DS-GA 1014: Final

Optimization and Computational Linear Algebra for Data Science (NYU, Fall 2018)

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The exam ends at 11:50. Please justify your answers, proving the statements you make.

If you cite a result from the lecture, lab, or homework, you must state the entire result (with its hypotheses and conclusions) to receive credit.

Make sure to cross out all statements or arguments that are not part of your final solution. Points can be deducted for extraneous statements or arguments that are not crossed out.

This exam is open book/notes. You are allowed to consult the notes and books you bring, but not allowed to use electronic devices.

If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n to be 2, for example), state explicitly that you have done so. Solutions where extra conditions were assumed, or where only special cases where treated, will also be graded (probably scored as a partial answer).

The exam has 4 pages (one of which is blank). It has 8 question groups that together total 100 points plus extra credit. Extra credit points will be added to your total score. If you have questions (or find a typo) let me know. Any eventual typo will be announced on the board.

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1. (10 pts) Suppose you have found two distinct solutions $u, v \in \mathbb{R}^n$ to the linear system $Ax = b$ where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$.

   (a) Prove that the system $Ax = b$ has infinitely many distinct solutions.

   (b) Give an explicit eigenvector $w \neq 0$ for A with corresponding eigenvalue 0, and justify your answer.

2. (18 pts) Let $A \in \mathbb{R}^{m\times n}$, let $b \in \mathbb{R}^m$, and assume that A^TA is diagonal with non-zero diagonal entries $d_1, \ldots, d_n$.

   (a) Prove that $m \geq n$.

   (b) Prove that $Ax = b$ has at most one solution $x \in \mathbb{R}^n$.

   (c) Compute $\|A(e_1 + \cdots + e_n)\|$ in terms of $d_1, \ldots, d_n$ and justify your answer. Here $e_i \in \mathbb{R}^n$ denotes the ith standard basis vector that has 1 in the ith position and 0 elsewhere.

3. (20 pts) We are interested in the solution x^* of the least squares problem

$$\min_{x\in\mathbb{R}^n} \|y - Ux\|,$$

   where $y \in \mathbb{R}^m$, $U \in \mathbb{R}^{m\times n}$, $m > n$, and $U^TU = I$ (i.e., the columns of U are orthonormal).

   (a) Compute $\text{Rank}(U)$ and justify your answer.

   (b) Compute $\dim(\ker(U))$ and justify your answer.

   (c) Suppose y is orthogonal to $\text{Im}(U)$ (i.e., $y \in \text{Im}(U)^\perp$). Determine x^*, the solution to the least squares problem, and justify your answer. You can assume the solution x^* is unique.

   (d) Let u_i denote the ith column of U for $i = 1, \ldots, n$. Give a formula for x_i^*, the ith entry of the least squares solution, in terms of the columns of U and justify your answer. You can assume the solution x^* is unique. [This gives a method for parallelizing the solution to the orthogonal least squares problem when m is very large.]

4. (20 pts) Let $A \in \mathbb{R}^{n\times n}$ with $n \geq 1$. For each of the following sets $S \subseteq \mathbb{R}^n$, determine if it is always a convex set, or not necessarily convex. In each case justify your answer by either proving the set S is always convex, or showing it is not convex for some choice of A and some $n \geq 1$.

   (a) $S = \{x \in \mathbb{R}^n : \|Ax\| = 0\}$

   (b) $S = \{x \in \mathbb{R}^n : \|Ax\| \leq 1\}$

   (c) $S = \{x \in \mathbb{R}^n : \|Ax\| \geq 1\}$

   (d) $S = \{x \in \mathbb{R}^{2n} : \sum_{k=1}^{n} x_k^2 \leq \sum_{k=n+1}^{2n} x_k^2\}$

   (e) **(Extra Credit)** $S = \{x \in \mathbb{R}^n :\ A - xx^T \succeq 0\}$ where $A \in \mathbb{R}^{n\times n}$ is symmetric.

5. (12 pts) Let $A \in \mathbb{R}^{m \times n}$ and let $a_i \in \mathbb{R}^m$ denote the ith column of A. Let $B \in \mathbb{R}^{m \times n}$ have the same columns as A, but in a different order. For example, if $n = 4$ we may have

$$B = \begin{bmatrix} | & | & | & | \\ a_3 & a_1 & a_4 & a_2 \\ | & | & | & | \end{bmatrix}.$$

Determine **all** of the following statements that must hold for any such A and B (**no justification required**).

(a) $\text{Im}(A) = \text{Im}(B)$

(b) $\text{Im}(A^T) = \text{Im}(B^T)$

(c) $\ker(A) = \ker(B)$

(d) $\ker(A^T) = \ker(B^T)$

(e) $\|A\|_F = \|B\|_F$

(f) $\text{Rank}(A) = \text{Rank}(B)$

6. (20 pts) Let $A, B \in \mathbb{R}^{2 \times 2}$ be given by

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Define $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = x^T Ax + x_1 - x_2$ and $g(x) = x^T Bx + 3x_1 + x_2$, where x_1, x_2 are the entries of x.

(a) Determine whether f is convex and justify your answer.

(b) Determine whether g is convex and justify your answer.

(c) Find any critical points of f (points x where $\nabla f(x) = 0$), and determine if they are local maxima, local minima, or saddle points. Justify your answer.

(d) Find any critical points of g (points x where $\nabla g(x) = 0$), and determine if they are local maxima, local minima, or saddle points. Justify your answer.

7. **(Extra credit)** (10 pts) Let $v_1, ..., v_n$ and $u_1, ..., u_n$ be orthonormal bases of $\mathbb{R}^n$.

(a) Prove that $\max_{i,j} |v_i^T u_j| \geq \frac{1}{\sqrt{n}}$.

(b) The above orthonormal bases are called mutually unbiased if $\max_{i,j} |v_i^T u_j| = \frac{1}{\sqrt{n}}$. Give an explicit example of u_1, u_2 and v_1, v_2 that are mutually unbiased (where $n = 2$).

8. **(Extra credit)** (10 pts) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Prove that if there exists a $C \in \mathbb{R}$ such that $f(x) \leq C$ for all $x \in \mathbb{R}^n$ then f is a constant function (i.e., that $f(x) = a$ for some $a \in \mathbb{R}$ and all $x \in \mathbb{R}^n$).