University of Science and Technology of China

Combinatorics, 2022 Fall

Exercises

1. Which of these sentences are propositions? What are the truth values of those that are propositions?
    i. Nanjing (南京) is the capital of Jiangsu (江苏).
    ii. Chongqing (重庆) is the capital of Sichuan (四川).
    iii. $2 + 3 = 5$.
    iv. $5 + 7 = 10$.
    v. $x + 2 = 11$.
    vi. Answer this question.
    vii. $x + y = y + x$ for every pair of real number x and y.
2. Let $Q(x, y)$ denote the statement "x is the capital of y." What are these truth values?
    i. Q("Hangzhou (杭州)", "Zhejiang (浙江)")
    ii. Q("Shenzhen (深圳)", "Guangdong (广东)")
    iii. Q("Qingdao (青岛)", "Shandong (山东)")
    iv. Q("Yinchuan (银川)", "Ningxia (宁夏)")
3. (Rosen, 2003, pp.15-20:1) Which of these sentences are propositions? What are the truth values of those that are propositions?
    i. Boston is the capital of Massachusetts.
    ii. Miami is the capital of Florida.
    iii. $2 + 3 = 5$.
    iv. $5 + 7 = 10$.
    v. $x + 2 = 11$.
    vi. Answer this question.

vii. $x + y = y + x$ for every pair of real numbers x and y.

4. (Rosen, 2003, pp.26-28:1) Use truth tables to verify these equivalences.

   i. $p \wedge \mathrm{T} \equiv p$

   ii. $p \vee \mathrm{F} \equiv p$

   iii. $p \wedge \mathrm{F} \equiv \mathrm{F}$

   iv. $p \vee \mathrm{T} \equiv \mathrm{T}$

   v. $p \vee p \equiv p$

   vi. $p \wedge p \equiv p$

5. (Rosen, 2003, pp.26-28:2) Show that $\neg(\neg p)$ and p are logically equivalent.

6. (Rosen, 2003, pp.26-28:3) Use truth tables to verify the commutative laws

   i. $p \wedge q \equiv q \wedge p$

   ii. $p \vee q \equiv q \vee p$

7. (Rosen, 2003, pp.26-28:4) Use truth table to verify the associative laws

   i. $(p \vee q) \vee r \equiv p \vee (q \vee r)$

   ii. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

8. (Rosen, 2003, pp.26-28:5) Use a truth table to verify the distributive law

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r).$$

9. (Rosen, 2003, pp.26-28:6) Use a truth table to verify the equivalence

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

10. (Rosen, 2003, pp.26-28:7) Show that each of these implications is a tautology by using truth tables.

i. $(p \wedge q) \to p$

ii. $p \to (p \vee q)$

iii. $\neg p \to (p \to q)$

iv. (p $\wedge q) \to (p \to q)$

v. $\neg(p \to q) \to p$

vi. $\neg(p \to q) \to \neg q$

11. (Rosen, 2003, pp.26-28:8) Show that each of these implications is a tautology by using truth tables.

i. $[\neg p \wedge (p \vee q)] \to q$

ii. $[(p \to q) \wedge (q \to r)] \to (p \to r)$

iii. $[p \wedge (p \to q)] \to q$

iv. $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$

12. (Rosen, 2003, pp.26-28:9) Show that each implication in Exercise 10 is a tautology without using truth tables.

13. (Rosen, 2003, pp.26-28:10) Show that each implication in Exercise 11 is a tautology without using truth tables.

14. (Rosen, 2003, pp.26-28:11) Use truth tables to verify the absorption laws.

i. $p \vee (p \wedge q) \equiv p$

ii. $p \wedge (p \vee q) \equiv p$

15. (Rosen, 2003, pp.26-28:12) Determine whether $(\neg p \wedge (p \to q)) \to \neg q$ is a tautology.

16. (Rosen, 2003, pp.26-28:13) Determine whether $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

17. (Rosen, 2003, pp.40-44:1) Let $P(x)$ denote the statement "$x \leq 4$." What are the truth values?

i. $P(0)$

ii. $P(4)$

iii. $P(6)$

18. (Rosen, 2003, pp.40-44:9) Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++". Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The universe of discourse for quantifiers consists of all students at your school.

    i. There is a student at your school who can speak Russian and who knows C++.
    ii. There is a student at your school who can speak Russian but who doesn't know C++.
    iii. Every student at your school either can speak Russian or knows C++.
    iv. No student at your school can speak Russian or knows C++.

19. (Rosen, 2003, pp.51-56:1) Translate these statements into Englis, where the universe od discourse for each variable consists of all real numbers.

    i. $\forall x \exists y (x < y)$
    ii. $\forall x \forall y (((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$
    iii. $\forall x \forall y \exists z (xy = z)$

20. (Rosen, 2003, pp.73-77:1) What rule of inference is used in each of these arguments?

    i. Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.
    ii. Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.
    iii. If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.
    iv. If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.
    v. If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, If I go swimming, then I will sunburn.

21. (Rosen, 2003, pp.85-86:1) List the members of these sets.

    i. $\{x|\ x \text{ is a real number such that } x^2 = 1\}$
    ii. $\{x|\ x \text{ is a positive integer less than } 12\}$
    iii. $\{x|\ x \text{ is a square of an integer and } x < 100\}$
    iv. $\{x|\ x \text{ is a integer such that } x^2 = 2\}$

22. (Rosen, 2003, pp.94-97:17) Let A, B, and C be sets. Show that

    i. $A \cup (B \cup C) = (A \cup B) \cup C$.
    ii. $A \cap (B \cap C) = (A \cap B) \cap C$.
    iii. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

23. (Rosen, 2003, pp.108-111:1) Why is f not a function from $\boldsymbol{R}$ to $\boldsymbol{R}$ if

    i. $f(x) = \frac{1}{x}$?
    ii. $f(x) = \sqrt{x}$?
    iii. $f(x) = \pm\sqrt{x^2+1}$?

24. (Rosen, 2003, pp.108-111:15) Determine whether the function $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

    i. $f(m, n) = m + n$.
    ii. $f(m, n) = m^2 + n^2$.
    iii. $f(m, n) = m$.
    iv. $f(m, n) = |n|$.
    v. $f(m, n) = m - n$.

25. (Rosen, 2003, pp.236-238:1) Find these terms of the sequence $\{a_n\}$ where $a_n = 2 \cdot (-3)^n + 5^n$.

    i. a_0
    ii. a_1
    iii. a_4
    iv. a_5

26. (Rosen, 2003, pp.236-238:13) What are the values of these sums?

    i. $\sum_{k=1}^{5}(k+1)$
    ii. $\sum_{j=0}^{4}(-2)^j$

iii. $\sum_{i=1}^{10} 3$

iv. $\sum_{j=0}^{8}(2^{j+1} - 2^j)$

27. (Brualdi, 2004, pp.20-25:1) Show that an m-by-n chessboard has a perfect cover by dominoes if and only if at least one of m and n is even.

28. (Brualdi, 2004, pp.20-25:2) Consider an m-by-n chessboard with m and n both odd. To fix the notation, suppose that the square in the upper left-hand corner is colored white. Show that if a white square is cut out anywhere on the board, the resulting pruned board has a perfect cover by dominoes.

29. (Brualdi, 2004, pp.20-25:3) Imagine a prison consisting of 64 cells arranged like the squares of an 8-by-8 chessboard. There are doors between all adjoining cells. A prisoner in one of the corner cells is told that he will be released, provided he can get into the diagonally opposite corner cell after passing through every other cell exactly once. Can the prisoner obtain his freedom?

30. (Brualdi, 2004, pp.20-25:4) (a) Let $f(n)$ count the number of different perfect covers of a 2-by-n chessboard by dominoes. Evaluate $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$. Try to find (and verify) a simple relation that the counting function f satisfies. Use this relation to compute $f(12)$. (b) Let $g(n)$ be the number of different perfect covers of a 3-by-n chessboard by dominoes. Evaluate $g(1)$, $g(2)$, $\cdots$, $g(6)$.

31. (Brualdi, 2004, pp.20-25:5) Find the number of different perfect covers of a 3-by-4 chessboard by dominoes.

32. (Brualdi, 2004, pp.20-25:6) Show how to cut a cube, 3 feet on an edge, into 27 cubes, 1 foot on an edge, using exactly 6 cuts, but making a nontrivial rearrangement of the pieces between two of the cuts.

33. (Brualdi, 2004, pp.20-25:7) Consider the following three-dimensional version of the chessboard problem: A *three-dimensional domino* is defined to be the geometric figure that results when two cubes, 1 unit on an edge, are joined along a face. Show that it is possible to construct a cube n units on an edge from dominoes if and only if n is even. If n is odd, is it possible to construct a cube n units on an edge with a 1-by-1 hole in the middle? (Hint: Think of a cube n units on an edge as being composed of n^3 cubes 1 unit on an edge. Color the cubes alternatively black and white.)

34. (Brualdi, 2004, pp.20-25:8) Let a and b be positive integers with a a factor of b. Show that an m-by-n board has a perfect cover by a-by-b pieces if and only if a is a factor of both m and n and b is a factor of either m or n. (Hint: Partition the a-by-b pieces into a 1-by-b pieces.)

35. (Brualdi, 2004, pp.20-25:9) Use Exercise 34 to conclude that when a is a factor of b, an m-by-n board has a perfect cover by a-by-b pieces if and only if it has a trivial perfect cover in which all the pieces are oriented the same way.

36. (Brualdi, 2004, pp.20-25:10) Show that the conclusions of Exercises 34 and 35 need not hold when a is not a factor of b.

37. (Brualdi, 2004, pp.20-25:11) Verify that there is no magic square of order 2.

38. (Brualdi, 2004, pp.20-25:12) Use de la Loubère's method to construct a magic square of order 7.

39. (Brualdi, 2004, pp.20-25:13) Use de la Loubère's method to construct a magic square of order 9.

40. (Brualdi, 2004, pp.20-25:14) Construct a magic square of order 6.

41. (Brualdi, 2004, pp.20-25:15) Show that a magic square of order 3 must have a 5 in the middle position. Deduce that there are exactly 8 magic squares of order 3.

42. (Brualdi, 2004, pp.20-25:16) Can the partial square below be completed to a magic square of order 4?

$$\begin{bmatrix} 2 & 3 & & \\ 4 & & & \\ & & & \\ & & & \end{bmatrix}$$

43. (Brualdi, 2004, pp.20-25:17) Show that the result of replacing every integer a in a magic square of order n with n^2+1-a is a magic square of order n.

44. (Brualdi, 2004, pp.20-25:18) Let n be a positive integer divisible by 4, say $n = 4m$. Consider the following construction of an n-by-n array:

    i. Proceeding from left to right and from first row to nth row, fill in the places of the array with the integers $1, 2, \cdots, n^2$ in order.

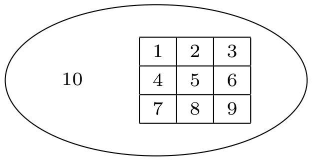
    ii. Partition the resulting square array into m^2 4-by-4 smaller arrays. Replace each number a on the two diagonals of each of the 4-by-4 arrays with its "complement" n^2+1-a.

    Verify that this construction produces a magic square of order n when $n = 4$ and $n = 8$. (Actually it produces a magic square for each n divisible by 4.)

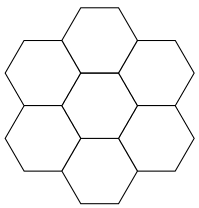
45. (Brualdi, 2004, pp.20-25:19) Show that there is no magic cube of order 2.

46. (Brualdi, 2004, pp.20-25:20) Show that there is no magic cube of order 4.

47. (Brualdi, 2004, pp.20-25:21) Show that the following map of 10 countries (1, 2, $\cdots$, 10) can be colored with three but no fewer colors. If the colors used are red, white, and blue, determine the number of different colorings.



48. (Brualdi, 2004, pp.20-25:22) (a) Does there exist a *magic hexagon* of order 2? That is, is it possible to arrange the numbers 1, 2, $\cdots$, 7 in the hexagonal array below so that all of the nine "line" sums (the sum of the numbers in the hexagonal boxes penetrated by a line through midpoints of opposite sides) are the same?



(b) Construct a magic hesagon of order 3, that is, arrange the integers 1, 2, $\cdots$, 19 in a hexagonal array (three integers on a side) in such a way that all of the fifteen "line" sums are the same (namely, 38).

49. (Brualdi, 2004, pp.20-25:23) Construct a pair of orthogonal Latin squares of order 4.

50. (Brualdi, 2004, pp.20-25:24) Construct Latin squares of order 5 and 6.

51. (Brualdi, 2004, pp.20-25:25) Find a general method for constructing a Latin square of order n.

52. (Brualdi, 2004, pp.20-25:26) A 6-by-6 chessboard is perfectly covered with 18 dominoes. Prove that it is possible to cut it either horizontally or vertically into two nonempty pieces without cutting through a domino; that is, prove that there must be a fault-line.

53. (Brualdi, 2004, pp.20-25:27) Construct a perfect cover of an 8-by-8 chessboard with domnioes having no fault-line.

54. (Brualdi, 2004, pp.20-25:28) Determine all shortest routes from A to B in the system of intersections and streets (graph) in the figure shown. The numbers on the streets represent the lengths of the streets measured in terms of some unit.

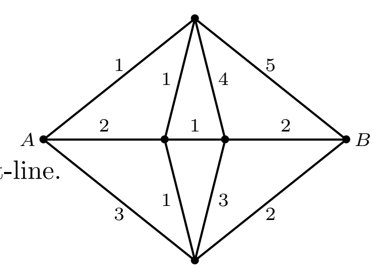


Figure in Question 54

55. (Brualdi, 2004, pp.20-25:29) Consider 3-heap Nim with piles of sizes 1, 2, and 4. Show that this game is unbalanced and determine a first move for player I.

56. (Brualdi, 2004, pp.20-25:30) Is 4-pile Nim with heaps of sizes 22, 19, 14, and 11 balanced or unbalanced? Player I's first move is to remove 6 coins from the heap of size 19. What should player II's first move be?

57. (Brualdi, 2004, pp.20-25:31) Consider 5-pile Nim with heaps of sizes 10, 20, 30, 40, and 50. Is this game balanced? Determine a first move for player I.

58. (Brualdi, 2004, pp.20-25:32) Show that player I can always win a Nim game in which the number of heaps with an odd number of coins is odd.

59. (Brualdi, 2004, pp.20-25:33) Show that in an unbalanced game of Nim in which the largest unbalanced bit is the jth bit, player I can always balance the game by removing coins from any heap the base 2 numeral of whose number has a 1 in the jth bit.

60. (Brualdi, 2004, pp.20-25:34) Suppose we change the object of Nim so that the player who takes the last coin loses (the misère version). Show that the following is a winning strategy: Play as in ordinary Nim until all but exactly on heap contains a single coin. Then remove either all or all but one of the coins of the exceptional heap so as to leave an *odd* number of heaps of size 1.

61. (Brualdi, 2004, pp.20-25:35) A game is played between two players, alternating turns as follows. The game starts with an empty pile. When it is his turn a player may add either 1, 2, 3, or 4 coins to the pile. The person who adds the 100th coin to the pile is the winner. Determine whether it is the first or second player who can guarantee a win in this game. What it the winning strategy to follow?

62. (Brualdi, 2004, pp.20-25:36) Suppose that in Exercise 61, the player who adds the 100th coin loses. Now who wins and how?

63. (Brualdi, 2004, pp.20-25:37) Eight people are at a party and pair off to form four teams of two. In how many ways can this be done? (This is sort of an "unstructured" domino covering problem.)

64. (Brualdi, 2004, pp.20-25:38) A Latin square of order n is *idempotent* provided the integers $1, 2, \cdots, n$ occur, in this order, in the diagonal positions $(1,1)$, $(2,2)$, $\cdots$, (n,n), and is *symmetric* provided the integer in position (i,j) equals the integer in position (j,i) whenever $i \neq j$. There is no symmetric, idempotent Latin square of order 3. Show that there is no symmetric, idempotent Latin square of order 4. What about order n in general, where n is even?

65. (Brualdi, 2004, pp.20-25:39) Take an set of $2n$ points in a plane with no three collinear, and then arbitrarily color each point red or blue. Prove that it is always possible to pair up the red points with the blue points by drawing line segments connecting them so that no two of the line segments intersect.

66. (Brualdi, 2004, pp.20-25:40) Consider an n-by-n board and L-tetrominos (4 squares joins in the shape of an L). Show that if there is a perfect cover of the n-by-n board with L-tetrominos, then n is divisible by 4. What about m-by-n boards?

67. (Brualdi, 2004, pp.39-43:1) Concerning Application 4, show that there is a succession of days during which the chess master will have played exactly k games, for each $k = 1, 2, \ldots, 21$. (The case $k = 21$ is the case treated in Application 4.) Is it possible to conclude that there is a succession of days during which the chess master will have played exactly 22 games?

68. (Brualdi, 2004, pp.39-43:2) Concerning Application 5, show that if 100 integers are chosen from 1, 2, $\cdots$, 200, and one of the integers chosen is less than 16, then there are two chosen numbers such that one of hem is divisible by the other.

69. (Brualdi, 2004, pp.39-43:3) Generalize Application 5 by choosing (how many?) integers from the set

$$\{1, 2, \cdots, 2n\}.$$

70. (Brualdi, 2004, pp.39-43:4) Show that if $n+1$ integers are chosen from the set $\{1, 2, \cdots, 2n\}$, then there are always two which differ by 1.

71. (Brualdi, 2004, pp.39-43:5) Show that if $n+1$ integers are chosen from the set $\{1, 2, \cdots, 3n\}$ then there are always two which differ by at most 2.

72. (Brualdi, 2004, pp.39-43:6) Generalize Exercises 70 and 71.

73. (Brualdi, 2004, pp.39-43:7) Show that for any given 52 integers there exist two of them whose sum, or else whose difference, if divisible by 100.

74. (Brualdi, 2004, pp.39-43:8) Use the pigeonhole principle to prove that the decimal expansion of a rational number m/n eventually is repeating. For example,

$$34,478/99,900 = .34512512512512512\cdots.$$

75. (Brualdi, 2004, pp.39-43:9) In a room there are 10 people, none of whom are older than 60 (ages are given in whole numbers only) but each of whom is at least 1 year old. Prove that one can always find two groups of people (with no common person) the sum of whose ages is the same. Can 10 be replaced by a smaller number?

76. (Brualdi, 2004, pp.39-43:10) A child watches TV at least one hour each day for 7 weeks but never more than 11 hours in any one week. Prove that there is some period of consecutive days in which the child watches excactly 20 hours of TV. (It is assumed that the child watches TV for a whole number of hours each day.)

77. (Brualdi, 2004, pp.39-43:11) A student has 37 days to prepare for an examination. From past experience she knows that she will require no more than 60 hours of study. She also wishes to study at least 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day, however), there is a succession of days during which she will have studied exactly 13 hours.

78. (Brualdi, 2004, pp.39-43:12) Show by example that the conclusion of the Chinese remainder theorem (Application 6) need not hould when m and n are not relatively prime.

79. (Brualdi, 2004, pp.39-43:13) Let S be a set of 6 points in the plane, with no 3 of the points collinear. Color either red or blue each of the 15 line segments determined by the points of S. Show that there are at least two triangles determined by points of S which are either red triangles or blue triangles. (Both may be red, or both may be blue, or one may be red and the other blue.)

80. (Brualdi, 2004, pp.39-43:14) A bag contains 100 apples, 100 bananas, 100 oranges, and 100 pears. If I pick one pieace of fruit out the bag every minute, how long will it be before I am assured of having picked at least a dozen pieces of fruit of the same kind?

81. (Brualdi, 2004, pp.39-43:15) Prove that, for any $n+1$ integers $a_1, a_2, \cdots, a_{n+1}$, there exist two of the integers a_i and a_j with $i \neq j$ such that $a_i - a_j$ is divisible by n.

82. (Brualdi, 2004, pp.39-43:16) Prove that in a group of $n > 1$ people there are two who have the same number of acquaintances in the group. (It is assumed that no one is acquainted with him or herself.)

83. (Brualdi, 2004, pp.39-43:17) There are 100 people at a party. Each person has an even number (possibly zero) of acquaintances. Prove that there are three people at the party with the same number of acquaintances.

84. (Brualdi, 2004, pp.39-43:18) Prove that of any five points chosen within a square of side length 2, there are two whose distance apart is at most $\sqrt{2}$.

85. (Brualdi, 2004, pp.39-43:19)

    i. Prove that of any five points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{2}$.

    ii. Prove that of any ten points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{3}$.

    iii. Determine an integer m_n such that if m_n points are chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{n}$.

86. (Brualdi, 2004, pp.39-43:20) Prove that $r(3,3,3) \leq 17$.

87. (Brualdi, 2004, pp.39-43:21) Prove that $r(3,3,3) \geq 17$ by exhibiting a coloring, with colors red, blue, and green, of the line segments joining 16 points with the property that there do not exist 3 points such that the 3 line segments joining them are all colored the same.

88. (Brualdi, 2004, pp.39-43:22) Prove that

$$r(\underbrace{3,3,\ldots,3}_{k+1}) \leq (k+1)(r(\underbrace{3,3,\ldots,3}_{k}) - 1) + 2.$$

Using this result to obtain an upper bound for

$$r(\underbrace{3,3,\ldots,3}_{n}).$$

89. (Brualdi, 2004, pp.39-43:23) The line segments joining 10 points are arbitrarily colored red or blue. Prove that there must exist 3 points such that the 3 line segments joining them are all red, or 4 points such that the 6 line segments joining them are all blue [that is, $r(3,4) \leq 10$].

90. (Brualdi, 2004, pp.39-43:24) Let q_3 and t be positive integers with $q_3 \geq t$. Determine the Ramsey number $r_t(t,t,q_3)$.

91. (Brualdi, 2004, pp.39-43:25) Let q_1, q_2, $\cdots$, q_k, t be positive integers, where $q_1 \geq t$, $q_2 \geq t$, $\cdots$, $q_k \geq t$. Let m be the largest of q_1, q_2, $\cdots$, q_k. Show that

$$r_t(m,m,\cdots,m) \geq r_t(q_1,q_2,\cdots,q_k).$$

Conclude that, to prove Ramsey's theorem, it is enough to prove it in the case that $q_1 = q_2 = \cdots = q_k$.

92. (Brualdi, 2004, pp.39-43:26) Suppose that the mn of a matching band are standing in a rectangular formation of m rows and n columns in such a way that in each row each person is taller than the one to her or his left. Suppose that the leader rearranges the people in each column in increasing order of height from front to back. Show that the rows are still arranged in increasing order of height from left to right.

