Problem 5.1.

Notice that $M.\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ so $\lambda_1 = 1$ is an eigenvalue of M.

By contradiction, assume that M is diagonalizable. Then there exist $P \in \mathbb{R}^{2\times 2}$ invertible and $\lambda_2 \in \mathbb{R}$ such that

$$M = P \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} P^{-1}$$

Since $Tr(M) = \lambda_1 + \lambda_2 = 2$, we get $\lambda_2 = 1$

So

$$M = P\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}P^{-1} = PP^{-1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

: we get a contradiction.

<u>Conclusion</u>: M is not diagonalizable

Problem 5.2

a) Compute $MM^T = (Id_n - 2P_S)(Id_n - 2P_S^T)$
(because $P_S^T = P_S$)
$$= (Id_n - 2P_S)(Id_n - 2P_S)$$
$$= Id_n - 4P_S + 4P_S^2$$
$$= Id_n$$
because $P_S^2 = P_S$.
M is therefore orthogonal.

b) M is symmetric ($M^T = M$) so we get from a) that $M^2 = Id_n$

Let $\lambda \in \mathbb{R}$ be an eigenvalue of M, and $v \in \mathbb{R}^n \setminus \{0\}$ an associated eigenvector.

We get $M^2 v = v$

so $M(\lambda v) = v$

$\lambda^2 v = v$ which gives $\lambda^2 = 1$

since $v \neq 0$: $\underline{\lambda = 1 \text{ or } \lambda = -1}$.

c) Let $k = \dim S$. Let $(v_1 \dots v_k)$ be a basis of S and $(v_{k+1}, \dots, v_n)$ be a basis of $S^\perp$ (recall that $\dim S^\perp = n-k$).
Then $(v_1, \dots v_k, v_{k+1}, \dots v_n)$ is a basis of $\mathbb{R}^n$

- For $i \in \{1, \dots k\}$, compute:
$M v_i = v_i - 2P_S v_i = v_i - 2v_i = -v_i$
because $P_S v_i = v_i$ since $v_i \in S$. We get that v_i is an eigenvector of M (with eigenvalue -1)

- For $i \in \{k+1, \dots, n\}$, compute
$M v_i = v_i - 2P_S v_i = v_i$ (because $P_S v_i = 0$, since $v_i \in S^\perp$). We get that v_i is an eigenvector of M (with eigenvalue 1).

$(v_1 \dots v_n)$ is therefore a basis of $\mathbb{R}^n$ consisting of eigenvectors of M.

Problem 5.3.

a) Compute: • $v_1^T A v_2 = v_1^T (\lambda_2 v_2) = \lambda_2 v_1^T v_2$

• $v_1^T A v_2 = (A^T v_1)^T v_2$

(because $A = A^T$) $= (A v_1)^T v_2$

$= \lambda_1 v_1^T v_2$.

We get that $\lambda_1 v_1^T v_2 = \lambda_2 v_1^T v_2$. Since $\lambda_1 \neq \lambda_2$, we deduce that $v_1^T v_2 = 0$: $v_1 \perp v_2$

b) Assume that A is diagonalizable.

For $\lambda \in Sp(A)$, we write

$$\begin{cases} E_\lambda = \ker(A - \lambda I_n) & \text{the corresponding eigenspace} \\ m_\lambda = \dim E_\lambda & \text{the multiplicity of } \lambda. \end{cases}$$

By the Gram-Schmidt procedure, we can find an orthonormal basis $(u_1^{(\lambda)}, \dots, u_{m_\lambda}^{(\lambda)})$ of E_λ

Let $\mathcal{B}$ be the family of vectors obtained by putting together all the vectors $u_1^{(\lambda)} \dots u_{m_\lambda}^{(\lambda)}$:

$$\mathcal{B} = \{ u_1^{(\lambda)}, \dots, u_{m_\lambda}^{(\lambda)} \mid \lambda \in Sp(A) \}.$$

- The vectors of $\mathcal{B}$ are all eigenvectors of A (because they all belong to E_λ for some $\lambda \in Sp(A)$)

- They all have unit norm (by construction).

- Let us now take two vectors $u \neq u' \in \mathcal{B}$ and verify that $u \perp u'$. We have $u \in E_\lambda$ and $u' \in E_{\lambda'}$ for some $\lambda, \lambda' \in Sp(A)$.

  - Case 1: $\lambda \neq \lambda'$. Then by ⓐ: $u \perp u'$.

  - Case 2: $\lambda = \lambda'$. Then by construction $u \perp u'$.

Conclusion: $\mathcal{B}$ is orthonormal.

