Problem 2.1.

a) T is not linear: $T((1,0)) = (1, 1)$

$T((2,0)) = (4,2) \neq 2 \cdot T((1,0))$.

b) T is not linear: $T((0,0)) = (1,0) \neq (0,0)$.

c) T is linear. Indeed, for all $A, B \in \mathbb{R}^{n \times m}$ and $\alpha \in \mathbb{R}$: we have for all $i \in \{1 \dots m\}$ and $j \in \{1, \dots n\}$

$$T(\alpha A + B)_{i,j} = (\alpha A + B)^T_{i,j}$$

$$= (\alpha A + B)_{j,i} = \alpha A_{j,i} + B_{j,i} = \alpha (A^T)_{ij} + (B^T)_{ij}$$

$$= \alpha T(A)_{i,j} + T(B)_{i,j}$$

Hence $T(\alpha A + B) = \alpha T(A) + T(B)$, which proves that T is linear (take $\alpha = 1$ to obtain that $T(A+B) = T(A) + T(B)$ and $B = 0$ to get $T(\alpha A) = \alpha T(A)$).

d) T is linear.
Let $A, B \in \mathbb{R}^{n \times n}$ and $\alpha \in \mathbb{R}$

$$T(\alpha A + B) = \sum_{i=1}^{n} (\alpha A + B)_{i,i}$$

$$= \sum_{i=1}^{n} \alpha A_{i,i} + B_{i,i} = \alpha \sum_{i=1}^{n} A_{i,i} + \sum_{i=1}^{n} B_{i,i}$$

$$= \alpha T(A) + T(B).$$

Problem 2.2.

a) We have to compute: by linearity of f.

$$f\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = f\left(\begin{pmatrix}2\\2\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix}\right) = f\left(\begin{pmatrix}2\\2\end{pmatrix}\right) - f\left(\begin{pmatrix}1\\2\end{pmatrix}\right)$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\-2\\-2\end{pmatrix}.$$

$$f\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = f\left(\frac{1}{2}\left(\begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right)\right) = \frac{1}{2} f\left(\begin{pmatrix}1\\2\end{pmatrix}\right) - \frac{1}{2} f\left(\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix}1\\2\\3\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\-2\\-2\end{pmatrix} = \begin{pmatrix}1/2\\2\\5/2\end{pmatrix}$$

The matrix of f is therefore $\begin{pmatrix}0 & 1/2\\ -2 & 2\\ -2 & 5/2\end{pmatrix}$

b) For $a,b,c \in \mathbb{R}$ we solve the sytem $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix}$:

$$\begin{cases} \frac{1}{2}y = a \\ -2x + 2y = b \\ -2x + \frac{5}{2}y = c \end{cases} \iff \begin{cases} y = 2a \\ x = y - \frac{b}{2} \\ x = \frac{5}{4}y - \frac{c}{2} \end{cases}$$

$$\iff \begin{cases} y = 2a \\ x = 2a - \frac{b}{2} \\ x = \frac{5}{2}a - \frac{c}{2}. \end{cases}$$

Consequently, the system has no solution if $2a - \frac{b}{2} \neq \frac{1}{2}(5a - c)$ and has a unique solution $(x,y) = (2a - \frac{b}{2}, 2a)$ otherwise.

In this question $a=1$, $b=4$, $c=5$ hence $2a - \frac{b}{2} = 2-2 = 0 = \frac{1}{2}(5a-c)$.

We conclude : $\{x \in \mathbb{R}^2 \mid f(x) = (1,4,5)\} = \{\binom{0}{2}\}$.

c) In this case $a=2, b=4, c=5$ so $2a - \frac{b}{2} = 2$ and $\frac{1}{2}(5a-c) = \frac{5}{2} \neq 2$: there is no solutions:
$\{x \in \mathbb{R}^2 \mid f(x) = (1,4,5)\} = \emptyset$

Problem 2.3.

