Probleme 12.4.

$$x_{t+1} = x_t - \beta(Mx_t - b) + \gamma x_t - \gamma x_{t-1}$$
$$= ((1+\gamma)Id - \beta M)\, x_t + \beta b - \gamma x_{t-1}$$

Subtracting $x^* = M^{-1}b$ on both sides gives:

$$x_{t+1} - x^* = ((1+\gamma)Id - \beta M)(x_t - x^*) - \gamma(x_{t-1} - x^*)$$

$(\alpha_1(t), \dots \alpha_d(t))$ are the coordinates of $x_t - x^*$ in the orthonormal basis $(v_1 \dots v_d)$. Hence

$$\alpha_i(t+1) = \langle v_i, x_{t+1} - x^* \rangle$$
$$= v_i^T((1+\gamma)Id - \beta M)(x_t - x^*) - \gamma \alpha_i(t-1)$$
$$= (1+\gamma-\beta\lambda_i)\, \alpha_i(t) - \gamma \alpha_i(t-1)$$

We have to study the order-2 recursion
$\alpha_i(t+1) - (1+\gamma-\beta\lambda_i)\, \alpha_i(t) + \gamma\, \alpha_i(t-1)$.

To do so, we have to compute the roots of $X^2 - (1+\gamma-\beta\lambda_i)X + \gamma$.

The discriminant of this quadratic function is

$$\Delta = (1+\gamma-\beta\lambda_i)^2 - 4\gamma.$$

$$1+\gamma-\beta\lambda_i = \frac{(\sqrt{L}+\sqrt{\mu})^2+(\sqrt{L}-\sqrt{\mu})^2-4\lambda_i}{(\sqrt{L}+\sqrt{\mu})^2}$$

$$= \frac{2}{(\sqrt{L}+\sqrt{\mu})^2}\left(L+\mu-2\lambda_i\right)$$

$$\Delta = \frac{4}{(\sqrt{L}+\sqrt{\mu})^4}\left(\left(L+\mu-2\lambda_i\right)^2-(\sqrt{L}-\sqrt{\mu})^2(\sqrt{L}+\sqrt{\mu})^2\right)$$

$$= \frac{4}{(\sqrt{L}-\sqrt{\mu})^4}\left((L+\mu-2\lambda_i)^2-(L-\mu)^2\right)$$

$$= \frac{4}{(\sqrt{L}-\sqrt{\mu})^4}\left((2\mu-2\lambda_i)(2L-2\lambda_i)\right)$$

$$= \frac{16}{(\sqrt{L}-\sqrt{\mu})^4}(\mu-\lambda_i)(L-\lambda_i) \leq 0.$$

The roots of $X^2-(1+\gamma-\beta\lambda_i)X+\gamma$ are therefore:

$$x = \frac{1}{2}\left((1+\gamma-\beta\lambda_i)+i\sqrt{|\Delta|}\right) \text{ and } x' = \frac{1}{2}\left((1+\gamma-\beta\lambda_i)-i\sqrt{|\Delta|}\right)$$

There exists two constants $C_1, C_2 \in \mathbb{R}$ such that for all $t \geq 0$:

$$\alpha_i(t) = C_1 x^t + C_2 (x')^t$$

Compute

$$|x| = |x'| = \frac{1}{2}\sqrt{(1+\gamma-\beta\lambda_i)^2-\Delta} = \sqrt{\gamma} = \frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}$$

Hence $|\alpha_i(t)| \leq (|C_1|+|C_2|)\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^t$

We conclude that

$$\|x_t - x^*\|^2 = \sum_{i=1}^{d} \alpha_i(t)^2$$

$$\leq \underbrace{\left(\sum_{i=1}^{d} c_i^2\right)}_{\overset{\text{def}}{=} C^2} \left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^{2t}$$

Hence $\|x_t - x^*\| \leq C \left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^{t}$