Further Consequences of the Colorful Helly Hypothesis: Beyond Point Transversals

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Helly's Theorem

Let $\mathcal{F}$ be a finite family of at least $d+1$ convex sets in $\mathbb{R}^d$.

Theorem (Helly's Theorem '23)

If each subfamily in $\binom{\mathcal{F}}{d+1}$ has non-empty intersection, then $\mathcal{F}$ has non-empty intersection.

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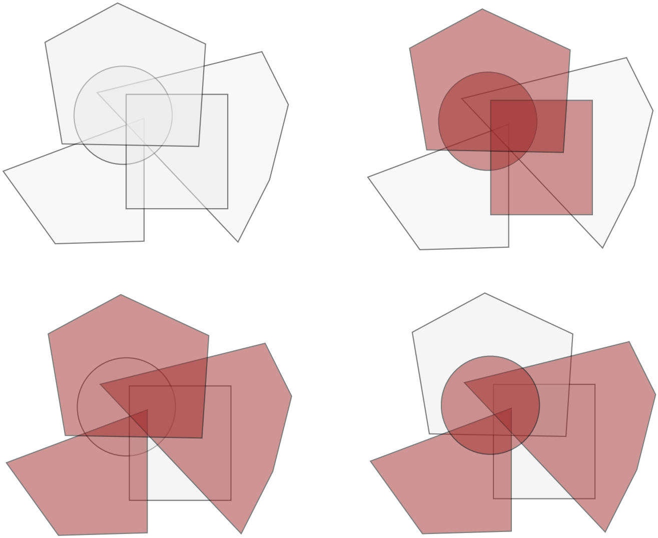
Note. Non-empty intersection $\iff$ single piercing point.

Helly's Theorem



Variations: Two of (many) possible directions

Problem (Weaker intersection hypothesis)

What can we say if we know that fewer of the subfamilies in $\binom{\mathcal{F}}{d+1}$ *have non-empty intersection?*

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Problem (Higher dimensional transversals)

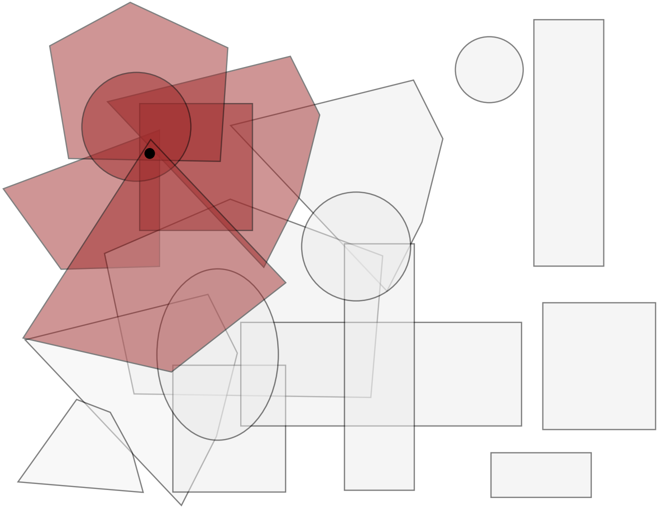
What happens if we replace piercing points with higher k*-dimensional transversal flats for* $1 \leq k \leq d-1$*?*

Fractional Helly's Theorem

Theorem (Fractional Helly's Theorem, Katchalski and Liu '79)

For each $\alpha \in (0,1)$ and $d \geq 1$ there is a $\beta = \beta(\alpha, d) > 0$ with the following property:
If at least $\alpha\binom{|\mathcal{F}|}{d+1}$ of the subfamilies in $\binom{\mathcal{F}}{d+1}$ have non-empty intersection, then there is a point that pierces at least $\beta|\mathcal{F}|$ sets of the family $\mathcal{F}$.

