Problem 3.1

a) $\left.\begin{array}{l} AB = \alpha\, Id_n B = \alpha B \\ BA = B(\alpha Id_n) = \alpha B Id_n = \alpha B \end{array}\right\} AB = BA.$

b) Let $k, l$ ~~…~~ $\in \{1, \ldots n\}$ and $B$ be the $n \times n$ matrix defined by:

$$B_{i,j} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise.} \end{cases}$$

Compute for $i, j \in \{1, \ldots n\}$:

$$(AB)_{i,j} = \sum_{m=1}^{n} A_{i,m} B_{m,j} = A_{i,k} B_{k,j}$$

$$(BA)_{i,j} = \sum_{m=1}^{n} B_{i,m} A_{m,j} = B_{i,l} A_{l,j}$$

Since $AB = BA$ we have $A_{i,k} B_{k,j} = B_{i,l} A_{l,j}$ for all $i, j$.

In particular, for $j = l$, we get: $A_{i,k} = B_{i,l} A_{l,l}$.

So $\left| \begin{array}{l} \text{if } i \neq k \text{ then } A_{i,k} = 0 . A_{l,l} = 0 \\ \text{if } i = k \text{ then } A_{k,k} = 1 . A_{l,l} = A_{l,l} . \end{array} \right.$

We conclude that all the coefficients of $A$ that are outside the diagonal are zero, and that the diagonal coefficients are all equal to some number that we call $\alpha$.

We have in other words $A = \alpha\, Id_n$.

Problem 3. 2.

By definition of the rank: $\dim \mathrm{Im}(M) = \mathrm{rank}(M) = r$.

Let $a_1, \ldots a_r \in \mathbb{R}^n$ be a basis of $\mathrm{Im}(M)$.
Let $c_1, \ldots c_m \in \mathbb{R}^n$ be the columns of $M$.
For $i = 1, \ldots m$, the vector $c_i$ belongs to $\mathrm{Im}(A)$.
Therefore, there exists scalars $b_{1,i} \ldots b_{r,i}$ such that

$$c_i = b_{1,i}\, a_1 + \ldots + b_{r,i}\, a_r .$$

Let $A$ be the matrix $A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_r \\ | & & | \end{pmatrix} \in \mathbb{R}^{n \times r}$.

and $B$ be the $r \times m$ matrix defined by
$B_{i,j} = b_{i,j}$ for all $(i,j) \in \{1 \ldots r\} \times \{1 \ldots m\}$.

By construction we have $M = AB$ because for all $i$, the $i^{th}$ column of $AB$ is:

$$b_{1,i}\, a_1 + \ldots + b_{r,i}\, a_r = c_i$$

Problem 3.3

a) Let us show that $\mathrm{Im}(AM) = \mathrm{Im}(A)$.

- $\mathrm{Im}(AM) \subset \mathrm{Im}(A)$. Indeed if $y \in \mathrm{Im}(AM)$ then there exists $x \in \mathbb{R}^m$ such that $y = AMx$.
Hence $y = A(Mx) \in \mathrm{Im}(A)$

- $\mathrm{Im}(A\cancel{M}) \subset \mathrm{Im}(AM)$. Indeed, let $y \in \mathrm{Im}(A)$.

There exists $x \in \mathbb{R}^m$ such that $y = Ax$.

Then $y = AM(M^{-1}x) \in \mathrm{Im}(A)$.

<u>Conclusion</u>: $\mathrm{Im}(A) = \mathrm{Im}(AM)$, hence $\mathrm{rank}(A) = \mathrm{rank}(MA)$.

b) Let us show that $\mathrm{Ker}(A) = \mathrm{Ker}(MA)$:

For $x \in \mathbb{R}^n$ we have:

$$
\begin{aligned}
x \in \mathrm{Ker}(A) &\iff Ax = 0 \\
&\iff MAx = 0 \\
&\iff x \in \mathrm{Ker}(MA)
\end{aligned}
$$

multiplication by $M$ ↓ ↑ multiplication by $M^{-1}$

Hence $\mathrm{Ker}(A) = \mathrm{Ker}(MA)$ and $\dim \mathrm{Ker}(A) = \dim \mathrm{Ker}(MA)$.

We conclude using the rank-nullity theorem:

$n - \mathrm{rank}(A) = n - \mathrm{rank}(MA)$, ie $\mathrm{rank}(A) = \mathrm{rank}(MA)$.

Problem 3.4

(a) $\mathrm{Tr}(AB) = \sum_{i=1}^{n} (AB)_{i,i} = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{i,j} B_{j,i}$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} B_{j,i} A_{i,j} = \sum_{j=1}^{m} (BA)_{j,j} = \mathrm{Tr}(BA)$$

(b). We have indeed $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$ since

$$\mathrm{Tr}(ABC) = \mathrm{Tr}((AB)C) \overset{\text{using (a)}}{=} \mathrm{Tr}(C(AB)) = \mathrm{Tr}(CAB)$$

- However, we do not necessary have $\mathrm{Tr}(ABC) = \mathrm{Tr}(ACB)$

Indeed for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

we have $ABC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $ACB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Problem 3.5

For $n \geqslant 1$ we define $u_n = \text{rank}(A^n)$.

We are going to show:

1. $u_1 \leqslant 9$
2. If $u_n > 0$ for some $n$, then $u_{n+1} < u_n$.

Clearly, 1 and 2 imply that $u_{10} = 0$ which gives $A^{10} = 0$.

1. By contradiction: if $\text{rank}(A) = 10$, then $A$ is invertible.
Multiplying $A^{2020} = 0$ by $(A^{-1})^{2020}$ on both sides gives

$Id_{10} = 0$ which is absurd. Hence $\text{rank}(A) \leqslant 9$.

(We knew that $\text{rank}(A) \leqslant 10$ because $A$ is $10 \times 10$).

2. Let $n \geqslant 1$ such that $\text{rank}(A^n) > 0$.

Clearly: $\text{Im}(A^{n+1}) \subset \text{Im}(A^n)$ which implies $\text{rank}(A^{n+1}) \leqslant \text{rank}(A^n)$

By contradiction, assume that $\text{rank}(A^{n+1}) = \text{rank}(A^n)$.

This implies that $\mathrm{Im}(A^{n+1}) = \mathrm{Im}(A^n)$

Let $v_0 \in \mathrm{Im}(A^n) \setminus \{0\}$ (~~the~~ such a vector exists since $\mathrm{rank}(A^n) > 0$)

Consider the following sequence $v_0, v_1, v_2 \dots \in \mathrm{Im}(A^n)$

~~Given $v_k$, we defined $v$~~

defined by: given $v_k$ in $\mathrm{Im}(A^n)$ we know that there exists $x \in \mathbb{R}^{110}$ such that $v_k = A^{n+1} x = A\, A^n x$

We define then $v_{k+1} = A^n x \in \mathrm{Im}(A^n)$.

By construction of the $v_0, v_1 \dots$ we have

$$v_0 = A v_1 = A^2 v_2 = \dots = A^{2020} v_k = 0$$

We get something absurd! Hence $\mathrm{rank}(A^{n+1}) < \mathrm{rank}(A^n)$.