

Algebra

June 17, 2017

1 Commutative Algebra

1.1 ED, PID and UFD (week 9)

We shall consider R to be an integral domain below.

Def 1. A function $N : R \rightarrow \mathbb{N}$ with $N(0) = 0$ is called a norm on R .

Def 2. R is called a Euclidean domain if exists a norm N on R satisfying

$$\forall a, b \in R, \exists q, r \in R \text{ s.t. } a = qb + r \text{ with } r = 0 \text{ or } N(r) < N(b)$$

Eg 1.1.1.

- \mathbb{Z} is a ED with $N(n) = |n|$.
- $K[x]$ is a ED with $N(f) = \deg f, \forall f \in K[x]$.

Def 3. A_d is defined to be the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{d})$ with $d \neq 1$ and d is square-free. That is,

$$A_d \triangleq \{\alpha \in \mathbb{Q}(\sqrt{d}) \mid \alpha \text{ is integral over } \mathbb{Z}\}$$

Theorem 1.

- If $d \equiv 1 \pmod{4}$, then

$$A_d = \left\{ a + b \frac{1 + \sqrt{d}}{2} : a, b \in \mathbb{Z} \right\}$$

- Else, $d \equiv 2, 3 \pmod{4}$, then

$$A_d = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$$

Proof. Let $\alpha = p + q\sqrt{d} \in A_d$ for $p, q \in \mathbb{Q}$ with $q \neq 0$. We have $\alpha - p = q\sqrt{d}$, then $(\alpha - p)^2 = q^2d$ and thus $\alpha^2 - 2p\alpha + (p^2 - q^2d) = 0$. Let $g(x) \triangleq x^2 - 2px + (p^2 - q^2d)$. Assume $f(x) \in \mathbb{Z}[x]$ with f monic and $f(\alpha) = 0$, then we could write $f(x) = q(x)g(x) + (ax + b)$. Since α is not rational, $a\alpha + b = 0 \implies a = b = 0$, so $f(x) = q(x)g(x)$ in $\mathbb{Q}[x]$. By gauss lemma, $g(x) \in \mathbb{Z}[x]$, so $2p \in \mathbb{Z}$ and $p^2 - q^2d \in \mathbb{Z}$.

If $2p$ is even, then $p \in \mathbb{Z}$, and $p^2 - q^2d \in \mathbb{Z}$ implies q is also an integer since d is square free.

If $2p$ is odd, say $2p = 2m + 1$, then $(2p)^2 \equiv (2m + 1)^2 \equiv 1 \pmod{4}$. Also, $4(p^2 - q^2d) \equiv 0 \pmod{4}$, so $4q^2d \equiv 4p^2 \equiv 1 \pmod{4}$. Since d is square free, so $4 \nmid d$, thus q has to be of the form $q = (2n + 1)/2$. Plug in the equation we get $d \equiv 1 \pmod{4}$. Thus in this case, p, q are half integer and $d \equiv 1 \pmod{4}$. \square

Theorem 2. A_d is a ED if $d = 2, 3, 5, -1, -2, -3, -7, -11$. Hence A_d is also PID and UFD for these value.

Proof. Let $N'(p + q\sqrt{d}) = (p + q\sqrt{d})(p - q\sqrt{d}) = p^2 - q^2d$. Define $N(\alpha) \triangleq |N'(\alpha)|$ which is positive since $p^2 - q^2d = 0 \iff p = q = 0$. Notice also N is multiplicative.

Now, for $\alpha, \beta \in A_d$, write $\alpha/\beta = x + y\sqrt{d}$. If we could find $\lambda = a + b\sqrt{d}$ such that $|\alpha/\beta - \lambda| < 1$, then $\alpha = \beta\lambda + \gamma$ with $N(\gamma) < N(\beta)$ which proves that A_d is an ED.

- $d = 2, 3, -2, -1$: Choose $a, b \in \mathbb{Z}$ such that $|x - a|, |y - b| \leq 1/2$. Then $N \triangleq N(\alpha/\beta - \lambda) = |(x - a)^2 - (y - b)^2d|$.

- If $d = 2, 3$, then $N \leq \max(|(x-a)^2|, |(y-b)^2d|) \leq \max(1/4, d/4) < 1$.
- If $d = -2, -1$, then $N \leq |(x-a)^2| + |(y-b)^2d| \leq 1/4 + |d|/4 < 1$.
- $d = 5, -3, -7, -11$: Similarly, but now $d \equiv 1 \pmod{4}$, so we could choose $\lambda = a + b(1 + \sqrt{d})/2 = (a+b/2) + b/2\sqrt{d}$. Thus let b be the one such that $|2y-b| \leq 1/2$, and then choose a so that $x-a-b/2 \leq 1/2$. We have $N(\alpha/\beta-\lambda) = |(x-a-b/2)^2 - d(y-b/2)^2| \leq 1/4 + d/16 < 1$.

□

Eg 1.1.2. A_{-5} is not a ED.

Proof. Consider $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. Notice that $1 + \sqrt{-5}$ is irreducible, since if $1 + \sqrt{-5} = \alpha\beta$, then $6 = N(1 + \sqrt{-5}) = N(\alpha)N(\beta)$. But this implies $a^2 + 5b^2 = 2$ or 3 which has no integer solution. Also $1 + \sqrt{-5} \nmid 2, 3$. Since if $(1 + \sqrt{-5})\alpha = 2$, then $N(1 + \sqrt{-5})N(\alpha) = N(2) = 4$, but $N(1 + \sqrt{-5}) = 6$. Similarly $1 + \sqrt{-5} \nmid 3$. So A_{-5} is not an UFD thus not an ED. □

1.1.1 A_{-1} and A_{-3}

Def 4. If p is odd and $a \not\equiv 0 \pmod{p}$, then

- If $x^2 \equiv a \pmod{p}$ is solvable, then define $\left(\frac{a}{p}\right) = 1$.
- Else $x^2 \equiv a \pmod{p}$ is not solvable and define $\left(\frac{a}{p}\right) = -1$.

Prop 1.1.1.

- $a \equiv b \pmod{p} \implies \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.
- $\left(\frac{a}{p}\right) = a^{(p-1)/2}$.

Proof. Consider the sequence:

$$\begin{array}{ccccccc} 1 & \longrightarrow & (\mathbb{F}_p^\times)^2 & \longrightarrow & \mathbb{F}_p^\times & \xrightarrow{\varphi} & \{\pm 1\} \longrightarrow 1 \\ & & y^2 & \longmapsto & y^2 = x & \longmapsto & (-1)^{(p-1)/2} \longmapsto 1 \end{array}$$

which is exact since $y^2 \mapsto y^2 \mapsto y^{2(p-1)/2} \equiv 1$. And since \mathbb{F}_p^\times is cyclic with even elements, $[\mathbb{F}_p^\times : (\mathbb{F}_p^\times)^2] = 2$, and $(\mathbb{F}_p^\times)^2 = \ker \varphi$. □

- $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \left(\frac{b}{p}\right)$.
- Let $t_k \equiv ka \pmod{p}$ with $0 \leq t_k < p$, for $1 \leq k \leq (p-1)/2$. Assume that $n = \#\{t_i \mid t_i > p/2\}$, then $\left(\frac{a}{p}\right) = (-1)^n$.

Proof. Define

$$|t_i| = \begin{cases} t_j & \text{If } 1 \leq t_j < p/2 \quad (t_j \equiv |t_j|) \\ p - t_j & \text{If } p/2 < t_j < p \quad (t_j \equiv -|t_j|) \end{cases}$$

Notice that $|t_i|$ takes value between 1 and $(p-1)/2$, and $|ra| \equiv |sa| \pmod{p} \implies ra \equiv \pm sa \pmod{p} \implies r \equiv \pm s \pmod{p}$ since $\gcd(a, p) = 1$. So $|t_k|$ would have distinct value for $1 \leq k \leq (p-1)/2$. Thus

$$\prod t_k \equiv \frac{p-1}{2}! a^{(p-1)/2} \equiv (-1)^n \frac{p-1}{2}! \implies a^{(p-1)/2} \equiv (-1)^n$$

□

- If p, q are odd primes, then we have:

$$\left(\frac{q}{p}\right) = (-1)^{\left(\sum_{k=1}^{(p-1)/2} \left\lfloor \frac{kq}{p} \right\rfloor\right)}$$

Proof. Write $kq = g_k p + t_k$ with $0 \leq t_k < p$ consistent with the previous definition. Then we have $\lfloor kq/p \rfloor = g_k$, and

$$\begin{aligned} \text{if } |t_k| = t_k & \rightsquigarrow qk = g_k p + |t_k| & \rightsquigarrow k \equiv g_k + |t_k| \pmod{2} \\ \text{if } |t_k| = p - t_k & \rightsquigarrow qk = (g_k + 1)p - |t_k| & \rightsquigarrow k \equiv g_k + 1 + |t_k| \pmod{2} \end{aligned}$$

So

$$\sum_{i=1}^{(p-1)/2} k \equiv n + \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{qk}{p} \right\rfloor + \sum_{k=1}^{(p-1)/2} |t_k| \pmod{2}$$

As in the previous proof, $\sum k = \sum |t_k|$, so $n \equiv \sum \lfloor qk/p \rfloor \pmod{2}$, which proves the statement. \square

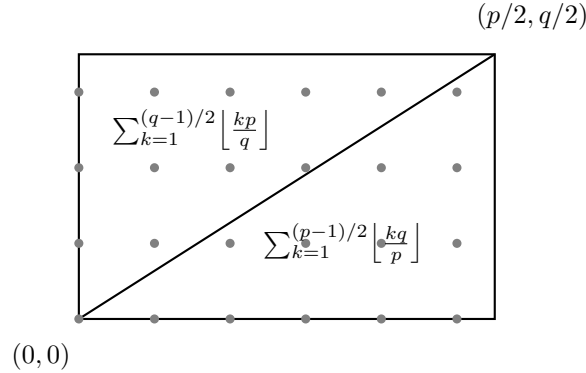
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$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}$$

By above,

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{\left(\sum_{k=1}^{(p-1)/2} \left\lfloor \frac{kq}{p} \right\rfloor\right)} (-1)^{\left(\sum_{k=1}^{(q-1)/2} \left\lfloor \frac{kp}{q} \right\rfloor\right)}$$

Which is the number of integer points in the rectangle:



And we know that there are $\frac{p-1}{2} \frac{q-1}{2}$ points in the rectangle.

Prop 1.1.2. • α is a unit $\iff N(\alpha) = 1$.

Proof. “ \Rightarrow ”: If $\alpha\beta = 1$, $N(\alpha)N(\beta) = 1$ so $N(\alpha) = 1$.

“ \Leftarrow ”: Immediately by $\alpha\bar{\alpha} = N(\alpha) = 1$. \square

- If α is a prime in A_d , then $N(\alpha) = p$ or p^2 for some prime integer p . Also $N(\alpha) = p^2 \implies \alpha \sim p$.

Proof. $\alpha\bar{\alpha} = N(\alpha) = p_1 \cdots p_n$ where p_i are primes in \mathbb{Z} . Continue using the fact that “If α is a prime and $\alpha \mid xy$, then $\alpha \mid x$ or $\alpha \mid y$ ”, we will get $\alpha \mid p_i$ for an i . Say $\alpha\beta = p_i$, then $\bar{\alpha}\bar{\beta} = \bar{p}_i = p_i$, so $N(\alpha)N(\beta) = p_i^2$ which means that $N(\alpha) = p_i$ or p_i^2 . Also, if $N(\alpha) = p_i^2$, then $N(\beta) = 1 \implies \beta$ is a unit. \square

By the proposition above we identify the unit in A_{-1}, A_{-3} .

- A_{-1} : $\pm 1, \pm i$.
- A_{-3} : $\pm 1, \pm \omega, \pm \omega^2$.

Now, notice that $2 = (1 + \sqrt{-1})(1 - \sqrt{-1})$, $3 = (1 - \omega)(1 - \omega^2)$, so 2, 3 are not prime in A_{-1}, A_{-3} respectively.

Let p be a prime in \mathbb{Z} .

- In A_{-1} :

$$\begin{aligned}
& p \text{ is a prime in } \mathbb{Z}[\sqrt{-1}] \\
\iff & \langle p \rangle \text{ is maximal ideal} \\
\iff & \frac{\mathbb{Z}[\sqrt{-1}]}{\langle p \rangle} \cong \frac{\mathbb{Z}[x]}{\langle p, x^2 + 1 \rangle} \cong \frac{\mathbb{Z}[x]/\langle p \rangle}{\langle p, x^2 + 1 \rangle/\langle p \rangle} \cong \frac{\mathbb{F}_p[x]}{x^2 + 1} \text{ is a field} \\
\iff & x^2 + 1 \text{ irreducible in } \mathbb{F}_p[x] \\
\iff & x^2 \equiv -1 \pmod{p} \text{ is not solvable} \\
\iff & \left(\frac{-1}{p} \right) = (-1)^{(p-1)/2} \neq 1 \\
\iff & p \not\equiv 1 \pmod{4}
\end{aligned}$$

So p is **not** a prime in $A_{-1} \iff p \equiv 1 \pmod{4}$.

- In A_{-3} : If a prime $p \neq 3$ in \mathbb{Z} is not a prime in $\mathbb{Z}[\omega]$, then it has a nontrivial factor $\alpha \mid p$. But $N(p) = p^2$, so we must have $N(\alpha) = p$, i.e. $\alpha\bar{\alpha} = p$. Let $\alpha = a + b\omega$, then $p = \alpha\bar{\alpha} = a^2 + b^2 - ab \implies 4p = (2a - b)^2 + 3b^2$, so $p \equiv (2a - b)^2 \equiv 1 \pmod{3}$. ($p \neq 0$ since $p \neq 3$)

Conversely, if $p \equiv 1 \pmod{3}$, then

$$\left(\frac{-3}{p} \right) = \left(\frac{-1}{p} \right) \left(\frac{3}{p} \right) = (-1)^{(p-1)/2} \left(\frac{p}{3} \right) (-1)^{\frac{p-1}{2} \cdot \frac{3-1}{2}} = \left(\frac{p}{3} \right) = \left(\frac{1}{3} \right) = 1$$

So exists $a \in \mathbb{Z}$ such that $a^2 \equiv -3 \pmod{p}$, say $pb = a^2 + 3 = (a + \sqrt{-3})(a - \sqrt{-3}) = (a + 1 + 2\omega)(a - 1 - 2\omega)$.

If p is a prime in $\mathbb{Z}[\omega]$, then $p \mid (a + 1 + 2\omega)$ or $p \mid (a - 1 - 2\omega)$, which implies that $p \mid 2$ (since $p \in \mathbb{Z}$, $p \mid a + b\omega \implies p \mid a, p \mid b$), which leads to a contradiction, thus p is not a prime.

Hence $p \neq 3$ is not a prime in $A_{-3} \iff p \equiv 1 \pmod{3}$.

1.2 Primary decomposition

Def 5.

- The radical of an ideal I is defined by $\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$.
- I is radical if $\sqrt{I} = I$.

Def 6. The **nilradical** is defined as $\sqrt{\langle 0 \rangle} \triangleq \{a \in R \mid a^n = 0 \text{ for some } n \in \mathbb{N}\}$. Elements in it are called nilpotent.

Prop 1.2.1. $\sqrt{\langle 0 \rangle} = \bigcap_{P \in \text{Spec } R} P$, where $\text{Spec } R$ is the set of prime ideals in R .

Proof. “ \subset ”: Notice that $a^n = 0 \in P$ for any prime ideal P . By the definition of prime ideal, either $a \in P$ or $a^{n-1} \in P$. No matter which, eventually we would get $a \in P$.

“ \supset ”: Let $\mathcal{S} \triangleq \{I : \text{ideal in } R \mid a^n \notin I, \forall n \in \mathbb{N}\}$. By the routine argument of Zorn’s lemma, exists maximal element Q in \mathcal{S} . We claim that Q is a prime ideal.

For each $x, y \notin Q$, we have $Q + Rx \supsetneq Q$ and $Q + Ry \supsetneq Q$. By the maximality of Q , these two ideals are not in \mathcal{S} . So exists n, m such that $a^n \in Q + Rx$, $a^m \in Q + Ry$ which implies $a^{n+m} \in Q + Rxy$, so $Q + Rxy \notin \mathcal{S}$, thus $xy \notin Q$, hence Q is prime. \square

Coro 1.2.1.

$$\sqrt{I} = \bigcap_{\substack{P \supset I \\ P \in \text{Spec } R}} P$$

Proof. Notice that $\text{Spec } R/I = \{P \in \text{Spec } R \mid R \subset I\}$. By the proposition above,

$$\sqrt{\langle 0 \rangle} = \bigcap_{\bar{P} \in \text{Spec } R/I} \bar{P} \implies \sqrt{I} = \bigcap_{\substack{P \supset I \\ P \in \text{Spec } R}} P \quad \square$$

Def 7. An ideal q of R is called primary if $q \neq R$ and “ $xy \in q$ and $x \notin q$ ” implies $y^n \in q$ for some $n \in \mathbb{N}$.

Prop 1.2.2.

- prime \implies primary.
- $\sqrt{\text{primary}} \implies$ prime. Also, if q is primary, then $p = \sqrt{q}$ is the smallest prime ideal containing q , we say q is p -primary.

Proof. The first one is obvious.

If q is primary and $\sqrt{q} = p$. For any $xy \in p$ and $x \notin p$, there exists n so that $x^n y^n \in q$, and for this n , $x^n \notin q$. Thus $(y^n)^m \in q$ for some m , hence $y \in p$. We conclude that p is a prime ideal.

Finally, by corollary 1.2.1,

$$p = \sqrt{q} = \bigcap_{\substack{P \supset q \\ P \in \text{Spec } R}} P \subset P, \quad \forall P \text{ prime},$$

thus p is indeed the smallest. \square

Eg 1.2.1. The primary ideals in \mathbb{Z} are $\langle 0 \rangle$ and $\langle p^m \rangle$ where p is a prime.

Proof. If $q = \langle a \rangle$ is primary, then $\sqrt{q} = \langle p \rangle$ is prime, and $p^n \in \langle a \rangle$. So $ab = p^n$ which implies $a = p^m$ for some m . \square

Def 8. An ideal I is said to be **irreducible** if $I = q_1 \cap q_2 \implies I = q_1 \vee I = q_2$.

Def 9. Define $(I : x) = \{a \in R \mid ax \in I\}$.

Theorem 3. In a Noetherian ring R , every irreducible ideal I is primary.

Proof. Let $xy \in I$ and $x \notin I$. Consider $(I : y) \subseteq (I : y^2) \subseteq \dots$. Since R is Noetherian, exists n such that $(I : y^n) = (I : y^m)$ for any $m \geq n$.

We claim that $I = (I + Ry^n) \cap (I + Rx)$.

- “ \subset ”: Obvious.
- “ \supset ”: For any $b \in (I + Ry^n) \cap (I + Rx)$, write $b = a_1 + r_1y^n = a_2 + r_2x$. Then $r_1y^{n+1} = a_2y - a_1y + r_2xy \in I$ since $a_1, a_2, xy \in I$. So $r_1 \in (I : y^{n+1}) = (I : y^n) \implies r_1y^n \in I$. Thus $b = a_1 + r_1y^n \in I$.

Now by the fact that I is irreducible and $I \neq I + Rx$ since $x \notin I$, thus $I = I + Ry^n \implies y^n \in I$. \square

Theorem 4. In a Noetherian ring R , every ideal is a finite intersection of irreducible ideals.

Proof. If not, let $\mathcal{I} \triangleq \{I : \text{ideal in } R \mid I \text{ is not a finite intersection of irreducible ideals}\}$ and \mathcal{I} is not an empty set. Since R is Noetherian, the set has a maximal element I_0 . Then I_0 is not irreducible (or else it is an intersection of itself, which is irreducible). Write $I_0 = I_1 \cap I_2$, with $I_1, I_2 \neq I_0$. Then $I_1, I_2 \notin \mathcal{I}$, so these two ideals could be written as a finite intersection of irreducible ideals, implying that I_0 could also be written as a finite intersection of irreducible ideals, which is a contradiction. \square

Prop 1.2.3. Let q be a p -primary ideal and $x \in R$.

1. If $x \in q$, then $(q : x) = R$.

Proof. In this case $1 \in (q : x)$, thus $(q : x) = R$. \square

2. If $x \notin q$, then $(q : x)$ is p -primary.

Proof. For any $y \in (q : x)$, $xy \in q$ but $x \notin q$, thus $y^n \in q \implies y \in p$. Hence

$$q \subset (q : x) \subset p \implies p = \sqrt{q} \subset \sqrt{(q : x)} \subset \sqrt{p} = p$$

and thus $(q : x)$ is p -primary.

For any y, z with $yz \in (q : x)$ but $y \notin (q : x)$, which is equivalent to $xyz \in q$ but $xy \notin q$. Since q primary, $z^n \in q \subset (q : x)$. \square

3. If $x \notin p$, then $(q : x) = q$.

Proof.

$$\begin{cases} y \in (q : x) \\ x \notin p \end{cases} \implies \begin{cases} xy \in (q : x) \\ x^n \notin q, \forall n \in \mathbb{N} \end{cases} \implies y \in q \quad \square$$

Prop 1.2.4. If each q_i are p -primary, then $q \triangleq \bigcap_{i=1}^n q_i$ is p -primary.

Proof. We check that $\sqrt{q} = \bigcap_{i=1}^n \sqrt{q_i} = \bigcap_{i=1}^n p = p$.

Also, if $xy \in q$ with $x \notin q$, then $x \notin q_k$ for some k . But $xy \in q_k$, thus $y^n \in q_k$. But $q_k \subseteq \sqrt{q_k} = p = \sqrt{q}$, so $(y^n)^{m'} = y^m \in q$, thus q is p -primary. \square

Def 10. A **primary decomposition** of $I = q_1 \cap \dots \cap q_n$ is **minimal** if $\sqrt{q_1}, \dots, \sqrt{q_n}$ are distinct and $q_i \not\supseteq \bigcap_{j \neq i} q_j$.

A minimal primary decomposition of an ideal always exists in Noetherian ring since by theorem 4, the ideal could be written as a finite intersection of irreducible ideals, and then by theorem 3, these ideals are primary. Now If $\sqrt{q_i} = \sqrt{q_j}$ happen in these ideals, we could remove these two ideals and add $q' = \sqrt{q_i} \cap \sqrt{q_j}$. By proposition 1.2.4, q' is also primary. And if $q_i \supseteq \bigcap_{j \neq i} q_j$, we could simply remove q_i .

Theorem 5 (Uniqueness of primary decomposition). Let $I = \cap_{i=1}^n q_i$ be a minimal decomposition of I . If $p_i = \sqrt{q_i}$, $\forall i$, then we have

$$\{p_i\} = \left\{ \sqrt{(I : x)} \mid x \in R \wedge \sqrt{(I : x)} \in \text{Spec } R \right\}$$

which is independent of the decomposition.

Proof. “ \supset ”: Let $x \in R \setminus I$, then $(I : x) = (\cap_{i=1}^n q_i : x) = \cap_{i=1}^n (q_i : x)$. By proposition 1.2.3, we have $\sqrt{(I : x)} = \cap \sqrt{(q_i : x)} = \cap_{x \notin q_i} p_i$.

Now, we have the following observation. “If $p \in \text{Spec } R$ with $p = \cap_{i=1}^n J_i$, then $p = J_j$ for some j .” If not, then $J_i \not\subset p$ for all i , so we could pick $x_i \in J_i \setminus p$. But then $x_1 x_2 \cdots x_n \in \cap J_i \in p$ since J_i are ideals, which leads to a contradiction since p is prime.

So if $\sqrt{(I : x)}$ is a prime, then it is equal to some p_i .

“ \subset ”: By assumption, $q_i \not\supseteq \cap_{j \neq i} q_j$ for each i , thus we could pick $x \in \cap_{j \neq i} q_j \setminus q_i$, then $\sqrt{(I : x)} = \cap_j \sqrt{(q_j : x)} = \sqrt{(q_i : x)} = p_i$. \square

Def 11. If $\{p_i\}$ is the unique prime ideals from the minimal primary decomposition of I .

- $\{p_i\}$ is said to be associated with I or to belong to I .
- The minimal elements in $\{p_i\}$ are called isolated primes.
- The other are called embedded primes.

Eg 1.2.2. Let $R = k[x, y]$ and $I = \langle x^2, xy \rangle$. If $P_1 = \langle x \rangle, P_2 = \langle x, y \rangle$, then $I = P_1 \cap P_2^2$. P_1 is isolated, while P_2 is embedded.

1.3 The equivalence of algebra and geometry (week 10)

In the following, k will be an algebraically closed field.

Def 12. The category of affine algebraic sets \mathcal{G} and its objects and morphisms are defined as following:

objects: The objects are affine algebraic sets in k^n .

An **affine algebraic set** is the common zero set of $\{F_i\}_{i \in \Lambda} \subset k[x_1, \dots, x_n]$ in k^n . We denote it by $V = \mathcal{V}(\{F_i\}_{i \in \Lambda}) \subset k^n$. (In fact, $I = \langle F_i : i \in \Lambda \rangle$ is Noetherian, so $I = \langle F_1, \dots, F_n \rangle$ and $V = \mathcal{V}(I)$.)

morphisms: The morphisms are the polynomial map from k^n to k^m .

A **polynomial map** is a mapping as following:

$$\begin{aligned} k^n &\longrightarrow k^m \\ \alpha &\longmapsto (F_1(\alpha), \dots, F_m(\alpha)) \end{aligned}$$

where each F_i is a polynomial in $K[x_1, \dots, x_n]$.

