An analytical model to estimate the state of charge and lifetime for batteries with energy harvesting capabilities

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Abstract

Energy limitation is one of the major bottlenecks during the operation of many emerging applications, such as electric vehicles, water and gas meters, and a number of sensors used in the context of the Internet of Things and cyber-physical systems. Energy harvesting techniques have arisen as a promising solution to minimize the energy issues found in these types of application domains. In energy harvesting systems, a critical challenge is the need to use battery models capable of accurately estimating both the input and output power of batteries. This paper proposes a temperature-dependent analytical battery model capable of estimating some output quantities – e.g., state of charge, voltage and lifetime – of batteries that use energy harvesting technologies. This model was validated by comparing its analytical results with a dataset called the Randomized Battery Usage Data Set, which is available at the data repository of the National Aeronautics and Space Administration (NASA) website. It is also presented a proof-of-concept application, demonstrating that the use of these technologies can serve as an effective means to extend the operating time of batteries, resulting in significant benefits for a number of applications. Full paper can be found at: http://dx.doi.org/10.1002/er.5269

1 Temperature-Dependent Kinetic Battery Model for Energy Harvesting (T-KiBaM$_{\mathrm{EH}}$)

This section introduces the necessary modifications to make the Temperature-Dependent Kinetic Battery Model (T-KiBaM) model compatible with energy harvesting devices, which means that batteries can be recharged while they are discharged by sensor nodes. Thus, we briefly present the definition of the extended battery model. Then, we derive the system of differential equations that allows calculating the behaviour of the battery charge over time.

1.1 Definition

This battery model, called T-KiBaM$_{\mathrm{EH}}$, extends the T-KiBaM model [1, 2]. In other words, all the concepts presented for T-KiBaM remain valid. The difference is the addition of a valve, J, that allows the entry of charge into the battery. It is precisely this charge that is provided by any energy harvesting device. Figure 1 depicts the mode of operation of the extended battery model.

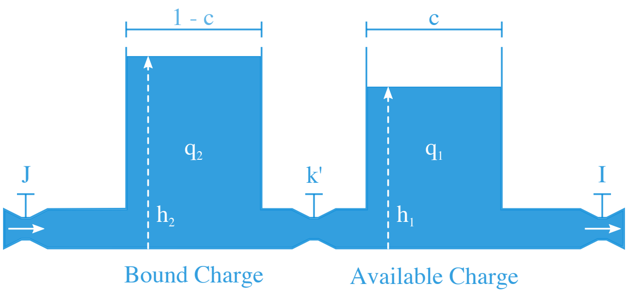

Figure 1: Kinetic Battery Model (KiBaM) [3] with charging capabilities.

In T-KiBaM$_{\mathrm{EH}}$, the charge regenerated through valve J is not immediately available for use by the connected device (which drains charge from valve I). According to the original idea of the KiBaM model [3, 4], the charge entering the battery is stored in the Bound Charge tank and only when there is a height difference between the tank charges (i.e. $h_2 - h_1 > 0$) is that the extra charge can be transferred from the Bound Charge tank to the Available Charge tank. In addition, note that if there is no recharge at all, i.e. $J = 0$, the model follows exactly the same behaviour as the original T-KiBaM model. Therefore, the main advantage of this approach lies in the fact that it is possible to both discharge and recharge the battery simultaneously.

1.2 Model Equations

The behaviour described above can be modelled through the equations presented in this section. Briefly, we present: (i) the main system of differential equations, which represents the behaviour of charge inside the tanks of the T-KiBaM$_{\mathrm{EH}}$ model; (ii) the new format of the equations after replacing the variables h_1, h_2 and k; and (iii) the final solution of the system of differential equations, after applying Laplace transforms.

1.2.1 System of Differential Equations

$$\begin{cases} \dfrac{dq_1}{dt} = -I + k' \cdot (h_2 - h_1) \\[2ex] \dfrac{dq_2}{dt} = +J - k' \cdot (h_2 - h_1) \end{cases} \tag{1}$$

$$\begin{cases} \dfrac{dq_1}{dt} = -I - k' \cdot (h_1 - h_2) \\[2ex] \dfrac{dq_2}{dt} = +J + k' \cdot (h_1 - h_2) \end{cases} \tag{2}$$

$$h_1 = \frac{q_1(t)}{c} \qquad h_2 = \frac{q_2(t)}{(1-c)} \qquad k = \frac{k'}{c \cdot (1-c)}$$

1.2.2 Replacing h_1, h_2 and k in Equation 2

$$\begin{cases} \dfrac{dq_1}{dt} = -I - k \cdot c \cdot (1-c) \left[\dfrac{q_1(t)}{c} - \dfrac{q_2(t)}{(1-c)} \right] \\[2ex] \dfrac{dq_2}{dt} = +J + k \cdot c \cdot (1-c) \left[\dfrac{q_1(t)}{c} - \dfrac{q_2(t)}{(1-c)} \right] \end{cases} \tag{3}$$

$$\begin{cases} \dfrac{dq_1}{dt} = -I - k \cdot \cancel{c \cdot (1-c)} \left[\dfrac{q_1(t) \cdot (1-c) - q_2(t) \cdot c}{\cancel{c \cdot (1-c)}} \right] \\[2ex] \dfrac{dq_2}{dt} = +J + k \cdot \cancel{c \cdot (1-c)} \left[\dfrac{q_1(t) \cdot (1-c) - q_2(t) \cdot c}{\cancel{c \cdot (1-c)}} \right] \end{cases} \tag{4}$$

$$\begin{cases} \dfrac{dq_1}{dt} = -I - k \cdot (1-c) \cdot q_1(t) + k \cdot c \cdot q_2(t) \\[2ex] \dfrac{dq_2}{dt} = +J + k \cdot (1-c) \cdot q_1(t) - k \cdot c \cdot q_2(t) \end{cases} \tag{5}$$

