Homework 1.
Answer key.

Exercise 1.1

Let $u, v \in \mathbb{R}^2$. We distinguish two cases:

- Case 1: u, v are linearly dependent.
  In that case, we are done!
- Case 2: u, v are linearly independent. We will show that they span $\mathbb{R}^2$. To do that, we will show that the vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of the canonical basis belong to $\mathrm{Span}(u, v)$.

Let us write $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. The coordinates u_1 and v_1 can not be both equal to 0, because this would imply that u, v are linearly dependent. Hence, we can assume (by symmetry between u and v) that $u_1 \neq 0$.

We now consider the vector
$$v' = v - \frac{v_1}{u_1} \cdot u = \begin{pmatrix} 0 \\ \underbrace{v_2 - \frac{v_1 u_2}{u_1}}_{\alpha} \end{pmatrix}.$$

$v' \neq 0$ because u, v are lin. indep.

Therefore $\alpha \neq 0$, and consequently:
$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\alpha} v' = \frac{1}{\alpha} v - \frac{v_1}{\alpha u_1} u \in \mathrm{Span}(u, v)$$

By the same reasoning, we can show that $e_1 \in \mathrm{Span}(u, v)$

As a result: $\mathbb{R}^2 = \mathrm{Span}(e_1, e_2) \subset \mathrm{Span}(u, v) \subset \mathbb{R}^2$

because (e_1, e_2) is a basis of $\mathbb{R}^2$

because $e_2, e_1 \in \mathrm{Span}(u, v)$

by definition of the Span.

$\boxed{\text{We conclude that } \mathrm{Span}(u,v) = \mathbb{R}^2}$

Problem 1.2.

a) Yes! • E_1 is indeed non-empty (because $(0,0,0) \in E_1$).

• If we take $v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \in E_1$ and $v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in E_1$

then the vector $v_1 + v_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$ belongs to E_1

because:

$$(x_1 + x_2) - 2(y_1 + y_2) + (z_1 + z_2)$$
$$= \underbrace{x_1 - 2y_1 + z_1}_{=0 \text{ because } v_1 \in E_1} + \underbrace{x_2 - 2y_2 + z_2}_{=0 \text{ because } v_2 \in E_1} = 0$$

• If we take $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_1$ and $\lambda \in \mathbb{R}$

then $\lambda v \in E_1$ because:

$$\lambda x - 2(\lambda y) + \lambda z = \lambda (x - 2y + z) = \lambda . 0 = 0.$$

b) No! E_2 does not contains 0: $0 - 2.0 + 0 = 0 \neq 3$.

c) No! The vector $v = (0, 1, -1)$ belongs to E_3, but not the vector $-v = (0, -1, 1)$.

Problem 1.3.

By contradiction, assume that $x, v_1, \ldots v_n$ are linearly dependent. This means that we can find $\lambda, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$, not all equal to zero, such that:

$$\lambda x + \lambda_1 v_1 + \ldots + \lambda_n v_n = 0.$$

We have necessarily $\lambda = 0$, because otherwise we get

$$x = -\frac{1}{\lambda} (\lambda_1 v_1 + \ldots + \lambda_n v_n) \in \text{Span}(v_1, \ldots v_n).$$

and this contradicts the assumption $x \notin \text{Span}(v_1 \ldots v_n)$.

We thus get (using that $\lambda = 0$): $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$.

Since $\lambda_1, \ldots \lambda_n$ are not all equal to zero, we obtain that $v_1, \dots, v_n$ are linearly dependent, which contradicts the hypotheses of the exercise.

Conclusion : $x, v_1, \dots, v_k$ are linearly independent.

Problem 1.4.

- If $\mathrm{Span}(v_1, \dots v_k) = S$, then we do not need to add any vector to $(v_1, \dots v_k)$: $(v_1, \dots v_k)$ is already a basis of S.

- Otherwise, we can find a vector $v_{k+1} \in S \setminus \mathrm{Span}(v_1 \dots v_k)$. Using the previous problem, $v_1 \dots v_k, v_{k+1}$ are linearly independent:
  - if $\mathrm{Span}(v_1, \dots, v_k, v_{k+1}) = S$ then we are done.
  - otherwise we can repeat the same procedure and find $v_{k+2} \in S$ such that $v_1, \dots v_k, v_{k+1}, v_{k+2}$ are linearly independent.

We iterate this procedure until $\mathrm{Span}(v_1, \dots v_{k+m}) = S$ for some m. (This will happen because otherwise we could construct ~~an~~ family $v_1, \dots v_k, \dots v_{n+1}$ of linearly independent vectors in $S \subset \mathbb{R}^n$ which is absurd because $n+1$ vectors in $\mathbb{R}^n$ have to be linearly dependent).

At the end of the procedure:

- $v_1, \dots v_{k+m}$ are linearly independent.
- $\mathrm{Span}(v_1, \dots, v_{k+m}) = S$.

Conclusion : $(v_1, \dots v_{k+m})$ is a basis of S.

Problem 1.5

By contradiction, let us assume that $U \cap V = \{0\}$.

Let us write $\left| \begin{array}{l} d_U = \dim(U) \\ d_V = \dim(V) \end{array} \right.$ and let $(u_1, \dots u_{d_U})$

and $(v_1, \dots, v_{d_V})$ be basis of (resp.) U and V.

We will show that $v_1, \dots v_{d_V}, u_1, \dots u_{d_U}$ are linearly independent. We do it by contradiction, assuming that there exists $\alpha_1 \dots \alpha_{d_V}, \lambda_1 \dots \lambda_{d_U} \in \mathbb{R}$, not all equal to 0 such that $\alpha_1 v_1 + \dots + \alpha_{d_V} v_{d_V} + \lambda_1 u_1 + \dots + \lambda_{d_U} u_{d_U} = 0$

This gives:

$$x \overset{\text{(definition of } x)}{=} \underbrace{\sum_{i=1}^{d_V} \alpha_i v_i}_{\in V} = \underbrace{\sum_{i=1}^{d_U} (-\lambda_i) u_i}_{\in \operatorname{Span}(u_1, \dots u_{d_U}) = U}$$

Consequently $x \in V$ and $x \in U$: ~~so~~ $x \in U \cap V = \{0\}$ which gives $x = 0$. We get:

$$\sum_{i=1}^{d_V} \alpha_i v_i = 0 \quad \text{and} \quad \sum_{i=1}^{d_U} \lambda_i u_i = 0.$$

We assumed $(v_1 \dots v_{d_V})$ and $(u_1, \dots u_{d_U})$ to be basis (and therefore linearly independent): we get that $\left\{ \begin{array}{l} \lambda_1 = \dots = \lambda_{d_U} = 0 \\ \alpha_1 = \dots = \alpha_{d_V} = 0 \end{array} \right.$

We get a contradiction: <u>$v_1, \dots v_{d_V}, u_1, \dots u_{d_U}$ are lin. indep.</u>

We now remark (using $\dim(U) + \dim(V) > n$) that the family $v_1, \dots v_{d_V}, u_1, \dots u_{d_U}$ has strictly more than n vectors in dimension n: ~~the this family~~ these vectors <u>can not be linearly independent</u>.

We get a contradiction: $U \cap V \neq \{0\}$, hence $U \cap V$ must contain a non-zero vector.