Let us look at the "target" matrix ABC:

$$ABC = \begin{pmatrix} B_{1,2} & B_{1,1} & B_{1,3} & B_{1,2} \\ B_{2,2}+B_{3,2} & B_{2,1}+B_{3,1} & B_{2,3}+B_{3,3} & B_{2,2}+B_{3,2} \\ B_{4,2} & B_{4,1} & B_{4,3} & B_{4,2} \end{pmatrix}$$

$$\underbrace{\qquad}_{} \qquad \underbrace{\qquad}_{B_{*,1}} \qquad \underbrace{\qquad}_{B_{*,3}} \qquad \underbrace{\qquad}_{B_{*,2}}$$

coeff. of the form $B_{*,2}$: they belong to the second column of B

We see that the order of the columns of B have changed, and that the second column of B has been duplicated. Such operation can be done by multiplying B with a matrix with 0 and 1 :

$$\left.\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right\} C.$$

$$\begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \\ B_{4,1} & B_{4,2} & B_{4,3} \end{pmatrix} \quad \begin{matrix} B_{1,2} & B_{1,1} & B_{1,3} & B_{1,2} \\ B_{2,2} & B_{2,1} & B_{2,3} & B_{2,2} \\ B_{3,2} & B_{3,1} & B_{3,3} & B_{3,2} \\ B_{4,2} & B_{4,1} & B_{4,3} & B_{4,2} \end{matrix}$$

Now, by looking at the rows of ABC we see rows 1 and 4 has been untouched but that the rows 2 and 3 have been added to_gether:

$$\left.\begin{pmatrix} B_{1,2} & B_{1,1} & B_{1,3} & B_{1,2} \\ B_{2,2} & B_{2,1} & B_{2,3} & B_{2,2} \\ B_{3,2} & B_{3,1} & B_{3,3} & B_{3,2} \\ B_{4,2} & B_{4,1} & B_{4,3} & B_{4,2} \end{pmatrix}\right\} BC$$

$$A \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right. \begin{matrix} B_{1,2} & B_{1,1} & B_{1,3} & B_{1,2} \\ B_{22}+B_{32} & B_{21}+B_{31} & B_{23}+B_{33} & B_{22}+B_{32} \\ B_{4,2} & B_{4,1} & B_{4,3} & B_{4,2} \end{matrix}$$

Problem 2.4

a) • $0 \in Im(A)$ because $A0 = 0$: $Im(A) \neq \emptyset$.

• Let $u, v \in Im(A)$ and $\alpha \in \mathbb{R}$. By definition there exists $u_0, v_0 \in \mathbb{R}^m$ such that $u = Au_0$ and $v = Av_0$.

Then: $\alpha u + v = \alpha A u_0 + A v_0 = A(\alpha u_0 + v_0) \in Im(A)$.

• $0 \in Ker(A)$ because $A.0 = 0$: $Ker(A) \neq \emptyset$.

• Let $u, v \in Ker(A)$ and $\alpha \in \mathbb{R}$.

$A(\alpha u + v) = \alpha Au + Av = 0 + 0 = 0$.

Hence $\alpha u + v \in Ker(A)$.

$Im(A)$ is therefore a subspace of $\mathbb{R}^n$ and $Ker(A)$ a subspace of $\mathbb{R}^m$.

ⓑ Let $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. We will solve

$$Ax = y \iff \begin{cases} x_1 + 2x_2 + x_3 + 2x_4 = y_1 \\ -x_1 + x_2 - x_3 + x_4 = y_2 \\ x_2 + 2x_4 = y_3 \end{cases}$$

$$\iff \begin{cases} x_1 + 2x_2 + x_3 + 2x_4 = y_1 \\ 3x_2 + 3x_4 = y_2 + y_1 & (R_2) + (R_1) \\ x_2 + 2x_4 = y_3 \end{cases}$$

$$\iff \begin{cases} x_1 + 2x_2 + x_3 + 2x_4 = y_1 \\ x_2 + 2x_4 = y_3 & (R_3) \\ -3x_4 = y_2 + y_1 - 3y_3 & (R_2) - 3(R_3) \end{cases}$$

- In order to find a basis of $Ker(A)$ we first consider the case where $y_1 = y_2 = y_3 = 0$.
In that case we obtain that
$$Ax = 0 \iff x_1 + x_3 = 0$$
$$\iff x \in Span\left(\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\right).$$

Hence $Ker(A) = Span\left(\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\right)$ and $\left(\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\right)$ is a basis of $Ker(A)$.

- In order to show that $Im(A) = \mathbb{R}^3$ we have to show that $Ax = y$ has at least 1 solution for all $y \in \mathbb{R}^3$.

