Problem 11.1.

- $\nabla f(x,y,z) = \begin{pmatrix} 4x+4 \\ 2y-6 \\ z-1 \end{pmatrix}$; $H_f\begin{pmatrix} x\\y\\z\end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\nabla f(x,y,z) = 0 \iff \begin{cases} x = -1 \\ y = 3 \\ z = 1 \end{cases}$.

Hence f has a single critical point at $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

The Hessian of f ~~at this point~~ is $\begin{pmatrix} 4&0&0 \\ 0&2&0 \\ 0&0&1 \end{pmatrix}$ which is positive definite because its eigenvalues are $\underbrace{1, 2 \text{ and } 4}_{>0}$.
Hence f is convex : $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ is a global minimizer.

- $g(x,y,z) = -xyz + x + y + z$.

$\nabla g(x,y,z) = \begin{pmatrix} 1-yz \\ 1-xz \\ 1-xy \end{pmatrix}$, $H_g(x,y,z) = \begin{pmatrix} 0 & -z & -y \\ -z & 0 & -x \\ -y & -x & 0 \end{pmatrix}$

$\nabla g(x,y,z) = 0 \iff \begin{cases} yz = 1 \\ xz = 1 \\ xy = 1 \end{cases}$

$\iff \begin{cases} y = 1/z \\ x = 1/z \\ z^2 = 1 \end{cases} \iff \left\{ \begin{array}{l} x = y = z = 1 \\ \text{or} \\ x = y = z = -1. \end{array} \right.$

Hence f has two critical point, at :

$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

$H_g(X_1) = -\begin{pmatrix} 0&1&1 \\ 1&0&1 \\ 1&1&0 \end{pmatrix} = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} - \begin{pmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{pmatrix}$

The ~~eigenvector~~ symmetric matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ has rank equal to 1:

Hence it has only one non-zero eigenvalue.

Notice that $M \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$: 3 is an eigenvalue of M.

<u>Conclusion</u>: the eigenvalues of M are 3 and 0.

Hence, the eigenvalues of $H_g(X_1) = Id - M$ are -2 and 1

X_1 is therefore a saddle-point.

$H_g(X_2) = M - Id$, which has 2 and -1 for eigenvalues

X_2 is also a saddle-point.

Problem 12.2.

Define: $u = (1,-1,1)$ and $v = (1,1,1)$.

For $X = (x,y,z) \in \mathbb{R}^3$ we define:

$f(X) = \langle X, u \rangle$

$g(X) = \|X\|^2 - 1$ and $h(X) = \langle X, v \rangle - 1$.

Compute $\nabla g(X) = 2X$ and $\nabla h(X) = v$.

Let now X be a solution of (1).

In order to be able to apply our theorem of Lagrange multipliers, we must verify that $\nabla g(X)$ and $\nabla h(X)$ are not colinear.

By contradiction, assume that $\nabla g(X)$ and $\nabla h(X)$ are colinear. This implies that the coordinates of X are all equal.

The constraint $x+y+z=1$ gives then $x=y=z=1/3$

This violates the second constraints since
$\left\| \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \right\|^2 = \frac{1}{3} \neq 1$. We get a contradiction.

We conclude that there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\nabla f(X) + \lambda_1 \nabla g(X) + \lambda_2 \nabla h(X) = 0$:

$u + 2\lambda_1 X + \lambda_2 v = 0$. $(*)$

We are now going to use the two constraints equations $\langle X, v\rangle = 1$ and $\|X\|^2 = 1$ to find X, λ_1 and λ_2

- We first take the inner product with v on both sides of $(*)$: $1 + 2\lambda_1 + 3\lambda_2 = 0$, where we used that $\langle X, v\rangle = 1$.

This gives: $2\lambda_1 = -3\lambda_2 - 1$.

- $(*)$ gives that $2\lambda_1 X = -u - \lambda_2 v$, hence

$(2\lambda_1)^2 \|X\|^2 = \|u + \lambda_2 v\|^2$.

