Lecture 3.2: Some properties of the rank

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

The rank-nullity theorem

Theorem

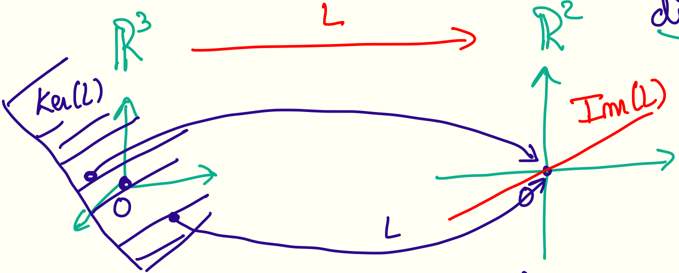
Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then

dim(Im L)

$$\operatorname{rank}(L) + \dim(\operatorname{Ker}(L)) = m.$$

Intuition: « the conservation of the dimension »

$$\underbrace{\dim \ker L}_{2} + \underbrace{\operatorname{rank}(L)}_{1} = 3$$



• "out of the 3 dimensions of $\mathbb{R}^3$, 2 are sent to zero by L"

Some inequalities

Proposition

Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times k}$. Then the following holds

1. $\mathrm{rank}(A) \leq \min(n, m)$.
2. $\mathrm{rank}(AB) \leq \min(\mathrm{rank}(A), \mathrm{rank}(B))$.

Proof.

$$A = \begin{pmatrix} | & & | \\ c_1 & \cdots & c_m \\ | & & | \end{pmatrix} = \begin{pmatrix} — & r_1 & — \\ & \vdots & \\ — & r_n & — \end{pmatrix}$$

- $\mathrm{rank}(A) = \mathrm{rank}(c_1, \dots, c_m) \leq m$
- $\mathrm{rank}(A) = \mathrm{rank}(r_1 \dots, r_n) \leq n$.

$\square$

Some inequalities

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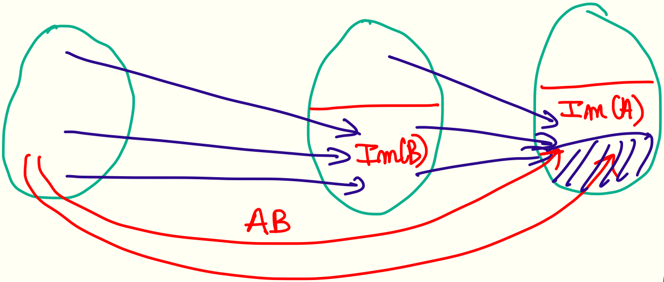
Proof. ① $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$

$$\mathbb{R}^k \xrightarrow{B} \mathbb{R}^m \xrightarrow{A} \mathbb{R}^n$$



$\operatorname{Im}(AB) \subset \operatorname{Im}(A)$

$\operatorname{rank}(AB) \leq \operatorname{rank}(A)$

$\square$

Some inequalities

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1. $\mathrm{rank}(A) \leq \min(n, m)$.
2. $\mathrm{rank}(AB) \leq \min(\mathrm{rank}(A), \mathrm{rank}(B))$.

Proof. 2 $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$

- $\underline{\mathrm{Ker}(B) \subset \mathrm{Ker}(AB)}$

If $x \in \mathrm{Ker}(B)$, $Bx = 0$, $ABx = 0$ $\therefore$ $x \in \mathrm{Ker}(AB)$

- $\underbrace{\dim \mathrm{Ker}\, B}_{k - \mathrm{rank}(B)} \leq \underbrace{\dim \mathrm{Ker}(AB)}_{k - \mathrm{rank}(AB)}$

$\longrightarrow$ $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$. $\square$