Optimization and Computational Linear Algebra for Data Science
Midterm solutions

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Problem 1

(a) $A0 = 0 \neq 3e_1$, hence the zero vector does not belong to E_1: E_1 is not a subspace.

(b) $E_2 = \mathrm{Ker}(v^\mathsf{T})$, where we see v^T as a $1 \times n$ matrix. E_2 is therefore a subspace.

Problem 2

(a) False. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $\mathrm{rank}(A) = 1 = \mathrm{rank}(B)$, however $\mathrm{Im}(A) = \mathrm{Span}(e_1)$ is not a subset of $\mathrm{Im}(B) = \mathrm{Span}(e_2)$.

(b) True. We have $Au = y$ and $Av = y$, hence $0 = Au - Av = A(u - v)$ which gives that $u - v \in \mathrm{Ker}(A)$.

(c) True. We have $\mathrm{rank}(\mathrm{Id}_4) = 4$ but $\mathrm{rank}(ABC) \leq \mathrm{rank}(C) \leq 3$ since C is 3×4.

(d) True. Consider

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The assumption on A gives that $Av = 2v$, hence 2 is an eigenvalue of A.

Problem 3 Compute

$$Ax = Av_1 + \cdots + Av_n = \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

The vectors x and Ax have coordinates $(1, \dots, 1)$ and $(\lambda_1, \dots, \lambda_n)$ respectively in the orthonormal basis $(v_1, \dots, v_n)$, hence:

$$\|x\|^2 = n \qquad \text{and} \qquad \|Ax\|^2 = \lambda_1^2 + \cdots + \lambda_n^2.$$

Problem 4

(a) We have $\langle u - v, x \rangle = 0$ for all $x \in \mathbb{R}^n$. Applying this for $x = u - v$ gives $\|u - v\|^2 = 0$, hence $u = v$.

(b) We have $\langle u - v, x \rangle = 0$ for all $x \in S$. Applying this for $x = P_S(u - v)$ gives

$$0 = \langle u - v, P_S(u - v) \rangle = \langle u - v - P_S(u - v), P_S(u - v) \rangle + \|P_S(u - v)\|^2 = \|P_S(u - v)\|^2,$$

where the last equality comes from the fact that for all vector $w \in \mathbb{R}^n$, $w - P_S(w) \perp S$. We conclude that $P_S(u - v) = 0$, hence $P_S(u) = P_S(v)$.

Other possible solution:

(a) Applying $\langle u, x \rangle = \langle v, x \rangle$ for $x = e_1, \dots, e_n$ gives that $u_i = v_i$ for all i hence $u = v$.

(b) Let $k = \dim(S)$ and $(w_1, \dots, w_k)$ an orthonormal basis of S. We have then

$$P_S(u) = \langle w_1, u \rangle w_1 + \cdots + \langle w_k, u \rangle w_k$$
$$P_S(v) = \langle w_1, v \rangle w_1 + \cdots + \langle w_k, v \rangle w_k.$$

Applying $\langle u, x \rangle = \langle v, x \rangle$ for $x = w_1, \dots, w_k$ gives that $\langle u, w_i \rangle = \langle v, w_i \rangle$ for all $i \in \{1, \dots, k\}$, hence $P_S(u) = P_S(v)$.

Problem 5 Let show that $\mathrm{Im}(A) = \mathrm{Im}(AB)$. First, the inclusion $\mathrm{Im}(AB) \subset \mathrm{Im}(A)$ is obvious. It remains to prove the reverse inclusion.

Notice that $\mathrm{Im}(B)$ is a subspace of $\mathbb{R}^r$ of dimension $\mathrm{rank}(B) = r$: we have $\mathrm{Im}(B) = \mathbb{R}^r$. Let $x \in \mathrm{Im}(A)$. By definition, there exists $u \in \mathbb{R}^r$ such that $Av = x$. Since $u \in \mathbb{R}^r = \mathrm{Im}(B)$, there exists a vector $v \in \mathbb{R}^n$ such that $Bv = u$. We get that $x = Au = ABv \in \mathrm{Im}(AB)$.