Given two affine algebraic sets $V \subset k^n$ and $W \subset k^m$, if a map $F : V \rightarrow W$ is the restriction of a polynomial map from k^n to k^m , then F is a morphism from V to W .

Moreover, if $F : V \rightarrow W$ and $G : W \rightarrow V$ satisfy $F \circ G = \text{Id}$ and $G \circ F = \text{Id}$, then we say $V \cong W$.

Def 13. The category of finitely generated reduced k -algebra \mathcal{A} and its objects and morphisms are defined as following:

objects: The objects are the reduced finitely generated k -algebra R .

A finitely generated k -algebra R is reduced if R has no non-zero nilpotent elements.

morphisms: The morphisms are the k -algebra homomorphisms.

Eg 1.3.1. It is easy to see that $\mathcal{V}(0) = k^n$ and $\mathcal{V}(1) = \emptyset$.

1.3.1 One-one correspondence between affine algebraic sets and radical ideals

Def 14. Define $\mathcal{I}(V) = \{f \in k[x_1, \dots, x_n] \mid f(\alpha) = 0, \forall \alpha \in V\}$.

The one-one correspondence is given by

$$\begin{aligned} \{\text{affine algebraic sets in } \mathbb{A}_k^n\} &\longleftrightarrow \{\text{radical ideals in } k[x_1, \dots, x_n]\} \\ V &\longmapsto \mathcal{I}(V) \\ \mathcal{V}(I) &\longleftarrow I \end{aligned}$$

Prop 1.3.1.

- $\sqrt{\mathcal{I}(V)} = \mathcal{I}(V)$.

Proof. For all $f^n \in \mathcal{I}(V)$, $f^n(\alpha) = 0, \forall \alpha \in V \implies f(\alpha) = 0, \forall \alpha \in V$. Thus $f \in \mathcal{I}(V)$. \square

- If V is an affine set, then $\mathcal{V}(\mathcal{I}(V)) = V$.

Proof. “ \supset ”: $\forall \alpha \in V, f \in \mathcal{I}(V), f(\alpha) = 0 \implies \alpha \in \mathcal{V}(\mathcal{I}(V))$.

“ \subset ”: Since V is an affine set, $V = \mathcal{V}(I)$, then $I \subset \mathcal{I}(V)$, so $\mathcal{V}(\mathcal{I}(V)) \subset \mathcal{V}(I) = V$. \square

Lemma 1. Given $T/S/R$, a tower of rings. If R is Noetherian, T/S is a module finite and T/R is a ring finite, then S/R is a ring finite.

Proof. Let $T = R[a_1, \dots, a_n] = S\omega_1 + \dots + S\omega_m$. Then $a_i = \sum_j r_{i,j} \omega_j$ for some $r_{i,j}$ and $\omega_{i,j} = \sum_k t_{i,j,k} w_k$ for some $t_{i,j,k}$.

Let $S' = R[\{r_{i,j}\}, \{t_{i,j,k}\}] \subseteq S$, which is Noetherian by the Hilbert basis theorem (R Noetherian $\implies R[x]$ Noetherian). Thus $T = S'\omega_1 + \dots + S'\omega_m$ is a Noetherian S' -module by the fact that finitely generated module over a Noetherian ring is a Noetherian module.

Since $S \subset T$, S is a finitely generated S' submodule, so $S = S'v_1 + \dots + S'v_r = R[\{r_{i,j}\}, \{t_{i,j,k}\}, \{v_i\}]$. \square

Lemma 2. If $S = k(z_1, \dots, z_p)$, $p > 0$ with each z_i transcendental, then S/k is not ring finite.

Proof. If not, say $S = k[f_1, \dots, f_n]$ with $f_i = g_i/h_i$, $g_i, h_i \in k[z_1, \dots, z_p]$. Then for any irreducible polynomial p such that $p \nmid h_i$ for each h_i (This polynomial exists since for each h_i there are only finite degree 1 factors). Then $1/p \notin k[f_1, \dots, f_n]$ by checking the divisibility of the denominator under addition and multiplication, which leads to a contradiction. \square

Lemma 3. If A/k is an extension of fields and ring finite, then A/k is algebraic.

Proof. If A/k is transcendental and let $\{z_1, \dots, z_t\}$ be a transcendental base. Then $A/k(z_1, \dots, z_t)$ is algebraic, thus a module finite. By lemma 1, $k(z_1, \dots, z_t)$ is ring finite, which contradict with lemma 2. \square

Theorem 6 (Weak form of Hilbert Nullstellensatz).

$$I \subsetneq k[x_1, \dots, x_n] \implies \mathcal{V}(I) \neq \emptyset$$

Proof. Since I proper, by lemma ??, exists a maximal ideal M such that $I \subseteq M$. Consider $K \triangleq k[x_1, \dots, x_n]/M = k[\bar{x}_1, \dots, \bar{x}_n]$. By proposition ??, K is a field, and by lemma 3, K/k is algebraic. Since k is already algebraically closed, $K = k$ and hence each $\bar{x}_i \in k$. Let $\alpha \triangleq (\bar{x}_1, \dots, \bar{x}_n) \in A_k^n$, then for any $f \in M$, $f(\alpha) = f(\bar{x}_1, \dots, \bar{x}_n) = \bar{f} = 0$, thus $\alpha \in \mathcal{V}(M) \subseteq \mathcal{V}(I)$. \square

Theorem 7 (Strong form of Hilbert Nullstellensatz). $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$

Proof. “ \supseteq ”: $f \in \sqrt{I} \implies f^n \in I$, then $f^n(\alpha) = 0, \forall \alpha \in \mathcal{V}(I) \implies f(\alpha) = 0, \forall \alpha \in \mathcal{V}(I)$, thus $f \in \mathcal{I}(\mathcal{V}(I))$.

“ \subseteq ”: If $\mathcal{I}(\mathcal{V}(I)) = 0$, then $I \subseteq \sqrt{I} \subseteq \mathcal{I}(\mathcal{V}(I)) = 0$, thus $I = 0$.

Otherwise, exists $0 \neq f \in \mathcal{I}(\mathcal{V}(I))$, Let $J = \langle I, ft - 1 \rangle \subset k[x_1, \dots, x_n, t]$. If (a_1, \dots, a_n, t_0) is a zero of J , then $ft - 1 \in J \implies -1 = f(a_1, \dots, a_n)t_0 - 1 = 0$, which is a contradiction, so by theorem 6, $J = k[x_1, \dots, x_n, t]$.

Write $1 = \sum h_i f_i + s(ft - 1)$, where each $f_i \in I$ and $h_i, s \in k[x_1, \dots, x_n, t]$. This is a equation of variables, so if we set $t = 1/f$, the equation still holds. Now each h_i would be the form $\sum p_i/f^{k_i}$, so we could multiply each side by a suitable f^ρ and get $f^\rho = \sum c_i f_i$ with each $c_i \in k[x_1, \dots, x_n]$. This implies $f^\rho \in I$, thus $f \in \sqrt{I}$. \square

Def 15. Let $V \in \mathcal{G}$, the coordinate ring of V is $k[V] \triangleq k[x_1, \dots, x_n]/\mathcal{I}(V)$

1.3.2 Equivalence of \mathcal{G} and \mathcal{A}

We define a functor F from \mathcal{G} to \mathcal{A} by

$$\begin{aligned} F : \quad \mathcal{G} &\longrightarrow \mathcal{A} \\ V &\longmapsto k[V] \end{aligned}$$

And For a polynomial map $f : V \rightarrow W$, define

$$\begin{aligned} F(f) = f^* : \quad k[W] &\longrightarrow k[V] \\ g &\longmapsto g \circ f \end{aligned}$$

Conversely, define a functor G by

$$\begin{aligned} G : \quad \mathcal{A} &\longrightarrow \mathcal{G} \\ k[x_1, \dots, x_n]/I &\longmapsto \mathcal{V}(I) \end{aligned}$$

Then if

$$\begin{aligned} \varphi : \quad k[\dots]/I &\longrightarrow k[\dots]/J \\ \bar{x}_i &\longmapsto \bar{f}_i \end{aligned}$$

Define

$$\begin{aligned} G(\varphi) = \psi : \quad \mathcal{V}(J) &\longrightarrow \mathcal{V}(I) \\ \alpha = (a_1, \dots, a_m) &\longmapsto (f_1(\alpha), \dots, f_n(\alpha)) \end{aligned}$$

1.4 Gröbner basis (week 11)

1.4.1 Division algorithm in $K[X_1, \dots, X_n]$

Eg 1.4.1. $I = \langle xy - 1, y^2 - 1 \rangle \subseteq K[x, y]$, $f_1 = xy - 1$ and $f_2 = y^2 - 1$ $G = \{f_1, f_2\}$. Does $f = x^2y + xy^2 + y^2 \in I$?

- Choose a lexicographic monomial ordering: $x > y$
- The multidegree $\partial(f) = (2, 1)$, $\partial(f_1) = (1, 1)$, $\partial(f_2) = (0, 2)$
- The leading term $\text{LT}(f) = x^2y$, $\text{LT}(f_1) = xy$, $\text{LT}(f_2) = y^2$
- $\text{LT}(f) = x \text{LT}(f_1) \Rightarrow f = x f_1 + xy^2 + y^2 + x \Rightarrow f = \underset{h_1}{(x+y)}f_1 + \underset{h_2}{(1)}f_2 + \underset{\bar{f}^G}{(x+y+1)}$ or

$$f = \underset{h_1}{x}f_1 + \underset{h_2}{(x+1)}f_2 + \underset{\bar{f}^G}{(2x+1)}.$$

Note: Divisor h_1 , h_2 and remainder \bar{f}^G are not unique!!

Def 16. Fix a monomial ordering and let I be an ideal of $K[X_1, \dots, X_n]$. The ideal of leading terms in I is defined to be $\text{LT}(I) = \langle \text{LT}(f) \mid f \in I \rangle$.

Remark 1. Let $I = \langle f_1, \dots, f_n \rangle$. In general, $\langle \text{LT}(f_1), \dots, \text{LT}(f_n) \rangle \subsetneq \text{LT}(I)$.

Eg 1.4.2. Let $f_1 = xy^2 + y$, $f_2 = x^2y$. And, $xf_1 - yf_2 = xy \in \langle f_1, f_2 \rangle$ but $xy \notin \langle xy^2, x^2y \rangle$.

Def 17. $G = \{g_1, \dots, g_m\}$ is called a Gröbner basis of I if $I = \langle g_1, \dots, g_m \rangle$ and $\text{LT}(I) = \langle \text{LT}(g_1), \dots, \text{LT}(g_m) \rangle$.

Prop 1.4.1. Let $g_1, \dots, g_m \in I$, then $\text{LT}(I) = \langle \text{LT}(g_1), \dots, \text{LT}(g_m) \rangle \implies I = \langle g_1, \dots, g_m \rangle$.

Proof. $\forall f \in I$, do the division process. Then $f = \sum_{i=1}^m h_i g_i + r$, either $r = 0$ or $\star = \text{no term of } r \text{ is divisible by any of } \text{LT}(g_1), \dots, \text{LT}(g_m)$. Assume $r \neq 0$, then $r = f - \sum_{i=1}^m h_i g_i \in I \Rightarrow \text{LT}(r) \in \text{LT}(I) = \langle \text{LT}(g_1), \dots, \text{LT}(g_m) \rangle$, which is a contradiction. Hence, $r = 0$ (i.e. $f \in \langle g_1, \dots, g_m \rangle$). \square

Theorem 8. Each ideal I has a Gröbner basis.

Proof. By Hilbert basis thm, $\text{LT}(I) = \langle f_1, \dots, f_m \rangle$ for some f_i 's. Write $f_i = \sum_{j=1}^{m_i} h_{ij} \text{LT}(g_{ij})$ with $h_{ij} \in K[X_1, \dots, X_n]$, $g_{ij} \in I$. Then $\text{LT}(I) = \langle \text{LT}(g_{ij}) \mid i = 1, \dots, m, j = 1, \dots, m_i \rangle$. By prop 1.4.1, This is Gröbner basis. \square

Theorem 9. Let $G = \{g_1, \dots, g_m\}$ be a Gröbner basis of I , then

- $\forall f \in K[X_1, \dots, X_n]$, $f = f_I + r$ where $f_I \in I, r = \star$ are unique.

Proof. By division algorithm, $f = f_I + \underset{\star}{r} = f'_I + \underset{\star}{r}'$, then $\underset{\star}{r} - \underset{\star}{r}' = f_I - f'_I$. But if $\underset{\star}{r} - \underset{\star}{r}' \neq 0$, then $\text{LT}(\underset{\star}{r} - \underset{\star}{r}') \in \text{LT}(I) = \langle \text{LT}(g_1), \dots, \text{LT}(g_m) \rangle$, which is a contradiction. Hence, $\underset{\star}{r} - \underset{\star}{r}' = 0 \Rightarrow f_I = f'_I$. \square

- $f \in I \iff r = 0$.

Proof. Suppose $f \in I$, then $f = f_I + \underset{\star}{r}$, and if $\underset{\star}{r} \neq 0$, $\underset{\star}{r} = f - f_I \in I$, which is a contradiction. Hence, $\underset{\star}{r} = 0$. Conversely, if $\underset{\star}{r} = 0$, $f = f_I \in I$. \square

1.4.2 Buchberger's algorithm

Def 18. Let $f, g \in K[x_1, \dots, x_n]$ and M be the monic least common multiple of $\text{LT}(f)$ and $\text{LT}(g)$. $S(f, g) = \frac{M}{\text{LT}(f)}f - \frac{M}{\text{LT}(g)}g$ is called an S-polynomial of f, g .

Let $I = \langle g_1, \dots, g_m \rangle$ and $G = \{g_1, \dots, g_m\}$. A Gröbner basis of I can be constructed by the following algorithm:

1. Initially let $G_0 \leftarrow G$.
2. Repeatly construct $G_{i+1} \leftarrow G_i \cup (\{S(f, g) \bmod G_i \mid f, g \in G_i\} \setminus \{0\})$, until once $G_{i+1} = G_i$, then G_i is a Gröbner basis of I .

Lemma 4. Let $f_1, \dots, f_m \in K[x_1, \dots, x_n]$ with $a_1, \dots, a_m \in K$ satisfying $\partial(f_1) = \partial(f_2) = \dots = \partial(f_m) = \alpha$ and $h = \sum_{i=1}^m a_i f_i$ with $\partial(h) < \alpha$. Then $h = \sum_{i=2}^m b_i S(f_{i-1}, f_i)$ for some $b_i \in K$.

Proof. Write $f_i = c_i f'_i$ with $c_i \in K$ and f'_i being monic of multidegree α . Note: $S(f_i, f_j) = f'_i - f'_j$ since all multidegree are equal. Then,

$$\begin{aligned} h &= \sum_{i=1}^m (a_i c_i f'_i) \\ &= a_1 c_1 (f'_1 - f'_2) + (a_1 c_1 + a_2 c_2)(f'_2 - f'_3) + \dots + (a_1 c_1 + \dots + a_{m-1} c_{m-1})(f'_{m-1} - f'_m) \\ &\quad + (a_1 c_1 + \dots + a_m c_m) f'_m \\ &= \sum_{i=2}^m b_i S(f_{i-1}, f_i) + b_{m+1} f'_m \text{ with } b_i = \sum_{j=1}^{i-1} a_j c_j. \end{aligned}$$

Also, in this equality, f'_m is the only term that has multidegree α (other terms have multidegree less than α). So $b_{m+1} = 0$ must hold. Then, we have $h = \sum_{i=2}^m b_i S(f_{i-1}, f_i)$. \square

Theorem 10 (Buchberger's criterion). Assume $I = \langle g_1, \dots, g_m \rangle$, then $G = \{g_1, \dots, g_m\}$ is a Gröbner basis of $I \iff S(g_i, g_j) \equiv 0 \pmod{G}$ for each i, j .

Proof.

- Suppose G is a Gröbner basis of I . $S(g_i, g_j) \in I \Rightarrow S(g_i, g_j) = 0$ by thm 9.
- Converely, suppose $S(g_i, g_j) \equiv 0 \pmod{G} \forall i, j$. For $f \in I$, $f \underset{\text{not division}}{=} \sum_{i=1}^m h_i g_i$ for some $h_i \in K[x_1, \dots, x_n]$. Define $\alpha = \max\{\partial(h_1 g_1), \dots, \partial(h_m g_m)\}$. We have $\partial(f) \leq \alpha$ and we can select an expression $f = \sum_{i=1}^m h_i g_i$ for f s.t α is minimal.
- Claim: $\partial(f) = \alpha$.
- (pf) If not, we rewrite f

$$\begin{aligned} f &= \sum_{i=1}^m h_i g_i \\ &= \sum_{\partial(h_i g_i) = \alpha} h_i g_i + \sum_{\partial(h_i g_i) < \alpha} h_i g_i \quad (\text{the first term} \neq 0 \text{ since } \alpha \text{ is minimal.}) \\ &= \sum_{\partial(h_i g_i) = \alpha} \text{LT}(h_i) g_i + \sum_{\partial(h_i g_i) = \alpha} (h_i - \text{LT}(h_i) g_i) + \sum_{\partial(h_i g_i) < \alpha} h_i g_i \end{aligned}$$

Let $\text{LT}(h_i) = a_i h_i^0$ with h_i^0 being a monic monomial. Comparing the multidegree on both side, $\partial\left(\sum_{\partial(h_i g_i) = \alpha} a_i h_i^0 g_i\right) < \alpha$ By lemma 4, $\sum_{\partial(h_i g_i) = \alpha} (a_i h_i^0 g_i) = c_{12} S(h_{i_1}^0 g_{i_1}, h_{i_2}^0 g_{i_2}) + \dots$ (finite)

where $\partial(h_{i_1}g_{i_1}) = \partial(h_{i_2}g_{i_2}) = \dots = \alpha$. By def, if we set $M_{st} = X_{st}^\beta$ = the monic LCM of $\text{LT}(g_{i_s}), \text{LT}(g_{i_t})$, then

$$\begin{aligned} S(h_{i_s}^0 g_{i_s}, h_{i_t}^0 g_{i_t}) &= \frac{X^\alpha}{\text{LT}(h_{i_s}^0 g_{i_s})} h_{i_s}^0 g_{i_s} - \frac{X^\alpha}{\text{LT}(h_{i_t}^0 g_{i_t})} h_{i_t}^0 g_{i_t} \\ &= X^{\alpha-\beta_{st}} \left(\frac{X^{\beta_{st}}}{h_{i_s}^0 \text{LT}(g_{i_s})} h_{i_s}^0 g_{i_s} - \frac{X^{\beta_{st}}}{h_{i_t}^0 \text{LT}(g_{i_t})} h_{i_t}^0 g_{i_t} \right) \\ &= X^{\alpha-\beta_{st}} S(g_{i_s}, g_{i_t}) \\ &= X^{\alpha-\beta_{st}} \sum_{j=1}^m l_j g_j \text{ (by division)} \end{aligned}$$

- Then, $\partial(l_j g_j) < \beta_{st} \implies$ we found a expression with multidegree less than α , which is a contradiction. Therefore, $\partial(f) = \alpha \implies \text{LT}(f) = \sum_{\partial(h_i g_i) = \alpha} \text{LT}(h_i) \text{LT}(g_i) \implies \text{LT}(f) \in \langle \text{LT}(g_1), \dots, \text{LT}(g_m) \rangle$.

□

Theorem 11. The Buchberger's algorithm will terminate

Proof. .

- $\langle \text{LT}(G_i) \rangle \subsetneq \langle \text{LT}(G_{i+1}) \rangle$ if $G_i \neq G_{i+1}$
 $G_i \neq G_{i+1} \implies \exists f, g \in G_i$ s.t. $S(f, g) \not\equiv 0 \pmod{G} \implies \text{LT}(S(f, g)) \notin \langle \text{LT}(G_i) \rangle$
- $\langle \text{LT}(G_0) \rangle \subsetneq \langle \text{LT}(G_1) \rangle \subsetneq \dots$ is not possible since $K[x_1, \dots, x_n]$ is a Noetherian ring. (Noetherian ACC condition).

□

1.5 Applications of Gröbner basis

Def 19. Let $I \subseteq K[x_1, \dots, x_n]$ and $x_1 > x_2 > \dots > x_n$. $I_i \triangleq I \cap K[x_{i+1}, \dots, x_n]$ is called the i -th elimination ideal of I .

Theorem 12 (Elimination theorem). Let $G = \{g_1, \dots, g_m\}$ be a Gröbner basis of $I \neq 0$ with ordering $x_1 > \dots > x_n$. Then $G_i \triangleq G \cap K[x_{i+1}, \dots, x_n]$ is a Gröbner basis of I_i (i.e., $\langle \text{LT}(G_i) \rangle = \text{LT}(I_i)$).

Proof. “ \subseteq ”: Obvious.

“ \supseteq ”: Let $f \in I_i$. Write

$$\text{LT}(f) = \sum h_i \text{LT}(g_i) = \sum a_k x^{\alpha_k} \text{LT}(g_{i_k})$$

Since $\text{LT}(f)$ involves only the variables x_{i+1}, \dots, x_n , and each terms of $x^{\alpha_k} \text{LT}(g_{i_k})$ which uses variables x_k with $k \leq i$ must sum to zero. Remove those term we could write $\text{LT}(f)$ as a combination of $\text{LT}(g_i)$ with $\text{LT}(g_i) \in K[x_{i+1}, \dots, x_n]$. But by the definition of leading term and the ordering $x_1 > \dots > x_n$, we have $g_i \in K[x_{i+1}, \dots, x_n] \implies g_i \in G_i$. Thus $\text{LT}(f) \in \langle \text{LT}(G_i) \rangle$. □

Eg 1.5.1. Find $V = \mathcal{V}(x + y - z, x^2 + y^2 - z^3, x^3 + y^3 - z^5)$.

We compute a Gröbner basis of $I = \langle f_1, \dots, f_3 \rangle$ with respect to the ordering $x > y > z$. The Gröbner basis is $\{x + y - z, 2y^2 - 2yz - z^3 + z^2, 2z^5 - 3z^4 + z^3\}$.

Eg 1.5.2.

$$\begin{aligned} f : \mathbb{A}^1 &\longrightarrow \mathbb{A}^3 \\ t &\longmapsto (t^4, t^3, t^2) \end{aligned}$$

We compute a Gröbner basis of $I = \langle t^4 - x, t^3 - y, t^2 - z \rangle$ with respect to $t > x > y > z$. The Gröbner basis is $\{-t^2 + z, ty - z^2, tz - y, x - z^2, y^2 - z^3\}$.

Eg 1.5.3.

$$\begin{aligned} f : V = \mathcal{V}(x^3 - x^2z - y^2z) &\longrightarrow \mathbb{A}^3 \\ (x, y, z) &\longmapsto (x^2z - y^2z, 2xyz, -z^3) \end{aligned}$$

The ideal is $\langle x^3 - x^2z - y^2z, u - x^2z + y^2z, v - 2xyz, w + z^3 \rangle$ has a Gröbner basis $\langle \dots, u^2 + v^2 - w^2 \rangle$.

Theorem 13. Let I, J be two ideals of $K[x_1, \dots, x_n]$, then $I \cap J = (t\tilde{I} + (1-t)\tilde{J}) \cap K[x_1, \dots, x_n]$, where $\tilde{I} \triangleq K[x_1, \dots, x_n, t]I$.

Proof. “ \subseteq ”: If $f \in I \cap J$, then $f = tf + (1-t)f \in \text{RHS}$.

“ \supseteq ”: If $f \in \text{RHS}$, then $f = t\tilde{f}_1 + (1-t)\tilde{f}_2$ with $\tilde{f}_1 \in \tilde{I}, \tilde{f}_2 \in \tilde{J}$. Write

$$\tilde{f}_1 = \sum (h_i t + r_i) f_i, \quad \tilde{f}_2 = \sum (h'_j t + r'_j) f_j$$

with each $r_i, r'_j \in K[x_1, \dots, x_n]$, $h_i, h'_j \in K[t, x_1, \dots, x_n]$. Take $t = 0$, $f = \sum r'_j f_j \in J$. Then take $t = 1$, $f = \sum (h_i(1, x_1, \dots, x_n) + r_i) f_i \in J$. Thus $f \in I \cap J$. \square

Eg 1.5.4. $I = \langle y^2, x - yz \rangle, J = \langle x, z \rangle$. We shall find $I \cap J$.

$tI + (1-t)J = \langle tx - tyz, ty^2, (1-t)x, (1-t)z \rangle$ has a Gröbner basis $\{f_1, f_2, f_3, f_4, xy, x - yz\}$, so $I \cap J = \langle xy, x - yz \rangle$.

Theorem 14. Let $L = \langle f_1, \dots, f_s \rangle \subsetneq K[x_1, \dots, x_n]$, then $f \in \sqrt{L} \iff \langle f_1, \dots, f_s, 1 - tf \rangle = K[x_1, \dots, x_n, t]$.

Proof. “ \Leftarrow ”: By theorem 6, $\langle f_1, \dots, f_s, 1 - tf \rangle = K[x_1, \dots, x_n, t]$ if and only if $\mathcal{V}(f_1, \dots, f_s, 1 - tf) = \emptyset$. Notice that $1 - tf$ has no zero if $f = 0$, which means that If \mathbf{x} is a common zero of f_1, \dots, f_s , then $f(\mathbf{x}) = 0$. So $f \in \mathcal{I}(\mathcal{V}(L)) \implies f \in \sqrt{L}$ by theorem 7.

