# Further Consequences of the Colorful Helly Hypothesis: Beyond Point Transversals

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# Helly's Theorem

Let $\mathcal{F}$ be a finite family of at least $d+1$ convex sets in $\mathbb{R}^d$.

## Theorem (Helly's Theorem '23)

*If each subfamily in* $\binom{\mathcal{F}}{d+1}$ *has non-empty intersection, then* $\mathcal{F}$ *has non-empty intersection.*

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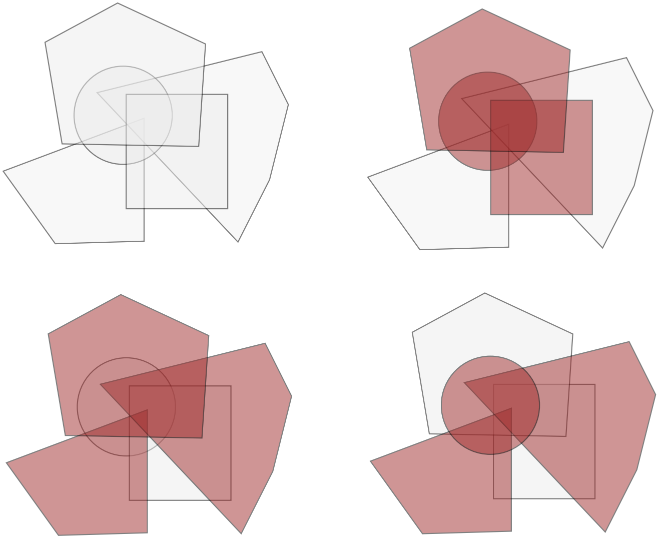
**Note.** Non-empty intersection $\iff$ single piercing point.

# Helly's Theorem



# Variations: Two of (many) possible directions

**Problem (Weaker intersection hypothesis)**

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**Problem (Higher dimensional transversals)**

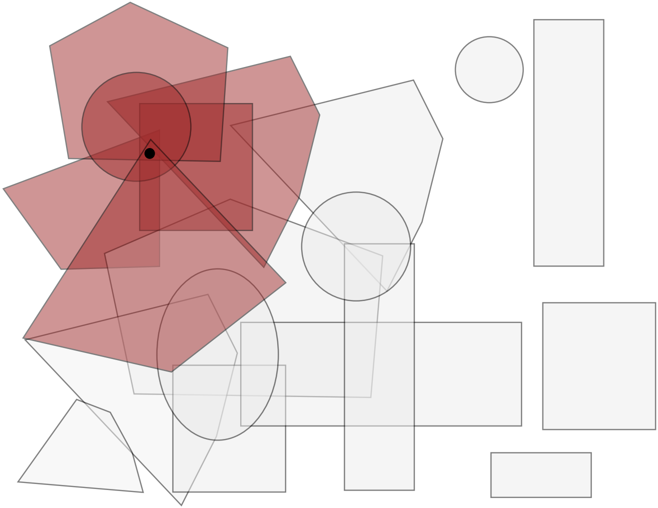
*What happens if we replace piercing points with higher* $k$*-dimensional transversal flats for* $1 \leq k \leq d-1$*?*

# Fractional Helly's Theorem

**Theorem (Fractional Helly's Theorem, Katchalski and Liu '79)**

*For each $\alpha \in (0, 1)$ and $d \geq 1$ there is a $\beta = \beta(\alpha, d) > 0$ with the following property:*
*If at least $\alpha\binom{|\mathcal{F}|}{d+1}$ of the subfamilies in $\binom{\mathcal{F}}{d+1}$ have non-empty intersection, then there is a point that pierces at least $\beta|\mathcal{F}|$ sets of the family $\mathcal{F}$.*

