DS GA 1014 Leo Miolane
Optimization and Computational Linear Algebra Ashwin Bhola

Midterm review questions

Questions with stars ($\star$) are a slightly outside of the scope of the midterm, they are simply here for your own general knowledge.

1. Let $A, B \in \mathbb{R}^{n\times m}$. For each one of the subsets of $\mathbb{R}^n$ below say whether it is a subspace of $\mathbb{R}^n$ and justify your answer.
   (a) $E_1 = \{x \in \mathbb{R}^n \mid Ax = 0\}$
   (b) $E_2 = \{x \in \mathbb{R}^n \mid Ax = Bx\}$
   (c) $E_3 = \{x \in \mathbb{R}^n \mid Ax = e_1\}$
   (d) $E_4 = \{x \in \mathbb{R}^n \mid Ax \in \mathrm{Span}(e_1)\}$
2. **True or False**: There exists matrices $M \in \mathbb{R}^{2\times 3}$ such that $\dim(\mathrm{Ker}(M)) = 1$ and $\mathrm{rank}(M) = 2$.
3. Let $n > m$ and let $A \in \mathbb{R}^{n\times m}$. Assume that A has "full rank", meaning that $\mathrm{rank}(A) = \min(n, m) = m$.
   (a) Does $Ax = b$ has a solution for all $b \in \mathbb{R}^n$? (Prove or give a counter example).
   (b) Can there exist a vector $b \in \mathbb{R}^n$ for which there exists two distinct solutions $x \neq x'$ such that $Ax = Ax' = b$? (Give an example of A,b for which it happens, or prove that it cannot happen).
4. Let now $n < m$ and let $A \in \mathbb{R}^{n\times m}$. Assume that A has "full rank", meaning that $\mathrm{rank}(A) = \min(n, m) = n$.
   (a) Does $Ax = b$ has a solution for all $b \in \mathbb{R}^n$? (Prove or give a counter example).
   (b) Can there exist a vector $b \in \mathbb{R}^n$ for which there exists two distinct solutions $x \neq x'$ such that $Ax = Ax' = b$? (Give an example of A,b for which it happens, or prove that it cannot happen).
5. **True or False:** There can exist a set of n non-zero orthogonal vectors in $\mathbb{R}^m$ for $n > m$.
6. Let $A \in \mathbb{R}^{m\times n}$.
   (a) Prove that $\mathrm{Ker}(A^T)$ and $\mathrm{Im}(A)$ are orthogonal to each other, i.e. for all $x \in \mathrm{Ker}(A^T)$ and all $y \in \mathrm{Im}(A)$, we have $x \perp y$.

(b) Prove that $\mathrm{Ker}(A^T) = \mathrm{Im}(A)^{\perp}$.

7. **True or False**: The matrix of an orthogonal projection is symmetric.

8. **True or False**: The matrix of an orthogonal projection is orthogonal.

9. Let S be a subspace of $\mathbb{R}^n$ and let P_S be the orthogonal projection onto S. Show that $\dim(S) = \mathrm{Tr}(P_S)$.

10. **True or False**: Let $A, B \in \mathbb{R}^{n \times n}$. If v is an eigenvector of A and B, then v is an eigenvector of $A + B$. Is v an eigenvector of AB ?

11. Let $A \in \mathbb{R}^{n \times n}$. Let $v_1, v_2 \in \mathbb{R}^n$ be two eigenvectors of A associated with the same eigenvalue λ. Show that any non-zero vector in $\mathrm{Span}(v_1, v_2)$ is an eigenvector of A associated with λ.

12. Let $x \in \mathbb{R}^n$ be a vector not equal to zero. Show that $M = xx^T$ is symmetric. What is the rank of M ? What are the eigenvalues of M and their multiplicity?

13. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Let $(u_1, \dots, u_n)$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A and let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues ($Au_i = \lambda_i u_i$ for all i). Give orthonormal basis of $\mathrm{Ker}(A)$ and $\mathrm{Im}(A)$ in terms of the vectors $u_1, \dots, u_n$.

14. ($\star$) For any matrix $A \in \mathbb{R}^{m \times n}$, give an expression for $P_{Im(A)}$ in terms of QR factorization of A.

15. ($\star$) True or False: Every matrix $A \in \mathbb{R}^{n \times n}$ can be written as A=LU where L is a lower triangular matrix and U is an upper triangular matrix