“ \Rightarrow ”: $f^m \in L \implies 1 = t^m f^m + 1 - t^m f^m = t^m f^m + (1 - tf)(1 + tf + \dots + t^{m-1} f^{m-1}) \in \langle f_1, \dots, f_s, 1 - tf \rangle$. \square

Eg 1.5.5. Let $I = \langle xy^2 + 2y^2, x^4 - 2x^2 + 1 \rangle$, and we are to determine $f = y - x^2 + 1$ is in \sqrt{I} or not.

Prop 1.5.1. An affine algebraic set V in \mathbb{A}_k^n has a unique minimal decomposition. $V = V_1 \cap V_2 \cap \dots \cap V_m$ with V_i irreducible and $V_i \not\subset V_j$.

Proof.

Existence: If not, then $V = V_1 \cup V'_1$, and one of V_1, V'_1 , say $V_1 = V_2 \cup V'_2, \dots$ So we would find

$$V \supsetneq V_1 \supsetneq V_2 \subsetneq \dots \implies \mathcal{I}(V) \subsetneq \mathcal{I}(V_1) \subsetneq \mathcal{I}(V_2) \subsetneq \dots \text{ in } k[x_1, \dots, x_n]$$

Which contradict that $k[x_1, \dots, x_n]$ is Noetherian.

- Uniqueness: If

$$V = V_1 \cup \dots \cup V_m = V'_1 \cup \dots \cup V'_m$$

then $V_i = (V_i \cap V'_1) \cup \dots \cup (V_i \cap V'_m)$. But V_i irreducible, so $V_i = V_i \cap V'_j \implies V_i \subset V'_j$. By symmetry we would find $V'_j \subset V_k$, then $V_i \subset V'_j \subset V_k \implies V_i = V_k$. Thus these two decomposition is equal. \square

Theorem 15 (Decomposition). Assume $\sqrt{I} = I$ and $I \subset J$, then $\mathcal{V}(I : J) = \mathcal{V}(\mathcal{I}(\mathcal{V}(I) \setminus \mathcal{V}(J)))$. and $\mathcal{V}(I) = \mathcal{V}(J) \cup \mathcal{V}(I : J)$.

Proof. Let $f \in \mathcal{I}(\mathcal{V}(I) \setminus \mathcal{V}(J))$ and $g \in J$, then $fg = \mathcal{I}(\mathcal{V}(I)) = \sqrt{I} = I$ since $f(\alpha) = 0$ for each $\alpha \in \mathcal{V}(I) \setminus \mathcal{I}(J)$ and $g(\alpha) = 0$ for each $\alpha \in \mathcal{V}(J)$. Thus $f \in (I : J)$. \square

Eg 1.5.6. Let $I = \langle xz - y^2, x^3 - yz \rangle$ and $V = \mathcal{V}(I)$.

Notice that $\langle xz - y^2, x^3 - yz \rangle \subseteq \langle x, y \rangle = J$, so $(I : J) = (I : \langle x \rangle) \cap (I : \langle y \rangle)$.

First we calculate $(I : x)$. Notice that we know how to calculate $I \cap \langle x \rangle$ now. After a calculation, $I \cap \langle x \rangle = \{x^2z - xy^2, x^4 - xyz, x^3y - xz^2\}$, so $(I : x) = \langle f_1/x, f_2/x, f_3/x \rangle = I + \langle x^2y - z^3 \rangle$. Similarly one could find that $(I : y) = (I : x)$, thus $(I : J) = (I : x)$.

Hence $V = \mathcal{V}(x, y) \cap \mathcal{V}(xz - y^2, x^3 - yz, x^2y - z^2)$.

Prop 1.5.2. Let $f : V \rightarrow W$, then $\overline{f(V)} = \mathcal{V}(\ker f^*)$ where $f^* : k[W] \rightarrow k[V]$.

Proof. We claim that $\ker f^* = \mathcal{I}(f(V))$, since

$$\bar{g} \in \mathcal{I}(f(V)) \iff \bar{g}(f(\alpha)) = 0, \forall \alpha \in V \iff \bar{g} \circ f \in \mathcal{I}(V) \iff f^*(\bar{g}) = \overline{g \circ f} = \bar{0} \iff \bar{g} \in \ker f^*$$

Thus $\mathcal{V}(\ker f^*) = \mathcal{V}(\mathcal{I}(f(V))) = \overline{f(V)}$. \square

Remark 2. In general, if $W \subseteq \mathbb{A}_k^n$ is an affine algebraic set defined by $x_i = f_i(t_1, \dots, t_m)$, then W is irreducible.

Proof. $f : \mathbb{A}_k^m \rightarrow W$ is onto, so $\overline{f(\mathbb{A}_k^m)} = W = \mathcal{V}(0)$. By the previous proposition, $\ker f^* = 0$, thus $f^* : K[W] \cong k[x_1, \dots, x_n]/\mathcal{I}(W) \hookrightarrow k[t_1, \dots, t_m]$. But $k[t_1, \dots, t_m]$ is an integer domain, so $\mathcal{I}(W)$ is a prime ideal, thus W is irreducible. \square

1.6 Local Rings (week 12)

From now on, R is a commutative ring with 1.

Def 20. R is called a local ring if it has a unique maximal ideal.

Prop 1.6.1.

- (1) R is a local ring.
- (2) The set of non-units forms an ideal.
- (3) $\exists M \in \text{Max } R$ s.t. $1 + m$ is a unit $\forall m \in M$.

Proof.

- (1) \Rightarrow (2): Let M be the unique maximal ideal of R . Then M couldn't contain any unit. For each non-unit x , $\langle x \rangle \neq R$ and is contained in a maximal ideal by lemma ??, thus $x \in M$. Hence $M = \{\text{non-units}\}$.
- (2) \Rightarrow (3): This ideal must be a maximal ideal M since it can't be extended. Now, $1 \notin M \rightsquigarrow 1 + m \notin M$. So $1 + m$ is a unit.
- (3) \Rightarrow (1): If there exists another maximal ideal N , then $M + N = R$. Say $m \in M, n \in N$ s.t. $m + n = 1$, then $n = 1 - m$ is a unit $\implies N = R$, which is a contradiction. \square

Eg 1.6.1. $k[[x]]$ is a local ring with the unique maximal ideal $\langle x \rangle$.

Proof. For each $f = \sum a_n x^n \in k[[x]]$, one could see that f is a unit if and only if $a_n \neq 0$, and the leftovers form an ideal $\langle x \rangle$. \square

Eg 1.6.2. Let $P \in \text{Spec } R$. If $S = R \setminus P$, then S is a multiplicatively closed set with $1 \in S$ and $R_P \triangleq R_S$ is a local ring.

Proof. S is a multiplicatively closed set simply follow from the definition of prime ideal. One could then easily see that $P_P \triangleq \left\{ \frac{x}{s} \mid x \in P, s \in S \right\}$ contains all non-unit, thus R_P is local. \square

Prop 1.6.2. The following sets are correspondent (k is algebraically closed):

- (1) \mathbb{A}_k^n
- (2) $\text{Max } k[x_1, \dots, x_n]$
- (3) $\text{Hom}_k(k[x_1, \dots, x_n], k)$

Proof. (1) \Rightarrow (2): For any $(a_1, \dots, a_n) \in \mathbb{A}_k^n$, $k[x_1, \dots, x_n] / \langle x_1 - a_1, \dots, x_n - a_n \rangle \cong k$ is a field, hence $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ is a maximal ideal.

(2) \Rightarrow (1): Let $M \in \text{Max } k[x_1, \dots, x_n]$, by theorem 6, $\mathcal{V}(M) \neq \emptyset$, so exists $(a_1, \dots, a_n) \in \mathcal{V}(M)$. Now $M = \sqrt{M} = \mathcal{I}(\mathcal{V}(M)) \subseteq \mathcal{I}((a_1, \dots, a_n)) = \langle \dots, x_i - a_i, \dots \rangle$ which is maximal, We conclude that (a_1, \dots, a_n) is the only element in $\mathcal{V}(M)$ and $M = \langle \dots, x_i - a_i, \dots \rangle$.

(1) \Rightarrow (3): For each (a_1, \dots, a_n) , define $\varphi \in \text{Hom}_k(\dots)$ by evaluation:

$$\begin{array}{ccc} \varphi : & k[x_1, \dots, x_n] & \longrightarrow k \\ & x_i & \longmapsto a_i \end{array}$$

(3) \Rightarrow (1): Similarly, for each $\varphi \in \text{Hom}_k(\dots)$, recover (a_1, \dots, a_n) by $(\varphi(x_1), \dots, \varphi(x_n))$. \square

Remark 3. Inspired by the correspondence,

Def 21. A property of an R -module M is said to be a local property if

$$M \text{ has this property} \iff M_P \text{ (as an } R_P\text{-module) has this property } \forall P \in \text{Spec } R$$

Prop 1.6.3. TFAE

- (1) $M = 0$
- (2) $M_P = 0 \quad \forall P \in \text{Spec } R$
- (3) $M_Q = 0 \quad \forall Q \in \text{Max } R$

Proof. (1) \Rightarrow (2) and (2) \Rightarrow (3) are clear.

(3) \Rightarrow (1): If $M \neq 0$, let $x \in M$ such that $x \neq 0$, then $\text{Ann}(x) \subsetneq R$ since $1 \notin \text{Ann}(x)$. Let $\text{Ann}(x) \subset Q \in \text{Max } R$. By assumption, $M_Q = 0$ implies $\frac{x}{1} = \frac{0}{1}$. By the definition of equal in localization, $\exists r \notin Q$ such that $rx = 0$, thus $r \in \text{Ann}(x)$ which leads to a contradiction. \square

Coro 1.6.1. Let $N \subseteq M$, TFAE (consider M/N)

- (1) $N = M$
- (2) $N_P = M_P \quad \forall P \in \text{Spec } R$
- (3) $N_Q = M_Q \quad \forall Q \in \text{Max } R$

Prop 1.6.4. TFAE

- (1) $0 \rightarrow M \xrightarrow{\phi} N \xrightarrow{\psi} L \rightarrow 0$ exact
- (2) $0 \rightarrow M_P \xrightarrow{\phi_P} N_P \xrightarrow{\psi_P} L \rightarrow 0$ exact $\forall P \in \text{Spec } R$
- (3) $0 \rightarrow M_Q \xrightarrow{\phi_Q} N_Q \xrightarrow{\psi_Q} L \rightarrow 0$ exact $\forall Q \in \text{Max } R$

Proof. (1) \Rightarrow (2): By the fact that localization preserves exact sequence.

(2) \Rightarrow (3): Obvious.

(3) \Rightarrow (1): Let $K = \ker \phi$, then $0 \rightarrow K \rightarrow M \rightarrow N$ exact. Since we just proved (1) \Rightarrow (3), $0 \rightarrow K_Q \rightarrow M_Q \rightarrow N_Q$ exact, but $K_Q = 0$, by proposition 1.6.3, $K = 0$.

We could prove the other half similarly by letting K to be the cokernel. \square

Def 22.

- Let $R \subseteq S$. $\bar{R} = \{x \in S \mid x \text{ is integral over } R\}$ is called the integral closure of R in S .
- R is integrally closed in S if $R = \bar{R}$.
- An integral domain R is called normal if R is integrally closed in its field of fractions.

Theorem 16. UFD is normal.

Proof. Let R be a UFD and K be its field of fractions. If $a \in K$ is integral over R and $a^n + r_1 a^{n-1} + \dots + r_n = 0$. Write $a = u/s$ with $\gcd(u, s) = 1$. Then $u^n + r_1 s u^{n-1} + \dots + r_n s^n = 0$. Now if s is a non-unit, says $p \mid s$ with p is a prime. Then $p \mid u$ obviously $\leadsto p \mid \gcd(u, s) = 1$, which is a contradiction. So s is a unit $\implies a \in R$. \square

Prop 1.6.5.

- Let S/R is an integral extension and $T \subset R$ be a m.c. set with $1 \in T$. Then S_T is also integral over R_T .

Proof. Let $a/t \in S_T$ with $a^n + r_1 a^{n-1} + \cdots + r_n = 0$, then we have

$$\left(\frac{a}{t}\right)^n + \frac{r_1}{t} \left(\frac{a}{t}\right)^{n-1} + \cdots + \frac{r_n}{t^n} \left(\frac{a}{t}\right)^n = 0$$

Thus a/t is integral over R_T . □

- Let S/R be an arbitrary extension and $T \subset R$ be m.c. with $1 \in T$. Then $(\bar{R})_T = \overline{(R_T)}$ in S_T .

Proof. By 1., $(\bar{R})_T$ is integral over R_T . If $a/t \in S_T$ is integral over R_T , say

$$\left(\frac{a}{t}\right)^n + \frac{r_1}{t_1} \left(\frac{a}{t}\right)^{n-1} + \cdots + \frac{r_n}{t_n} \left(\frac{a}{t}\right)^n = 0$$

If we let $v = t_1 t_2 \cdots t_n$, multiply the equation by $(tv)^n$, we get

$$(va)^n + (r_1 t t_2 \cdots t_n)(va)^{n-1} + \cdots = 0 \implies va \in \bar{R}$$

So $a/t = va/(vt) \in \bar{R}_T$. □

Prop 1.6.6. “Being normal” is a local property. TFAE

- (1) R is normal
- (2) R_P is normal $\forall P \in \text{Spec } R$
- (3) R_Q is normal $\forall Q \in \text{Max } R$

Proof. The key is to realize that if K is the field of fraction of R , then K is also the field of fraction of any R_P . Then by lemma 1.6.4,

$$0 \rightarrow R \rightarrow \bar{R} \rightarrow 0 \iff 0 \rightarrow R_P \rightarrow (\bar{R})_P \rightarrow 0, \forall P$$

By the previous proposition, $(\bar{R})_P = \overline{R_P}$ in S_P , this proves all. □

Def 23. An R -module F is flat if the functor $- \otimes_R M$ is exact (i.e., it preserves exact sequence).

Prop 1.6.7. Given an homomorphism $R_1 \rightarrow R_2$. If M is a flat R_1 -module, then $R_2 \otimes_{R_1} M$ is a flat R_2 module.

Proof. Notice that $N \otimes_{R_1} M \cong N \otimes_{R_2} (R_2 \otimes_{R_1} M)$, so

$$\begin{aligned} 0 \rightarrow N \rightarrow N' \text{ exact} &\implies 0 \rightarrow N \otimes_{R_1} M \rightarrow N' \otimes_{R_1} M \text{ exact} \\ &\implies 0 \rightarrow N \otimes_{R_2} (R_2 \otimes_{R_1} M) \rightarrow N' \otimes_{R_2} (R_2 \otimes_{R_1} M) \text{ exact} \end{aligned}$$

Which is to say that $R_2 \otimes_{R_1} M$ flat. □

Prop 1.6.8. TFAE

- (1) M is a flat R -module
- (2) M_P is a flat R -module $\forall P \in \text{Spec } R$
- (3) M_Q is a flat R -module $\forall Q \in \text{Max } R$

Proof. (1) \Rightarrow (2): By the previous proposition combined with the property of localization, $M_P \cong R_P \otimes_R M$ is a flat module.

(2) \Rightarrow (3): Obvious.

(3) \Rightarrow (1): If $0 \rightarrow N \rightarrow N'$ exact, then by prop 1.6.4, $0 \rightarrow N_Q \rightarrow N'_Q$ exact, so

$$0 \rightarrow N_Q \otimes_{R_Q} M_Q \rightarrow N'_Q \otimes_{R_Q} M_Q$$

is also exact. By the property of localization, $N_Q \otimes_{R_Q} M_Q \cong (N \otimes_R M)_Q$. Using prop 1.6.4, $0 \rightarrow N \otimes_R M \rightarrow N' \otimes_R M$ exact. \square

1.7 Krull dimension

Def 24.

- The Krull dimension of a topological space X is the supremum of the lengths n of all chains $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$, where X_i are closed irreducible subset of X .
- The Krull dimension of a commutative ring R with 1 is the supremum of the lengths n of all chains $P_0 \subsetneq \cdots \subsetneq P_n$ where $P_i \in \text{Spec } R$.

Prop 1.7.1. Let $R \subseteq S$ be two integral domains and S/R be integral. Then S is a field if and only if R is a field.

Proof. “ \Rightarrow ”: For each $a \neq 0$ in R , $a^{-1} \in S$, so we could write

$$(a^{-1})^n + r_1(a^{-1})^{n-1} + \cdots + r_n = 0, \quad r_i \in R$$

Which implies

$$a^{-1} = -(r_1 + \cdots + r_n a^{n-1}) \in R$$

“ \Leftarrow ”: For each $a \neq 0$ in S , write

$$a^n + r_1 a^{n-1} + \cdots + r_n = 0, \quad r_i \in R$$

Notice that we could assume $r_n \neq 0$, or else $a(a^{n-1} + r_1 a^{n-2} + \cdots + r_{n-1}) = 0$ and hence $a^{n-1} + r_1 a^{n-2} + \cdots + r_{n-1} = 0$ because R is an integral domain. Then

$$a^{-1} = -r_n^{-1}(a^{n-1} + r_1 a^{n-2} + \cdots + r_{n-1})$$

\square

Prop 1.7.2. Let S/R be integral.

1. If $q \in \text{Spec } S$ and $p = q \cap R \in \text{Spec } R$, then $q \in \text{Max } S \iff p \in \text{Max } R$.

Proof. It is easy to see that S/q is integral over R/p by the identification

$$\begin{aligned} R/p &\hookrightarrow S/p \\ r + p &\longmapsto r + q \end{aligned}$$

So

$$q \in \text{Max } S \iff S/q \text{ is a field} \iff R/p \text{ is a field} \iff p \in \text{Max } R$$

\square

2. If $q, q' \in \text{Spec } S$ with $q \subseteq q'$ and $q \cap R = p = q' \cap R$. Then $q = q'$.

Proof. We know that $S_P \triangleq S_{R \setminus P}$ is integral over R_P . Since $q_P \subseteq q'_P$ and both $q_P \cap R_P$ and $q'_P \cap R_P$ equal P_P is maximal in R_P . Using 1., q_P, q'_P are maximal in S_P , but $q_P \subseteq q'_P \implies q_P = q'_P$. By corollary 1.6.1, $q = q'$. \square

Theorem 17 (Going-up theorem). Let S/R be integral, then

- If $p \in \text{Spec } R$, then $\exists q \in \text{Spec } S$ such that $q \cap R = p$.

Proof. We have the diagram:

$$\begin{array}{ccccc} p & \rightsquigarrow & R & \hookrightarrow & S \\ & & \downarrow & & \downarrow \\ P_p & \rightsquigarrow & R_p & \hookrightarrow & S_p \end{array}$$

Pick $q_p = N \in \text{Max } S_p$, then $N \cap R_p \in \text{Max } R_p = \{P_p\}$ by 1. of proposition 1.7.2, so $N \cap R_p = P_p$, and $(q \cap R)_p = q_p \cap R_p = P_p$, thus $q \cap R = p$. \square

- If $p_1 \subset p_2$ in $\text{Spec } R$ and $q_1 \in \text{Spec } S$ with $q_1 \cap R = p_1$, then $\exists q_2 \in \text{Spec } S$ with $q_1 \subset q_2$ and $q_2 \cap R = p_2$.

Proof. Let $R' = R/p_1$ and $S' = S/q_1$. Then again, S'/R' is integral. By the previous statement, exists $q_2/q_1 \in \text{Spec } S'$ so that $q_2/q_1 \cap R' = p_2/p_1$, thus $q_2 \cap R = p_2$ and $q_2 \supseteq q_1$. \square

Theorem 18. If S/R is integral, then $\dim S = \dim R$.

Proof. For any chain $q_0 \subsetneq q_1 \subsetneq \dots \subsetneq q_n$ in $\text{Spec } S$, by prop 2., $q_0 \cap R \subsetneq q_1 \cap R \subsetneq \dots \subsetneq q_n \cap R$.

Conversely, given $p_0 \subsetneq p_1 \subsetneq \dots \subsetneq p_n$ in $\text{Spec } R$, there is $q_0 \subsetneq q_1 \subsetneq \dots \subsetneq q_n$ by the going up theorem (17). \square

Prop 1.7.3. Let S, R be integral domains and S/R be integral, assume R is normal with the field of fractions K . If $a \in S$ is integral over $I \subseteq R$, then $f = m_{\alpha, K} = x^n + r_1 x^{n-1} + \dots + r_n$ with $r_i \in \sqrt{I}$.

Proof. Assume $\deg f = n$ and $a_1, \dots, a_n \in \overline{K}$ are the zeros of f . By assumption, $a^m + t_1 a^{m-1} + \dots + t_m = 0$ with $t_i \in I \subset R \subset K$. For each i , exists $\varphi \in \text{Aut}(\overline{K}/K)$ such that $\varphi(a) = a_i$. Then $0 = \varphi(a^m + t_1 a^{m-1} + \dots + t_m) = a_i^m + t_1 a_i^{m-1} + \dots + t_m$, so a_i is integral over I . Moreover, the coefficients of f are the elementary symmetric polynomial of a_i , hence they are integral over I and lie in $\sqrt{IR} = \sqrt{IR} = \sqrt{I}$. \square

Theorem 19 (Going-down theorem). Let S, R be integral domains and S/R be integral, assume R is normal with the field of fractions K . If $p_1 \supset p_2$ in $\text{Spec } R$ and $q_1 \in \text{Spec } S$ with $q_1 \cap R = p_1$, then $\exists q_2 \in \text{Spec } S$ such that $q_1 \supset q_2$ and $q_2 \cap R = p_2$.

Proof. First we claim that $p_2 S_{q_1} \cap R = p_2$.

“ \supseteq ”: Obvious.

“ \subseteq ”: For $b/t \in p_2 S_{q_1} \cap R$, $b \in p_2 S \subset \sqrt{p_2 S} = \sqrt{p_2 R}$, which means that b is integral over p_2 and $t \in S \setminus q_1$. By proposition 1.7.3, if $m_{b, K} = x^l + r_1 x^{l-1} + \dots + r_l$, then $r_i \in \sqrt{p_2} = p_2$.

Now, $a = b/t \in R$, so $t = b/a \in S_{R \setminus \{0\}} = SK$, so

$$\left(\frac{b}{a}\right)^l + \left(\frac{r_1}{a}\right)\left(\frac{b}{a}\right)^{l-1} + \cdots + \left(\frac{r_l}{a}\right) \leftrightarrow b^l + r_1 b^{l-1} + \cdots + r_l = 0$$

is a correspondence. Thus we know that $m_{t,K} = x^l + (r_1/a)x^{l-1} + \cdots + (r_l/a)$.

Again by proposition 1.7.3, since t is integral over R , $u_i \triangleq r_i/a^i \in R$, and $u_i a^i = r_i$ for each i .

If $a \notin p_2$, then $u_i a^i = r_i \in p_2$, so $u_i \in p_2$. But with $m_{t,K}$ we will find that $t^l \in p_2 S \subseteq p_1 S \subseteq q_1$, so $t \in q_1$, which leads to a contradiction. Thus $a \in p_2$.

Now we've proved $p_2 S_{q_1} \cap R = p_2$, by exercise 12.4, $p_2 = Q \cap R$ for some $Q \in S_{q_1}$. Letting $q = Q \cap S$ and we're done. \square

Theorem 20. All maximal chain in $\text{Spec } K[x_1, \dots, x_n]$ have the same length n , and thus

$$\dim K[x_1, \dots, x_n] = n.$$

Proof. Let $P_0 \subset P_1 \subset \cdots \subset P_m$ in $\text{Spec } K[x_1, \dots, x_n]$ We shall use induction on n to prove $m = n$.

$n = 0$: Then $\langle 0 \rangle$ is a max chain in $\text{Spec } K$, so $m = 0 = n$.

$n > 0$: Let $K[y_1, \dots, y_n] \hookrightarrow K[x_1, \dots, x_n]$ be a strong Noether normalization with $P_1 \cap K[y_1, \dots, y_n] = \langle y_{d+1}, \dots, y_n \rangle$, then $h(p_1) = 1 \implies h(p_1 \cap k[y_1, \dots, y_n])$ by the going down theorem (19). (為什麼這樣就證完了???) \square

1.8 Artinian rings and DVR (week 13)

1.8.1 Artinian rings

Def 25. R is called an Artinian ring if one of the followings holds:

- any non-empty set of ideals has a minimal element.
- any descending chain of ideals is stationary (DCC).

Goal:

1. $R \cong R_1 \times \cdots \times R_l$ where R_i is an Artinian local rings.
2. Artinian \iff Noetherian + $\dim = 0$.

Prop 1.8.1.

$$\bullet \sqrt{\mathfrak{m}_i^{n_i} + \mathfrak{m}_j^{n_j}} = \sqrt{\sqrt{\mathfrak{m}_i^{n_i}} + \sqrt{\mathfrak{m}_j^{n_j}}}$$

Proof.

" \subseteq " $\forall a \in \text{LHS}$, that is, $a^n = b + c$ with $b \in \mathfrak{m}_i^{n_i} \subseteq \sqrt{\mathfrak{m}_i^{n_i}}$ and $c \in \mathfrak{m}_j^{n_j} \subseteq \sqrt{\mathfrak{m}_j^{n_j}}$ then $a \in \text{RHS}$.
" \supseteq " $\forall a \in \text{RHS}$, that is, $a^n = b + c$ with $b^k \in \mathfrak{m}_i^{n_i}$ and $c^t \in \mathfrak{m}_j^{n_j}$. Then $(a^n)^{k+t} = (b+c)^{k+t} = b^{k+t} + \cdots + C_t^k b^k c^t + \cdots + c^{k+t}$. Every term either in $\mathfrak{m}_i^{n_i}$ or $\mathfrak{m}_j^{n_j}$, then $(a^n)^{k+t} = c + d$ with $c \in \mathfrak{m}_i^{n_i}$ $d \in \mathfrak{m}_j^{n_j} \Rightarrow a \in \text{LHS}$ \square

- If m is prime, $\sqrt{m^n} = m$

Proof.

