

# Adaptivity versus Quarantine

leo.h

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## 1 Introduction

## 2 Results

Consider arguably the simplest adaptive version of the SIR model.

$$\frac{d}{dt}\rho_I = \beta\rho_{SI} - \gamma\rho_I \quad (1)$$

$$\frac{d}{dt}\rho_R = \gamma\rho_I \quad (2)$$

$$\frac{d}{dt}\rho_{SI} = -(\beta + \gamma + w)\rho_{SI} + \beta\rho_{SI}\frac{2\rho_{SS} - \rho_{SI}}{1 - \rho_I - \rho_R} \quad (3)$$

$$\frac{d}{dt}\rho_{SS} = -2\beta\frac{\rho_{SI}\rho_{SS}}{1 - \rho_I - \rho_R} + w\left[\frac{1 - \rho_I - \rho_R}{1 - \rho_I}\right]\rho_{SI} \quad (4)$$

We calculate the mean field critical point at which an epidemic has the chance to propagate. First we consider the disease-free state with  $\rho_I = \rho_R = \rho_{SI} = 0$  and with  $\rho_{SS} = \mu/2$ , i.e. all the links are of type  $SS$ . Either by looking at the Jacobian for that system at that given point or by looking directly at equation (3) together with an infinitesimal departure away from that point, we find that for all

$$\beta < \beta_c^{adp} = \frac{\gamma + w}{\mu - 1} \quad (5)$$

an epidemic dies out exponentially.

Would be great to have analytic expression for  $r_\infty$

We may now take a quarantine model, such as the one investigated by Brockman et al in ???. In order to compare these two models, we consider quarantining and rewiring only for the infected compartment. Here are the non-closed moment equations.

So now the equations would read:

$$\frac{d}{dt}[S] = -\beta[SI] \quad (6)$$

$$\frac{d}{dt}[I] = \beta[SI] - (\gamma + \kappa)[I] \quad (7)$$

$$\frac{d}{dt}[R] = \gamma[I] + \delta[X_I] \quad (8)$$

$$\frac{d}{dt}[X_I] = \kappa[I] - \delta[X_I] \quad (9)$$

$$\frac{d}{dt}[SI] = -(\beta + \gamma + w)[SI] + \beta[SSI] - \beta[ISI] - \kappa[SI] \quad (10)$$