Fractional Helly's Theorem

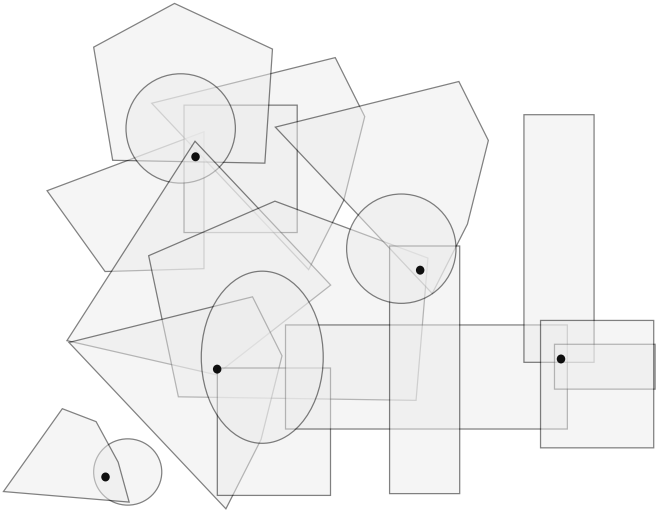


The (p, q)-theorem

Theorem (The (p, q)-theorem, Alon and Kleitman '92)

For each $p \geq q \geq d+1$ *there is a* $P = P(p, q, d)$ *with the following property:*
If any subfamily $\mathcal{F}' \in \binom{\mathcal{F}}{p}$ *contains an intersecting family* $\mathcal{F}'' \in \binom{\mathcal{F}'}{q}$*, then* $\mathcal{F}$ *can be pierced by* P *points.*

The (p, q)-theorem



Colorful Helly's Theorem

Definition

Let k be an integer. Let $\mathcal{F}$ be a family of convex sets split into k non-empty color classes $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$. We say that this (split) family has the colorful intersection hypothesis if every rainbow selection $K_i \in \mathcal{F}_i$ for $1 \leq i \leq k$, satisfies $\bigcap_{i=1}^{k} K_i \neq \emptyset$.

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Theorem (Colorful Helly, Lovász, '82)

A family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ split into $d+1$ color classes that satisfy the colorful intersection hypothesis has a class with non-empty intersection.

Colorful Helly's Theorem for Boxes

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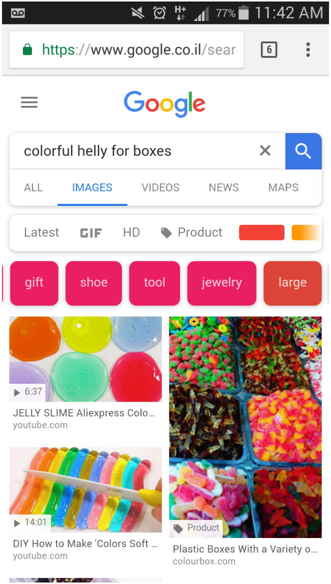

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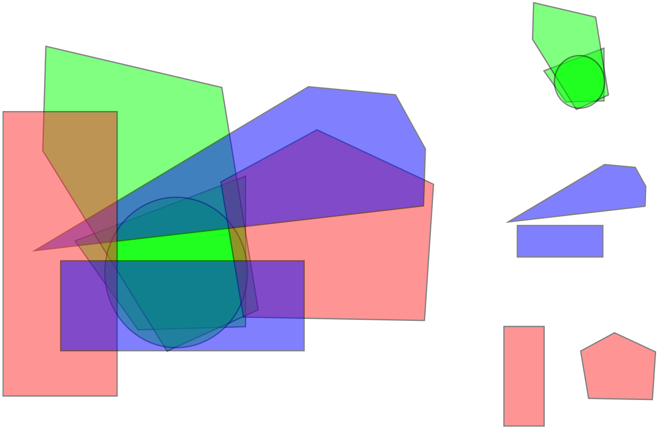

And the rest of them?

What happens with the rest of the colors?

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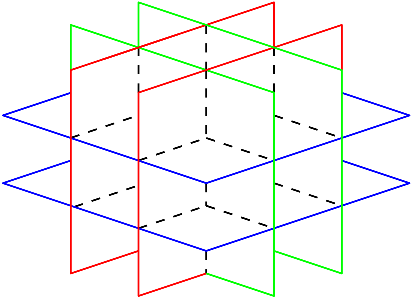

A cute but very easy result

Theorem

Let k be an integer in $[d+1]$. A family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ split into $d+1$ color classes that satisfy the colorful intersection hypothesis has k color classes all of whose sets can be pierced by a single $(k-1)$-flat.