Coming back to the resolotion of $Ax=y$ we have :

$$Ax=y \iff \begin{cases} x_1 = y_1 - x_3 - 2x_2 - 2x_4 \\ x_2 = y_3 - 2x_4 \\ x_4 = -\frac{1}{3}(y_2 + y_1 - 3y_3) \end{cases}$$

$$\iff \begin{cases} x_1 = y_1 - x_3 - 2\left(y_3 + \frac{2}{3}(y_2+y_1-3y_3)\right) + \frac{2}{3}(y_2+y_1-3y_3) \\ x_2 = y_3 + \frac{2}{3}(y_2+y_1-3y_3) \\ x_4 = -\frac{1}{3}(y_2+y_1-3y_3) \end{cases}$$

Consequently, the vector

$$\begin{pmatrix} y_1 - 2(y_3 + \frac{2}{3}(y_2+y_1-3y_3)) + \frac{2}{3}(y_2+y_1-3y_3) \\ y_3 + \frac{2}{3}(y_2+y_1-3y_3 \\ 0 \\ -\frac{1}{3}(y_2+y_1-3y_3 \end{pmatrix}$$

is a solution of $Ax=y$. We conclude that $\mathrm{Im}(A) = \mathbb{R}^3$.

Problem 5:

- Assume that there exists $C \in \mathbb{R}^{m \times k}$ such that $A = CB$.

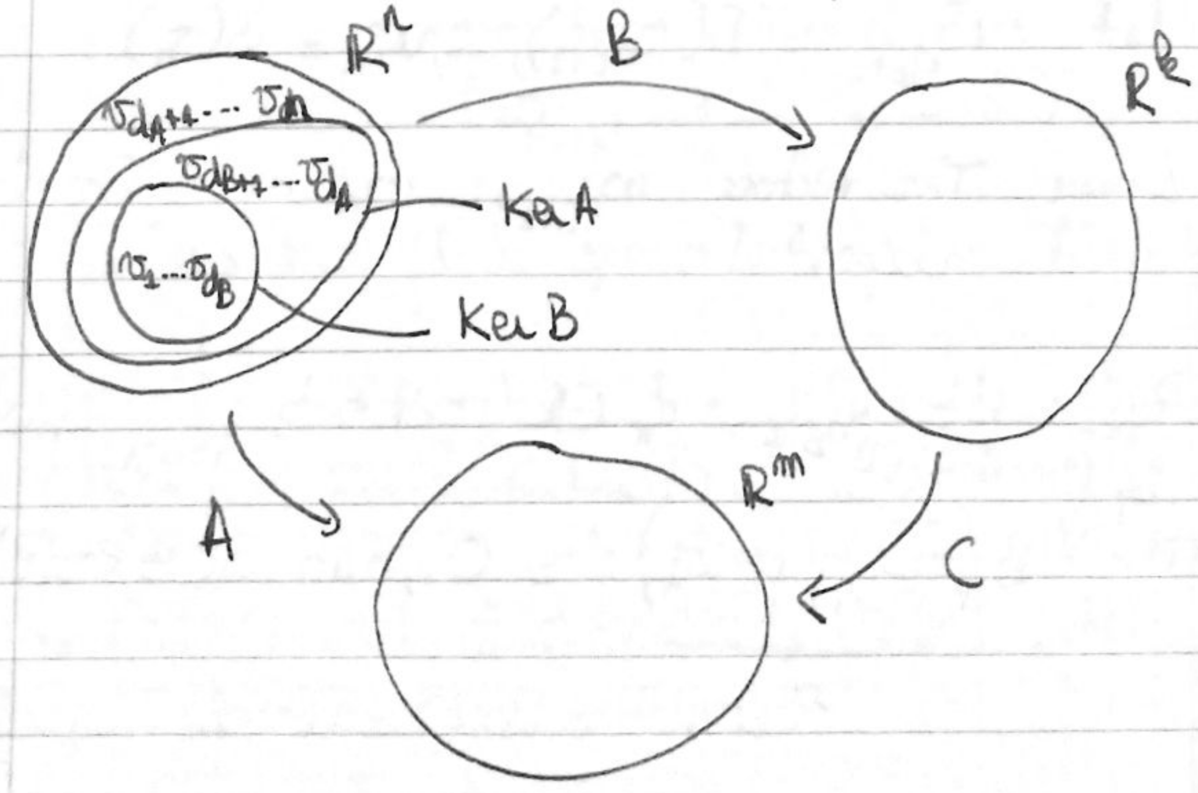
Then for all $x \in \mathrm{Ker}(B)$, $Ax = CBx$
$= C.0$ (because $Bx = 0$)
$= 0$.

Hence $x \in \mathrm{Ker}(A)$.

This gives that $\mathrm{Ker}(B) \subset \mathrm{Ker}(A)$. Because we know that $\mathrm{Ker}(B)$ is a subspace of $\mathbb{R}^n$, it is then of subspace of the vector space $\mathrm{Ker}(A)$.

- Assume that $\mathrm{Ker}(B) \subset \mathrm{Ker}(A)$. Let A, B be the linear maps associated ~~[illegible]~~ to A and B.

