Simulations of Photon Trajectory in Schwarzschild Spacetime

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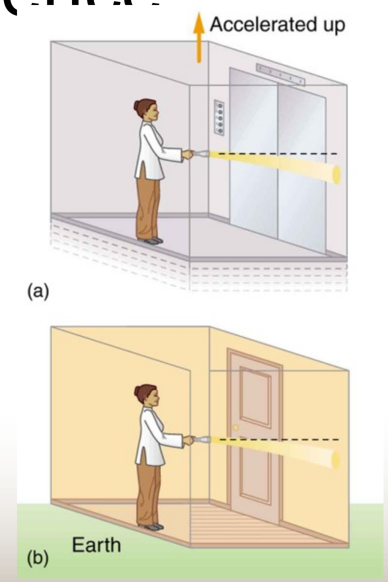
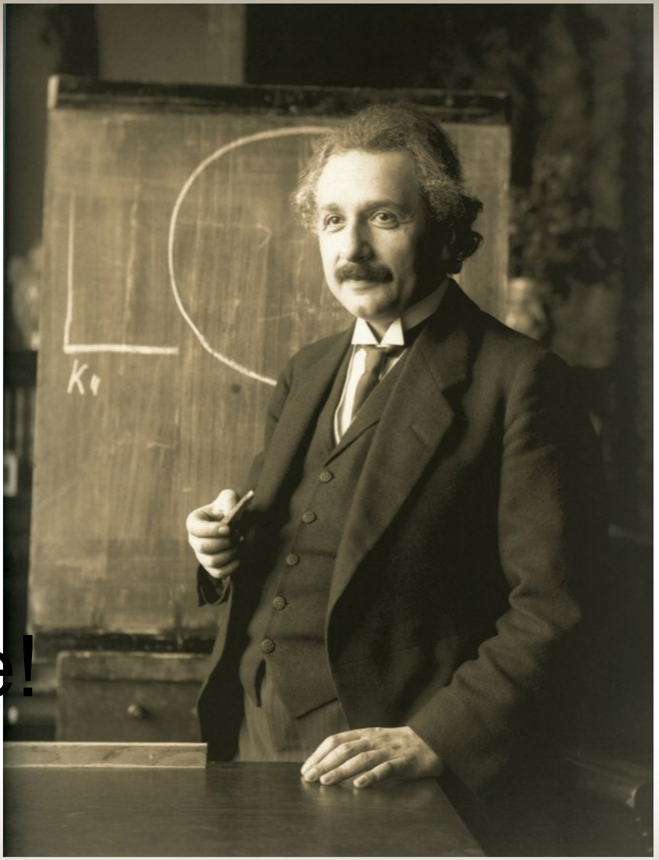
Failure of Newtonian Gravity

- For massless object, zero gravitational force
- Motion should be that of massive object under no force
- Einstein proposed Principle of Equivalence
- Gravity could be transformed away
- Light ray would be bent!
- Experimental results by Eddington in 1919 prove that GR is correct!



Foundation of GR

- What is Gravity?
- Gravity “Doesn’t exist”
- Spacetime is curved
- Particle’s motion follow Geodesic
- How? Minimize Proper time!



Geodesic in GR

- Consider line element ds^2
- In general $ds^2 = g_{\mu\nu}x^{\mu}x^{\nu}$
- $g_{\mu\nu}$is the component of metric tensor
- In GR,x^{ν} runs from 0-3
- x^0 usually refer to coordinate ct
- In our project, mainly focus on Schwarzschild space-time

Symmetry and conservation laws

- Similar to Hamiltonian Mechanics, there is also conserved quantities in GR.
- In particular, if the component of metric tensor of certain coordinate, is unchanged under translational transformation of certain coordinate, there is an associated ***Killing vector*** and Conserved quantities.
- The dot product between Killing Vectors and 4-velocity yields the quantities

Killing Vectors

- Two Killing Vectors associated to coordinate t and ϕ!

- That of $t : \varsigma = (1,0,0,0)$

- That of $\phi : \eta = (0,0,0,1)$

Killing Vectors

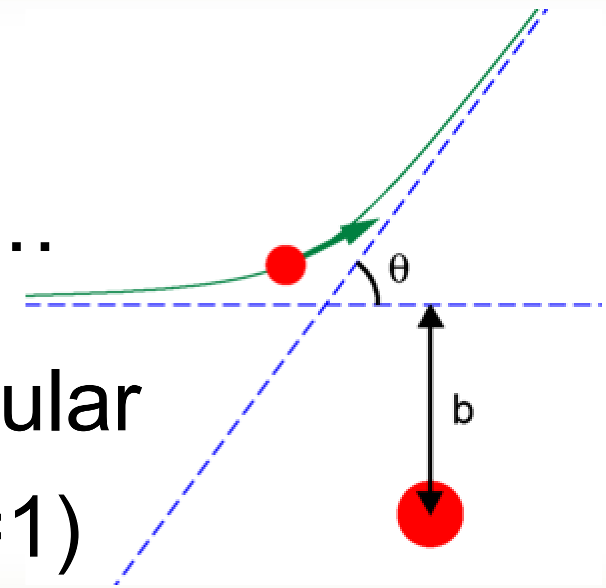
- Schwarzschild Spacetime
- $ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
- Set G,c = 1 to simplify calculation(Geometrize Units!)
- Independent of t and ϕ!

Equation of motion

- By conservation laws, one can obtain the followings
- $-\varsigma \cdot u = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} = e$
- $\eta \cdot u = r^2 \sin^2\theta \frac{d\phi}{d\lambda} = l$
- u is the 4-Velocity, λ is some free parameter, l and e are some conserved quantities

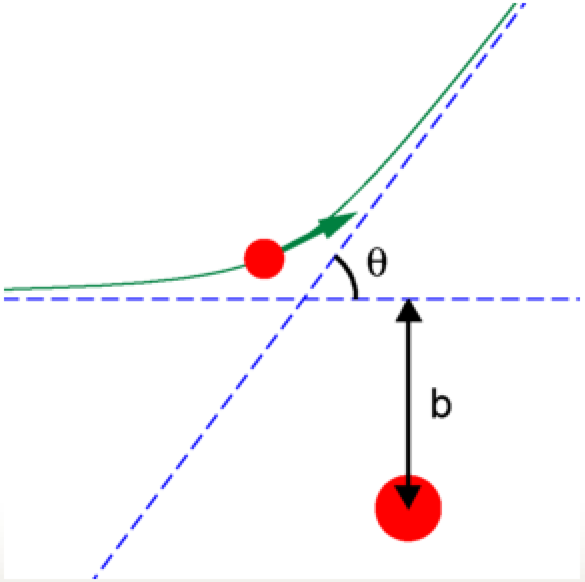
What are the quantities?