93. (Brualdi, 2004, pp.39-43:27) A collection of subsets of $\{1,2,\cdots,n\}$ has the property that each pair of subsets has at least one elements in common. Prove that there are at most 2^{n-1} subsets in the collection.

94. (Brualdi, 2004, pp.39-43:28) At a dance-hop there are 100 men and 20 women. For each i from 1, 2, $\cdots$, 100, the ith man selects a group of a_i women as potential dance partners (his dance list), but in such a way that given any group of 20 men, it is always possible to pair the 20 men up with the 20 women with each man paired up with a woman on his dance list. What is the smallest sum $a_1 + a_2 + \cdots + a_n$ that will *guarantee* this?

95. (Brualdi, 2004, pp.75-82:1) For each of the four combinations of the two properties i and ii, count the number of four-digit numbers whose digits are either 1, 2, 3, 4, or 5,

    i. The digits are distinct.

    ii. The number is even.

    Note that there are four problems here, $\emptyset$ (no further restriction), $\{a\}$ (property i holds), $\{b\}$ (property ii holds), $\{a, b\}$ (both properties i and ii hold).

96. (Brualdi, 2004, pp.75-82:2) How many orderings are there for a deck of 52 cards if all the cards of the same suit are together?

97. (Brualdi, 2004, pp.75-82:3) In how many ways a poker hand (5 cards) be dealt? How many different poker hands are there?

98. (Brualdi, 2004, pp.75-82:4) How many distinct positive divisors do each of the following numbers have?

    i. $3^4 \times 5^2 \times 7^6 \times 11$

    ii. 620

    iii. 10^{10}

99. (Brualdi, 2004, pp.75-82:5) Determine the largest power of 10 that is a factor of the following numbers (equivalently, the number of terminal 0's, using ordinary base 10 representation),

    i. 50!

    ii. 1000!

100. (Brualdi, 2004, pp.75-82:6) How many integers greater than 5400 have both of the following properties?

    i. The digits are distinct

    ii. The digits 2 and 7 do not occur

101. (Brualdi, 2004, pp.75-82:7) Determine the number of poker hands of the following types,

    i. full houses (3 cards of one rank and 2 of a different rank).

    ii. straights (5 consecutive ranks).

iii. flushes (5 cards of the same suit).

iv. straight flushes (5 consecutive cards of the same suit).

v. exactly two pairs (2 cards of one rank, 2 cards of another rank, and 1 card of a third rank).

vi. exactly one pair (2 cards of one rank, and 3 cards of three other and different ranks).

102. (Brualdi, 2004, pp.75-82:8) In how many ways can six men and six women be seated at a round table if the men and women are to sit in alternate seats?

103. (Brualdi, 2004, pp.75-82:9) In how many ways can 15 people be seated at a round table if B refuses to sit next to A? What if B only refuses to sit on A's right?

104. (Brualdi, 2004, pp.75-82:10) A committee of 5 is to be chosen from a club that boasts a membership of 10 men and 12 women. How many ways can the committee to formed if it is to contain at least 2 women? How many ways if, in addition, one particular man and one particular woman who are members of the club refuse to serve together on the committee?

105. (Brualdi, 2004, pp.75-82:11) How many sets of 3 numbers each can be formed from the numbers $\{1, 2, 3, \cdots, 20\}$ if no 2 consecutive numbers are to be in a set?

106. (Brualdi, 2004, pp.75-82:12) A football team of 11 players is to be selected from a set of 15 players, 5 of whom can play only in the backfield, 8 of whom can play only on the line, and 2 of whom can play either in the backfield or on the line. Assuming a football team has 7 men on the line and 4 in the backfield, determine the number of football teams possible.

107. (Brualdi, 2004, pp.75-82:13) There are 100 students at a school and three dormitories, A, B and C, with capacities 25, 35, and 40, respectively.

i. How many ways are there to fill the dormitories?

ii. Suppose that, of the 100 students, 50 are men and 50 are women and that A is an all-men's dorm, B is an all-women's dorm, and C is co-ed. How many ways are there to fill the dormitories?

108. (Brualdi, 2004, pp.75-82:14) A classroom has 2 rows of 8 seats each. There are 14 students, 5 of whom always sit in the front row and 4 of whom always sit in the back row. In how many ways can the students be seated?

109. (Brualdi, 2004, pp.75-82:15) At a party there are 15 men and 20 women.

i. How many ways are there to form 15 couples consisting of one man and one woman?

ii. How many ways are there to form 10 couples consisting of one man and one woman?

110. (Brualdi, 2004, pp.75-82:16) Prove that

$$\binom{n}{r} = \binom{n}{n-r}$$

by using a combinatorial argument and not the values of these numbers as given in Theorem 3.3.1.

111. (Brualdi, 2004, pp.75-82:17) In how many ways can 6 indistinguishable rooks be placed on a 6-by-6 board so that no two rooks can attack one another? In how many ways if there are 2 red and 4 blue rooks?

112. (Brualdi, 2004, pp.75-82:18) In how many ways can 2 red and 4 blue rooks be placed on an 8-by-8 board so that no two rooks can attack one another?

113. (Brualdi, 2004, pp.75-82:19) We are given 8 rooks, 5 of which are red and 3 of which are blue.

i. In how many ways can the 8 rooks be placed on an 8-by-8 chessboard so that no two rooks can attack on another?

ii. In how many ways can the 8 rooks be placed on a 12-by-12 chessboard so that no two rooks can attack one another?

114. (Brualdi, 2004, pp.75-82:20) Determine the number of circular permutations of $\{0, 1, 2, \cdots, 9\}$ in which 0 and 9 are not opposite. (Hint: Count those in which 0 and 9 are opposite.)

115. (Brualdi, 2004, pp.75-82:21) How many permutations are there of the letters of the word ADDRESSES? How many 8-permutations are there of these 9 letters?

116. (Brualdi, 2004, pp.75-82:22) A footrace takes place between 4 runners. If ties are allowed (even all 4 runners finishing at the same time), how many ways are there for the race to finish?

117. (Brualdi, 2004, pp.75-82:23) Bridge is played with 4 players and an ordinary deck of 52 cards. Each player begins with a hand of 13 cards. In how many ways can a bridge game start?

118. (Brualdi, 2004, pp.75-82:24) A roller coaster has 5 cars, each containing 4 seats, two in front and two in back. There are 20 people ready for a ride. In how many ways can the ride begin? Same question but now a certain two people want to sit in different cars.

119. (Brualdi, 2004, pp.75-82:25) A ferris wheel has 5 cars, each containing 4 seats in a row. There are 20 people ready for a ride. In how many ways can the ride begin? Same question but now a certain two people want to sit in different cars.

120. (Brualdi, 2004, pp.75-82:26) A group of mn people are to be arranged into m teams each with n players.

   i. Determine the number of ways if each team has a different name.

   ii. Determine the number of ways if the teams don't have names.

121. (Brualdi, 2004, pp.75-82:27) In how many ways can 5 indistinguishable rooks be placed on an 8-by-8 chessboard so that no rook can attack another and neither the first row nor the first column is empty?

122. (Brualdi, 2004, pp.75-82:28) A secretary works in a building located 9 blocks east and 8 blocks north of his home. Every day he walks 17 blocks to work (the map shown).

   i. How many different routes are possible for him?

   ii. How many different routes if the street in the easterly direction, which begins 4 blocks east and 3 blocks north of his home, is under water (and he can't swim)? (Hint: Count the routes that use the block under water.)

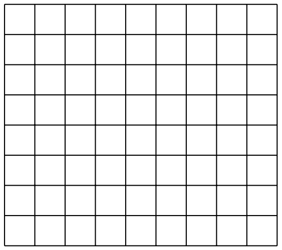


Figure in Question 122

123. (Brualdi, 2004, pp.75-82:29) Let S be a multiset with repetition numbers n_1, n_2, $\cdots$, n_k where $n_1 = 1$. Let $n = n_2 + \cdots + n_k$. Prove that the number of circular permutations of S equals

$$\frac{n!}{n_2! \cdots n_k!}.$$

124. (Brualdi, 2004, pp.75-82:30) We are to seat 5 men, 5 women, and 1 dog in a circular arrangement around a table. In how many ways can this be done if no man is to sit next to a man and no woman is to sit next to a woman?

125. (Brualdi, 2004, pp.75-82:31) In a soccer tournament of 15 teams, the top 3 teams are awarded gold, silver, and bronze cups, and the last 3 teams are dropped to a lower league. We regard two outcomes of the tournament as the same if the teams which receive the gold, silver, and bronze cups, respectively, are identical and the teams which drop to a lower league are also identical. How many different possible outcomes are there for the tournament?

126. (Brualdi, 2004, pp.75-82:32) Determine the number of 11-permutations of the multiset

$$S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}.$$

127. (Brualdi, 2004, pp.75-82:33) Determine the number of 10-permutations of the multiset

$$S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}.$$

128. (Brualdi, 2004, pp.75-82:34) Determine the number of 11-permutations of the multiset

$$\{3 \cdot a, 3 \cdot b, 3 \cdot c, 3 \cdot d\}.$$

129. (Brualdi, 2004, pp.75-82:35) List all 3-combinations and 4-combinations of the multiset

$$\{2 \cdot a, 1 \cdot b, 3 \cdot c\}.$$

130. (Brualdi, 2004, pp.75-82:36) Determine the total number of combinations (of any size) of a multiset of objects of k different types with finite repetition numbers n_1, n_2, $\cdots$, n_k, respectively.

131. (Brualdi, 2004, pp.75-82:37) A bakery sells 6 different kinds of pastry. If the bakery has at least a dozen of each kind, how many different options for a dozen of pastry are there? What if a box is to contain at least one of each kind of pastry?

132. (Brualdi, 2004, pp.75-82:38) How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 30$$

satisfy $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, and $x_4 \geq 8$?

133. (Brualdi, 2004, pp.75-82:39) There are 20 identical sticks lined up in a row occupying 20 distinct places as follows

$$||||||||||||||||||||.$$

Six of them are to be chosen.

i. How many choices are there?

ii. How many choices are there if no two of the chosen sticks can be consecutive?

iii. How many choices are there if there must be at least two sticks between each pair of chosen sticks?

134. (Brualdi, 2004, pp.75-82:40) There are n sticks lined up in a row and k of them are to be chosen.

i. How many choices are there?

ii. How many choices are there if no two of the chosen sticks can be consecutive?

iii. How many choices are there if there must be at least l sticks between each pair of chosen sticks?

135. (Brualdi, 2004, pp.75-82:41) In how many ways can 12 indistinguishable apples and 1 orange be distributed among three children in such a way that each child gets at least on piece of fruit?

136. (Brualdi, 2004, pp.75-82:42) Determine the number of ways to distribute 10 orange drinks, 1 lemon drink, and 1 lime drink to 4 thirsty students so that each student gets at least 1 drink, and the lemon and lime drinks go to different students.

137. (Brualdi, 2004, pp.75-82:43) Determine the number of r-combinations of the multiset

$$\{1 \cdot a_1, \infty \cdot a_2, \cdots, \infty \cdot a_k\}.$$

138. (Brualdi, 2004, pp.75-82:44) Prove that the number of ways to distribute n different objects among k children equals k^n.

139. (Brualdi, 2004, pp.75-82:45) Twenty different books are to be put on five book shelves, each of which holds at least twenty books.

i. How many different arrangements are there if you only care about the number of books on the shelves (and not which book is where)?

ii. How many different arrangements are there if you care about which books are where but the order of the books on the shelves doesn't matter?

iii. How many different arrangements are there if the order on the shelves does matter?

140. (Brualdi, 2004, pp.75-82:46)

i. There is an even number $2n$ of people at a party, and they talk together in pairs with everyone talking with someone (so n pairs). In how many different ways can the $2n$ people be talking like this?

ii. Now suppose that there is an odd number $2n + 1$ of people at the party with everyone but one person talking with someone. How many different pairings are there?

141. (Brualdi, 2004, pp.75-82:47) There are $2n + 1$ identical books to be put in a bookcase with three shelves. In how many ways can this be done if each pair of shelves together contain more books than the other shelf?

142. (Brualdi, 2004, pp.75-82:48) Prove that the number of permutations of m A's and at most n B's equals

$$\binom{m+n+1}{m+1}.$$

143. (Brualdi, 2004, pp.75-82:49) Prove that the number of permutations of at most m A's and at most n B's equals

$$\binom{m+n+2}{m+1} - 2.$$

144. (Brualdi, 2004, pp.75-82:50) In how many ways can five identical rooks be placed on the square of an 8-by-8 board so that four of them form the corners of a rectangle?

145. (Brualdi, 2004, pp.75-82:51) Consider the multiset $\{n \cdot a, 1, 2, 3, \cdots, n\}$ of size $2n$. Determine the number of its n-combinations.

146. (Brualdi, 2004, pp.75-82:52) Consider the multiset $\{n \cdot a, n \cdot b, 1, 2, 3, \cdots, n+1\}$ of size $3n+1$. Determine the mumber of its n-combinations.

147. (Brualdi, 2004, pp.75-82:53) Establish a one-to-one correspondence between the permutations of the set $\{1, 2, \cdots, n\}$ and the towers $A_0 \subset A \subset A_2 \subset \cdots \subset A_n$ where $|A_k| = k$ for $k = 0, 1, 2, \cdots, n$.

148. (Brualdi, 2004, pp.75-82:54) Determine the number of towers of the form $\emptyset \subseteq A \subseteq B \subseteq \{1, 2, \cdots, n\}$.

149. (Brualdi, 2004, pp.75-82:55) How many permutations are there of the letters in the word PNEUMONOULTRAMICROSCOPICSILICOVOLCANOCONIOSIS? This word is, by some accounts, the longest word in the English language.

150. (Brualdi, 2004, pp.117-123:1) Which permutation of $\{1, 2, 3, 4, 5\}$ follows 31524 in using the algorithm described in Section 4.1? Which permutation comes before 31524?

151. (Brualdi, 2004, pp.117-123:2) Determine the mobile integers in

$$\overrightarrow{4}\ \overleftarrow{8}\ \overrightarrow{3}\ \overleftarrow{1}\ \overrightarrow{6}\ \overleftarrow{7}\ \overleftarrow{2}\ \overrightarrow{5}.$$

152. (Brualdi, 2004, pp.117-123:3) Use the algorithm of Section 4.1 to generate the permutations $\{1, 2, 3, 4, 5\}$, starting with $\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{4}\,\overleftarrow{5}$.

153. (Brualdi, 2004, pp.117-123:4) Prove, that in the algorithm of Section 4.1, which generates directly the permutations of $\{1, 2, \cdots, n\}$, the directions of 1 and 2 never change.

154. (Brualdi, 2004, pp.117-123:5) Let $i_1 i_2 \cdots i_n$ be a permutation of $\{1, 2, \cdots, n\}$ with inversion sequence $b_1, b_2, \cdots, b_n$, and let $k = b_1 + b_2 + \cdots + b_n$. Show by induction that one cannot bring $i_1 i_2 \cdots i_n$ to $12 \cdots n$ by fewer than k successive switches of adjacent numbers.

155. (Brualdi, 2004, pp.117-123:6) Determine the inversion sequences of the following permutations of $\{1, 2, \cdots, 8\}$

  i. 35168274

  ii. 83476215

156. (Brualdi, 2004, pp.117-123:7) Construct the permutations of $\{1, 2, \ldots, 8\}$ whose inversion sequences are

  i. 2, 5, 5, 0, 2, 1, 1, 0

  ii. 6, 6, 1, 4, 2, 1, 0, 0

157. (Brualdi, 2004, pp.117-123:8) How many permutations of $\{1, 2, 3, 4, 5, 6\}$ have

  i. exactly 15 inversions?

  ii. exactly 14 inversions?

  iii. exactly 13 inversions?

158. (Brualdi, 2004, pp.117-123:9) Show that the largest number of inversions of a permutation of $\{1, 2, \cdots, n\}$ equals $\frac{n(n-1)}{2}$. Determine the unique permutation with $\frac{n(n-1)}{2}$ inversions. Also determine all those permutations with one fewer inversion.

159. (Brualdi, 2004, pp.117-123:10) Bring the permutations 256143 and 436251 to 123456 by successive switches of adjacent numbers.

160. (Brualdi, 2004, pp.117-123:11) Let $S = \{x_7, x_6, \cdots, x_1, x_0\}$. Determine the 8-tuples of 0's and 1's corresponding to the following combinations of S

  i. $\{x_5, x_4, x_3\}$

  ii. $\{x_7, x_5, x_3, x + 1\}$

  iii. $\{x_6\}$

161. (Brualdi, 2004, pp.117-123:12) Let $S = \{x_7, x_6, \cdots, x_1, x_0\}$. Determine the combinations of S corresponding to the following 8-tuples

  i. 00011011

ii. 01010101

iii. 00001111

162. (Brualdi, 2004, pp.117-123:13) Generating the 5-tuples of 0's and 1's by using the base 2 arithmetic generating scheme and identify them with combinations of the set $\{x_4, x_3, x_2, x_1, x_0\}$.

163. (Brualdi, 2004, pp.117-123:14) Repeat Exercise 162 for the 6-tuples of 0's and 1's.

164. (Brualdi, 2004, pp.117-123:15) For each of the following combinations of $\{x_7, x_6, \ldots, x_1, x_0\}$, determine the combination that immediately follows it by using the base 2 arithmetic generating scheme,

   i. $\{x_4, x_1, x_0\}$

   ii. $\{x_7, x_5, x_3\}$

   iii. $\{x_7, x_5, x_4, x_3, x_2, x_1, x_0\}$

   iv. $\{x_0\}$

165. (Brualdi, 2004, pp.117-123:16) For each of the combinations in the preceding exercise, determine the combination that immediately *prcedes* it in the base 2 arithmetic generating scheme.

166. (Brualdi, 2004, pp.117-123:17) Which combination of $\{x_7, x_6, \cdots, x_1, x_0\}$ is 150th of the list of combinations of S when the base 2 arithmetic generating scheme is used? 200th? 250th? (As in Section 4.3, the places on the list are numbered beginning with 0.)

167. (Brualdi, 2004, pp.117-123:18) Build (the corners and edges of) the 4-cube, and indicate the reflected Gray code on it.

168. (Brualdi, 2004, pp.117-123:19) Give an example of a noncyclic Gray code of order 3.

169. (Brualdi, 2004, pp.117-123:20) Give an example of a cyclic Gray code of order 3 that is not the reflected Gray code.

170. (Brualdi, 2004, pp.117-123:21) Construct the reflected Gray code of order 5 by

   i. using the inductive definition, and

   ii. using the Gray code algorithm.

171. (Brualdi, 2004, pp.117-123:22) Determine the reflected Gray code of order 6.

172. (Brualdi, 2004, pp.117-123:23) Determine the immediate successors of the following 9-tuples in the reflected Gray code of order 9,

   i. 010100110
   ii. 110001100
   iii. 111111111

173. (Brualdi, 2004, pp.117-123:24) Determine the precedecessors of each of the 9-tuples of the previous Exercise 172 in the reflected Gray code of order 9.

174. (Brualdi, 2004, pp.117-123:25) The reflected Gray code of order n is properly called the reflected *binary* Gray code, since it is a listing of the n-tuples of 0's and 1's. It can be generalized to any base system, in particular the ternary and decimal system. Thus, the reflected decimal Gray code of order n is a listing of all the decimal numbers of n digits such that consecutive numbers in the list differ in only one place and the absolute value of the difference is 1. Determine the reflected decimal Gray codes of orders 1 and 2. (Note we have not said precisely what a reflected decimal Gray code is. Part of the problem is to discover what it is.) Also, determine the reflected ternary Gray codes of orders 1, 2, and 3.

175. (Brualdi, 2004, pp.117-123:26) Generate the 2-combinations of $\{1, 2, 3, 4, 5\}$ in lexicographic order by using the algorithm described in Section 4.4.

176. (Brualdi, 2004, pp.117-123:27) Generate the 3-combinations of $\{1, 2, 3, 4, 5, 6\}$ in lexicographic order by using the algorithm described in Section 4.4.

177. (Brualdi, 2004, pp.117-123:28) Determine the 6-combination of $\{1, 2, \cdots, 10\}$ that immediately follows 2, 3, 4, 6, 9, 10 in the lexicographic order. Determine the 6-combination that immediately precedes 2, 3, 4, 6, 9, 10?

178. (Brualdi, 2004, pp.117-123:29) Determine the 7-combination of $\{1, 2, \cdots, 15\}$ that immediately follows 1, 2, 4, 6, 8, 14, 15 in the lexicographic order. Determine the 7-combination that immediately precedes 1, 2, 4, 6, 8, 14, 15.

179. (Brualdi, 2004, pp.117-123:30) Generate the inversion sequences of the permutations of $\{1, 2, 3\}$ in the lexicographic order, and write down the corrresponding permutations. Repeat for the inversion sequences of permutations of $\{1, 2, 3, 4\}$.

180. (Brualdi, 2004, pp.117-123:31) Generate the 3-permutations of $\{1, 2, 3, 4, 5\}$.

181. (Brualdi, 2004, pp.117-123:32) Generate the 4-permutations of $\{1, 2, 3, 4, 5, 6\}$.

182. (Brualdi, 2004, pp.117-123:33) In which position does the combination 2489 occur in the lexicographic order of the 4-combinations of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

183. (Brualdi, 2004, pp.117-123:34) Consider the r-combinations of $\{1, 2, \cdots, n\}$ in lexicographic order.

    i. What are the first $(n - r + 1)$ r-combinations?

    ii. What are the last $(r + 1)$ r-combinations?

184. (Brualdi, 2004, pp.117-123:35) The *complement* of an r-combination A of $\{1, 2, \cdots, n\}$ is the $(n-r)$-combination $\bar{A}$ of $\{1, 2, \cdots, n\}$, consisting of all those elements that do not belong to A. Let $M = \binom{n}{r}$, the number of r-combinations, and the number of $(n-r)$-combinations of $\{1, 2, \cdots, n\}$. Prove that, if

$$A_1, A_2, A_3, \cdots, A_M$$

are the r-combinations in lexicographic order, then

$$\overline{A_M}, \cdots, \overline{A_3}, \overline{A_2}, \overline{A_1}$$

are the $(n - r)$-combinations in lexicographic order.

185. (Brualdi, 2004, pp.117-123:36) Let X be a set of n elements. How many different relations on X are there? How many of these are reflexive? Symmetric? Antisymmetric? Reflexive and symmetric? Reflexive and antysymmetric?