It remains to check that $\mathcal{B}$ is a basis of $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$. Since A is diagonalizable x can be expressed as a linear combination of eigenvectors of A:

$$x = \sum_{\lambda \in Sp(A)} \alpha_\lambda v_\lambda$$

for some $\alpha_\lambda \in \mathbb{R}$, $v_\lambda \in E_\lambda$.
Since $E_\lambda \subset \text{Span}(\mathcal{B})$ we get that $x \in \text{Span}(\mathcal{B})$: hence $\mathcal{B}$ spans $\mathbb{R}^n$ (and is hence a basis because it is orthonormal).

Problem 5.4

a) $$x_t = \frac{A x_{t-1}}{\|A x_{t-1}\|} = \frac{A \frac{A x_{t-2}}{\|A x_{t-2}\|}}{\left\| A \frac{A x_{t-2}}{\|A x_{t-2}\|} \right\|}$$

$$= \frac{A^2 x_{t-2}}{\|A^2 x_{t-2}\|}$$

$$= \frac{A^2 \frac{A x_{t-3}}{\|A x_{t-3}\|}}{\left\| A^2 \frac{A x_{t-3}}{\|A x_{t-3}\|} \right\|} = \frac{A^3 x_{t-3}}{\|A^3 x_{t-3}\|}$$

$$= \cdots = \frac{A^t x_0}{\|A^t x_0\|}$$

b) The set of vectors that have their first coordinate in the basis $(v_1, \dots v_n)$ equal to zero is $\mathrm{Span}(v_2, \dots v_n)$.
This is an hyperplane of $\mathbb{R}^n$: a randomly chosen vector has zero probability to belong to it (for instance in $\mathbb{R}^3$ a randomly chosen vector will be outside of the horizontal plane with probability 1).

c) $x_0 = \alpha_1 v_1 + \dots + \alpha_n v_n$ so

$$A^t x_0 = \alpha_1 A^t v_1 + \dots + \alpha_n A^t v_n$$
$$= \alpha_1 \lambda_1^t v_1 + \dots + \alpha_n \lambda_n^t v_n .$$

Consequently:

$$x_t = \frac{A^t x_0}{\|A^t x_0\|} = \frac{\alpha_1 v_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^t \alpha_2 v_2 + \dots + \left(\frac{\lambda_n}{\lambda_1}\right)^t \alpha_n v_n}{\left\|\alpha_1 v_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^t \alpha_2 v_2 + \dots + \left(\frac{\lambda_n}{\lambda_1}\right)^t \alpha_n v_n\right\|}$$

Since $\left(\frac{\lambda_i}{\lambda_1}\right)^t \xrightarrow[t\to+\infty]{} 0$ for all $i \in \{2, \dots n\}$, we get

$$x_t \xrightarrow[t\to+\infty]{} \frac{\alpha_1 v_1}{\|\alpha_1 v_1\|} = \frac{\alpha_1}{|\alpha_1|} v_1 .$$

and

$$\|A x_t\| \xrightarrow[t\to+\infty]{} \left\|A \frac{\alpha_1}{|\alpha_1|} v_1\right\| = \|A v_1\|$$
$$= \|\lambda_1 v_1\| = \lambda_1$$

Problem: 5.5

The function f is continuous over the unit sphere $S = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ and admits therefore a maximum on S, at some x_*

Since for all $r \neq 0$ and $x \in S$, $f(rx) = f(x)$, x_* is a global maximizer of f on $\mathbb{R}^n \setminus \{0\}$.

Hence $\nabla f(x_*) = 0$. By definition:

$$f(x) = \frac{\sum_{i,j=1}^{n} A_{i,j} x_i x_j}{\sum_{i=1}^{n} x_i^2} \quad \text{so}$$

$$\frac{\partial}{\partial x_k} f(x) = \frac{\left(2 A_{k,k} x_k + \sum_{i \neq k} (A_{i,k} + A_{k,i}) x_i\right)\|x\|^2 - 2x_k\, x^T A x}{\|x\|^4}$$

For $x \in S$, this simplifies to (using that $A_{ik} = A_{ki}$)

$$\frac{\partial}{\partial x_k} f(x) = 2 \sum_{i=1}^{n} A_{k,i}\, x_i - 2\, x_k\, x^T A x.$$

Hence, $\frac{\partial}{\partial x_k} f(x_*) = 0$ gives that for all k:

$$\sum_{i=1}^{n} A_{k,i} (x_*)_i = (x_*^T A x_*) (x_*)_k$$

ie : $(A x_*)_k = (x_*^T A x_*)\ (x_*)_k$

We conclude that $A x_* = \lambda x_*$, where $\lambda = x_*^T A x_*$