- Divide l by e yields
- $\frac{d\phi}{dt} r^2 \left(1 - \frac{2M}{r}\right)^{-1} sin^2 \theta = \frac{l}{e}$
- We can set $\frac{l}{e} = b$, For large r…
- $b \approx r^2 \frac{d\phi}{dt}$, very similar to angular momentum!(Where we set c=1)
- b is the impact parameter



What are the quantities?

- b carries the meaning of impact parameter



Trajectory of photon

- Choose $\theta = \frac{\pi}{2}$ and rewrite the Schwarzschild metric using l and e as defined, we have:

$$\mathrm{ds}^2 = -\left(1 - \frac{2M}{r}\right)e^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{l^2}{r^2}$$

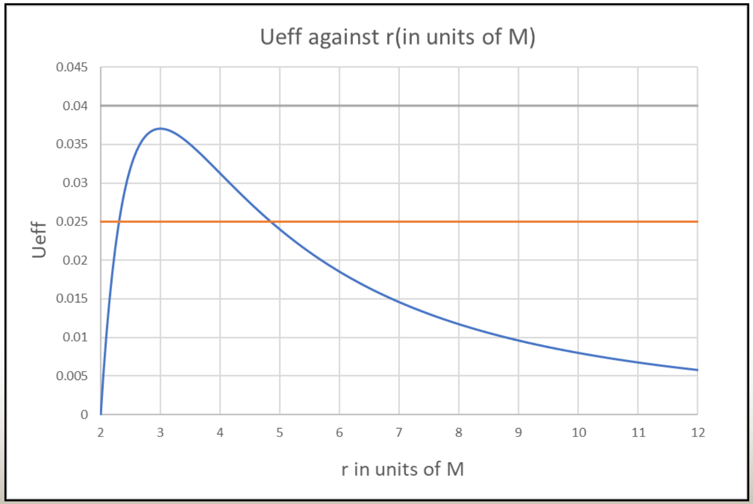
- Since photon follow null-geodesics, we set the term on the right hand side equals to zero
- Radial direction

Trajectory of photon

- $-\left(1-\frac{2M}{r}\right)e^2+\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2+\frac{l^2}{r^2}=0$
- Multiply both side by $\frac{1-\frac{2M}{r}}{l^2}$ and let $b=\frac{l^2}{e^2}$
- Let $W_{eff}=\frac{1}{r^2}\left(1-\frac{2M}{r}\right)$
- $\frac{1}{b^2}=\frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2+W_{eff}$(r)

Effective potential

- The nature of orbit can be deduced form effective potential
- For orbit to exist, $\frac{1}{b^2} \geq W_{eff}$ for any *r*
- Values of $\frac{1}{b^2}$ is important
- Very similar to Classical Orbiting motion



Effective potential

- There is a turning point of effective potential
- Obtained by differentiation
- $W_{eff}(maxmiun) = \frac{1}{27M^2}$
- $r = 3M$
- Values of $\frac{1}{b^2}$ with respect to this value determine to nature of orbits

Special properties

- We can separate into different case
- For ***incoming photon*** infinity…
- If $\frac{1}{b^2} < W_{eff}(maxmium)$, the photon will reach a turning point, and escape to infinity, denote as escaping case.
- If $\frac{1}{b^2} > W_{eff}(maximum)$, the photon will go towards the black hole, denote as absorbing case

Special properties

- For ***outgoing photon*** at some point near black hole…
- If $\frac{1}{b^2} < W_{eff}(maxmium)$, the photon will reach a turning point, being absorbed to black hole, denote as absorbing case.
- If $\frac{1}{b^2} > W_{eff}(maximum)$, the photon will escape to infinity, denote as escaping case.

Special properties

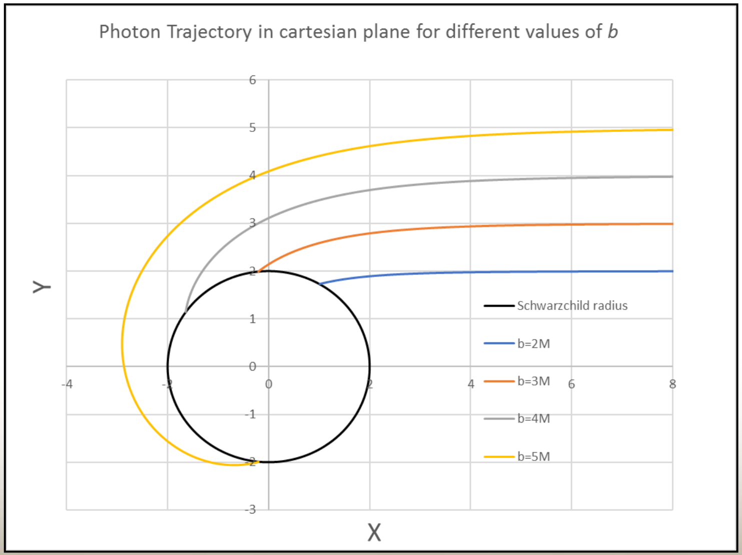
- For both case, if $\frac{1}{b^2} \approx \frac{1}{27M^2}$,
- We expect the photon will reach the unstable maximum, performing some circular orbit
- Then either go towards black hole or escape
- The transition must occur around this value of *b*, which is $b \approx 5.19615$

Circular orbit

- There is a unstable maximum of effective potential.
- If $r = 3M\ and\ \frac{1}{b^2} = \frac{1}{27M^2}$, we expect a circular orbit exists
- But it is unstable!
- So a small perturbation $\delta = \pm 0.0001$ will cause drastic change in orbits

Incoming Photon

- Numerical simulation for $\frac{1}{b^2} > Weff(maxmimum)$ is performed
- All of them absorbed to black hole
- Agrees with prediction!