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In particular, there is an additional class that can be pierced by a single line, a third that can be pierced by a plane, etc.

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Proof.

We perform a generic projection to $\mathbb{R}^{d-k+1}$. We use very colorful Helly, (Arocha et al.): if we have $m+\ell$ color classes in $\mathbb{R}^m$ and the colorful intersection hypothesis holds, then there are ℓ of them that can be simultaneously pierced by a single point. $\square$

Change the dimension of transversals

Problem

Let $1 \leq k \leq d$ be an integer and $\mathcal{F}$ a family of convex sets in $\mathbb{R}^d$. Suppose that each subfamily in $\binom{\mathcal{F}}{d+1}$ has a single k-flat transversal. Can we find a transversal for $\mathcal{F}$ with one (or few) k-flats? Can we find a k-flat transversal to a positive fraction of the sets?

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Problem (On the plane, and $k = 1$)

Suppose that each 3 sets of $\mathcal{F}$ have a transversal line. Is it true that $\mathcal{F}$ has a transversal line?

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Suppose that each 3 sets of $\mathcal{F}$ have a transversal line. Is it true that $\mathcal{F}$ has a transversal line? No

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Problem (On the plane, and $k = 1$)

Suppose that each 3 sets of $\mathcal{F}$ have a transversal line. Is it true that $\mathcal{F}$ has a transversal line? No Can it be pierced with few lines? Is there a line that pierces a positive fraction?

Change the dimension of transversals

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Let $1 \leq k \leq d$ be an integer and $\mathcal{F}$ a family of convex sets in $\mathbb{R}^d$. Suppose that each subfamily in $\binom{\mathcal{F}}{d+1}$ has a single k-flat transversal. Can we find a transversal for $\mathcal{F}$ with one (or few) k-flats? Can we find a k-flat transversal to a positive fraction of the sets?

Problem (On the plane, and $k = 1$)

Suppose that each 3 sets of $\mathcal{F}$ have a transversal line. Is it true that $\mathcal{F}$ has a transversal line? No Can it be pierced with few lines? Is there a line that pierces a positive fraction? Yes, yes

Piercing by few hyperplanes

Theorem (Eckhoff '93, Holmsen '13)

On the plane, if each 3 sets can be pierced with a line then:

- *There is a transversal set of 4 lines that pierce $\mathcal{F}$.*
- *There is a line through at least $\frac{1}{3}|\mathcal{F}|$ of the sets of $\mathcal{F}$*

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Theorem (Alon and Kalai '95)

On $\mathbb{R}^d$, if each $d+1$ sets can be pierced with one hyperplane then:

- *$\mathcal{F}$ admits a transversal of $h := h(d)$ hyperplanes.*
- *There is a hyperplane through at least $\delta|\mathcal{F}|$ of the sets of $\mathcal{F}$.*

Transversal lines in high dimensions

What happens for $1 \leq k \leq d - 2$?

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Theorem (Alon et al. '02)

For every integers $d \geq 3$, m and sufficiently large $n_0 > m+4$ there is a family of at least n_0 convex sets so that any m of the sets can be pierced with a line but no $m+4$ of them can.

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For every integers $d \geq 3$, m and sufficiently large $n_0 > m + 4$ there is a family of at least n_0 convex sets so that any m of the sets can be pierced with a line but no $m + 4$ of them can.

In particular, no (p, q)-theorem and not even a fractional theorem.

Our main result

We go back to the Colorful Helly's Theorem context.

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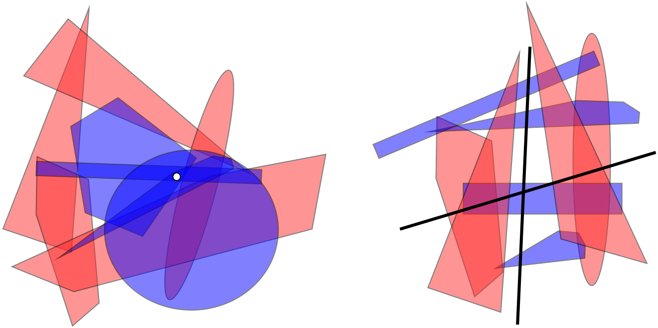
Theorem (MSRPR, '18+)

