Recitation 1

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Concept Review: Vector Spaces

Definition

A **vector space** is a set V endowed with two 'nice and compatible' operations $+$ and $\cdot$ that verify:

- For all $u, v \in V$, $u + v \in V$.
- For all $u \in V$ and all $\lambda \in \mathbb{R}$, $\lambda \cdot u \in V$.

Example: $V = \mathbb{R}^n$, with the usual vector addition $+$ and scalar multiplication $\cdot$ is a vector space.

Concept Review: Vector Spaces

In this class:

- We will always consider *real* scalars. Note that it is also possible to consider *complex* scalars.
- V is (usually) $\mathbb{R}^n$, or (sometimes) $\mathbb{R}^{n \times m}$ (set of $n \times m$ matrices).

Concept Review: Subspaces

Definition (Subspace)

A non-empty subset S of a vector space V is called a *subspace* if it is closed under addition and scalar multiplication:

1. Closure under Addition: $x + y \in S$ for all $x, y \in S$.
2. Closure under Scalar Multiplication: $\alpha x \in S$, for all $x \in S$ and $\alpha \in \mathbb{R}$.

- A subspace is also a vector space!
- A subspace always contains the zero vector.

Indeed if S is a sobspace, then S contains some vector. x

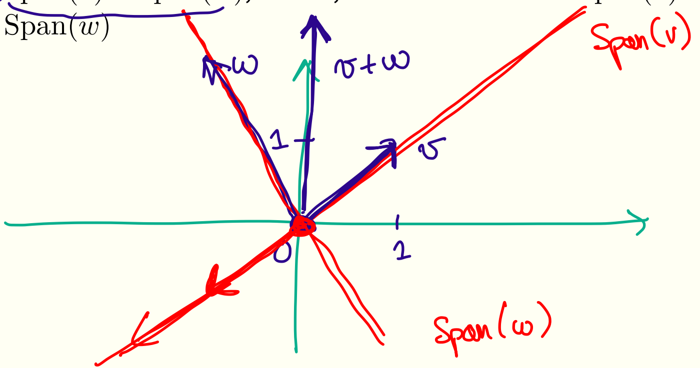
Then $\underbrace{0 \cdot x}_{=0} \in S$.

Questions 1: Subspaces, Span

Consider the two vectors $v = (1,1)$ and $w = (-1,2)$. Describe the following sets geometrically. Which are subspaces of $\mathbb{R}^2$?

1. $\mathrm{Span}(v)$
2. $\mathrm{Span}(v, w) = \mathbb{R}^2$
3. $\boxed{\mathrm{Span}(v) \cup \mathrm{Span}(w)}$, that is, the vectors in $\mathrm{Span}(v)$ or $\mathrm{Span}(w)$
4. $\mathrm{Span}(v) \cap \mathrm{Span}(w)$, that is, the vectors in both $\mathrm{Span}(v)$ and $\mathrm{Span}(w)$

$\{0\}$



Questions 1: Subspaces, Span

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4. $\mathrm{Span}(v) \cap \mathrm{Span}(w)$, that is, the vectors in both $\mathrm{Span}(v)$ and $\mathrm{Span}(w)$
5. $\{(1-t)v + tw | t \in [0,1]\}$
6. $\{(1-t)v + tw | t \in \mathbb{R}\}$.
7. $\{\alpha v + \beta w | \alpha, \beta \geq 0\}$
8. $\mathrm{Span}(v, w, u)$ where $u = (0,5)$.
9. $\{(a,b) \in \mathbb{R}^2 | a^2 + b^2 \leq 25\}$
10. $\{(a,a) \in \mathbb{R}^2 | a \in \mathbb{R}\}$

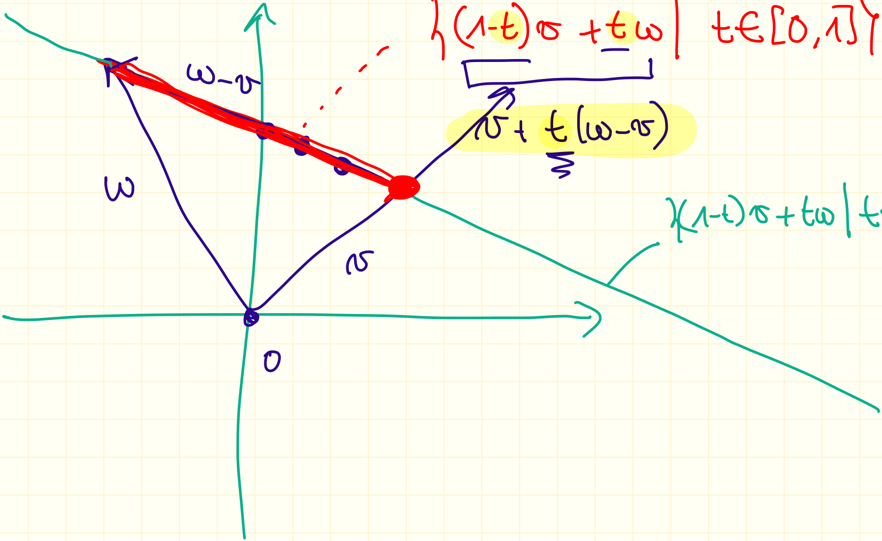
$\{\underbrace{(1-t)v+tw}\ |\ t\in[0,1]\}$

the set of " elements of the form

"such that"

this is verified.