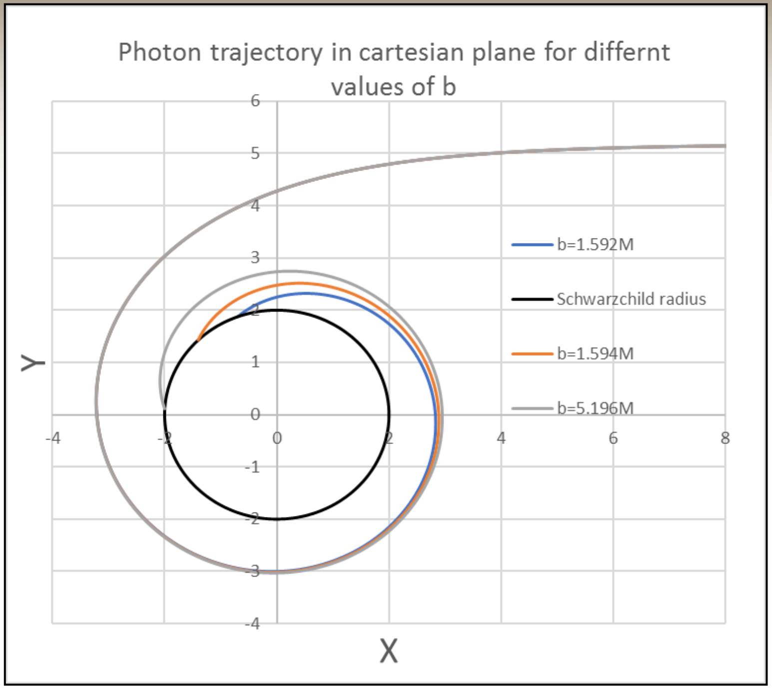


Incoming photon

- What if $\frac{1}{b^2}$ decrease?

- The photon will have more time to “survive” before reaching black hole

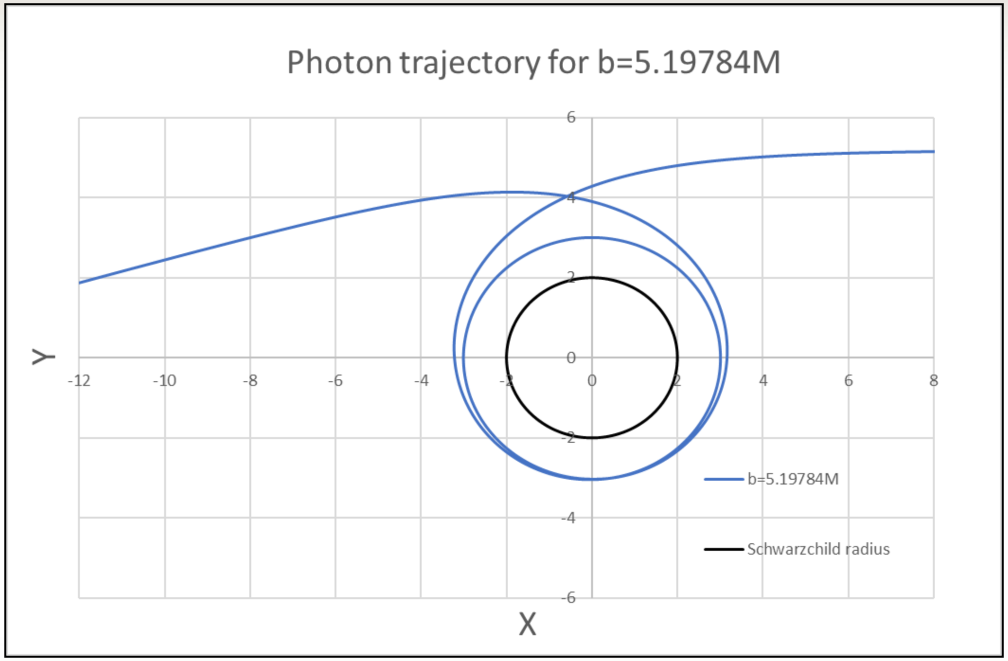
- Some rounding orbit before reaching its destiny, number of revolving increase as $\frac{1}{b^2}$ decrease

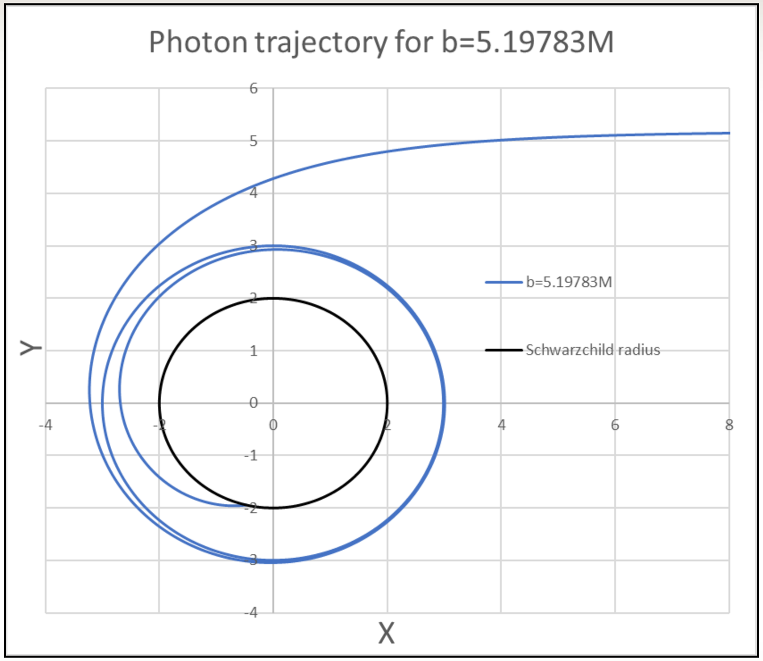


Transition case

- When the $\frac{1}{b^2}$ is very near to transition
- Photon perform more rounding circular orbit before absorbing to black hole or escape
- Agrees with prediction
- But the numerical results give transition between b=5.19783 and b=5.19784
- Deviates with prediction