" \subseteq " $a \in \text{LHS} \Rightarrow a^k \in m^n$ and m is prime. $\Rightarrow a \in m$.
" \supseteq " $a \in \text{RHS} \Rightarrow a^n \in m \Rightarrow a \in m$. \square

- If $m, m_i, i = 1, \dots, n$ are prime and $m \supseteq m_1 \cap \cdots \cap m_n$, then $m \supseteq m_i$ for some i .

Proof.

Suppose not, then we pick $a_i \in m_i \setminus m$. $b = a_1 \cdots a_n \in m_i \forall i$. $\rightsquigarrow b \in m_1 \cap \cdots \cap m_n \subseteq m$. But, m is prime, exist $a_i \in m$, a contradiction. \square

Prop 1.8.2. Let R be an Artinian ring

- (1) $I \subseteq R \rightsquigarrow R/I$ is also Artinian.
- (2) If R is an integral domain, then R is a field.

Proof. $\forall \begin{smallmatrix} a \\ \neq 0 \end{smallmatrix} \in R, \langle a \rangle \supseteq \langle a^2 \rangle \supseteq \cdots$ is a descending chain of ideals $\implies \langle a^l \rangle = \langle a^{l+1} \rangle = \cdots$ for some $l \in \mathbb{N} \implies a^l = ba^{l+1} \implies a^l(1 - ab) = 0 \implies ab = 1$ since $a^l \neq 0$. \square

- (3) $\text{Spec } R = \text{Max } R$. ($\implies \dim R = 0$)

Proof. $\forall p \in \text{Spec } R, R/p$ is an integral domain $\rightsquigarrow R/p$ is a field $\rightsquigarrow p \in \text{Max } R$. \square

- (4) $|\text{Max } R| < \infty$.

Proof. Consider the set $\left\{ \bigcap_{\text{finite}} \mathfrak{m} \mid \mathfrak{m} \in \text{Max } R \right\} \neq \emptyset$. So there exists a minimal element in this set (R is Artinian), say $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k$. Now, for $\mathfrak{m} \in \text{Max } R$, we have $\mathfrak{m} \cap \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k$ since the latter is minimal $\implies \mathfrak{m} \supseteq \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k \rightsquigarrow \mathfrak{m} \supseteq \mathfrak{m}_i$ for some i , by Prop 1.8.1. $\rightsquigarrow m = m_i$, since m_i is max. So $\text{Max } R = \{\mathfrak{m}_1, \dots, \mathfrak{m}_k\}$. \square

$$(5) \exists n_1, \dots, n_k \in \mathbb{N} \text{ s.t. } \langle 0 \rangle = \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} = \mathfrak{m}_1^{n_1} \cap \mathfrak{m}_2^{n_2} \cap \cdots \cap \mathfrak{m}_k^{n_k}.$$

Proof.

$$\bullet \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} = \mathfrak{m}_1^{n_1} \cap \mathfrak{m}_2^{n_2} \cap \cdots \cap \mathfrak{m}_k^{n_k}.$$

Recall I_i, I_j are coprime for $i \neq j \rightsquigarrow \prod_{i=1}^n I_i = \bigcap_{i=1}^n I_i$. And, by Prop 1.8.1

$$\sqrt{\mathfrak{m}_i^{n_i} + \mathfrak{m}_j^{n_j}} = \sqrt{\sqrt{\mathfrak{m}_i^{n_i}} + \sqrt{\mathfrak{m}_j^{n_j}}} = \sqrt{\mathfrak{m}_i + \mathfrak{m}_j} = \sqrt{R} = R \rightsquigarrow \mathfrak{m}_i^{n_i} + \mathfrak{m}_j^{n_j} = R.$$

$$\bullet \langle 0 \rangle = \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} \text{ for suitable } \{n_i\} \text{ that } \mathfrak{m}_i^{n_i} = \mathfrak{m}_i^{n_i+1}$$

Let $\mathcal{S} = \{J \subseteq R \mid J \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} \neq 0\}$. If $\langle 0 \rangle \neq \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k}$, then $\mathfrak{m}_i \in \mathcal{S}$. $\rightsquigarrow \mathcal{S} \neq \emptyset$. Since R is Artinian, there exists a minimal element $J_0 \in \mathcal{S}$. By definition of \mathcal{S} , $\exists x \in J_0$, $x \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} \neq 0 \rightsquigarrow \langle x \rangle \in \mathcal{S}$ and $\langle x \rangle \subseteq J_0 \Rightarrow \langle x \rangle = J_0$.

Also, $x \mathfrak{m}_1^{n_1+1} \mathfrak{m}_2^{n_2+1} \cdots \mathfrak{m}_k^{n_k+1} = x \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} \neq 0 \rightsquigarrow I = x \mathfrak{m}_1 \cdots \mathfrak{m}_k \in \mathcal{S}$ and $I \subseteq J_0 = xR \rightsquigarrow I = xR$.

$$(\mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_k) xR = xR \rightsquigarrow (\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_k) xR = xR \rightsquigarrow (\text{Jac } R) xR = xR$$

By Nakayama's lemma, $xR = 0 \implies x = 0$, which is a contradiction. \square

(6) The nilradical \mathfrak{n}_R of R is nilpotent.

Proof. By (3), $\mathfrak{n}_R = \text{Jac } R$. Let $n = \max\{n_1, \dots, n_k\}$ in (5), then $\mathfrak{n}_R^n = (\mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_k)^n = 0$. \square

Goal 1: $R \cong R_1 \times R_k$ where R_i is Artinian local ring.

Proof. By Chinese Remainder theorem,

$$R \cong R/\langle 0 \rangle = R/\mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} \cong R/\mathfrak{m}_1^{n_1} \times R/\mathfrak{m}_2^{n_2} \times \cdots \times R/\mathfrak{m}_k^{n_k}$$

Let $R_i = R/\mathfrak{m}_i^{n_i}$, then $\bar{\mathfrak{m}} \in \text{Max } R_i$ if $\mathfrak{m} \in \text{Max } R$ and $\mathfrak{m} \supset \mathfrak{m}_i^{n_i} \rightsquigarrow \mathfrak{m} = \mathfrak{m}_i$. So $\text{Max } R_i = \{\bar{\mathfrak{m}}_i\} \implies R_i$ is a local ring. \square

Lemma 5. Let V be a K -vector space, TFAE

- (1) $\dim_k V < \infty$
- (2) V has DCC on subspaces.
- (3) V has ACC on subspaces.

Proof.

Fact : If $V_1 \subseteq V_2$ is finite dimensional vector space over K , then $V_1 = V_2 \iff \dim_k V_1 = \dim_k V_2$. Otherwise, $\dim_k V_1 < \dim_k V_2$.

(1) \Leftrightarrow (3)

" \Rightarrow " Suppose there exists a chain in vector space V with strictly increasing and infinite length,

$$V_1 \subset V_2 \subset \cdots \subseteq V \Rightarrow \dim_k V_1 < \dim_k V_2 < \cdots \leq \dim_k V$$

Then, $\dim_k V$ must be infinite.

" \Leftarrow " If $\dim_k V$ is infinite, let $S = \{b_1, b_2, \dots\}$ be basis of V .

$$\langle b_1 \rangle_K \subset \langle b_1, b_2 \rangle_K \subset \cdots$$

is a infinite ascending chain.

Similarly, (1) \Leftrightarrow (2). □

Observation: If R is Northrian and $\dim R = 0$, then $\langle 0 \rangle = \bigcap_{i=1}^k q_i$ (primary decomposition) and $\sqrt{q_i} = \mathfrak{m}_i \in \text{Spec } R = \text{Max } R$. Also, $\exists n_i \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} = \langle 0 \rangle$

Since \mathfrak{m}_i is finitely generated, $\exists n_i$ s.t. $\mathfrak{m}_i^{n_i} \subseteq q_i$. Hence

$$\mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} = \mathfrak{m}_1^{n_1} \cap \mathfrak{m}_2^{n_2} \cap \cdots \cap \mathfrak{m}_k^{n_k} \subseteq q_1 \cap q_2 \cap \cdots \cap q_k = \langle 0 \rangle$$

$$\Rightarrow \mathfrak{m}_1^{n_1} \mathfrak{m}_2^{n_2} \cdots \mathfrak{m}_k^{n_k} = \langle 0 \rangle$$

Goal 2: In a ring R , let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be, not necessarily different, maximal ideals in R s.t. $\mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_n = 0$. Then R is Artinian $\iff R$ is Noetherian.

Proof. We have a chain of ideals in R : $\mathfrak{m}_0 = R \supset \mathfrak{m}_1 \supset \mathfrak{m}_1 \mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_n = 0$.

Let $M_i = \mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_{i-1} / \mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_i$ as R -module. Notice that $\mathfrak{m}_i M_i = 0$, we can treat M_i as R/\mathfrak{m}_i -module. But R/\mathfrak{m}_i is a field, so M_i can be regarded as a vector space. Hence, by lemma 5

$$M_i \text{ is Artinian } \iff M_i \text{ is Noetherian.}$$

By definition,

$$0 \rightarrow \mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_i \rightarrow \mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_{i-1} \rightarrow M_i \rightarrow 0$$

By Ex1,

$$\begin{aligned} \mathfrak{m}_0 = R \text{ Artinian} &\iff \mathfrak{m}_1, M_1 \text{ Artinian} \\ &\iff \mathfrak{m}_1 \mathfrak{m}_2, M_1, M_2 \text{ Artinian} \\ &\vdots \\ &\iff \mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_n, M_1, \dots, M_n \text{ Artinian} \\ &\quad \text{is } 0 \\ &\iff \mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_n, M_1, \dots, M_n \text{ Noetherian} \\ &\quad \text{is } 0 \\ &\vdots \\ &\iff \mathfrak{m}_1 \mathfrak{m}_2, M_1, M_2 \text{ Noetherian} \\ &\iff \mathfrak{m}_1, M_1 \text{ Noetherian} \iff \mathfrak{m}_0 = R \text{ Noetherian} \end{aligned}$$

Note: Goal 2 is accomplished by recongnizing that,

- R is Artinian $\Rightarrow \mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_k^{n_k} = \langle 0 \rangle$ by prop 1.8.2 (4).
- R is Noetherian + $\dim 0 \Rightarrow \mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_k^{n_k} = \langle 0 \rangle$ by Observation.

□

1.8.2 DVR (Discrete Valuation Ring)

Def 26.

(1) Let K be a field. A discrete valuation of K is $\nu : K^\times \rightarrow \mathbb{Z}$ ($\nu(0) = \infty$) s.t.

- $\nu(xy) = \nu(x) + \nu(y)$.
- $\nu(x \pm y) = \min\{\nu(x), \nu(y)\}$.

(2) The valuation ring of ν is $R = \{x \in K \mid \nu(x) \geq 0\}$, called a DVR.

• Fact

$$\nu(1) = 0 : \nu(1) = \nu(1) + \nu(1) \implies \nu(1) = 0$$

$$\nu(x) = -\nu(x^{-1}) : 0 = \nu(xx^{-1}) = \nu(x) + \nu(x^{-1})$$

- $\mathfrak{m} = \{x \in R \mid \nu(x) > 0\}$ is the unique maximal ideal in R since $\nu(x) = 0 \iff x$ is a unit.

Proof.

$$" \Rightarrow " : \nu(x) = 0 \rightsquigarrow \nu(x^{-1}) = 0 \rightsquigarrow x^{-1} \in R$$

$$" \Leftarrow " : \nu(x^{-1}), \nu(x) \geq 0. \text{ And, } \nu(x) = -\nu(x^{-1}) \leq 0 \rightsquigarrow \nu(x) = 0 \quad \square$$

- Let $t \in R$ with $\nu(t) = 1$, then $\mathfrak{m} = \langle t \rangle$.

$$\forall x \in \mathfrak{m}, \nu(x) = k > 0. \rightsquigarrow \nu(x(t^k)^{-1}) = \nu(x) - k\nu(t) = 0 \rightsquigarrow x = t^k u, u \text{ is unit in } R.$$

- Let $I \subseteq \mathfrak{m}$ and define $m = \min\{l \in \mathbb{N} \mid x = t^l u \ \forall x \in I\}$. Then $I = \langle t^m \rangle$.

Prop 1.8.3. R is a DVR $\iff R$ is 1-dimensional normal, Noetherian local domain.

Proof.

$$" \Rightarrow ": \text{DVR} \implies \text{PID} \begin{matrix} \searrow \\ \swarrow \end{matrix} \begin{matrix} \text{UFD} \\ \text{Noetherian} \end{matrix} \implies \text{normal}$$

$$\forall P \neq 0 \in \text{Spec } R, P = \langle t^k \rangle = m^k \text{ for some } k \in \mathbb{N} \rightsquigarrow P = \sqrt{P} = \sqrt{m^k} = m \rightsquigarrow P = m \rightsquigarrow \langle 0 \rangle \subset m \rightsquigarrow \dim R = 1.$$

" \Leftarrow ":

- $\mathfrak{m} \neq \mathfrak{m}^2$:

If $\mathfrak{m} = \mathfrak{m}^2$ and $\mathfrak{m} = \text{Jac } R$, then $m = \langle 0 \rangle$ by Nakayama's lemma, which is a contradiction to $\dim R = 1$.

- Let $t \in \mathfrak{m} - \mathfrak{m}^2$ and $\mathfrak{m} = \langle t \rangle$

Consider $M = \mathfrak{m}/\langle t \rangle$ and assume $M \neq 0$

Fact: $I = \text{Ann}(\bar{x}) \ \bar{x} \in M \implies I \in \text{Spec } R$ Since $ab \in I, a, b \notin I$, then $ab\bar{x} = 0$, and $b\bar{x} \neq 0$.

Suppose I is maximal, $\text{Ann}(b\bar{x}) \supseteq \text{Ann}(\bar{x}) \rightsquigarrow \text{Ann}(b\bar{x}) = \text{Ann}(\bar{x})$ Then, $a \in \text{Ann}(b\bar{x}) = \text{Ann}(\bar{x})$, which is a contradiction.

By the Fact, $\exists \bar{x} \neq 0 \in M$ s.t. $\text{Ann}(\bar{x}) = \mathfrak{m} \rightsquigarrow x\mathfrak{m} = \langle t \rangle = tR \rightsquigarrow \frac{x}{t}\mathfrak{m} \subseteq R$.

(1) If $\frac{x}{t} \in R \rightsquigarrow \frac{xy}{t} = 1$ for some $y \in \mathfrak{m} \rightsquigarrow t = xy \in \mathfrak{m}$, which is a contradiction.

(2) If $\frac{x}{t} \in \mathfrak{m}$, let $\mathfrak{m} = \langle y_1, \dots, y_n \rangle_R$ Write $\frac{x}{t}y_i = \sum_{j=1}^l a_{ij}y_j \ \forall i = 1, \dots, n$ By using determinant trick, we have $\frac{x}{t}$ is integral over R , but R is normal $\rightsquigarrow \frac{x}{t} \in R \rightsquigarrow x \in \langle t \rangle \rightsquigarrow \bar{x} = \bar{0}$, which is a contradiction.

Therefore, $\mathfrak{m} = \langle t \rangle$.

- By Ex3, $\bigcap_{n=0}^{\infty} m^n = 0$. Thus, $\forall x \in R, \exists! k$ s.t. $x \in m^k$ and $x \notin m^{k+1}$. $\rightsquigarrow x = t^k u, u$ is units.

- Define $\nu(x) = k$ and $\forall \frac{x}{y} \in \text{Frac } R \nu(\frac{x}{y}) = \nu(x) - \nu(y)$.

(1) $\frac{x}{y} = \frac{x'}{y'} \rightsquigarrow xy' = x'y \rightsquigarrow \nu(xy') = \nu(x'y)$.

$$\begin{aligned}
(2) \quad & \nu\left(\frac{a}{b} \frac{c}{d}\right) = \nu(ac) - \nu(bd) = [\nu(a) - \nu(b)] - [\nu(c) - \nu(d)] = \nu\left(\frac{a}{b}\right) - \nu\left(\frac{c}{d}\right) \\
(3) \quad & \nu\left(\frac{a}{b} + \frac{c}{d}\right), \nu(a) = v_a, \nu(b) = v_b, \nu(c) = v_c, \nu(d) = v_d. \quad \nu\frac{a}{b} + \frac{c}{d} = \min\left\{\nu\left(\frac{a}{b}\right), \nu\left(\frac{c}{d}\right)\right\} = \\
& \min\left\{\nu\left(\frac{ad}{bd}\right), \nu\left(\frac{bc}{bd}\right)\right\} = \nu\left(\frac{ad+bd}{bd}\right).
\end{aligned}$$

Therefore, R is DVR. \square

1.8.3 Dedekind domains

Def 27. A Dedekind domain is a Noetherian normal domain of dim 1.

Def 28. Let R be an integral domain and $K = \text{Frac}(R)$. A nonzero R -submodule I of K is called a fractional ideal of R if $\exists 0 \neq a \in R$ s.t. $aI \subset R$.

Eg 1.8.1. If $I = \langle f_1, \dots, f_n \rangle_R$ with $f_i = \frac{a_i}{b_i} \in K$, then $a = b_1 b_2 \cdots b_n$ and $aI \subset R \implies I$ is fractional.

In general, if R is a Noetherian, then every fractional ideal I of R is finitely generated.

Def 29. A fractional ideal I of R is invertible if $\exists J$: a fractional ideal of R s.t. $IJ = R$.

Prop 1.8.4.

1. If I is invertible, then $J = I^{-1}$ is unique and equals $J = (R : I) \triangleq \{a \in K \mid aI \subset R\}$.

$$\text{Proof. } J \subseteq (R : I) \subseteq (R : I)R \subseteq (R : I)IJ \subseteq RJ = J \rightsquigarrow J = (R : I) \quad \square$$

2. If I is invertible, then I is a finitely generated R -module.

$$\begin{aligned}
\text{Proof. } I(R : I) &= R \rightsquigarrow 1 = \sum_{i=0}^k x_i y_i, \quad x_i \in I \text{ and } y_i \in (R : I). \quad \text{Then, } \forall x \in I, \quad x = \\
& \sum_{i=0}^k \underbrace{(x y_i)}_{\in R} x_i \rightsquigarrow I = \langle x_0, \dots, x_k \rangle_R. \quad \square
\end{aligned}$$

3. Let R be a local domain but not a field, $K = \text{Frac}(R)$. Then R is a DVR \iff every nonzero fractional ideal I of R is invertible.

Proof.

" \Rightarrow ": Let I be fractional ideal of R , then $\exists a \in R \rightsquigarrow aI \subseteq R$. And, $\mathfrak{m} = \langle t \rangle$, since R is not a field $t \neq 0$. $\rightsquigarrow a = t^k u$ where u is a unit in R . If $aI = R$, then $J = \langle a \rangle_R \rightsquigarrow IJ = R$. If not, $aI \subsetneq R \rightsquigarrow aI \subseteq \mathfrak{m} \rightsquigarrow aI = \langle t^l \rangle \rightsquigarrow I = \langle t^{l-k} \rangle_R$. Let $J = \langle t^{k-l} \rangle_R$, then $IJ = R$.

" \Leftarrow ":

- R is Noetherian: $\forall I \subsetneq R$ and I is invertible $\rightsquigarrow I$ is f.g. R -module.
- $\mathfrak{m} \neq \mathfrak{m}^2$, \mathfrak{m} is the unique maximal in R : $\mathfrak{m} = \mathfrak{m}^2$ and $\mathfrak{m} = \text{Jac } R \rightsquigarrow \mathfrak{m} = 0 \rightsquigarrow R$ is a field, which is a contradiction.
- $\mathfrak{m} = \langle t \rangle_R$: Pick $t \in \mathfrak{m} - \mathfrak{m}^2$ and let $\mathfrak{m}\mathfrak{m}^{-1} = R \rightsquigarrow t\mathfrak{m}^{-1} \subseteq R$. If $t\mathfrak{m} \subsetneq$, then $t\mathfrak{m}\mathfrak{m}^{-1} = tR \subseteq \mathfrak{m}^2$, a contradiction. Therefore, $\mathfrak{m} \subset t\mathfrak{m} \rightsquigarrow t\mathfrak{m} = R \rightsquigarrow t\mathfrak{m}\mathfrak{m}^{-1} = tR = \mathfrak{m} \rightsquigarrow \mathfrak{m} = \langle t \rangle_R$.
- Using the same construction ν in prop 1.8.3 we have the R is DVR. \square

Theorem 21. Let R be an integral domain and $K = \text{Frac}(R)$. TFAE

- (a) R is a Dedekind domain.

- (b) R is Noetherian and R_P is a DVR for all $P \in \text{Spec } R$.
- (c) Every nonzero fractional ideal of R is invertible.
- (d) Every nonzero proper ideal of R can be written (uniquely) as a product of powers of prime ideals.

Proof.

(a) \Leftrightarrow (b):

- R is normal $\iff R_P$ is normal for all $P \in \text{Spec } R$.
- $\dim R_P = 1 \quad \forall P \in \text{Spec } R \iff h(P) = 1 \quad \forall 0 \neq P \in \text{Spec } R \iff \dim R = 1$.

(b) \Leftrightarrow (c):

- (b) and (c) $\implies R$ is Noetherian
- $I_P(R : I)_P = I_P(R_P : I_P)$ (Hint: I is f.g.)
- R_P is DVR $\forall P \in \text{Spec } R \iff$ Every nonzero fractional ideal of R is invertible.

$$\forall P \in \text{Spec } R \quad R_P = I_P(R_P : I_P) = I_P(R : I)_P = I(R : I)_P \iff R = I(R : I)$$

(a)(b)(c) \Rightarrow (d):

Existence:

- $I = q_1 \cap \cdots \cap q_n$ and $\sqrt{q_i} = P_i \in R$ by primary decomposition thm.
- $q_1 \cap \cdots \cap q_n = q_1 \cdots q_n$
Since $\dim R = 1$, $P_i \in \text{Max } R$. And, R is Noetherian, $\exists n_i \in \mathbb{N}$ s.t. $m_i^{n_i} \subseteq q_i$. Then, $m_i^{n_i} + m_j^{n_j} = R \forall i \neq j \rightsquigarrow q_i + q_j = R \rightsquigarrow q_1 \cap \cdots \cap q_n = q_1 \cdots q_n$
- $I = m_i^{r_i} \cdots m_n^{r_n}$
Since R_{m_i} is DVR $\rightsquigarrow (q_i)_{m_i} = (m_i^{r_i})_{m_i} \rightsquigarrow q_i = m_i^{r_i}$ by prime ideals have 1-1 correspondence in localization. Therefore, $I = m_i^{r_i} \cdots m_n^{r_n}$.

Uniqueness:

- $P_1 \cdots P_k = Q_1 \cdots Q_r$ P_i, Q_i is prime. Then, $P_1 \cdots P_k \subseteq Q_1 \rightsquigarrow P_i \subseteq Q_1$, say $i = 1$ by prop 1.8.1.
Since Q_1 is invertible, then $P_2 \cdots P_k = Q_2 \cdots Q_r$. By induction by hypothesis, we have the uniqueness result.

(d) \Rightarrow (c):

- Every invertible prime is maximal:
If not, let $p + aR = P_1 \cdots P_k$ and $p + a^2R = Q_1 \cdots Q_r$. $\rightsquigarrow p \subseteq P_i$ and Q_j
Claim $(p + aR)^2 = (p + a^2R)$:
In R/p , $\langle \bar{a} \rangle = (P_1/p) \cdots (P_k/p)$ and $\langle \bar{a}^2 \rangle = (Q_1/p) \cdots (Q_r/p)$. And, $\langle \bar{a} \rangle = (P_1/p)^2 \cdots (P_k/p)^2 = (Q_1/p) \cdots (Q_r/p)$
- $P = P^2 + aP$

$$P \subseteq P + a^2R = (P + aR)^2 \subseteq P^2 + aR$$

Then,

$$\forall x \in P, \quad x = \underset{\in P}{y} + a \underset{\in R}{z} \rightsquigarrow z \in P$$

Therefore, $P \subseteq P^2 + aP \rightsquigarrow P = P^2 + aP$. By invertibility of P , we have $R = P + aR$, which is a contradiction.

- Every nonzero prime is invertible:
Let $0 \neq a \in P$, and $P \supseteq \langle a \rangle$ is invertible. $\langle a \rangle = P_1 \cdots P_n$ and P_i is invertible. And, $P \subseteq P_i \rightsquigarrow P = P_i$ since P_i is maximal.

- \forall ideal $0 \neq I \subseteq R$, $I = P_1 \cdots P_m \rightsquigarrow I$ is invertible.
- If I is fractional ideal of R , say $aI \subseteq R \rightsquigarrow \exists J$ ideal in $R \rightsquigarrow aIJ = R \rightsquigarrow I(aJ) = R$. I is invertible.

□

2 Introduction to Homological Algebra

2.1 Projective, Injective and Flat modules (week 14)

Def 30.

- $M \in \mathbf{Mod}_R$ is **projective** if $\text{Hom}(M, \cdot)$ preserves the *right* exactness.
- $N \in \mathbf{Mod}_R$ is **injective** if $\text{Hom}(\cdot, N)$ preserves the *right* exactness.
- $M \in \mathbf{Mod}_R$ is **flat** if $M \otimes \cdot$ preserves the *left* exactness.