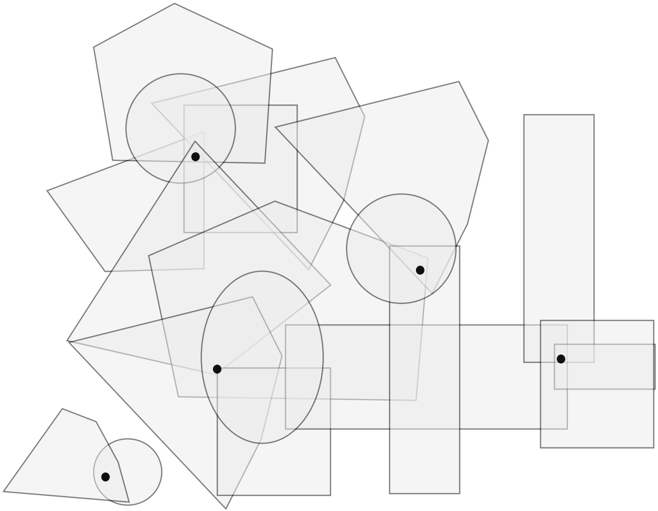
# Fractional Helly's Theorem



# The $(p, q)$-theorem

**Theorem (The $(p, q)$-theorem, Alon and Kleitman '92)**
*For each $p \geq q \geq d + 1$ there is a $P = P(p, q, d)$ with the following property:*
*If any subfamily $\mathcal{F}' \in \binom{\mathcal{F}}{p}$ contains an intersecting family $\mathcal{F}'' \in \binom{\mathcal{F}'}{q}$, then $\mathcal{F}$ can be pierced by $P$ points.*

# The $(p, q)$-theorem



# Colorful Helly's Theorem

## Definition

*Let $k$ be an integer. Let $\mathcal{F}$ be a family of convex sets split into $k$ non-empty color classes $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$. We say that this (split) family has the colorful intersection hypothesis if every rainbow selection $K_i \in \mathcal{F}_i$ for $1 \leq i \leq k$, satisfies $\bigcap_{i=1}^{k} K_i \neq \emptyset$.*

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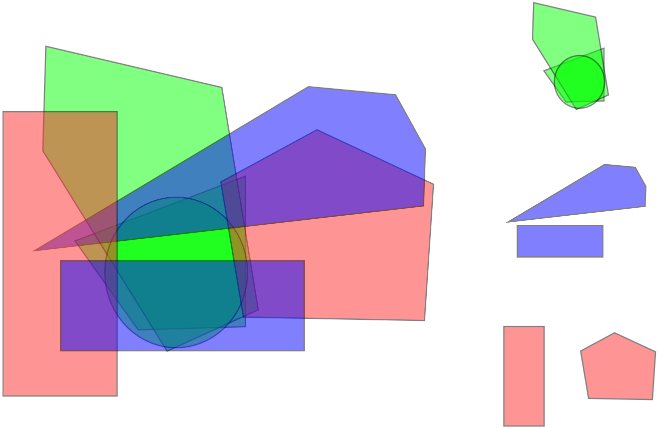
## Theorem (Colorful Helly, Lovász, '82)

*A family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ split into $d+1$ color classes that satisfy the colorful intersection hypothesis has a class with non-empty intersection.*

# Colorful Helly's Theorem



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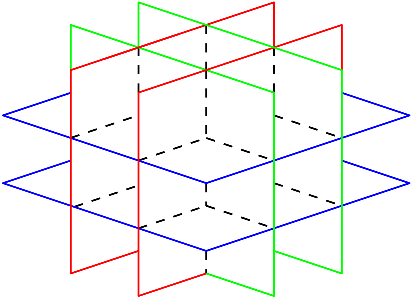

# Change the dimension of transversals

## Problem

*Let $1 \leq k \leq d$ be an integer and $\mathcal{F}$ a family of convex sets in $\mathbb{R}^d$. Suppose that each subfamily in $\binom{\mathcal{F}}{d+1}$ has a single $k$-flat transversal. Can we find a transversal for $\mathcal{F}$ with one (or few) $k$-flats? Can we find a $k$-flat transversal to a positive fraction of the sets?*

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**Problem (On the plane, and $k = 1$)**

*Suppose that each $3$ sets of $\mathcal{F}$ have a transversal line. Is it true that $\mathcal{F}$ has a transversal line?*