Transition case

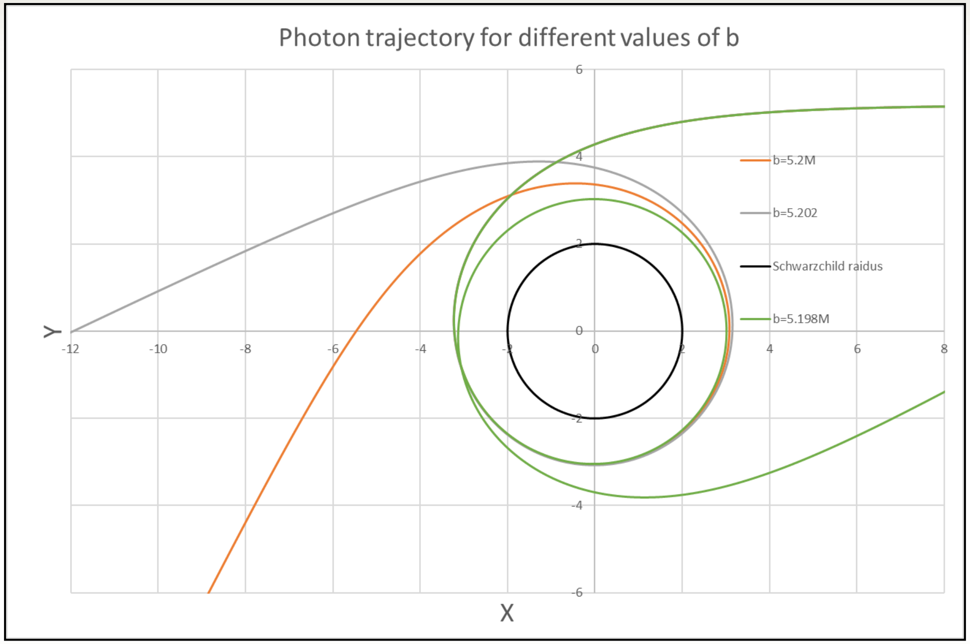


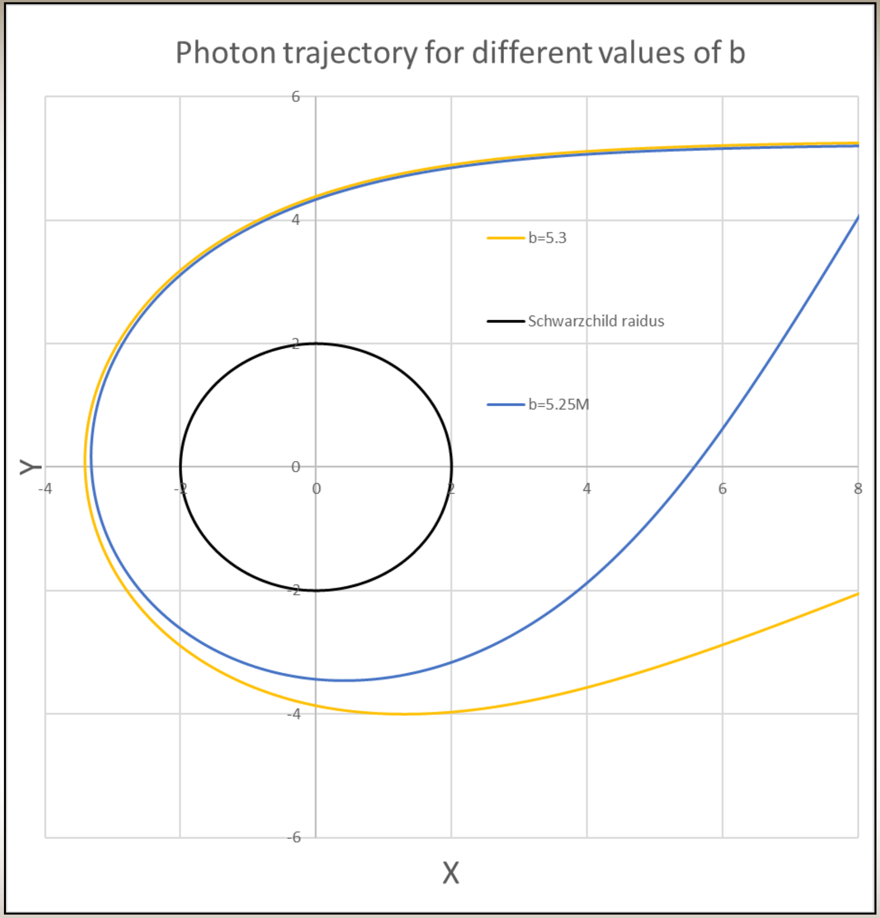


Incoming photon

- Upon transition, the photon are able to escape to infinity
- As $\frac{1}{b^2}$ decrease, the photon are more able to escape to infinity, without rounding around the black hole
- Number of revolve decrease, the direction of escape turns clockwise

Incoming photon





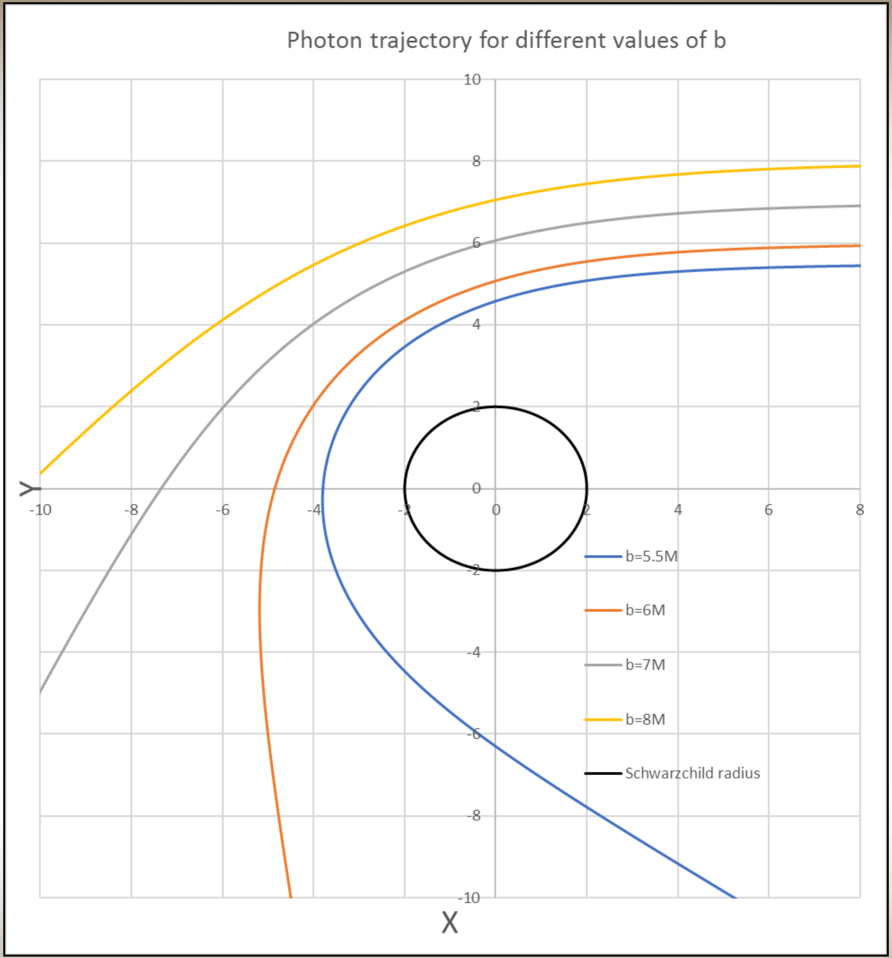
Some interesting fact

- Since the escape direction turns clockwise
- There is some discrete range of *b* such that the escape direction do not point to Quadrant III
- Observers lying on negative x-axis may observe some dark rings around black hole!

