

Problem M.1.

- $\nabla f(x, y, z) = \begin{pmatrix} 4x + 4 \\ 2y - 6 \\ z - 1 \end{pmatrix}; H_f(x) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\nabla f(x, y, z) = 0 \Leftrightarrow \begin{cases} x = -1 \\ y = 3 \\ z = 1 \end{cases}$$

Hence f has a single critical point at $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

The Hessian ~~of f~~ is $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which is positive definite because its eigenvalues are 1, 2 and 4. Hence f is convex: $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ is a global minimizer.

- $g(x, y, z) = -xyz + xy + z.$

$$\nabla g(x, y, z) = \begin{pmatrix} 1-yz \\ 1-xz \\ 1-xy \end{pmatrix}, H_g(x, y, z) = \begin{pmatrix} 0 & -z & -y \\ -z & 0 & -x \\ -y & -x & 0 \end{pmatrix}$$

$$\begin{aligned} \nabla g(x, y, z) = 0 &\Leftrightarrow \begin{cases} yz = 1 \\ xz = 1 \\ xy = 1 \end{cases} \\ &\Leftrightarrow \begin{cases} y = 1/z \\ x = 1/z \\ z^2 = 1 \end{cases} \Leftrightarrow \begin{cases} x = y = z = 1 \\ \text{or} \\ x = y = z = -1. \end{cases} \end{aligned}$$

Hence f has two critical points, at:

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$H_g(X_1) = - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The symmetric matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ has rank equal to 1:

Hence it has only one non-zero eigenvalue.

Notice that $M \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$: 3 is an eigenvalue of M .

Conclusion: the eigenvalues of M are 3 and 0.

Hence, the eigenvalues of $Hg(x_1) = Id - M$ are -3 and 1
 x_1 is therefore a saddle-point.

$Hg(x_2) = M - Id$, which has 3 and -1 for eigenvalue
 x_2 is also a saddle-point.

Problem 12.2.

Define: $u = (1, -1, 1)$ and $v = (1, 1, 1)$.

For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we define:

$$f(x) = \langle x, u \rangle$$

$$g(x) = \langle x \rangle^2 - 1 \quad \text{and} \quad h(x) = \langle x, v \rangle - 1.$$

Compute $\nabla g(x) = 2x$ and $\nabla h(x) = v$.

Let now x be a solution of (1).

In order to be able to apply our theorem of Lagrange multipliers, we must verify that $\nabla g(x)$ and $\nabla h(x)$ are not colinear.

By contradiction, assume that $\nabla g(x)$ and $\nabla h(x)$ are colinear. This implies that the coordinates of x are all equal.

The constraint $x_1 + x_2 + x_3 = 1$ gives then $x_1 = x_2 = x_3 = \frac{1}{3}$.
This violates the second constraint since

$$\left\| \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \right\|^2 = \frac{1}{3} \neq 1. \quad \text{We get a contradiction.}$$

We conclude that there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\nabla f(x) + \lambda_1 \nabla g(x) + \lambda_2 \nabla h(x) = 0$:

$$u + 2\lambda_1 x + \lambda_2 v = 0. \quad (\dagger)$$

We are now going to use the two constraints

equations $\langle X, v \rangle = 1$ and $\|X\|^2 = 1$ to find X , λ_1 and λ_2

- We first take the inner product with v on both sides of (*): $1 + 2\lambda_1 + 3\lambda_2 = 0$, where we used that $\langle X, v \rangle = 1$.

This gives : $2\lambda_1 = -3\lambda_2 - 1$.

- (*) gives that $2\lambda_1 X = -u - \lambda_2 v$, hence $(2\lambda_1)^2 \|X\|^2 = \|u + \lambda_2 v\|^2$.

Since $\|X\|^2 = 1$ and $2\lambda_1 = -3\lambda_2 - 1$, we get :

$$(1+3\lambda_2)^2 = \left\| \begin{pmatrix} -1+\lambda_2 \\ -1+\lambda_2 \\ 1+\lambda_2 \end{pmatrix} \right\|^2 = 2(1+\lambda_2)^2 + (\lambda_2 - 1)^2.$$

Hence $(3\lambda_2 + 1)^2 - (\lambda_2 - 1)^2 = 2(1+\lambda_2)^2$
 $(2\lambda_2 + 2)(4\lambda_2) = 2(\lambda_2 + 1)^2$
 $(\lambda_2 + 1)(3\lambda_2 - 1) = 0$

We get that:

$$\begin{array}{ll} \lambda_2 = -1 & \text{OR} \\ \rightarrow \lambda_1 = 1 & \lambda_2 = \frac{1}{3} \\ X = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} & \left| \begin{array}{l} \lambda_1 = -1 \\ X = \begin{pmatrix} -2/3 \\ 1/3 \\ -2/3 \end{pmatrix} \end{array} \right. \end{array}$$

CONCLUSION : if X is a solution of (1) then
 $X = (0, -1, 0)$ or $X = (-2/3, 1/3, -2/3)$

This two points satisfy the constraints. However
 $f(0, -1, 0) = 1$ and $f(-2/3, 1/3, -2/3) = -\frac{5}{3} < 1$.