$$\frac{d}{dt}[SS] = -\beta[SSI] + w * \frac{[S]}{[R] + [S]}[SI] \quad (11)$$

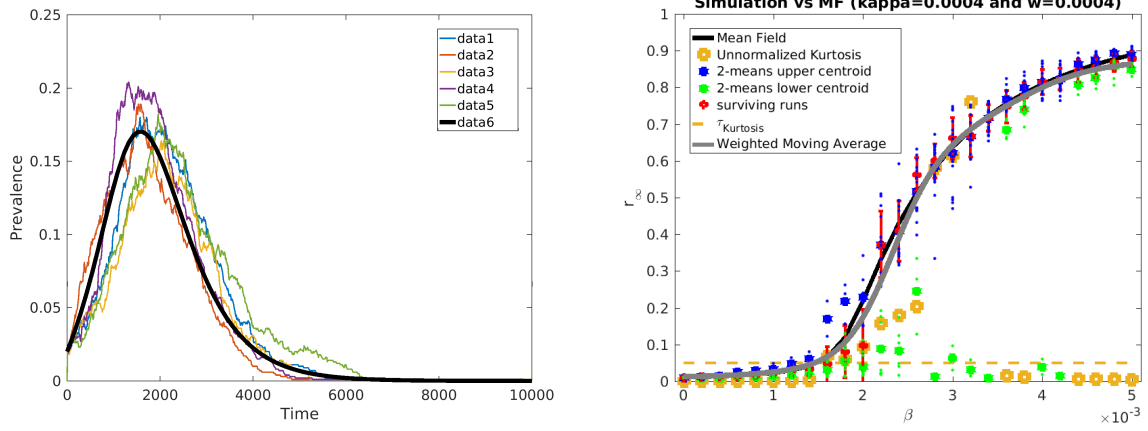


Figure 1: We show the comparison between the mean field model (16) and the numerical simulation. We fixed the parameters to  $N = 500, \mu = 15, \gamma = 1/40, \delta = 1/100$  and  $I(0) = 5$ . On the left we see sample paths for the disease prevalence (coloured) and the mean field approximation (black) at  $w = 0.0004, \kappa = 0.0004$  and  $\beta = 0.0025$ . On the right we see the values of  $r_\infty$  coming from the mean field approximation (black) and from the simulations (grey). The data points are divided into surviving runs and epidemic runs. I can explain this on our next skype meeting. It's not meant for presentation, just for you. We compare the simulations (left) with the mean field analysis (right) and indicate the critical curve, as calculated from (17). Again  $N = 500, \mu = 15, \gamma = 1/40, \delta = 1/100$  and  $I(0) = 5$ .

So putting these into 5 independent ODEs

$$\dot{\rho}_S = -\beta \rho_{SI} \quad (12)$$

$$\dot{\rho}_I = -(\kappa + \gamma)\rho_I + \beta \rho_{SI} \quad (13)$$

$$\dot{\rho}_R = +\gamma \rho_I + \delta(1 - \rho_S - \rho_I - \rho_R) \quad (14)$$

$$\dot{\rho}_{SI} = -(\beta + \gamma + w + \kappa)\rho_{SI} + \beta \rho_{SI} \frac{2\rho_{SS} - \rho_{SI}}{\rho_S} \quad (15)$$

$$\dot{\rho}_{SS} = -2\beta \frac{\rho_{SI}\rho_{SS}}{\rho_S} + w \frac{\rho_S}{\rho_S + \rho_R} \rho_{SI} \quad (16)$$

A basic calculation reveals:

$$\beta < \beta_c^{qurt+adp} = \frac{\gamma + w + \kappa}{\mu - 1} \quad (17)$$

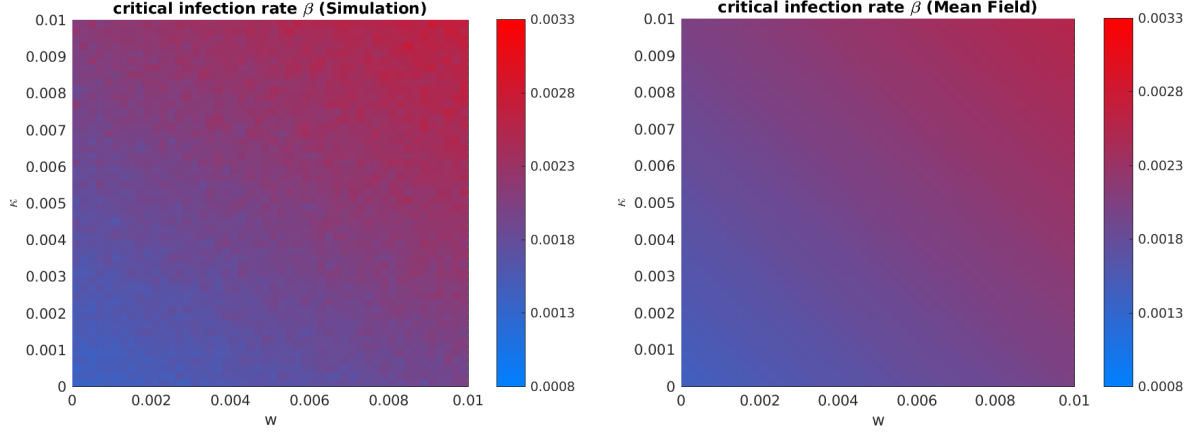


Figure 2: We depict the rate  $\beta$  at which the epidemics surpasses a threshold of  $r_\infty = 0.05$ . We compare the simulations (left) with the mean field analysis (right).  $N = 500, \mu = 15, \gamma = 1/40, \delta = 1/100$  and  $I(0) = 5$ .