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### Problem (On the plane, and $k = 1$)

*Suppose that each 3 sets of $\mathcal{F}$ have a transversal line. Is it true that $\mathcal{F}$ has a transversal line? No*

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*Suppose that each 3 sets of $\mathcal{F}$ have a transversal line. Is it true that $\mathcal{F}$ has a transversal line? No Can it be pierced with few lines? Is there a line that pierces a positive fraction?*

# Change the dimension of transversals

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### Problem (On the plane, and $k = 1$)

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# Piercing by few hyperplanes

Theorem (Eckhoff '93, Holmsen '13)

*On the plane, if each 3 sets can be pierced with a line then:*

- *There is a transversal set of 4 lines that pierce $\mathcal{F}$.*
- *There is a line through at least $\frac{1}{3}|\mathcal{F}|$ of the sets of $\mathcal{F}$*

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**Theorem (Alon and Kalai '95)**

*On $\mathbb{R}^d$, if each $d+1$ sets can be pierced with one hyperplane then:*

- *$\mathcal{F}$ admits a transversal of $h := h(d)$ hyperplanes.*
- *There is a hyperplane through at least $\delta|\mathcal{F}|$ of the sets of $\mathcal{F}$.*

# Transversal lines in high dimensions

What happens for $1 \leq k \leq d - 2$?

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Theorem (Alon et al. '02)

*For every integers $d \geq 3$, $m$ and sufficiently large $n_0 > m+4$ there is a family of at least $n_0$ convex sets so that any $m$ of the sets can be pierced with a line but no $m+4$ of them can.*

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In particular, no $(p, q)$-theorem and not even a fractional theorem.

# Our main result

We go back to the Colorful Helly's Theorem context.

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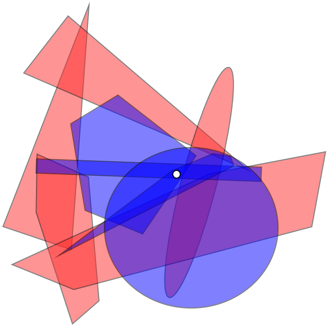
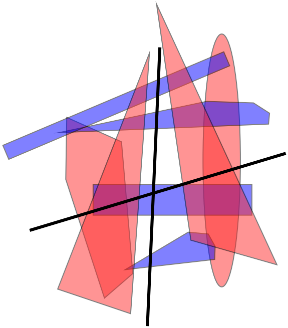
**Theorem (MSRPR, '18+)**

*For each dimension $d$ there exist $f(d)$ and $g(d)$ for which:*
*If $\mathcal{F}$ is split into $d+1$ color classes with the colorful intersection hypothesis and $\mathcal{F}_{d+1}$ is the intersecting class given by CHT, then either*

- *an additional $\mathcal{F}_i$ for $i \in [d]$ can be pierced by $f(d)$ points or*
- *the entire family $\mathcal{F}$ admits a transversal by $g(d)$ lines.*

# The 2-colored picture





# The Transversal Step-Down Lemma

## Theorem (MSRPR, '18+)

*For each dimension $d$, every postive integer $m$ and every $k \in [d+1]$ there exist numbers $F(m,k,d)$ and $G(m,k,d)$ for which:*
*If $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ and the family of bicolorful intersections*

$$\mathcal{I}(\mathcal{A},\mathcal{B}) := \{A \cap B : \ A \in \mathcal{A}, B \in \mathcal{B}\}$$

*can be crossed by $m$ $k$-flats then either:*

- *$\mathcal{A}$ can be pierced by $F(m,k,d)$ points, or*
- *$\mathcal{B}$ can be crossed by $G(m,k,d)$ $(k-1)$-flats*

# Reminder of the Alon and Kleitman framework

Sketch

- Set-up a useful hypergraph $\mathcal{H}$
- Bound $\nu^*(\mathcal{H})$: Use (weighted) Fractional Helly
- Linear duality: Conclude $\tau^*(H) = \nu^*(H)$ is small
- Break the integrality gap: Use small weak $\epsilon$-nets to bound $\tau(H)$ in terms of $\tau^*(H)$ and $d$.