186. (Brualdi, 2004, pp.117-123:37) Let R' and R'' be two partial orders on a set X. Define a new relation R on X by xRy if and only if both $xR'y$ and $xR''y$ hold. Prove that R is also a partial order on X. (R is called the *intersection* of R' and R''.)

187. (Brualdi, 2004, pp.117-123:38) Let $(X_1, \leq_1)$ and $\{X_2, \leq_2\}$ be partially ordered sets. Define a relation T on the set

$$X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$$

by

$$(x_1, x_2)T(x_1', x_2') \text{ if and only if } x_1 \leq_1 x_1' \text{ and } x_2 \leq_2 x_2'.$$

Prove that $(X_1 \times X_2, T)$ is a partially ordered set. $(X_1 \times X_2, T)$ is called the *direct product* of $(X_1, \leq_1)$ and $(X_2, \leq_2)$ and is also denoted by $(X_1, \leq_1) \times (X_2, \leq_2)$. More generally, prove that the direct product $(X_1, \leq_1) \times (X_2, \leq_2) \times \cdots \times (X_m, \leq_m)$ of partially ordered sets is also a partially ordered set.

188. (Brualdi, 2004, pp.117-123:39) Let $(J, \leq)$ be the partially ordered set with $J = \{0, 1\}$ and with $0 < 1$. By identifying the combinations of a set X of n elements with the n-tuples of 0's and 1's, prove that the partially ordered set $(X, \subseteq)$ can be identified with the n-fold direct product $(J, \leq) \times (J, \leq) \times \cdots \times (J, \leq)$ (n factors).

189. (Brualdi, 2004, pp.117-123:40) Generalize Exercise 188 to the multiset of all combinations of the multiset $X = \{n_1 \cdot a_1, n_2 \cdot a_2, \cdots, n_m \cdot a_m\}$. (Part of this exercise is to determine the "natural" partical order on these multisets.)

190. (Brualdi, 2004, pp.117-123:41) Prove that a partial order on a finite set is uniquely determined by its cover relation.

191. (Brualdi, 2004, pp.117-123:42) Describe the cover relation for the partial order $\subseteq$ on the collection $\mathcal{P}(X)$ of all subsets of a set X.

192. (Brualdi, 2004, pp.117-123:43) Let $X = \{a, b, c, d, e, f\}$ and let the relation R on X be defined by aRb, bRc, cRd, aRe, eRf, fRd. Verify that R is the cover relation of a partially ordered set, and determine all the linear extensions of this partial order.

193. (Brualdi, 2004, pp.117-123:44) Let $A_1, A_2, \cdots, A_s$ be a partition of a set X. Define a relation R on X by xRy if and only if x and y belong to the same part of the partition. Prove that R is an equivalence relation.

194. (Brualdi, 2004, pp.117-123:45) Define a realtion R on the set Z of all integers by aRb if and only if $a = \pm b$. Is R an equivalence relation on Z?

195. (Brualdi, 2004, pp.117-123:46) Let m be a positive integer and define a relation R on the set X of all nonegative integers by: aRb if and only if a and b have the same remainder when divided by m. Prove that R is an equivalence relation on X. How many different equivalence classed does this equivalence relation have?

196. (Brualdi, 2004, pp.117-123:47) Let Π_n denote the set of all partitions of the set $\{1, 2, \cdots, n\}$. Given two partitions π and σ in Π_n, define $\pi \leq \sigma$, provided each part of π is contained in a part of σ. Thus, the partition π can be obtained by partitioning the parts of σ. This relation is usually expressed by saying that π is a *refinement* of σ.

    i. Prove that this relation is a partial order on Π_n.

    ii. By Theorem 4.5.3, we know that there is a one-to-one correspondence between Π_n and the set Λ_n of all equivalence relations on $\{1, 2, \cdots, n\}$. What is the partial order on Λ_n that corresponds to this partial order on Π_n?

    iii. Construct the diagram of $(\Pi_n, \leq)$ for $n = 1$, 2, 3, and 4.

197. (Brualdi, 2004, pp.117-123:48) Consider the partial order $\leq$ on the set X of positive integers given by "is a divisor of". Let a and b be two integers. Let c be the largest integer such that $c \leq a$ and $c \leq b$, and let d be the smallest integer such that $a \leq d$ and $b \leq d$. What are c and d?

198. (Brualdi, 2004, pp.117-123:49) Prove that the intersection $R \cap S$ of two equivalence relations R and S on a set X is also an equivalence relation on X. Is the union of two equivalence relations on X always an equivalence relation?

199. (Brualdi, 2004, pp.117-123:50) Consider the partially ordered set $(X, \subseteq)$ of subsets of the set $X = \{a, b, c\}$ of 3 elements. How many linear extensions are there?

200. (Brualdi, 2004, pp.117-123:51) Let n be a positive integers, and let X_n be the set of $n!$ permutations of $\{1, 2, \cdots, n\}$. Let π and σ be two permutations in X_n, and define $\pi \leq \sigma$ provided the set of inversions of π is a subset of the set of inversions of σ. Verify that this defines a partial order on X_n, called the *inversion poset*. Describe the cover relation for this partial order and then draw the diagram for the inversion poset $(H_4, \leq)$.

201. (Brualdi, 2004, pp.117-123:52) Verify that a binary n-tuple $a_{n-1} \cdots a_1 a_0$ is in place k in the Gray code order list where k is determined as follows: For $i = 0, 1, \cdots, n-1$, let

$$b_i = \begin{cases} 0 & \text{if } a_{n-1} + \cdots + a_i \text{ is even, and} \\ 1 & \text{if } a_{n-1} + \cdots + a_i \text{ is odd.} \end{cases}$$

Then

$$k = b_{n-1} \times 2^{n-1} + \cdots + b_1 \times 2 + b_0 \times 2^0.$$

Thus, $a_{n-1} \cdots a_1 a_0$ is in the same place in the Gray code order list of binary n-tuples as $b_{n-1} \cdots b_1 b_0$ is in the lexicographic order list of binary n-tuples.

202. (Brualdi, 2004, pp.117-123:53) Referring to Exercise 201, show that $a_{n-1} \cdots a_1 a_0$ can be recovered from $b_{n-1} \cdots b_1 b_0$ by $a_{n-1} = b_{n-1}$, and for $i = 0, 1, \cdots, n-1$,

$$a_i = \begin{cases} \\ 0 & \text{if } b_i + b_{i+1} \text{ is even, and} \\ 1 & \text{if } b_i + b_{i+1} \text{ is odd.} \end{cases}$$

203. (Brualdi, 2004, pp.117-123:54) Let $(X, \leq)$ be a finite partially ordered set. By Theorem 4.5.2 we know that $(X, \leq)$ has a linear extension. Let a and b be incomparable elements of X. Modify the proof of Theorem 4.5.2 to obtain a linear extension of $(X, \leq)$ such that $a < b$. Hint: First find a partial order $\leq'$ on X such that whenever $x \leq y$ then $x \leq' y$ and, in addition, $a \leq' b$.

204. (Brualdi, 2004, pp.117-123:55) Use Exercise 203 to prove that a finite partially ordered set is the intersection of all its linear extensions (see Exercise 186.)

205. (Brualdi, 2004, pp.117-123:56) The *dimension* of a finite partially ordered set $(X, \leq)$ is the smallest number of its linear extensions whose intersection is $(X, \leq)$. By Exercise 204, every partially ordered set has a dimension. Those that have dimension 1 are the linear orders. Let n be a positive integer and let $i_1, i_2, \cdots, i_n$ be a permutation σ of $\{1, 2, \cdots, n\}$ that is different from $1, 2, \cdots, n$. Let $X = \{(1, i_1), (2, i_2), \cdots, (n, i_n)\}$. Now define a relation R on X by $(k, i_k)R(l, i_l)$ if and only if $k \leq l$ (ordinary integer inequality) and $i_k \leq i_l$ (again ordinary inequality); that is, (i_k, i_l) is not a inversion of σ. Thus, for instance, if $n = 3$ and $\sigma = 2, 3, 1$, then $X = \{(1, 2), (2, 3), (3, 1)\}$, and $(1, 2)R(2, 3)$, but $(1, 2)\not R(3, 1)$. Prove that R is a partial order on X and that the dimension of the partially ordered set (X, R) is 2, provided that $i_1, i_2, \cdots, i_n$ is not the identity permutation $1, 2, \cdots, n$.

206. (Brualdi, 2004, pp.153-159:1) Prove Pascal's formula by substituting the values of the binomial coefficients are given in equation

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}.$$

207. (Brualdi, 2004, pp.153-159:2) Fill in the rows of Pascal's triangle corresponding to $n = 9$ and 10.

208. (Brualdi, 2004, pp.153-159:3) Consider the sum of the binomial coefficients along the diagonals of Pascal's triangle running upward from the left. The first few are: 1, 1, $1 + 1 = 2$, $1 + 2 = 3$, $1 + 3 + 1 = 5$, $1 + 4 + 3 = 8$. Compute several more of these diagonal sums, and determine how these sums are related. (Compare them with the values of the counting function f in Exercise 30.)

209. (Brualdi, 2004, pp.153-159:4) Expand $(x + y)^5$ and $(x + y)^6$, using the binomial theorem.

210. (Brualdi, 2004, pp.153-159:5) Expand $(2x - y)^7$, using the binomial theorem.

211. (Brualdi, 2004, pp.153-159:6) What is the coefficient of x^5y^{13} in the expansion of $(3x - 2y)^{18}$? What is the coefficient fo x^8y^9? (There is not a misprint in this question!)

212. (Brualdi, 2004, pp.153-159:7) Use the binomial theorem to prove that

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Generalize to find the sum

$$\sum_{k=0}^{n} \binom{n}{k} r^k$$

for any real number r.

213. (Brualdi, 2004, pp.153-159:8) Use the binomial theorem to prove that

$$2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}.$$

214. (Brualdi, 2004, pp.153-159:9) Evaluate the sum

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^k.$$

215. (Brualdi, 2004, pp.153-159:10) Use *combinatorial* reasoning to prove the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}, \quad (n \text{ and } k \text{ positive integers.})$$

(Hint: Think of choosing a team with one person designated as captain.)

216. (Brualdi, 2004, pp.153-159:11) Use *combinatorial* reasoning to prove the identity (in the form given)

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}.$$

(Hint: Let S be a set with three distinguished elements a, b, and c and count certain k-combinations of S.)

217. (Brualdi, 2004, pp.153-159:12) Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^m \binom{2m}{m} & \text{if } n = 2m. \end{cases}$$

(Hint: For $n = 2m$, consider the coefficient of x^n in $(1-x^2)^n = (1+x)^n(1-x)^n$.)

218. (Brualdi, 2004, pp.153-159:13) Find one binomial coefficient equal to the following expression

$$\binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}.$$

219. (Brualdi, 2004, pp.153-159:14) Prove that

$$\binom{r}{k} = \frac{r}{r-k}\binom{r-1}{k}$$

for r a real number and k an integer with $r \neq k$.

220. (Brualdi, 2004, pp.153-159:15) Prove, that for every integer $n > 1$,

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} + \cdots + (-1)^{n-1}n\binom{n}{n} = 0.$$

221. (Brualdi, 2004, pp.153-159:16) By integrating the binomial expansion, prove that, for a positive integer n,

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}.$$

222. (Brualdi, 2004, pp.153-159:17) Prove the identity in the previous exercise by using

$$k\binom{n}{k} = n\binom{n-1}{k-1}, \quad (n \text{ and } k \text{ positive integers.})$$

and

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n, \quad (n \geq 0)$$

223. (Brualdi, 2004, pp.153-159:18) Evaluate the sum

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \cdots + (-1)^n\frac{1}{n+1}\binom{n}{n}.$$

224. (Brualdi, 2004, pp.153-159:19) Sum the series $1^2 + 2^2 + 3^2 + \cdots + n^2$ by observing that

$$m^2 = 2\binom{m}{2} + \binom{m}{1}$$

and using the identity

$$\binom{0}{k} + \binom{1}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}.$$

225. (Brualdi, 2004, pp.153-159:20) Find integers a, b, and c such that

$$m^3 = a\binom{m}{3} + b\binom{m}{2} + c\binom{m}{1}$$

for all m. Then sum the series $1^3 + 2^3 + 3^3 + \cdots + n^3$.

226. (Brualdi, 2004, pp.153-159:21) Prove that, for all real numbers r and all integers k,

$$\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}.$$

227. (Brualdi, 2004, pp.153-159:22) Prove that, for all real number r and all integers k and m,

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}.$$

228. (Brualdi, 2004, pp.153-159:23) Every day a student walks from her home to school, which is located 10 blocks east and 14 blocks north from home. She always takes a shortest walk of 24 blocks.

   i. How many different walks are possible?
   ii. Suppose that 4 blocks east and 5 blocks north of her home lives her best friend, whom she meets each day on her way to school. Now how many different walks are possible?
   iii. Suppose, in addition, that 3 blocks east and 6 blocks north of her friend's house there is a park where the two girls stop each day to rest and play. Now how many different walks are there?
   iv. Supposing at a park to rest and play, the two students often get to school late. To avoid the temptation of the park, our two students decide never to pass the intersection where the park is. Now how many different walks are there?

229. (Brualdi, 2004, pp.153-159:24) Consider a three-dimensioal grid whose dimensions are 10 by 15 by 20. You are at the front lower left corner of the grid and wish to get to the back upper right corner 45 "blocks" away. How many different routes are there in which you walk exactly 45 blocks?

230. (Brualdi, 2004, pp.153-159:25) Use a combinatorial argument to prove the *Vandermonde Convolution* for the binomial coefficients: For all positive integers m_1, m_2, and n,

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}.$$

Deduce the indentity

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n},\quad (n\geq 0)$$

as a special case.

231. (Brualdi, 2004, pp.153-159:26) Let n be a positive integer. Verify by substitution that

$$\binom{2n}{n+1}+\binom{2n}{n}=\frac{1}{2}\binom{2n+2}{n+1}.$$

Then give a combinatorial proof.

232. (Brualdi, 2004, pp.153-159:27) Let n and k be integers with $1\leq k\leq n$. Prove that

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}=\frac{1}{2}\binom{2n+1}{n+1}-\binom{2n}{n}.$$

233. (Brualdi, 2004, pp.153-159:28) Let n and k be positive integers. Give a combinatorial proof of the identity

$$n(n+1)2^{n-2}=\sum_{k=1}^{n}k^2\binom{n}{k},\quad (n\geq 1).$$

234. (Brualdi, 2004, pp.153-159:29) Let n and k be positive integers. Give a combinatorial proof that

$$\sum_{k=1}^{n}k\binom{n}{k}^2=n\binom{2n-1}{n-1}.$$

235. (Brualdi, 2004, pp.153-159:30) Find and prove a formula for

$$\sum_{\substack{r,s,t\geq 0\\ r+s+t=n}}\binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t}$$

where the summation extends over all nonnegative integers r, s and t with sum $r+s+t=n$.

236. (Brualdi, 2004, pp.153-159:31) Prove that the only clutter of $S = \{1, 2, 3, 4\}$ of size 6 is the clutter of all 2-combinations of S.

237. (Brualdi, 2004, pp.153-159:32) Prove that there are only two clutters of $S = \{1, 2, 3, 4, 5\}$ of size 10 (10 is maximum by Sperner's Theorem), namely, the clutter of all 2-combinations of S and the clutter of all 3-combinations.

238. (Brualdi, 2004, pp.153-159:33) Let S be a set of n elements. Prove that, if n is even, the only clutter of size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is the clutter of all $\frac{n}{2}$-combinations; if n is odd, prove that the only clutter of this size are the clutter of all $\frac{n-1}{2}$-combinations and the clutter of all $\frac{n+1}{2}$-combinations.

239. (Brualdi, 2004, pp.153-159:34) Construct a partition of the combinations of $\{1, 2, 3, 4, 5\}$ into symmetric chains.

240. (Brualdi, 2004, pp.153-159:35) In a partition of the combinations of $\{1, 2, \cdots, n\}$ into symmetric chains, how many chains have only one combination in them? two combinations? k combinations?

241. (Brualdi, 2004, pp.153-159:36) A talk show host has just bought 10 new jokes. Each night he tells some the jokes. What is the largest number of nights on which you can tune in so that you never hear on on night at least all the jokes you heard on *one* of the other nights? (Thus, for instance, it is acceptable that you hear jokes 1, 2, and 3 on one night, jokes 3 and 4 on another, and jokes 1, 2, and 4 on a third. It is not acceptable that you hear jokes 1 and 2 on one night and joke 2 on another night.)

242. (Brualdi, 2004, pp.153-159:37) Prove that identity of Exercise 230, using the binomial theorem and the relation $(1+x)^{m_1}(1+x)^{m_2} = (1+x)^{m_1+m_2}$.

243. (Brualdi, 2004, pp.153-159:38) Use the multinomial theorem to show that, for positive integers n and t,

$$t^n = \sum \binom{n}{n_1\ n_2\ \cdots\ n_t},$$

where the summation extends over all nonnegative integral solutions $n_1, n_2, \cdots, n_t$ of $n_1 + n_2 + \cdots + n_t = n$.

244. (Brualdi, 2004, pp.153-159:39) Use the multinomial theorem to expand $(x_1 + x_2 + x_3)^4$.

245. (Brualdi, 2004, pp.153-159:40) Determine the coefficient fo $x_1^3 x_2 x_3^4 x_5^2$ in the expansion of

$$(x_1 + x_2 + x_3 + x_4 + x_5)^{10}.$$

246. (Brualdi, 2004, pp.153-159:41) What is the coefficient of $x_1^3 x_2^3 x_3 x_4^2$ in the expression of

$$(x_1 - x_2 + 2x_3 - 2x_4)^9?$$

247. (Brualdi, 2004, pp.153-159:42) Expand $(x_1+x_2+x_3)^n$ by observing that

$$(x_1+x_2+x_3)^n = [(x_1+x_2)+x_3]^n$$

and then using the binomail theorem.

248. (Brualdi, 2004, pp.153-159:43) Prove the identity

$$\binom{n}{n_1\ n_2\ \cdots\ n_t} = \binom{n-1}{n_1-1\ n_2\ \cdots\ n_t} + \binom{n-1}{n_1\ n_2-1\ \cdots\ n_t} + \cdots + \binom{n-1}{n_1\ n_2\ \cdots\ n_r-1}$$

by a combinatorial argument. (Hint: Consider the premutations of a multiset of objects of t different types with repetion numbers $n_1, n_2, \cdots, n_t$, respectively. Partition these permutations according to what type of object is in the first position.)

249. (Brualdi, 2004, pp.153-159:44) Prove by induction on n that, for n a positive integer,

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k, \quad |z|<1.$$

Assume the validity of

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \quad |z|<1.$$

250. (Brualdi, 2004, pp.153-159:45) Use Newton's binomial theorem to approximate $\sqrt{30}$.

251. (Brualdi, 2004, pp.153-159:46) Use Newton's binomial theorem to approximate $10^{\frac{1}{3}}$.

252. (Brualdi, 2004, pp.153-159:47) Use Theorem 5.7.1 to show that, if m and n are positive integers, then a partially ordered set of $mn+1$ elements has a chain of size $m+1$ or an antichain of size $n+1$.

253. (Brualdi, 2004, pp.153-159:48) Use the result of the previous exercise to show that a sequence of $mn+1$ real numbers either contains an increasing subsequence of $m+1$ numbers or a decreasing subsequence of $n+1$ numbers (see Application 9 of Section 2.2).

254. (Brualdi, 2004, pp.153-159:49) Consider the partially ordered set $(X, |)$ on the set $X=\{1,2,\cdots,12\}$ of the first 12 positive integers, partially ordered by "is divisible by."

i. Determine a chain of largest size and a partition of X into the smallest number of antichains.

ii. Determine an antichain of largest size and a partition of X into the smallest number of chains.

255. (Brualdi, 2004, pp.153-159:50) Let R and S be two partial orders on the same set X. Considering R and S as subsets of $X \times X$, we assume that $R \subseteq S$ but $R \neq S$. Show that there exists an ordered pair (p, q) where $(p, q) \in S$ and $(p, q) \notin R$ such that $R' = R \cup (p, q)$ is also a paired order on X. Show by example that not every such (p, q) has the property that R' is a partial order on X.

256. (Brualdi, 2004, pp.200-205:1) Find the number of integers between 1 and 10,000 inclusive that are not divisible by 4, 5, or 6.

257. (Brualdi, 2004, pp.200-205:2) Find the number of integers between 1 and 10,000 inclusive that are not divisible by 4, 6, 7, or 10.

258. (Brualdi, 2004, pp.200-205:3) Find the number of integers between 1 and 10,000 that are neither perfect squares nor perfect cubes.

259. (Brualdi, 2004, pp.200-205:4) Determine the number of 12-combinations of the multiset

$$S = \{4 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\}.$$

260. (Brualdi, 2004, pp.200-205:5) Determine the number of 10-combinations of the multiset

$$S = \{\infty \cdot a, 4 \cdot b, 5 \cdot c, 7 \cdot d\}.$$

261. (Brualdi, 2004, pp.200-205:6) A bakery sells chocolate, cinnamon, and plain doughnuts and at a particular time has 6 chocolate, 6 cinnamon, and 3 plain. If a box contains 12 doughnuts, how many different options are there for a box of doughnuts?

262. (Brualdi, 2004, pp.200-205:7) Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 14$ in nonnegative integers x_1, x_2, x_3, and x_4 not exceeding 8.

263. (Brualdi, 2004, pp.200-205:8) Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 14$ in positive integers x_1, x_2, x_3, and x_4 not exceeding 8.

264. (Brualdi, 2004, pp.200-205:9) Determine the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

that satisfy

$$1 \leq x_1 \leq 6, \quad 0 \leq x_2 \leq 7, \quad 4 \leq x_3 \leq 8, \quad 2 \leq x_4 \leq 6.$$

265. (Brualdi, 2004, pp.200-205:10) Let S be a multiset with k distinct objects whose repetition numbers are $n_1, n_2, \cdots, n_k$, respectively. Let r be a positive integer such that there is at least one r-combination of S. Show that, in applying the inclusion-exclusion principle to determine the number of r-combinations of S, one has $A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset$.

266. (Brualdi, 2004, pp.200-205:11) Determine the number of permutations of $\{1, 2, \cdots, 8\}$ in which no even integer is in its natural position.

267. (Brualdi, 2004, pp.200-205:12) Determine the number of permutations of $\{1, 2, \cdots, 8\}$ in which exactly four integers are in their natural positions.

