Problem 5.1

a) Compute $MM^T = (Id_n - 2P_S)(Id_n - 2P_S^T)$

(because $P_S^T = P_S$)

$$= (Id_n - 2P_S)(Id_n - 2P_S)$$
$$= Id_n - 4P_S + 4P_S^2$$
$$= Id_n$$

because $P_S^2 = P_S$.

M is therefore orthogonal.

b) M is symmetric ($M^T = M$) so we get from a) that $M^2 = Id_n$

Let $\lambda \in \mathbb{R}$ be an eigenvalue of M, and $v \in \mathbb{R}^n \setminus \{0\}$ an associated eigenvector.

We get $M^2 v = v$

so $M(\lambda v) = v$

$\lambda^2 v = v$ which gives $\lambda^2 = 1$

since $v \neq 0$: $\boxed{\lambda = 1 \text{ or } \lambda = -1}$.

Problem 5.2

$v \in \mathbb{R}^{n\times 1}$

- $\text{rank}(M) \leq \text{rank}(v) \leq 1$, hence $\text{rank}(M)$ is equal to 0 or 1.

Since $M \neq 0$ (because for instance we have $\text{Tr}(M) = \|v\|^2 \neq 0$) we have necessarily

$$\boxed{\text{rank}(M) = 1}$$

- The rank-nullity theorem gives then that $\dim \text{Ker}(M) = n-1$. In other words: 0 is an eigenvalue of M, of multiplicity $n-1$.

- Notice now that $Mv = v \underbrace{v^T v}_{= \|v\|^2 \in \mathbb{R}}$

$$= \|v\|^2 v$$

hence $\lambda = \|v\|^2$ is another eigenvalue of M ($\lambda \neq 0$ because $v \neq 0$) (note that $v \neq 0$).

- Since the sum of multiplicities of distinct eigenvalues is less or equal to n:

$$\underbrace{\dim \text{Ker}(M)}_{= n-1} + \dim \text{Ker}(M - \lambda \, \text{Id}) \leq n$$

We get that $\dim \text{Ker}(M - \lambda \, \text{Id}) = 1$: the eigenvalue $\lambda = \|v\|^2$ has multiplicity 1

- There is no other eigenvalue because the sum of the multiplicities of 0 and $\|v\|^2$ equals n.

Problem 5.3

Compute: • $v_1^T A v_2 = v_1^T (\lambda_2 v_2) = \lambda_2 v_1^T v_2$

• $v_1^T A v_2 = (A^T v_1)^T v_2$

(because $A = A^T$) $= (A v_1)^T v_2$

$= \lambda_1 v_1^T v_2$.

We get that $\lambda_1 v_1^T v_2 = \lambda_2 v_1^T v_2$. Since $\lambda_1 \neq \lambda_2$, we deduce that $v_1^T v_2 = 0$: $v_1 \perp v_2$

Problem 5.4

a) $$x_t = \frac{A x_{t-1}}{\|A x_{t-1}\|} = \frac{A \frac{A x_{t-2}}{\|A x_{t-2}\|}}{\left\| A \frac{A x_{t-2}}{\|A x_{t-2}\|} \right\|}$$

$$= \frac{A^2 x_{t-2}}{\|A^2 x_{t-2}\|}$$

$$= \frac{A^2 \frac{A x_{t-3}}{\|A x_{t-3}\|}}{\left\| A^2 \frac{A x_{t-3}}{\|A x_{t-3}\|} \right\|} = \frac{A^3 x_{t-3}}{\|A^3 x_{t-3}\|}$$

$$= \cdots = \frac{A^t x_0}{\|A^t x_0\|}$$

b) The set of vectors that have their first coordinate in the basis $(v_1, \dots v_n)$ equal to zero is $\mathrm{Span}(v_2, \dots v_n)$.
This is an hyperplane of $\mathbb{R}^n$: a randomly chosen vector has zero probability to belong to it (for instance in $\mathbb{R}^3$ a randomly chosen vector will be outside of the horizontal plane with probability 1).

c) $x_0 = \alpha_1 v_1 + \dots + \alpha_n v_n$ so

$$A^t x_0 = \alpha_1 A^t v_1 + \dots + \alpha_n A^t v_n$$
$$= \alpha_1 \lambda_1^t v_1 + \dots + \alpha_n \lambda_n^t v_n .$$

Consequently:

$$x_t = \frac{A^t x_0}{\|A^t x_0\|} = \frac{\alpha_1 v_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^t \alpha_2 v_2 + \dots + \left(\frac{\lambda_n}{\lambda_1}\right)^t \alpha_n v_n}{\left\|\alpha_1 v_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^t \alpha_2 v_2 + \dots + \left(\frac{\lambda_n}{\lambda_1}\right)^t \alpha_n v_n\right\|}$$

Since $\left(\frac{\lambda_i}{\lambda_1}\right)^t \xrightarrow[t \to +\infty]{} 0$ for all $i \in \{2, \dots n\}$, we get

$$\boxed{x_t \xrightarrow[t \to +\infty]{} \frac{\alpha_1 v_1}{\|\alpha_1 v_1\|} = \frac{\alpha_1}{|\alpha_1|} v_1 .}$$

and

$$\boxed{\begin{aligned} \|A x_t\| \xrightarrow[t \to +\infty]{} \left\|A \frac{\alpha_1}{|\alpha_1|} v_1\right\| &= \|A v_1\| \\ &= \|\lambda_1 v_1\| = \lambda_1 \end{aligned}}$$

Problem: 5.5

The function f is continuous over the unit sphere $S = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ and admits therefore a maximum on S, at some x_*

Since for all $r \neq 0$ and $x \in S$, $f(rx) = f(x)$, x_* is a global maximizer of f on $\mathbb{R}^n \setminus \{0\}$.

(optional)

Hence $\nabla f(x_*) = 0$. By definition:

$$f(x) = \frac{\sum_{i,j=1}^{n} A_{i,j} x_i x_j}{\sum_{i=1}^{n} x_i^2} \quad \text{so}$$

$$\frac{\partial}{\partial x_k} f(x) = \frac{\left(2 A_{k,k} x_k + \sum_{i \neq k} (A_{i,k} + A_{k,i}) x_i\right)\|x\|^2 - 2x_k \, x^T A x}{\|x\|^4}$$

For $x \in S$, this simplifies to (using that $A_{ik} = A_{ki}$)

$$\frac{\partial}{\partial x_k} f(x) = 2 \sum_{i=1}^{n} A_{k,i} x_i - 2 x_k \, x^T A x.$$

Hence, $\frac{\partial}{\partial x_k} f(x_*) = 0$ gives that for all k:

$$\sum_{i=1}^{n} A_{k,i} (x_*)_i = (x_*^T A x_*)(x_*)_k$$

ie: $(A x_*)_k = (x_*^T A x_*)(x_*)_k$

We conclude that $A x_* = \lambda x_*$, where $\lambda = x_*^T A x_*$