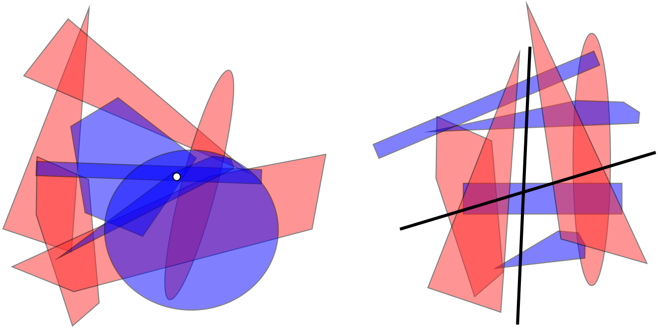
# Bi-colored Lemma

### Theorem (MSRPR, '18+)

*If $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ has the colorful intersection hypothesis then either*

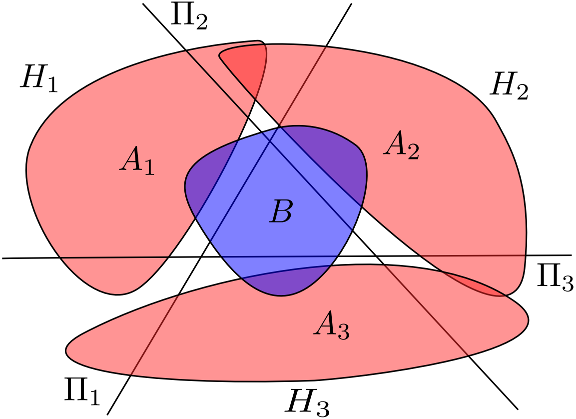
- *$\mathcal{A}$ can be pierced by a single point or*
- *$\mathcal{B}$ can be crossed by $d$ hyperplanes*

# The 2-colored picture



# Bi-colored Lemma Proof

**Proof.**




$\square$

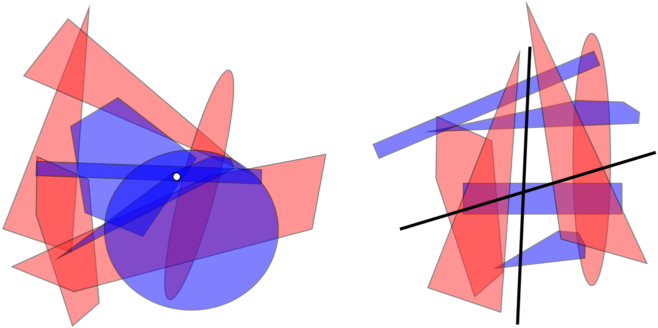
# Fractional Bi-colored Lemma

**Theorem**

*For each dimension $d$, and $0 < \alpha \leq 1$ there exist numbers $\gamma := \gamma(\alpha, d)$ and $\lambda := \lambda(\alpha, d)$ for which:*
*If $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ satisfies that at least $\alpha|A||B|$ of the pairs $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are intersecting then either:*

- *it is possible to pierce $\gamma|\mathcal{A}|$ sets of $\mathcal{A}$ by a single point or*
- *it is possible to cross $\lambda|\mathcal{B}|$ sets of $\mathcal{B}$ by a single hyperplane.*

# The 2-colored picture



# Our main result

Once again, we want to prove the following:

**Theorem**

*For each dimension $d$ there exist $f(d)$ and $g(d)$ for which:*
*If $\mathcal{F}$ is split into $d+1$ color classes with the colorful intersection hypothesis and $\mathcal{F}_{d+1}$ is the intersecting class given by CHT, then either*

- *an additional $\mathcal{F}_i$ for $i \in [d]$ can be pierced by $f(d)$ points or*
- *the entire family $\mathcal{F}$ admits a transversal by $g(d)$ lines.*