1.2.3 Applying Laplace Transforms

Initial conditions: $q_1(0) = c \cdot q_0$ and $q_2(0) = (1-c) \cdot q_0$.

$$\begin{cases} \mathscr{L}\left\{\dfrac{dq_1}{dt}\right\} = \mathscr{L}\{-I - k\cdot(1-c)\cdot q_1(t) + k\cdot c\cdot q_2(t)\} \\ \mathscr{L}\left\{\dfrac{dq_2}{dt}\right\} = \mathscr{L}\{+J + k\cdot(1-c)\cdot q_1(t) - k\cdot c\cdot q_2(t)\} \end{cases} \tag{6}$$

Knowing that $\mathscr{L}\left\{\frac{dq_1}{dt}\right\} = s\cdot Q_1(s) - q_1(0)$ and $\mathscr{L}\left\{\frac{dq_2}{dt}\right\} = s\cdot Q_2(s) - q_2(0)$, then:

$$\begin{cases} s\cdot Q_1(s) - q_1(0) = -I\cdot\mathscr{L}\{1\} - k\cdot(1-c)\cdot\mathscr{L}\{q_1(t)\} + k\cdot c\cdot\mathscr{L}\{q_2(t)\} \\ s\cdot Q_2(s) - q_2(0) = +J\cdot\mathscr{L}\{1\} + k\cdot(1-c)\cdot\mathscr{L}\{q_1(t)\} - k\cdot c\cdot\mathscr{L}\{q_2(t)\} \end{cases} \tag{7}$$

$$\begin{cases} s\cdot Q_1(s) - q_1(0) = -I\cdot\dfrac{1}{s} - k\cdot(1-c)\cdot Q_1(s) + k\cdot c\cdot Q_2(s) \\ s\cdot Q_2(s) - q_2(0) = +J\cdot\dfrac{1}{s} + k\cdot(1-c)\cdot Q_1(s) - k\cdot c\cdot Q_2(s) \end{cases} \tag{8}$$

$$\begin{cases} s\cdot Q_1(s) + k\cdot(1-c)\cdot Q_1(s) - k\cdot c\cdot Q_2(s) = -\dfrac{I}{s} + q_1(0) \\ s\cdot Q_2(s) - k\cdot(1-c)\cdot Q_1(s) + k\cdot c\cdot Q_2(s) = \dfrac{J}{s} + q_2(0) \end{cases} \tag{9}$$

$$\begin{cases} [s + k\cdot(1-c)]\cdot Q_1(s) - k\cdot c\cdot Q_2(s) = -\dfrac{I}{s} + q_1(0) \\ -k\cdot(1-c)\cdot Q_1(s) + [s + k\cdot c]\cdot Q_2(s) = \dfrac{J}{s} + q_2(0) \end{cases} \tag{10}$$

1.2.4 Eliminating $Q_1(s)$

For this, it becomes necessary to multiply the second term of Equation 10 by $\left[\frac{s}{k\cdot(1-c)} + 1\right]$. Thus:

$$\begin{cases} [s + k\cdot(1-c)]\cdot Q_1(s) - k\cdot c\cdot Q_2(s) = -\dfrac{I}{s} + q_1(0) \\ [-k\cdot(1-c)\cdot Q_1(s) + [s + k\cdot c]\cdot Q_2(s)]\cdot\left[\dfrac{s}{k\cdot(1-c)} + 1\right] = \left[\dfrac{J}{s} + q_2(0)\right]\cdot\left[\dfrac{s}{k\cdot(1-c)} + 1\right] \end{cases} \tag{11}$$

$$\begin{cases} [s + k\cdot(1-c)]\cdot Q_1(s) - k\cdot c\cdot Q_2(s) = -\dfrac{I}{s} + q_1(0) \\ \dfrac{-\cancel{k\cdot(1-c)}\cdot s\cdot Q_1(s)}{\cancel{k\cdot(1-c)}} - k\cdot(1-c)\cdot Q_1(s) + \dfrac{[s + k\cdot c]\cdot s\cdot Q_2(s)}{k\cdot(1-c)} + [s + k\cdot c]\cdot Q_2(s) = \\ \qquad \dfrac{J\cdot\cancel{s}}{k\cdot(1-c)\cdot\cancel{s}} + \dfrac{J}{s} + \dfrac{s\cdot q_2(0)}{k\cdot(1-c)} + q_2(0) \end{cases} \tag{12}$$

