Lecture 1.3: Span, Linear dependency and dimension

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

Contents

1. Linear combination and span
2. Linear dependency
3. Basis and dimension

Linear combination & Span

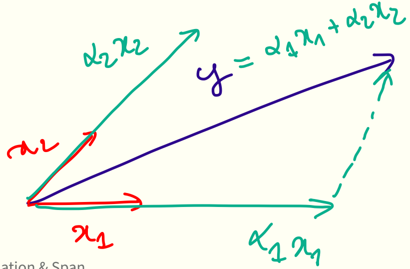
Linear combination

Let V be a vector space (think for instance $V = \mathbb{R}^n$).

Definition

We say that $y \in V$ is a *linear combination* of the vectors $x_1, \ldots, x_k \in V$ if there exists $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ such that

$$y = \sum_{i=1}^{k} \alpha_i x_i = \alpha_1 x_1 + \cdots + \alpha_k x_k.$$



Remarks

- A linear combination is always a finite sum.
- If S is a subspace of V, then any linear combination of vectors $x_1, \ldots, x_k$ of S is also in S:

$$\alpha_1 x_1 + \cdots + \alpha_k x_k \in S, \qquad \text{for all } \alpha_1, \ldots, \alpha_k \in \mathbb{R}.$$

« Subspaces are closed under linear combinations. »

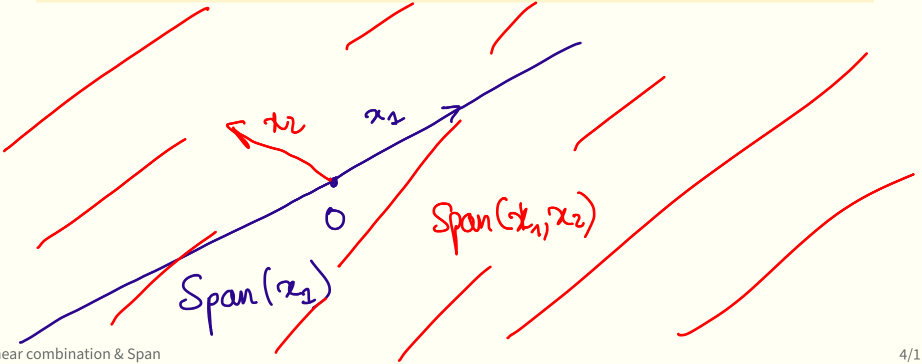
Exercise: Prove it !

Span

Definition

Let $x_1, \ldots, x_k$ be vectors of V. We define the *linear span* of $x_1, \ldots, x_k$ as the set of all linear combinations of these vectors:

$$\mathrm{Span}(x_1, \ldots, x_k) \overset{\text{def}}{=} \left\{ \alpha_1 x_1 + \cdots + \alpha_k x_k \;\middle|\; \alpha_1, \ldots, \alpha_k \in \mathbb{R} \right\}.$$



Linear dependency

Linear dependency

Definition

Vectors $x_1, \dots x_k \in V$ are *linearly dependent* is there exists $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ **that are not all zero** such that

$$\alpha_1 x_1 + \cdots + \alpha_k x_k = 0.$$

They are said to be *linearly independent* otherwise.

Key observation: « $x_1, \dots, x_k$ are linearly dependent » *is equivalent to* « one of the vectors $x_1, \dots, x_k$ can be obtained as a linear combination of the others.»

Why ?

- Assume that $x_1 \dots x_k$ are lin. dep.

There exists $\alpha_1 \dots \alpha_k \in \mathbb{R}$ $\Big|$ $\alpha_1 x_1 + \dots + \alpha_k x_k = 0$

$\alpha_i \neq 0$ for some i

without the i^{th} term

$$\alpha_i x_i = -\alpha_1 x_1 - \dots - \alpha_k x_k$$

$$\rightarrow \quad x_i = -\frac{\alpha_1}{\alpha_i} x_1 - \dots - \frac{\alpha_k}{\alpha_i} x_k$$

because $\alpha_i \neq 0$

- If $x_i = \beta_1 x_1 + \dots + \beta_k x_k$ for some $\beta_1 \dots \beta_k \in \mathbb{R}$

no i^{th} term

then $$\underbrace{\beta_1 x_1 + \dots - x_i \dots + \beta_k x_k}_{\text{"non trivial lin comb."}} = 0$$

Basis, dimension

Basis

Definition

A family $(x_1, \ldots, x_n)$ of vectors of V is a basis of V if

1. $x_1, \ldots, x_n$ are linearly independent,
2. $\mathrm{Span}(x_1, \ldots, x_n) = V$.

This means that $(x_1, \ldots, x_n)$ is a basis of V if

1. None of the x_i is a linear combination of the others $(x_j)_{j \neq i}$.
2. Any vector of V can be expressed as a linear combination of $(x_1, \ldots, x_n)$.

Example: the canonical basis of $\mathbb{R}^n$

Let us define the vectors $e_1, \ldots, e_n \in \mathbb{R}^n$ by

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ e_n &= (0, 0, 0, \ldots, 1). \end{aligned}$$

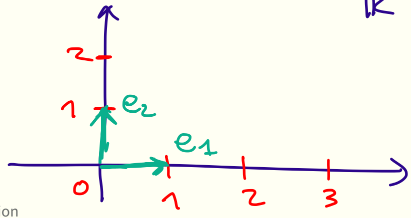
n vectors

if $v = (v_1 \ldots v_n) \in \mathbb{R}^n$

then $v = \sum_{i=1}^{n} v_i e_i$

One can verify (exercise!) that the family $(e_1, \ldots, e_n)$ is a basis of $\mathbb{R}^n$. This basis is called the "canonical basis" of $\mathbb{R}^n$. We conclude that $\mathbb{R}^n$ has dimension n.

$\mathbb{R}^2$



Dimension

Theorem

Let V be a vector space.

- If V admits a basis $(v_1, \ldots, v_n)$, then every basis of V has also n vectors. We say that V has dimension n and write $\dim(V) = n$.
- Otherwise, we say that V has infinite dimension: $\dim(V) = +\infty$.

Example:

- $\mathbb{R}^n$ has dimension n, because the canonical basis $(e_1, \ldots, e_n)$ is a basis of $\mathbb{R}^n$ with n vectors.
- $\{f \mid f : \mathbb{R} \to \mathbb{R}\}$ has infinite dimension.