268. (Brualdi, 2004, pp.200-205:13) Determine the number of permutations of $\{1, 2, \cdots, 9\}$ in which at least one odd integer is in its natural position.

269. (Brualdi, 2004, pp.200-205:14) Determine a general formula for the number of permutations of the set $\{1, 2, \cdots, n\}$ in which exactly k integers are in their natural positions.

270. (Brualdi, 2004, pp.200-205:15) At a party seven gentlemen check their hats. In how many ways can their hats be returned so that

    i. no gentleman receives his own hat?

    ii. at least one of the gentlemen receives his own hat?

    iii. at least two of the gentlemen receive their own hats?

271. (Brualdi, 2004, pp.200-205:16) Use combinatorial reasoning to derive the identity

$$n! = \binom{n}{0} D_n + \binom{n}{1} D_{n-1} + \binom{n}{2} D_{n-2} + \cdots + \binom{n}{n-1} D_1 + \binom{n}{n} D_0.$$

(Here, D_0 is defined to be 1.)

272. (Brualdi, 2004, pp.200-205:17) Determine the number of permutations of the multiset

$$S = \{3 \cdot a, 4 \cdot b, 2 \cdot c\},$$

where, for each type of letter, the letters of the same type do not appear consecutively. (Thus *abbbbcaca* is not alllowed, but *abbbacacb* is.)

273. (Brualdi, 2004, pp.200-205:18) Verify the factorial formula

$$n! = (n-1)[(n-2)! + (n-1)!], \quad (n = 2, 3, 4, \cdots).$$

274. (Brualdi, 2004, pp.200-205:19) Using the evaluation of the derangement numbers as given in Theorem 6.3.1, provide a proof of the relation

$$D_n = (n-1)(D_{n-2} + D_{n-1}), \quad (n = 3, 4, 5, \cdots).$$

275. (Brualdi, 2004, pp.200-205:20) Starting from the formula $D_n = nD_{n-1} + (-1)^n$, $(n = 2, 3, 4, \cdots)$, give a proof of Theorem 6.3.1.

276. (Brualdi, 2004, pp.200-205:21) Prove that D_n is an even number if and only if n is an odd number.

277. (Brualdi, 2004, pp.200-205:22) Show that the numbers Q_n of Section 6.5 can be rewritten in the form

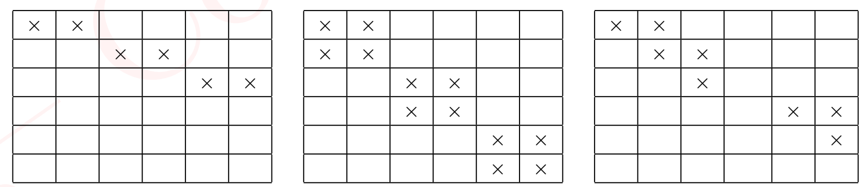
$$Q_n = (n-1)!\left[n - \frac{n-1}{1!} + \frac{n-2}{2!} - \frac{n-3}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right].$$

278. (Brualdi, 2004, pp.200-205:23) (Continuation of Excercise 277.) Verify the identity

$$(-1)^k \frac{n-k}{k!} = (-1)^k \frac{n}{k!} + (-1)^{k-1} \frac{1}{(k-1)!},$$

and use it to prove that $Q_n = D_n + D_{n-1}$, $(n = 2, 3, \cdots)$.

279. (Brualdi, 2004, pp.200-205:24) What is the number of ways to place six nonattacking rooks on the 6-by-6 boards with forbidden positions as shown?



(a) (b) (c)

Figure in Question 279

280. (Brualdi, 2004, pp.200-205:25) Count the permutations $i_1i_2i_3i_4i_5i_6$ of $\{1, 2, 3, 4, 5, 6\}$, where $i_1 \neq 1, 5$; $i_3 \neq 2, 3, 5$; $i_4 \neq 4$ and $i_6 \neq 5, 6$.

281. (Brualdi, 2004, pp.200-205:26) Count the permutations $i_1i_2i_3i_4i_5i_6$ of $\{1, 2, 3, 4, 5, 6\}$, where $i_1 \neq 1, 2, 3$; $i_2 \neq 1$; $i_3 \neq 1$; $i_5 \neq 5, 6$ and $i_6 \neq 5, 6$.

282. (Brualdi, 2004, pp.200-205:27) A carousel has eight seats, each representing a different animal. Eight girls are seated on the carousel facing forward (each girl looks at another girl's back). In how many ways can they change seats so that each has a different girl in front of her? How does the problem change if all the seats are identical?

283. (Brualdi, 2004, pp.200-205:28) A carousel has eight seats each representing a different animal. Eight boys are seated on the carousel but facing inward, so that each boy faces another (each boy looks at nother boy's front). In how many ways can they change seats so that each faces a different boy? How does the problem change if all the seats are identical?

284. (Brualdi, 2004, pp.200-205:29) How many circular permutations are there of the multiset

$$\{3 \cdot a, 4 \cdot b, 2 \cdot c, 1 \cdot d\},$$

where, for each type of letter, all letters of that type do not appear consecutively?

285. (Brualdi, 2004, pp.200-205:30) How many circular permutations are there of the multiset

$$\{2 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\},$$

where, for each type of letter, all letters of that type do not appear consecutively?

286. (Brualdi, 2004, pp.200-205:31) Let n be a positive integer and let $p_1, p_2, \cdots, p_k$ be all the different prime numbers that divide n. Consider the Euler function ϕ defined by

$$\phi(n) = |\{k : 1 \leq k \leq n, \mathrm{GCD}(k, n) = 1\}|.$$

Use the inclusion-exclusion principle to show that

$$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

287. (Brualdi, 2004, pp.200-205:32) Let n and k be positive integers with $k \leq n$. Let $a(n, k)$ be the number of ways to place k nonattacking rooks on an n-by-n board in which the positions $(1,1), (2,2), \cdots, (n,n)$ and $(1,2), (2,3), \cdots, (n-1,n), (n,1)$ are forbidden. For example, if $n = 6$ the board is

×	×				
	×	×			
		×	×		
			×	×	
				×	×
×					×

Prove that

$$a(n, k) = \frac{n}{n-k} \binom{n-k}{k}.$$

Not that $a(n,k)$ is the number of ways to choose k children from a group of $2n$ children arranged in a circle so that no two consecutive children are chosen.

288. (Brualdi, 2004, pp.200-205:33) Prove that the convolution product satisfies the associative law

$$f * (g * h) = (f * g) * h.$$

289. (Brualdi, 2004, pp.200-205:34) Consider the linearly ordered set $1 < 2 < \cdots < n$. Let $F : \{1, 2, \cdots, n\} \to \mathcal{R}$ be a function and let $G : \{1, 2, \cdots, n\} \to \mathcal{R}$ be defined by

$$G(m) = \sum_{k=1}^{m} F(k), \quad (1 \le k \le n).$$

Apply Möbius inversion to get F in terms of G.

290. (Brualdi, 2004, pp.200-205:35) Consider the board with forbidden positions as shown

	×	×	
×			
			×
	×		

Use formula

$$F(X_n) = \sum_{S \subseteq X_n} (-1)^{n-|S|} \prod_{i=1}^{n} \left(\sum_{j \in S} a_{ij} \right)$$

to compute the number of ways to place 4 nonattacking rooks on this board.

291. (Brualdi, 2004, pp.200-205:36) Consider the partially ordered set $(\mathcal{P}(X_3), \subseteq)$ of subsets of $\{1, 2, 3\}$ partially ordered by containment. Let a function f in $\mathcal{F}[\mathcal{P}(X)]$ be defined by

$$f(A, B) = \begin{cases} 1, & \text{if } A = B, \\ 2, & \text{if } A \subset B \text{ and } |B| - |A| = 1, \\ 1, & \text{if } A \subset B \text{ and } |B| - |A| = 2, \\ -1, & \text{if } A \subset B \text{ and } |B| - |A| = 3. \end{cases}$$

Find the inverse of f with respect to the convolution product.

292. (Brualdi, 2004, pp.200-205:37) Recall the partially ordered set Π_n of all partitions of $\{1, 2, \cdots, n\}$, where the partial order is that of refinement (See Exercise 196). Determine the Möbius functions of Ω_3 and Π_4.

293. (Brualdi, 2004, pp.200-205:38) Let n be a positive integer and consider the partially ordered set $(X_n, |)$. Let a and b be positive integers in X_n, where $a|b$. Prove that $\mu(a, b) = \mu(1, b/a)$.

294. (Brualdi, 2004, pp.200-205:39) Consider the multiset $X = \{n_1 \cdot a_1, n_2 \cdot a_2, \cdots, n_k \cdot a_k\}$ of k distinct elements with positive repetition numbers $n_1, n_2, \cdots, n_k$. We introduce a partial order on the submultisets of X by stating the following relationship: If $A = \{p_1 \cdot a_1, p_2 \cdot a_2, \cdots, p_k \cdot a_k\}$ and $B = \{q_1 \cdot a_1, q_2 \cdot a_2, \cdots, q_k \cdot a_k\}$ are submultisets of X, then $A \le B$ provided that $p_i \le q_i$ for $i = 1, 2, \cdots, k$. Prove that this statement defines a partial order on X and then compute its Möbius function.

295. (Brualdi, 2004, pp.259-266:1) Let $f_0, f_1, f_2, \ldots, f_n, \ldots$ donote the Fibonacci sequence. By evaluating each of the following expressions for small values of n, conjecture a general formula and then prove it, using mathematical induction and the Fibonacci recurrence,

   i. $f_1 + f_3 + \cdots + f_{2n-1}$
   ii. $f_0 + f_2 + \cdots + f_{2n}$
   iii. $f_0 - f_1 + f_2 - \cdots + (-1)^n f_n$
   iv. $f_0^2 + f_1^2 + \cdots + f_n^2$

296. (Brualdi, 2004, pp.259-266:2) Prove that the nth Fibonacci number f_n is the integer that is closest to the number

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

297. (Brualdi, 2004, pp.259-266:3) Prove the following about the Fibonacci numbers,

   i. f_n is even if and only if n is divisible by 3.
   ii. f_n is divisible by 3 if and only if n is divisible by 4.
   iii. f_n is divisible by 4 if and only if n is divisible by 6.

298. (Brualdi, 2004, pp.259-266:4) Prove that the Fibonacci sequence is the solution of the recurence relation

$$a_n = 5a_{n-4} + 3a_{n-5}, \quad (n \ge 5),$$

where $a_0 = 0$, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, and $a_4 = 3$. Then use this formula to show that the Fibonacci numbers satisfy the condition that f_n is divisible by 5 if and only if n is divisible by 5.

299. (Brualdi, 2004, pp.259-266:5) By examing the Fibonacci sequence, make a conjecture about when f_n is divisible by 7 and then prove your conjecture.

300. (Brualdi, 2004, pp.259-266:6) Let m and n be positive integers. Prove that, if m is divisible by n, then f_m is divisible by f_n.

301. (Brualdi, 2004, pp.259-266:7) Let m and n be positive integers whose greatest common divisor is d. Prove that the greatest common divisor of the Fibonacci numbers f_m and f_n is the Fibonacci number f_d.

302. (Brualdi, 2004, pp.259-266:8) Consider a 1-by-n chessboard. Suppose we color each square of the chessboard with one of the two colors red and blue. Let h_n be the number of colorings in which no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then derive a formula for h_n.

303. (Brualdi, 2004, pp.259-266:9) Let h_n equal the number of differenet ways in which the squares of a 1-by-n chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n.

304. (Brualdi, 2004, pp.259-266:10) Suppose that, in his problem, Fibonacci has placed two pairs of rabbits in the enclosure at the beginning of a year. Find the numbers of pairs of rabbits in the enclosure after one year. More generally, find the number of pairs of rabbits in the enclosure are n months.

305. (Brualdi, 2004, pp.259-266:11) The *Lucas numbers* $l_0, l_1, \cdots, l_n, \cdots$ are defined on the basis of the same recurrence relation defining the Fibonacci numbers, but with different initial conditions

$$l_n = l_{n-1} + l_{n-2}, \quad (n \geq 2), \quad l_0 = 2, \quad l_1 = 1.$$

Prove that

i. $l_n = f_{n-1} + f_{n+1}$ for $n \geq 1$.

ii. $l_0^2 + l_1^2 + \cdots + l_n^2 = l_n l_{n+1} + 2$ for $n \geq 0$.

306. (Brualdi, 2004, pp.259-266:12) Solve the recurrence relation $h_n = 4h_{n-2}$, $(n \geq 2)$ with initial values $h_0 = 0$ and $h_1 = 1$.

307. (Brualdi, 2004, pp.259-266:13) Solve the recurrence relation $h_n = (n+2)h_{n-1}$, $(n \geq 1)$ with initial value $h_0 = 2$.

308. (Brualdi, 2004, pp.259-266:14) Solve the recurrence relation $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $(n \geq 3)$ with the initial values $h_0 = 0$, $h_1 = 1$, and $h_2 = 2$.

309. (Brualdi, 2004, pp.259-266:15) Solve the recurrence relation $h_n = 8h_{n-1} - 16h_{n-2}$, $(n \geq 2)$ with initial values $h_0 = -1$ and $h_1 = 0$.

310. (Brualdi, 2004, pp.259-266:16) Solve the recurrence relation $h_n = 3h_{n-2} - 2h_{n-3}$, $(n \geq 3)$ with initial values $h_0 = 1$, $h_1 = 0$, and $h_2 = 0$.

311. (Brualdi, 2004, pp.259-266:17) Solve the recurrence relation $h_n = 5h_{n-1} - 6h_{n-2} - 4h_{n-3} + 8h_{n-4}$, $(n \geq 4)$ with initial values $h_0 = 0$, $h_1 = 1$, $h_2 = 1$, and $h_3 = 2$.

312. (Brualdi, 2004, pp.259-266:18) Determine a recurrence relation for the number a_n of ternary strings (made up of 0's, 1's, and 2's) of length n that do not contain two consecutive 0's or two consecutive 1's. Then, find a formula for a_n.

313. (Brualdi, 2004, pp.259-266:19) Solve the following recurrence relations by examing the first few values for a formula and then proving your conjectured formula by induction.

   i. $h_n = 3h_{n-1}$, $(n \geq 1)$; $h_0 = 1$
   ii. $h_n = h_{n-1} - n + 3$, $(n \geq 1)$; $h_0 = 2$
   iii. $h_n = -h_{n-1} + 1$, $(n \geq 1)$; $h_0 = 0$
   iv. $h_n = -h_{n-1} + 2$, $(n \geq 1)$; $h_0 = 1$
   v. $h_n = 2h_{n-1} + 1$, $(n \geq 1)$; $h_0 = 1$

314. (Brualdi, 2004, pp.259-266:20) Let h_n denote the number of ways to perfectly cover a 1-by-n board with monominoes and dominoes in such a way that no two dominoes are consecutive. Find, bu do not solve, a recurrence relation and initial conditions satisfied by h_n.

315. (Brualdi, 2004, pp.259-266:21) Let a_n equal the number of ternary strings of length n made up of 0's, 1's, and 2's, such that a 0 and a 1 are never adjacent (01 and 10 never occur). Prove that

$$a_n = a_{n-1} + 2a_{n-1}, \quad (n \geq 2),$$

withe $a_0 = 1$ and $a_1 = 3$. Then find a formula for a_n.

316. (Brualdi, 2004, pp.259-266:22) Let $2n$ equally spaced points be chosen on a circle. Let h_n denote the number of ways to join these points in pairs so that the resulting line segments do not interact. Establish a recurrence relation for h_n.

317. (Brualdi, 2004, pp.259-266:23) Solve the nonhomogeneous recurrence relation

$$h_n = 4h_{n-1} + 3 \times 2^n, \quad (n \geq 1)$$
$$h_0 = 1.$$

318. (Brualdi, 2004, pp.259-266:24) Solve the nonhomogeneous recurrence relation

$$h_n = 3h_{n-1} - 2, \quad (n \geq 1)$$
$$h_0 = 1.$$

319. (Brualdi, 2004, pp.259-266:25) Solve the nonhomogeneous recurrence relation

$$h_n = 2h_{n-1} + n, \qquad (n \geq 1), \qquad h_0 = 1.$$

320. (Brualdi, 2004, pp.259-266:26) Solve the nonhomogeneous recurrence relation

$$h_n = 6h_{n-1} - 9h_{n-2} + 2n, \quad (n \geq 2)$$
$$h_0 = 1, \quad h_1 = 0.$$

321. (Brualdi, 2004, pp.259-266:27) Solve the nonhomogeneous recurrence relation

$$h_n = 4h_{n-1} - 4h_{n-2} + 3n + 1, \quad (n \geq 2)$$
$$h_0 = 1, \quad h_1 = 2.$$

322. (Brualdi, 2004, pp.259-266:28) Determine the generating function for each of the following sequences

i. $c^0 = 1, c, c^2, \cdots, c^n, \cdots$.

ii. $1, -1, 1, -1, \cdots, (-1)n, \cdots$.

iii. $\binom{\alpha}{0}, -\binom{\alpha}{1}, \binom{\alpha}{2}, \cdots, (-1)^n\binom{\alpha}{n}, \cdots$, ($\alpha$ is a real number.)

iv. $1, \frac{1}{1!}, \frac{1}{2!}, \cdots, \frac{1}{n!}, \cdots$.

v. $1, -\frac{1}{1!}, \frac{1}{2!}, \cdots, (-1)^n \frac{1}{n!}, \cdots$.

323. (Brualdi, 2004, pp.259-266:29) Let S be the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \infty \cdot e_3, \infty \cdot e_4\}$. Determine the generating function for the sequence $h_0, h_1, h_2, \cdots, h_n, \cdots$ where h_n is the number of n-combinations of S with the following added restrictions

i. Each e_i occurs an odd number of times.

ii. Each e_i occurs a multiple-of-3 number of times.

iii. The element e_1 does not occur, and e_2 occurs at most once.

iv. The element e_1 occurs 1, 3, or 11 times, and the element e_2 occurs 2, 4, or 5 times.

v. Each e_i occurs at least 10 times.

324. (Brualdi, 2004, pp.259-266:30) Solve the following recurrence relations by using the method of generating functions as described in Section 7.5,

i. $h_n = 4h_{n-2}$, $(n \geq 2)$; $h_0 = 0$, $h_1 = 1$

ii. $h_n = h_{n-1} + h_{n-2}$, $(n \geq 2)$; $h_0 = 1$, $h_1 = 3$

iii. $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $(n \geq 3)$; $h_0 = 0$, $h_1 = 1$, $h_2 = 2$

iv. $h_n = 8h_{n-1} - 16h_{n-2}$, $(n \geq 2)$; $h_0 = -1$, $h_1 = 0$

v. $h_n = 3h_{n-2} - 2h_{n-3}$, $(n \geq 3)$; $h_0 = 1$, $h_1 = 0$, $h_2 = 0$

vi. $h_n = 5h_{n-1} - 6h_{n-2} - 4h_{n-3} + 8h_{n-4}$, $(n \geq 4)$; $h_0 = 0$, $h_1 = 1$, $h_2 = 1$, $h_3 = 2$

325. (Brualdi, 2004, pp.259-266:31) Solve the nonhomogeneous recurrence relation

$$h_n = 4h_{n-1} + 4^n, \quad (n \geq 1)$$
$$h_0 = 3.$$

326. (Brualdi, 2004, pp.259-266:32) Determine the generating function for the sequence of cubes

$$0, 1, 8, \cdots, n^3, \cdots.$$

327. (Brualdi, 2004, pp.259-266:33) Let $h_0, h_1, h_2, \cdots, h_n, \cdot$ be the sequence defined by

$$h_n = n^3, \quad (n \geq 0).$$

Show that $h_n = h_{n-1} + 3n^2 - 3n + 1$ is the recurrence relation for the sequence.

328. (Brualdi, 2004, pp.259-266:34) Formulate a combinatorial problem that leads to the following generating function

$$(1 + x + x^2)(1 + x^2 + x^4 + x^6)(1 + x^2 + x^4 + \cdots)(x + x^2 + x^3 + \cdots).$$

329. (Brualdi, 2004, pp.259-266:35) Determine the generating function for the number h_n of bags of fruit of apples, oranges, bananas, and pears in which there are an even number of apples, at most two oranges, a multiple of three number of bananas, and at most one pear. Then find a formula for h_n from the generating function.

330. (Brualdi, 2004, pp.259-266:36) Determine the generating function for the number h_n of nonnegative integral solutions of

$$2e_1 + 5e_2 + e_3 + 7e_4 = n.$$

331. (Brualdi, 2004, pp.259-266:37) Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be the sequence defined by $h_n = \binom{n}{2}$, $(n \geq 0)$. Determine the generating function for the sequence.

332. (Brualdi, 2004, pp.259-266:38) Let $h_0, h_1, h_2, \cdots, h_n, \cdots$ be the sequence defined by $h_n = \binom{n}{3}$, $(n \geq 0)$. Determine the generating function for the sequence.

333. (Brualdi, 2004, pp.259-266:39) Let h_n denote the number of regions into which a convex polygonal region with $n + 2$ sides is divided by its diagonals, assuming no three diagonals have a common point. Define $h_0 = 0$. Show that

$$h_n = h_{n-1} + \binom{n+1}{3} + n, \quad (n \geq 1).$$

Then determine the generating function and from it obtain a a formula for h_n.

334. (Brualdi, 2004, pp.259-266:40) Determine the exponential generating function for the sequence of factorials: $0!, 1!, 2!, 3!, \cdots, n!, \cdots$.

335. (Brualdi, 2004, pp.259-266:41) Let α be a real number. Let the sequence $h_0, h_1, h_2, \cdots, h_n, \cdots$ be defined by $h_0 = 1$, and $h_n = \alpha(\alpha - 1) \cdots (\alpha - n + 1)$, $(n \geq 1)$. Determine the exponential generating function for the sequence.

336. (Brualdi, 2004, pp.259-266:42) Let S denote the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \cdots, \infty \cdot e_k\}$. Determine the exponential generating function for the sequence $h_0, h_1, h_2, \cdots, h_n, \cdots$ where $h_0 = 1$ and, for $n \geq 1$:

   i. h_n equals the number of n-permutations of S in which each object occurs an odd number of times.

   ii. h_n equals the number of n-permutations of S in which each object occurs at least four times.

   iii. h_n equals the number of n-permutations of S in which e_1 occurs at least once, e_2 occurs at least twice, $\cdots$, e_k occurs at least k times.

   iv. h_n equals the number of n-permutations of S in which e_1 occurs at most once, e_2 occurs at most twice, $\cdots$, e_k occurs at most k times.