Fact 2.1.1.

- M is projective \iff

$$\begin{array}{ccc} & M & \\ \exists \tilde{f} \swarrow & \downarrow f & \\ M_2 & \longrightarrow & M_3 \longrightarrow 0 \\ 0 \longrightarrow M_1 & \longrightarrow & M_2 \\ & \downarrow g & \nwarrow \exists \tilde{g} \\ & N & \end{array}$$
- N is injective \iff
- free \implies projective: If $X = \{x_i \mid i \in \Lambda\}$ and $f : x_i \mapsto a_i$. Since β onto, exists b_i so that $\beta(b_i) = a_i$. we can then set $\tilde{f} : x_i \mapsto b_i$ by the universal property of free module.

$$\begin{array}{ccc} & F(X) & \\ \exists \tilde{f} \swarrow & \downarrow f & \\ M_2 & \xrightarrow{\beta} & M_3 \longrightarrow 0 \end{array}$$

- free \implies flat: Let $F \cong R^{\oplus \Lambda}$ be a free module, and M_1, M_2 be two modules such that $0 \rightarrow M_1 \rightarrow M_2$. Since $R \otimes_R M \cong M$, we have

$$\begin{aligned} 0 \rightarrow M_1 \rightarrow M_2 & \quad \text{exact} \\ \implies 0 \rightarrow R \otimes M_1 \rightarrow R \otimes M_2 & \quad \text{exact} \\ \implies 0 \rightarrow \bigoplus_{i \in \Lambda} R \otimes M_1 \rightarrow \bigoplus_{i \in \Lambda} R \otimes M_2 & \quad \text{exact} \\ \stackrel{(a)}{\implies} 0 \rightarrow R^{\oplus \Lambda} \otimes M_1 \rightarrow R^{\oplus \Lambda} \otimes M_2 & \quad \text{exact} \\ \implies 0 \rightarrow F \otimes M_1 \rightarrow F \otimes M_2 & \quad \text{exact} \end{aligned}$$

Where (a) is by the fact that $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$. Thus F flat.

- If S is a multiplication closed set in R with $1 \in S$, then

$$0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0 \implies 0 \rightarrow M_S \rightarrow N_S \rightarrow L_S \rightarrow 0.$$

We know that $M_S \cong R_S \otimes_R M$. So R_S is a flat R -module. e.g. \mathbb{Q} is a flat \mathbb{Z} -module.

For any $M \in \mathbf{Mod}_R$, a projective module N such that $N \rightarrow M \rightarrow 0$ could be easily found: Simply let $N = F$, a free module on the generating set of M .

Now we shall ask for any module M , does there exist $N \in \mathbf{Mod}_R$ such that N is injective and $0 \rightarrow M \rightarrow N$?

Theorem 22 (Baer's criterion). N is injective $\iff \forall I \subset R$, and a homomorphism f , there exists a homomorphism h such that the following diagram commutes:

$$\begin{array}{ccc} 0 \longrightarrow & I & \longrightarrow R \\ & \downarrow f & \nearrow \exists h \\ & N & \end{array}$$

Proof. “ \Rightarrow ”: See I as an R module, then it is immediate by the definition of injective module.

“ \Leftarrow ”: Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 \\ & & \downarrow g & & \\ & & N & & \end{array}$$

Let $S \triangleq \{(M, \rho) \mid M_1 \subseteq M \subseteq M_2 \text{ and } \rho : M \rightarrow N \text{ extends } g\} \neq \emptyset$ since $(M_1, g) \in S$.

By the routinely proof using Zorn's lemma, exists a maximal element $(M^*, \mu) \in S$.

We claim that $M^* = M_2$. If not, pick $a \in M_2 \setminus M^*$ and let $M' \triangleq M^* + Ra \subsetneq M^*$, $I \triangleq \{r \in R \mid ra \in M^*\}$. Define $f : I \rightarrow N$ with $r \mapsto \mu(ra)$. Then we have an extension $h : R \rightarrow N$ of f .

Now, let $\mu' : M' \rightarrow N = x + ra \mapsto \mu(x) + h(r)$. We shall prove that this map is well-defined: If $x_1 + r_1a = x_2 + r_2a$, then $(r_1 - r_2)a = x_2 - x_1 \in M \implies r_1 - r_2 \in I$. So $h(r_1) - h(r_2) = f(r_1 - r_2) = \mu((r_1 - r_2)a) = \mu(x_2) - \mu(x_1)$, which prove μ' is well defined, and the existence of μ' contradicts the fact that (M^*, μ) is maximal. \square

Def 31. M is **divisible** if $\forall x \in M, r \in R \setminus \{0\}$, there exists $y \in M$ such that $x = ry$, i.e. $rM = M \quad \forall r \in R \setminus \{0\}$.

Prop 2.1.1.

1. Every injective module N over an integral domain is divisible.

Proof. For any $x_0 \in N$ and $r_0 \in R \setminus \{0\}$. Let $I = \langle r_0 \rangle \subset R$. As long as R is an integral domain, $I \cong R$ as an R -module, so the R -module homomorphism $f : I \rightarrow N = rr_0 \mapsto rx_0$ is well-defined. Since N injective, this map extends to $h : R \rightarrow N$. Let $y_0 \triangleq h(1)$, then $r_0y_0 = r_0h(1) = h(r_0) = x_0$. Thus N injective. \square

2. Every divisible module N over an PID is injective.

Proof. For any $I \subseteq R$ and a homomorphism $f : I \rightarrow N$, if $I = 0$ then $h = x \mapsto 0$ is always an extension of f . So assume $I \neq 0$. Since R is a PID, $I = \langle r_0 \rangle$ for some $r_0 \neq 0 \in R$. By the fact that N divisible, exists $y_0 \in N$ such that $r_0y_0 = x_0 \triangleq f(r_0)$.

Now we could define $h : R \rightarrow N$ by $1 \mapsto y_0$. Then $h(r_0) = r_0h(1) = r_0y_0 = x_0$, thus h is an extension of f and N injective. \square

3. If R is a PID, then any quotient N of a injective R -module M is injective.

Proof. By 2., $rM = M$ for any $r \neq 0$, thus $rN = N$ for any $r \neq 0$, and hence N injective. \square

Theorem 23. For any $M \in \mathbf{Mod}_R$, there exists an injective module N containing M .

Proof.

Case 1: $R = \mathbb{Z}$.

Let $X = \{x_i\}_{i \in \Lambda}$ be a generating set for M and F is free on X . Let f be the natural map from F to M . then $M \cong F/\ker f$.

Define $F' = \bigoplus_{i \in \Lambda} \mathbb{Q}e_i \supset F$, which is obviously a divisible \mathbb{Z} -module. Then $M \subseteq F'/\ker f \triangleq M'$, where M' is injective by proposition 2.1.1.

Case 2: R arbitrary.

We can regard any M as a \mathbb{Z} -module, then there exists an injective module $N_0 \supset M$. Now, we have an R -module $N \triangleq \text{Hom}_{\mathbb{Z}}(R, N_0)$ with multiplication $rf \triangleq x \mapsto f(xr)$.

We claim that N is injective. For any $f : M_1 \rightarrow N$, and a homomorphism $\alpha : M_1 \rightarrow M_2$, first we can regard α as a \mathbb{Z} -module homomorphism, then we define $f' : M_1 \rightarrow N_0$ as $x \mapsto f(x)(1)$. Since N_0 injective (in $\mathbf{Mod}_{\mathbb{Z}}$), there exists a \mathbb{Z} -module homomorphism h' from M_2 to N_0 .

$$\begin{array}{ccc}
 0 \longrightarrow M_1 & \xrightarrow{\alpha} & M_2 \\
 \downarrow f & \nearrow \exists h(?) & \downarrow f' \\
 N & \xleftarrow{\cong} & \text{Hom}_{\mathbb{Z}}(R, N_0) \\
 \downarrow \psi & & \downarrow \psi \\
 f(x) & & f(x)(1)
 \end{array}$$

Now, define

$$\begin{aligned}
 h : M_2 &\longrightarrow N \\
 y &\longmapsto h(y) : R \longrightarrow N_0 \\
 1 &\longmapsto h'(y) \\
 r &\longmapsto h'(ry)
 \end{aligned}$$

We check that h is well-defined.

- $h(y) \in \text{Hom}_{\mathbb{Z}}(R, N_0)$

$$h(y)(r_1 + r_2) = h'((r_1 + r_2)y) = h'(r_1y + r_2y) = h'(r_1y) + h'(r_2y) = h(y)(r_1) + h(y)(r_2)$$

- $h \in \text{Hom}_R(M_2, N)$

$$\begin{aligned}
 h(r_1y_1 + y_2)(r) &= h'(r(r_1y_1 + y_2)) = h'(rr_1y_1 + ry_2) \\
 &= h'(rr_1y_1) + h'(ry_2) \\
 &= h(y)(rr_1) + h(y_2)(r) \\
 &= (r_1h(y))(r) + h(y_2)(r)
 \end{aligned}$$

- Show diagram commute $f = h \circ \alpha$. Fix $y \in M_1$, then $\forall r \in R$:

$$\begin{aligned}
 (h \circ \alpha)(y)(r) &= h(\alpha(y))(r) = h'(r\alpha(y)) \\
 &= h'(\alpha(ry)) = f'(ry) \\
 &= f(ry)(1) = rf(y)(1) \\
 &= f(y)(r)
 \end{aligned}$$

Thus N injective.

Now, notice that $\text{Hom}_{\mathbb{Z}}(R, \cdot)$ is a left exact functor, so $M \hookrightarrow N_0$ implies $\text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, N_0)$, thus $M \cong \text{Hom}_R(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, N_0) = N$. \square

Prop 2.1.2. TFAE

1. M is projective.
2. Every exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$ split.

3. $\exists M'$ s.t. $M \oplus M' \cong F$: free.

Proof.

(1) \Rightarrow (2) : Since M projective, the map λ with $\beta \circ \lambda = \text{Id}$ exists in the following diagram:

$$\begin{array}{ccc} & M & \\ \swarrow \exists \lambda & \downarrow \text{Id} & \\ M_2 & \xrightarrow{\beta} & M \longrightarrow 0 \end{array}$$

Then λ is a lifting, so $M_2 \cong M_1 \oplus M$ and $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$ split.

(2) \Rightarrow (3): Let F be a free module on a generating set of M , and $\beta :: F \rightarrow M$ be the natural map, then $0 \rightarrow \ker \beta \rightarrow F \rightarrow M \rightarrow 0$ split, so $F \cong \ker \beta \oplus M$.

(3) \Rightarrow (1): For any $M_2 \rightarrow M_3 \rightarrow 0$, since $M' \oplus M$ free and thus projective, λ' exists in the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \longrightarrow & M' \oplus M & \xleftarrow[\pi]{\mu} & M \longrightarrow 0 \\ & & & & \downarrow \exists \lambda' & & \downarrow f \\ & & & & M_2 & \xrightarrow{\beta} & M_3 \longrightarrow 0 \end{array}$$

Define $\lambda = \lambda' \circ \mu$. Then $\beta \circ \lambda = \beta \circ \lambda' \circ \mu = f \circ \pi \circ \mu = f$. □

Prop 2.1.3. TFAE

1. M is injective.
2. Each exact sequence $0 \rightarrow M \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ split.

Proof. (1) \Rightarrow (2): Similar to the projective case, μ exists in the following diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & M \xrightarrow{\alpha} M_2 \\ & & \downarrow \text{Id} \\ & & M \end{array} \quad \begin{array}{c} \swarrow \exists \mu \\ \end{array}$$

So $M_2 = M \oplus M_3$.

(2) \Rightarrow (1): By theorem 23, there is a module $N \subset M$ so that N is injective.

Consider $0 \longrightarrow M \xrightleftharpoons[\exists \mu]{i} N \longrightarrow \text{coker } i \longrightarrow 0$ split exact and $\mu \circ i = \text{Id}_M$. Since N injective, h' exists in the following diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & M_1 \xrightarrow{\alpha} M_2 \\ & & \downarrow f \\ & & M \\ & \nearrow i \circ f & \uparrow \mu \\ & & N \end{array} \quad \begin{array}{c} \swarrow \exists h' \\ \end{array}$$

Let $h = \mu \circ h'$, then $h \circ \alpha = \mu \circ h' \circ \alpha = \mu \circ i \circ f = f$. □

Prop 2.1.4. projective \implies flat.

Proof. Observe that $\bigoplus_{i \in \Lambda} M_i$ is flat if and only if M_i is flat for each i , since if $0 \rightarrow N_1 \xrightarrow{\alpha} N_2$ exact, then

$$\begin{array}{ccc} 0 & \longrightarrow & (\bigoplus M_i) \otimes N_1 \xrightarrow{1 \otimes \alpha} (\bigoplus M_i) \otimes N_2 \\ & & \parallel \\ 0 & \longrightarrow & \bigoplus (M_i \otimes N_1) \xrightarrow{\bigoplus (1 \otimes \alpha)} \bigoplus (M_i \otimes N_2) \\ & & \parallel \\ 0 & \longrightarrow & M_i \otimes N_1 \xrightarrow{1 \otimes \alpha} M_i \otimes N_2 \quad \forall i \in \Lambda \end{array}$$

If M is projective, then by proposition 2.1.2 $\exists M'$ such that $M \oplus M' \cong F$ is free. Since free implies flat, by above, M is flat. \square

Def 32.

- A chain complex C_\bullet of R -modules is a sequence and maps:

$$C_\bullet : \cdots \rightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \rightarrow \cdots \rightarrow C_1 \xrightarrow{d_1} C_0 \rightarrow 0$$

with $d_n \circ d_{n+1} = 0$, $\forall n$. (i.e. $\text{Im } d_{n+1} \subseteq \ker d_n$)

Then define

- $Z_n(C_\bullet) \triangleq \ker d_n$ is the n -cycle.
- $B_n(C_\bullet) \triangleq \text{Im } d_{n+1}$ is the n -boundary.
- $H_n(C_\bullet) \triangleq Z_n(C_\bullet)/B_n(C_\bullet)$ is called the n -th homology.

- A cochain complex C^\bullet of R -modules is a sequence and maps:

$$C^\bullet : 0 \rightarrow C^0 \xrightarrow{d^1} C^1 \rightarrow \cdots \rightarrow C^{n-1} \xrightarrow{d^n} C^n \xrightarrow{d^{n+1}} C^{n+1} \rightarrow \cdots$$

with $d^{n+1} \circ d^n = 0$, $\forall n$. (i.e. $\text{Im } d^n \subseteq \ker d^{n+1}$)

Then define

- $Z^n(C^\bullet) \triangleq \ker d^{n+1}$ is the n -cocycle.
- $B^n(C^\bullet) \triangleq \text{Im } d^n$ is the n -coboundary.
- $H^n(C^\bullet) \triangleq Z^n(C^\bullet)/B^n(C^\bullet)$ is called the n -th cohomology.

- $\varphi : C_\bullet \rightarrow \tilde{C}_\bullet$ is a chain map if the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} \longrightarrow \cdots \\ & & \downarrow \varphi_{n+1} & & \downarrow \varphi_n & & \downarrow \varphi_{n-1} \\ \cdots & \longrightarrow & \tilde{C}_{n+1} & \xrightarrow{\tilde{d}_{n+1}} & \tilde{C}_n & \xrightarrow{\tilde{d}_n} & \tilde{C}_{n-1} \longrightarrow \cdots \end{array}$$

Observe that $\varphi_n(\ker d_n) \subseteq \ker \tilde{d}_n$ and $\varphi_n(\text{Im } d_{n+1}) \subseteq \text{Im } \tilde{d}_{n+1}$. This will induce the following maps:

$$\begin{aligned} \varphi_* : H_n(C_\bullet) &\rightarrow H_n(\tilde{C}_\bullet) \\ x + B_n(C_\bullet) &\mapsto \varphi_n(x) + B_n(\tilde{C}_\bullet) \end{aligned}$$

- $f : C_\bullet \rightarrow \tilde{C}_\bullet$ is null homotopic if $\exists s_n : C_n \rightarrow \tilde{C}_{n+1}$ s.t. $f_n = \tilde{d}_{n+1} \circ s_n + s_{n-1} \circ d_n$, $\forall n$.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} \longrightarrow \cdots \\ & & \downarrow f_{n+1} & \nearrow s_n & \downarrow f_n & \nearrow s_{n-1} & \downarrow f_{n-1} \\ \cdots & \longrightarrow & \tilde{C}_{n+1} & \xrightarrow{\tilde{d}_{n+1}} & \tilde{C}_n & \xrightarrow{\tilde{d}_n} & \tilde{C}_{n-1} \longrightarrow \cdots \end{array}$$

Prop 2.1.5. If f is null homotopic, then $f_* = 0$.

Proof. $f_*(x) = \tilde{d}_{n+1}s_n(x) + s_{n-1}d_n(x) = \tilde{d}_{n+1}s_n(x) \in B_n(\tilde{C}_\bullet) \implies f_*(\bar{x}) = 0$. \square

- Two chain map $f, g : C_\bullet \rightarrow \tilde{C}_\bullet$ are homotopic if $f - g$ is null homotopic. ($f_* = g_*$)
- Let $M \in \mathbf{Mod}_R$. A projection resolution of M is an exact sequence:

$$\cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\alpha} M \rightarrow 0$$

where P_i is projective for all i .

For any M , projection resolution always exists. Let P_0 be a free module on the generators of M . We get $P_0 \xrightarrow{\alpha} M \rightarrow 0$. Similarly, let P_1 be free on $\ker \alpha$, then we could extend the map to $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$. Continue the process we would get a diagram as below, where K_i are the kernels:

$$\begin{array}{ccccccc} \cdots & \rightarrow & P_2 & \xrightarrow{\quad} & P_1 & \xrightarrow{\quad} & P_0 \rightarrow M \rightarrow 0 \\ & & \searrow & & \swarrow & & \swarrow \\ & & & K_1 & & & K_0 \\ & \nearrow & & \searrow & \nearrow & & \searrow \\ 0 & & & & 0 & & 0 \end{array}$$

Theorem 24 (Comparison theorem). Given two chain as following:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 \xrightarrow{\alpha} M \longrightarrow 0 & \text{(projective resolution)} \\ & & \downarrow \exists f_2 & & \downarrow \exists f_1 & & \downarrow \exists f_0 & \downarrow f \\ \cdots & \longrightarrow & \tilde{C}_2 & \xrightarrow{d'_2} & \tilde{C}_1 & \xrightarrow{d'_1} & \tilde{C}_0 \xrightarrow{\alpha'} N \longrightarrow 0 & \text{(exact sequence)} \end{array}$$

Then $\exists f_i : P_i \rightarrow C_i$ s.t. $\{f_i\}$ forms a chain map making the completed diagram commutes. And any two such chain maps are homotopic.

Proof. Using induction on n .

For $n = 0$, the existence of f_0 is guaranteed by the definition of projective module.

$$\begin{array}{ccc} & P_0 & \\ \exists f_0 \swarrow & \downarrow f \circ \alpha & \\ C_0 & \longrightarrow & N \longrightarrow 0 \end{array}$$

For $n > 0$, we claim that $f_{n-1}d_n(P_n) \subseteq \text{Im } d'_n$, since $d'_{n-1}f_{n-1}d_n(x) = f_{n-2}d_{n-1}d_n(x) = 0$ and by the fact that C is exact, $f_{n-1}d_n(x) \in \ker d'_{n-1} = \text{Im } d'_n$. So using the diagram and again by the definition of projective module, f_n exists.

$$\begin{array}{ccc} & P_n & \\ \exists f_n \swarrow & \downarrow f_{n-1} \circ d_n & \\ C_n & \longrightarrow & \text{Im } d'_n \longrightarrow 0 \end{array}$$

Now, for another chain map $\{g_i : P_i \rightarrow C_i\}$, we shall construct suitable $\{s_n\}$ to prove they are homotopic. For $s_{-1} : M \rightarrow C_0$ we could simply pick the zero map. Again, if we could prove that $\text{Im}(g_n - f_n - s_{n-1}d_n) \subseteq \ker d'_n$, then by the definition of projective module, we would obtain s_n with

$$d_{n+1}s_n = g_n - f_n - s_{n-1}d_n \implies g_n - f_n = d_{n+1}s_n + s_{n-1}d_n$$

immediately. For this we calculate $d'_n(g_n - f_n - s_{n-1}d_n) = g_{n-1}d_n - f_{n-1}d_n - d'_n s_{n-1}d_n$. Notice that $d'_n s_{n-1} = g_{n-1} - f_{n-1} - s_{n-2}d_{n-1}$, and with $d_{n-1}d_n = 0$, we get $d'_n(g_n - f_n - s_{n-1}d_n) = 0$. \square

Def 33. Let $M \in \mathbf{Mod}_R$ and $\cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\alpha} M \rightarrow 0$ be a projective resolution of M . Fix $N \in \mathbf{Mod}_R$. Applying $\text{Hom}_R(\cdot, N)$ will get a complex:

$$0 \rightarrow \text{Hom}_R(M, N) \xrightarrow{\bar{\alpha}} \text{Hom}_R(P_0, N) \xrightarrow{\bar{d}_1} \text{Hom}_R(P_1, N) \rightarrow \cdots$$

Define

- $\text{Ext}_R^0(M, N) = \ker \bar{d}_1 = \text{Im } \bar{\alpha} \cong \text{Hom}_R(M, N)$.
- $\text{Ext}_R^n(M, N) = H^n(\text{Hom}(P_\bullet, N))$, $\forall n \geq 1$.

Theorem 25 (Indenpedency of the choice of projective resolutions). $\text{Ext}^n(M, N)$ is independent of the choice of the projective resolution used.

Proof. First, consider two projective resolutions of M, \tilde{M} , and map $f : M \rightarrow \tilde{M}$, and two liftings $\{f_i\}, \{g_i\}$. Use $\bar{\cdot}$ to denote the natural transformation from $X \rightarrow Y$ to $\text{Hom}(Y, N) \rightarrow \text{Hom}(X, N)$ by $\bar{f} \triangleq g \mapsto g \circ f$. Then we shall prove that $\bar{f}_\bullet^* = \bar{g}_\bullet^*$, which is to say \bar{f}_\bullet^* is independent of the lifting used.

By comparison theorem (24), $\{f_i\}, \{g_i\}$ are homotopic, and we could write down the diagram below:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\alpha} & M & \longrightarrow & 0 \\ & & f_2 \downarrow & g_2 & \swarrow s_1 & f_1 \downarrow & g_1 & \swarrow s_0 & f_0 \downarrow & g_0 & \\ \cdots & \longrightarrow & \tilde{P}_2 & \xrightarrow{\tilde{d}_2} & \tilde{P}_1 & \xrightarrow{\tilde{d}_1} & \tilde{P}_0 & \xrightarrow{\tilde{\alpha}} & \tilde{M} & \longrightarrow & 0 \end{array}$$

Notice that $\bar{\cdot}$ act linearly, that is, $\overline{f+g} = \bar{f} + \bar{g}$, and $\overline{fg} = \bar{g}\bar{f}$. So we have

$$g_n - f_n = s_{n-1}d_n + \tilde{d}_{n+1}s_n \implies \bar{g}_n - \bar{f}_n = \bar{d}_n \bar{s}_{n-1} + \bar{s}_n \bar{d}_{n+1}$$

and \bar{f}_n, \bar{g}_n are homotopic. Thus by proposition 2.1.5, $\bar{f}_\bullet^* = \bar{g}_\bullet^*$.

Now, let P_\bullet, P'_\bullet be two projective resolutions. Consider the diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M \longrightarrow 0 \\ & & \text{Id} \downarrow & f_1 & \text{Id} \downarrow & f_0 & \downarrow \text{Id} \\ \cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow g_1 & & \downarrow g_0 & & \downarrow \text{Id} \\ \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M \longrightarrow 0 \end{array}$$

Then $g_i \circ f_i$ and Id are two liftings, and thus by previous we have $\bar{g}_i^* \circ \bar{f}_i^* = \text{Id}^*$. By symmetry, $\bar{f}_i^* \circ \bar{g}_i^* = \text{Id}^*$, which means that the homology calculated using different resolution are isomorphic. \square

Theorem 26 (Horseshoe Lemma). Given $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ and projective resolutions $P_\bullet \rightarrow L \rightarrow 0, \tilde{P}_\bullet \rightarrow N \rightarrow 0$. Then there is a projective resolution for M such that the following

diagram commutes:

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & P_1 & \longrightarrow & \bar{P}_1 & \longrightarrow & \tilde{P}_1 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_0 & \longrightarrow & \bar{P}_0 & \longrightarrow & \tilde{P}_0 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

Proof. Let $\bar{P}_n \triangleq P_n \oplus \tilde{P}_n$. \bar{P}_n is projective by the fact that direct sum of projective modules are projective. Also $0 \rightarrow P_n \rightarrow P_n \oplus \tilde{P}_n \rightarrow \tilde{P}_n \rightarrow 0$ by injection and projection. It remains to show that the maps in the middle column exists.