$(0, -1, 0)$ can therefore not be solution of (1).

$\rightarrow (-2/3, 1/3, -2/3)$ is the unique solution of (1).

Problem M.3.

a) Let $i^* \in \{1, \dots, n\}$ such that $|u_{i^*}| > |u_{ij}|$ for all $j \neq i^*$.

(Such i^* exists since we assumed that the $|u_{ij}|$ were all distinct).

Let $x \in \mathbb{R}^n$ such that $\|x\|_1 \leq 1$

$$\langle u, x \rangle = \sum_{i=1}^n x_i u_i \leq \sum_{i=1}^n |x_i| |u_i| \leq |u_{i^*}| \sum_{i=1}^n |x_i| = |u_{i^*}|$$

Define $x^* \in \mathbb{R}^n$ by:
$$x_i^* = \begin{cases} 0 & \text{if } i \neq i^* \\ x_{i^*} & \text{if } i = i^* \end{cases}$$

Then $\langle u, x^* \rangle = |u_{i^*}| x_{i^*}^* = |u_{i^*}|$: x^* is a solution to our constrained maximization problem because we showed that for all x such that $\|x\|_1 \leq 1$:

$$\langle u, x \rangle \leq |u_{i^*}| = \langle x^*, u \rangle.$$

Let us show that x^* is the unique solution.

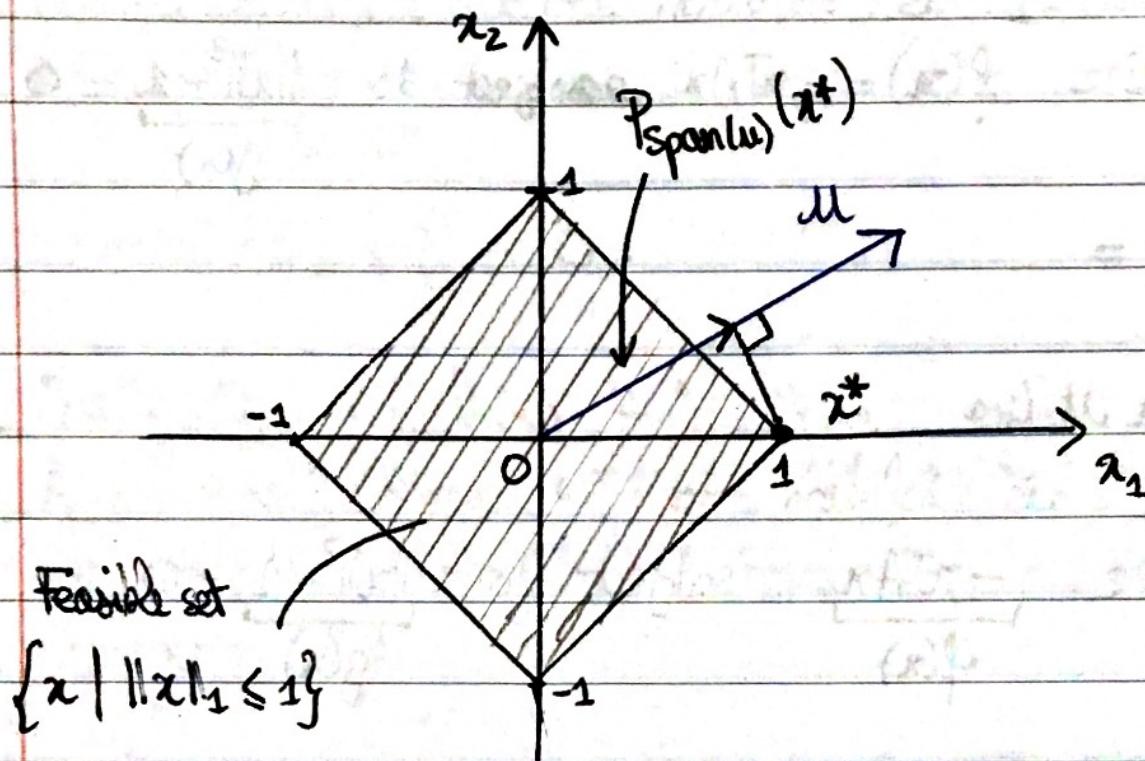
Let $x \neq x^*$ such that $\|x\|_1 = 1$. In that case, there exists $j \neq i^*$ such that $x_j \neq 0$

