Lecture 2.1: Linear transformations

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

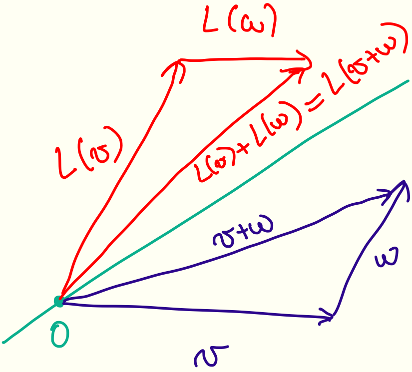
Contents

Definition

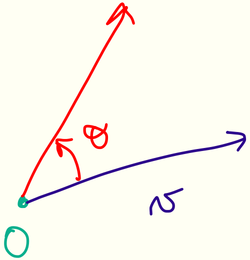
Examples

You already know some linear transformations from high-school !

Symmetry



Rotation



Definition

Symmetries (about a line passing through the origin) and rotations (about the origin) are mappings

$$\begin{aligned} L : \mathbb{R}^2 &\to \mathbb{R}^2 \\ v &\mapsto L(v), \end{aligned}$$

that are “linear”:

Definition

A function $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear if

1. for all $v, w \in \mathbb{R}^m$ we have $L(v+w) = L(v) + L(w)$ and
2. for all $v \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$ we have $L(\alpha v) = \alpha L(v)$.

An example

- $$L : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (v_1, v_2) & \mapsto & (5v_1,\ 0,\ v_1 + v_2) \end{array}$$ is linear

- Let $v = (v_1, v_2)$ and $w = (w_1, w_2)$ in $\mathbb{R}^2$

$$L(v+w) = L((v_1+v_2, w_1+w_2)) = (5(v_1+v_2), 0, v_1+v_2+w_1+w_2)$$

$$= (5v_1, 0, v_1+v_2) + (5w_1, 0, w_1+w_2)$$

$$= L(v) + L(w)$$

- Similarly $L(\alpha v) = \alpha L(v)$

$\to$ check this!

An example of a non-linear map

The function $\begin{array}{rccc} F: & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & x^2 \end{array}$ is **not** linear.

- $F(1+1) = F(2) = 4$
$$\neq$$
- $F(1) + F(1) = 1 + 1 = 2$

hence F is not linear

Properties

Composition of linear maps

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "L"'] \arrow[rr, bend right, "M \circ L"'] & \mathbb{R}^n \arrow[r, "M"'] & \mathbb{R}^k \end{tikzcd}$$

Proposition

If $L : \mathbb{R}^m \to \mathbb{R}^n$ and $M : \mathbb{R}^n \to \mathbb{R}^k$ are both linear, then the composite function

$$\begin{array}{rccc} M \circ L : & \mathbb{R}^m & \to & \mathbb{R}^k \\ & v & \mapsto & M(L(v)) \end{array}$$

is also linear.

Proof.

- Let $v, w \in \mathbb{R}^m$,

$$\begin{aligned} M \circ L(v+w) &= M(L(v+w)) = M(L(v)+L(w)) \\ &= M(L(v)) + M(L(w)) \\ &= M \circ L(v) + M \circ L(w) \end{aligned}$$

- Analogously: $M \circ L(\alpha v) = \alpha\, M \circ L(v)$

$\square$

Basic properties

Proposition

If $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then

- $L(0) = 0.$
- $L\left(\sum_{i=1}^{k} \alpha_i v_i\right) = \sum_{i=1}^{k} \alpha_i L(v_i)$, for all $\alpha_i \in \mathbb{R}$, $v_i \in \mathbb{R}^m$.

"compatible with linear combination"

Proof.

- $L(0) = L(0+0) = L(0) + L(0)$ ∴ $L(0) = 0$
- $L(\alpha_1 v_1 + \dots + \alpha_k v_k) = L(\alpha_1 v_1) + L(\alpha_2 v_2 + \dots + \alpha_k v_k)$

$= \alpha_1 L(v_1) + L(\alpha_2 v_2 + \dots + \alpha_k v_k)$

⋮ repeat

$= \alpha_1 L(v_1) + \alpha_2 L(v_2) + \dots + \alpha_k L(v_k)$

□