# Proof of the Step-Down Lemma

- We setup two simultaneous hypergraphs $\mathcal{H}_0 := \mathcal{H}_0(\mathcal{A})$ and $\mathcal{H}_{k-1} := \mathcal{H}_{k-1}(\mathcal{B})$. We suppose that $\tau(H_0)$ is unbounded.
- We use the Alon-Kleitmain scheme to conclude that there is a bad weight function for $\mathcal{H}_0$.
- We give a weight function for $\mathcal{H}_{k-1}$. By pidgeon-hole principle in the heaviest $m$-flat $\Pi$ we have a positive fraction of bicolored intersections.
- We apply the fractional bicolored version (in $\Pi \approx \mathbb{R}^k$). We get a positive fraction piercing point for $\mathcal{H}_{k-1}$. Thus, we have bounded $\nu^*(\mathcal{H}_{k-1})$.
- We apply linear duality.
- We finish by using $m$ small hyperplane weak $\epsilon$-nets.

# Proof of Main Theorem

$$
\begin{tikzcd}
 & & \mathcal{I}(\mathcal{F}_1,\ldots,\mathcal{F}_d) \arrow[dl] \arrow[d] & \text{Can be pierced by } \mathbb{R}^d \\
\mathcal{F}_1 & \text{Can be pierced by few points} & \mathcal{I}(\mathcal{F}_2,\ldots,\mathcal{F}_d) \arrow[dl] \arrow[d] & \text{Can be pierced by few hyperplanes} \\
\mathcal{F}_2 & \text{Can be pierced by few points} & \mathcal{I}(\mathcal{F}_3,\ldots,\mathcal{F}_d) \arrow[dl] \arrow[d] & \text{Can be pierced by few } (d-2)\text{-flats} \\
 & \cdots & \cdots \arrow[dl] \arrow[d] & \\
\mathcal{F}_{d-1} & \text{Can be pierced by few points} & \mathcal{F}_d & \text{Can be pierced by few lines}
\end{tikzcd}
$$

$\square$

# Characterization up to transversal dimension

**Theorem**

*For all $1 \leq i \leq d$ there exist numbers $f(i,d)$ and $g(i,d)$ for which: Let $\mathcal{F}$ be a finite $(d+1)$-colored family of convex sets that satisfies the colorful intersection hypothesis. Then there exist $k \in [d]$ and a re-labeling of the color classes $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ of $\mathcal{F}$ so that*

1. $\bigcup_{1 \leq i \leq k} \mathcal{F}_i$ *can be pierced by $f(k,d)$ points, and*
2. $\bigcup_{k < i \leq d+1} \mathcal{F}_i$ *can be crossed by $g(k,d)$ $k$-flats.*

# Conjecture

## Conjecture

*For all* $1 \leq k \leq d$ *there exist numbers* $h(k, d)$ *with the following property. For any* $d$*-colored family* $\mathcal{F}$ *of convex sets with the colorful intersection hypothesis there exist numbers* $k_1, \ldots, k_d$ *so that*

1. $\sum_i k_i \leq d$*, and*
2. *each color class* $\mathcal{F}_i$*, can be crossed by* $h(k_i, d)$ $k_i$*-flats.*