$$\langle x, u \rangle \leq \sum_{i=1}^n |x_i| |u_i| < \sum_{i=1}^n |x_i| |u_{i^*}| = |u_{i^*}|$$

(because $|x_j| |u_j| < |x_i| |u_{i^*}|$ because $|u_j| \neq 0$)

Conclusion: x^* is the unique solution

b) We illustrate the problem for $n=2$:



When maximizing $\langle u, x \rangle$ subject to $\|x\|_1 \leq 1$, we are looking for the vector in $\{x \mid \|x\|_1 \leq 1\}$ ~~that has the largest correlation with u~~

for which $\langle x, u \rangle = \langle P_{\text{Span}(u)}(x), u \rangle$ is maximal.

On the picture we see that ~~not~~ this is achieved at a point x^* whose coordinates are all 0 except 1, as proved in a).

Problem M.4

a) v_1 is solution to:

$$\text{minimize}_{\frac{-x^T A x}{f(x)}} \text{ subject to } \frac{\|x\|^2 - 1}{g(x)} = 0$$

$$\nabla g(v_1) = 2v_1 \neq 0 \text{ since } \|v_1\| = 1.$$

Hence there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(v_1) + \lambda \nabla g(v_1) = 0,$$

$$-2Ax_1 + 2\lambda v_1 = 0$$

Hence $Ax_1 = \lambda v_1$: v_1 is an eigenvector of A .

b) v_2 is solution of:

$$\text{minimize}_{\frac{-x^T A x}{f(x)}} \text{ subject to } \frac{\|x\|^2 - 1}{g(x)} = 0 \text{ and } \frac{\langle x, v_1 \rangle}{h(x)} = 0$$

$\nabla g(v_2) = 2v_2$ and $\nabla h(v_2) = v_1$ are linearly independent because $v_1 \perp v_2$ and $\|v_1\| = \|v_2\| = 1$.

Hence there exists $\lambda, \lambda' \in \mathbb{R}$ such that

$$\nabla f(v_2) + \lambda \nabla g(v_2) + \lambda' \nabla h(v_2) = 0.$$

$$\text{Hence } -2A\mathbf{v}_2 + 2\mathbf{v}_2 \cdot \lambda + \lambda' \mathbf{v}_1 = 0$$

Taking inner product with \mathbf{v}_1 on both sides and using that $\|\mathbf{v}_1\| = 1$ we get:

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$$

$$-2\mathbf{v}_2^T A \mathbf{v}_2 + \lambda' = 0.$$

Using ①, we know that there exists $\alpha \in \mathbb{R}$ such that $A\mathbf{v}_2 = \alpha \mathbf{v}_2$. Hence $\mathbf{v}_2^T A \mathbf{v}_2 = \langle \mathbf{v}_2, A\mathbf{v}_2 \rangle = \langle \mathbf{v}_2, \alpha \mathbf{v}_2 \rangle = 0$. We conclude that $\lambda' = 0$, hence $A\mathbf{v}_2 = \lambda \mathbf{v}_2$: \mathbf{v}_2 is an eigenvector of A .

c) Using the same argument as in b) we get that there exists $\lambda, \lambda', \lambda'' \in \mathbb{R}$ such that

$$-2A\mathbf{v}_3 + 2\lambda\mathbf{v}_3 + \lambda'\mathbf{v}_1 + \lambda''\mathbf{v}_2 = 0$$

Taking the inner product with \mathbf{v}_1 , we get

$$\lambda' = 2\mathbf{v}_1^T A \mathbf{v}_3 = 2 \langle A\mathbf{v}_1, \mathbf{v}_3 \rangle = 0.$$

Similarly, taking now the inner product with \mathbf{v}_2 we get

$$\lambda'' = 2\mathbf{v}_2^T A \mathbf{v}_3 = 0.$$

Hence $A\mathbf{v}_3 = \lambda \mathbf{v}_3$: \mathbf{v}_3 is an eigenvector of A .

Problem M.5.

The function $V(\omega)$ is convex because Σ is positive semi-definite and the constraints of (5) are affine: (5) is a convex constrained problem.