Since $\|X\|^2 = 1$ and $2\lambda_1 = -3\lambda_2 - 1$, we get:

$$(1+3\lambda_2)^2 = \left\| \begin{pmatrix} 1+\lambda_2 \\ -1+\lambda_2 \\ 1+\lambda_2 \end{pmatrix} \right\|^2 = 2(1+\lambda_2)^2 + (\lambda_2 - 1)^2 .$$

Hence
$$(3\lambda_2+1)^2 - (\lambda_2-1)^2 = 2(1+\lambda_2)^2$$
$$(2\lambda_2+2)(4\lambda_2) = 2(\lambda_2+1)^2$$
$$(\lambda_2+1)(3\lambda_2-1) = 0$$

We get that

$$\lambda_2 = -1 \qquad \text{OR} \qquad \lambda_2 = \frac{1}{3}$$

$\lambda_2 = -1$	$\lambda_2 = \frac{1}{3}$
$\lambda_1 = 1$	$\lambda_1 = -1$
$X = \frac{1}{2\lambda_1}(-u - \lambda_2 v)$	$X = -\frac{1}{2}(-u - \lambda_2 v)$
$= \frac{1}{2}\begin{pmatrix} -1+1 \\ 1+1 \\ -1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$= \frac{1}{2}\begin{pmatrix} 1+1/3 \\ -1+1/3 \\ 1+1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix}$

<u>CONCLUSION</u>: if X is a solotion of (1) then
$X = (0,1,0)$ or $X = (2/3, -1/3, 2/3)$.
Theses two points satisfy the constraints. However
$f(0,1,0) = -1$ and $f(2/3, -1/3, 2/3) = +\frac{5}{3} > -1$
$(2/3, -1/3, 2/3)$ can not therefore be solution of (1):

$(0,1,0)$ is the unique solotion of (1).

Problem 11.3

a) Let $i^* \in \{1,\dots n\}$ such that
$|u_{i^*}| > |u_j|$ for all $j \neq i^*$.
(Such i^* exists since we assumed that the $|u_j|$ were all distinct).

Let $x \in \mathbb{R}^n$ such that $\|x\|_1 \leq 1$

$$\langle u, x\rangle = \sum_{i=1}^{n} x_i u_i \leq \sum_{i=1}^{n} |x_i| \, |u_i| \leq |u_{i^*}| \sum_{i=1}^{n} |x_i| = |u_{i^*}|$$

Define $x^* \in \mathbb{R}^n$ by:
$$\begin{cases} x^*_i = 0 & \text{if } i \neq i^* \\ x^*_{i^*} = \begin{cases} 1 & \text{if } u_{i^*} > 0 \\ -1 & \text{otherwise.} \end{cases} \end{cases}$$

Then $\langle u, x^*\rangle = u_{i^*} x^*_{i^*} = |u_{i^*}|$: x^* is a solution to our constrained maximization problem because we showed that for all x such that $\|x\|_1 \leq 1$:
$$\langle x, u\rangle \leq |u_{i^*}| = \langle x^*, u\rangle.$$

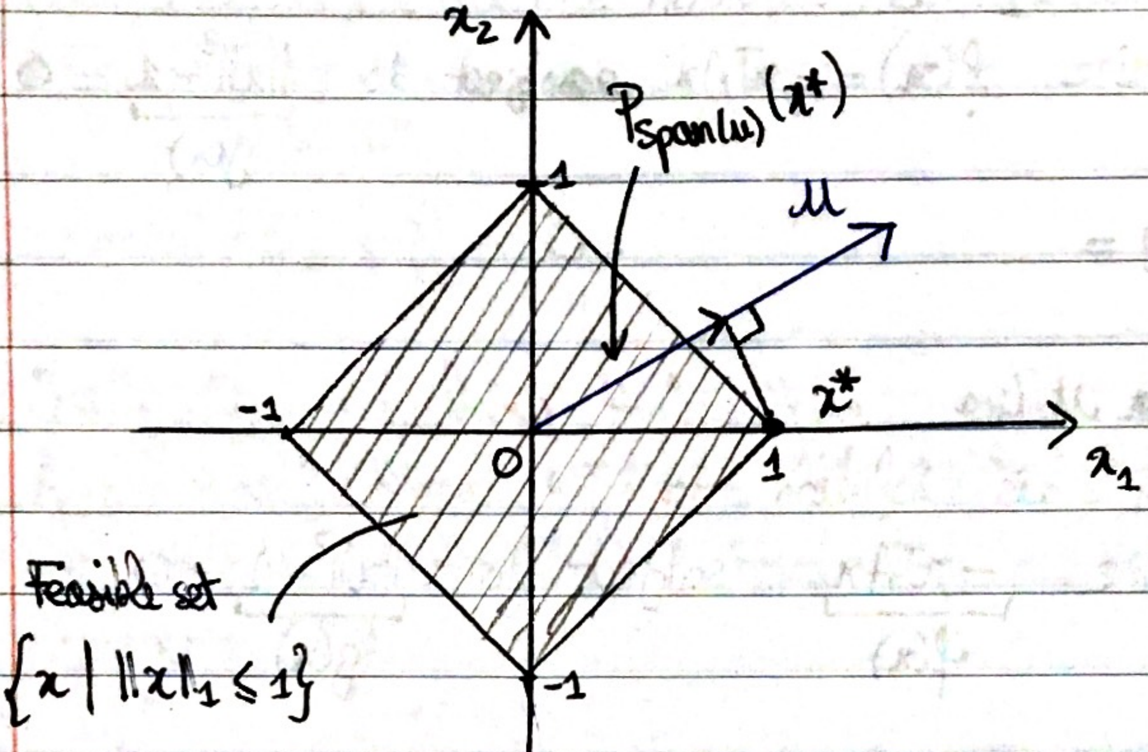
Let us show that x^* is the unique solution.
Let $x \neq x^*$ such that $\|x\|_1 = 1$. In that case, there exists $j \neq i^*$ such that $x_j \neq 0$

