Lecture 6.1: What we will not be talking about

Optimization and Computational Linear Algebra for Data Science

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Warning

The determinant

There exists a function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ called *the determinant* that verifies

$$\boxed{\det(M) = 0 \quad \Longleftrightarrow \quad M \quad \text{is not invertible.}}$$

The determinant can be computed using the following formula:

$$\det(M) = \sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma) \underbrace{\prod_{i=1}^{n} M_{i,\sigma(i)}}_{= M_{1,\sigma(1)} M_{2,\sigma(2)} \cdots M_{n,\sigma(n)}}$$

Sum over all "reordering" σ of the numbers $1, 2 \ldots n$

$\{ -1, 1$ depending on σ.

Ex: $n = 4$, $2\,3\,4\,1$ is a reodering of $1, \ldots 4$

$\sigma(1)$ $\sigma(2)$ $\sigma(3)$ $\sigma(4)$

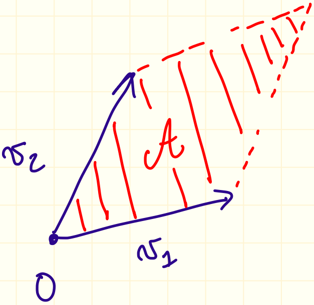
Geometrical interpretation

$n=2$

$$A = \begin{pmatrix} \boxed{\begin{matrix} a \\ c \end{matrix}} & \boxed{\begin{matrix} b \\ d \end{matrix}} \end{pmatrix}$$

$v_1 \quad v_2$

$$\boxed{\det(A) = ad - bc}$$



$$\mathcal{A} = |\det(A)|$$

$$= |ad - bc|$$

$\det(A) = 0$

$\iff \mathcal{A} = 0$

$\iff v_1, v_2$ lin. dep.

$\iff A$ is not invertible

Link with eigenvalues

λ is an eigenvalue of A

$v \in \mathrm{Ker}(A - \lambda \mathrm{Id})$

$\iff$ there exists $v \neq 0$ such that $Av = \lambda v$.

$\iff \mathrm{Ker}(A - \lambda \mathrm{Id}) \neq \{0\}$

$\iff A - \lambda \mathrm{Id}$ is not invertible.

$$\iff \underbrace{\det(A - \lambda \mathrm{Id})}_{\text{function of } \lambda \text{ that we write } \boxed{P_A(\lambda)}} = 0$$

The characteristic polynomial

- $P_A(x)$ is a polynomial in x.

Ex: Let's consider $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$$P_A(x) = \det(A - x\,\mathrm{Id}) = \det\begin{pmatrix} 1-x & 1 \\ 1 & 2-x \end{pmatrix}$$
$$= (1-x)(2-x) - 1 = \boxed{x^2 - 3x + 1}$$

- P_A is called the characteristic polynomial of A

its roots are the eigenvalues of A.

- $\deg(P_A) \leq n$: hence A has at most n distincts eigenvalues.

Example

Let's take $\boxed{A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$ $= R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $\theta = \pi/2$

$$P_A(x) = \det(A - x\,\mathrm{Id})$$

$$= \det\begin{pmatrix} -x & -1 \\ 1 & -x \end{pmatrix} = \boxed{x^2 + 1}$$

- For all $x \in \mathbb{R}$, $P_A(x) = x^2 + 1 \geq 1 > 0$

  Hence A does not have any real eigenvalues.

- $P_A(i) = i^2 + 1 = (-1) + 1 = 0$

i is a complex eigenvalue of A. ($-i$ is the other one)