Lecture 1.2: Vector Spaces

Optimization and Computational Linear Algebra for Data Science

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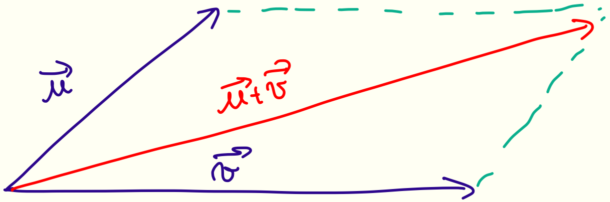
Contents

Introduction

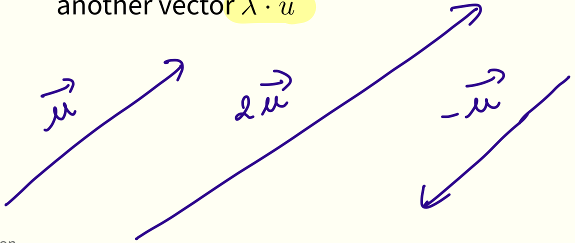
So far, « Vectors = arrows »

Two fundamental operations:

1. Add two vectors $\vec{u}$ and $\vec{v}$ to obtain another vector $\vec{u} + \vec{v}$

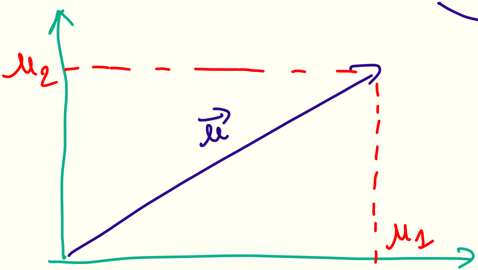


2. Multiply a vector $\vec{u}$ by a «scalar» (= a real number) λ to get another vector $\lambda \cdot \vec{u}$



Coordinate representation

- One can represent vectors using coordinates
- 2D vectors in the plane $\vec{u} = (u_1, u_2) \in \mathbb{R}^2$
- 3D vectors in space $\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$
- n-dimensional vectors $\vec{u} = \underbrace{(u_1, u_2, \ldots, u_n)}_{\text{data point with } n \text{ features}} \in \mathbb{R}^n$



- $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$
- $\lambda \cdot \vec{u} = (\lambda u_1, \lambda u_2, \ldots, \lambda u_n)$

Vector Spaces

Abstract definition

Definition (simplified)

A vector space consists of a set V (whose elements are called vectors) and two operations $+$ and $\cdot$ such that

- The sum of two vectors is a vector: for $\vec{x}, \vec{y} \in V$, the sum $\vec{x} + \vec{y}$ is a vector, i.e. $\vec{x} + \vec{y} \in V$.
- Multiplying a vector $\vec{x} \in V$ by a scalar $\lambda \in \mathbb{R}$ gives a vector $\lambda \cdot \vec{x} \in V$.
- The operations $+$ and $\cdot$ are "nice and compatible".

« Nice and compatible » ?

1. The vector sum is commutative and associative. For all $\vec{x}, \vec{y}, \vec{z} \in V$:

$$\vec{x} + \vec{y} = \vec{y} + \vec{x} \quad \text{and} \quad \vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}.$$

2. There exists a zero vector $\vec{0} \in V$ that verifies $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in V$.
3. For all $\vec{x} \in V$, there exists $\vec{y} \in V$ such that $\vec{x} + \vec{y} = \vec{0}$. Such $\vec{y}$ is called the additive inverse of $\vec{x}$ and is written $-\vec{x}$.
4. Identity element for scalar multiplication: $1 \cdot \vec{x} = \vec{x}$ for all $\vec{x} \in V$.
5. Distributivity: for all $\alpha, \beta \in \mathbb{R}$ and all $\vec{x}, \vec{y} \in V$,

$$(\alpha + \beta) \cdot \vec{x} = \alpha \cdot \vec{x} + \beta \cdot \vec{y} \quad \text{and} \quad \alpha \cdot (\vec{x} + \vec{y}) = \alpha \cdot \vec{x} + \alpha \cdot \vec{y}.$$