Incoming photon

- If $\frac{1}{b^2}$ is small enough, the photon will barley curved around black hole
- Image from far side behind black hole will have apparent location different than the actual location
- Gravitational lensing effects!

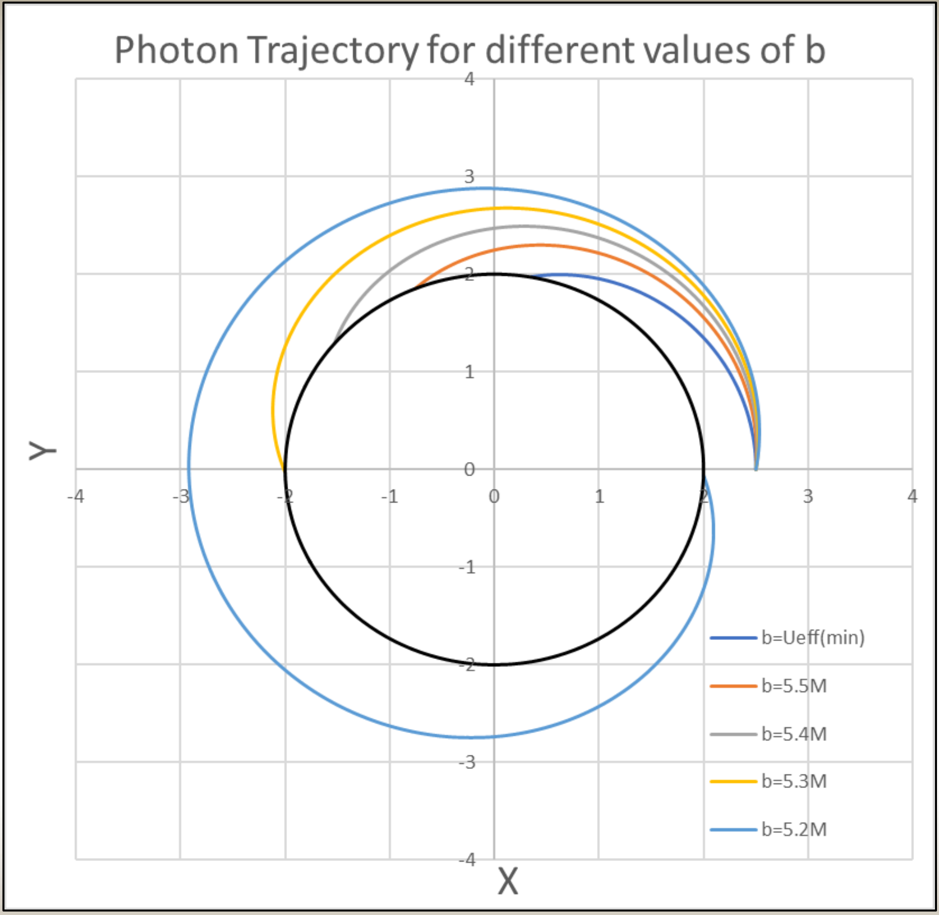
Incoming photon

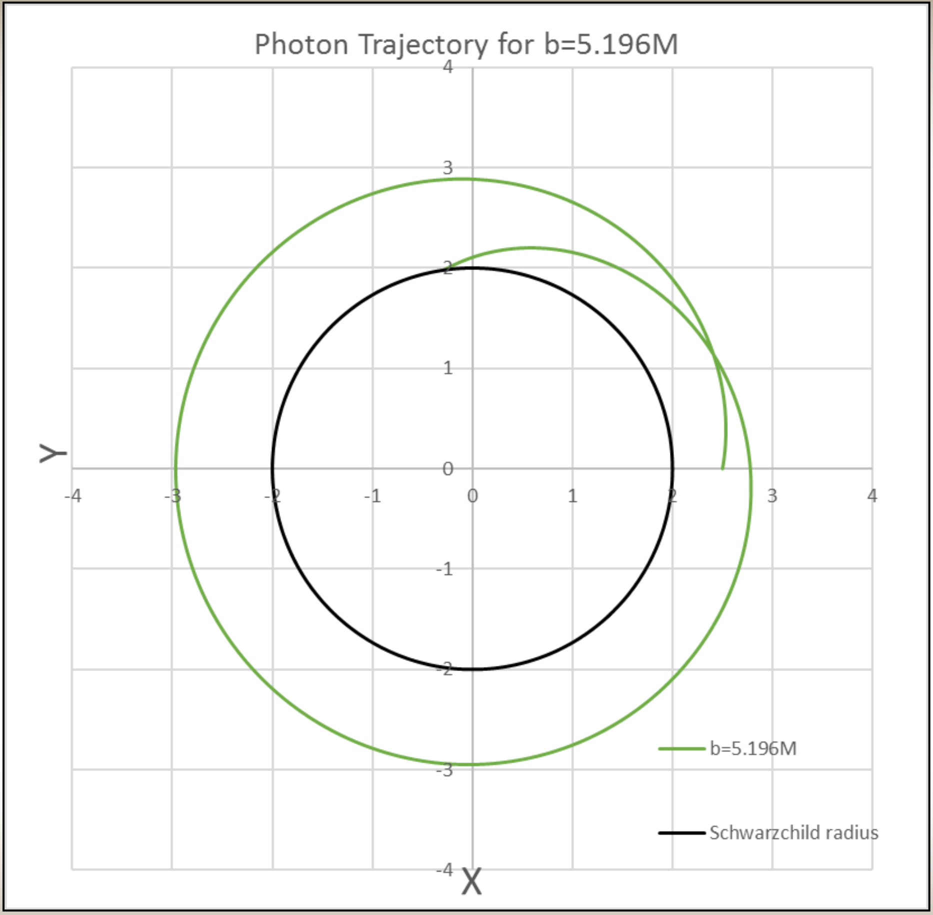


Outgoing photon

- Numerical simulation is performed for $\frac{1}{b^2} < W_{eff}(maximum)$
- As $\frac{1}{b^2}$ increase, photon, has more time to survive
- More rounding orbits around the black hole before reaching its destiny
- The value b indicates the direction of projection, the photon is more outward so that it can escape.