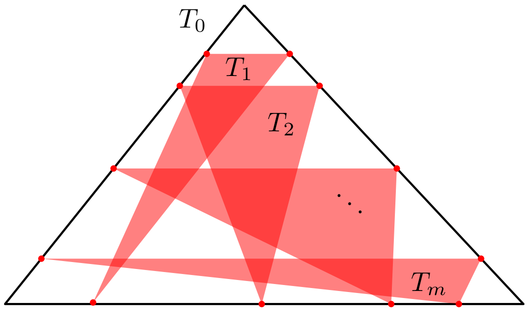
# Qualitative lower bounds

**Theorem (MSRPR, '18+)**

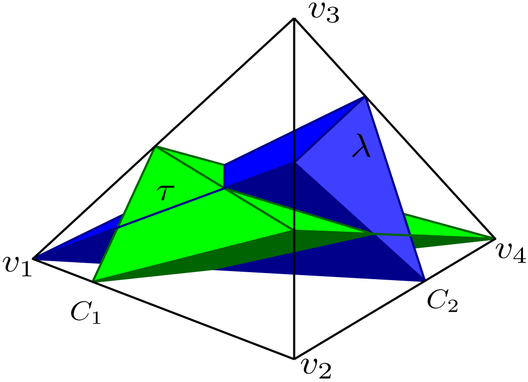
*For every $d \geq 2$ and integer $f \geq 1$ there exists a $d$-colored family $\mathcal{F}$ in $\mathbb{R}^d$ with the colorful intersection hypothesis and the following additional properties:*

- *For every $1 \leq i \leq d$, one needs at least $f$ points to pierce the color class $\mathcal{F}_i$.*
- *At least $\lceil \frac{d+1}{2} \rceil$ lines are necessary to cross $\bigcup \mathcal{F}_i$.*

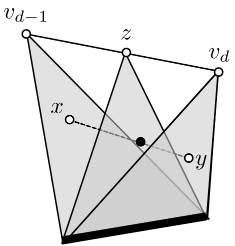
# Example on the plane



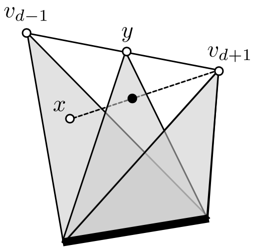
# Example in high dimensions



# Proof that the example works




$\mathrm{conv}(\{v_1, \ldots, v_{d-2}\})$

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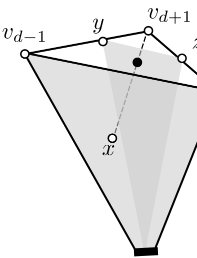
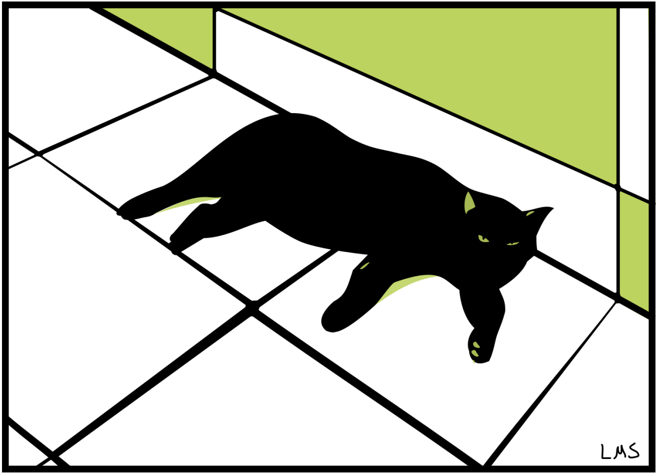



$\mathrm{conv}(\{v_1, \ldots, v_{d-2}\} \setminus \{v_i\})$

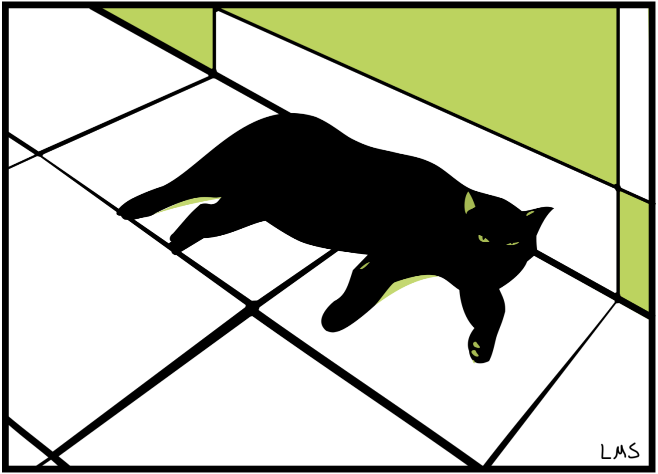
# Thank you!

Time to wake up!



# Thank you!

Time to wake up!




**Thank you for your attention!**