Problem 4.1. Let $M^* = \max\{ v^T x \mid v \in \mathbb{R}^n, \|v\|=1 \}$

- If $x=0$ then obviously $M^* = 0$

- If now $x \neq 0$, then we can consider $v = \frac{x}{\|x\|}$ which has unit norm to get:

$$M^* \geqslant v^T x = \frac{x^T x}{\|x\|} = \frac{\|x\|^2}{\|x\|} = \|x\|.$$

To obtain the converse bound, we use the Cauchy-Schwarz inequality which gives that for all $v \in \mathbb{R}^n$ such that $\|v\|=1$:

$$x^T v \leqslant \|x\| \|v\| = \|x\|.$$

So we get that $M^* \leqslant \|x\|$.

CONCLUSION: $M^* = \|x\|$.

Problem 4.2. Let $x \in \mathbb{R}^n$.

- $\|x\|_1 = \sum_{i=1}^{n} |x_i| = u \cdot v$ where:

$$u = \begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \end{pmatrix} \text{ and } v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

By the Cauchy-Schwarz inequality:

$$u \cdot v \leqslant \|u\|_2 \|v\|_2 = \|x\|_2 \times \sqrt{n}.$$

So we obtain the inequality on the left.

- $\|x\|_2^2 = x_1^2 + \dots + x_n^2$

$$\|x\|_1^2 = (|x_1| + \dots + |x_n|)^2$$

$$= x_1^2 + \dots + x_n^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} |x_i||x_j|$$

$$\geqslant \|x\|_2^2$$

Since $\|x\|_1$ and $\|x\|_2$ are non-negative, we conclude $\|x_1\| \geqslant \|x\|_2$

Problem 4.3.

a) Let $x, y \in \mathbb{R}^n$.

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
(by linearity)
$$= \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$$
$$= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2.$$

Similarly:
$$\|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2.$$

By soming the two equalities above we get:
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

b) Let $x = e_1$ and $y = e_2$.

$$\|x+y\|_1^2 + \|x-y\|_1^2 = 4+4 = 8$$
$$\neq$$
$$2\|x\|_1^2 + 2\|y\|_1^2 = 2+2 = 4$$

Consequently there exists no inner product that induces the ℓ_1 norm.

Analogously $\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 1+1 = 2$
$$2\|x\|_\infty^2 + 2\|y\|_\infty^2 = 2+2 = 4.$$

The $\|\cdot\|_\infty$ norm is not cinduced by an inner product.

Problem 4.4.

a) Let P_S denotes the orthogonal projection on S.

We notice that $S^\perp = \mathrm{Ker}(P_S)$: $S^\perp$ is therefore a sobspace of $\mathbb{R}^n$.

b) Notice that $S = \mathrm{Im}(P_S)$. The rank-nullity theorem applied to P_S gives:

$$\dim S + \dim S^\perp = n.$$

c) By (b), $\dim H = 2$. The vectors

$$v_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

forms an orthonormal family. Since they belong to H, (v_1, v_2) is an orthonormal basis of H.

d) Let $V = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ -2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} \end{pmatrix}$

$$P_H = VV^T$$

$$= \begin{array}{cc} & \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \\ \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ -2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} \end{pmatrix} & \begin{matrix} \frac{1}{6}+\frac{1}{2} & -\frac{1}{3} & \frac{1}{6}-\frac{1}{2} \\ -1/3 & 2/3 & -1/3 \\ \frac{1}{6}-\frac{1}{2} & -1/3 & \frac{1}{6}+\frac{1}{2} \end{matrix} \end{array}$$

$$= \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

Problem 4.5.

Let $S = \mathrm{Im}(P)$ and let P_S be the orthogonal projection on S.

- Let $x \in S$. By definition, there exists $u \in \mathbb{R}^n$ such that $x = Pu$. Hence:

$$Px = P^2 u = Pu = x \qquad (1)$$

- Let $x \in S^\perp$. Since $Px \in \mathrm{Im}(P) = S$, we have $x^T P x = 0$.

Now, $P = PP = P^T P$, hence $0 = x^T P x = x^T P^T P x = \|Px\|^2$

We get $Px = 0$. $\qquad (2)$

Let $x \in \mathbb{R}^n$.

Notice that $x = \underbrace{P_S x}_{\in S} + \underbrace{x - P_S x}_{\in S^\perp}$

By linearity:

$$\begin{aligned} Px &= PP_S x + P(x - P_S x) \\ &= \underbrace{P_S x}_{(\text{by } (1))} + \underbrace{0}_{(\text{by } (2))} \\ &= P_S x. \end{aligned}$$

<u>Conclusion</u>: P is the orthogonal projection on S.