Lecture 2.2: Matrices

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

The key observation

- Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.
- Let $(e_1, \dots, e_m)$ be the canonical basis of $\mathbb{R}^m$.

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ pos.}$$

Then, for all $x = (x_1, \dots, x_m) \in \mathbb{R}^m$:

$$L(x) = L\Big(\sum_{i=1}^m x_i e_i\Big) = \sum_{i=1}^m x_i L(e_i).$$

L is a linear map

The key observation

- Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.
- Let $(e_1, \ldots, e_m)$ be the canonical basis of $\mathbb{R}^m$.

Then, for all $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$:

$$L(x) = L\Big(\sum_{i=1}^{m} x_i e_i\Big) = \sum_{i=1}^{m} x_i L(e_i).$$

Conclusion: if you give me the vectors $L(e_1), \ldots, L(e_m) \in \mathbb{R}^n$ then, I am able to compute $L(x)$ for any $x \in \mathbb{R}^m$.

« One needs $n \times m$ numbers to store the linear map L on a computer »

Matrices

Definition

A $n \times m$ matrix is an array (of real numbers) with n rows and m columns. We denote by $\mathbb{R}^{n\times m}$ the set of all $n \times m$ matrices.

m col.

n rows

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix}$$

row col.

Canonical matrix of a linear map

We can encode a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ by a $n \times m$ matrix.

Definition

The canonical matrix of L is the $n \times m$ matrix (that we will write also L) whose columns are $L(e_1), \dots, L(e_m)$:

$$L = \underline{\begin{pmatrix} | & | & & | \\ L(e_1) & L(e_2) & \cdots & L(e_m) \\ | & | & & | \end{pmatrix}} = \begin{pmatrix} L_{1,1} & L_{1,2} & \cdots & L_{1,m} \\ L_{2,1} & L_{2,2} & \cdots & L_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n,1} & L_{n,2} & \cdots & L_{n,m} \end{pmatrix}$$

where we write $L(e_j) = \begin{pmatrix} L_{1,j} \\ L_{2,j} \\ \vdots \\ L_{n,j} \end{pmatrix}$.

jth column

Example #1: identity matrix

The Identity map $\begin{array}{rccc} \mathrm{Id} : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & x \end{array}$ is linear.

Exercise: what is the canonical matrix of Id ?

$\mathrm{Id}(e_i) = e_i$

Hence :

$$\mathrm{Id} = \begin{pmatrix} | & & | \\ \mathrm{Id}(e_1) & \cdots & \mathrm{Id}(e_n) \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}}$$

Example #2: Homothety

Let $\lambda \in \mathbb{R}$. The homothety map of ratio λ:

$$H_\lambda : \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & \lambda x \end{array}$$

is linear.

Exercise: what is the canonical matrix of H_λ ?

$$H_\lambda(e_i) = \lambda e_i$$

$$H_\lambda = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda \end{pmatrix}$$

Example #3: rotations in $\mathbb{R}^2$

Let $\theta \in \mathbb{R}$. The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ of angle θ about the origin is linear.

Exercise: what is the canonical matrix of R_θ ?