Outgoing photon





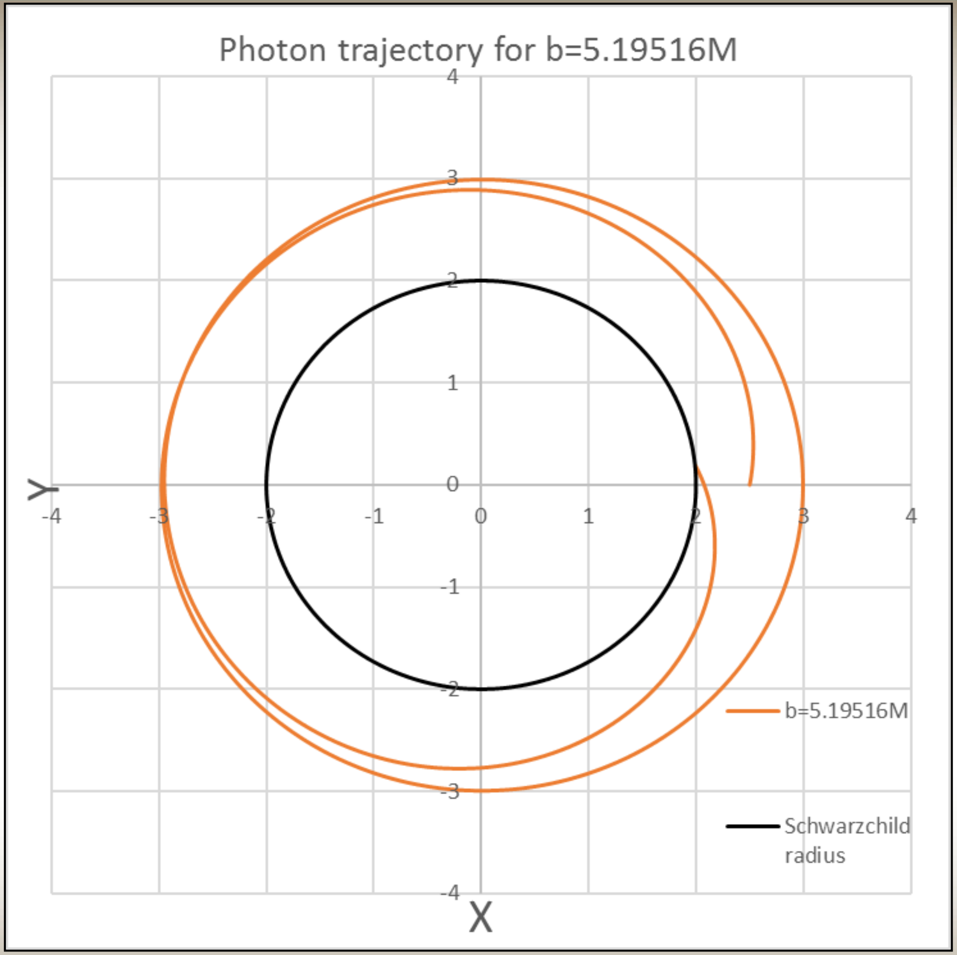
Interesting fact

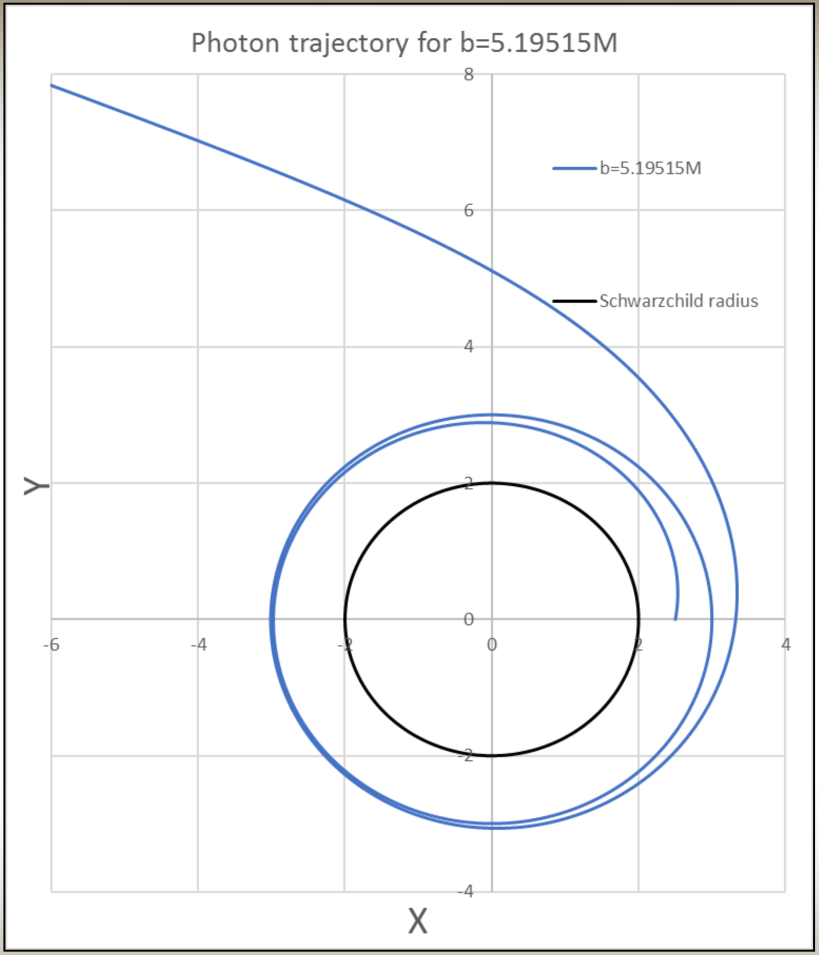
- b represent the initial direction of emission
- But there is a limited value of b
- $\frac{1}{b^2} \geq W_{eff}(at\ emission\ point)$ is to be satisfied
- Thus there is some direction which light could not be emitted
- If you are close to black hole and look outward, you would not able to see the object that is in front of you!

