Further Consequences of the Colorful Helly Hypothesis: Beyond Point Transversals

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Helly's Theorem

Let $\mathcal{F}$ be a finite family of at least $d+1$ convex sets in $\mathbb{R}^d$.

Theorem (Helly's Theorem '23)

If each subfamily in $\binom{\mathcal{F}}{d+1}$ *has non-empty intersection, then* $\mathcal{F}$ *has non-empty intersection.*

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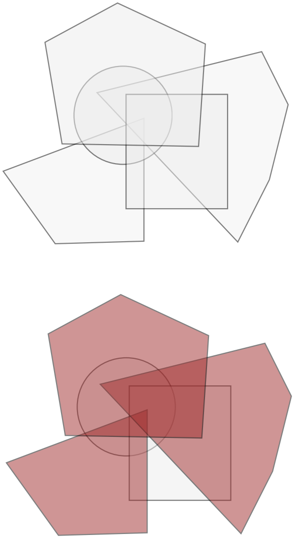
Note. Non-empty intersection $\iff$ single piercing point.

Helly’s Theorem



Variations: Two of (many) possible directions

Problem (Weaker intersection hypothesis)

What can we say if we know that fewer of the subfamilies in $\binom{\mathcal{F}}{d+1}$ *have non-empty intersection?*

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Problem (Higher dimensional transversals)

What happens if we replace piercing points with higher k*-dimensional transversal flats for* $1 \leq k \leq d-1$*?*

Colorful Helly's Theorem

Definition

Let k be an integer. Let $\mathcal{F}$ be a family of convex sets split into k non-empty color classes *$\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$. We say that this (split) family has the* colorful intersection hypothesis *if every rainbow selection $K_i \in \mathcal{F}_i$ for $1 \leq i \leq k$, satisfies $\bigcap_{i=1}^{k} K_i \neq \emptyset$.*

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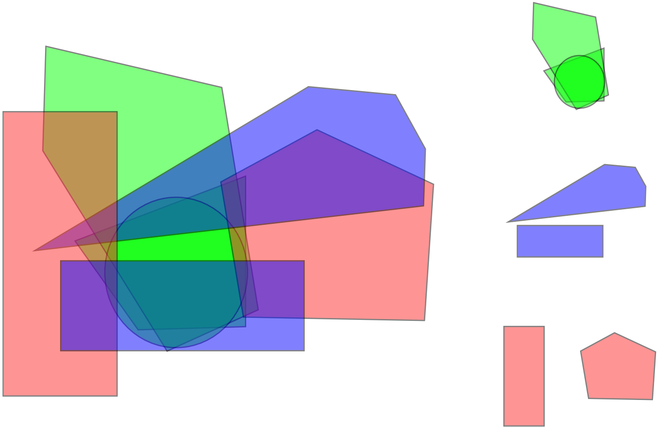
Theorem (Colorful Helly, Lovász, '82)

A family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ split into $d+1$ color classes that satisfy the colorful intersection hypothesis has a class with non-empty intersection.

Colorful Helly's Theorem



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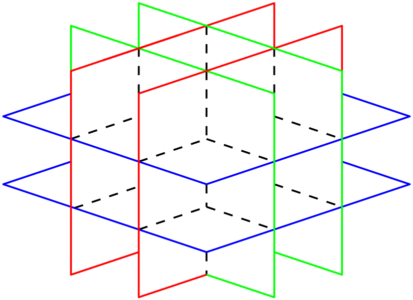

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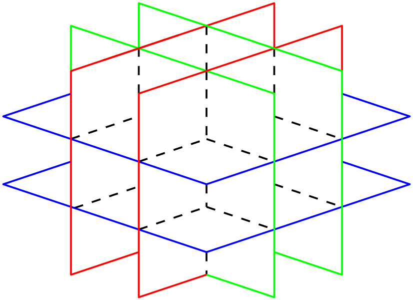

Then what?