We conclude that $\mathrm{Im}(AB) = \mathrm{Im}(A)$ and therefore $\mathrm{rank}(AB) = \mathrm{rank}(A) = r$.

Problem 6

(a) The assumption on M gives that $\mathrm{rank}(M) = m$, hence the rank-nullity theorem gives that $\dim(\mathrm{Ker}(M)) = m - m = 0$. Let $x \in \mathrm{Ker}(M^\top M)$. We have $M^\top M x = 0$, therefore

$$0 = x^\top M^\top M x = \|Mx\|^2,$$

which gives $Mx = 0$. Since we know that $\mathrm{Ker}(M) = \{0\}$, we get that $\mathrm{Ker}(M^\top M) = \{0\}$: $M^\top M$ is invertible.

(b) Let $x \in \mathbb{R}^m$. By definition $Mx \in \mathrm{Im}(M)$ and the properties of orthogonal projections give that $u - w \perp \mathrm{Im}(M)$, so we conclude that $(Mx) \perp (u - w)$.

(c) This gives that for all $x \in \mathbb{R}^m$,

$$0 = x^\top M^\top (u - w) = \langle x, M^\top (u - w)\rangle.$$

Taking $x = M^\top (u - w)$ gives that $\|M^\top (u - w)\|^2 = 0$: $M^\top u = M^\top v$.

(d) Since $u \in \mathrm{Im}(M)$, there exists $v \in \mathbb{R}^m$ such that $u = Mv$. We get:

$$M^\top M v = M^\top w.$$

From **(a)**, we know that that $M^\top M$ is invertible, hence $v = (M^\top M)^{-1} M^\top w$. We conclude that $u = M(M^\top M)^{-1} M^\top w$.

Problem 7 M is symmetric so the spectral theorem gives that there exists $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and an orthogonal matrix $P \in \mathbb{R}^{n \times n}$ such that

$$M = P\mathrm{Diag}(\lambda_1, \dots, \lambda_n)P^\top.$$

Compute now

$$M^{2020} = P\mathrm{Diag}(\lambda_1, \dots, \lambda_n)P^\top P\mathrm{Diag}(\lambda_1, \dots, \lambda_n)P^\top \cdots P\mathrm{Diag}(\lambda_1, \dots, \lambda_n)P^\top = P\mathrm{Diag}(\lambda_1^{2020}, \dots, \lambda_n^{2020})P^\top,$$

since $P^\top P = \mathrm{Id}_n$. We get that

$$P\mathrm{Diag}(\lambda_1^{2020}, \dots, \lambda_n^{2020})P^\top = \mathrm{Id}_n$$

which gives (using $PP^\top = \mathrm{Id}_n$) that $\mathrm{Diag}(\lambda_1^{2020}, \dots, \lambda_n^{2020}) = \mathrm{Id}_n$. We get that for all $i \in \{1, \dots, n\}$, λ_i is either 1 or -1. Consequently:

$$M^2 = P\mathrm{Diag}(\lambda_1^2, \dots, \lambda_n^2)P^\top = P\mathrm{Diag}(1, \dots, 1)P^\top = PP^\top = \mathrm{Id}_n.$$

Problem 8 The equality $A^2 = 0$ implies $\mathrm{Im}(A) \subset \mathrm{Ker}(A)$. Indeed, for $y \in \mathrm{Im}(A)$ there exists $x \in \mathbb{R}^{10}$ such that $Ax = y$ hence

$$Ay = AAx = A^2 x = 0.$$

Consequently $\mathrm{rank}(A) \leq \dim(\mathrm{Ker}(A))$. The rank-nullity theorem gives that $\dim(\mathrm{Ker}(A)) + \mathrm{rank}(A) = 10$ which, combined to the previous inequality, implies $\mathrm{rank}(A) \leq 5$.