Problem 6.1 A is symmetric so we know that by the spectral theorem there exists an orthogonal matrix P such that

$$A = P\,\mathrm{Diag}(\lambda_1 \dots \lambda_n) P^T$$

where $\lambda_1, \dots \lambda_n$ are the eigenvalues of A.

A is orthogonal $\iff AA^T = Id_n \iff P\,\mathrm{Diag}(\lambda_1 \dots \lambda_n)P^TP\,\mathrm{Diag}(\lambda_1 \dots \lambda_n)P^T = Id_n$

$$\iff \mathrm{Diag}(\lambda_1^2, \dots, \lambda_n^2) = Id_n$$

$$\iff |\lambda_i| = 1 \text{ for all } i.$$

Problem 6.2

a) Let $x \in \mathbb{R}^n$. $x^TAA^Tx = (A^Tx)^TA^Tx = \|A^Tx\|^2 \geq 0$.
Hence AA^T is positive semidefinite.

b) Assume that M is positive semidefinite. Let λ be an eigenvalue of M and $x \in \mathbb{R}^n$ be an associated eigenvector: $Mx = \lambda x$.
Compute $x^TMx = \lambda x^Tx = \lambda \|x\|^2$. $x \neq 0$ so we get

$$\lambda = \frac{x^TMx}{\|x\|^2} \geq 0.$$

Conversely, assume that all the eigenvalues of M are non-negative. By the spectral theorem, there exists a orthonormal basis $(v_1, \dots v_n)$ of $\mathbb{R}^n$ such that all the v_i are eigenvectors of M: $Mv_i = \lambda_i v_i$, for some $\lambda_i \in \mathbb{R}$.

By assumption we get $\lambda_i \geq 0$ for all i. Let $x \in \mathbb{R}^n$ and let $(\alpha_1 \dots \alpha_n)$ be the coordinates of x in the basis $(v_1 \dots v_n)$

$x = \alpha_1 v_1 + \dots + \alpha_n v_n$.

Multiplying by M gives: $Mx = \alpha_1 M v_1 + \dots + \alpha_n M v_n$

$$= \alpha_1 \lambda_1 v_1 + \dots + \alpha_n \lambda_n v_n$$

Hence $(\alpha_1\lambda_1, \dots \alpha_n\lambda_n)$ are the coordinates of Mx in the orthonormal basis $(v_1, \dots v_n)$:

$$x^T M x = \langle x, Mx \rangle = \sum_{i=1}^{n} \alpha_i \times \alpha_i \lambda_i$$

$$= \sum_{i=1}^{n} \alpha_i^2 \lambda_i \geq 0$$

M is positive semidefinite.

c) Let M be a symmetric positive semidefinite matrix. By the spectral theorem we know that

$$M = P \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix} P^T$$ for some orthogonal

matrix P, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are the eigenvalues of M. (We can assume that $\lambda_1 \dots \lambda_n$ are ordered in that way, otherwise it suffices to permute the columns of P).

From (b) we know that $\lambda_1 \dots \lambda_n \geq 0$. Since M is diagonalizable, we know also that

$$r = \text{rank}(M) = \#\{i \mid \lambda_i \neq 0\}$$

We get that $\begin{cases} \lambda_1 \geq \lambda_2 \dots \geq \lambda_r > 0 \\ \lambda_{r+1} = \dots = \lambda_n = 0. \end{cases}$

Hence $M = P \begin{pmatrix} \lambda_1 & & & & (0) \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ (0) & & & & \ddots \; 0 \end{pmatrix} P^T$.

Define $B = \begin{pmatrix} \sqrt{\lambda_1} & & (0) \\ (0) & \ddots & \sqrt{\lambda_r} \\ & (0) & \end{pmatrix} \in \mathbb{R}^{n \times r}$ we get

$M = PBB^TP^T = PB(PB)^T = AA^T$ where $A = PB \in \mathbb{R}^{n \times r}$.

Problem 6.4. For all $x \in \mathbb{R}^n$, $x^TAx = (x^TAx)^T = x^TA^Tx$.

Hence $x^TAx = x^T\underbrace{\left(\frac{A+A^T}{2}\right)}_{M}x = x^TMx$. $(*)$

M is symmetric.

Since $x^TAx \geq 0$ for all x, we get that M is positive semi-definite.

By probleme 6.2, there exists $B \in \mathbb{R}^{n \times \text{rank}(M)}$ such that $M = BB^T$

Let $x \in \text{Ker}(A)$. From $(*)$ we get $x^TMx = 0$ so $x^TBB^Tx = 0$ hence $B^Tx = 0$, $BB^Tx = 0$ so $Mx = 0$.

So

$$0 = Mx = \frac{1}{2}(\underbrace{Ax}_{=0} + A^Tx)$$, which gives that $A^Tx = 0$: $x \in \text{Ker}(A^T)$

We get that $\text{Ker}(A) \subset \text{Ker}(A^T)$. The inclusion $\text{Ker}(A^T) \subset \text{Ker}(A)$ follows by applying the result to A^T which verifies $x^TAx \geq 0$ for all $x \in \mathbb{R}^n$.