337. (Brualdi, 2004, pp.259-266:43) Let h_n denote the number of ways to color the squares of a 1-by-n board with the colors red, white, blue, and green in such a way that the number of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence $h_0, h_1, \cdots, h_n, \cdots$, and then find a simple formula for h_n

338. (Brualdi, 2004, pp.259-266:44) Determine the number of ways to color the squares of a 1-by-n chessboard, using the colors red, blue, green, and orange if an even number of squares is to be colored red and an even number is to be colored green.

339. (Brualdi, 2004, pp.259-266:45) Determine the number of n digit numbers with all digits odd, such that 1 and 3 each occur a nonzero, even number of times.

340. (Brualdi, 2004, pp.259-266:46) Determine the number of n digit numbers with all digits at least 4, such that 4 and 6 each occur an even number of times, and 5 and 7 each occur at least once, there being no restriction on the digits 8 and 9.

341. (Brualdi, 2004, pp.259-266:47) We have used exponential generating functions to show that the number h_n of n digit numbers with each digit odd, where the digits 1 and 3 occur an even number of times, satisfies the formula

$$h_n = \frac{5^n + 2 \times 3^n + 1}{4}, \quad (n \geq 0).$$

   Obtain an alternative derivation of this formula by finding a recurrence relation satisfied by h_n and then solving the recurrence relation.

342. (Brualdi, 2004, pp.259-266:48) We have used exponential generating functions to show that the number h_n of ways to color the squares of a 1-by-n board with the colors red, white, and blue, where the number of red squares is even and there is at least one blue square, satisfies the formula

$$h_n = \frac{3^n - 2^n + 1}{2}, \quad (n \geq 1)$$

with $h_0 = 0$. Obtain an alternative derivation of this formula by finding a recurrence relation satisfied by h_n and then solving the recurrence relation.

343. (Brualdi, 2004, pp.317-321:1) Let $2n$ (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the nth Catalan number C_n.

344. (Brualdi, 2004, pp.317-321:2) Prove that the number of 2-by-n arrays

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \end{bmatrix}$$

that can be made from the numbers $1, 2, \cdots, 2n$ such that

$$x_{11} < x_{12} < \cdots < x_{1n},$$
$$x_{21} < x_{22} < \cdots < x_{2n}$$
$$x_{11} < x_{21}, \quad x_{12} < x_{22}, \quad \cdots, \quad x_{1n} < x_{2n},$$

equals the nth Catalan numbers, C_n.

345. (Brualdi, 2004, pp.317-321:3) Write out all of the multiplication schemes for four numbers and the triangularization of a convex polygonal region of five sides corresponding to them.

346. (Brualdi, 2004, pp.317-321:4) Determine the triangularization of a convex polygonal region corresponding to the following multiplication scheme

i. $(a_1 \times ((a_2 \times a_3) \times (a_4 \times a_5) \times a_6))$

ii. $(((a_1 \times a_2) \times (a_3 \times (a_4 \times a_5)) \times ((a_6 \times a_7) \times a_8)))$

347. (Brualdi, 2004, pp.317-321:5) Let m and n be nonnegative integers with $n \geq m$. There are $m + n$ people in line to get into a theatre for which admission is 50 cents. Of the $m + n$ people, n have a 50-cent piece and m have a \$1 dollar bill. The box office opens with an empty cash register. Show that the number of ways the people can line up so that change is available when needed is

$$\frac{n - m + 1}{n + 1}\binom{m + n}{m}.$$

(The case $m = n$ is the case treated in Section 8.1.)

348. (Brualdi, 2004, pp.317-321:6) Let the sequence $h_0, h_1, \cdots, h_n, \cdots$ be defined by $h_n = 2n^2 - n + 3$, $(n \geq 0)$. Determine the difference table, and find a formula for $\sum_{k=0}^{n} h_k$.

349. (Brualdi, 2004, pp.317-321:7) The general term h_n of a sequence is a polynomial in n of degree 3. If the first four entries of the 0th row of its difference table are 1, -1, 3, 10, determine h_n and a formula for $\sum_{k=0}^{n} h_k$.

350. (Brualdi, 2004, pp.317-321:8) Find the sum of the fifth powers of the first n positive integers.

351. (Brualdi, 2004, pp.317-321:9) Prove the following formula for the kth-order differences of a sequence $h_0, h_1, \cdots, h_n, \cdots$:

$$\Delta^k h_n = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} h_{n+j}.$$

352. (Brualdi, 2004, pp.317-321:10) If h_n is a polynomial in n of degree m, prove that the constants $c_0, c_1, \cdots, c_m$ such that

$$h_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + \cdots + c_m \binom{n}{m}$$

are uniquely determined. (Cf. Theorem 8.2.2.)

353. (Brualdi, 2004, pp.317-321:11) Compute the Stirling numbers of the seconed kind $S(8, k)$, $(k = 0, 1, \cdots, 8)$.

354. (Brualdi, 2004, pp.317-321:12) Prove that the Stirling numbers of the second kind satisfy the following relations,

i. $S(n, 1) = 1$, $(n \geq 1)$

ii. $S(n, 2) = 2^{n-1} - 1$, $(n \geq 2)$

iii. $S(n, n-1) = \binom{n}{2}$, $(n \geq 1)$

iv. $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$, $(n \geq 2)$

355. (Brualdi, 2004, pp.317-321:13) Let X be a p-element set and Let Y be a k-element set. Prove that the number of functions $f : X \to Y$ which map X *onto* Y equals

$$k! S(p, k) = S^{\#}(p, k).$$

356. (Brualdi, 2004, pp.317-321:14) Find and verify a general formula for

$$\sum_{k=0}^{n} k^p$$

involving Stirling numbers of the second kind.

357. (Brualdi, 2004, pp.317-321:15) The number of partitions of a set of n elements into k distinguishable boxes (some of which may be empty) is k^n. By counting in a different way, prove that

$$k^n = \binom{k}{1} 1! S(n,1) + \binom{k}{2} 2! S(n,2) + \cdots + \binom{k}{n} k! S(n,n).$$

(If $k > n$, define $S(n,k)$ to be 0.)

358. (Brualdi, 2004, pp.317-321:16) Compute the Bell number B_8. (Cf. Exercise 353.)

359. (Brualdi, 2004, pp.317-321:17) Compute the triangle of Stirling numbers of the first kind $s(n,k)$ up to $n = 7$.

360. (Brualdi, 2004, pp.317-321:18) Write $[n]_k$ as a polynomial in n for $k = 1, 2, \cdots, 7$.

361. (Brualdi, 2004, pp.317-321:19) Prove that the Stirling numbers of the first kind satisfy the following formulas

i. $s(n,1) = (n-1)!$, $(n \geq 1)$

ii. $s(n,n-1) = \binom{n}{2}$, $(n \geq 1)$.

362. (Brualdi, 2004, pp.317-321:20) Verify that $[n]_n = n!$, and write $n!$ as a polynomial in n using the Stirling numbers of the first kind. Do this explicitly for $n = 6$.

363. (Brualdi, 2004, pp.317-321:21) For each integer $n = 1, 2, 3, 4, 5$, construct the diagram of the set $\mathcal{P}_n$ of partitions of n partially ordered by majorization.

364. (Brualdi, 2004, pp.317-321:22)

i. Calculate $p(6)$ and construct the diagram of the set $\mathcal{P}_6$ partially ordered by majorization.

ii. Calculate $p(7)$ and construct the diagram of the set $\mathcal{P}_7$ partially ordered by majorization.

365. (Brualdi, 2004, pp.317-321:23) A total order on a finite set has a unique maximal element (a largest element) and a unique minimal element (a smallest element). What are the largest partition and smallest partition in the lexicographic order on $\mathcal{P}(n)$?

366. (Brualdi, 2004, pp.317-321:24) A partial order on a finite set may have many maximal elements and minimal elements. In the set $\mathcal{P}_n$ of partitions of n partially ordered by majorization, prove that there is a unique maximal element and a unique minimal element.

367. (Brualdi, 2004, pp.317-321:25) Let $t_1, t_2, \cdots, t_m$ be distinct positive integers, and let

$$q_n = q_n(t_1, t_2, \cdots, t_n)$$

equal the number of partitions of n in which all parts are taken from $t_1, t_2, \cdots, t_m$. Define $q_0 = 1$. Show that the generating function for $q_0, q_1, \cdots, q_n, \cdots$ is

$$\prod_{k=1}^{m} \left(1 - x^{t_k}\right)^{-1}.$$

368. (Brualdi, 2004, pp.317-321:26) Determine the conjugate of each of the following partitions,

i. $12 = 5 + 4 + 2 + 1$

ii. $15 = 6 + 4 + 3 + 1 + 1$

iii. $20 = 6 + 6 + 4 + 4$

iv. $21 = 6 + 5 + 4 + 3 + 2 + 1$

v. $29 = 8 + 6 + 6 + 4 + 3 + 2$

369. (Brualdi, 2004, pp.317-321:27) For each integer $n > 2$, determine a self-conjugate partition of n that has at least two parts.

370. (Brualdi, 2004, pp.317-321:28) Prove that conjugation reverses the order of majorization, that is, if λ and μ are partitions of n and λ is majorized by μ, then μ^* is majorized by λ^*.

371. (Brualdi, 2004, pp.317-321:29) Evaluate $h_{k-1}^{(k)}$, the number of regions into which k-dimensional space is partitioned by $k - 1$ hyperplanes in general position.

372. (Brualdi, 2004, pp.317-321:30) Use the recurrence relation

$$(n+2)s_{n+2} - 3(2n+1)s_{n+1} + (n-1)s_n = 0, \quad (n \geq 1)$$

to compute the small Schröder numbers s_8 and s_9.

373. (Brualdi, 2004, pp.317-321:31) Use the recurrence relation

$$R_n = R_{n-1} + \sum_{k=0}^{n-1} R_k R_{n-1-k}, \quad (n \geq 1)$$

to compute the large Schröder numbers R_7 and R_8. Verify that $R_7 = 2s_8$ and $R_8 = 2s_9$, as stated in Corollary 8.5.8.

374. (Brualdi, 2004, pp.317-321:32) Use the generating function for the large Schröder numbers to compute the first few large Schröder numbers.

375. (Brualdi, 2004, pp.317-321:33) Use the gerenating function for the small Schröder numbers to comput the first new small Schröder numberts.

376. (Brualdi, 2004, pp.317-321:34) Prove that the large Schröder number R_n equals the number of lattic paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ that never go above the horizontal axix. (These are sometimes called *Dyck paths.*)

377. (Brualdi, 2004, pp.317-321:35) The large Schröder number R_n counts the number of subdiagonal lattice paths from $(0,0)$ to (n,n). The small Schröder number counts the number of dissections of a convex polygonal region of $n+1$. Since $R_n = 2s_{n+1}$ for $n \geq 1$, there are as many subdiagonal lattic paths from $(0,0)$ to (n,n) as there are dissections of a convex polygonal region of $n+1$ sides. Find a one-to-one correspondence between these lattic paths and these dissections.

378. (Brualdi, 2004, pp.358-362:1) Consider the chessboard B with forbidden positions shown in Figure. Construct the rook-bipartite graph G associated with B. Find 6 positions for 6 nonattacking rooks on B, and determine the corresponding matching in G.

379. (Brualdi, 2004, pp.358-362:2) Construct the domino-bipartite graph G associated with the board B in Figure in Exercise 378. Determine a matching of 10 edges in G and the associated perfect cover of the board by dominoes.

380. (Brualdi, 2004, pp.358-362:3) Show that every bipartite graph is the rook-bipartite graph of some board.

			×	×	
×			×		
×					×
×	×	×	×	×	
×	×	×			
		×	×		

Figure in Question 378

381. (Brualdi, 2004, pp.358-362:4) Give an example of a bipartite graph that is not the domino-bipartite graph of any board.

382. (Brualdi, 2004, pp.358-362:5) Consider an m-by-n chessboard in which both m and n are odd. The board has one more square of one color, say, black, than of white. Show that, if exactly one black square is forbidden on the board, the resulting board has a perfect cover with dominoes.

383. (Brualdi, 2004, pp.358-362:6) Considere an m-by-n chessboard, where at least one of m and n is even. The board has an equal number of white and black squares. Show that if m and n are at least 2 and if exactly one white and exactly one black square are forbidden, the resulting board has a perfect cover with dominoes.

384. (Brualdi, 2004, pp.358-362:7) Let $G = (X, \Delta, Y)$ be a bipartite graph. Suppose that there is a positive integer p such that each vertex in X meets at least p edges, and each vertex in Y meets at most p edges. By counting the total number of edges in G, prove that Y has at least as many vertices as X.

385. (Brualdi, 2004, pp.358-362:8) Let $G = (X, \Delta, Y)$ be a bipartite graph that is regular of degree $p \geq 1$. Use Theorem 9.2.5 and induction to show that the edges of G can be partitioned into p perfect matchings.

386. (Brualdi, 2004, pp.358-362:9) Consider an n-by-n chessboard with forbidden positions for which there exists a positive integer p such that each row and each column contains exactly p allowed squares. Prove that it is possible to place n nonattacking rooks on the board.

387. (Brualdi, 2004, pp.358-362:10) Use the matching algorithm to determine the largest number of edges in a matching M of the bipartite graphs in Figure. In each case, find a cover S with $|S| = |M|$.

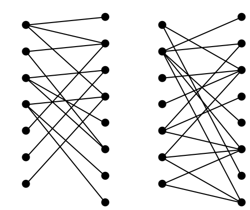


Figure in Question 387

388. (Brualdi, 2004, pp.358-362:11) A corporation has 7 available positions $y_1, y_2, \cdots, y_7$ and 10 applicants $x_1, x_2, \cdots, x_{10}$. The set of positions each applicant is qualified for is given, respectively, by $\{y_1, y_2, y_6\}$, $\{y_2, y_6, y_7\}$, $\{y_3, y_4\}$, $\{y_1, y_5\}$, $\{y_6, y_7\}$, $\{y_3\}$, $\{y_2, y_3\}$, $\{y_1, y_3\}$, $\{y_1\}$, $\{y_5\}$. Determine the largest number of positions that can be filled by the qualified applicants and justify your answer.

389. (Brualdi, 2004, pp.358-362:12) Let $\mathcal{A} = (A_1, A_2, A_3, A_4, A_5, A_6)$, where

$$A_1 = \{a, b, c\}, \quad A_2 = \{a, b, c, d, e\}, \quad A_3 = \{a, b\}, \quad A_4 = \{b, c\}, \quad A_5 = \{a\}, \quad A_6 = \{a, c, e\}.$$

390. (Brualdi, 2004, pp.358-362:13) Let $\mathcal{A} = (A_1, A_2, A_3, A_4, A_5, A_6)$, where

$$A_1 = \{1, 2\}, \quad A_2 = \{2, 3\}, \quad A_3 = \{3, 4\}, \quad A_4 = \{4, 5\}, \quad A_5 = \{5, 6\}, \quad A_6 = \{6, 1\}.$$

Determine the number of different SDR's that $\mathcal{A}$ has. Generalize to n sets.

391. (Brualdi, 2004, pp.358-362:14) Let $\mathcal{A} = (A_1, A_2, \cdots, A_n)$ be a family of sets with an SDR. Let x be an element of A_1. Prove that there is an SDR containing x, but show by example that it may not be possible to find an SDR in which x represents A_1.

392. (Brualdi, 2004, pp.358-362:15) Suppose $\mathcal{A} = (A_1, A_2, \cdots, A_n)$ is a family of sets that "more than satisfies" the Marriage Condition. More precisely, suppose that

$$|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| \geq k + 1$$

for each $k = 1, 2, \cdots, n$ and each choice of k distinct indices $i_1, i_2, \cdots, i_k$. Let x be an element of A_1. Prove that $\mathcal{A}$ has an SDR in which x represents A_1.

393. (Brualdi, 2004, pp.358-362:16) Let $n > 1$, and let $\mathcal{A} = (A_1, A_2, \cdots, A_n)$ be the famil of subsets of $\{1, 2, \cdots, n\}$, where

$$\boldsymbol{A}_i = \{1, 2, \cdots, n\} - \{i\}, \quad (i = 1, 2, \cdots, n).$$

Prove that $\mathcal{A}$ has an SDR and that the number of SDR's is the nth derangement number D_n.

394. (Brualdi, 2004, pp.358-362:17) Consider a chessboard with forbidden positions which has the property that, if a square is forbidden, so is every square to its right and every square below it. Prove that the chessboard has a perfect cover by dominoes if and only if the number of allowable white squares equals the number of allowable black squares.

395. (Brualdi, 2004, pp.358-362:18) Let A be a matrix with n columns, with integer entries taken from the set $S = \{1, 2, \cdots, k\}$. Assume that each integer i in S occrus exactly nr_i times in A, where r_i is an integer. Prove that it is possible to permute the entries in each row of A to obtain a matrix B in which each integer i in S appears r_i times i each column (Kramer et al., 1991).

396. (Brualdi, 2004, pp.358-362:19) Find a 2-by-2 preferential ranking matrix for which both complete marriages are stable.

397. (Brualdi, 2004, pp.358-362:20) Consider a preferential ranking matrix in which woman A ranks man a first, and man a ranks A first. Show that, in every stable marriage, A is paired with a.

398. (Brualdi, 2004, pp.358-362:21) Consider the preferential ranking matrix

$$\begin{bmatrix} 1,n & 2,n-1 & 3,n-2 & \cdots & n,1 \\ n,1 & 1,n & 2,n-1 & \cdots & n-1,2 \\ n-1,2 & n,1 & 1,n & \cdots & n-2,3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 3,n-3 & 4,n-3 & 5,n-4 & \cdots & 2,n-1 \\ 2,n-2 & 3,n-2 & 4,n-3 & \cdots & 1,n \end{bmatrix}.$$

Prove that, for each $k = 1, 2, \cdots, n$, the complete marriage in which each woman get her kth choice is stable.

399. (Brualdi, 2004, pp.358-362:22) Use the deferred acceptance algorithm to obtain both the women-optimal and men-optimal stable complete marriages for the preferential ranking matrix. Conclude that, for the given preferential ranking matrix, there is only one stable complete marriage.

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline A & 1,3 & 2,3 & 3,2 & 4,3 \\ B & 1,4 & 4,1 & 3,3 & 2,2 \\ C & 2,2 & 1,4 & 3,4 & 4,1 \\ D & 4,1 & 2,2 & 3,1 & 1,4 \end{array}$$

Figure in Question 399

400. (Brualdi, 2004, pp.358-362:23) Prove that in every application of the deferred acceptance algorithm with n women and n men, there are at most $n^2 - n + 1$ proposals.

401. (Brualdi, 2004, pp.358-362:24) Extend the deferred acceptance algorithm to the case in which there more men than women. In such a case, not all of the men will get partners.

402. (Brualdi, 2004, pp.358-362:25) Show, by using Exercise 399, that it is possible that no complete marriage does any person get his or her first choice.

403. (Brualdi, 2004, pp.358-362:26) Apply the deferred acceptance algorithm to obtain a stable complete marriage for the preferential ranking matrix as shown.

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline A & 1,3 & 2,2 & 3,1 & 4,3 \\ B & 1,4 & 2,3 & 3,2 & 4,4 \\ C & 3,1 & 1,4 & 2,3 & 4,2 \\ D & 2,2 & 3,1 & 1,4 & 4,1 \end{array}$$

Figure in Question 403

404. (Brualdi, 2004, pp.358-362:27) Consider an n-by-n board in which there is a nonnegative number a_{ij} in the square in row i and column j, $(1 \le i, j \le n)$. Assume that the sum of the numbers in each row and in eah column equals 1. Prove that it is possible to place n nonattacking rooks on the board at positions occupied by positive numbers.

405. (Brualdi, 2004, pp.415-421:1) Compute the addition table and the multiplication table for the integers mode 4.

406. (Brualdi, 2004, pp.415-421:2) Compute the substraction table for the integers mod 4. How does it compare with the addition table computed in Exercise 405?

407. (Brualdi, 2004, pp.415-421:3) Compute the addition table and the multiplication table for the integers mod 5.

408. (Brualdi, 2004, pp.415-421:4) Compute the substraction table of the integers mod 5. How does it compare with the addition table computed in Exercise 407?

409. (Brualdi, 2004, pp.415-421:5) Prove that no two integers in Z_n, arithmetic mod n, have the same additive inverse. Conclude from the pigeonhole principle that

$$\{-0, -1, -2, \cdots, -(n-1)\} = \{0, 1, 2, \cdots, n-1\}.$$

(Remember that $-a$ is the integer which, when added to a in Z_n, gives 0.)

410. (Brualdi, 2004, pp.415-421:6) Prove that the columns of the substraction table of Z_n are a rearrangement of the columns of the addition table of Z_n (Cf. Exercises 406 and 408).

411. (Brualdi, 2004, pp.415-421:7) Compute the addition table and multiplication table for the integers mod 6.

412. (Brualdi, 2004, pp.415-421:8) Determine the additive inverse of the integers in Z_8, with arithmetic mod 8.

413. (Brualdi, 2004, pp.415-421:9) Determine the additive inverse of 3, 7, 8, and 19 in the integers mod 20.

414. (Brualdi, 2004, pp.415-421:10) Determine which integers in Z_{12} have multiplicative inverses, and find the multiplicative inverses when they exist.

415. (Brualdi, 2004, pp.415-421:11) For each of the following integers in Z_{24}, determine the multiplicative inverse if a multiplicative inverse exists

$$4, 9, 11, 15, 17, 23.$$

416. (Brualdi, 2004, pp.415-421:12) Prove that $n-1$ always has a multiplicative inverse in Z_n, $(n \geq 2)$.

417. (Brualdi, 2004, pp.415-421:13) Let $n = 2m+1$ be an odd integer with $m \geq 2$. Prove that the multiplicative inverse of $m+1$ in Z_n is 2.

418. (Brualdi, 2004, pp.415-421:14) Use the algorithm in Section 10.1 to find the GCD of the following pairs of integers:

   i. 12 and 31
   ii. 24 and 82
   iii. 26 and 97

iv. 186 and 334

v. 423 and 618

419. (Brualdi, 2004, pp.415-421:15) For each of the pairs of integers in Exercise 418, let m denote the first integer and let n denote the second integer of the pair. When it exists, determine the multiplicative inverse of m in Z_n.

420. (Brualdi, 2004, pp.415-421:16) Apply the algorithm for the GCD in Section 10.1 to 15 and 46, and then use the results to determine the multiplicative inverse of 15 in Z_{46}.