Consider the following diagram:

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus \tilde{P}_0 & \longrightarrow & \tilde{P}_0 \longrightarrow 0 \\
& & \downarrow \alpha & & \downarrow \bar{\alpha} & \nearrow \exists \sigma & \downarrow \tilde{\alpha} \\
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

σ exists because \tilde{P}_0 is projective. Define

$$\begin{aligned}
\bar{\alpha} : P_0 \oplus \tilde{P}_0 &\longrightarrow M \\
(z, y) &\longmapsto f \circ \alpha(z) + \sigma(y)
\end{aligned}$$

It easy to see that $\bar{\alpha}$ let the diagram commutes. So we show that $P_0 \oplus \tilde{P}_0 \xrightarrow{\bar{\alpha}} M \rightarrow 0$:

For any $x \in M$, consider $g(x) \in N$. Since $\tilde{P}_0 \xrightarrow{\tilde{\alpha}} N \rightarrow 0$, there exists $y \in \tilde{P}_0$ such that $\tilde{\alpha}(y) = g(x) \implies g \circ \sigma(y) = g(x)$. Then $x - \sigma(y) \in \ker g = \text{Im } f$, so there exists $w \in L$ such that $f(w) + \sigma(y) = x$. Now, since $P_0 \xrightarrow{\alpha} L \rightarrow 0$, there exists $z \in P_0$ such that $\alpha(z) = w$. Then we have $\bar{\alpha}(z, y) = x$. So $P_0 \oplus \tilde{P}_0 \xrightarrow{\bar{\alpha}} M \rightarrow 0$.

By induction on n , but we use $\ker d_{n-1}, \ker \bar{d}_{n-1}, \ker \tilde{d}_{n-1}$ to replace L, M, N ($d_{-1} = \alpha$ and so on). Then we are done. \square

Theorem 27 (Long exact sequence for Ext). If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ exact, then there is a long exact sequence:

$$\begin{aligned}
0 \rightarrow \text{Hom}(N, K) &\rightarrow \text{Hom}(M, K) \rightarrow \text{Hom}(L, K) \\
&\rightarrow \text{Ext}^1(N, K) \rightarrow \text{Ext}^1(M, K) \rightarrow \text{Ext}^1(L, K) \rightarrow \text{Ext}^2(N, K) \rightarrow \dots
\end{aligned}$$

Proof. Taking $\text{Hom}(-, K)$ in the diagram of Horseshoe' lemma (26) and delete the first row, we get

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 0 & \longleftarrow & \text{Hom}(P_1, K) & \longleftarrow & \text{Hom}(\bar{P}_1, K) & \longleftarrow & \text{Hom}(\tilde{P}_1, K) \longleftarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 0 & \longleftarrow & \text{Hom}(P_0, K) & \longleftarrow & \text{Hom}(\bar{P}_0, K) & \longleftarrow & \text{Hom}(\tilde{P}_0, K) \longleftarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

Notice that $\text{Hom}(M \oplus N, K) \cong \text{Hom}(M, K) \oplus \text{Hom}(N, K)$, so each row is indeed exact.

By exercise 14.7, the long exact sequence in the statement exists. (one can check the kernels of the first row are indeed $\text{Hom}(N, K), \text{Hom}(M, K), \text{Hom}(L, K)$.) \square

2.2 Ext and Tor (week 15)

Given $M, N \in \mathbf{Mod}_R$, there are two ways to define $\text{Ext}^n(M, N)$:

Def 34 (Ext functor).

- Find any projective resolution $P_\bullet \xrightarrow{\alpha} M \rightarrow 0$, and let $P_M : P_\bullet \rightarrow 0$ (called a *deleted resolution*). We can define $\text{Ext}_{\text{proj}}^n(M, N) = H^n(\text{Hom}(P_M, N))$.
- Find any injective resolution $0 \xrightarrow{\alpha} N \rightarrow E^\bullet$, and let $E_N : 0 \rightarrow E^\bullet$. We can define $\text{Ext}_{\text{inj}}^n(M, N) = H^n(\text{Hom}(M, E_N))$.

Prop 2.2.1. $\text{Ext}_{\text{proj}}^0(M, N) \cong \text{Ext}_{\text{inj}}^0(M, N) \cong \text{Hom}(M, N)$.

Proof.

$$\text{Hom}(P_M, N) : 0 \xrightarrow{\bar{d}_0} \text{Hom}(P_0, N) \xrightarrow{\bar{d}_1} \text{Hom}(P_1, N) \rightarrow \dots$$

$$\text{so } \text{Ext}_{\text{proj}}^0(M, N) = \ker \bar{d}_1 / \text{im } \bar{d}_0 = \ker \bar{d}_1 = \text{im } \alpha = \text{Hom}(M, N). \quad \square$$

Similarly, $\text{Ext}_{\text{inj}}^0(M, N) = \text{Hom}(M, N)$.

Lemma 6.

- If M is projective, then $\text{Ext}_{\text{proj}}^n(M, N) = 0$ for all $n > 0, N \in \mathbf{Mod}_R$.
- If N is injective, then $\text{Ext}_{\text{inj}}^n(M, N) = 0$ for all $n > 0, M \in \mathbf{Mod}_R$.

Proof. If M is projective, then $0 \rightarrow M \xrightarrow{1_M} M \rightarrow 0$ is a projective resolution of M . Its deleted resolution is then $P_M : 0 \rightarrow M \rightarrow 0$. Hence for $n > 0$, $\text{Ext}_{\text{proj}}^n(M, N) = H^n(\text{Hom}(P_M, N)) = 0$.

The argument applies similarly to injective case. \square

Theorem 28 (Equivalence of Ext_{proj} and Ext_{inj}).

$$\text{Ext}_{\text{proj}}^n(M, N) \cong \text{Ext}_{\text{inj}}^n(M, N).$$

Proof. Let $P_\bullet \rightarrow M \rightarrow 0$ and $0 \rightarrow N \rightarrow E^\bullet$ be projective and injective resolutions, then we have $0 \rightarrow K_0 \rightarrow P_0 \rightarrow M \rightarrow 0$ and $0 \rightarrow N \rightarrow E^0 \rightarrow L^1 \rightarrow 0$ exact.

$$\begin{array}{ccccccc} \cdots & \rightarrow & P_2 & \xrightarrow{\quad} & P_1 & \xrightarrow{\quad} & P_0 \rightarrow M \rightarrow 0 \\ & & \searrow & & \nearrow & & \searrow \\ & & & & K_1 & & K_0 \\ & & \nearrow & & \searrow & & \nearrow \\ 0 & & & & & & 0 \end{array} \quad \begin{array}{ccccccc} 0 \rightarrow N \rightarrow E^0 & \xrightarrow{\quad} & E^1 & \xrightarrow{\quad} & E^2 \rightarrow \cdots \\ & & \searrow & & \nearrow \\ & & L^1 & & L^2 \\ & & \nearrow & & \searrow \\ 0 & & & & 0 \end{array}$$

We can construct long exact sequences of homology of $\text{Hom}(\cdot, E_N)$:

$$0 \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(M, E^0) \rightarrow \text{Hom}(M, L^1) \rightarrow \text{Ext}_{\text{inj}}^1(M, N) \rightarrow \text{Ext}_{\text{inj}}^1(M, E^0) = 0$$

$$0 \rightarrow \text{Hom}(P_0, N) \rightarrow \text{Hom}(P_0, E^0) \rightarrow \text{Hom}(P_0, L^1) \rightarrow 0$$

$$0 \rightarrow \text{Hom}(K_0, N) \rightarrow \text{Hom}(K_0, E^0) \rightarrow \text{Hom}(K_0, L^1) \rightarrow \text{Ext}_{\text{inj}}^1(K_0, N) \rightarrow \text{Ext}_{\text{inj}}^1(K_0, E^0) = 0$$

The second sequence is short because P_0 is projective (so $\text{Hom}(P_0, \cdot)$ preserves exactness).

Similarly, for $\text{Hom}(P_M, \cdot)$ we have:

$$0 \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(P_0, N) \rightarrow \text{Hom}(K_0, N) \rightarrow \text{Ext}_{\text{proj}}^1(M, N) \rightarrow \text{Ext}_{\text{proj}}^1(P_0, N) = 0$$

$$\begin{aligned}
0 &\rightarrow \text{Hom}(M, E^0) \rightarrow \text{Hom}(P_0, E^0) \rightarrow \text{Hom}(K_0, E^0) \rightarrow 0 \\
0 &\rightarrow \text{Hom}(M, L^1) \rightarrow \text{Hom}(P_0, L^1) \rightarrow \text{Hom}(K_0, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(M, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(P_0, L^1) = 0
\end{aligned}$$

Combining these sequences together, we got the following 2D diagram:

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{Hom}(M, N) & \longrightarrow & \text{Hom}(M, E^0) & \xrightarrow{\phi} & \text{Hom}(M, L^1) \longrightarrow \text{Ext}_{\text{inj}}^1(M, N) \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{Hom}(P_0, N) & \longrightarrow & \text{Hom}(P_0, E^0) & \xrightarrow{\sigma} & \text{Hom}(P_0, L^1) \longrightarrow 0 \\
& \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\
0 & \longrightarrow & \text{Hom}(K_0, N) & \longrightarrow & \text{Hom}(K_0, E^0) & \xrightarrow{\tau} & \text{Hom}(K_0, L^1) \longrightarrow \text{Ext}_{\text{inj}}^1(K_0, N) \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& \text{Ext}_{\text{proj}}^1(M, N) & & 0 & & \text{Ext}_{\text{proj}}^1(M, L^1) & \\
& \downarrow & & & & \downarrow & \\
& 0 & & & & 0 &
\end{array}$$

By Snake lemma, there is an exact sequence

$$(\ker \alpha \rightarrow) \ker \beta \rightarrow \ker \gamma \rightarrow \text{coker } \alpha \rightarrow \text{coker } \beta (\rightarrow \text{coker } \gamma)$$

and this reads

$$\text{Hom}(M, E^0) \xrightarrow{\phi} \text{Hom}(M, L^1) \rightarrow \text{Ext}_{\text{proj}}^1(M, N) \rightarrow 0$$

Thus $\text{Ext}_{\text{proj}}^1(M, N) \cong \text{coker } \phi \cong \text{Ext}_{\text{inj}}^1(M, N)$.

(From now on, we don't need to distinguish proj/inj for Ext^1 !)

Since σ is onto, $\text{im } \gamma = \text{im}(\gamma \circ \sigma)$. Similarly, $\text{im } \tau = \text{im}(\tau \circ \beta)$.

By the commutativity of the diagram, $\text{im } \gamma = \text{im } \tau$, so

$$\text{Ext}^1(K_0, N) \cong \text{coker } \gamma = \text{Hom}(K_0, L^1) / \text{im } \gamma \cong \text{coker } \tau \cong \text{Ext}^1(M, L^1).$$

Write $K_{-1} := M, L^0 := N$, then $\text{Ext}^1(K_0, L^0) = \text{Ext}^1(K_{-1}, L^1)$ (\star).

Similarly, from the exact sequences

$$0 \rightarrow K_j \rightarrow P_j \rightarrow K_{j-1} \rightarrow 0$$

$$0 \rightarrow L^i \rightarrow E^i \rightarrow L^{i+1} \rightarrow 0$$

, we can obtain $\text{Ext}^1(K_j, L^i) \cong \text{Ext}^1(K_{j-1}, L^{i+1})$ for $i, j \geq 0$.

Now, observe that

$$0 \rightarrow L^{n-1} \rightarrow E^{n-1} \xrightarrow{d_{n-1}} E^n \xrightarrow{d_n} \dots$$

is an injective resolution of L^{n-1} , and $\text{Ext}^1(M, L^{n-1}) \cong \ker \bar{d}_n / \text{im } \bar{d}_{n-1} \cong \text{Ext}_{\text{inj}}^n(M, N)$.

Similarly, for projective resolution we have $\text{Ext}^1(K_{n-2}, N) \cong \text{Ext}_{\text{proj}}^n(M, N)$.

Finally, by (\star),

$$\text{Ext}_{\text{inj}}^n(M, N) \cong \text{Ext}^1(K_{-1}, L^{n-1}) \cong \text{Ext}^1(K_0, L^{n-2}) \cong \dots \cong \text{Ext}^1(K_{n-2}, L^0) \cong \text{Ext}_{\text{proj}}^n(M, N).$$

□

Def 35 (Tor functor). Let $M, N \in \mathbf{Mod}_R$, and $P_\bullet \rightarrow M \rightarrow 0$ be a projective resolution of M , similar to the Ext case, for $n \geq 0$ we can define

$$\mathrm{Tor}_n(M, N) = H_n(P_M \otimes N).$$

Fact 2.2.1. By Horseshoe lemma, short exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ induces a long exact sequence:

$$\cdots \rightarrow \mathrm{Tor}_1(M_1, N) \rightarrow \mathrm{Tor}_1(M_2, N) \rightarrow \mathrm{Tor}_1(M_3, N) \rightarrow M_1 \otimes N \rightarrow M_2 \otimes N \rightarrow M_3 \otimes N \rightarrow 0$$

Prop 2.2.2. If M is flat, then $\mathrm{Tor}_n(M, N) = 0$ for $n > 0, N \in \mathbf{Mod}_R$.

Proof. M is flat $\implies M \otimes \cdot$ is an exact functor. If $Q_\bullet \rightarrow N \rightarrow 0$ is a projective resolution of N , then $\cdots \rightarrow M \otimes Q_1 \rightarrow M \otimes Q_0 \rightarrow M \otimes N \rightarrow 0$ is also exact. By Exercise 15-1, we have

$$\mathrm{Tor}_n(M, N) \cong H_n(M \otimes Q_N) = 0. \quad \square$$

Theorem 29 (Tor for flat resolutions). Let $U_\bullet \rightarrow M \rightarrow 0$ be a flat resolution of M , then for $n \geq 0$,

$$\mathrm{Tor}_n(M, N) \cong H_n(U_M \otimes N).$$

Proof.

$$\begin{array}{ccccccc} \cdots & \rightarrow & U_2 & \xrightarrow{\quad} & U_1 & \xrightarrow{\quad} & U_0 \rightarrow M \rightarrow 0 \\ & & \searrow & & \nearrow & & \nearrow \\ & & & W_1 & & & W_0 \\ & \nearrow & & \searrow & \nearrow & & \searrow \\ 0 & & & & 0 & & 0 \end{array}$$

$$H_\bullet(U_M \otimes N) : \cdots \rightarrow U_2 \otimes N \xrightarrow{d_2 \otimes 1} U_1 \otimes N \xrightarrow{d_1 \otimes 1} U_0 \otimes N \rightarrow 0$$

- $n = 0$:

Since tensor is right exact, $U_1 \otimes N \xrightarrow{d_1 \otimes 1} U_0 \otimes N \xrightarrow{\alpha \otimes 1} M \otimes N \rightarrow 0$ is exact. Hence

$$H_0(U_M \otimes N) = (U_0 \otimes N) / \mathrm{im}(d_1 \otimes 1) = (U_0 \otimes N) / \ker(\alpha \otimes 1) \cong M \otimes N$$

Any projective resolution also has this property, so $\mathrm{Tor}_0(M, N) = H_0(U_M \otimes N)$.

- $n = 1$:

$0 \rightarrow W_0 \rightarrow U_0 \rightarrow M \rightarrow 0$ induces a long exact sequence:

$$\cdots \rightarrow \mathrm{Tor}_1(W_0, N) \rightarrow 0 \rightarrow \mathrm{Tor}_1(M, N) \rightarrow W_0 \otimes N \xrightarrow{i \otimes 1} U_0 \otimes N \rightarrow M \otimes N \rightarrow 0$$

where $\mathrm{Tor}_1(U_0, N) = 0$ because U_0 is flat. We can see that $\mathrm{Tor}_1(M, N) \cong \ker(i \otimes 1)$.

$$\begin{array}{ccccc} U_2 \otimes N & \xrightarrow{d_2 \otimes 1} & U_1 \otimes N & \xrightarrow{d_1 \otimes 1} & U_0 \otimes N \\ & \searrow \beta' \otimes 1 & \nearrow j \otimes 1 & \searrow \alpha' \otimes 1 & \nearrow i \otimes 1 \\ & & W_1 \otimes N & & W_0 \otimes N \\ & & \searrow & & \searrow \\ & & & 0 & 0 \end{array}$$

Since $\alpha' \otimes 1$ is onto, $W_0 \otimes N \cong (U_1 \otimes N) / \ker(\alpha' \otimes 1)$. Also, $x \in \ker(d_1 \otimes 1) \iff (d_1 \otimes 1)(x) = 0 \iff (\alpha' \otimes 1)(x) \in \ker(i \otimes 1)$, so $\ker(i \otimes 1) = (\alpha' \otimes 1)(\ker(d_1 \otimes 1)) \cong \ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1)$. ($\alpha' \otimes 1$ can be considered a quotient map, then $\ker(d_1 \otimes 1)$ descends to $\ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1)$.)

Now, in the diagram $W_1 \otimes N \rightarrow U_1 \otimes N \rightarrow W_0 \otimes N \rightarrow 0$ exact, so $\ker(\alpha' \otimes 1) = \text{im}(j \otimes 1)$. But $\beta' \otimes 1$ is onto, thus $\text{im}(j \otimes 1) = \text{im}(d_2 \otimes 1)$.

Finally,

$$\text{Tor}_1(M, N) \cong \ker(i \otimes 1) \cong \ker(d_1 \otimes 1) / \ker(\alpha' \otimes 1) = \ker(d_1 \otimes 1) / \text{im}(d_2 \otimes 1) = H_1(U_M \otimes N).$$

- $n \geq 2$:

Let's see further in the previous long exact sequences:

$$\cdots \rightarrow \text{Tor}_2(W_0, N) \rightarrow 0 \rightarrow \text{Tor}_2(M, N) \xrightarrow{\sim} \text{Tor}_1(W_0, N) \rightarrow 0 \rightarrow \text{Tor}_1(M, N) \rightarrow \cdots$$

we can see that $\text{Tor}_n(M, N) \cong \text{Tor}_{n-1}(W_0, N)$ for $n \geq 2$.

Now,

$$\cdots U_3 \xrightarrow{d_3} U_2 \xrightarrow{d_2} U_1 \rightarrow W_0 \rightarrow 0$$

is a flat resolution of W_0 , and its homology is $H_{n-1}(U_{W_0} \otimes N) = \ker(d_n \otimes 1) / \text{im}(d_{n-1} \otimes 1) = H_n(U_M \otimes N)$.

By induction, assume it's true for $n - 1$, then

$$H_n(U_M \otimes N) = H_{n-1}(U_{W_0} \otimes N) \cong \text{Tor}_{n-1}(W_0, N) \cong \text{Tor}_n(M, N).$$

□

Eg 2.2.1. $R = \mathbb{Z}, M = \mathbb{Z}/m\mathbb{Z}$ with $m \geq 2$. Then

$$P : 0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{/m\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \rightarrow 0$$

is a projective resolution of $\mathbb{Z}/m\mathbb{Z}$. So for any $N \in \mathbf{Mod}_{\mathbb{Z}}$,

$$H^n(\text{Hom}_{\mathbb{Z}}(P_{\mathbb{Z}/m\mathbb{Z}}, N)) : 0 \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) \xrightarrow{\overline{m}} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) \rightarrow 0,$$

$$\begin{aligned} \text{Ext}_{\mathbb{Z}}^0(\mathbb{Z}/m\mathbb{Z}, N) &= \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, N) \cong {}_mN := \{a \in N \mid ma = 0\} \\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m\mathbb{Z}, N) &\cong N/mN \\ \text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/m\mathbb{Z}, N) &= 0 \quad (\text{for } n \geq 2) \end{aligned}$$

Eg 2.2.2. $\mathbb{Q} = \mathbb{Z}_{(0)}$ is a localization, thus a flat \mathbb{Z} module. Then

$$U : 0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

is a flat resolution of \mathbb{Q}/\mathbb{Z} . For $G \in \mathbf{Mod}_{\mathbb{Z}}$ (i.e. an abelian group),

$$H_n(G \otimes U_{\mathbb{Q}/\mathbb{Z}}) : 0 \rightarrow G \otimes \mathbb{Z} \xrightarrow{1 \otimes i} G \otimes \mathbb{Q} \rightarrow 0$$

$$\begin{aligned} \text{Tor}_0(G, \mathbb{Q}/\mathbb{Z}) &\cong G \otimes \mathbb{Q}/\mathbb{Z} \\ \text{Tor}_1(G, \mathbb{Q}/\mathbb{Z}) &= \ker(1 \otimes i) \cong t(G) := \{a \in G \mid ma = 0 \text{ for some } m \in \mathbb{N}\} \\ \text{Tor}_n(G, \mathbb{Q}/\mathbb{Z}) &= 0 \quad (\text{for } n \geq 2) \end{aligned}$$

Def 36. Let M be a left R -module, then define $M^* := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ as a right R -module by

$$\begin{aligned} fr &: M \rightarrow \mathbb{Q}/\mathbb{Z} \\ x &\mapsto f(rx) \end{aligned}$$

Fact 2.2.2.

1. \mathbb{Q}/\mathbb{Z} is injective.
2. $A = 0 \iff A^* = 0$.
3. $B \hookrightarrow C \iff C^* \twoheadrightarrow B^*$.

Proof.

1. For $m \in \mathbb{Z} \setminus \{0\}$, $m(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$ by $m(\frac{a}{mb} + \mathbb{Z}) \hookrightarrow \frac{a}{b} + \mathbb{Z}$, so \mathbb{Q}/\mathbb{Z} is divisible. But \mathbb{Z} is a PID, so \mathbb{Q}/\mathbb{Z} is injective.
2. $(\Rightarrow) A^* = \text{Hom}_{\mathbb{Z}}(0, \mathbb{Q}/\mathbb{Z}) = 0$.

(\Leftarrow) If $A \neq 0$, then $\exists a \in A, a \neq 0$, so $0 \rightarrow \mathbb{Z}a \xrightarrow{i} A$ is an inclusion.

Since $\mathbb{Z}a$ is a cyclic abelian group, there is a nonzero $g : \mathbb{Z}a \rightarrow \mathbb{Q}/\mathbb{Z}$. (If $\mathbb{Z}a \cong \mathbb{Z}/m\mathbb{Z}$, let $g : a \mapsto \frac{1}{m}$; if $\mathbb{Z}a \cong \mathbb{Z}$, let $g : a \mapsto \frac{1}{2}$.)

But \mathbb{Q}/\mathbb{Z} is injective, so $\exists f : A \rightarrow \mathbb{Q}/\mathbb{Z}$ (i.e. $f \in A^*$), and $f \circ i = g \neq 0$ so $f \neq 0$, thus $A^* \neq 0$.

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z}a & \xrightarrow{i} & A \\ & & \downarrow g & \swarrow \exists f & \\ & & \mathbb{Q}/\mathbb{Z} & & \end{array}$$

3. Since \mathbb{Q}/\mathbb{Z} is injective, $\text{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ is exact. Let $0 \rightarrow \ker f \rightarrow B \xrightarrow{f} C$ exact, applying $\text{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ results in $C^* \xrightarrow{f^*} B^* \rightarrow (\ker f)^* \rightarrow 0$ exact. Thus $\text{coker } f^* = (\ker f)^*$.
By 2., $B \hookrightarrow C \iff \ker f = 0 \iff (\ker f)^* = 0 \iff \text{coker } f^* = 0 \iff C^* \twoheadrightarrow B^*$.

□

Prop 2.2.3. Let M be an R -module, then TFAE

1. M is flat.
2. M^* is injective (as a R -module).
3. $\text{Tor}_1(R/I, M) = 0$ for all ideal $I \subseteq R$.
4. $I \otimes_R M \cong IM$ for all ideal $I \subseteq R$.

Proof.

- 3. \iff 4.

For any ideal $I \subseteq R$, $0 \rightarrow I \xrightarrow{i} R \xrightarrow{q} R/I \rightarrow 0$ is exact. This induces a long exact sequence:

$$\text{Tor}_1(R, M) \rightarrow \text{Tor}_1(R/I, M) \rightarrow I \otimes_R M \xrightarrow{i \otimes \mathbf{1}} R \otimes_R M \xrightarrow{q \otimes \mathbf{1}} R/I \otimes_R M \rightarrow 0$$

- $\text{Tor}_1(R, M) = 0$ since R is a flat R -module.
- $R \otimes_R M \cong M$.
- $R/I \otimes_R M \cong M/IM$ by $(r + I) \otimes a \mapsto (ra + IM)$.

So we have

$$0 \rightarrow \text{Tor}_1(R/I, M) \rightarrow I \otimes_R M \xrightarrow{i'} M \xrightarrow{q'} M/IM \rightarrow 0$$

exact, with $q' : M \rightarrow M/IM$ being exactly the quotient map (one can check that $q \otimes \mathbf{1} \cong q'$).

Now it's clear that $\text{Tor}_1(R/I, M) = 0 \iff I \otimes_R M \cong \ker(q') \cong IM$.

(The reverse direction requires $I \otimes_R M \cong IM$ being the natural isomorphism $r \otimes b \mapsto rb$, so $i' : IM \rightarrow M$ can then be the natural inclusion.)

- 1. \iff 2.

Let $0 \rightarrow N' \xrightarrow{f} N$, then $\text{Hom}_R(N, M^*) \xrightarrow{\bar{f}} \text{Hom}_R(N', M^*)$.

By the adjoint relation,

$$\text{Hom}_R(N, M^*) = \text{Hom}_R(N, \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(N \otimes_R M, \mathbb{Q}/\mathbb{Z}) = (N \otimes_R M)^*,$$

we have another map $(N \otimes_R M)^* \xrightarrow{(f \otimes 1)^*} (N' \otimes_R M)^*$ isomorphic to the previous one, with its unstarred map $N' \otimes_R M \xrightarrow{f \otimes 1} N \otimes_R M$.