Solution

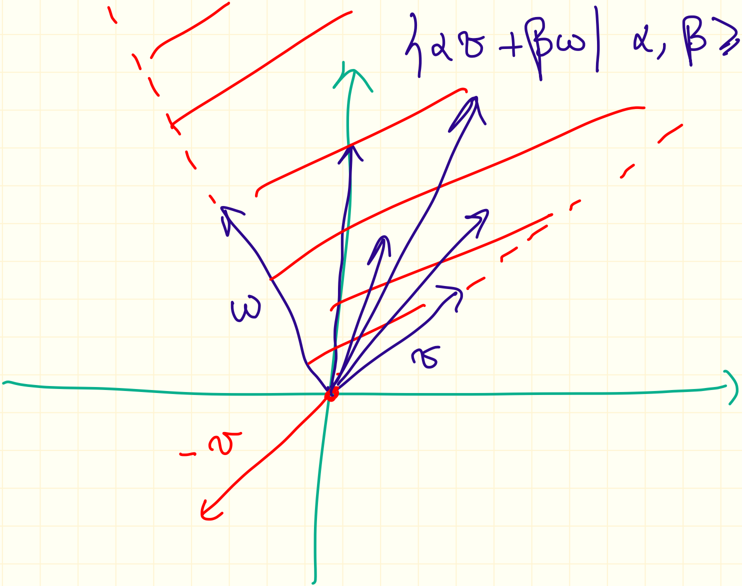


$\{\underbrace{(1-t)v + tw}_{v + t(w-v)} \mid t \in [0,1]\}$

$\{(1-t)v + tw \mid t \in \mathbb{R}\}$

Solution

$\{\alpha v + \beta w \mid \alpha, \beta \geq 0\}$

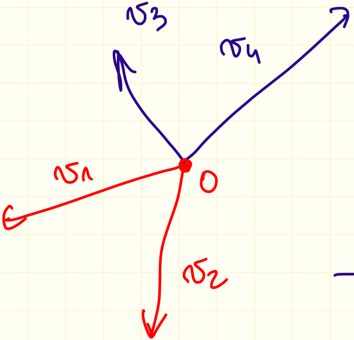


Solution

Solution

Questions 2: Linear Independence

1. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$. Let $C_1 = \{v_1, v_2\}$; $C_2 = \{v_3, v_4\}$. If C_1 and C_2 are both linearly independent, what are the possible values for $\dim(\operatorname{Span}(v_1, v_2, v_3, v_4))$? (No formal proof necessary)



- it can equal to 3 if for instance

$$v_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = e_2, \quad v_3 = e_3, \quad v_4 = e_1$$

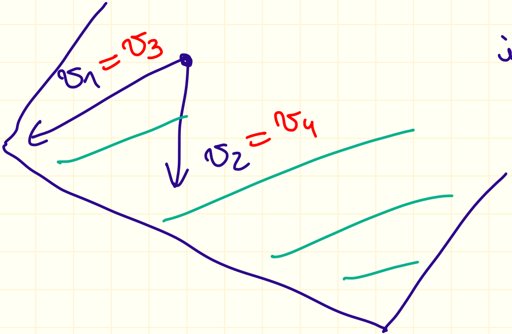
Then $\operatorname{Span}(v_1 \dots v_4) = \mathbb{R}^3$

Questions 2: Linear Independence

1. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$. Let $C_1 = \{v_1, v_2\}$; $C_2 = \{v_3, v_4\}$.
   If C_1 and C_2 are both linearly independent, what are the possible values for $\dim(\text{Span}(v_1, v_2, v_3, v_4))$? (No formal proof necessary)

It is also possible to have dimension 2

if $\begin{cases} v_1 = v_3 = e_1 \\ v_2 = v_4 = e_4 \end{cases}$



Questions 2: Linear Independence

2. Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be linearly dependent.
Prove that for $x \in \text{Span}(v_1, \ldots, v_m)$, there exist infinitely many $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{m} \alpha_i v_i. = \alpha_1 v_1 + \ldots + \alpha_n v_n$$