421. (Brualdi, 2004, pp.415-421:17) Start with the field Z_2 and show that $x^3 + x + 1$ cannot be factored in a nontrivial way (into polynomials with coefficients in Z_2), and then use this polynomial to construct a field with $2^3 = 8$ elements. Let i be the root of this polynomial adjoined to Z_2, and then do the following computations

i. $(1 + i) + (1 + i + i^2)$

ii. $(1 + i^2) + (1 + i^2)$

iii. i^{-1}

iv. i$^2 \times (1 + i + i^2)$

v. (1 + i) (1 + i + i^2)

vi. $(1 + i)^{-1}$

422. (Brualdi, 2004, pp.415-421:18) Does there exist a BIBD with parameters $b = 10$, $v = 8$, $r = 5$, and $k = 4$?

423. (Brualdi, 2004, pp.415-421:19) Does there exist a BIBD whose parameters satisfy $b = 20$, $v = 18$, $k = 9$, and $r = 10$?

424. (Brualdi, 2004, pp.415-421:20) Let $\mathcal{B}$ be a BIBD with parameters b, v, k, r, λ whose set of varieties is $X = \{x_1, x_2, \cdots, x_v\}$ and whose blocks are $B_1, B_2, \cdots, B_b$. For each block B_i, let $\overline{B_i}$ denote the set of varieties which do not belong to B_i. Let $\mathcal{B}^c$ be the collection of subsets $\overline{B_1}, \overline{B_2}, \cdots, \overline{B_b}$ of X. Prove that $\mathcal{B}^c$ is a block design with parameters

$$b' = b, \quad v' = v, \quad k' = v - k, \quad r' = b - r, \quad \lambda' = b - 2r + \lambda,$$

provided that we have $b - 2r + \lambda > 0$. The BIBD $\mathcal{B}^c$ is called the *complementary design* of $\mathcal{B}$.

425. (Brualdi, 2004, pp.415-421:21) Determine the complementary design of the BIBD with parameters $b = v = 7$, $k = r = 3$, $\lambda = 1$ in Section 10.2

426. (Brualdi, 2004, pp.415-421:22) Determine the complementary design of the BIBD with parameters $b = v = 16$, $k = r = 6$, $\lambda = 2$ in Section 10.2

427. (Brualdi, 2004, pp.415-421:23) How are the incidence matrices of a BIBD and its complement related?

428. (Brualdi, 2004, pp.415-421:24) Show that a BIBD, with v varieties whose block size k equals $v - 1$, does not have a complementary design.

429. (Brualdi, 2004, pp.415-421:25) Prove that a BIBD with parameters b, v, k, r, λ has a complementary design if and only if $2 \le k \le v - 2$ (Cf. Exercise 424 and 428).

430. (Brualdi, 2004, pp.415-421:26) Let B be a difference set in Z_n. Show that, for each integer k in Z_n, $B + k$ is also a difference set. (This implies that we can always assume without loss of generality that a difference set contains 0 for, if it did not, we can replace it by $B + k$, where k is the additive inverse of any integer in B.)

431. (Brualdi, 2004, pp.415-421:27) Prove that Z_v is itself a difference set in Z_v. (These are *trivial* difference sets.

432. (Brualdi, 2004, pp.415-421:28) Show that $B = \{0, 1, 3, 9\}$ is a difference set in Z_{13}, and use this difference set as a starter block to construct an SBIBD. Identify the parameters of the block design.

433. (Brualdi, 2004, pp.415-421:29) Is $B = \{0, 2, 5, 11\}$ a difference set in Z_{12}?

434. (Brualdi, 2004, pp.415-421:30) Show that $B = \{0, 2, 3, 4, 8\}$ is a difference set in Z_{11}. What are the parameters of the SBIBD developed from B?

435. (Brualdi, 2004, pp.415-421:31) Prove that $B = \{0, 3, 4, 9, 11\}$ is a difference set in Z_{21}.

436. (Brualdi, 2004, pp.415-421:32) Use Theorem 10.3.2 to construct a Steiner triple system of index 1 having 21 varieties.

437. (Brualdi, 2004, pp.415-421:33) Let t be a positive integer. Use Theorem 10.3.2 to prove that there exists a Steiner triple system of index 1 having 3^t varieties.

438. (Brualdi, 2004, pp.415-421:34) Let t be a positive integer. Prove that, if there exists a Steiner triple system of index 1 having v varieties, then there exists a Steiner triple system having v^t varieties (Cf. Exercise 437).

439. (Brualdi, 2004, pp.415-421:35) Assume a Steiner triple system exists with parameters b, v, k, r, λ where $k = 3$. Let a be the remainder when λ is divided by 6. Use Theorem 10.3.1 to show the following:

   i. If $a = 1$ or 5, then v has remainder 1 or 3 when divided by 6.

ii. If $a = 2$ or 4, then v has remainder 0 or 1 when divided by 3.

iii. If $a = 3$, then v is odd.

440. (Brualdi, 2004, pp.415-421:36) Verify that the following three steps construct a Steiner triple system of index 1 with 13 varieties (we begin with Z_{13}).

   i. Each of the integers 1, 3, 4, 9, 10, 12 occurs exactly once as a difference of two integers in $B_1 = \{0, 1, 4\}$.

   ii. Each of the integers 2, 5, 6, 7, 8, 11 occurs exactly once as a difference of two integers in $B_2 = \{0, 2, 7\}$.

   iii. The 12 blocks developed from B_1 together with the 12 blocks developed from B_2 are the blocks of a Steiner triple system of index 1 with 13 varieties.

441. (Brualdi, 2004, pp.415-421:37) Prove that, if we interchange the rows of a Latin square in any way and interchange the columns in any way, the result is always a Latin square.

442. (Brualdi, 2004, pp.415-421:38) Use Theorem 10.4.2 with $n = 6$ and $r = 5$ to construct a Latin square of order 6.

443. (Brualdi, 2004, pp.415-421:39) Let n be a positive integer and let r be a nonzero integer in Z_n such that the GCD of r and n is not 1. Prove that the array constructed using the prescription in Theorem 10.4.2 is not a Latin square.

444. (Brualdi, 2004, pp.415-421:40) Let n be a positive integer and let r and r' be distinct nonzero integers in Z_n such that the GCD of r and n is 1 and the GCD of r' and n is 1. Show that the Latin squares constructed by using Theorem 10.4.2 need not be orthogonal.

445. (Brualdi, 2004, pp.415-421:41) Use Theorem 10.4.2 with $n = 8$ and $r = 3$ to construct a Latin square of order 8.

446. (Brualdi, 2004, pp.415-421:42) Construct 4 MOLS of order 5.

447. (Brualdi, 2004, pp.415-421:43) Construct 3 MOLS of order 7.

448. (Brualdi, 2004, pp.415-421:44) Construct 2 MOLS of order 9.

449. (Brualdi, 2004, pp.415-421:45) Construct 2 MOLS of order 15.

450. (Brualdi, 2004, pp.415-421:46) Construct 2 MOLS of order 8.

451. (Brualdi, 2004, pp.415-421:47) Let A be a Latin square of order n for which there exists a Latin square B of order n such that A and B are orthogonal. B is called an *orthogonal mate* of A. Think of the 0's in A as rooks of color red, the 1's as rooks of color white, the 2's as rooks of color blue, and so on. Prove that there are n nonattacking rooks in A, no two of which have the same color. Indeed, prove that the entire set of n^2 rooks can be partitioned into n sets of n nonattacking rooks each, with no two rooks in the same set having the same color.

452. (Brualdi, 2004, pp.415-421:48) Prove that the addition table of Z_4 is a Latin square without an orthogonal mate (Cf. Exercise 451).

453. (Brualdi, 2004, pp.415-421:49) First construct 4 MOLS of order 5, and then construct the resolvable BIBD corresponding to them as given in Theorem 10.4.10.

454. (Brualdi, 2004, pp.415-421:50) Let A_1 and A_2 be MOLS of order m and let B_1 and B_2 be MOLS of order n. Prove that $A_1 \otimes B_1$ and $A_2 \otimes B_2$ are MOLS of order mn.

455. (Brualdi, 2004, pp.415-421:51) Fill in the details in the proof of Theorem 10.4.10.

456. (Brualdi, 2004, pp.415-421:52) Construct a completion of the 3-by-6 Latin retangle

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 & 0 \\ 5 & 4 & 3 & 0 & 1 & 2 \end{bmatrix}.$$

457. (Brualdi, 2004, pp.415-421:53) Construct a completion of the 3-by-7 Latin rectangle

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 0 & 6 & 5 & 4 & 1 \\ 1 & 4 & 6 & 0 & 2 & 3 & 5 \end{bmatrix}.$$

458. (Brualdi, 2004, pp.415-421:54) How many 2-by-n Latin rectangles have first row equal to

$$0\ 1\ 2\ \cdots\ n-1?$$

459. (Brualdi, 2004, pp.415-421:55) Construct a completion of the semi-Latin square

$$\begin{bmatrix} & 2 & 0 & & & 1 \\ 2 & 0 & & & 1 & \\ 0 & & 2 & 1 & & \\ & & 1 & 2 & & 0 \\ & 1 & & & 0 & 2 \\ 1 & & & 0 & 2 & \end{bmatrix}.$$

460. (Brualdi, 2004, pp.415-421:56) Construct a completion of the semi-Latin square

$$\begin{bmatrix} 0 & 2 & 1 & & & & 3 \\ 2 & 0 & & 1 & & 3 & \\ 3 & & 0 & 2 & 1 & & \\ & 3 & 2 & 0 & & 1 & \\ & & 3 & & 0 & 2 & 1 \\ 1 & & & & 3 & 0 & 2 \\ & 1 & & 3 & 2 & & 0 \end{bmatrix}.$$

461. (Brualdi, 2004, pp.415-421:57) Let $n \geq 2$ be an integer. Prove that an $(n-2)$-by-n Latin rectangle has at least 2 completions, and, for each n, find an example that has exactly 2 completions.

462. (Brualdi, 2004, pp.415-421:58) A Latin square A of order n is *symmetric*, provided the entry a_{ij} at row i, column j equals the entry a_{ji} at row j, column i for all $i \neq j$. Prove that the addition table of Z_n is a symmetri Latin square.

463. (Brualdi, 2004, pp.415-421:59) A Latin square of order n (based on Z_n) is *idempotent*, provided that its entries on the diagonal running from upper left to lower right are $0, 1, 2, \cdots, n-1$.

   i. Construct an example of an idempotent Latin square of order 5.

   ii. Construct an example of a symmetric, idempotent Latin square of order 5.

464. (Brualdi, 2004, pp.415-421:60) Prove that a symmetric, idempotent Latin square has odd order.

465. (Brualdi, 2004, pp.415-421:61) Let $n = 2m+1$, where m is a positive integer. Prove that the n-by-n array A whose entry a_{ij} in row i, column j satisfies

$$a_{ij} = (m+1) \times (i+j) \quad \text{(arithmetic mod } n\text{)}$$

is a symmetric, idempotent Latin square of order n. [Remark: The integer $m+1$ is the multiplicative inverse of 2 in Z_n. Thus, our prescription for a_{ij} is to "average" i and j.]

466. (Brualdi, 2004, pp.415-421:62) Let L be an m-by-n Latin rectangle (based on Z_n) and let the entry in row i, column j be denoted by a_{ij}. We define an n-by-n array B whose entry b_{ij} in position row i, column j satisfies

$$b_{ij} = k, \quad \text{provided } a_{kj} = i$$

and is blank otherwise. Prove that B is a semi-Latin square of order n and index m. In particular, if A is a Latin square of order n so is B.

467. (Brualdi, 2004, pp.482-493:1) How many nonisomorphic graphs of order 1 are there? of order 2? of order 3? Explain why the answer to each of the preceding questions is ∞ for general graphs.

468. (Brualdi, 2004, pp.482-493:2) Determine each of the 11 nonisomorphic graphs of order 4, and give a planar representation of each.

469. (Brualdi, 2004, pp.482-493:3) Does there exist a graph of order 5 whose degree sequence equals $(4, 4, 3, 2, 2)$?

470. (Brualdi, 2004, pp.482-493:4) Does there exist a graph of order 5 whose degree sequence equals $(4, 4, 4, 2, 2)$? a multigraph?

471. (Brualdi, 2004, pp.482-493:5) Use the pigeonhole principle to prove that a graph of order $n \geq 2$ always has two vertices of the same degree. Does the same conclusion hold for multigraphs?

472. (Brualdi, 2004, pp.482-493:6) Let $(d_1, d_2, \cdots, d_n)$ be a sequence of n nonnegative integers. Prove that there exists a general graph with this sequence as its degree sequence.

473. (Brualdi, 2004, pp.482-493:7) Let $(d_1, d_2, \cdots, d_n)$ be a sequence of n nonnegative integers whos sum $d_1 + d_2 + \cdots + d_n$ is even. Prove that there exists a general graph with this sequence as its degree sequence. Devise an algorithm to construct such a general graph.

474. (Brualdi, 2004, pp.482-493:8) Let G be a graph with degree sequence $(d_1, d_2, \cdots, d_n)$. Prove that, for each k with $0 < k < n$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

475. (Brualdi, 2004, pp.482-493:9) Draw a connected graph whose degree sequence equals

$$(5, 4, 3, 3, 3, 3, 3, 2, 2).$$

476. (Brualdi, 2004, pp.482-493:10) Prove that any two connected graphs of order n with degree sequence $(2, 2, \cdots, 2)$ are isomorphic.

477. (Brualdi, 2004, pp.482-493:11) Determine which pairs of the general graphs as shown are isomorphic and, if isomorphic, find an isomorphism.

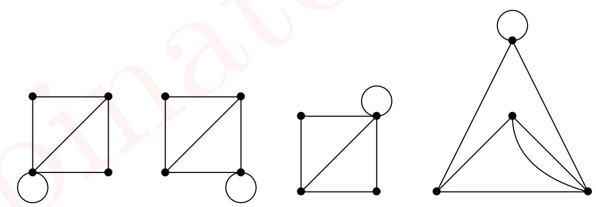


Figure in Question 477

478. (Brualdi, 2004, pp.482-493:12) Determine which pairs of the multigraphs as shown are isomorphic, and for those that are isomorphic, find an isomorphism.

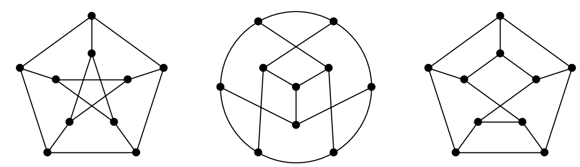


Figure in Question 478

479. (Brualdi, 2004, pp.482-493:13) Prove that, if two vertices of a general graph are joined by a walk, then they are joined by a path.

480. (Brualdi, 2004, pp.482-493:14) Let x and y be vertices of a general graph and suppose that there is a closed walk containing both x and y. Must there be a closed trail containing both x and y?

481. (Brualdi, 2004, pp.482-493:15) Let x and y be vertices of a general graph and suppose that there is a closed trail containing both x and y. Must there be a cycle containing both x and y?

482. (Brualdi, 2004, pp.482-493:16) Let G be a connected graph of order 6 with degree sequence $(2, 2, 2, 2, 2, 2)$.

   i. Determine all the nonisomorphic induced subgraphs of G.

   ii. Determine all the nonisomorphic spanning subgraphs of G.

   iii. Determine all the nonisomorphic subgraphs of order 6 of G.

483. (Brualdi, 2004, pp.482-493:17) First, prove that any two multigraphs G of order 3 with degree sequence $(4, 4, 4)$ are isomorphic. Then

   i. determine all the nonisomorphic induced subgraphs of G.

   ii. determine all the nonisomorphic spanning subgraphs of G.

   iii. determine all the nonisomorphic subgraphs of order 3 of G.

484. (Brualdi, 2004, pp.482-493:18) Let γ be a trail joining vertices x and y in a general graph. Prove that the edges of γ can be partitioned so that one part of the partition determines a path joining x and y and the other parts determine cycles.

485. (Brualdi, 2004, pp.482-493:19) Let G be a general graph and let G' be the graph obtained from G by deleting all loops and all but one copy of each edge with multiplicity greater than 1. Prove that G is connected if and only if G' is connected. Also prove that G is planar if and only if G' is planar.

486. (Brualdi, 2004, pp.482-493:20) Prove that a graph of order n with at least

$$\frac{(n-1)(n-2)}{2}+1$$

edges must be connected. Give an example of a disconnected graph of order n with one fewer edge.

487. (Brualdi, 2004, pp.482-493:21) Let G be a general graph with exactly two vertices x and y of odd degree. Let G^* be the general graph obtained by putting a new edge $\{x, y\}$ joining x and y. Prove that G is connected if and only if G^* is connected.

488. (Brualdi, 2004, pp.482-493:22) (This and the following two exercises prove Theorem 11.1.3.) Let $G = (V, E)$ be a general graph. If x and y are in V, degine $x \sim y$ to mean that either $x = y$ or there is a walk joining x and y. Prove that, for all vertices x, y, and z, we have

   i. $x \sim x$.
   ii. $x \sim y$ if and only if $y \sim x$.
   iii. if $x \sim y$ and $y \sim z$, then $x \sim z$.

489. (Brualdi, 2004, pp.482-493:23) (Continuation of Exercise 488.) For each vertex x, let

$$C(x) = \{z : x \sim z\}.$$

Prove the following:

   i. For all vertices x and y, either $C(x) = C(y)$ or else $C(x) \cap C(y) = \emptyset$. In other words two of the sets $C(x)$ and $C(y)$ cannot intersect unless they are equal.
   ii. if $C(x) \cap C(y) = \emptyset$, then there does not exist an edge joining a vertex in $C(x)$ to a vertex in $C(y)$.

490. (Brualdi, 2004, pp.482-493:24) (Continuation of Exercise 489.) Let $V_1, V_2, \cdots, V_k$ be the different sets that occur among the $C(x)$'s. Prove that

i. $V_1, V_2, \cdots, V_k$ form a partition of the vertex set V of G.

ii. the general subgraphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \cdots, G_k = (V_k, E_k)$ of G induced by $V_1, V_2, \cdots, V_k$, respectively, are connected.

The induced subgraphs $G_1, G_2, \cdots, G_k$ are the *connected components* of G.

491. (Brualdi, 2004, pp.482-493:25) Prove Theorem 11.1.4

492. (Brualdi, 2004, pp.482-493:26) Determine the adjacency matrices of the first and second general graphs shown in figure

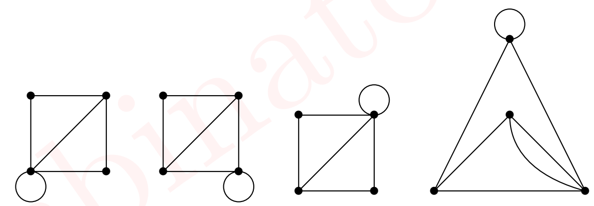


Figure in Question 492

493. (Brualdi, 2004, pp.482-493:27) Determine the adjacency matrices of the first and second multigraphs in figure as shown

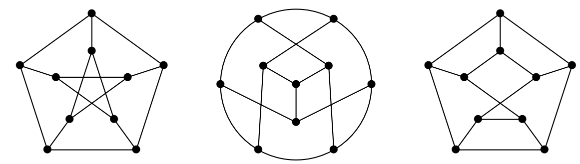


Figure in Question 493

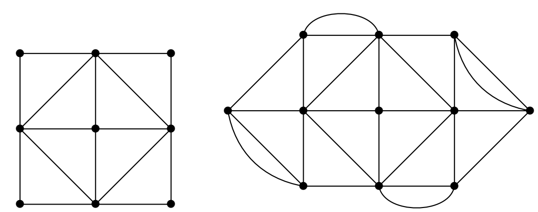
494. (Brualdi, 2004, pp.482-493:28) Let A and B be two n-by-n matrices of numbers whose entries are denoted by a_{ij} and b_{ij}, $(1 \le i, j \le n)$, respectively. Define the product $A \times B$ to be the n-by-n matrix C whose entry C_{ij} in row i and column j is given by

$$c_{ij} = \sum_{p=1}^{n} a_{ip} b_{pj}, \quad (1 \le i, j \le n).$$

If k is a positive integer, define

$$A^k = A \times A \times \cdots \times A \quad (k\ A\text{'s}).$$

Now let A denote the adjacency matrix of a general graph of order n with vertices $a_1, a_2, \cdots, a_n$. Prove that the entry in row i, column j of A^k equals the number of walks of length k in G joining vertices a_i and a_j.



(a) (b)

Figure in Question 495

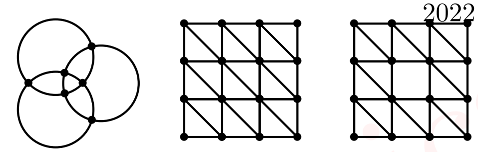
495. (Brualdi, 2004, pp.482-493:29) Determine if the multigraphs shown in Figure have Eulerian trails (closed or open). In case there is an Eulerian trail, use our algorithms to construct one.

496. (Brualdi, 2004, pp.482-493:30) Which complete graph K_n have closed Eulerian trails? open Eulerian trails?

497. (Brualdi, 2004, pp.482-493:31) Prove Theorem 11.2.4.

498. (Brualdi, 2004, pp.482-493:32) What is the fewest number of open trails in to which the edges of GraphBuster can be partitioned?

499. (Brualdi, 2004, pp.482-493:33) Show how, removing one's pencil from the paper the fewest number of times, to trace the plane graphs as shown



(a) (b) (c)

Figure in Question 499

500. (Brualdi, 2004, pp.482-493:34) Determine all nonisomorphic graphs of order at most 6 that have a closed Eulerian trail.

501. (Brualdi, 2004, pp.482-493:35) Show how, removing one's pencil from the paper the fewest number of times, to trace out the graph of the regular dodecahedron as shown

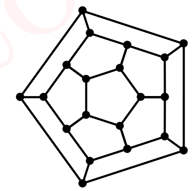


Figure in Question 501

502. (Brualdi, 2004, pp.482-493:36) Let G be a connected graph. Let γ be a closed walk that contains each edge of G at least once. Let G^* be the multigraph obtained from G by increasing the multiplicity of each edge from 1 to the number of times ir occurs in γ. Prove that γ is a closed Eulerian trail in G^*. Conversely, suppose we increase the multiplicity of some of the edges of G and obtain a multigraph with m edges, each of whose vertices has even degree. Prove that there is a closed walk in G of length m which contains each edge of G at least once. This exercise shows tht the Chinese postman problem for G is equivalent to determining the smallest number of copies of the edges of G that need to be inserted so as to obtain a multigraph all of whose vertices have even degree.