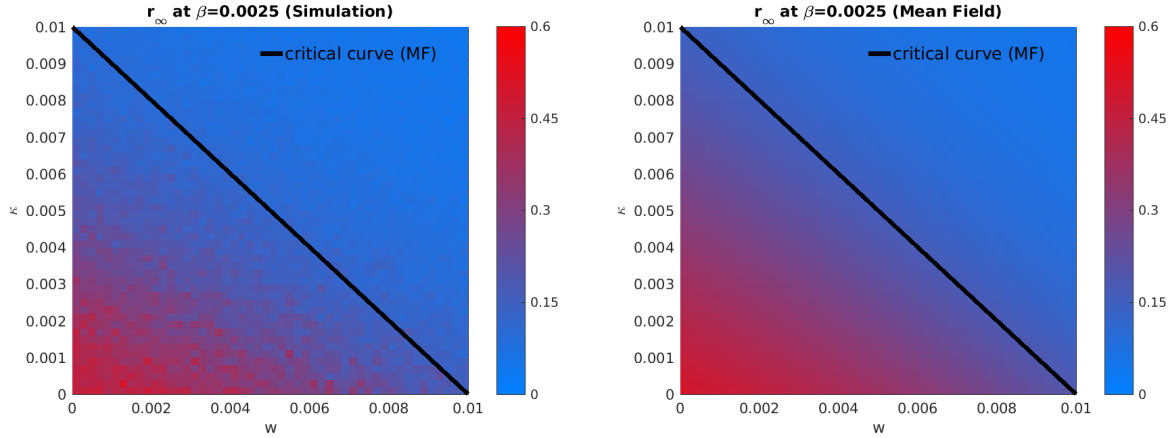


Figure 3: We depict the final fraction of recovered individuals  $r_\infty$  at an infection rate of  $\beta = 0.0025$ . We compare the simulations (left) with the mean field analysis (right) and indicate the critical curve, as calculated from (17). Again  $N = 500, \mu = 15, \gamma = 1/40, \delta = 1/100$  and  $I(0) = 5$ .