Finding a matrix $C \in \mathbb{R}^{m \times k}$ such that $A = CB$ is equivalent to find a linear map $C : \mathbb{R}^k \to \mathbb{R}^m$ such that $A = C \circ B$, ie such that for all $x \in \mathbb{R}^n$, $Ax = C(B(x))$.



Let $d_B = \dim(\mathrm{Ker}\, B)$ $\quad d_A = \dim(\mathrm{Ker}\, A)$.

Since Ker B is a subspace of Ker A (which is itself a subspace of $\mathbb{R}^n$) we have $d_B \leq d_A \leq n$.

Let $(v_1, \dots v_{d_B})$ be a basis of Ker B. By Pb 1.4, one can ~~find~~ add vectors $v_{d_B+1}, \dots, v_{d_A}$ to ~~this~~ it to obtain a basis of Ker A.
Applying Pb 1.4 again we can add vectors $v_{d_A+1} \dots v_n$ to obtain a basis of $\mathbb{R}^n$.
To summarize we have:

$$\underbrace{\underbrace{\underbrace{v_1 \dots v_{d_B}}_{\text{Basis of Ker(B)}}\ v_{d_B+1} \dots v_{d_A}}_{\text{Basis of Ker(A)}}\ v_{d_A+1} \dots v_n}_{\text{Basis of } \mathbb{R}^n}$$

Let $w_{d_B+1} = B(v_{d_B+1}), \dots, w_n = B(v_n)$.

Lemma: The vectors $w_{d_B+1}, \dots w_n$ are linearly independent.

<u>Proof</u>: Let $\alpha_{d_B+1} \dots \alpha_n \in \mathbb{R}$ such that $\sum_{i=d_B+1}^{n} \alpha_i w_i = 0$
by linearity of B
Then $B\Big(\underbrace{\sum_{i=d_B+1}^{n} \alpha_i v_i}_{u}\Big) = 0$, ie $u \in \mathrm{Ker}(B)$.

Hence $u = \sum_{i=1}^{d_B} \alpha_i v_i$ for some $\alpha_1 \dots \alpha_{d_B} \in \mathbb{R}$ (because $v_1 \dots v_{d_B}$ is a basis of $\mathrm{Ker}(B)$). This gives
$\sum_{i=1}^{d_B} \alpha_i v_i - \sum_{i=d_{B+1}}^{n} \alpha_i v_i = 0$ and we obtain that

$\alpha_1 = \dots = \alpha_{d_{B+1}} = \dots = \alpha_n = 0$ because $(v_1 \dots v_n)$ are ~~lin. dep.~~ linearly independent. $\blacksquare$

Using the Lemma and Pb. 1.4 we can find vectors $w'_1, \dots w'_\ell$ (with $\ell = k-(n-d_B)$) such that $w_{d_{B+1}}, \dots w_n, w'_1, \dots, w'_\ell$ is a basis of $\mathbb{R}^k$.

We define now the linear map C:

$$C : \mathbb{R}^k \longrightarrow \mathbb{R}^m$$

$$x \longmapsto \alpha_{d_{B+1}} A(v_{d_{B+1}}) + \dots + \alpha_n A(v_n)$$

where $(\alpha_{d_{B+1}}, \dots, \alpha_n, \alpha'_1 \dots \alpha'_\ell)$ are the coordinates of x in the basis $(w_{d_{B+1}}, \dots w_n, w'_1, \dots w'_\ell)$.

Let us verify that for all $i = 1 \dots n$, $A(v_i) = C(B(v_i))$.

- If $i \in \{1 \dots d_B\}$ then $B(v_i) = 0$ and $A(v_i) = 0$ because $\mathrm{Ker}(B) \subset \mathrm{Ker}(A)$. Hence $A(v_i) = C(B(v_i))$.
- If $i \in \{d_B+1, \dots n\}$, then by definition $B(v_i) = w_i$ and by definition of C: $C(w_i) = A(v_i)$. Hence $C(B(v_i)) = C(w_i) = A(v_i)$.

The linear transformations A and $C \circ B$ are thus equal over the vectors of the basis $(v_1, \dots v_n)$; they are therefore equal because for all $x \in \mathbb{R}^n$ we have

$$C(B(x)) = \sum_{i=1}^{n} \lambda_i \, C \circ B(v_i)$$
$$= \sum_{i=1}^{n} \lambda_i \, A(v_i) = A(x), \text{ where}$$

$(\lambda_1, \dots \lambda_n)$ denotes the coordinates of x in the basis $(v_1, \dots v_n)$.