$$
\begin{cases}
[s + k\cdot(1-c)]\cdot Q_1(s) - k\cdot c\cdot Q_2(s) = -\dfrac{I}{s} + q_1(0) \\[2ex]
-s\cdot Q_1(s) - k\cdot(1-c)\cdot Q_1(s) + \dfrac{s^2\cdot Q_2(s) + k\cdot c\cdot s\cdot Q_2(s)}{k\cdot(1-c)} + s\cdot Q_2(s) + k\cdot c\cdot Q_2(s) = \\[2ex]
\qquad\qquad \dfrac{J}{k\cdot(1-c)} + \dfrac{J}{s} + \dfrac{s\cdot q_2(0)}{k\cdot(1-c)} + q_2(0)
\end{cases}
\tag{13}
$$

$$
\begin{cases}
[s + k\cdot(1-c)]\cdot Q_1(s) - k\cdot c\cdot Q_2(s) = -\dfrac{I}{s} + q_1(0) \\[2ex]
-[s + k\cdot(1-c)]\cdot Q_1(s) + \dfrac{s^2\cdot Q_2(s)}{k\cdot(1-c)} + \dfrac{\cancel{k}\cdot c\cdot s\cdot Q_2(s)}{\cancel{k}\cdot(1-c)} + s\cdot Q_2(s) + k\cdot c\cdot Q_2(s) = \\[2ex]
\qquad\qquad \dfrac{J}{k\cdot(1-c)} + \dfrac{J}{s} + \dfrac{s\cdot q_2(0)}{k\cdot(1-c)} + q_2(0)
\end{cases}
\tag{14}
$$

$$
\begin{cases}
\cancel{[s + k\cdot(1-c)]\cdot Q_1(s)} - k\cdot c\cdot Q_2(s) = -\dfrac{I}{s} + q_1(0) \\[2ex]
\cancel{-[s + k\cdot(1-c)]\cdot Q_1(s)} + \left[\dfrac{s^2}{k\cdot(1-c)} + \dfrac{c\cdot s}{(1-c)} + s + k\cdot c\right]\cdot Q_2(s) = \\[2ex]
\qquad\qquad \dfrac{J}{k\cdot(1-c)} + \dfrac{J}{s} + \dfrac{s\cdot q_2(0)}{k\cdot(1-c)} + q_2(0)
\end{cases}
\tag{15}
$$

$$
\begin{gathered}
\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \; + \\
-k\cdot c\cdot Q_2(s) + \left[\frac{s^2}{k\cdot(1-c)} + \frac{c\cdot s}{(1-c)} + s + k\cdot c\right]\cdot Q_2(s) = \\
-\frac{I}{s} + \frac{J}{k\cdot(1-c)} + \frac{J}{s} + \frac{s\cdot q_2(0)}{k\cdot(1-c)} + q_1(0) + q_2(0)
\end{gathered}
$$

Considering that $q_0 = q_1(0) + q_2(0)$, it becomes possible to obtain the following:

$$
\left[\cancel{-k\cdot c} + \frac{s^2}{k\cdot(1-c)} + \frac{c\cdot s}{(1-c)} + s + \cancel{k\cdot c}\right]\cdot Q_2(s) = -\frac{I}{s} + \frac{J}{k\cdot(1-c)} + \frac{J}{s} + \frac{s\cdot q_2(0)}{k\cdot(1-c)} + q_0 \tag{16}
$$

$$
\begin{gathered}
\left[\frac{s^2}{k\cdot(1-c)} + \frac{c\cdot s}{(1-c)} + s\right]\cdot Q_2(s) = \\
\frac{-I\cdot k\cdot(1-c) + J\cdot s + J\cdot k\cdot(1-c) + s^2\cdot q_2(0) + q_0\cdot s\cdot k\cdot(1-c)}{s\cdot k\cdot(1-c)}
\end{gathered}
\tag{17}
$$

$$
Q_2(s) = \frac{-I\cdot k\cdot(1-c) + J\cdot s + J\cdot k\cdot(1-c) + s^2\cdot q_2(0) + q_0\cdot s\cdot k\cdot(1-c)}{s\cdot k\cdot(1-c)\cdot\left[\frac{s^2}{k\cdot(1-c)} + \frac{c\cdot s}{(1-c)} + s\right]} \tag{18}
$$

$$
Q_2(s) = \frac{-I\cdot k\cdot(1-c) + J\cdot s + J\cdot k\cdot(1-c) + s^2\cdot q_2(0) + q_0\cdot s\cdot k\cdot(1-c)}{\left[\frac{s^3\cdot\cancel{k\cdot(1-c)}}{\cancel{k\cdot(1-c)}} + \frac{s^2\cdot k\cdot c\cdot\cancel{(1-c)}}{\cancel{(1-c)}} + s^2\cdot k\cdot(1-c)\right]} \tag{19}
$$

$$Q_2(s) = \frac{-I \cdot k \cdot (1-c) + J \cdot s + J \cdot k \cdot (1-c) + s^2 \cdot q_2(0) + q_0 \cdot s \cdot k \cdot (1-c)}{[s^3 + s^2 \cdot k \cdot c + s^2 \cdot k \cdot (1-c)]} \tag{20}$$

$$Q_2(s) = \frac{-I \cdot k \cdot (1-c) + J \cdot s + J \cdot k \cdot (1-c) + s^2 \cdot q_2(0) + q_0 \cdot s \cdot k \cdot (1-c)}{s^3 + s^2 \cdot k \cdot (\cancel{c} + 1 - \cancel{c})} \tag{21}$$

$$Q_2(s) = \frac{-I \cdot k \cdot (1-c) + J \cdot s + J \cdot k \cdot (1-c) + s^2 \cdot q_2(0) + q_0 \cdot s \cdot k \cdot (1-c)}{s^2 \cdot (s+k)} \tag{22}$$