For each dimension d there exist $f(d)$ and $g(d)$ for which:
If $\mathcal{F}$ is split into $d+1$ color classes with the colorful intersection hypothesis and $\mathcal{F}_{d+1}$ is the intersecting class given by CHT, then either

- *an additional $\mathcal{F}_i$ for $i \in [d]$ can be pierced by $f(d)$ points or*
- *the entire family $\mathcal{F}$ admits a transversal by $g(d)$ lines.*

The 2-colored picture



The Transversal Step-Down Lemma

Theorem (MSRPR, '18+)

For each dimension d, every postive integer m and every $k \in [d+1]$ there exist numbers $F(m,k,d)$ and $G(m,k,d)$ for which:
If $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ and the family of bicolorful intersections

$$\mathcal{I}(\mathcal{A},\mathcal{B}) := \{A \cap B : \ A \in \mathcal{A}, B \in \mathcal{B}\}$$

can be crossed by m k-flats then either:

- *$\mathcal{A}$ can be pierced by $F(m,k,d)$ points, or*
- *$\mathcal{B}$ can be crossed by $G(m,k,d)$ $(k-1)$-flats*

Reminder of the Alon and Kleitman framework

Sketch

- Set-up a useful hypergraph $\mathcal{H}$
- Bound $\nu^*(\mathcal{H})$: Use (weighted) Fractional Helly
- Linear duality: Conclude $\tau^*(H) = \nu^*(H)$ is small
- Break the integrality gap: Use small weak ϵ-nets to bound $\tau(H)$ in terms of $\tau^*(H)$ and d.

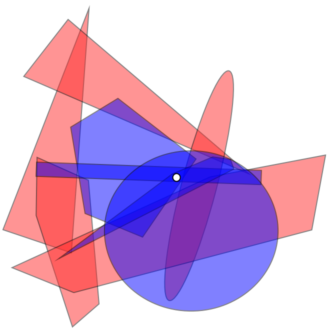
Bi-colored Lemma

Theorem (MSRPR, '18+)

If $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ has the colorful intersection hypothesis then either

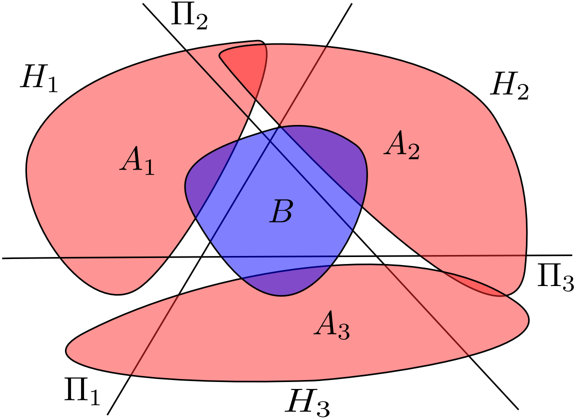
- $\mathcal{A}$ *can be pierced by a single point or*
- $\mathcal{B}$ *can be crossed by d hyperplanes*

The 2-colored picture



Bi-colored Lemma Proof

Proof.



□

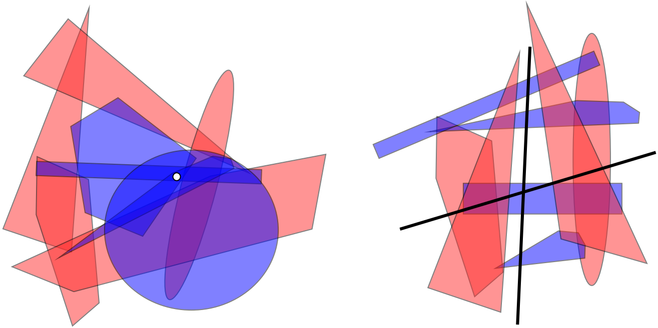
Fractional Bi-colored Lemma

Theorem

For each dimension d, and $0 < \alpha \leq 1$ there exist numbers $\gamma := \gamma(\alpha, d)$ and $\lambda := \lambda(\alpha, d)$ for which:
If $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ satisfies that at least $\alpha|A||B|$ of the pairs $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are intersecting then either:

- *it is possible to pierce $\gamma|\mathcal{A}|$ sets of $\mathcal{A}$ by a single point or*
- *it is possible to cross $\lambda|\mathcal{B}|$ sets of $\mathcal{B}$ by a single hyperplane.*

The 2-colored picture



Our main result

Once again, we want to prove the following:

Theorem

For each dimension d there exist $f(d)$ and $g(d)$ for which:
If $\mathcal{F}$ is split into $d+1$ color classes with the colorful intersection hypothesis and $\mathcal{F}_{d+1}$ is the intersecting class given by CHT, then either

- *an additional $\mathcal{F}_i$ for $i \in [d]$ can be pierced by $f(d)$ points or*
- *the entire family $\mathcal{F}$ admits a transversal by $g(d)$ lines.*

Proof of the Step-Down Lemma

- We setup two simultaneous hypergraphs $\mathcal{H}_0 := \mathcal{H}_0(\mathcal{A})$ and $\mathcal{H}_{k-1} := \mathcal{H}_{k-1}(\mathcal{B})$. We suppose that $\tau(H_0)$ is unbounded.
- We use the Alon-Kleitmain scheme to conclude that there is a bad weight function for $\mathcal{H}_0$.
- We give a weight function for $\mathcal{H}_{k-1}$. By pidgeon-hole principle in the heaviest m-flat Π we have a positive fraction of bicolored intersections.
- We apply the fractional bicolored version (in $\Pi \approx \mathbb{R}^k$). We get a positive fraction piercing point for $\mathcal{H}_{k-1}$. Thus, we have bounded $\nu^*(\mathcal{H}_{k-1})$.
- We apply linear duality.
- We finish by using m small hyperplane weak ϵ-nets.

Proof of Main Theorem

$\mathcal{I}(\mathcal{F}_1, \ldots, \mathcal{F}_d)$ Can be pierced by $\mathbb{R}^d$

$\mathcal{F}_1$ Can be pierced by few points

$\mathcal{I}(\mathcal{F}_2, \ldots, \mathcal{F}_d)$ Can be pierced by few hyperplanes

$\mathcal{F}_2$ Can be pierced by few points

$\mathcal{I}(\mathcal{F}_3, \ldots, \mathcal{F}_d)$ Can be pierced by few $(d-2)$-flats

$\cdots$ $\cdots$

$\mathcal{F}_{d-1}$ Can be pierced by few points

$\mathcal{F}_d$ Can be pierced by few lines

$\square$

Characterization up to transversal dimension

Theorem

For all $1 \leq i \leq d$ there exist numbers $f(i, d)$ and $g(i, d)$ for which: Let $\mathcal{F}$ be a finite $(d+1)$-colored family of convex sets that satisfies the colorful intersection hypothesis. Then there exist $k \in [d]$ and a re-labeling of the color classes $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$ of $\mathcal{F}$ so that

1. *$\bigcup_{1 \leq i \leq k} \mathcal{F}_i$ can be pierced by $f(k, d)$ points, and*
2. *$\bigcup_{k < i \leq d+1} \mathcal{F}_i$ can be crossed by $g(k, d)$ k-flats.*

Conjecture

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For all $1 \leq k \leq d$ *there exist numbers* $h(k, d)$ *with the following property. For any* d*-colored family* $\mathcal{F}$ *of convex sets with the colorful intersection hypothesis there exist numbers* $k_1, \ldots, k_d$ *so that*

1. $\sum_i k_i \leq d$, *and*
2. *each color class* $\mathcal{F}_i$, *can be crossed by* $h(k_i, d)$ k_i*-flats.*

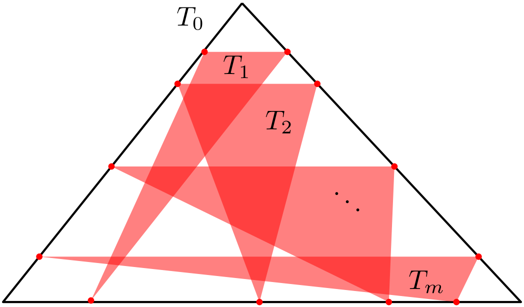
Qualitative lower bounds

Theorem (MSRPR, '18+)