Transition case

- When the $\frac{1}{b^2}$ is very near to transition
- Photon perform more rounding circular orbit before absorbing to black hole or escape
- Agrees with prediction
- But the numerical results give transition between b=5.19515 and b=5.19516
- Deviates with prediction

Outgoing photon

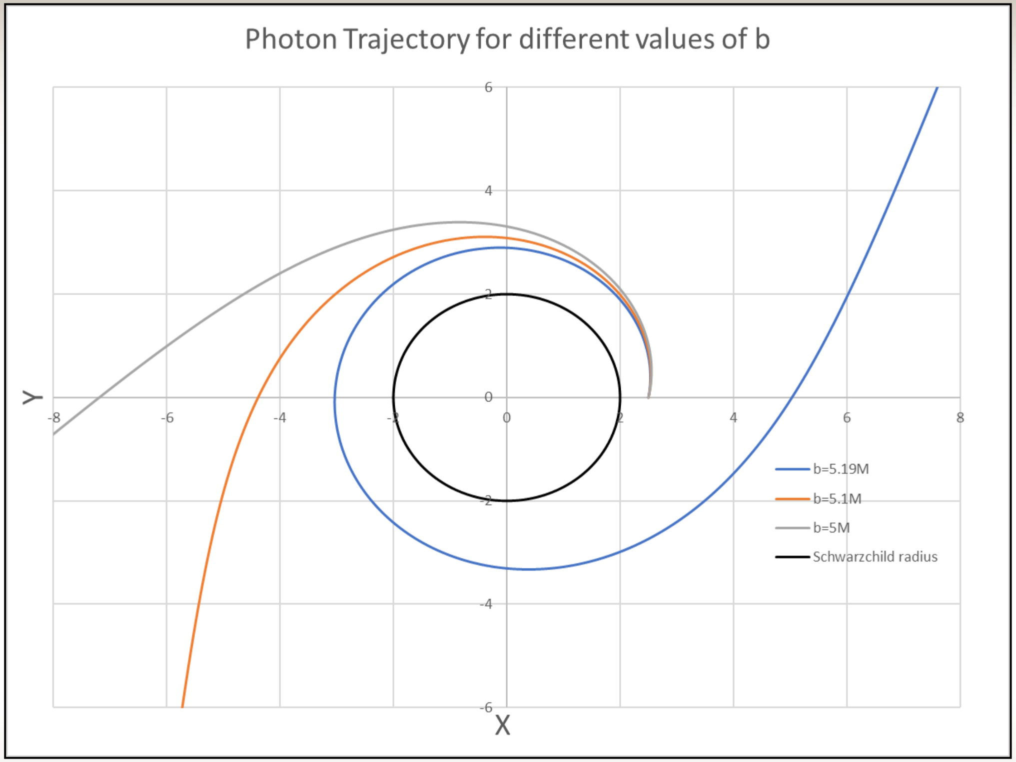


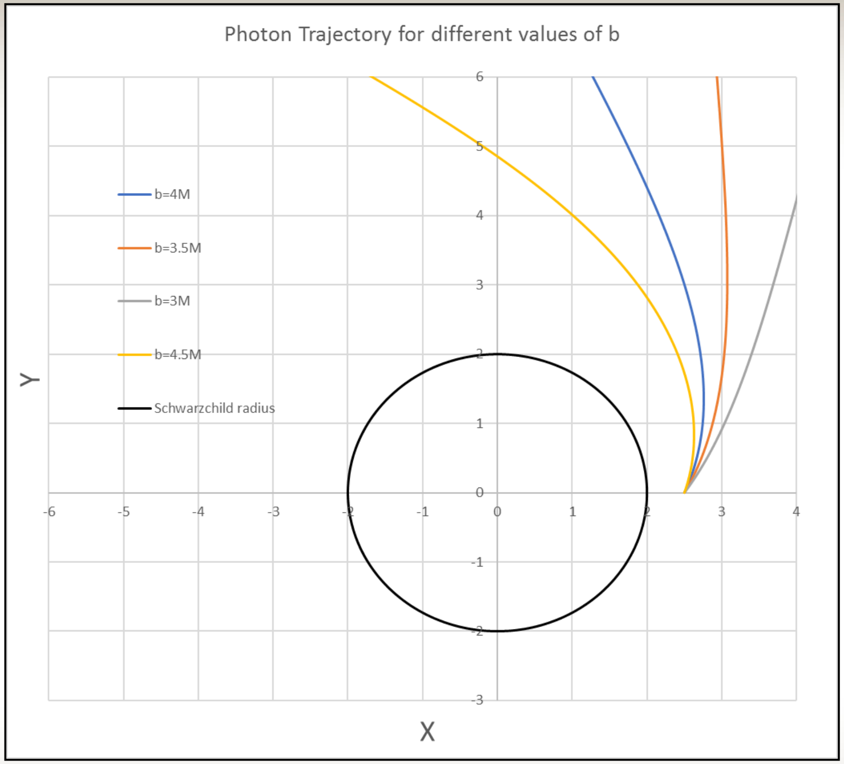


Outgoing photon

- Upon transition, the photon are able to escape to infinity
- As $\frac{1}{b^2}$ increase, the photon are more able to escape to infinity, without rounding around the blackhole for several time
- Number of revolve decrease, the direction of escape turn clock-wise

