_1 \triangleleft G_0 = G$ s.t. G_i/G_{i+1} is cyclic. we call G a solvable group.

pf: By induction on n , $n = 1$ is trivial. For $n > 1$, assume that the statement holds for $n - 1$. By prop 1.6.1, $Z_G \neq \{1\}$. $\exists a \in Z_G, a \neq 1$. Let $\text{ord}(a) = p^l$, then $\text{ord}(a^{p^{l-1}}) = p$. \implies in any case, $\exists a \in Z_G$ with $\text{ord}(a) = p$.

Now $|G/\langle a \rangle| = p^{n-1}$, so by induction hypothesis, $\forall 0 \leq k \leq n - 1, \exists \overline{G_k} \triangleleft G/\langle a \rangle$ s.t. $|\overline{G_k}| = p^k, \overline{G_i} \leq \overline{G_{i+1}}$.

By 3rd isom. thm., $\exists G_{k+1} \triangleleft G$ s.t. $\overline{G_k} = G_{k+1}/\langle a \rangle, G_j \leq G_{j+1}$ and $|G_{k+1}| = p^{k+1}$. □

Prop 1.6.5. Let a p -group $G \curvearrowright X$ with $|X| < \infty$. Then $|X| \equiv |\text{Fix } G| \pmod{p}$.

Theorem 12 (Cauchy theorem). Let $p \mid |G|$. Then $\exists a \in G$ s.t. $\text{ord}(a) = p$. Consider

$$X = \{ (a_1, \dots, a_p) \mid a_i \in G, a_1 a_2 \dots a_p = 1 \}$$

and the action $\mathbb{Z}/p\mathbb{Z} \times X \rightarrow X$:

$$(\bar{k}, (a_1, \dots, a_p)) \mapsto (a_{k+1}, \dots, a_p, a_1, \dots, a_k)$$

(This is well-defined since $ab = 1 \implies ba = 1$ in a group.) We find that $(a_1, \dots, a_p) \in \text{Fix } \mathbb{Z}/p\mathbb{Z} \iff a_1 = a_2 \dots a_p$. By prop 1.6.5, $|\text{Fix } \mathbb{Z}/p\mathbb{Z}| \equiv |X| \pmod{p}$. And $|X| = |G|^{p-1} \equiv 0 \pmod{p}$. Since $(1, \dots, 1) \in \text{Fix } \mathbb{Z}/p\mathbb{Z}$, $|\mathbb{Z}/p\mathbb{Z}| \neq 0 \implies |\mathbb{Z}/p\mathbb{Z}| \geq p$. So $\exists (a, \dots, a) \in \text{Fix } \mathbb{Z}/p\mathbb{Z} \implies a^p = 1$.

Application: Let $|G| = p^3$ and G be non-abelian (p is odd). By prop 1.6.3, $|G/Z_G| = p^2$. Since G is non-abelian, we have $G/Z_G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. That is, $\forall a \in G, a^p \in Z_G$.

So,

$$\exists \varphi : G \rightarrow Z_G \cong C_p \text{ with } \varphi : a \mapsto a^p$$

Since G/Z_G is abelian, $[G, G] \leq Z_G$. And

$$\begin{cases} |[G, G]| \mid |Z_G| = p \\ G \text{ is non-abelian} \end{cases} \implies [G, G] = Z_G$$

Def 34. $[x, y] = x^{-1}y^{-1}xy \in [G, G], [x, y]^p = 1$.

So $a^p b^p = a^p b^p [b, a]^p \dots$ 換換換總共需要 $p(p-1)/2$

$$a^p b^p = (ab)^p [b, a]^{\frac{p(p-1)}{2}} = (ab)^p$$

So φ is a group homo.

Now if $\ker \varphi = G$ ($\forall a \in G, a^p = 1$), i.e. φ is trivial, then φ is useless. Else, $\exists a \in G$ s.t. $\text{ord}(a) = p^2$, then $H = \langle a \rangle \triangleleft G$. ($[G : H] = p$ is the smallest prime dividing $|G|$)

Also, in this case, $\varphi : G \rightarrow Z_G \implies G/\ker \varphi \cong Z_G$. Let $E = \ker \varphi$, $|E| = p^2$. By the def. of $\ker \varphi$, $E \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

We find that $H \cap E = \langle a^p \rangle$. Pick $b \in E \setminus H$ and let $K = \langle b \rangle \implies |K| = p, H \cap K = \{1\}, HK = G$.