Lecture 4.1: Norms

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

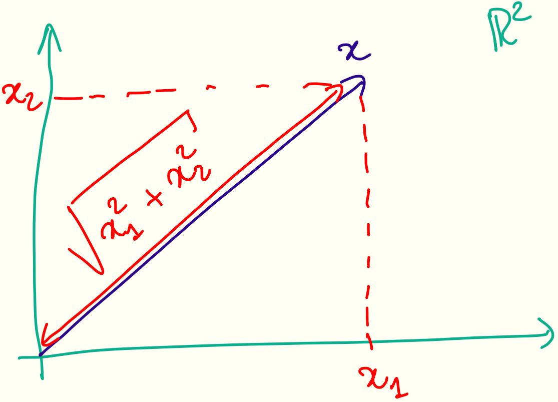
Introduction: the Euclidean norm

Definition

We define the Euclidean norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as:

ℓ_2 - norm

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}.$$

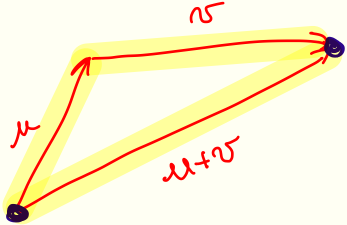


Introduction: the Euclidean norm

Definition

We define the Euclidean norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as:

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}.$$

- For all $\alpha \in \mathbb{R}$, $\|\alpha x\|_2 = |\alpha| \, \|x\|_2$
- If $\|x\|_2 = 0$, then $\underline{x = 0}$
- 

$\|u+v\|_2 \leq \|u\|_2 + \|v\|_2$

General norms

Let V be a vector space.

$\{x \in \mathbb{R} \mid x \geq 0\}$

Definition

A norm $\|\cdot\|$ on V is a function from V to $\mathbb{R}_{\geq 0}$ that verifies:

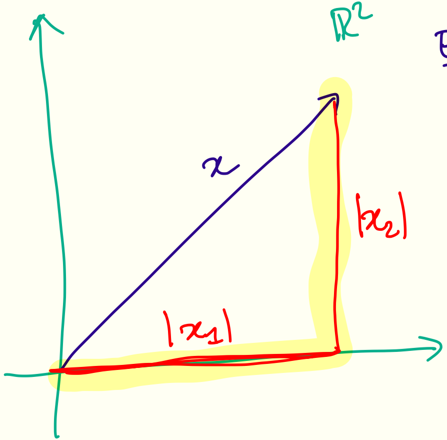
1. *Homogeneity*: $\|\alpha v\| = |\alpha| \times \|v\|$ for all $\alpha \in \mathbb{R}$ and $v \in V$.
2. *Positive definiteness*: if $\|v\| = 0$ for some $v \in V$, then $v = 0$.
3. *Triangular inequality*: $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

Ex: For $V = \mathbb{R}^n$, the Euclidean norm $\|\cdot\|_2$ is a norm on V.

Other examples

- The ℓ_1 norm

$$\|x\|_1 \stackrel{\text{def}}{=} \sum_{i=1}^{n} |x_i| = |x_1| + \cdots + |x_n|.$$



Ex: check that $\|\cdot\|_1$ verifies the 3 points of the definition.

Other examples

- The infinity-norm

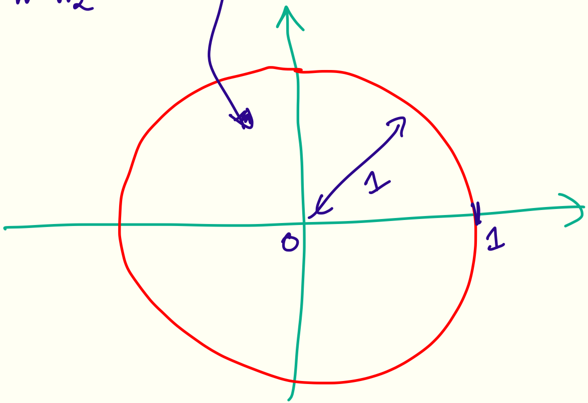
$$\|x\|_\infty \stackrel{\text{def}}{=} \max(|x_1|, \ldots, |x_n|).$$

Exercise: Balls drawing

For each of the norms $\|\cdot\|_2$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$, draw the «ball»:

$$B = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}.$$

- $\|\cdot\|_2$



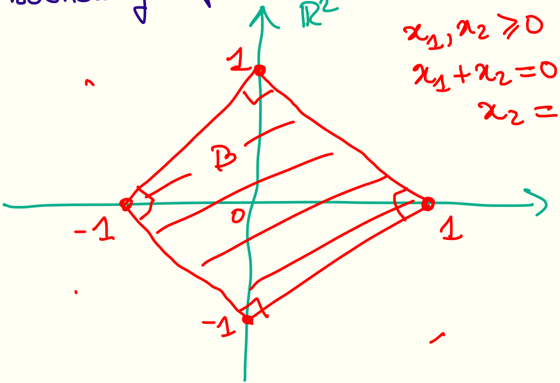
Exercise: Balls drawing

For each of the norms $\|\cdot\|_2$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$, draw the «ball»:

$$B = \{x \in \mathbb{R}^2 \mid \|x\| \underset{=}{\leq} 1\}.$$

• $\|\cdot\|_1$.

We are going to investigate the "shape" of the boundary of B: the x, such that $|x_1| + |x_2| = 1$



$x_1, x_2 \geqslant 0$

$x_1 + x_2 = 0$

$x_2 = 1 - x_1$

Exercise: Balls drawing

For each of the norms $\|\cdot\|_2$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$, draw the «ball»:

$$B = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}.$$

$$= \left\{ (x_1, x_2) \;\middle|\; \begin{array}{l} |x_1| \leq 1 \\ |x_2| \leq 1 \end{array} \right\}$$