Lecture 4.1: Inner product

Optimization and Computational Linear Algebra for Data Science

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The Euclidean dot product

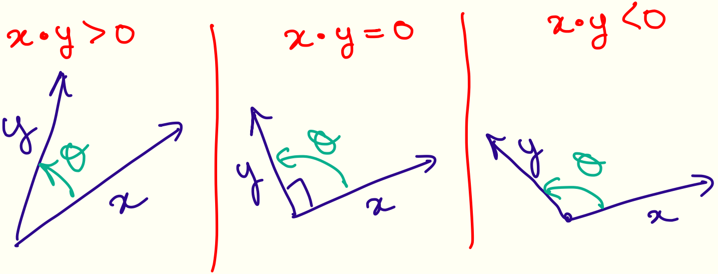
Definition

We define the Euclidean dot product of two vectors x and y of $\mathbb{R}^n$ as:

$$\underline{x \cdot y} = \sum_{i=1}^{n} x_i \, y_i = \underline{x_1 y_1} + \cdots + \underline{x_n y_n}.$$

Remarks:

- $x \cdot x = \|x\|_2^2$
- $x \cdot y = \|x\|_2 \, \|y\|_2 \cos\theta$



Inner product

Let V be a vector space.

Definition

An inner product on V is a function $\langle \cdot, \cdot \rangle$ from $V \times V$ to $\mathbb{R}$ that verifies the following points:

1. *Symmetry*: $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$.
2. *Linearity*: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ and $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$.
3. *Positive definiteness*: $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

Ex: for $V = \mathbb{R}^n$, the Euclidean dot prod. $\langle x, y \rangle = x \cdot y$, is an inner product on $\mathbb{R}^n$

Other example

If V is the set of all random variables (on a probability space Ω) that have a finite second moment, then

rand. var.

$$\langle X, Y \rangle \stackrel{\text{def}}{=} \mathbb{E}[XY]$$

mean value of XY

is an inner product on V.

- Symmetry $\langle X, Y \rangle = \mathbb{E}[XY] = \mathbb{E}[YX] = \langle Y, X \rangle$
- Linearity: exercise!
- $\langle X, X \rangle = \mathbb{E}[X^2] \geq 0$

if $\mathbb{E}[X^2] = 0$ then $X = 0$

Norm induced by an inner product

Proposition

If $\langle \cdot, \cdot \rangle$ is an inner product on V then

$$\|v\| \stackrel{\text{def}}{=} \sqrt{\langle v, v \rangle}$$

is a norm on V. We say that the norm $\| \cdot \|$ is induced by the inner product $\langle \cdot, \cdot \rangle$.

Example: On $V = \mathbb{R}^n$, the Euclidean norm $\|\cdot\|_2$ is induced by the Euclidean dot product:

$$\|x\|_2 = \sqrt{x \cdot x}$$

Example

Consider again the set V of all random variables (on a probability space Ω) that have a finite second moment, with the inner product:

$$\langle X, Y \rangle \stackrel{\text{def}}{=} \mathbb{E}[XY].$$

$$\rightarrow \quad \|X\| = \sqrt{\langle X, X \rangle} = \sqrt{\mathbb{E}[X^2]}$$

second moment of X.

Cauchy Schwarz inequality

Theorem (Cauchy-Schwarz inequality)

Let $\|\cdot\|$ be the norm induced by the inner product $\langle\cdot,\cdot\rangle$ on the vector space V. Then for all $x, y \in V$:

$$|\langle x, y\rangle| \leq \|x\|\,\|y\|. \tag{1}$$

Moreover, there is equality in (1) if and only if x and y are linearly dependent, i.e. $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{R}$.

Ex: $V = \mathbb{R}^n$

$$|x \cdot y| = \big|\, \|x\|_2 \, \|y\|_2 \cos(\theta) \big| \leq \|x\|_2 \|y\|_2$$

Examples

In the case $\langle X, Y \rangle = \mathbb{E}[XY]$:

Cauchy - Scharz gives:

$$\left|\mathbb{E}[XY]\right| \leq \sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{E}[Y^2]}$$

Replace: X by $X - \mathbb{E}X$ and Y by $Y - \mathbb{E}Y$

$$\left|\mathbb{E}\left[(X-\mathbb{E}X)(Y-\mathbb{E}Y)\right]\right| \leq \sqrt{\mathbb{E}\left[(X-\mathbb{E}X)^2\right]}\sqrt{\mathbb{E}\left[(Y-\mathbb{E}Y)^2\right]}$$

$$\left|\mathrm{Cov}(X,Y)\right| \leq \mathrm{Var}(X)\,\mathrm{Var}(Y)$$

Examples