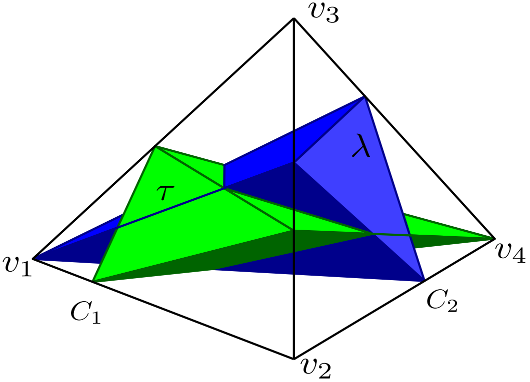
For every $d \geq 2$ and integer $f \geq 1$ there exists a d-colored family $\mathcal{F}$ in $\mathbb{R}^d$ with the colorful intersection hypothesis and the following additional properties:

- *For every $1 \leq i \leq d$, one needs at least f points to pierce the color class $\mathcal{F}_i$.*
- *At least $\lceil \frac{d+1}{2} \rceil$ lines are necessary to cross $\bigcup \mathcal{F}_i$.*

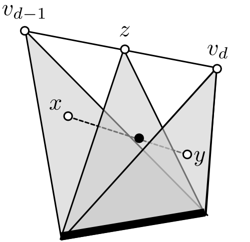
Example on the plane



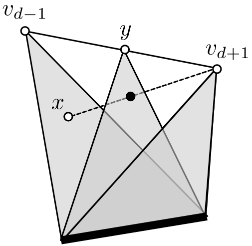
Example in high dimensions



Proof that the example works



$\mathrm{conv}(\{v_1, \ldots, v_{d-2}\})$



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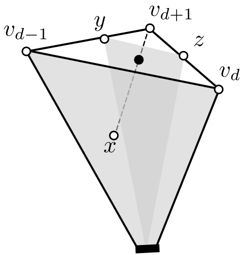
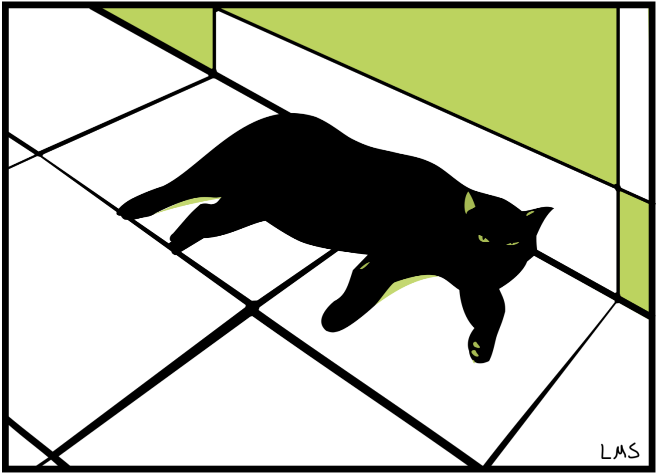


$\mathrm{conv}(\{v_1, \ldots, v_{d-2}\} \setminus \{v_i\})$

Thank you!

Time to wake up!



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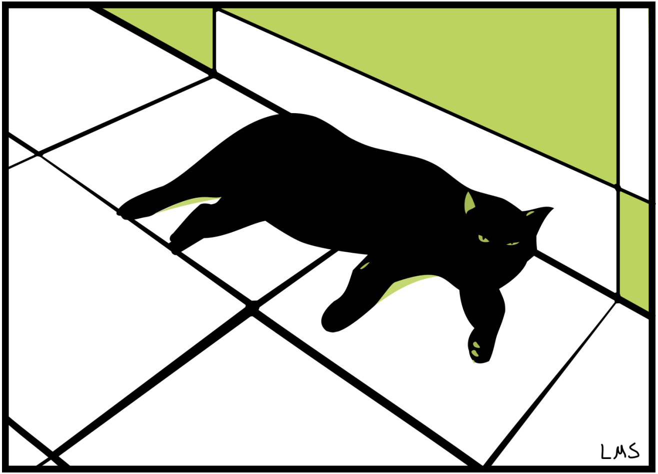

Thank you for your attention!