503. (Brualdi, 2004, pp.482-493:37) Solve the Chinese postman problem for the complete graph K_6.

504. (Brualdi, 2004, pp.482-493:38) Solve the Chinese postman problem for the graph obtained from K_6 by removing any edge.

505. (Brualdi, 2004, pp.482-493:39) Call a graph *cubic* if each vertex has degree equal to 3. The complete graph K_4 is the smallest example of a cubic graph. Find an example of a connected, cubic graph that does not have a Hamilton path.

506. (Brualdi, 2004, pp.482-493:40) Let G be a graph of order n having at least

$$\frac{(n-1)(n-2)}{2}+2$$

edges. Prove that G has a Hamilton cycle. Exhibit a graph of order n with one fewer edge that does not have a Hamilton cycle.

507. (Brualdi, 2004, pp.482-493:41) Let $n \geq 3$ be an integer. Let G_n be the graph whose vertices are the $n!$ permutations of $\{1, 2, \cdots, n\}$, wherein two permutations are joined by an edge if and only if one can be obtained from the other by the interchange of two numbers (an arbitrary transposition). Deduce from the results of Section 4.1 that G_n has a Hamilton cycle.

508. (Brualdi, 2004, pp.482-493:42) Prove Theorem 11.3.4.

509. (Brualdi, 2004, pp.482-493:43) Devise an algorithm analogous to our algorithm for a Hamilton cycle that constructs a Hamilton path in graphs satisfying the condition given in Theorem 11.3.4.

510. (Brualdi, 2004, pp.482-493:44) Which complete bipartite graphs $K_{m,n}$ have Hamilton cycles? Which have Hamilton paths?

511. (Brualdi, 2004, pp.482-493:45) Prove that a multigraph is bipartite if and only if each of its connected components is.

512. (Brualdi, 2004, pp.482-493:46) Prove that $K_{m,n}$ is isomorphic to $K_{n,m}$.

513. (Brualdi, 2004, pp.482-493:47) Prove that a bipartite multigraph with an odd number of vertices does not have a Hamilton cycle.

514. (Brualdi, 2004, pp.482-493:48) Is GraphBuster a bipartite graph? If so, find a bipartition of its vertices. What if we delete the loops?

515. (Brualdi, 2004, pp.482-493:49) Let $V = \{1, 2, \cdots, 20\}$ be the set of the first 20 positive integers. Consider the graphs whose vertex set is V and whose edge sets are defined below. For each graph, investigate whether the graph (a) is connected (if not connected, determine the connected components), (b) is bipartite, (c) has an Eulerian trail, and (d) has a Hamilton path.

   i. $\{a, b\}$ is an edge if and only if $a + b$ is even.
   ii. $\{a, b\}$ is an edge if and only if $a + b$ is odd.
   iii. $\{a, b\}$ is an edge if and only if $a \times b$ is even.
   iv. $\{a, b\}$ is an edge if and only if $a \times b$ is odd.
   v. $\{a, b\}$ is an edge if and only if $a \times b$ is a perfect square.
   vi. $\{a, b\}$ is an edge if and only if $a - b$ is divisible by 3.

516. (Brualdi, 2004, pp.482-493:50) What is the smallest number of edges that can be removed from K_5 in order to leave a bipartite graph?

517. (Brualdi, 2004, pp.482-493:51) Find a knight's tour on the boards of the following sizes:

   i. 5-by-5
   ii. 6-by-6
   iii. 7-by-7

518. (Brualdi, 2004, pp.482-493:52) Prove that there does not exist a knight's tour on a 4-by-4 board.

519. (Brualdi, 2004, pp.482-493:53) Prove that a graph is a tree if and only if it does not contain any cycles, but the insertion of any new edge always creates exactly one cycle.

520. (Brualdi, 2004, pp.482-493:54) Which trees have an Eulerian path?

521. (Brualdi, 2004, pp.482-493:55) Which trees have a Hamilton path?

522. (Brualdi, 2004, pp.482-493:56) Grow all the nonisomorphic trees of order 7.

523. (Brualdi, 2004, pp.482-493:57) Let $(d_1, d_2, \cdots, d_n)$ be a sequence of integers.

   i. Prove that there is a tree of order n with this degree sequence if and only if $d_1, d_2, \cdots, d_n$ are positive integers with sum $d_1 + d_2 + \cdots + d_n = 2(n-1)$.

   ii. Write an algorithm that, starting with a sequence $(d_1, d_2, \cdots, d_n)$ of positive integers, either constructs a tree with this degree sequence or concludes that none is possible.

524. (Brualdi, 2004, pp.482-493:58) A *forest* is a graph each of whose connected components is a tree. In particular, a tree is a forest. Prove that a graph is a forest if and only if it does not have any cycles.

525. (Brualdi, 2004, pp.482-493:59) Prove that the removal of an edge from a tree leaves a forest of two trees.

526. (Brualdi, 2004, pp.482-493:60) Let G be a forest of k trees. What is the fewest number of edges that can be inserted in G in order to obtain a tree?

527. (Brualdi, 2004, pp.482-493:61) Determine a spanning tree for GraphBuster.

528. (Brualdi, 2004, pp.482-493:62) Prove that, if a tree has a vertex of degree p, then it has at least p pendent vertices.

529. (Brualdi, 2004, pp.482-493:63) Determine a spanning tree for each of the graphs as shown.

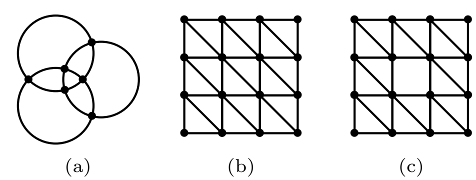


Figure in Question 529

530. (Brualdi, 2004, pp.482-493:64) For each integers $n \geq 3$ and for each integer k with $2 \leq k \leq n-1$, construct a tree of order n with exactly k pendent vertices.

531. (Brualdi, 2004, pp.482-493:65) Use the algorithm for a spanning tree in Section 11.5 in order to construct a spanning tree of the graph of the dodecahedron.

532. (Brualdi, 2004, pp.482-493:66) How many cycles does a connected graph of order n with n edges have?

533. (Brualdi, 2004, pp.482-493:67) Let G be a graph of order n that is not necessarily connected. A forest is defined in Exercise 524. A *spanning forest* of G is a forest consisting of a spanning tree of each of the connected components of G. Modify the algorithm for a spanning tree given in Section 11.5 so that it constructs a spanning forest of G.

534. (Brualdi, 2004, pp.482-493:68) Determine whether the Shannon switching games played on the graphs in Figure are positive, negative or neutral games.

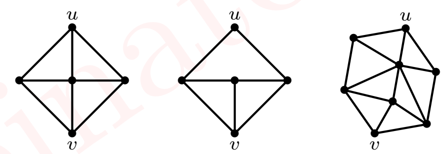


Figure in Question 534

535. (Brualdi, 2004, pp.482-493:69) Let G be a connected multigraph. An *edge-cut* of G is a set F of edges whose removal disconnects G. an edge-cut F is *minimal*, provided the no subset of F other than F itself is an edge-cut. Prove that a bridge is always a minimal edge-cut, and conclude that the only minimal edge-cuts of a tree are the sets containing of a single edge.

536. (Brualdi, 2004, pp.482-493:70) Let G be a connected multigraph having a vertex of degree k, Prove that G has a minimal edge-cut F with $|F| \leq k$.

537. (Brualdi, 2004, pp.482-493:71) Let F be a minimal edge-cut of a connected multigraph $G = (V, E)$. Prove that there exists a subset U of V such that F is precisely the set of edges that join a vertex in U to a vertex in the complement $\overline{U}$ of U.

538. (Brualdi, 2004, pp.482-493:72) Continuation of Excercise 537.) Prove that a spanning tree of a connected multigraph contains at leat one edge of every edge-cut.

539. (Brualdi, 2004, pp.482-493:73) Use the algorithm for growing a spanning tree in Section 11.7 in order to grow a spanning tree of GraphBuster. (Note: GraphBuster is a general graph and has loops and edges of multiplicity greater than 1. The loops can be igonored and only one copy of each edge need be considered.)

540. (Brualdi, 2004, pp.482-493:74) Use the algorithm for growing a spanning tree in order to grow a spanning tree of the graph of the regular dodecahedron.

541. (Brualdi, 2004, pp.482-493:75) Apply the BF-algorithm of Section 11.7 to determine a BFS-tree for the following:

i. The graph of the regular dodecahedron (any root).
ii. GraphBuster (any root).
iii. A graph of order n whose edges are arranged in a cycle (any root).

iv. A complete graph K_n (any root).

v. A compete bipartite graph $K_{m,n}$ (a left-vertex root and a right-vertex root).

542. (Brualdi, 2004, pp.482-493:76) Apply the DF-algorithm of Section 11.7 to determine a DFS-tree for graphs as in Exercise 541. In each case, determine the depth-first numbers.

543. (Brualdi, 2004, pp.482-493:77) Let G be a graph that has a Hamilton path which joins two vertices u and v. Is the Hamilton path a DFS-tree rooted at u for G? Could there be other DFS-trees?

544. (Brualdi, 2004, pp.482-493:78) (Solution of the Chinese postman problem for trees.) Let G be a tree of order n. Prove that the length of a shortest closed walk that includes each edge of G at least once is $2(n-1)$. Show how the depth-first algorithm finds a walk of length $2(n-1)$ that includes each edge exactly twice.

545. (Brualdi, 2004, pp.482-493:79) Use Dijkstra's algorithm in order to construct a distance tree for u for the weighted graph, with specified vertex u as shown.

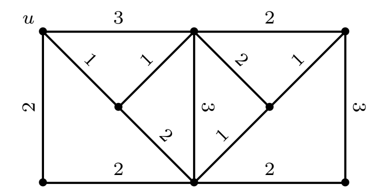


Figure in Question 545

546. (Brualdi, 2004, pp.482-493:80) Consider the complete graph K_n with labeled vertices $1, 2, \cdots, n$, in which the edge joining vertices i and j is weighted by $c(i, j) = i + j$ for all $i \neq j$. Use Dijkstra's algorithm to construct a distance tree rooted at vertex $u = 1$ for

i. K_4.

ii. K_6.

iii. K_8.

547. (Brualdi, 2004, pp.482-493:81) Consider the complete graph K_n with labeled vertices $1, 2, \cdots, n$, with the weighted function $c(i, j) = |i - j|$ for all $i \neq j$. Use Dijkstra's algorithm to construct a distance tree rooted at vertex $u = 1$ for

i. K_4.

ii. K_6.

iii. K_8.

548. (Brualdi, 2004, pp.482-493:82) Consider the complete graph K_n whose edges are weighted as in Exercise 546. Apply the greedy algorithm to determine a minimum-weighted spanning tree for

i. K_4.

ii. K_6.

iii. K_8.

549. (Brualdi, 2004, pp.482-493:83) Consider the complete graph K_n whose edges are weighted as in Exercise 547. Apply the greedy algorithm to determine a minimum-weighted spanning tree for

i. K_4.

ii. K_6.

iii. K_8.

550. (Brualdi, 2004, pp.482-493:84) Same as Exercise 548, using Prim's algorithm in place of the greedy algorithm.

551. (Brualdi, 2004, pp.482-493:85) Same as Exercise 549, using Prim's algorithm in place of the greedy algorithm.

552. (Brualdi, 2004, pp.482-493:86) Let G be a weighted connected graph in which all edge weghts are different. Prove that there is exactly one spanning tree of minimum weight.

553. (Brualdi, 2004, pp.482-493:87) Define a *caterpillar* to be a tree T that has a path γ such that every edge of T is either an edge of γ or has one of its vertices on γ.

i. Verify that all trees with 6 or fewer vertices are caterpillars.

ii. Let T_7 be the tree on 7 vertices consisting of three paths of length 2 meeting at a central vertex c. Prove that T_7 is the only tree on 7 vertices that is not a caterpillar.

iii. Prove that a tree is a caterpillar if and only if it does not contain T_7 as a spanning subgraph.

554. (Brualdi, 2004, pp.482-493:88) Let $d_1, d_2, \cdots, d_n$ be positive integers. Prove that there is a caterpillar with degree sequence $(d_1, d_2, \cdots, d_n)$ if and only if $d_1 + d_2 + \cdots + d_n = 2(n-1)$. Compare with Exercise 523.

555. (Brualdi, 2004, pp.482-493:89) A *graceful labeling* of a graph G with vertex set V and with m edges is an injective function $g : V \to \{0, 1, 2, \cdots, m\}$ such that the labels $|g(x) - g(y)|$ corresponding to the m edges $\{x, y\}$ of G are $1, 2, \cdots, m$ in some order. It has been *conjectured* by Kotzig and Ringel (1964) that every tree has a graceful labeling. Find a graceful labeling of the tree T_7 in the previous exercise, any path, and the graph $K_{1,n}$.

556. (Brualdi, 2004, pp.482-493:90) Verify that cycles of lengths 5 and 6 cannot be gracefully labeled. Then find graceful labelings of cycles of lengths 7 and 8.

557. (Brualdi, 2004, pp.514-518:1) Prove Theorem 12.1.2.

558. (Brualdi, 2004, pp.514-518:2) Prove Theorem 12.1.3.

559. (Brualdi, 2004, pp.514-518:3) Prove that an orientation of K_n is a transitive tournament if and only if it does not have any directed cycles of length 3.

560. (Brualdi, 2004, pp.514-518:4) Give an example of a digraph that does not have a closed Eulerian directed trail but whose underlying general graph has a closed Eulerian trail.

561. (Brualdi, 2004, pp.514-518:5) Prove that a digraph has no directed cycles if and only if its vertices can be labeled from 1 up to n so that the terminal vertex of each arc has a larger label than the initial vertex.

562. (Brualdi, 2004, pp.514-518:6) Prove that a digraph is strongly connected if and only if there is a closed, directed walk that contains each vertex at least once.

563. (Brualdi, 2004, pp.514-518:7) Let T be any tournament. Prove that it is possible to change the direction of at most one arc in order to obtain a tournament with a directed Hamilton cycle.

564. (Brualdi, 2004, pp.514-518:8) Use the proof of Theorem 12.1.5 in order to write an algorithm for determining a Hamilton path in a tournament.

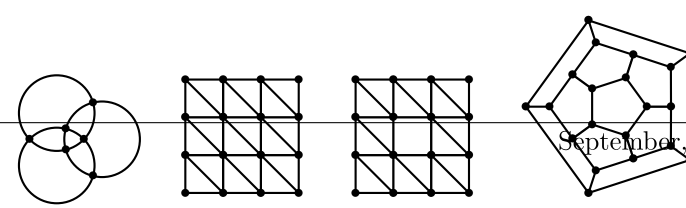
565. (Brualdi, 2004, pp.514-518:9) Prove that a tournament is strongly connected if and only if it has a directed Hamilton cycle.

566. (Brualdi, 2004, pp.514-518:10) Prove that every tournament contains a vertex u such that, for every other vertex x, there is a path from u to x of length at most 2.

567. (Brualdi, 2004, pp.514-518:11) Prove that every graph has the property that it is possible to orient each of its edges so that, for each vertex x, the indegree and outdegree of x differ by at most 1.

568. (Brualdi, 2004, pp.514-518:12) Devise an algorithm for constructing a directed Hamilton cycle in a strongly connected tournament.

569. (Brualdi, 2004, pp.514-518:13) Apply the algorithm in Section 12.1 and determine a strongly connected orientation of the graphs as shown.



(a) (b) (c) (d)

Figure in Question 569

570. (Brualdi, 2004, pp.514-518:14) Prove the following generalization of Theorem 12.1.6: Let G be a connected graph. Then, after replacing each bridge $\{a, b\}$ by the two arcs (a, b) and (b, a), one in each direction, it is possible to give the remaining edges of G an orientation so that the resulting digraph is strongly connected.

571. (Brualdi, 2004, pp.514-518:15) Modify the algorithm for constructing a strongly connected oriantation of a bridgeless connected graph in order to accommodate the situation described in Exercise 568.

572. (Brualdi, 2004, pp.514-518:16) Consider a trader problem in which trader t_1 ranks his item number 1. Prove that, in every core allocatio, t_1 gets to keep his own item.

573. (Brualdi, 2004, pp.514-518:17) Construct an example of a trading problem, with n traders, with the preoperty that, in each core allocation, exactly one trader get the item he ranks first.

574. (Brualdi, 2004, pp.514-518:18) Show that, for the trading problem in which the preferences are given by the table

	t_1	t_2	t_3
t_1	2	1	3
t_2	3	2	1
t_3	1	3	2

there are exactly two core allocations. Which of these results from applying the constructive proof of Theorem 12.1.9?

575. (Brualdi, 2004, pp.514-518:19) Suppose that, in a trading problem, some trader ranks his own item number k. Prove that, in each core allocation, that player obtains an item he ranks no lower than k. (Thus, a player never leaves with an item that he values less than the item he brought to trade.)

576. (Brualdi, 2004, pp.514-518:20) Prove that, in the core allocation obtained by applying the constructive proof of Theorem 12.1.9, at least one player gets an item he ranks number 1. Show by example that there may be core allocations in which no player gets his first choice.

577. (Brualdi, 2004, pp.514-518:21) Prove that, in a trading problem, there is a core allocation in which every trader gets the item he ranks number 1 if and only if the digraph D^1 constructed in the proof of Theorem 12.1.9 consists of directed cycles, no two of which have a vertex in common.

t_1	2	3	1	4	7	5	6
t_2	1	6	4	3	2	7	5
t_3	2	7	3	5	1	4	6
t_4	3	4	2	7	1	6	5
t_5	1	3	4	2	5	7	6
t_6	2	4	1	5	3	7	6
t_7	7	3	4	2	1	6	5

Figure in Question 578

578. (Brualdi, 2004, pp.514-518:22) Construct a core allocation for the trading problem in which the preferences are given by the figure shown.

579. (Brualdi, 2004, pp.514-518:23) Explicitly write the algorithm for a core allocation that is implicit in the proof of Theorem 12.1.9.

580. (Brualdi, 2004, pp.514-518:24) Determine a maximum flow and a minimum cut in each of the networks $N = (V, A, s, t, c)$ as shown. (The numbers near arcs are their capacities.)

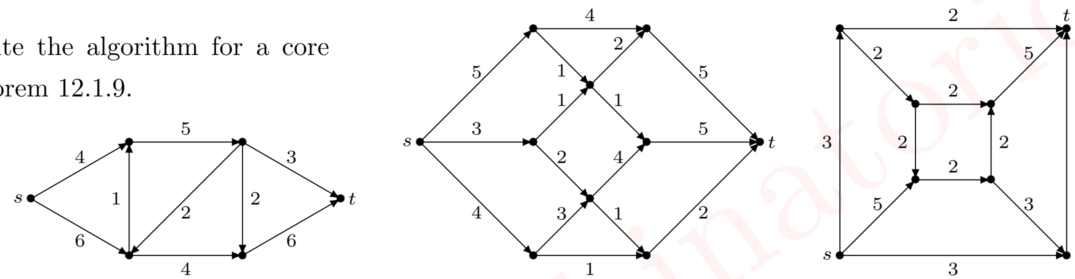


Figure in Question 580

581. (Brualdi, 2004, pp.514-518:25) Determine the maximum number of pairwise arc-disjoint paths from s to t in the digraphs of the networks in Exercise 580. Verify that the number is maximum by exhibiting an st-separating set with the same number of arcs. (Cf. Theorem 12.2.4.)

582. (Brualdi, 2004, pp.514-518:26) Consider the network as shown, where there are *three* sources s_1, s_2, and s_3 for a certain commodity and *three* targets t_1, t_2, and t_3. Each source has a certain supply of the commodity, and each target has a certain demand for the commodity. These supplies and demands are the numbers in brackets next to the sources and sinks. The supplies are to flow from the sources to the targets, subject to the flow capacities on each arc. Determine whether all the demands can be met simultaneously with the availabel supplies. (One possible way to approach this problem is to intorduce an auxiliary source s and an auxiliary target t, arcs from s to each s_i with capacity equal to s_i's supply, and arcs from each t_j to t with capacity equal to t_j's demand, and then find a maximum flow from s to t in the augmented network and check whether all demands are met.)

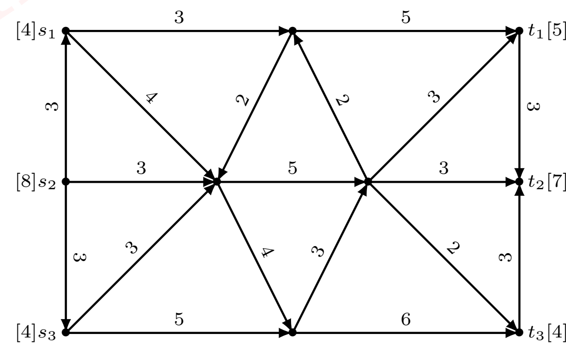


Figure in Question 582

583. (Brualdi, 2004, pp.514-518:27) In Exercise 582, change the supplies at s_1, s_2, and s_3 to a, b, and c, respectively, and determine again whether all the demands can be met simultaneously with the available supplies.

584. (Brualdi, 2004, pp.514-518:28) Formulate and prove a theorem that gives necessary and sufficient conditions for a network with multipe sources and sinks, with prescribed supplies and demands, respectively, to have a flow that simultaneously meets all demands with the available supplies.

585. (Brualdi, 2004, pp.514-518:29) Consider the set A of the 2^n binary sequences of length n. This exercise concerns the existence of a circular arrangement γ_n of 2^n 0's and 1's, so that the 2^n sequences of n consecutive bits of γ give all of A, that it, are all distinct. Such a circular

arrangement is called a *de Bruijn cycle*. For example, if $n = 2$, the circular arrangement 0, 0, 1, 1 (regarding the first 0 as following the last 1) gives 0, 0; 0, 1; 1, 1; and 1, 0. For $n = 3$, 0, 0, 0, 1, 0, 1, 1, 1 (regarded cyclically) is a de Bruijn cycle. Define a digraph Γ_n whose vertices are the 2^{n-1} binary sequences of length $n-1$. Given two such binary sequences x and y, we put an arc e from x to y, provided that the last $n-2$ bits of x agree with the first $n-2$ bits of y, and then we label the arc e with the first bit of x.

i. Prove that every vertex of Γ_n has indegree and outdegree equal to 2. Thus, Γ_n has a total of $2 \cdot 2^{n-1} = 2^n$ arcs.

ii. Prove that Γ_n is strongly connected, and hence Γ_n has a closed Eulerian directed trail (of length 2^n).

iii. Let $b_1, b_2, \cdots, b_{2^n}$ be the labels of the arcs (considered as a circular arrangement) as one traverses an Eulerian directed trail of Γ_n. Prove that $b_1, b_2, \cdots, b_{2^n}$ is a de Bruijn cycle.

iv. Prove that, given any two vertices x and y of the digraph Γ_n, there is a path from x to y of length at most $n-1$.