6. Compatibility between scalar multiplication and the usual multiplication: for all $\alpha, \beta \in \mathbb{R}$ and all $\vec{x} \in V$, we have

$$\alpha \cdot (\beta \cdot \vec{x}) = (\alpha\beta) \cdot \vec{x}.$$

Example 1: $\mathbb{R}^n$

The set $V = \mathbb{R}^n$ endowed with the usual vector addition $+$

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

and the usual scalar multiplication $\cdot$

$$\alpha \cdot (x_1, \dots, x_n) = (\alpha x_1, \dots, \alpha x_n)$$

is a vector space.

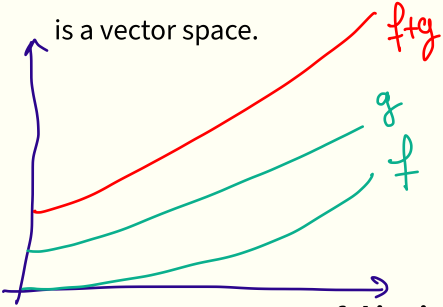
We will work in $\mathbb{R}^n$ 99% ot the time !

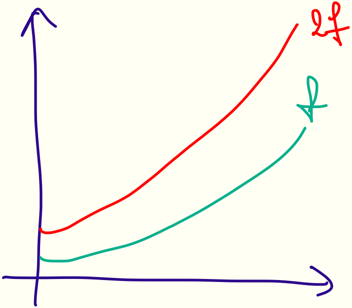
Example 2: functions

The set $V \overset{\text{def}}{=} \{f \mid f : \mathbb{R} \to \mathbb{R}\}$ of all functions from $\mathbb{R}$ to itself endowed with the addition $+$ and the scalar multiplication $\cdot$ defined by

$$\begin{aligned} f+g : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto f(t)+g(t) \end{aligned} \quad \text{and} \quad \begin{aligned} \alpha \cdot f : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto \alpha f(t) \end{aligned}$$

is a vector space.





Useful in signal processing.

Example 3: random variables

The set of random variables on a given probability space Ω is a vector space:
If X and Y are two random variables and $\alpha \in \mathbb{R}$, $X + Y$ and αX are also random variables.

Important to have this in mind when doing stats/probabilities!

Why do we need all this?

- **Get geometric intuition.**

  We will see for instance that the notion of length in $\mathbb{R}^n$ is deeply connected to the notion of variance of random variables.

- **Save time.**

  A theorem that applies to vector spaces will in particular be true for all the examples we listed before.

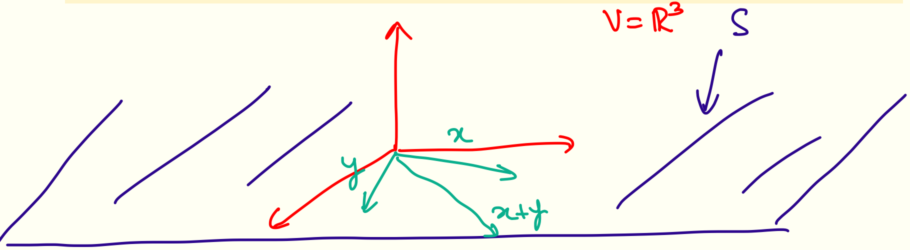
Subspaces

Definition

Definition

We say that a non-empty subset S of a vector space V is a *subspace* if it is closed under addition and multiplication by a scalar, that is if

1. for all $x, y \in S$ we have $x + y \in S$,
2. for all $x \in S$ and all $\alpha \in \mathbb{R}$ we have $\alpha x \in S$.



S is a subspace of $V = \mathbb{R}^3$

Remark: a subspace is a also vector space.

Examples

- $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. obvious

- $\{0\}$ is a subspace of $\mathbb{R}^n$. if $x, y \in \{0\}$, then $x = y = 0$, hence $x+y = 0+0 \in \{0\}$

- Any line that contains the origin is subspace of $\mathbb{R}^2$.