1.2.5 Applying the Partial-Fractions Decomposition

$$Q_2(s) = \frac{-I \cdot k \cdot (1-c) + J \cdot s + J \cdot k \cdot (1-c) + s^2 \cdot q_2(0) + q_0 \cdot s \cdot k \cdot (1-c)}{s^2 \cdot (s+k)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+k)} \tag{23}$$

$$\begin{aligned} &\frac{-I \cdot k \cdot (1-c) + J \cdot s + J \cdot k \cdot (1-c) + s^2 \cdot q_2(0) + q_0 \cdot s \cdot k \cdot (1-c)}{\cancel{s^2 \cdot (s+k)}} = \\ &\qquad \frac{A \cdot s \cdot (s+k) + B \cdot (s+k) + C \cdot s^2}{\cancel{s^2 \cdot (s+k)}} \end{aligned} \tag{24}$$

$$\begin{aligned} &-I \cdot k \cdot (1-c) + J \cdot s + J \cdot k \cdot (1-c) + s^2 \cdot q_2(0) + q_0 \cdot s \cdot k \cdot (1-c) = \\ &\qquad A \cdot s \cdot (s+k) + B \cdot (s+k) + C \cdot s^2 \end{aligned} \tag{25}$$

- $s = 0$:

$$-I \cdot \cancel{k} \cdot (1-c) + 0 + J \cdot \cancel{k} \cdot (1-c) + 0 + 0 = 0 + B \cdot \cancel{k} + 0 \tag{26}$$

$$\boxed{B = -I \cdot (1-c) + J \cdot (1-c)} \tag{27}$$

- $s = -k$:

$$\begin{aligned} &-I \cdot k \cdot (1-c) + J \cdot (-k) + J \cdot k \cdot (1-c) + (-k)^2 \cdot q_2(0) + q_0 \cdot (-k) \cdot k \cdot (1-c) = \\ &\qquad A \cdot (-k) \cdot ((-k) + k) + B \cdot ((-k) + k) + C \cdot (-k)^2 \end{aligned} \tag{28}$$

$$\begin{aligned} &-I \cdot k \cdot (1-c) - J \cdot k + J \cdot k \cdot (1-c) + k^2 \cdot q_2(0) - q_0 \cdot k^2 \cdot (1-c) = \\ &\qquad -A \cdot k \cdot (\cancel{-k} + \cancel{k}) + B \cdot (\cancel{-k} + \cancel{k}) + C \cdot k^2 \end{aligned} \tag{29}$$

$$-I \cdot \cancel{k} \cdot (1-c) - J \cdot \cancel{k} + J \cdot \cancel{k} \cdot (1-c) + k^{\cancel{2}} \cdot q_2(0) - q_0 \cdot k^{\cancel{2}} \cdot (1-c) = C \cdot k^{\cancel{2}} \tag{30}$$

$$-I \cdot (1-c) - J + J \cdot (1-c) + k \cdot q_2(0) - q_0 \cdot k \cdot (1-c) = C \cdot k \tag{31}$$

$$C = \frac{-I \cdot (1-c) - J + J \cdot (1-c) + k \cdot q_2(0) - q_0 \cdot k \cdot (1-c)}{k} \tag{32}$$

$$C = \frac{-I\cdot(1-c) - J + J\cdot(1-c) + k\cdot q_2(0) - q_0\cdot k\cdot(1-c)}{k} \tag{33}$$

$$C = \frac{-I\cdot(1-c)\cancel{-J} + \cancel{J} - J\cdot c + k\cdot q_2(0) - q_0\cdot k\cdot(1-c)}{k} \tag{34}$$

$$\boxed{C = \frac{-I\cdot(1-c) - J\cdot c + k\cdot q_2(0) - q_0\cdot k\cdot(1-c)}{k}} \tag{35}$$

