Homework 1

Problem 1.1

a) E_1 is a sobspace of $\mathbb{R}^3$. Indeed, E_1 is non-empty because $(0,0,0) \in E_1$ and:

- if $u \in E_1, v \in E_1$ with $\begin{cases} u = (u_1, u_2, u_3) \\ v = (v_1, v_2, v_3) \end{cases}$ then

$$(u_1+v_1) - 2(u_2+v_2) + (u_3+v_3) = \underbrace{u_1 - 2u_2 + u_3}_{=0} + \underbrace{v_1 - 2v_2 + v_3}_{=0}$$

$= 0$ because $u, v \in E_1$

hence $\underline{u+v \in E_1}$

- if $u \in E_1$ and $\alpha \in \mathbb{R}$, with $u = (u_1, u_2, u_3)$ then

$$(\alpha u_1) - 2(\alpha u_2) + (\alpha u_3) = \alpha(u_1 - 2u_2 + u_3) = 0$$

because $u \in E_1$

hence $\underline{\alpha u \in E_1}$

b) $0 - 20 + 0 \neq 3$ hence $(0,0,0) \notin E_2$: E_2 is not a sobspace of $\mathbb{R}^3$

c) E_3 is not a sobspace of $\mathbb{R}^3$. Indeed: $(0,1,-1) \in E_3$ but $(-1) \cdot (0,1,-1) = (0,-1,1) \notin E_3$.

Problem 1.2 In order to show $\mathrm{Span}(x_1 - x_k) = \mathrm{Span}(x_2 - x_k)$ we show successively:

① $\mathrm{Span}(x_2 - x_k) \subset \mathrm{Span}(x_1 - x_k)$: This is obvious. if we want to give the details:

Let $u \in \mathrm{Span}(x_2 - x_k)$. Then, there exists $\alpha_2 \dots \alpha_k \in \mathbb{R}$ such that

$$u = \alpha_2 x_2 + \dots + \alpha_k x_k$$
$$= 0 \cdot x_1 + \alpha_2 x_2 \dots + \alpha_k x_k \in \mathrm{Span}(x_1 \dots x_k)$$

② $\mathrm{Span}(x_1 - x_k) \subset \mathrm{Span}(x_2 - x_k)$.

Let $u \in \mathrm{Span}(x_1 - x_k)$. There exists $\alpha_1 \dots \alpha_k \in \mathbb{R}$ such that

$$u = \alpha_1 x_1 + \dots + \alpha_k x_k = \underbrace{\alpha_1 x_1}_{\in \mathrm{Span}(x_2 - x_k)} + \underbrace{\alpha_2 x_2 + \dots + \alpha_k x_k}_{\in \mathrm{Span}(x_2 - x_k)}$$

~~Since $x_1 \in \mathrm{Span}$~~

Since $\mathrm{Span}(x_2 - x_k)$ is a subspace of $\mathbb{R}^n$, we get that it is closed under vector addition, hence $u \in \mathrm{Span}(x_2 - x_k)$.

Problem 1.3

Let $\alpha_1, \dots \alpha_{k+1} \in \mathbb{R}$ such that

$$\alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} x = 0$$

We are going to show that $\alpha_1 = \dots = \alpha_{k+1} = 0$, which gives that $(v_1 \dots v_k, x)$ is linearly independent.

We have two cases:

Case 1: $\alpha_{k+1} = 0$. In that case we get $\alpha_1 v_1 + \dots + \alpha_k v_k = 0$.
Since $(v_1 \dots v_k)$ is linearly independent this implies that $\alpha_1 = \dots = \alpha_k = 0$ and we are done.

Case 2: $\alpha_{k+1} \neq 0$. In that case, we get:

$$x = -\frac{\alpha_1}{\alpha_{k+1}} v_1 - \dots - \frac{\alpha_k}{\alpha_{k+1}} v_k \in \text{Span}(v_1 \dots v_k)$$

This is not possible because $x \notin \text{Span}(v_1 \dots v_k)$.

We conclude that only Case 1 happens: $\alpha_1 = \dots = \alpha_{k+1} = 0$
The vectors $v_1 \dots v_k, x$ are linearly independent.

Problem 1.4

a) By contradiction, assume that $(x_1 \dots x_k)$ are not a basis of S. Since they are linearly independent, we have then: $\mathrm{Span}(x_1 \dots x_k) \neq S$.

- Since $x_1 \dots x_k \in S$, $\mathrm{Span}(x_1 \dots x_k) \subset S$: hence there exists $v \in S \setminus \mathrm{Span}(x_1 \dots x_k)$.
- Problem 1.3 gives that $(x_1 \dots x_k, v)$ is linearly independent.
- This is absurd, because $\dim(S) = k$: any family of linearly indep vectors of S contains at most k vectors!

We conclude that $(x_1 \dots x_k)$ is a basis of S.

b) • By contradiction, assume that $x_1 \dots x_k$ are linearly dependent.

- Hence, one of the vectors $x_1 \dots x_k$ belongs to the span of the others. We can assume that this vector is x_1 (up to a relabelling of the vectors) : $x_1 \in \mathrm{Span}(x_2 - x_k)$
- Now, Pb 1.2 Gives : $\mathrm{Span}(x_2 - x_k) = \mathrm{Span}(x_1 - x_k) = S$ (by assumption)

we get that S is spanned by $k-1$ vectors $(x_2 \dots x_k)$.
This is absurd because $\dim S = k$! We conclude that $(x_1 \dots x_k)$ is a basis of S.

<u>Problem 1.5</u> Let $k = \dim U$ and $l = \dim V$.

Let $(u_1 \dots u_k)$ be a basis of U and $(v_1 \dots v_l)$ be a basis of V.

By contradiction, assume that $U \cap V = \{0\}$.

. We are going to show that $(u_1 \dots u_k, v_1 \dots v_l)$ are linearly independent.

. Let $\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_l \in \mathbb{R}$ such that

$$\alpha_1 u_1 + \dots + \alpha_k u_k + \beta_1 v_1 + \dots + \beta_l v_l = 0$$

Then : $x \overset{def}{=} \underbrace{\alpha_1 u_1 + \dots + \alpha_k u_k}_{\in U} = \underbrace{-\beta_1 v_1 - \dots - \beta_l v_l}_{\in V}$

hence $x \in U \cap V = \{0\}$. We get $x = 0$ and therefore :

$$\left| \begin{array}{l} \alpha_1 u_1 + \dots + \alpha_k u_k = 0 \\ \beta_1 v_1 + \dots + \beta_l v_l = 0 \end{array} \right.$$

$(u_1 \dots u_k)$ is linearly independent, hence $\alpha_1 = \dots = \alpha_k = 0$

$(v_1 \dots v_l)$ ———————————————— $\beta_1 = \dots = \beta_l = 0$

we conclude that $(u_1 \dots u_k, v_1 \dots v_l)$ is a family of $k + l > n = \dim \mathbb{R}^n$ vectors of $\mathbb{R}^n$: this is <u>absurd</u>!
linearly indep.

<u>Conclusion</u> : $U \cap V \neq \{0\}$.