$$\langle x, u\rangle \leq \sum_{i=1}^{n} |x_i| \, |u_i| < \sum_{i=1}^{n} |x_i| \, |u_{i^*}| = |u_{i^*}|$$

(because $|x_j| \, |u_j| < |x_j| \, |u_{i^*}|$ because $|x_j| \neq 0$)

Conclusion: x^* is the unique solution

b) We illustrate the problem for $n=2$:



When maximizing $\langle u, x\rangle$ subject to $\|x\|_1 \leq 1$, we are looking for the vector in $\{x \mid \|x\|_1 \leq 1\}$ ~~that has the largest correlation with u~~ for which $\langle x, u\rangle = \langle P_{span(u)}(x), u\rangle$ is maximal.
On the picture we see that this it achieved at a point x^* whose coordinates are all 0 except 1, as proved in a).

Problem 11.4

a) v_1 is solution to:

minimize $\underbrace{-x^T A x}_{f(x)}$ subject to $\underbrace{\|x\|^2 - 1}_{g(x)} = 0$

$\nabla g(v_1) = 2 v_1 \neq 0$ since $\|v_1\| = 1$.

Hence there exists $\lambda \in \mathbb{R}$ such that

$\nabla f(v_1) + \lambda \nabla g(v_1) = 0$,

$-2Av_1 + 2\lambda v_1 = 0$

Hence $Av_1 = \lambda v_1$: v_1 is an eigenvector of A.

b) v_2 is solution of:

minimize $\underbrace{-x^T A x}_{f(x)}$ subject to $\underbrace{\|x\|^2 - 1}_{g(x)} = 0$ and $\underbrace{\langle x, v_1 \rangle}_{h(x)} = 0$

$\nabla g(v_2) = 2v_2$ and $\nabla h(v_2) = v_1$ are linearly independent because $v_1 \perp v_2$ and $\|v_1\| = \|v_2\| = 1$.

Hence there exists $\lambda, \lambda' \in \mathbb{R}$ such that

$\nabla f(v_2) + \lambda \nabla g(v_2) + \lambda' \nabla h(v_2) = 0$.

Hence $-2Av_2 + 2v_2 \cdot \lambda + \lambda' v_1 = 0$

Taking inner product with v_1 on both sides and using that $\left| \begin{array}{l} \|v_1\| = 1 \\ \langle v_1, v_2 \rangle = 0 \end{array} \right.$ we get:

$$-2 v_1^T A v_2 + \lambda' = 0.$$

Using ⓐ, we know that there exists $\alpha \in \mathbb{R}$ such that $Av_1 = \alpha v_1$. Hence $v_1^T A v_2 = \langle v_2, Av_1 \rangle = \langle v_2, \alpha v_1 \rangle = 0$. We conclude that $\lambda' = 0$, hence $Av_2 = \lambda v_2$: v_2 is an eigenvector of A.

c) Using the same arguments as in b) we get that there exists $\lambda, \lambda', \lambda'' \in \mathbb{R}$ such that

$$-2Av_3 + 2\lambda v_3 + \lambda' v_1 + \lambda'' v_2 = 0$$

Taking the inner product with v_1, we get

$$\lambda' = 2 v_1^T A v_3 = 2 \langle Av_1, v_3 \rangle = 0.$$

Similarly, taking now the inner product with v_2 we get

$$\lambda'' = 2 v_2^T A v_3 = 0.$$

Hence $Av_3 = \lambda v_3$: v_3 is an eigenvector of A.

Problem 11.5.

The function $V(\omega)$ is convex because Σ is positive semi-definite and the constraints of (5) are affine: (5) is a convex constrained problem.