- $s = k$:

$$\begin{aligned}-I\cdot k\cdot(1-c) + J\cdot k + J\cdot k\cdot(1-c) + k^2\cdot q_2(0) + q_0\cdot k\cdot k\cdot(1-c) =\\ A\cdot k\cdot(k+k) + B\cdot(k+k) + C\cdot k^2\end{aligned} \tag{36}$$

$$\begin{aligned}-I\cdot\cancel{k}\cdot(1-c) + J\cdot\cancel{k} + J\cdot\cancel{k}\cdot(1-c) + k^{\cancel{2}}\cdot q_2(0) + q_0\cdot k^{\cancel{2}}\cdot(1-c) =\\ 2\cdot A\cdot k^{\cancel{2}} + 2\cdot B\cdot\cancel{k} + C\cdot k^{\cancel{2}}\end{aligned} \tag{37}$$

$$-I\cdot(1-c) + J + J\cdot(1-c) + k\cdot q_2(0) + q_0\cdot k\cdot(1-c) = 2\cdot A\cdot k + 2\cdot B + C\cdot k \tag{38}$$

$$-I\cdot(1-c) + J + J\cdot(1-c) + k\cdot q_2(0) + q_0\cdot k\cdot(1-c) - 2\cdot B - C\cdot k = 2\cdot A\cdot k \tag{39}$$

$$\begin{aligned}-I\cdot(1-c) + J + J\cdot(1-c) + k\cdot q_2(0) + q_0\cdot k\cdot(1-c) - 2\cdot[-I\cdot(1-c) + J\cdot(1-c)]\\ -\left[\frac{-I\cdot(1-c) - J\cdot c + k\cdot q_2(0) - q_0\cdot k\cdot(1-c)}{\cancel{k}}\right]\cdot\cancel{k} = 2\cdot A\cdot k\end{aligned} \tag{40}$$

$$\begin{aligned}-I\cdot(1-c) + J + J\cdot(1-c) + k\cdot q_2(0) + q_0\cdot k\cdot(1-c) - 2\cdot[-I\cdot(1-c) + J\cdot(1-c)]\\ +I\cdot(1-c) + J\cdot c - k\cdot q_2(0) + q_0\cdot k\cdot(1-c) = 2\cdot A\cdot k\end{aligned} \tag{41}$$

$$\begin{aligned}\cancel{-I\cdot(1-c)} + J + J\cdot(1-c) + k\cdot q_2(0) + q_0\cdot k\cdot(1-c) + 2\cdot I\cdot(1-c) - 2\cdot J\cdot(1-c)\\ \cancel{+I\cdot(1-c)} + J\cdot c - k\cdot q_2(0) + q_0\cdot k\cdot(1-c) = 2\cdot A\cdot k\end{aligned} \tag{42}$$

$$\begin{aligned}J + J\cdot(1-c)\cancel{+k\cdot q_2(0)} + q_0\cdot k\cdot(1-c) + 2\cdot I\cdot(1-c) - 2\cdot J\cdot(1-c)\\ +J\cdot c\cancel{-k\cdot q_2(0)} + q_0\cdot k\cdot(1-c) = 2\cdot A\cdot k\end{aligned} \tag{43}$$

$$\begin{aligned}\boxed{J + J}\cancel{-J\cdot c} + q_0\cdot k\cdot(1-c) + 2\cdot I\cdot(1-c) - 2\cdot J\cdot(1-c)\\ \cancel{+J\cdot c} + q_0\cdot k\cdot(1-c) = 2\cdot A\cdot k\end{aligned} \tag{44}$$

$$2\cdot J + \boxed{q_0\cdot k\cdot(1-c)} + 2\cdot I\cdot(1-c) - 2\cdot J\cdot(1-c) + \boxed{q_0\cdot k\cdot(1-c)} = 2\cdot A\cdot k \tag{45}$$

$$2 \cdot J + 2 \cdot q_0 \cdot k \cdot (1-c) + 2 \cdot I \cdot (1-c) - 2 \cdot J \cdot (1-c) = 2 \cdot A \cdot k \tag{46}$$

$$\cancel{2 \cdot J} + 2 \cdot q_0 \cdot k \cdot (1-c) + 2 \cdot I \cdot (1-c) \cancel{- 2 \cdot J} + 2 \cdot J \cdot c = 2 \cdot A \cdot k \tag{47}$$

$$\cancel{2} \cdot q_0 \cdot k \cdot (1-c) + \cancel{2} \cdot I \cdot (1-c) + \cancel{2} \cdot J \cdot c = \cancel{2} \cdot A \cdot k \tag{48}$$

$$\boxed{A = \frac{q_0 \cdot k \cdot (1-c) + I \cdot (1-c) + J \cdot c}{k}} \tag{49}$$

Therefore, it becomes possible to unify all results as follows:

$$Q_2(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+k)} \tag{50}$$

$$\begin{aligned} Q_2(s) = \frac{q_0 \cdot k \cdot (1-c) + I \cdot (1-c) + J \cdot c}{k \cdot s} + \frac{-I \cdot (1-c) + J \cdot (1-c)}{s^2} \\ + \frac{-I \cdot (1-c) - J \cdot c + k \cdot q_2(0) - q_0 \cdot k \cdot (1-c)}{k \cdot (s+k)} \end{aligned} \tag{51}$$

$$\boxed{\begin{aligned} Q_2(s) = \frac{q_0 \cdot k \cdot (1-c) + I \cdot (1-c) + J \cdot c}{k \cdot s} + \frac{-I \cdot (1-c) + J \cdot (1-c)}{s^2} \\ + \frac{-I \cdot (1-c) - J \cdot c - q_0 \cdot k \cdot (1-c)}{k \cdot (s+k)} + \frac{q_2(0)}{(s+k)} \end{aligned}} \tag{52}$$