We would like to apply the Kuhn-Tucker theorem.

Hence, we need to show that there exists a strictly feasible point. Let j such that $r_j = \max_{k=1\dots n} r_k$

$r_i \neq r_j$ because $r \notin \text{Span}(\mathbf{1})$.

Let $\omega = m(t r_i + (1-t) r_j)$ with $t = \frac{N - r_j}{r_i - r_j}$ (strictly)

One can verify easily that ω is feasible.

We can then apply the Kuhn-Tucker theorem to get that

$$\omega \text{ solution of (5)} \Leftrightarrow \begin{cases} \omega \text{ feasible} \\ \text{there exists } \lambda_1, \lambda_2 \in \mathbb{R} \text{ such that} \\ 2\sum \omega + \lambda_1 \mathbf{1} + \lambda_2 r = 0 \quad (*) \end{cases}$$

Let's solve (*) for ω feasible, $\lambda_1, \lambda_2 \in \mathbb{R}$: assume that ω feasible, λ_1, λ_2 verify (*) and that ω is feasible. Then we get:

$$\cancel{\begin{cases} 2\sum \omega = -\lambda_1 \mathbf{1} - \lambda_2 r \\ \omega \text{ feasible} \end{cases}} \quad (**)$$

(Σ is invertible because positive definite)

Taking the inner-product with \mathbf{M} on both sides:

$$-2\mathbf{m} = \lambda_1 \mathbf{M}^T \Sigma^{-1} \mathbf{M} + \lambda_2 \mathbf{M}^T \Sigma^{-1} \mathbf{r}$$

Same with \mathbf{r} :

$$-2\mathbf{p} = \lambda_1 \mathbf{r}^T \Sigma^{-1} \mathbf{M} + \lambda_2 \mathbf{r}^T \Sigma^{-1} \mathbf{r}$$

Hence we get that: $M \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = -2 \begin{pmatrix} \mathbf{m} \\ \mathbf{p} \end{pmatrix}$

Assume for a moment that M is invertible (we will prove it at the end): we get:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = -2M^{-1} \begin{pmatrix} \mathbf{m} \\ \mathbf{p} \end{pmatrix}$$

Using (*) we conclude: $\omega = \Sigma^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} M^{-1} \begin{pmatrix} \mathbf{m} \\ \mathbf{p} \end{pmatrix}$

Hence if ω is feasible and if $\lambda_1, \lambda_2, \omega$ satisfy (*), then:

$$\left. \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = -2M^{-1} \begin{pmatrix} \mathbf{m} \\ \mathbf{p} \end{pmatrix} \right] (***)$$

$$\left. \omega = \Sigma^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} M^{-1} \begin{pmatrix} \mathbf{m} \\ \mathbf{p} \end{pmatrix} \right]$$

It is easy ~~to~~ to verify the converse, that is that if w, λ_1, λ_2 are given by $(***)$ then they verify $(*)$ and w is feasible.

We conclude that w is optimal if and only if $w = w^* \stackrel{\text{def}}{=} \Sigma^{-1} \begin{pmatrix} \mathbf{1} & \mathbf{r} \end{pmatrix} M^{-1} \begin{pmatrix} \mathbf{m} \\ \mathbf{p} \end{pmatrix}$.

w^* is the unique solution of (S) .

It remains to prove that M is invertible by contradiction
~~as symmetric~~

Assume that there exists $(a, b) \neq (0, 0)$ such that $M \begin{pmatrix} a \\ b \end{pmatrix} = 0$.

This implies $\begin{pmatrix} a \\ b \end{pmatrix}^T M \begin{pmatrix} a \\ b \end{pmatrix} = 0$, hence:

$$\underbrace{a^2 \mathbf{1}^T \Sigma^{-1} \mathbf{1} + 2ab \mathbf{1}^T \Sigma^{-1} \mathbf{r} + b^2 \mathbf{r}^T \Sigma^{-1} \mathbf{r}}_V = 0$$

$$(a\mathbf{1} + b\mathbf{r})^T \Sigma^{-1} (a\mathbf{1} + b\mathbf{r}) = 0$$

definite.

Σ is assumed to be positive^V, hence Σ^{-1} is also positive ~~definite~~. This implies that $a\mathbf{1} + b\mathbf{r} = 0$

which then give that $a=b=0$ because $\mathbf{1} \neq 0$ and $\mathbf{r} \notin \text{Span}(\mathbf{1})$

This is a contradiction. We conclude that M is invertible