Colorful Helly's Theorem for Boxes

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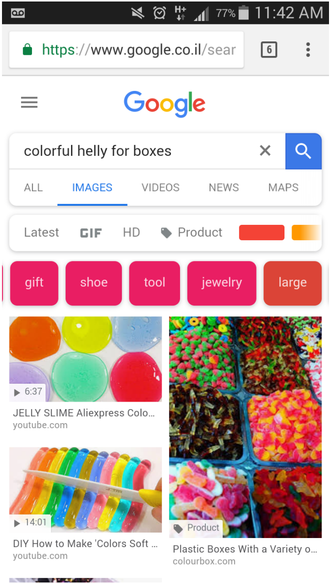
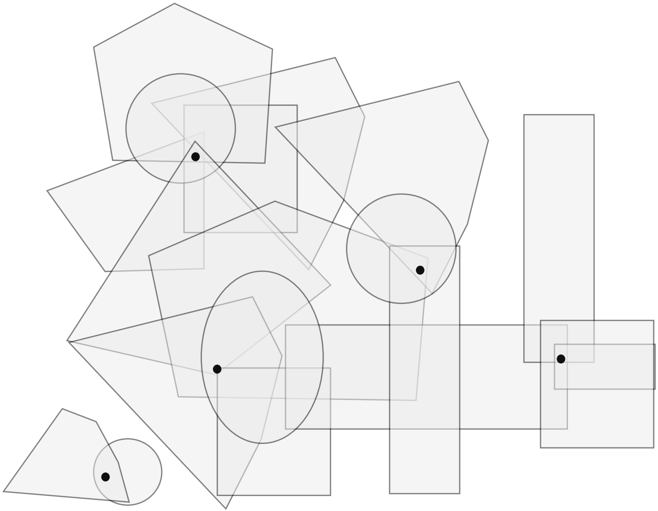


The (p, q)-theorem

Theorem (The (p, q)-theorem, Alon and Kleitman '92)

For each $p \geq q \geq d+1$ *there is a* $P = P(p, q, d)$ *with the following property:*
If any subfamily $\mathcal{F}' \in \binom{\mathcal{F}}{p}$ *contains an intersecting family* $\mathcal{F}'' \in \binom{\mathcal{F}'}{q}$*, then* $\mathcal{F}$ *can be pierced by* P *points.*

The (p, q)-theorem



Change the dimension of transversals

Problem

Let $1 \leq k \leq d$ be an integer and $\mathcal{F}$ a family of convex sets in $\mathbb{R}^d$. Suppose that each subfamily in $\binom{\mathcal{F}}{d+1}$ has a single k-flat transversal. Can we find a transversal for $\mathcal{F}$ with one (or few) k-flats? Can we find a k-flat transversal to a positive fraction of the sets?

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Problem (On the plane, and $k = 1$)

Suppose that each 3 sets of $\mathcal{F}$ have a transversal line. Is it true that $\mathcal{F}$ has a transversal line?

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Suppose that each 3 sets of $\mathcal{F}$ have a transversal line. Is it true that $\mathcal{F}$ has a transversal line? No

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Problem (On the plane, and $k = 1$)

Suppose that each 3 sets of $\mathcal{F}$ have a transversal line. Is it true that $\mathcal{F}$ has a transversal line? No Can it be pierced with few lines?

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Suppose that each 3 sets of $\mathcal{F}$ have a transversal line. Is it true that $\mathcal{F}$ has a transversal line? No Can it be pierced with few lines? Yes

Piercing by few hyperplanes

Theorem (Eckhoff '93, Holmsen '13)

On the plane, if each 3 *sets can be pierced with a line then:*

- *There is a transversal set of* 4 *lines that pierce* $\mathcal{F}$.
- *There is a line through at least* $\frac{1}{3}|\mathcal{F}|$ *of the sets of* $\mathcal{F}$

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Theorem (Alon and Kalai '95)

On $\mathbb{R}^d$, if each $d+1$ sets can be pierced with one hyperplane then:

- *$\mathcal{F}$ admits a transversal of $h := h(d)$ hyperplanes.*
- *There is a hyperplane through at least $\delta|\mathcal{F}|$ of the sets of $\mathcal{F}$.*

Transversal lines in high dimensions

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Theorem (Alon et al. '02)

For every integers $d \geq 3$, m and sufficiently large $n_0 > m+4$ there is a family of at least n_0 convex sets so that any m of the sets can be pierced with a line but no $m+4$ of them can.

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In particular, no (p, q)-theorem.