1.2.6 Applying Inverse Laplace Transform

$$\begin{aligned} \mathscr{L}^{-1}\{Q_2(s)\} = \left[\frac{q_0 \cdot k \cdot (1-c) + I \cdot (1-c) + J - J \cdot (1-c)}{k}\right] \cdot \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} \\ + [-I \cdot (1-c) + J \cdot (1-c)] \cdot \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} \\ + \left[\frac{-I \cdot (1-c) - J + J \cdot (1-c) - q_0 \cdot k \cdot (1-c)}{k}\right] \cdot \mathscr{L}^{-1}\left\{\frac{1}{(s+k)}\right\} \\ + q_2(0) \cdot \mathscr{L}^{-1}\left\{\frac{1}{(s+k)}\right\} \end{aligned} \tag{53}$$

Since $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$, $\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$, and $\mathscr{L}^{-1}\left\{\frac{1}{(s+k)}\right\} = e^{-k \cdot t}$, then:

$$\begin{aligned} q_2(t) = \left[\frac{q_0 \cdot k \cdot (1-c) + I \cdot (1-c) + J - J \cdot (1-c)}{k}\right] \cdot 1 + [-I \cdot (1-c) + J \cdot (1-c)] \cdot t \\ + \left[\frac{-I \cdot (1-c) - J + J \cdot (1-c) - q_0 \cdot k \cdot (1-c)}{k}\right] \cdot e^{-k \cdot t} + q_2(0) \cdot e^{-k \cdot t} \end{aligned} \tag{54}$$

$$q_2(t) = -\left[\frac{k \cdot q_0 \cdot (c-1) + c \cdot (I-J) - I}{k}\right] - I \cdot (1-c) \cdot t + J \cdot (1-c) \cdot t$$
$$\boxed{-\left[\frac{I \cdot (1-c) + J - J \cdot (1-c) + q_0 \cdot k \cdot (1-c)}{k}\right] \cdot e^{-k \cdot t}} + q_2(0) \cdot e^{-k \cdot t} \tag{55}$$

$$q_2(t) = -\left[\frac{k \cdot q_0 \cdot (c-1) + c \cdot (I-J) - I}{k}\right] - I \cdot (1-c) \cdot t + J \cdot (1-c) \cdot t$$
$$\boxed{+\left[\frac{e^{-k \cdot t} \cdot [k \cdot q_0 \cdot (c-1) + c \cdot (I-J) - I]}{k}\right] + q_2(0) \cdot e^{-k \cdot t}} \tag{56}$$

$$q_2(t) = \frac{e^{-k \cdot t} \cdot [q_2(0) \cdot k + k \cdot q_0 \cdot (c-1) + c \cdot (I-J) - I]}{k}$$
$$\boxed{-\left[\frac{k \cdot q_0 \cdot (c-1) + c \cdot (I-J) - I}{k}\right] - I \cdot (1-c) \cdot t + J \cdot (1-c) \cdot t} \tag{57}$$

$$\boxed{\begin{gathered} q_2(t) = \frac{e^{-k \cdot t} \cdot [q_2(0) \cdot k + k \cdot q_0 \cdot (c-1) + c \cdot (I-J) - I]}{k} \\ -\left[\frac{k \cdot q_0 \cdot (c-1) + c \cdot (1 - k \cdot t) \cdot (I-J) + I \cdot (k \cdot t - 1) - J \cdot k \cdot t}{k}\right] \end{gathered}} \tag{58}$$

1.2.7 Applying the Derivative in $q_2(t)$

$$\frac{dq_2(t)}{dt} = \frac{d[e^{-k \cdot t}]}{dt} \cdot \frac{[q_2(0) \cdot k + k \cdot q_0 \cdot (c-1) + c \cdot (I-J) - I]}{k}$$
$$-\frac{d}{dt} \cdot \frac{k \cdot q_0 \cdot (c-1)}{k} - \frac{c \cdot (I-J)}{k} \cdot \frac{d[1 - k \cdot t]}{dt} - \frac{I}{k} \cdot \frac{d[k \cdot t - 1]}{dt} + \frac{J}{k} \cdot \frac{d[k \cdot t]}{dt} \tag{59}$$

$$\frac{dq_2(t)}{dt} = [-k \cdot e^{-k \cdot t}] \cdot \frac{[q_2(0) \cdot k + k \cdot q_0 \cdot (c-1) + c \cdot (I-J) - I]}{k}$$
$$-\frac{c \cdot (I-J)}{k} \cdot [-k] - \frac{I}{k} \cdot [k] + \frac{J}{k} \cdot [k] \tag{60}$$

$$\frac{dq_2(t)}{dt} = [-\cancel{k} \cdot e^{-k \cdot t}] \cdot \frac{[q_2(0) \cdot k + k \cdot q_0 \cdot (c-1) + c \cdot (I-J) - I]}{\cancel{k}}$$
$$-\frac{c \cdot (I-J)}{\cancel{k}} \cdot [-\cancel{k}] - \frac{I}{\cancel{k}} \cdot [\cancel{k}] + \frac{J}{\cancel{k}} \cdot [\cancel{k}] \tag{61}$$

$$\boxed{\begin{aligned} \frac{dq_2(t)}{dt} &= [-e^{-k \cdot t}] \cdot [q_2(0) \cdot k + k \cdot q_0 \cdot (c-1) + c \cdot (I-J) - I] \\ &\qquad + c \cdot (I-J) - I + J \end{aligned}} \tag{62}$$