586. (Brualdi, 2004, pp.556-561:1) Prove that isomorphic graphs have the same chromatic number and the same chromatic polynomail.

587. (Brualdi, 2004, pp.556-561:2) Prove that the chromatic number of a disconnected graph is the largest of the chromatic numbers of its connected components.

588. (Brualdi, 2004, pp.556-561:3) Prove that the chromatic polynomial of a disconnected graph equals the product of the chromatic polynomials of its connected components.

589. (Brualdi, 2004, pp.556-561:4) Prove that the chromatic number of a cycle graph C_n of odd length equals 3.

590. (Brualdi, 2004, pp.556-561:5) Determine the chromatic numbers of the graphs as shown.

591. (Brualdi, 2004, pp.556-561:6) Prove that the greedy with chromatic nuber equal to k has at least $\binom{k}{2}$ edges.

592. (Brualdi, 2004, pp.556-561:7) Prove that the greedy algorithm always produces a coloring of the vertices of $K_{m,n}$ in 2 colors ($m, n \geq 1$).

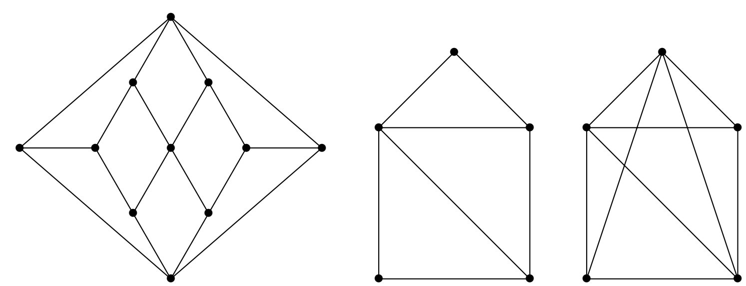


Figure in Question 590

593. (Brualdi, 2004, pp.556-561:8) Let G be a graph of order $n \geq 1$ with chromatic polynomial $p_G(k)$.

i. Prove that the constant term of $p_G(k)$ equals 0.

ii. Prove that the coefficient of k in $p_G(k)$ is nonzero if and only if G is connected.

iii. Prove that the coefficient of k^{n-1} in $p_G(k)$ equals $-m$, where m is the number of edges of G.

594. (Brualdi, 2004, pp.556-561:9) Let G be a graph of order n whose chromatic polynomial is $p_G(k) = k(k-1)^{n-1}$ (i.e., the chromatic polynomial of G is the same as that of a tree of order n). Prove that G is a tree.

595. (Brualdi, 2004, pp.556-561:10) What is the chromatic number of the graph obtained from K_n by removing one edge?

596. (Brualdi, 2004, pp.556-561:11) Prove that the chromatic polynomial of the graph obtained from K_n by removing an edge equals

$$[k]_n + [k]_{n-1}.$$

597. (Brualdi, 2004, pp.556-561:12) What is the chromatic number of the graph obtained from K_n by removing two edges with a common vertex?

598. (Brualdi, 2004, pp.556-561:13) What is the chromatic number of the graph obtained from K_n by removing two edges without a common vertex?

599. (Brualdi, 2004, pp.556-561:14) Prove that the chromatic polynomial of a cycle graph C_n equals

$$(k-1)^n + (-1)^n(k-1).$$

600. (Brualdi, 2004, pp.556-561:15) Prove that the chromatic number of a graph that has exactly one cycle of odd length is 3.

601. (Brualdi, 2004, pp.556-561:16) Prove that the polynomial $k^4 - 4k^3 + 3k^2$ is not the chromatic polynomial of any graph.

602. (Brualdi, 2004, pp.556-561:17) Use Theorem 13.1.10 to determine the chromatic number of the graph as shown

603. (Brualdi, 2004, pp.556-561:18) Use the algorithm for computing the chromatic polynomial of a graph to determine the chromatic polynomial of the graph Q_3 of vertices and edges of a three-dimensional cube.

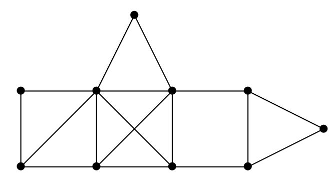


Figure in Question 602

604. (Brualdi, 2004, pp.556-561:19) Find a planar graph that has two different planar representations such that, for some integer f, one has a region bounded by f edge-curves and the other has no such region.

605. (Brualdi, 2004, pp.556-561:20) Give an example of a planar graph with chromatic number 4 that does not contain a K_4 as an induced subgraph.

606. (Brualdi, 2004, pp.556-561:21) A plane is divided into regions by a finite number of straight lines. Prove that the regions can be colored with two colors in such a way that regions which share a boundary are colored differently.

607. (Brualdi, 2004, pp.556-561:22) Repeat Exercise 606, with circles replacing straight lines.

608. (Brualdi, 2004, pp.556-561:23) Let G be a connected planar graph of order n having $e = 3n - 6$ edges. Prove that, in any planar representation of G, each region is bounded by exactly 3 edge-curves.

609. (Brualdi, 2004, pp.556-561:24) Prove that a connected graph can always be contracted to a single vertex.

610. (Brualdi, 2004, pp.556-561:25) Verify that a contraction of a planar graph is planar.

611. (Brualdi, 2004, pp.556-561:26) Let G be a planar graph of order n in which every vertex has the same degree k. Prove that $k \leq 5$.

612. (Brualdi, 2004, pp.556-561:27) Let G be a planar graph of order $n \geq 2$. Prove that G has at least two vertices whose degrees are at most 5.

613. (Brualdi, 2004, pp.556-561:28) A graph is called *color-critical* provided each subgraph obtained by removing a vertex has a smaller chromatic number. Let $G = (V, E)$ be a color-critical graph. Prove the following:

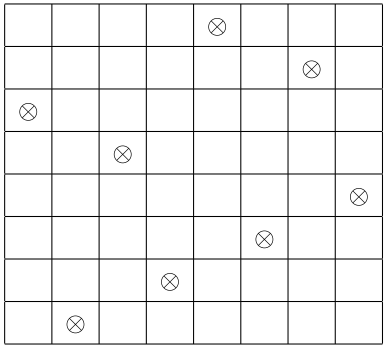
    i. $\chi\left(G_{V-\{x\}}\right) = \chi(G) - 1$ for every vertex x.
    ii. G is connected.
    iii. Each vertex of G has degree at least equal to $\chi(G) - 1$.
    iv. G does not have an articulation set U such that G_U is a complete graph.
    v. Every graph H has an induced subgraph G such that $\chi(G) = \chi(H)$ and G is color-critical.

614. (Brualdi, 2004, pp.556-561:29) Let $p \geq 3$ be an integer. Prove that a graph, each of whose vertices has degree at least $p - 1$, contains a cycle of length greater than or equal to p. Then use Exercise 613 to show that a graph with chromatic number equal to p contains a cycle of length at least p.

615. (Brualdi, 2004, pp.556-561:30) Let G be a graph without any articulation vertices such that each vertex has degree at least 3. Prove that G contains a subgraph that can be contracted to a K_4. (Hint: Begin with a cycle of largest length p. By Exercise 613, we have $p \geq 4$.) Now use Exercise 613 to obtain a proof of Hadwiger's conjecture for $p = 4$.

616. (Brualdi, 2004, pp.556-561:31) Let G be a connected graph. Let T be a spanning tree of G. Prove that T contains a spanning subgraph T' such that, for each vertex v, the degree of v in G and the degree of v in T' are equal modulo 2.

617. (Brualdi, 2004, pp.556-561:32) Find a solution to the problem of the 8 queens that is different from that given in Figure.



618. (Brualdi, 2004, pp.556-561:33) Prove that the independence number of a tree of order n is at least $\lceil \frac{n}{2} \rceil$.

619. (Brualdi, 2004, pp.556-561:34) Prove that the complement of a disconnected graph is connected.

620. (Brualdi, 2004, pp.556-561:35) Let H be a spanning subgraph of a graph G. Prove that $\text{dom}(G) \leq \text{dom}(H)$.

621. (Brualdi, 2004, pp.556-561:36) For each integer $n \geq 2$, determine a tree of order n whose domination number equals $\lfloor \frac{n}{2} \rfloor$.

622. (Brualdi, 2004, pp.556-561:37) Determine the domination number of the graph Q_3 of vertices and edges of a three-dimensional cube.

623. (Brualdi, 2004, pp.556-561:38) Determine the domination number of a cycle graph C_n.

624. (Brualdi, 2004, pp.556-561:39) For $n = 5$ and 6, show that the domination number of the queens graph of an n-by-n chessboard is, at most, 3 by finding 3 squares on which to place queens so that every other square is attacked by at least one of the queens.

625. (Brualdi, 2004, pp.556-561:40) Show that the domination number of the queens graph of a 7-by-7 chessboard is, at most, 4.

626. (Brualdi, 2004, pp.556-561:41) Show that the domination number of the queens graph of an 8-by-8 chessboard is, at most, 5.

627. (Brualdi, 2004, pp.556-561:42) Prove that an induced subgraph of an interval graph is an interval graph.

628. (Brualdi, 2004, pp.556-561:43) Prove that an induced subgraph of a chordal graph is chordal.

629. (Brualdi, 2004, pp.556-561:44) Prove that the only connected bipartite graphs that are chordal are trees.

630. (Brualdi, 2004, pp.556-561:45) Prove that all bipartite graphs are perfect.

631. (Brualdi, 2004, pp.556-561:46) Let G be a graph such that either G or its complement $\overline{G}$ has an induced subgraph equal to a chordless cycle of odd length greater than 3. Prove that G is not perfect.

632. (Brualdi, 2004, pp.556-561:47) Prove that the edge-connectivity of K_n equals $n-1$.

633. (Brualdi, 2004, pp.556-561:48) Give an example of a graph G different from a complete graph for which $\kappa(G) = \lambda(G)$.

634. (Brualdi, 2004, pp.556-561:49) Give an example of a graph G for which $\kappa(G) < \lambda(G)$.

635. (Brualdi, 2004, pp.556-561:50) Give an example of a graph G for which $\kappa(G) < \lambda(G) < \delta(G)$.

636. (Brualdi, 2004, pp.556-561:51) Determine the edge-connectivity of the complete bipartite graphs $K_{m,n}$.

637. (Brualdi, 2004, pp.556-561:52) Let G be a graph of order n with vertex degrees of $d_1, d_2, \cdots, d_n$. Assume that the degrees have been arranged so that $d_1 \leq d_2 \leq \cdots \leq d_n$. Prove that, if $d_k \geq k$ for all $k \leq n - d_n - 1$, then G is a connected graph.

638. (Brualdi, 2004, pp.556-561:53) Let G be a graph of order n in which every vertex has degree equal to d.

   i. How large must d be in order to *guarantee* that G is connected?

   ii. How large must d be in order to *guarantee* that G is 2-connected?

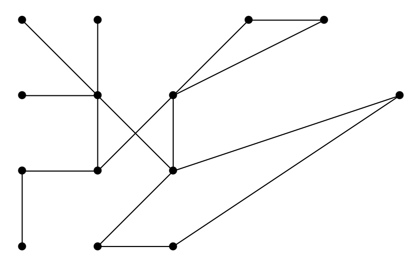


Figure in Question 639

639. (Brualdi, 2004, pp.556-561:54) Determine the blocks of the graph as shown.

640. (Brualdi, 2004, pp.556-561:55) Prove that the blocks of a tree are all K_2's.

641. (Brualdi, 2004, pp.556-561:56) Let G be a connected graph. Prove that an edge of G is a bridge if and only if it is the edge of a block equal to a K_2.

642. (Brualdi, 2004, pp.556-561:57) Let G be a graph. Prove that G is 2-connected if and only if, for each vertex x and each edge α, there is a cycle that contains both the vertex x and the edge α.

643. (Brualdi, 2004, pp.556-561:58) Let G be a graph each of whose vertices has positive degree. Prove that G is 2-connected if and only if, for each pair of edges α_1, α_2, there is a cylce containing both α_1 and α_2.

644. (Brualdi, 2004, pp.556-561:59) Prove that a connected graph of order $n \geq 2$ has at least two vertices that are note articulation vertices. (Hint: Take the two end vertices of a longest path.)

645. (Brualdi, 2004, pp.601-605:1) Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 1 & 5 & 3 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 2 & 4 & 1 \end{pmatrix}$$

Determine

i. $f \circ g$ and $g \circ f$

ii. f^{-1} and g^{-1}

iii. f^2, f^5

iv. $f \circ g \circ f$

v. g^3 and $f \circ g^3 \circ f^{-1}$.

646. (Brualdi, 2004, pp.601-605:2) Prove that permutation composition is associative:

$$(f \circ g) \circ h = f \circ (g \circ h).$$

647. (Brualdi, 2004, pp.601-605:3) Determine the symmetry group and corner-symmetry group of an equilateral triangle.

648. (Brualdi, 2004, pp.601-605:4) Determine the symmetry group and corner-symmetry group of a triangle that is isoceles but not equlateral.

649. (Brualdi, 2004, pp.601-605:5) Determine the symmetry group and corner-symmetry group of a triangle that is neither equilateral nor isocels.

650. (Brualdi, 2004, pp.601-605:6) Determine the symmetry group of a regular tetrahedron. (Hint: There are 12 symmetries.)

651. (Brualdi, 2004, pp.601-605:7) Determine the corner-symmetry group of a regular tetrahedron.

652. (Brualdi, 2004, pp.601-605:8) Determine the edge-symmetry group of a regular tetrahedron.

653. (Brualdi, 2004, pp.601-605:9) Determine the face-symmetry group of a regular tetrahedron.

654. (Brualdi, 2004, pp.601-605:10) Determine the symmetry group and the corner-symmetry group of a rectangle that is not a square.

655. (Brualdi, 2004, pp.601-605:11) Compute the corner-symmetry group of a regular hexagon (the dihedral group D_6 of order 12).

656. (Brualdi, 2004, pp.601-605:12) Determine all the permutations in the edge-symmetry group of a square.

657. (Brualdi, 2004, pp.601-605:13) Let f and g be the permutations

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 1 & 5 & 3 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 2 & 4 & 1 \end{pmatrix}$$

Consider the coloring $\boldsymbol{c} = (R, B, B, R, R, R)$ of 1, 2, 3, 4, 5, 6 with the colors R and B. Determine the following actions on $\boldsymbol{c}$:

i. $f * \mathbf{c}$

ii. $f^{-1} * \mathbf{c}$

iii. $g * \mathbf{c}$

iv. $(g \circ f) * \mathbf{c}$ and $(f \circ g) * \mathbf{c}$

v. $(g^2 \circ f) * \mathbf{c}$.

658. (Brualdi, 2004, pp.601-605:14) By examing all possibilities, determine the number of nonequivalent colorings of the corners of an equilateral triangle with the colors red and blue. (With the colors red, white, and blue.)

659. (Brualdi, 2004, pp.601-605:15) By examing all possibilities, determine the number of nonequivalent colorings of the corners of an regular tetrahedron with the colors red and blue. (With the colors red, white, and blue.)

660. (Brualdi, 2004, pp.601-605:16) Characterize the cycle factorizations of those permutations f in S_n for which $f^{-1} = f$, that is, for which $f^2 = \iota$.

661. (Brualdi, 2004, pp.601-605:17) In section 14.2 it is established that there are 8 nonequivalent colorings of the corners of a regular pertagon with the colors red and blue. Explicitly determine 8 nonequivalent colorings.

662. (Brualdi, 2004, pp.601-605:18) Using Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of a square with p colors.

663. (Brualdi, 2004, pp.601-605:19) Use Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of an equilateral triangle with the colors red and blue. With p colors (Cf. Exercise 647).

664. (Brualdi, 2004, pp.601-605:20) Use Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of an triangle that is isoceles, but not equilateral, with the colors red and blue. With p colors (Cf. Exercise 648).

665. (Brualdi, 2004, pp.601-605:21) Use Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of an triangle that is neither equilateral nor isoceles, with the colors red and blue. With p colors (Cf. Exercise 649).

666. (Brualdi, 2004, pp.601-605:22) Use Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of a rectangle that is not a square with the colors red and blue. With p colors (Cf. Exercise 654).

667. (Brualdi, 2004, pp.601-605:23) A (one-sided) *marked-domino* is a piece consisting of two squares joined along an edge, where each square on on side of the piece is marked with 0, 1, 2, 3, 4, 5, or 6 dots. The two squares of a marked-domino may receive the same number of dots.

   i. Use Theorem 14.2.3 to determine the number of different marked-dominoes.

   ii. How many different marked-domnioes are there if we are allowed to mark the squares with $0, 1, \cdots, p-1$, or p dots?

668. (Brualdi, 2004, pp.601-605:24) A *two-sided marked-domino* is a piece consisting of two squares joined along an edge, where each square on both sides of the piece is marked with 0, 1, 2, 3, 4, 5, or 6 dots.

   i. Use Theorem 14.2.3 to determine the number of different two-sided marked-dominoes.

   ii. How many different two-sided marked-domnioes are there if we are allowed to mark the squares with $0, 1, \cdots, p-1$, or p dots?

669. (Brualdi, 2004, pp.601-605:25) How many different necklaces are there that contain 3 red and 2 blue beads?

670. (Brualdi, 2004, pp.601-605:26) How many different necklaces are there that contain 4 red and 3 blue beads?

671. (Brualdi, 2004, pp.601-605:27) Determine the cycle factorization of the permutations

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 1 & 5 & 3 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 2 & 4 & 1 \end{pmatrix}$$

672. (Brualdi, 2004, pp.601-605:28) Let f be a permutation of a set X. Give a simple algorithm for finding the cycle factorization of f^{-1} from the cycle factorization of f.

673. (Brualdi, 2004, pp.601-605:29) Determine the cycle factorization of each permutation in the dihedral group D_6 (Cf. Exercise 655).

674. (Brualdi, 2004, pp.601-605:30) Determine permutations f and g of the same set X such that f and g each have 2 cycles in their cycle factorizations but $f \circ g$ has only one.

675. (Brualdi, 2004, pp.601-605:31) Determine the number of nonequivalent colorings of the corners of a regular 5-gon with k colors.

676. (Brualdi, 2004, pp.601-605:32) Determine the number of nonequivalent colorings of the corners of a regular hexagon with the colors red, white and blue (Cf. Exercise 673).

677. (Brualdi, 2004, pp.601-605:33) Prove that a permutation and its inverse have the same type (Cf. Exercise 672).

678. (Brualdi, 2004, pp.601-605:34) Let $e_1, e_2, \cdots, e_n$ be a nonnegative integers such the $1e_1 + 2e_2 + \cdots + ne_n = n$. Show how to construct a permutation f of the set $\{1, 2, \cdots, n\}$ such that $\text{type}(f) = (e_1, e_2, \cdots, e_n)$.

679. (Brualdi, 2004, pp.601-605:35) Determine the number of nonequivalent colorings of the corners of a regular 6-gon with k colors (Cf. Exercise 673).

680. (Brualdi, 2004, pp.601-605:36) Determine the number of nonequivalent colorings of the corners of a regular 5-gon with colors red, white, and blue in which two corners are colored red, two are colored white, and one is colored blue.

681. (Brualdi, 2004, pp.601-605:37) Determine the cycle index of the dihedral group D_6 (Cf. Exercise 673).

682. (Brualdi, 2004, pp.601-605:38) Determine the generating function for nonequivalent colorings of the corners of a regular hexagon with 2 colors and also with 3 colors (Cf. Exercise 681.)

683. (Brualdi, 2004, pp.601-605:39) Determine the cycle index of the edge-symmetry group of a square.

684. (Brualdi, 2004, pp.601-605:40) Determine the generating function for nonequivalent colorings of the edges of a square with the colors red and blue. How many nonequivalent colorings are there with k colors? (Cf. Exercise 683).

685. (Brualdi, 2004, pp.601-605:41) Let n be an odd prime number. Prove that each of the permutaions, $\rho_n, \rho_n^2, \cdots, \rho_n^{n-1}$, of $\{1, 2, \cdots, n\}$ is an n-cycle. (Recall that ρ_n is the permutation that sends 1 to 2, 2 to 3, $\cdots$, $n-1$ to n, and n to 1.)

686. (Brualdi, 2004, pp.601-605:42) Let n be a prime number. Determine the number of different necklaces that can be made from n beads of k different colors.

687. (Brualdi, 2004, pp.601-605:43) The nine squares of a 3-by-3 chessboard are to be colored red and blue. The chessboard is free to rotate but cannot be flipped over. Determine the generating funciton for the number of nonequivalent colorings and the total number of nonequivalent colorings.

688. (Brualdi, 2004, pp.601-605:44) A stained glass window in the form of a 3-by-3 chessboard has 9 squares, each of which is colored red or blue (the colors are tranparent, and the window can be looked at from either side). Determine the generating function for the number of different stained galss windows and the total number of stained glass windows.

689. (Brualdi, 2004, pp.601-605:45) Repeat Exercise 688 for stained glass windows in the form of a 4-by-4 chessboard with 16 squares.

690. (Brualdi, 2004, pp.601-605:46) Find the generating function for the different necklaces that can be made with p beads of color red or blue if p is a prime number (cf. Exercise 686).

691. (Brualdi, 2004, pp.601-605:47) Determine the cycle index of the dihedral group D_{2p}, where p is a prime number.

692. (Brualdi, 2004, pp.601-605:48) Find the generating function for the different necklaces that can be made with $2p$ beads each of color red or blue if p is a prime number.

693. (Brualdi, 2004, pp.601-605:49) Ten balss are stacked in a triangular array with 1 atop 2 atop 3 atop 4. (Think of billiards.) The triangular array is free to rotate. Find the generating function for the number of nonequivalent colorings with the colors red and blue. Find the generating function if we are also allowed to turn over the array.

694. (Brualdi, 2004, pp.601-605:50) Use Theorem 14.3.3 to determine the generating function for nonisomorphic graphs of order 5. (Hint: This exercise will require some work and is a fitting last exercis. One needs to obtain the cycle index of the group $S_5^{(2)}$ of permutations of the set X of 10 unordered parts of distinct integers from $\{1, 2, 3, 4, 5\}$ (the possible edges of a graph of order 5). First, compute the number of permutations f of S_5 of each type. Then use the fact that the type of f as a permutation of X depends only on the type of f as a permutation of $\{1, 2, 3, 4, 5\}$.)

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