Our main result

We go back to the Colorful Helly's Theorem context.

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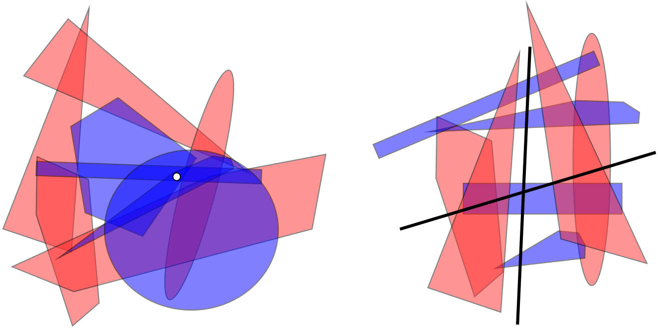
Theorem (-, Roldán-Pensado, Rubin, '18+)

For each dimension d there exist $f(d)$ and $g(d)$ for which:
If $\mathcal{F}$ is split into $d+1$ color classes with the colorful intersection hypothesis and $\mathcal{F}_{d+1}$ is the intersecting class given by CHT, then either

- *an additional $\mathcal{F}_i$ for $i \in [d]$ can be pierced by $f(d)$ points or*
- *the entire family $\mathcal{F}$ admits a transversal by $g(d)$ lines.*

The 2-colored picture



Some words on the proof

Blackboard and diagram time!

The Transversal Step-Down Lemma

Theorem (-, Roldán-Pensado, Rubin, '18+)

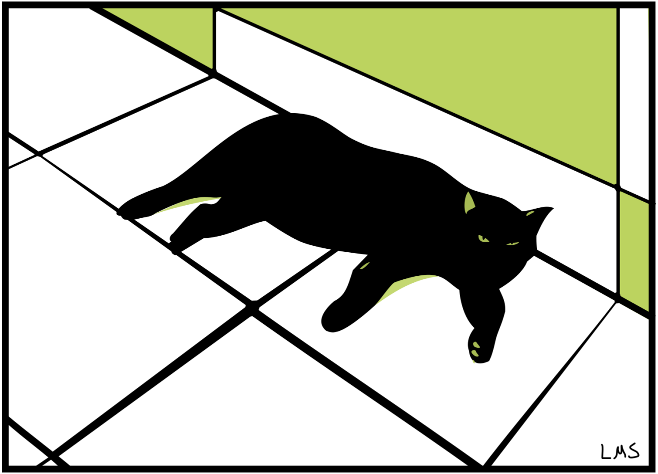
For each dimension d, every postive integer m and every $k \in [d+1]$ there exist numbers $F(m,k,d)$ and $G(m,k,d)$ for which:
If $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ and the family of bicolorful intersections

$$\mathcal{I}(\mathcal{A},\mathcal{B}) := \{A \cap B : \ A \in \mathcal{A}, B \in \mathcal{B}\}$$

can be crossed by m k-flats then either:

- *$\mathcal{A}$ can be pierced by $F(m,k,d)$ points, or*
- *$\mathcal{B}$ can be crossed by $G(m,k,d)$ $(k-1)$-flats*

Thank you!



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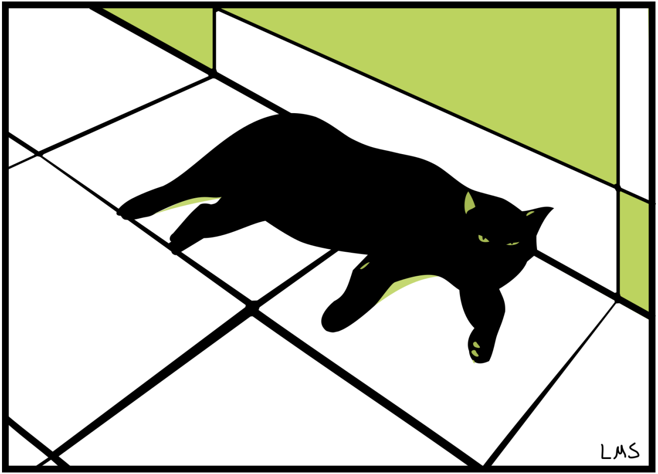

Thank you for your attention!