Remember that $q_2'(t) = J + k \cdot (1-c) \cdot q_1(t) - k \cdot q_2(t) \cdot c$ and $q_0 = q_1(0) + q_2(0)$.

$$
\begin{aligned}
&-k \cdot e^{-k\cdot t} \cdot q_2(0) + k \cdot e^{-k\cdot t} \cdot q_0 \cdot (1-c) + I \cdot (1-c) \cdot (-1 + e^{-k\cdot t}) + J \cdot (1-c) \cdot (1 - e^{-k\cdot t}) \\
&+ J \cdot e^{-k\cdot t} = J + \boxed{k \cdot (1-c) \cdot q_1(t)} - k \cdot c \Bigg[q_2(0) \cdot e^{-k\cdot t} + q_0 \cdot (1-c) \cdot (1 - e^{-k\cdot t}) \\
&+ \frac{I \cdot (1-c) \cdot [1 - k \cdot t - e^{-k\cdot t}]}{k} + \frac{J \cdot (1-c) \cdot [-1 + k \cdot t + e^{-k\cdot t}]}{k} + \frac{J \cdot (1 - e^{-k\cdot t})}{k} \Bigg]
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
&\boxed{k \cdot (1-c)} \cdot q_1(t) = -k \cdot e^{-k\cdot t} \cdot q_2(0) + k \cdot e^{-k\cdot t} \cdot q_0 \cdot (1-c) + I \cdot (1-c) \cdot (-1 + e^{-k\cdot t}) \\
&+ J \cdot (1-c) \cdot (1 - e^{-k\cdot t}) + J \cdot e^{-k\cdot t} - J + k \cdot c \cdot \Bigg[q_2(0) \cdot e^{-k\cdot t} + q_0 \cdot (1-c) \cdot (1 - e^{-k\cdot t}) \\
&+ \frac{I \cdot (1-c) \cdot [1 - k \cdot t - e^{-k\cdot t}]}{k} + \frac{J \cdot (1-c) \cdot [-1 + k \cdot t + e^{-k\cdot t}]}{k} + \frac{J \cdot (1 - e^{-k\cdot t})}{k} \Bigg]
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
q_1(t) &= \frac{-k \cdot e^{-k\cdot t} \cdot q_2(0) + k \cdot e^{-k\cdot t} \cdot q_0 \cdot (1-c) + I \cdot (1-c) \cdot \left(-1 + e^{-k\cdot t}\right)}{k\,(1-c)} \\
&+ \frac{J \cdot (1-c) \cdot \left(1 - e^{-k\cdot t}\right) + J \cdot e^{-k\cdot t} - J}{k\,(1-c)} + \boxed{\frac{k \cdot c \cdot \left[q_2(0) \cdot e^{-k\cdot t} + q_0 \cdot (1-c) \cdot \left(1 - e^{-k\cdot t}\right)\right]}{k\,(1-c)}} \\
&+ \boxed{\frac{k \cdot c \cdot \left[\frac{I\cdot(1-c)\cdot\left(1-k\cdot t-e^{-k\cdot t}\right)+J\cdot(1-c)\cdot\left(-1+k\cdot t+e^{-k\cdot t}\right)+J\cdot\left(1-e^{-k\cdot t}\right)}{k}\right]}{k\,(1-c)}}
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
q_1(t) &= \frac{\boxed{-k \cdot e^{-k\cdot t} \cdot q_2(0) + k \cdot e^{-k\cdot t} \cdot q_0 \cdot (1-c)} + \boxed{I \cdot (1-c) \cdot \left(-1 + e^{-k\cdot t}\right)}}{k\,(1-c)} \\
&+ \frac{\boxed{J \cdot (1-c) \cdot \left(1 - e^{-k\cdot t}\right) + J \cdot e^{-k\cdot t} - J}}{k\,(1-c)} \\
&+ \frac{k \cdot c \left[e^{-k\cdot t} \cdot \left(q_2(0) + q_0 \cdot (c-1) + \frac{c\cdot(I-J)-I}{k}\right) + q_0 \cdot (1-c) + \frac{c\cdot(I-J)\cdot(k\cdot t-1)+I\cdot(1-k\cdot t)+J\cdot k\cdot t}{k}\right]}{k\,(1-c)}
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
q_1(t) &= \frac{-e^{-k\cdot t} \cdot [k \cdot q_2(0) + k \cdot q_0 \cdot (c-1) + c \cdot (I-J) - I] + c \cdot (I-J) - I}{k\,(1-c)} \\
&+ \frac{\cancel{k} \cdot c \left[e^{-k\cdot t} \cdot \left(q_2(0) + q_0 \cdot (c-1) + \frac{c\cdot(I-J)-I}{\cancel{k}}\right) + q_0 \cdot (1-c) + \frac{c\cdot(I-J)\cdot(k\cdot t-1)+I\cdot(1-k\cdot t)+J\cdot k\cdot t}{\cancel{k}}\right]}{k\,(1-c)}
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
q_1(t) = {} & \frac{\boxed{-e^{-k\cdot t}\cdot[k\cdot q_2(0)+k\cdot q_0\cdot(c-1)+c\cdot(I-J)-I]}+c\cdot(I-J)-I}{k\,(1-c)} \\
& + \frac{\boxed{c\cdot e^{-k\cdot t}\cdot[k\cdot q_2(0)+k\cdot q_0\cdot(c-1)+c\cdot(I-J)-I]}}{k\,(1-c)} \\
& - c\cdot\frac{[k\cdot q_0\cdot(c-1)+c\cdot(1-k\cdot t)\cdot(I-J)+I\cdot(k\cdot t-1)-J\cdot k\cdot t]}{k\,(1-c)}
\end{aligned} \tag{68}
$$