Now, M^* is injective $\iff \bar{f}$ is surjective $\forall N, N' \iff (f \otimes 1)^*$ is surjective $\forall N, N' \iff f \otimes 1$ is injective $\forall N, N' \iff M$ is flat.

- 2. \iff 4.

Similar to the previous section, by Baer's criterion,

$$\begin{aligned} M^* \text{ is injective} &\iff \text{Hom}_R(R, M^*) \twoheadrightarrow \text{Hom}_R(I, M^*), \forall I \subseteq R \\ &\iff (R \otimes_R M)^* \twoheadrightarrow (I \otimes_R M)^*, \forall I \subseteq R \\ &\iff I \otimes_R M \hookrightarrow R \otimes_R M \cong M, \forall I \subseteq R \\ &\iff I \otimes_R M \cong IM, \forall I \subseteq R. \end{aligned}$$

Similarly, this requires the isomorphism of $I \otimes_R M \cong IM$ be natural (the following f).

The map $f : I \otimes_R M \rightarrow IM$
 $r \otimes a \mapsto ra$ is always onto, but may not be 1-1. If it is, $I \otimes_R M \cong IM$.

□

Prop 2.2.4. For $I, J \subseteq R$ being ideals, then $\text{Tor}_1(R/I, R/J) \cong (I \cap J)/IJ$.

Proof. $0 \rightarrow I \xrightarrow{i} R \rightarrow R/I \rightarrow 0$ induces a long exact sequence

$$0 \rightarrow \text{Tor}_1(R/I, R/J) \rightarrow I \otimes_R R/J \xrightarrow{i \otimes 1} R \otimes_R R/J \rightarrow R/I \otimes_R R/J \rightarrow 0,$$

where $\text{Tor}_1(R, R/J) = 0$ since R is flat.

Also $I \otimes_R R/J \cong I/IJ, R \otimes_R R/J \cong R/J$, so we have $I/IJ \xrightarrow{i'} R/J$ with $\text{Tor}_1(R/I, R/J) \cong \ker(i \otimes 1) \cong \ker i'$.

But $i' : I/IJ \rightarrow R/J$
 $x + IJ \mapsto x + J$, so $\bar{x} \in \ker i' \iff x \in I \text{ and } x \in J \iff x \in I \cap J$, hence $\ker i' \cong (I \cap J)/IJ$.

□

2.3 Koszul complex (week 16)

In this section, we assume that R is commutative with 1.

Def 37. Let $L \in \mathbf{Mod}_R$, with $f : L \rightarrow R$ an R -linear map, define

$$\begin{aligned} d_f : \Lambda^n L &\rightarrow \Lambda^{n-1} L \\ x_1 \wedge \cdots \wedge x_n &\mapsto \sum_{i=1}^n (-1)^{i+1} f(x_i) x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \end{aligned}$$

where $\Lambda^n L$ is the n -th exterior power of L , and \hat{x}_i means omitting x_i .

Then we can define a chain complex called **Koszul complex**:

$$K_\bullet(f) : \cdots \rightarrow \Lambda^n L \xrightarrow{d_f} \Lambda^{n-1} L \rightarrow \cdots \rightarrow \Lambda^2 L \xrightarrow{d_f} L \xrightarrow{f} R$$

Also, d_f can be considered as a graded R -homomorphism of degree -1 :

$$\begin{aligned} d_f : \Lambda L &\rightarrow \Lambda L \\ x \wedge y &\mapsto d_f(x) \wedge y + (-1)^{\deg x} \cdot x \wedge d_f(y) \end{aligned}$$

where ΛL is the exterior algebra of L , and x, y are any homogeneous elements of ΛL .

Def 38. Let $(C_\bullet, d), (C'_\bullet, d')$ be chain complexes of R -modules, define their *tensor product* to be a chain complex $C_\bullet \otimes C'_\bullet$ with

$$(C_\bullet \otimes C'_\bullet)_n = \bigoplus_{i=0}^n (C_i \otimes_R C'_{n-i})$$

with the boundary maps being

$$\begin{aligned} d \otimes d' : (C_\bullet \otimes C'_\bullet)_n &\rightarrow (C_\bullet \otimes C'_\bullet)_{n-1} \\ \sum_{i=0}^n x_i \otimes y_{n-i} &\mapsto \sum_{i=0}^n (d(x_i) \otimes y_{n-i} + (-1)^i \cdot x_i \otimes d'(y_{n-i})) \end{aligned}$$

One can verify that

$$\begin{aligned} (d \otimes d') \circ (d \otimes d')(x \otimes y) &= (d \otimes d')(d(x) \otimes y + (-1)^{\deg x} \cdot x \otimes d'(y)) \\ &= d \circ d(x) \otimes y + (-1)^{\deg x-1} \cdot d(x) \otimes d'(y) \\ &\quad + (-1)^{\deg x} \cdot d(x) \otimes d'(y) + x \otimes d' \circ d'(y) \\ &= 0 \end{aligned}$$

Prop 2.3.1. Let $L_1, L_2 \in \mathbf{Mod}_R$, $f_1 \in \text{Hom}_R(L_1, R)$, $f_2 \in \text{Hom}_R(L_2, R)$. Define

$$\begin{aligned} f = f_1 + f_2 : L_1 \oplus L_2 &\rightarrow R \\ (x, y) &\mapsto f_1(x) + f_2(y) \end{aligned}$$

then

$$\begin{aligned} K_\bullet(f_1) \otimes K_\bullet(f_2) &\cong K_\bullet(f) \\ \bigoplus_{i=0}^n (\Lambda^i L_1 \otimes_R \Lambda^{n-i} L_2) &\cong \Lambda^n(L_1 \oplus L_2) \end{aligned}$$

with $d_{f_1} \otimes d_{f_2} = d_f$.

Proof. Exercise 16-1(2). □

Def 39. Let $L = \bigoplus_{i=1}^n Re_i$ be a free R -module, and $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i \in R$, define

$$K_{\bullet}(\mathbf{x}) := K_{\bullet}(f), \text{ with } \begin{array}{l} f : L \rightarrow R \\ e_i \mapsto x_i \end{array}.$$

Coro 2.3.1. $K_{\bullet}(\mathbf{x}) \cong K_{\bullet}(x_1) \otimes \dots \otimes K_{\bullet}(x_n)$ with $K_{\bullet}(x_i) : 0 \rightarrow R \xrightarrow{x_i} R$.

Prop 2.3.2. Let $x \in R$ and (C_{\bullet}, ∂) be a chain complex of R -modules, then there exist ρ, π s.t.

$$0 \rightarrow C_{\bullet} \xrightarrow{\rho} C_{\bullet} \otimes K_{\bullet}(x) \xrightarrow{\pi} C_{\bullet}(-1) \rightarrow 0$$

is exact, where $(C_{\bullet}(-1))_n = C_{n-1}$.

Proof. Since $K_{\bullet}(x) : 0 \rightarrow R \xrightarrow{x} R$, so

$$(C_{\bullet} \otimes K_{\bullet}(x))_n = (C_i \otimes_R R) \oplus (C_{i-1} \otimes_R R),$$

and the boundary map is

$$\begin{array}{ccc} d : (C_i \otimes_R R) \oplus (C_{i-1} \otimes_R R) & \rightarrow & (C_{i-1} \otimes_R R) \oplus (C_{i-2} \otimes_R R) \\ (z_1 \otimes r_1, z_2 \otimes r_2) & \mapsto & (\partial z_1 \otimes r_1 + (-1)^{i-1} z_2 \otimes x r_2, \partial z_2 \otimes r_2) \end{array}.$$

Under the isomorphism $C_i \otimes_R R \cong C_i$, the boundary map become

$$\begin{array}{ccc} d : C_i \oplus C_{i-1} & \rightarrow & C_{i-1} \oplus C_{i-2} \\ \begin{pmatrix} r_1 z_1 \\ r_2 z_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \partial & (-1)^{i-1} x \\ 0 & \partial \end{pmatrix} \begin{pmatrix} r_1 z_1 \\ r_2 z_2 \end{pmatrix} \end{array}$$

Let

$$\begin{array}{ccc} \rho_i : C_i \rightarrow C_i \oplus C_{i-1} & \text{and} & \pi_i : C_i \oplus C_{i-1} \rightarrow C_{i-1} \\ z_1 \mapsto (z_1, 0) & & (z_1, z_2) \mapsto z_2 \end{array}$$

then

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_i & \xrightarrow{\rho_i} & C_i \oplus C_{i-1} & \xrightarrow{\pi_i} & C_{i-1} \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow d & & \downarrow \partial \\ 0 & \longrightarrow & C_{i-1} & \xrightarrow{\rho_{i-1}} & C_{i-1} \oplus C_{i-2} & \xrightarrow{\pi_{i-1}} & C_{i-2} \longrightarrow 0 \end{array}$$

commutes and exact:

- $d \circ \rho(z_1) = d(z_1, 0) = (\partial z_1, 0)$
- $\rho \circ \partial(z_1) = \rho(\partial z_1) = (\partial z_1, 0)$
- $\partial \circ \pi(z_1, z_2) = \partial(z_2) = \partial z_2$
- $\pi \circ d(z_1, z_2) = \pi(\partial z_1 + (-1)^{i-1} x z_2, \partial z_2) = \partial z_2$

□

Coro 2.3.2. This induces a long exact sequence

$$\dots \rightarrow H_i(C_{\bullet}) \xrightarrow{\rho_*} H_i(C_{\bullet} \otimes K_{\bullet}(x)) \xrightarrow{\pi_*} H_i(C_{\bullet}(-1)) \xrightarrow{\pm x} H_{i-1}(C_{\bullet}) \rightarrow \dots$$

Proof. We only need to show the connection homomorphism is indeed $\pm x$.

Given $z \in C_{i-1}$ with $\partial z = 0$,

$$z \xrightarrow{\pi^{-1}} (0, z) \xrightarrow{d} ((-1)^{i-1} x z, 0) \xrightarrow{\rho^{-1}} (-1)^{i-1} x z.$$

□

Def 40. We call x to be C_\bullet -regular, if x is not a zero divisor of C_i and $C_i/xC_i \neq 0$, for all $i \geq 0$.

Prop 2.3.3. If x is C_\bullet -regular, then $H_i(C_\bullet \otimes K_\bullet(x)) \cong H_i(C_\bullet/xC_\bullet)$ for all $i \geq 0$.

Proof. Let

$$\begin{aligned} \phi_i : C_i \oplus C_{i-1} &\rightarrow C_i/xC_i \\ (z_1, z_2) &\mapsto \overline{z_1}, \end{aligned}$$

then

$$\begin{array}{ccc} C_i \oplus C_{i-1} & \xrightarrow{\phi_i} & C_i/xC_i \\ \downarrow d_i & & \downarrow \bar{\partial}_i \\ C_{i-1} \oplus C_{i-2} & \xrightarrow{\phi_{i-1}} & C_{i-1}/xC_{i-1} \end{array}$$

commutes.

- $\bar{\partial} \circ \phi_i(z_1, z_2) = \bar{\partial}(z_1) = \overline{\partial z_1}$.
- $\phi_{i-1} \circ d(z_1, z_2) = \phi_{i-1}(\partial z_1 + (-1)^{i-1}xz_2, \partial z_2) = \overline{\partial z_1}$, since $xz_2 \in xC_{i-1}$.

Now we need to show the induced maps

$$\begin{aligned} \phi_{*i} : \ker d_i / \text{im } d_{i+1} &\rightarrow \ker \bar{\partial}_i / \text{im } \bar{\partial}_{i+1} \\ \overline{(z_1, z_2)} &\mapsto \overline{z_1} = \overline{z_1} + \text{im } \bar{\partial}_{i+1} \end{aligned}$$

are isomorphisms.

- **Onto:**

For $\bar{z} \in \ker \bar{\partial}_i$ with $\partial z = xz' \in xC_{i-1}$, $z' \in C_{i-1}$. Then $\phi_i(z, (-1)^i z') = \bar{z}$, and $d(z, (-1)^i z') = (\partial z - xz', (-1)^i \partial z') = (0, 0)$, so $(z, (-1)^i z') \in \ker d_i$. (Since $x\partial z' = \partial(xz') = \partial^2 z = 0$, and x is not a zero divisor of C_i , so $\partial z' = 0$.)

Now, $\phi_{*i}(\overline{(z, (-1)^i z')}) = \bar{z}$, so ϕ_{*i} is onto.

- **1-1:**

Let $(z, z') \in \ker d_i$ with $\phi_i(z, z') = \bar{z} \in \text{im } \bar{\partial}_{i+1}$, i.e. $\bar{z} = \overline{\partial z''}$ with $z'' \in C_{i+1}$. This means $z - \partial z'' = xz'''$ with $z''' \in C_i$, so $\partial(z - \partial z'') = \partial z = x\partial z'''$.

On the other hand, $d(z, z') = (\partial z + (-1)^{i-1}xz', \partial z') = (0, 0)$, so $\partial z = (-1)^i xz'$, $\partial z' = 0$.

So $d(z'', (-1)^i z''') = (\partial z'' + (-1)^{2i}xz''', (-1)^i \partial z''') = (z, z')$, i.e. $(z, z') \in \text{im } d_{i+1}$. ($\partial z = x\partial z''' = (-1)^i xz'$, since x is not a zero divisor, so $\partial z''' = (-1)^i z'$.)

Hence, $\phi_{*i}(\overline{(z_1, z_2)}) = \bar{0}$ implies $\overline{(z_1, z_2)} = \bar{0}$, so ϕ_{*i} is 1-1.

□

Def 41. Let $M \in \mathbf{Mod}_R$. A sequence $\{a_1, \dots, a_m\}$, $m \geq 0$ is said to be M -regular if

- $M/\langle a_1, \dots, a_m \rangle M \neq 0$.
- a_{i+1} is not a zero divisor of $M/\langle a_1, \dots, a_i \rangle M$ for $0 \leq i \leq m-1$.

Theorem 30. If $\mathbf{x} = (x_1, \dots, x_n)$ is an R -regular sequence, then $K_\bullet(\mathbf{x}) \rightarrow R/\langle x_1, \dots, x_n \rangle \rightarrow 0$ is a free resolution of $R/\langle x_1, \dots, x_n \rangle$.

Proof. Since its modules are $K_i(\mathbf{x}) = \Lambda^i R^n \cong R^{\binom{n}{i}}$, i.e. free R -modules, so we only need to show the exactness.

By induction on n ,

- $n = 1$: $K_\bullet(x_1) : 0 \rightarrow R \xrightarrow{x_1} R \rightarrow R/\langle x_1 \rangle \rightarrow 0$ exact.

- $n > 1$: Assume that $\mathbf{x}' = (x_1, \dots, x_{n-1})$ and $K_\bullet(\mathbf{x}') \rightarrow R/\langle x_1, \dots, x_{n-1} \rangle \rightarrow 0$ exact, i.e. $H_i(K_\bullet(\mathbf{x}')) = 0$ for $i > 0$.

Since we have $K_\bullet(\mathbf{x}) \cong K_\bullet(\mathbf{x}') \otimes K_\bullet(x_n)$ and a long exact sequence

$$\cdots \rightarrow H_i(K_\bullet(\mathbf{x}')) \rightarrow H_i(K_\bullet(\mathbf{x})) \rightarrow H_i(K_\bullet(\mathbf{x}')(-1)) \xrightarrow{\pm x_n} H_{i-1}(K_\bullet(\mathbf{x})) \rightarrow \cdots$$

where $H_i(K_\bullet(\mathbf{x}')(-1)) = H_{i-1}(K_\bullet(\mathbf{x}'))$.

For $i > 1$, the sequence becomes

$$\cdots \rightarrow 0 \rightarrow H_i(K_\bullet(\mathbf{x})) \rightarrow 0 \xrightarrow{\pm x_n} \cdots,$$

so $H_i(K_\bullet(\mathbf{x})) = 0$.

For $i = 1$, we have $H_0(K_\bullet(\mathbf{x})) \cong R/\langle x_1, \dots, x_{n-1} \rangle$, so

$$0 \rightarrow H_1(K_\bullet(\mathbf{x})) \rightarrow R/\langle x_1, \dots, x_{n-1} \rangle \xrightarrow{\pm x_n} R/\langle x_1, \dots, x_{n-1} \rangle$$

But x_n is not a zero divisor of $R/\langle x_1, \dots, x_{n-1} \rangle$, so the map $\pm x_n$ is 1-1, then $H_1(K_\bullet(\mathbf{x})) \cong \ker(\pm x_n) = 0$.

□

Eg 2.3.1. Let $\mathbf{x} = (x_1, x_2)$, then

$$K_\bullet(\mathbf{x}) : 0 \rightarrow R \xrightarrow{\alpha} R^2 \xrightarrow{\beta} R \xrightarrow{q} R/\langle x_1, x_2 \rangle \rightarrow 0$$

with $\alpha : r \mapsto (-x_2r, x_1r)$ and $\beta : (r_1, r_2) \mapsto x_1r_1 + x_2r_2$.

Coro 2.3.3. Let $I = \langle x_1, \dots, x_n \rangle \subset R$ be an ideal with $\{x_1, \dots, x_n\}$ be R -regular, then R/I has *projective dimension* $\text{pd}(R/I) = n$, i.e. the shortest projective resolution of R/I has length n .

Proof. $K_\bullet(\mathbf{x})$ is already a projective resolution of length N , so we only need to show that there's no shorter ones.

The left side of $K_\bullet(\mathbf{x})$ reads

$$0 \rightarrow \Lambda^n R^n \xrightarrow{d_n} \Lambda^{n-1} R^n \rightarrow \cdots$$

But

$$\Lambda^n R^n = R(e_1 \wedge \cdots \wedge e_n) \cong R, \quad \Lambda^{n-1} R^n = \bigoplus_{i=1}^n R(e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e^n) \cong R^n$$

so

$$d_n : R \rightarrow R^n \\ r \mapsto (x_1r, -x_2r, \dots, (-1)^{n-1}x_nr)$$

Taking tensor with R/I , we get

$$0 \rightarrow R \otimes_R R/I \xrightarrow{d_n \otimes 1} R^n \otimes_R R/I \rightarrow \cdots$$

but $R \otimes_R R/I \cong R/I$, $R^n \otimes_R R/I \cong (R/I)^n$, so

$$d_n \otimes 1 : R/I \rightarrow \overline{(R/I)^n} \\ \bar{r} \mapsto (\overline{x_1r}, \overline{-x_2r}, \dots, \overline{(-1)^{n-1}x_nr})$$

Now,

$$\text{Tor}_n(R/I, R/I) = H_n(K_\bullet(\mathbf{x}) \otimes R/I) = \ker(d_n \otimes 1) = \text{Ann}_{R/I} I = \{\bar{r} \in R/I \mid rI = I\} = R/I \neq 0.$$

($R/I \neq 0$ is because $\{x_1, \dots, x_n\}$ is R -regular.) Thus, any projective resolution can't have length shorter than n since that will imply $\text{Tor}_n(R/I, R/I) = 0$. □

Remark 4. Let $I = \langle x_1, \dots, x_n \rangle$ generated by R -regular sequence $\{x_1, \dots, x_n\}$, then

- $\text{Tor}_n(R/I, M) \cong \text{Ann}_M I$.
- $\text{Ext}^n(R/I, M) \cong M/IM$.

2.4 Derived category

Def 42.

- \mathcal{C} is a pre-additive category if $\text{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group $\forall X, Y \in \mathcal{C}$ s.t.

$$X \xrightarrow{u} Y \xrightleftharpoons[g]{f} Z \xrightarrow{v} T$$

with $(f + g)u = fu + gu$ and $v(f + g) = vf + vg$.

- additive category: a pre-additive category \mathcal{C} s.t.
 - There exists a zero object 0 s.t. $\forall X, \text{Hom}_{\mathcal{C}}(0, X) = \{0\} = \text{Hom}_{\mathcal{C}}(X, 0)$.
 - Finite sum and finite products exist.

Def 43.

- $f \in \text{Hom}(B, C)$ is called a monomorphism if $\forall X \xrightarrow{g} B \xrightarrow{f} C$ with $f \circ g = 0 \implies g = 0$.
- $f \in \text{Hom}(B, C)$ is called an epimorphism if $\forall B \xrightarrow{f} C \xrightarrow{h} D$ with $h \circ f = 0 \implies h = 0$.
- a kernel of $f \in \text{Hom}(B, C)$ is a morphism $i : A \rightarrow B$ s.t. $f \circ i = 0$ and $\forall g : X \rightarrow B$ with $f \circ g = 0$, we have

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{f} & C \\ & \nwarrow \exists! & \uparrow g & \searrow 0 & \\ & & X & & \end{array}$$

- a cokernel of $f \in \text{Hom}(B, C)$ is a morphism $p : C \rightarrow D$ s.t. $p \circ f = 0$ and $\forall h : C \rightarrow Y$ with $h \circ f = 0$, we have

$$\begin{array}{ccccc} B & \xrightarrow{f} & C & \xrightarrow{p} & D \\ & \searrow 0 & \downarrow h & \swarrow \exists! & \\ & & Y & & \end{array}$$

Remark 5.

- If i is a kernel of f , then i is a monomorphism.
- If p is a cokernel of f , then p is an epimorphism.

Remark 6. An epimorphism may not be a cokernel. Consider $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$ which is an epimorphism in the category of f.g. free \mathbb{Z} -modules. If $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$ is the cokernel of $G \xrightarrow{f} \mathbb{Z}$, then

$$\begin{array}{ccccc} G & \xrightarrow{f} & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} \\ & \searrow 0 & \downarrow \times 2 & \swarrow \exists! \tilde{f} & \\ & & \mathbb{Z} & & \end{array}$$

This implies $\tilde{f} : 1 \mapsto \frac{2}{3}$, which is impossible.

Def 44. \mathcal{A} is an **abelian category** if it is an additive category s.t.

- kernels and cokernels always exist in \mathcal{A} .
- every monomorphism is a kernel and every epimorphism is a cokernel.

Fact 2.4.1. If \mathcal{A} is an abelian category, then:

- every morphism is expressible as the composite of an epimorphism and a monomorphism. Given $f : B \rightarrow C$, we have

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ & \searrow & \nearrow \\ & \text{Im } f & \end{array}$$

where $\text{Im } f$ is unique up to isomorphism.

Proof. Consider the following diagram:

$$\begin{array}{ccccccc} \ker f & \xleftarrow{i} & B & \xrightarrow{f} & C & \xrightarrow{p} & \text{coker } f \\ & & \downarrow p' & \swarrow \exists \mu & \searrow \exists \nu & & \uparrow i' \\ & & \text{coker } i & \xrightarrow[\exists \sigma]{\quad} & \ker p & & \end{array}$$

where μ, ν exist because i', p' are kernel and cokernel. Now, $i'\mu i = fi = 0$, and since i' is a monomorphism, $\mu i = 0$. Moreover, since p' is the cokernel of i , there exists a unique σ letting the diagram commute.

By exercise, σ is both a monomorphism and epimorphism. In an abelian category, this implies that σ is actually an isomorphism (i.e., σ^{-1} exists). \square

- $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is exact if f is monomorphism, g is epimorphism and $\text{Im } f = \ker g$.

Theorem 31 (Freyd-Mitchell theorem). A small abelian category is equivalent to a full subcategory of a category of R -modules.

Def 45.

- $I \in \text{Obj } \mathcal{A}$ is injective if the functor $\text{Hom}(-, I)$ is exact.
- An abelian category is said to be **enough injectives** if for any $A \in \text{Obj } \mathcal{A}$, there exists an injective object I such that $A \hookrightarrow I$.

Def 46. Given a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ satisfy:

1. F is additive, which is to say F is a group homomorphism $\text{Hom}(A, A') \rightarrow \text{Hom}(FA, FA')$.
2. F is left exact. If $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$, then $0 \rightarrow FA' \rightarrow FA \rightarrow FA''$.

Then the derived functor $R^i F : \mathcal{A} \rightarrow \mathcal{B}$ is defined as

$$R^i F(A) = \begin{cases} F(A), & \text{if } i = 0 \\ H^i(F(I^\bullet)), & \text{else} \end{cases}$$

Our goal is to construct the derived category $D^+(\mathcal{A})$ and $D^+(\mathcal{B})$ letting RF be a exact functor.

Def 47. Let \mathcal{A} be an abelian category.

- $\text{Kom}(\mathcal{A})$ is the category of complexes over \mathcal{A} .

- $K(\mathcal{A})$ is the homotopy category of \mathcal{A} , defined by $\text{Obj}(K(\mathcal{A})) = \text{Obj}(\text{Kom}(\mathcal{A}))$ and

$$\text{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \text{Hom}_{\text{Kom}(\mathcal{A})}(A^\bullet, B^\bullet) / \sim,$$

where \sim indicates homotopy equivalences.

Remark 7.

- $\text{Hom}_{K(\mathcal{A})}(I_A^\bullet, I_B^\bullet) \cong \text{Hom}_{\mathcal{A}}(A, B)$ by comparison theorem (24).
- It could be shown that $K(\mathcal{A})$ is additive but may not be abelian.

Def 48. $f \in \text{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet)$ is called a quasi-isomorphism if $H^n(f)$ is an isomorphism between $H^n(A^\bullet)$ and $H^n(B^\bullet)$ for each n .

Eg 2.4.1. • A quasi-isomorphism is often not invertible. For example:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & 0 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \longrightarrow \dots \end{array}$$

- Given $0 \rightarrow A \rightarrow I^\bullet$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 \longrightarrow \dots \end{array}$$

are two quasi-isomorphic complexes.

