Problem 3.1

a)
$$\left.\begin{aligned} AB &= \alpha \, Id_n B = \alpha B \\ BA &= B(\alpha Id_n) = \alpha B Id_n = \alpha B \end{aligned}\right\} AB = BA.$$

b) Let $k, l \in \{1, \dots n\}$ and B be the $n \times n$ matrix defined by:

$$B_{i,j} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise.} \end{cases}$$

Compute for $i, j \in \{1, \dots n\}$:

$$(AB)_{i,j} = \sum_{m=1}^{n} A_{i,m} B_{m,j} = A_{i,k} B_{k,j}$$

$$(BA)_{i,j} = \sum_{m=1}^{n} B_{i,m} A_{m,j} = B_{i,l} A_{l,j}$$

Since $AB = BA$ we have $A_{i,k} B_{k,j} = B_{i,l} A_{l,j}$ for all i, j.

In particular, for $j = l$, we get: $A_{i,k} = B_{i,l} A_{l,l}$.

So $\left| \begin{array}{l} \text{if } i \neq k \text{ then } A_{i,k} = 0 . A_{l,l} = 0 \\ \text{if } i = k \text{ then } A_{k,k} = 1 . A_{l,l} = A_{l,l} . \end{array} \right.$

We conclude that all the coefficients of A that are outside the diagonal are zero, and that the diagonal coefficients are all equal to some number that we call α.

We have in other words $A = \alpha \, Id_n$.

Problem 3. 2.

By definition of the rank: $\dim \operatorname{Im}(M) = \operatorname{rank}(M) = r$.

Let $a_1, \dots a_r \in \mathbb{R}^n$ be a basis of $\operatorname{Im}(M)$.
Let $c_1, \dots c_m \in \mathbb{R}^n$ be the columns of M.
For $i = 1, \dots m$, the vector c_i belongs to $\operatorname{Im}(A)$.
Therefore, there exists scalars $b_{1,i} \dots b_{r,i}$ such that

$$c_i = b_{1,i}\, a_1 + \dots + b_{r,i}\, a_r \, .$$

Let A be the matrix $A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_r \\ | & & | \end{pmatrix} \in \mathbb{R}^{n \times r}$.

and B be the $r \times m$ matrix defined by
$B_{i,j} = b_{i,j}$ for all $(i,j) \in \{1 \dots r\} \times \{1 \dots m\}$.

By construction we have $M = AB$ because for all i, the i^{th} column of AB is:

$$b_{1,i}\, a_1 + \dots + b_{r,i}\, a_r = c_i$$

Problem 3.3

a) Let us show that $\operatorname{Im}(AM) = \operatorname{Im}(A)$.

- $\operatorname{Im}(AM) \subset \operatorname{Im}(A)$. Indeed if $y \in \operatorname{Im}(AM)$ then there exists $x \in \mathbb{R}^m$ such that $y = AMx$.
Hence $y = A(Mx) \in \operatorname{Im}(A)$

• $\mathrm{Im}(A) \subset \mathrm{Im}(AM)$. Indeed, let $y \in \mathrm{Im}(A)$.
There exists $x \in \mathbb{R}^m$ such that $y = Ax$.
Then $y = AM(M^{-1}x) \in \mathrm{Im}(A)$.

<u>Conclusion</u>: $\mathrm{Im}(A) = \mathrm{Im}(AM)$, hence $\mathrm{rank}(A) = \mathrm{rank}(MA)$.

b) Let us show that $\mathrm{Ker}(A) = \mathrm{Ker}(MA)$:
For $x \in \mathbb{R}^n$ we have:

$$
\begin{aligned}
x \in \mathrm{Ker}(A) &\iff Ax = 0 \\
&\iff MAx = 0 \\
&\iff x \in \mathrm{Ker}(MA)
\end{aligned}
$$

multiplication by M ↓ ↑ multiplication by M^{-1}

Hence $\mathrm{Ker}(A) = \mathrm{Ker}(MA)$ and $\dim \mathrm{Ker}(A) = \dim \mathrm{Ker}(MA)$.
We conclude using the rank-nullity theorem:
$n - \mathrm{rank}(A) = n - \mathrm{rank}(MA)$, ie $\mathrm{rank}(A) = \mathrm{rank}(MA)$.

<u>Problem 3.4.</u>
Assume that for all $i \in \{1, \dots n\}$, $a_{i,i} \neq 0$.
We are going to show that $\mathrm{Ker}(A) = \{0\}$.
Let $x \in \mathrm{Ker}(A)$. By contradiction, suppose that $x \neq 0$.
Let i^* be the largest index i for which $x_i \neq 0$, ie:
$$i^* = \max\{ i \in \{1, \dots n\} \mid x_i \neq 0 \}.$$

~~If $i^* = 0$, then the first column of A is zero: A is not invertible.~~

We have :

$$\begin{pmatrix} a_{1,1} & & & ? & \\ & \ddots & & & \\ & & a_{i^*,i^*} & & \\ & (0) & & \ddots & \\ & & & & a_{n,n} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_{i^*} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} ? \\ \vdots \\ ? \\ a_{i^*,i^*}\, x_{i^*} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Indeed, $(Ax)_{i^*} = \sum_{i=1}^{n} a_{i^*,i}\, x_i$

$= \sum_{i=i^*}^{n} a_{i^*,i}\, x_i$ (because $a_{i^*,i} = 0$ for $i < i^*$)

$= a_{i^*,i^*}\, x_{i^*}$

because, by definition of i^*, we have $x_i = 0$ for $i > i^*$.

Recall that $x \in \ker(A)$, so we get $a_{i^*,i^*}\, x_{i^*} = 0$ which implies that $a_{i^*,i^*} = 0$ because $x_{i^*} \neq 0$.
This is a contradiction!
We conclude that $x = 0$ which gives $\ker(A) = \{0\}$:
A is invertible.

Conversely, Assume that A is invertible.
By contradiction, suppose that $a_{i,i} = 0$ for some $i \in \{1 .. n\}$

Let $v_1, \dots v_n$ be the columns of A and $(e_1, \dots e_n)$ be the canonical basis of $\mathbb{R}^n$.

We have $i > 1$ otherwise $v_1 = 0$ and A is not invertible.
Notice now that

$$v_1, \dots v_i \in \mathrm{Span}(e_1, \dots e_{i-1})$$

and that $\mathrm{Span}(e_1, \dots e_{i-1})$ is a subspace of dimension $i-1$.

The vectors $v_1, \dots v_i$ are thus linearly dependent: the columns of A are linearly dependent; A is not invertible. This is a contradiction.

Conclusion : $\forall i \in \{1, \dots n\}$, $a_{ii} \neq 0$.