Lecture 5.2: Orthogonal matrices

Optimization and Computational Linear Algebra for Data Science

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Orthogonal matrices

Definition

A matrix $A \in \mathbb{R}^{n\times n}$ is called an *orthogonal matrix* if its columns are an orthonormal family.

Example:

$$Id_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

is an orthogonal matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

matrix of the rotation of angle θ

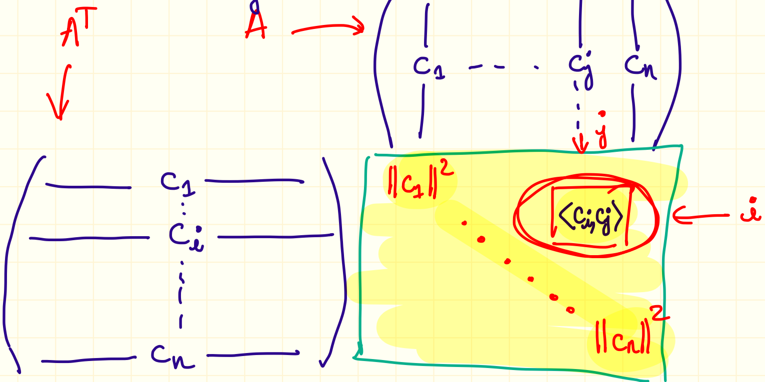
is orthogonal.

A proposition

Proposition

Let $A \in \mathbb{R}^{n\times n}$. The following points are equivalent:

1. A is orthogonal.
2. $A^\mathsf{T} A = \mathrm{Id}_n$. $\iff$ A is invertible and $A^{-1} = A^T$
3. $AA^\mathsf{T} = \mathrm{Id}_n$ $\iff$ A is invertible and $A^{-1} = A^T$
4. A^T is orthogonal



Orthogonal matrices & norm

Proposition

Let $A \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Then A preserves the dot product in the sense that for all $x, y \in \mathbb{R}^n$,

$$\langle Ax, Ay\rangle = \langle x, y\rangle.$$

In particular if we take $x = y$ we see that A preserves the Euclidean norm: $\|Ax\| = \|x\|$.

Proof: $\langle Ax, Ay\rangle = (Ax)^T (Ay) = x^T \underbrace{A^T A}_{= Id_n} y$

$= x^T y = \langle x, y\rangle$ ∎

Orthogonal matrices $= \left\{ \begin{array}{l} \text{rotations} \\ \text{symmetries} \end{array} \right.$ of $\mathbb{R}^n$