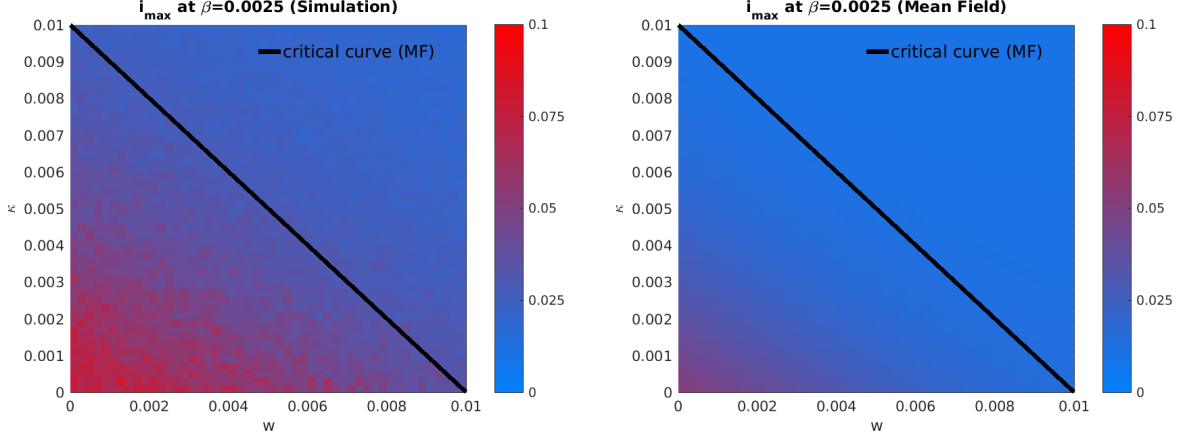


Figure 4: We depict the maximal fraction of infected individuals  $\widehat{[I]}/N$  at an infection rate of  $\beta = 0.0025$ . We compare the simulations (left) with the mean field analysis (right) and indicate the critical curve, as calculated from (17). We see here a strong difference. This can be explained by the fact that the simulations are random processes. The average of sample path maxima over many sample paths is not the same as the maximum of the average of sample paths. The former overestimates the expectation value of  $\rho_I$ . Again  $N = 500, \mu = 15, \gamma = 1/40, \delta = 1/100$  and  $I(0) = 5$ .

## A Moment Equations for the Adaptive and Quarantine Epidemics

### A.1 Adaptive SIR Model

$$\begin{aligned}
\frac{d}{dt}[S] &= -\beta[SI] \\
\frac{d}{dt}[I] &= \beta[SI] - \gamma[I] \\
\frac{d}{dt}[R] &= \gamma[I] \\
\frac{d}{dt}[SI] &= -(\beta + \gamma + w)[SI] + \beta[SSI] - \beta[ISI] \\
\frac{d}{dt}[SS] &= -\beta[SSI] + w * \frac{[S]}{[R] + [S]}[SI]
\end{aligned}$$

Eventually, after moment closure approximation this yields the following system for the densities:

$$\begin{aligned}
\frac{d}{dt}\rho_I &= \beta\rho_{SI} - \gamma\rho_I \\
\frac{d}{dt}\rho_R &= \gamma\rho_I \\
\frac{d}{dt}\rho_{SI} &\approx -(\beta + \gamma + w)\rho_{SI} + \beta\rho_{SI} \frac{2\rho_{SS} - \rho_{SI}}{1 - \rho_I - \rho_R} \\
\frac{d}{dt}\rho_{SS} &\approx -2\beta \frac{\rho_{SI}\rho_{SS}}{1 - \rho_I - \rho_R} + w \left[ \frac{1 - \rho_I - \rho_R}{1 - \rho_I} \right] \rho_{SI}
\end{aligned}$$

We have omitted  $\rho_S$  due to node conservation.

### A.2 adaptive SIR with infinite quarantine a la Brockmann 2020

Then imagine instead that not the links are removed, but that there is a state  $[X]$  which represents the quarantined state. In that state disease cannot be transmitted along the link. So all those links attached to

that node are removed from the pool of transmittable links. Effectively this behaves like link-removal.

$$\begin{aligned}
\frac{d}{dt}[S] &= -\beta[SI] - \kappa_0[S] \\
\frac{d}{dt}[I] &= \beta[SI] - \gamma[I] - (\kappa_0 + \kappa)[I] \\
\frac{d}{dt}[R] &= \gamma[I] \\
\frac{d}{dt}[X] &= \kappa_0[S] + (\kappa_0 + \kappa)[I] \\
\frac{d}{dt}[SI] &= -(\beta + \gamma + w)[SI] + \beta[SSI] - \beta[ISI] - 2\kappa_0[SI] - \kappa[SI] \\
\frac{d}{dt}[SS] &= -\beta[SSI] + w * \frac{[S]}{[R] + [S]}[SI] - 2\kappa_0[SS]
\end{aligned}$$

This latter model has the following representation:

$$\begin{aligned}
\frac{d}{dt}\rho_S &= -\beta\rho_{SI} - \kappa_0\rho_S \\
\frac{d}{dt}\rho_I &= \beta\rho_{SI} - \gamma\rho_I - (\kappa_0 + \kappa)\rho_I \\
\frac{d}{dt}\rho_R &= \gamma\rho_I \\
\frac{d}{dt}\rho_{SI} &\approx -(\beta + \gamma + w + 2\kappa_0 + \kappa)\rho_{SI} + \beta\rho_{SI} \frac{2\rho_{SS} - \rho_{SI}}{\rho_S} \\
\frac{d}{dt}\rho_{SS} &\approx -2\beta \frac{\rho_{SI}\rho_{SS}}{\rho_S} + w \left[ \frac{\rho_S}{\rho_S + \rho_R} \right] \rho_{SI} - 2\kappa_0\rho_{SS}
\end{aligned}$$

