

# Lecture 2.2: Matrices

Optimization and Computational Linear Algebra for Data Science

# The key observation

- Let  $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a linear transformation.
- Let  $(e_1, \dots, e_m)$  be the canonical basis of  $\mathbb{R}^m$ .

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th pos.}$$

Then, for all  $\underline{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ :

$$\underline{L(x)} = L\left(\sum_{i=1}^m x_i e_i\right) = \sum_{i=1}^m x_i L(e_i).$$

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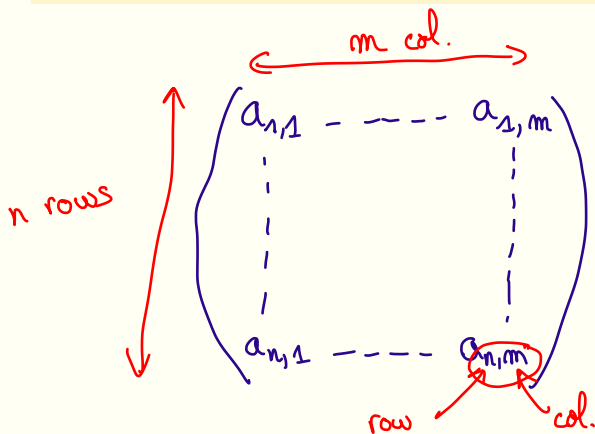
**Conclusion:** if you give me the vectors  $L(e_1), \dots, L(e_m) \in \mathbb{R}^n$  then, I am able to compute  $L(x)$  for any  $x \in \mathbb{R}^m$ .

« One needs  $n \times m$  numbers to store the linear map  $L$  on a computer »

# Matrices

## Definition

A  $n \times m$  matrix is an array (of real numbers) with  $n$  rows and  $m$  columns. We denote by  $\mathbb{R}^{n \times m}$  the set of all  $n \times m$  matrices.



# Canonical matrix of a linear map

We can encode a linear map  $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$  by a  $n \times m$  matrix.

## Definition

The canonical matrix of  $L$  is the  $n \times m$  matrix (that we will write also  $L$ ) whose columns are  $L(e_1), \dots, L(e_m)$ :

$$L = \left( \begin{array}{c|c|c} L(e_1) & L(e_2) & \cdots & L(e_m) \end{array} \right) = \begin{pmatrix} L_{1,1} & L_{1,2} & \cdots & L_{1,m} \\ L_{2,1} & L_{2,2} & \cdots & L_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n,1} & L_{n,2} & \cdots & L_{n,m} \end{pmatrix}$$

where we write  $L(e_j) = \begin{pmatrix} L_{1,j} \\ L_{2,j} \\ \vdots \\ L_{n,j} \end{pmatrix}$  *j<sup>th</sup> column*

# Example #1: identity matrix

The Identity map  $\text{Id} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is linear.  
 $\underline{x} \mapsto \underline{x}$

**Exercise:** what is the canonical matrix of Id ?

$$\text{Id}(e_i) = e_i$$

Hence :

$$\begin{aligned} \text{Id} &= \left( \begin{array}{c|c|c} | & & | \\ \text{Id}(e_1) & \dots & \text{Id}(e_n) \\ | & & | \end{array} \right) \\ &= \left( \begin{array}{c|c|c} | & & | \\ e_1 & \dots & e_n \\ | & & | \end{array} \right) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & \dots & 0 & 1 \end{pmatrix} \end{aligned}$$

# Example #2: Homothety

Let  $\lambda \in \mathbb{R}$ . The homothety map of ratio  $\lambda$ :

$$H_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$$
$$\begin{array}{ccc} \mathbb{R}^n & \rightarrow & \mathbb{R}^n \\ x & \mapsto & \lambda x \end{array}$$

is linear.

**Exercise:** what is the canonical matrix of  $H_\lambda$ ?

$$H_\lambda(e_i) = \lambda e_i$$

$$H_\lambda = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda \end{pmatrix}$$

## Example #3: rotations in $\mathbb{R}^2$

Let  $\theta \in \mathbb{R}$ . The rotation  $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of angle  $\theta$  about the origin is linear.

**Exercise:** what is the canonical matrix of  $R_\theta$ ?