Def 49. Let \mathcal{B} be a category. A class of morphism $S \subset \text{Mor}(\mathcal{B})$ is said to be **localizing** if

1. S is closed under composition with $\text{Id}_X \in S$ for each object X in \mathcal{B} .
2. Extension condition holds: For each $f \in \text{Mor } \mathcal{B}$, $s \in S$, exists $g \in \text{Mor } \mathcal{B}$, $t \in S$ such that $ft = sg$. The dual version should hold as well.
3. For any $f, g \in \text{Hom}(X, Y)$,

$$\exists s \in S \text{ s.t. } sf = sg \iff \exists t \in S \text{ s.t. } ft = gt.$$

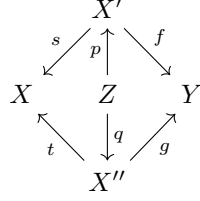
Theorem 32. If S is localizing, then there exists a category $\mathcal{B}[S^{-1}]$ with a functor $Q : \mathcal{B} \rightarrow \mathcal{B}[S^{-1}]$ such that

1. $Q(s)$ is an isomorphism for each $s \in S$.
2. Given another functor $F : \mathcal{B} \rightarrow \mathcal{B}'$ satisfy condition 1, there exists a unique functor $G : \mathcal{B}[S^{-1}] \rightarrow \mathcal{B}'$ such that $F = G \circ Q$.

Proof. Define a roof to be a pair (s, t) with

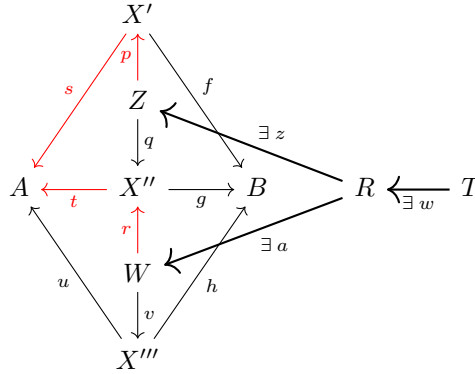
$$\begin{array}{ccc} & X' & \\ s \ni s \swarrow & & \searrow t \\ X & & Y \end{array}$$

Also, define $(s, f) \sim (t, g)$ if there exists Z such that



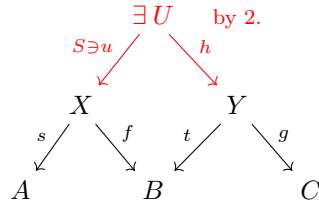
with $sp = tq \in S$ and $fp = gq$.

First we check that “ \sim ” is indeed an equivalence relation. $(s, f) \sim (s, f)$ and $(s, f) \sim (t, g) \implies (t, g) \sim (s, f)$ are trivial. If $(s, f) \sim (t, g)$ and $(t, g) \sim (u, h)$, then we have the following diagram:



Using definition 2. on $tr \in S$ and sp , there are morphism z, a with $z \in S$ and $spz = tra$. Moreover, $tqz = spz = tra$, if we let $b = qz, c = ra$, then by 3., morphism $w \in S$ exists with $bw = cw$. Define $x = pzw, y = vaw$, we have $sx = spzw = tqzw = tbw = tcw = traw = uvaw = uy$ and $sx \in S$ since $sx = spzw$ and sp, z, w are all in S . Similarly, $fx = hy$, thus $(s, f) \sim (u, h)$. Hence we've just proved that \sim is an equivalence relation.

Now we could construct the localized category as following: The objects are $\text{Obj}(\mathcal{B}[S^{-1}]) = \mathcal{B}$ and $\text{Mor}(\mathcal{B}[S^{-1}]) = \{ \text{equivalence classes under } \sim \}$. $[(t, g)] \circ [(s, f)] = [(su, gh)]$ could be defined as in the following diagram:



□

Finally, define functor Q by $Q(X) = X, \forall X \in \text{Obj}(\mathcal{B})$ and $Q(f) = [(\text{Id}_X, f)]$. For the universal property, if F is another functor making every morphism in S be invertible, then the functor G exists uniquely by $G([(s, f)]) = F(f)F(s)^{-1}$.

Def 50. The mapping cone of a chain map f between two chain $X^\bullet \xrightarrow{f} Y^\bullet$ is defined as a chain with $\text{cone}(f)^n = X^{n+1} \oplus Y^n$, and the chain map is defined as

$$d_{\text{cone}(f)} : \text{cone}(f)^n = X^{n+1} \oplus Y^n \longrightarrow \text{cone}(f)^{n+1} = X^{n+2} \oplus Y^{n+1}$$

$$(x_{n+1}, y_n) \longmapsto \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix} (-d_X(x_{n+1}), f(x_{n+1}) + d_Y(y_n))$$

It is easy to see that $d_{\text{cone}(f)}^2 = 0$.

Prop 2.4.1. Suppose that $f : X^\bullet \rightarrow Y^\bullet$ is a chain map, then there is a short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \longrightarrow & \text{cone}(f) & \longrightarrow & X[+1] \longrightarrow 0 \\ & & y & \longmapsto & (0, y) & & \\ & & & & (x, y) & \longmapsto & x \end{array}$$

Proof. It is easy to see that the rows are exact. Tracing the diagram shows that the diagram commutes. \square

Coro 2.4.1. There exists a long exact sequence of homology:

$$\cdots \rightarrow H^m(Y^\bullet) \rightarrow H^m(\text{cone}(f)) \rightarrow H^{m+1}(X^\bullet) \xrightarrow{\delta} H^{m+1}(Y^\bullet) \rightarrow H^{m+1}(\text{cone}(f)) \rightarrow \cdots$$

Where the connecting homomorphism $\delta = f^*$.

Proof. Tracing the diagram below as in the snake lemma,

$$\begin{array}{ccccc} X^m \oplus Y^{m-1} & \longrightarrow & X^m & & \\ \downarrow & & \downarrow & & \\ Y^m & \longrightarrow & X^{m+1} \oplus Y^m & \longrightarrow & X^{m+1} \end{array}$$

Suppose $\bar{x} \in H^m(X^\bullet)$, then $d_X(x) = 0$, so $d(x, 0) = (-d_X(x), f(x)) = (0, f(x))$, which implies $f(x) :: Y^m \mapsto d(x, 0) :: X^{m+1} \oplus Y^m$, then $\delta(\bar{x}) = \overline{f(x)}$, so $\delta = f^*$. \square

Coro 2.4.2. $\text{cone}(f)$ acyclic (exact) $\iff f$ quasi-isomorphic.

Proof. Directly by the exact sequence

$$H^{m-1}(\text{cone}(f)) \rightarrow H^m(X^\bullet) \rightarrow H^m(Y^\bullet) \rightarrow H^m(\text{cone}(f))$$

\square

Notice that $X[-k]$ is defined as $X[-k]^n = X^{n-k}$ with $d_{X[-k]} = (-1)^k d_X$ below.

Theorem 33. Let \mathcal{A} be an abelian category and $K(\mathcal{A})$ be the homotopy category. Then the class of quasi-isomorphisms are localizing.

Proof. We check that:

1. It is closed under composition: If f, g are quasi-isomorphic, then $(fg)^* = f^*g^*$ is a isomorphism since both f^*, g^* are, thus fg is quasi-isomorphic.

2. The diagram could be completed:

$$\begin{array}{ccc} \exists W^\bullet & \dashrightarrow & Z^\bullet \\ \downarrow & & \downarrow g: \text{q-iso} \\ X^\bullet & \xrightarrow{f} & Y^\bullet \end{array}$$

Consider the following diagram:

$$\begin{array}{ccccc} \text{cone}(\pi f)[-1] & \xrightarrow[\substack{(x_n, z_n, y_{n-1}) \mapsto x_n}]{k} & X^\bullet & \xrightarrow{\pi f} & \text{cone}(g) \\ \downarrow \substack{(x_n, z_n, y_{n-1}) \mapsto z_n} h[-1] & & \downarrow f & & \parallel \\ Z^\bullet & \xrightarrow[\substack{z_n \mapsto g(z_n)}]{k} & Y^\bullet & \xrightarrow[\substack{y_n \mapsto (0, y_n)}]{\pi} & \text{cone}(g) \end{array}$$

Where $\text{cone}(\pi f)^n \cong X^{n+1} \oplus \text{cone}(g)^n \cong X^{n+1} Z^{n+1} Y^n$.

We claim that $fk \simeq gh[-1]$. Since $(fk - gh[-1])(x_n, z_n, y_{n-1}) = f(x_n) + g(z_n)$. Define

$$\begin{aligned} \varphi : \text{cone}(\pi f)[-1]^n &= \text{cone}(\pi f)^{n-1} \longrightarrow Y^{n-1} \\ (x_n, z_n, y_{n-1}) &\longmapsto -y_{n-1} \end{aligned}$$

Then

$$\begin{aligned} \varphi d_{C(\pi f)[-1]}(x_n, (z_n, y_{n-1})) &= \varphi(d(x_n), -\pi f(x_n) - d(z_n, y_{n-1})) \\ &= \varphi(d(x_n), -(0, f(x_n)) - (d(z_n), g(z_n) + d(y_{n-1}))) \\ &= \varphi(d(x_n), -d(z_n), -f(x_n) - g(z_n) - d(y_{n-1})) \\ &= f(x_n) + g(z_n) + d(y_{n-1}) \end{aligned}$$

and $d_Y \varphi(x_n, z_n, y_{n-1}) = -d(y_{n-1})$, so $\varphi d_{C(\pi f)[-1]} + d_Y \varphi = fk - gh[-1]$, thus $fk \simeq gh[-1]$.

3. Let $f : X^\bullet \rightarrow Y^\bullet$ in $K(\mathcal{A})$. We shall prove that

$$\exists s : Y^\bullet \rightarrow Z^\bullet \text{ s.t. } sf = 0 \iff \exists t : W^\bullet \rightarrow X^\bullet \text{ s.t. } ft = 0$$

Let $h^i : X^i \rightarrow Z^{i-1}$ be a homotopy between sf and 0. Consider the diagram:

$$\begin{array}{ccccccc} \text{cone}(s)[-1] & \xleftarrow[\substack{(f(x_n), -h(x_n)) \mapsto x_n}]{g} & X^\bullet & \xleftarrow{t} & \text{cone}(g)[-1] = W^\bullet & & \\ \parallel & & \downarrow f & & & & \\ \text{cone}(s)[-1] & \xrightarrow{p[-1]} & Y^\bullet & \xrightarrow{s} & Z^\bullet & \xrightarrow{\pi} & \text{cone}(s) \end{array}$$

One can check that g is a chain map. Now, we have $ft = p[-1]gt$, but $gt \simeq 0$ by

$$\begin{aligned} k_n : \quad W^n &= X^n \oplus Y^{n-1} \oplus Z^{n-2} \longrightarrow C(s)[-1]^{n-1} = Y^{n-1} \oplus Z^{n-2} \\ (x_n, y_{n-1}, z_{n-2}) &\longmapsto (y_{n-1}, z_{n-2}) \end{aligned}$$

since

$$\begin{aligned} kd(x_n, y_{n-1}, z_{n-2}) &= k(-(dx_n, g(x_n) + d(y_{n-1}, z_{n-2}))) \\ &= k(-dx_n, -(f(x_n), -h(x_n)) + (-dy_{n-1}, g(y_{n-1}) + dz_{n-2})) \\ &= (-f(x_n) - dy_{n-1}, h(x_n) + g(y_{n-1}) + dz_{n-2}) \end{aligned}$$

and $dk(x_n, y_{n-1}, z_{n-2}) = d(y_{n-1}, z_{n-2}) = (dy_{n-1}, -g(y_{n-1}) - dz_{n-2})$. Thus $dk + kd = -gt \implies gt \simeq 0$.

Now, since s is quasi-isomorphic, by corollary 2.4.2, $\text{cone}(s)$ is acyclic, and thus t is quasi-isomorphic. Hence we've find t so that $ft \simeq 0$.

We could then define the derived category as $D(\mathcal{A}) = K(\mathcal{A})[S^{-1}]$ now. \square

Prop 2.4.2. The derived category is additive.

Proof. Let $\varphi, \varphi' : X \rightarrow Y$ in $D(\mathcal{A})$ with $\varphi = [(s, f)]$, $\varphi' = [(s', f')]$, that is, we have the following two diagram

$$\begin{array}{ccc} & Z & \\ s \swarrow & & \searrow f \\ X & & Y \end{array} \quad \begin{array}{ccc} & Z' & \\ s' \swarrow & & \searrow f' \\ X & & Y \end{array}$$

using 2. in the definition of localizing, exists U so that

$$\begin{array}{ccc} \exists U & \xrightarrow{r'} & Z' \\ \downarrow r & & \downarrow s' \\ Z & \xrightarrow{s} & X \end{array}$$

with one of r, r' is guaranteed to be quasi-isomorphic, say r . But then $H^n(U) \cong H^n(Z) \cong H^n(X) \cong H^n(Z')$ since r, s, s' are all quasi-isomorphic. This implies r' is also quasi-isomorphic, so we'll have the new roof for φ

$$\begin{array}{ccc} & U & \\ & \swarrow r & \downarrow g \\ & Z & \\ s \swarrow & & \searrow f \\ X & & Y \end{array}$$

Similarly, this applies to φ' . Since $rs = r's'$, we could define $\varphi + \varphi' = [(rs, g + g')]$. \square

Def 51. Let \mathcal{A}, \mathcal{B} be abelian categories, $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor.

- Define $D^+(\mathcal{A})$ as a subcategory of $D(\mathcal{A})$ consist of all the objects (chains) X^\bullet in $D(\mathcal{A})$ such that $X^i = 0$ for all $i \leq i_0(X^\bullet)$. $K^+(\mathcal{A})$ is defined similarly.
- Assume that F act on complexes component wise. $K^+(F) : K^+(\mathcal{A}) \rightarrow K^+(\mathcal{B})$.
- A triangle in $K^+(\mathcal{A})$ is a diagram of the form $\triangle : X^\bullet \rightarrow Y^\bullet \rightarrow Z^\bullet \rightarrow X^\bullet[1]$
- \triangle is said to be distinguished if

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f} & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X^\bullet[1] \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \bar{X}^\bullet & \xrightarrow{\bar{f}} & \bar{Y}^\bullet & \longrightarrow & \text{cone}(\bar{f}) & \longrightarrow & \bar{X}^\bullet[1] \end{array}$$

In this case, we denote it as \triangleleft .

Recall that $\bar{Y}^\bullet \rightarrow \text{cone}(\bar{f}) \rightarrow \bar{X}^\bullet$ induces a long exact sequence

$$\cdots \rightarrow H^i(\bar{Y}) \rightarrow H^i(\text{cone}(\bar{f})) \rightarrow H^i(\bar{X}[1]) \rightarrow H^{i+1}(\bar{Y}) \rightarrow \cdots$$

Prop 2.4.3. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor, then

1. The exact functor $D^+(F) : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ exists.

2. $D^+(F)$ preserves distinguished triangle, (i.e., $\triangle \mapsto \triangle$.)

Proof. First, we have the following observation:

- F sends acyclic chain to acyclic chain: If X^\bullet acyclic, then X^\bullet could be decomposed to many short exact sequence:

$$0 \rightarrow \ker d_X^i \rightarrow X^i \rightarrow \ker d_X^{i+1} \rightarrow 0$$

Apply F we would then get

$$0 \rightarrow F(\ker d_X^i) \rightarrow F(X^i) \rightarrow \ker d_X^{i+1} \rightarrow 0$$

which we could connect them and get the desired exact sequence

$$\dots \rightarrow F(X^{i-1}) \rightarrow F(X^i) \rightarrow F(X^{i+1}) \rightarrow \dots$$

- If $f : X^\bullet \rightarrow Y^\bullet$, then $F(f) : F(X)^\bullet \rightarrow F(Y)^\bullet$, and we have $F(\text{cone}(f)) \cong \text{cone}(F(f))$, since $F(\text{cone}(f))^n = F(X^{n+1} \oplus Y^n) \cong F(X^{n+1}) \oplus F(Y^n) = \text{cone}(F(f))^n$ because F is additive. Moreover, the boundary map $d_{\text{cone}(F(f))}$ is

$$\begin{pmatrix} -F(d_X) & 0 \\ F(f) & F(d_Y) \end{pmatrix} = F \begin{pmatrix} d_X & 0 \\ f & d_Y \end{pmatrix} = d_{F(\text{cone}(f))}$$

Since F transform the object and morphisms consistently, thus $F(\text{cone}(f)) \cong \text{cone}(F(f))$. Similarly we have $F(\text{cyl}(f)) \cong \text{cyl}(F(f))$.

Now, return to our proof:

1. If f quasi-isomorphic, then $\text{cone}(f)$ acyclic by corollary 2.4.2, and $F(\text{cone}(f)) \cong \text{cone}(F(f))$ acyclic by the discussion above, and finally $F(f)$ acyclic by the same corollary. Thus F preserves quasi-isomorphisms.

Moreover, we could complete the following diagram

$$\begin{array}{ccc} K^+(\mathcal{A}) & \xrightarrow{K^+(F)} & K^+(\mathcal{B}) \\ \downarrow Q_A & & \downarrow Q_B \\ K^+(\mathcal{A})[S_A^{-1}] & \xrightarrow{\exists ! D^+(F)} & K^+(\mathcal{B})[S_B^{-1}] \end{array}$$

since F send quasi-isomorphisms to quasi-isomorphism and by the universal property of the localized category. Thus $D^+(f)$ exists.

2. Apply $D^+(F)$ to the diagram

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f} & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X^\bullet[1] \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \bar{X}^\bullet & \xrightarrow{\bar{f}} & \bar{Y}^\bullet & \longrightarrow & \text{cone}(\bar{f}) & \longrightarrow & \bar{X}^\bullet[1] \end{array}$$

We get

$$\begin{array}{ccccccc} FX^\bullet & \xrightarrow{Ff} & FY^\bullet & \longrightarrow & FZ^\bullet & \longrightarrow & FX^\bullet[1] \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ F\bar{X}^\bullet & \xrightarrow{F\bar{f}} & F\bar{Y}^\bullet & \longrightarrow & F\text{cone}(\bar{f}) & \longrightarrow & F\bar{X}^\bullet[1] \end{array}$$

Where the quasi-isomorphisms are preserved by the discussion above.

□

Def 52. A class R of objects in $\text{Obj } \mathcal{A}$ is said to be adapted to a left exact functor F if

1. It is stable under finite direct sums
2. F sends acyclic chain in $\text{Kom}^+(R)$ to acyclic chain (in $\text{Kom}^+(\mathcal{B})$).
3. For each $X \in \text{Obj } \mathcal{A}$, exists $I \in R$ such that $0 \rightarrow X \rightarrow I$.

Theorem 34. Let F be a left exact functor, R be a class of objects adapted to F . Define S_R to be the class of quasi-isomorphisms on $K^+(R)$ which is localizing since it is stable with the construction of mapping cones. Then $D^+(\mathcal{A}) \cong K^+(R)[S_R^{-1}]$.

Proof. First we claim that for all $C^\bullet \in D^+(\mathcal{A})$ (which we assume $C^i = 0, \forall i < 0$), There exists $I^\bullet \in K^+(R)$ such that $C^\bullet \cong I^\bullet$.

We shall construct quasi-isomorphism $t^n : C^n \rightarrow I^n$. Using induction on n :

$n = 0$: By the definition of adapting class we have $0 \rightarrow C^0 \xrightarrow{t^0} I^0$ for some I^0 . Consider the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 0 & \longrightarrow & C^0 & \xrightarrow{d_C} & C^1 & \xrightarrow{t^1=ca} & I^1 \\
 & & \downarrow t^0 & \searrow d_I & \downarrow a & \nearrow c & \\
 & & I^0 & \xrightarrow{b} & I^0 \amalg_{C^0} C^1 & & \\
 & & & \nearrow & & & \\
 & & 0 & & & &
 \end{array}$$

Where $I^0 \amalg_{C^0} C^1 \triangleq (I^0 \oplus C^1) / \{(t^0(x), -d_C(x)) \mid x \in C^0\}$.

We shall prove that t^0 is an isomorphism between $H^0(C^\bullet) = \ker d_C^1$ and $H^0(I^\bullet) = \ker d_I^1$. It is obviously 1-1 since $0 \rightarrow C^0 \xrightarrow{t^0} I^0$, so we need to check it is onto. For any $y \in \ker d_I^1 = \ker b$ since c is monomorphism. Then $b(y) = 0 \implies (y, 0) = (t^0(x), -d_C^1(x))$ for some $x \in C^0$. So $y = t^0(x)$ with $d_C^1(x) = 0 \implies x \in \ker d_C^1$.

$n = 1$: Consider the diagram now:

$$\begin{array}{ccccccc}
 & & C^1 & \xrightarrow{d_C^2} & C^2 & \xrightarrow{t^2} & I^2 \\
 & & \downarrow & \searrow d_I^2 & \downarrow a' & \nearrow c' & \\
 I^0 & \xrightarrow{d_I^1} & I^1 & \xrightarrow{f} & \text{coker } d_I^1 & \xrightarrow{b'} & \text{coker } d_I^1 \amalg_{C^1} C^2 \\
 & & & \nearrow & & & \\
 & & 0 & & & &
 \end{array}$$

Similarly, we shall prove that

$$H^1(t) : \frac{\ker d_C^2}{\text{Im } d_C^1} \xrightarrow{\sim} \frac{\ker d_I^2}{\text{Im } d_I^1}$$

is an isomorphism.

- 1-1: Let $t^1(x) \in \text{Im } d_I^1$. Since $t^1 = ca$ and $d_I^1 = cb$, there is y such that $ca(x) = cb(y)$. Since c 1-1, $a(x) = b(y) \implies (0, x) = (y, 0)$. in the pushout, so $(y, -x) = (t^0(z), -d_C^1(z))$ for some $z \in C^0$. Thus $x = d_C^1(z) \in \text{Im } d_C^1$.
- onto: For each $y \in \ker d_I^2 = \ker b'p$ since c' 1-1. Then

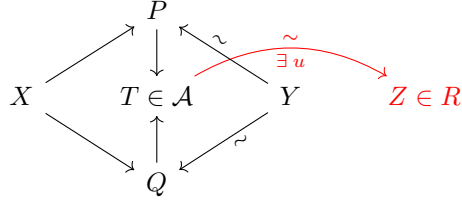
$$b'p(y) = 0 \implies (y + \text{Im } d_I^1, 0) = (t'(x) + \text{Im } d_I^1, -d_C^2(x)) \text{ for some } x \in C^1$$

in the pushout, so we have $y - t'(x) \in \text{Im } d_I^1$ and $x \in \ker d_C^2$ and thus $H^1(t)(\bar{x}) = \bar{y}$.

$n > 1$: Similar as $n = 1$.

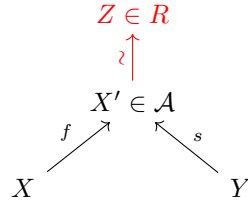
After proving this claim, we shall show that $\text{Hom}_{K^+(R)[S_R^{-1}]}(X^\bullet, Y^\bullet) \cong \text{Hom}_{K^+(A)[S_A^{-1}]}(X^\bullet, Y^\bullet)$.
We will use left roofs instead of right roofs defined before here.

- 1-1: If $(f, s) \cong (g, t)$ in $K^+(\mathcal{A})[S_{\mathcal{A}}^{-1}]$, then



where u exists by the previous claim.

- onto: Given a root in \mathcal{A}



We could find a root in R which is equivalent to it again by the previous claim.

□

Finally, if $F : \mathcal{A} \rightarrow \mathcal{B}$ is an additive left exact functor, then we will have $K^+(F) : K^+(A) \rightarrow K^+(B)$ which sends acyclic chain in $K^+(R)$ to acyclic chain in $K^+(B)$. This implies that $K^+(F)$ sends quasi-isomorphism in $K^+(R)$ to quasi-isomorphism in $K^+(B)$. So we have the following diagram:

$$\begin{array}{ccc}
 K^+(R) & \xrightarrow{K^+(F)} & K^+(B) \\
 \downarrow Q_R & & \downarrow Q_R \\
 I^\bullet \in K^+(R)[S_R^{-1}] & \xrightarrow{\exists ! \bar{F}} & D^+(B) \\
 \uparrow \wr & \nearrow RF & \\
 D^+(\mathcal{A}) & &
 \end{array}$$

Where \bar{F} exists by the universal property of localization. Then the derived functor RF could be defined with $R^i F(C^\bullet) = H^i(RF(C^\bullet))$.

The universal property of RF is as following: $RF : D^+(A) \rightarrow D^+(B)$ is exact and the diagram commutes:

$$\begin{array}{ccccc}
 & & D^+(A) & & \\
 & \nearrow Q_A & & \searrow RF & \\
 K^+(A) & & & & D^+(B) \\
 & \searrow K^+(F) & & \nearrow Q_B & \\
 & & K^+(B) & &
 \end{array}$$

with $\epsilon_F : Q_B \circ K^+(F) \rightarrow RF \circ Q_A$ being a morphism of functors (???).

Moreover, if $G : D^+(A) \rightarrow D^+(B)$ is another exact functor with $\epsilon_G : Q_B \circ K^+(F) \rightarrow G \circ Q_A$, then

there is an unique $y : RF \rightarrow G$ such that

$$\begin{array}{ccc} & Q_B \circ K^+(F) & \\ \epsilon_F \swarrow & & \searrow \epsilon_G \\ RF \circ Q_A & \xrightarrow{y \circ Q_A} & G \circ Q_A \end{array}$$

Now, one may ask that whether $RG \circ RF \cong R(G \circ F)$, the answer is no in generally, but there are spectral sequences so that ... Which is another story then...

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