### A.3 adaptive SIR with quarantine an return rate *delta*

Now let's consider, as before, that there is a quarantined compartment, but quarantine doesn't last for ever. We still allow for both, the susceptible population as well as the infected population to be quarantined at a rate  $\kappa_0$  and  $\kappa_0 + \kappa$  respectively. If they are healthy quarantinees, then they'd go back into the healthy compartment. If they are infected, they go into the recovered compartment, respectively at a rate  $\delta$

So now the equations would read:

$$\begin{aligned}
\frac{d}{dt}[S] &= -\beta[SI] - \kappa_0[S] + \delta[X_S] \\
\frac{d}{dt}[I] &= \beta[SI] - \gamma[I] - (\kappa_0 + \kappa)[I] \\
\frac{d}{dt}[R] &= \gamma[I] + \delta[X_I] \\
\frac{d}{dt}[X_S] &= \kappa_0[S] - \delta[X_S] \\
\frac{d}{dt}[X_I] &= (\kappa_0 + \kappa)[I] - \delta[X_I] \\
\frac{d}{dt}[SI] &= -(\beta + \gamma + w)[SI] + \beta[SSI] - \beta[ISI] - 2\kappa_0[SI] - \kappa[SI] + \delta[X_SI] \\
\frac{d}{dt}[SS] &= -\beta[SSI] + w * \frac{[S]}{[R] + [S]}[SI] - 2\kappa_0[SS] + \delta[X_SS] \\
\frac{d}{dt}[X_SS] &= -\delta[X_SS] + 2\kappa_0[SS] - \beta[X_S SI] \\
\frac{d}{dt}[X_SI] &= -\delta[X_SI] - \gamma[X_SI] - (\kappa_0 + \kappa)[X_SI] + \kappa_0[SI]
\end{aligned}$$

So putting these into 8 independent ODEs is a bit more involved:

$$\begin{aligned}
\frac{d}{dt}\rho_S &= -\beta\rho_{SI} - \kappa_0\rho_S + \delta\rho_{X_s} \\
\frac{d}{dt}\rho_R &= +\gamma(1 - \rho_R - \rho_S - \rho_{X_s} - \rho_{X_i}) + \delta\rho_{X_i} \\
\frac{d}{dt}\rho_{X_s} &= +\kappa_0\rho_S - \delta\rho_{X_s} \\
\frac{d}{dt}\rho_{X_i} &= +(\kappa_0 + \kappa)(1 - \rho_S - \rho_R - \rho_{X_s} - \rho_{X_i}) - \delta\rho_{X_i} \\
\\
\frac{d}{dt}\rho_{SI} &= -(\beta + \gamma + w + 2\kappa_0 + \kappa)\rho_{SI} + \beta\rho_{SI}\frac{2\rho_{SS} - \rho_{SI}}{\rho_S} + \delta\rho_{IX_s} \\
\frac{d}{dt}\rho_{SS} &= -2\beta\frac{\rho_{SI}\rho_{SS}}{\rho_S} + w\frac{\rho_S}{\rho_S + \rho_R}\rho_{SI} - 2\kappa_0\rho_{SS} + \delta\rho_{SX_s} \\
\frac{d}{dt}\rho_{SX_s} &= -\delta\rho_{SX_s} + 2\kappa_0\rho_{SS} - \beta\rho_{SX_s}\frac{\rho_{SI}}{\rho_S} \\
\frac{d}{dt}\rho_{IX_s} &= -\delta\rho_{IX_s} - \gamma\rho_{IX_s} - (\kappa_0 + \kappa)\rho_{IX_s} + \kappa\rho_{SI}
\end{aligned}$$