- Since $x \in \text{Span}(v_1 \ldots v_m)$, by definition there exists $\alpha_1 \ldots \alpha_m$ such that $x = \alpha_1 v_1 + \ldots + \alpha_m v_m$
- $v_1 \ldots v_m$ are lin. dep. hence there exist $\beta_1 \ldots \beta_m$ that are not all zero such that

$$\beta_1 v_1 + \ldots + \beta_m v_m = 0$$

Questions 2: Linear Independence

2. Let $v_1, ..., v_m \in \mathbb{R}^n$ be linearly dependent.
   Prove that for $x \in \mathrm{Span}(v_1, \ldots, v_m)$, there exist infinitely many $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{m} \alpha_i v_i.$$

By suming, we get that for all $r \in \mathbb{R}$

$$\boxed{x + 0 = (\alpha_1 + r\beta_1)v_1 + \ldots + (\alpha_m + r\beta_m)v_m}$$

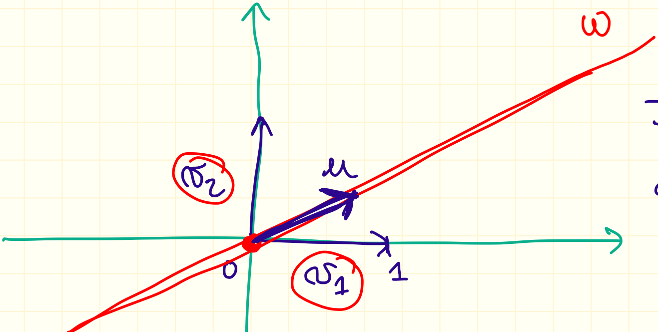
Since I can take any r that I want, I obtain infinitely many ways of decomposing x. (since one of the β_i is not zero)

Questions 2: Linear Independence

3. True or False: If $B = (v_1, \ldots, v_n)$ is a basis for $\mathbb{R}^n$, and W is a subspace of $\mathbb{R}^n$, then some subset of B is a basis for W.

Let see in $\mathbb{R}^2$

v_1 and v_2 do not belong to W, I can not build a basis of W out of them.



$\{0\}$ $\qquad \dim(\{0\}) = 0$

Questions 3: Bases, Dimension

Let V be the set of functions

$= a_0 1 + a_1 x + \dots + a_n x^n$

$$V \stackrel{\text{def}}{=} \left\{ p : \mathbb{R} \to \mathbb{R} \;\middle|\; p(x) = \sum_{k=0}^{n} a_k x^k, \text{ for some } a_0, \dots, a_n \in \mathbb{R} \right\}$$

1. What kind of functions does this set contain?
2. Define an addition operation $+ : V \times V \to V$, and a scalar multiplication operation $\cdot : \mathbb{R} \times V \to V$, such that the triple $(V, +, \cdot)$ is a vector space.
3. What is the zero vector of this vector space?
4. Find a basis for this vector space.
5. What is the dimension of this vector space? $\dim(V) = n+1$

Solution

② Given two polynomials p and q in V

$$p(x) = \sum_{i=0}^{n} a_i x^i \qquad q(x) = \sum_{i=0}^{n} b_i x^i$$

we define: $p+q : \mathbb{R} \longrightarrow \mathbb{R}$

$$x \longmapsto p(x) + q(x) = \sum_{i=0}^{n} (a_i + b_i) x^i$$

similarly, we define

for $\lambda \in \mathbb{R}$, $\lambda \cdot p : \mathbb{R} \longrightarrow \mathbb{R}$

$$x \longmapsto \lambda p(x) = \sum_{i=0}^{n} (\lambda a_i) x^i$$

Solution

(3) What is the zero vector of V ?

$\rightarrow$ this is the function : $p_0 : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto 0$

Indeed:

$(p_0 + q)(x) = \underbrace{p_0(x)}_{=0} + q(x) = q(x)$ for any x

hence $p_0 + q = q$ for any $q \in V$.

Solution

(4) Let's define, for $k \in \{0, 1, \dots, n\}$

$$q_k : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^k \qquad \left(q_k(x) = x^k \right)$$

Claim: $(q_0, q_1, \dots, q_n)$ is a basis of V.

- $\boxed{\mathrm{Span}(q_0, q_1, \dots, q_n) = V}$

Let $p \in V$ of the form $p(x) = \sum_{k=0}^{n} a_k x^k$

Hence $p = a_0 q_0 + a_1 q_1 + \dots + a_n q_n$

Hence $q_0 \dots q_n$ spans V.

Solution

- Let's show that $q_0, \dots, q_n$ are lin. indep.

Let take $\alpha_0 \dots \alpha_n \in \mathbb{R}$ such that

$$\alpha_0 q_0 + \dots + \alpha_n q_n = 0$$

we will show that this implies $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$

this implies that for all $x \in \mathbb{R}$,

$$P(x) \overset{(\text{def})}{=} \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n = 0$$

$P(x)$ is always zero hence all of its coefficients are zero: $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$