$$
\begin{aligned}
q_1(t) = {} & \frac{e^{-k\cdot t}\cdot\cancel{(c-1)}\cdot[k\cdot q_2(0)+k\cdot q_0\cdot(c-1)+c\cdot(I-J)-I]-c\cdot[k\cdot q_0\cdot\cancel{(c-1)}]}{k\cdot\cancel{(1-c)}} \\
& + \frac{c^2\cdot(I-J)\cdot(k\cdot t-1)-c\cdot[I\cdot(k\cdot t-2)+J\cdot(1-k\cdot t]-I}{k\cdot(1-c)}
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
q_1(t) = {} & -\frac{e^{-k\cdot t}\cdot[k\cdot q_2(0)+k\cdot q_0\cdot(c-1)+c\cdot(I-J)-I]}{k} + \frac{c\cdot\cancel{k}\cdot q_0}{\cancel{k}} \\
& + \boxed{\frac{c^2\cdot(I-J)\cdot(k\cdot t-1)-c\cdot[I\cdot(k\cdot t-2)+J\cdot(1-k\cdot t]-I}{k\cdot(1-c)}}
\end{aligned} \tag{70}
$$

$$
\begin{aligned}
q_1(t) = {} & \frac{c\cdot k\cdot q_0-\cancel{(1-c)}\cdot c\cdot[(k\cdot t-1)\cdot(I-J)-I]}{k\cdot\cancel{(1-c)}} \\
& - \frac{e^{-k\cdot t}\cdot[k\cdot q_2(0)+k\cdot q_0\cdot(c-1)+c\cdot(I-J)-I]}{k}
\end{aligned} \tag{71}
$$

Replace $q_2(0)$ to $q_0 - q_1(0)$:

$$
\begin{aligned}
q_1(t) = {} & \frac{c\cdot k\cdot q_0+c\cdot(1-k\cdot t)\cdot(I-J)-I}{k} \\
& - \boxed{\frac{e^{-k\cdot t}\cdot[k\cdot(q_0-q_1(0))+k\cdot q_0\cdot(c-1)+c\cdot(I-J)-I]}{k}}
\end{aligned} \tag{72}
$$

$$
q_1(t) = \frac{c\cdot k\cdot q_0+c\cdot(1-k\cdot t)\cdot(I-J)-I}{k} - \frac{e^{-k\cdot t}\cdot[c\cdot k\cdot q_0+c\cdot(I-J)-I-k\cdot q_1(0)]}{k} \tag{73}
$$

1.2.8 Final Solution

$$
\begin{cases}
q_1(t) = & \dfrac{c\cdot k\cdot q_0+c\cdot(1-k\cdot t)\cdot(I-J)-I}{k} \\
& - \dfrac{e^{-k\cdot t}\cdot[c\cdot k\cdot q_0+c\cdot(I-J)-I-k\cdot q_1(0)]}{k} \\
q_2(t) = & \dfrac{e^{-k\cdot t}\cdot[q_2(0)\cdot k+k\cdot q_0\cdot(c-1)+c\cdot(I-J)-I]}{k} \\
& - \dfrac{k\cdot q_0\cdot(c-1)+c\cdot(1-k\cdot t)\cdot(I-J)+I\cdot(k\cdot t-1)-J\cdot k\cdot t}{k}
\end{cases} \tag{74}
$$

References

[1] L. Rodrigues, C. Montez, R. Moraes, P. Portugal, and F. Vasques, "A Temperature-Dependent Battery Model for Wireless Sensor Networks," *Sensors*, vol. 17, p. 422, feb 2017.

[2] L. Rodrigues, C. Montez, G. Budke, F. Vasques, and P. Portugal, "Estimating the Lifetime of Wireless Sensor Network Nodes through the Use of Embedded Analytical Battery Models," *Journal of Sensor and Actuator Networks*, vol. 6, no. 2, p. 8, 2017.

[3] J. F. Manwell and J. G. McGowan, "Lead Acid Battery Storage Model for Hybrid Energy Systems," *Solar Energy*, vol. 50, pp. 399–405, May 1993.

[4] J. F. Manwell and J. G. McGowan, "Extension of the Kinetic Battery Model for Wind/Hybrid Power Systems," in *Proceedings of the European Wind Energy Association Conference (EWEC)*, pp. 284—-289, 1994.