We would like to apply the Kuhn-Tucker theorem. Hence, we need to show that there exists a strictly feasible points. Let $\begin{cases} i \text{ such that } r_i = \max\limits_{k=1\dots n} r_k \\ j \text{ such that } r_j = \min\limits_{k=1\dots n} r_k \end{cases}$

$r_i \neq r_j$ because $r \notin \mathrm{Span}(\mathbb{1})$.

Let $\omega = m\left(t r_i + (1-t) r_j\right)$ with $t = \dfrac{\mu - r_j}{r_i - r_j}$

One can verify easily that ω is (strictly) feasible.

We can then apply the Kuhn-Tucker theorem to get that

ω solution of (5) $\Longleftrightarrow \begin{cases} \omega \text{ feasible} \\ \text{there exists } \lambda_1, \lambda_2 \in \mathbb{R} \text{ such that} \\ 2\Sigma\omega + \lambda_1 \mathbb{1} + \lambda_2 r = 0 \quad (*) \end{cases}$

Let's solve $(*)$ for ω feasible, $\lambda_1, \lambda_2 \in \mathbb{R}$: assume that ω ~~feasi~~, λ_1, λ_2 verify $(*)$ and that ω is feasible.
Then ~~to~~ we get:

$$\begin{cases} -2\omega = +\lambda_1 \Sigma^{-1}\mathbb{1} + \lambda_2 \Sigma^{-1} r \quad (**) \\ (\Sigma \text{ is invertible because positive definite}) \end{cases}$$

Taking the inner-product with $\mathbb{1}$ on both sides:

$$-2m = \lambda_1 \mathbb{1}^T \Sigma^{-1} \mathbb{1} + \lambda_2 \mathbb{1}^T \Sigma^{-1} r$$

Same with r:

$$-2\mu = \lambda_1 r^T \Sigma^{-1} \mathbb{1} + \lambda_2 r^T \Sigma^{-1} r$$

Hence we get that: $M . \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = -2 \begin{pmatrix} m \\ \mu \end{pmatrix}$

Assume for a moment that M is invertible (we will prove it at the end): we get:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = -2M^{-1} \begin{pmatrix} m \\ \mu \end{pmatrix}$$

Using (**) we conclude: $w = \Sigma^{-1} \begin{pmatrix} \vdots & \vdots \\ \mathbb{1} & r \\ \vdots & \vdots \end{pmatrix} M^{-1} \begin{pmatrix} m \\ \mu \end{pmatrix}$

Hence if w is feasible and if λ_1, λ_2, w satisfy (*), then:

$$\left| \begin{array}{l} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = -2M^{-1} \begin{pmatrix} m \\ \mu \end{pmatrix} \\ w = \Sigma^{-1} \begin{pmatrix} \vdots & \vdots \\ \mathbb{1} & r \\ \vdots & \vdots \end{pmatrix} M^{-1} \begin{pmatrix} m \\ \mu \end{pmatrix} . \end{array} \right] \quad (***)$$

It is easy ~~the~~ to verify the converse, that is that if $\omega, \lambda_1, \lambda_2$ are given by $(***)$ then they verify $(*)$ and ω is feasible.

We conclude that ω is optimal if and only if $\omega = \omega^* \overset{\text{def}}{=} \Sigma^{-1} \begin{pmatrix} \mathbb{1} & r \end{pmatrix} M^{-1} \begin{pmatrix} m \\ p \end{pmatrix}$.

ω^* is the unique solution of (S).

It remains to prove that M is invertible by contradiction
~~M is symmetric~~
Assume that there exists $(a,b) \neq (0,0)$ such that $M \begin{pmatrix} a \\ b \end{pmatrix} = 0$.
This implies $\begin{pmatrix} a \\ b \end{pmatrix}^T M \begin{pmatrix} a \\ b \end{pmatrix} = 0$, hence:

$$\underbrace{a^2 \mathbb{1}^T \Sigma^{-1} \mathbb{1} + 2ab\, \mathbb{1}^T \Sigma^{-1} r + b^2 r^T \Sigma^{-1} r}_{} = 0$$

$$(a\mathbb{1} + br)^T \Sigma^{-1} (a\mathbb{1} + br) = 0$$

Σ is assumed to be positive definite, hence Σ^{-1} is also positive definite. This implies that $a\mathbb{1} + br = 0$

which then give that $a = b = 0$ because $\mathbb{1} \neq 0$ and $r \notin \mathrm{Span}(\mathbb{1})$
This is a contradiction. We conclude that M is invertible