Outgoing photon

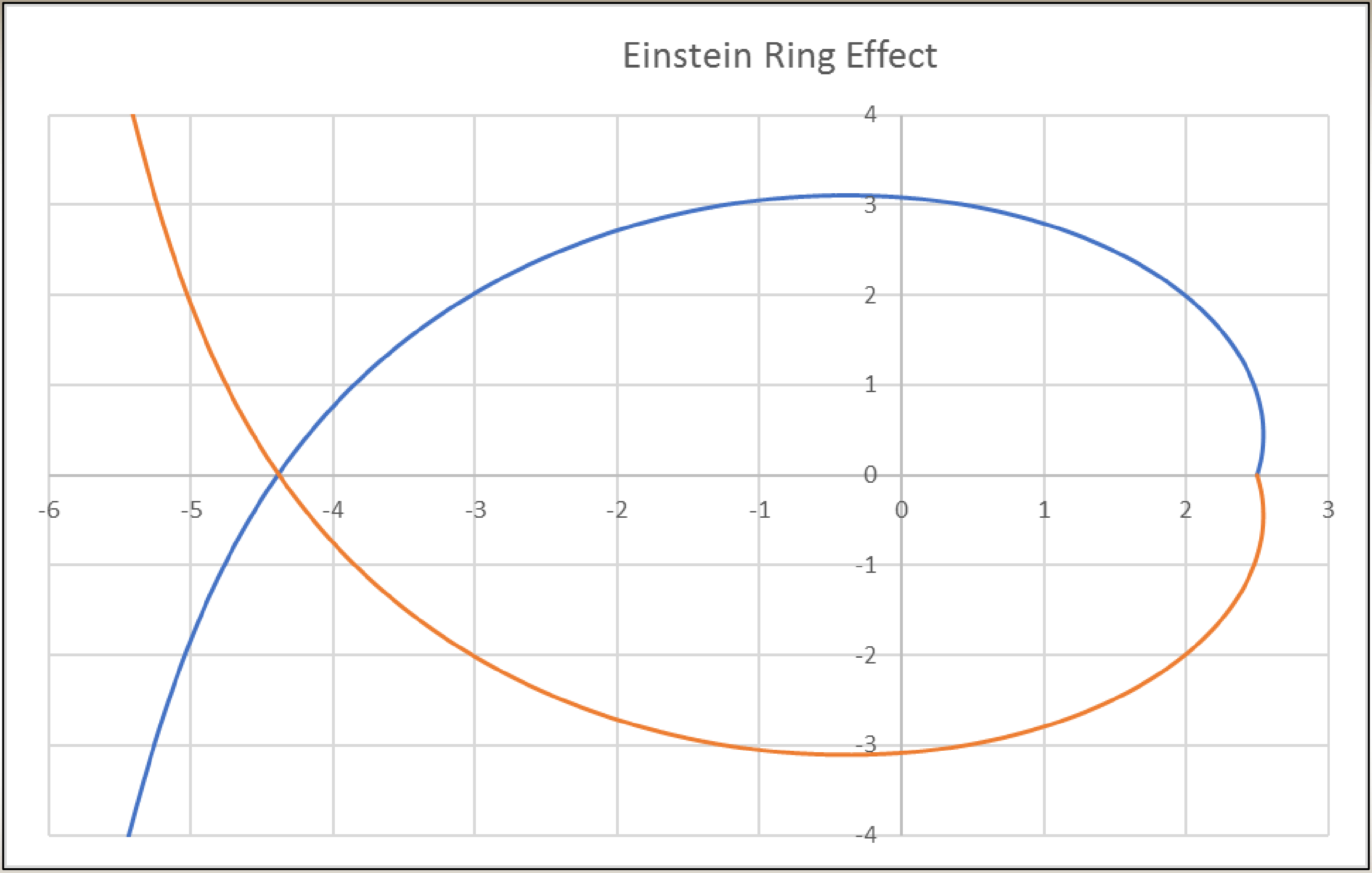




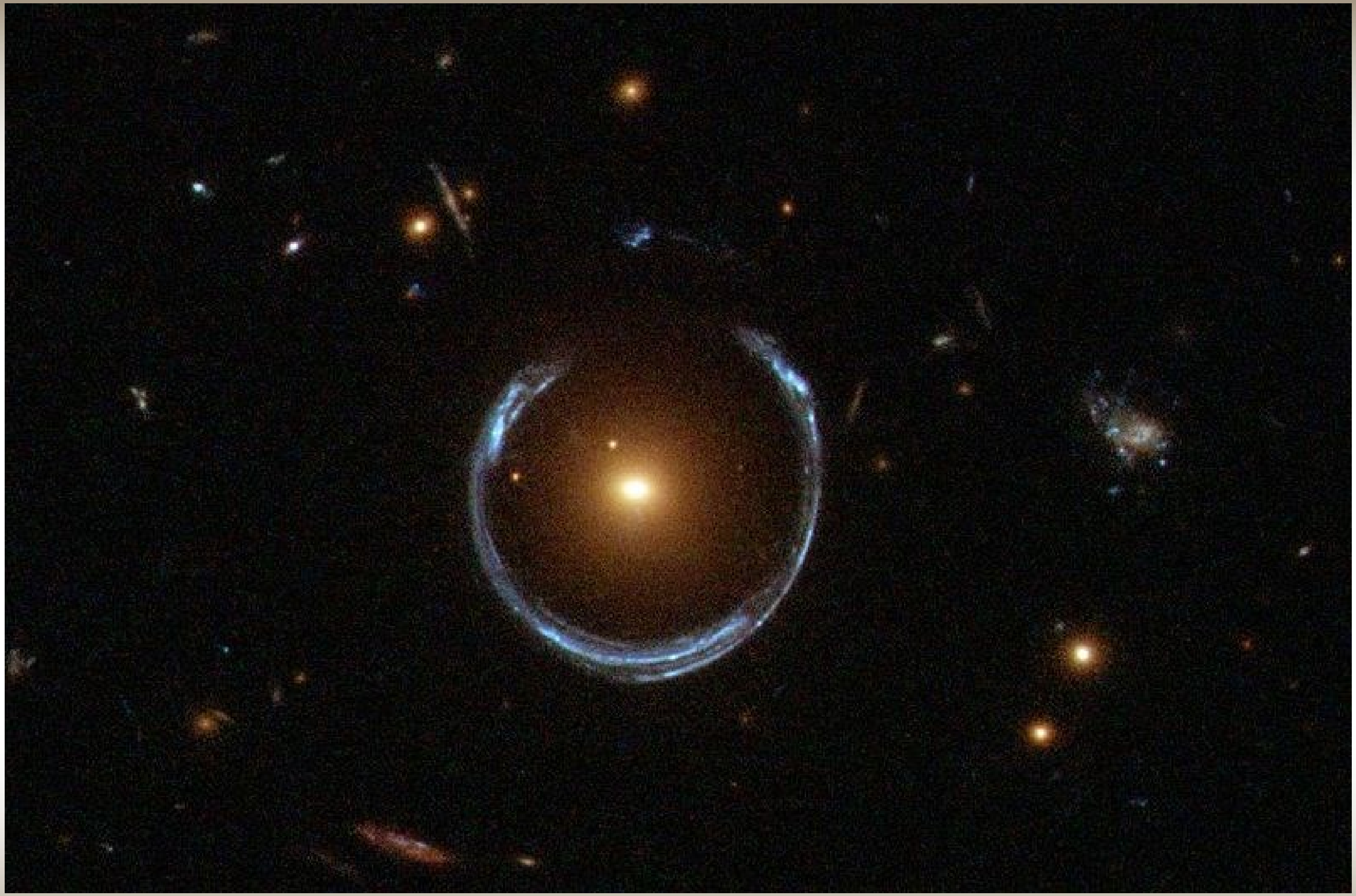
Some interesting fact

- For some b, the same light ray with same initial condition but with emission direction symmetric about x-axis, would cross each other at the negative x-axis
- Object block by the black hole could be seen by observer!
- The object appeared to be distorted around black hole, or multiple image is produce around black hole.
- It is the Einstein ring!

Einstein Ring effect



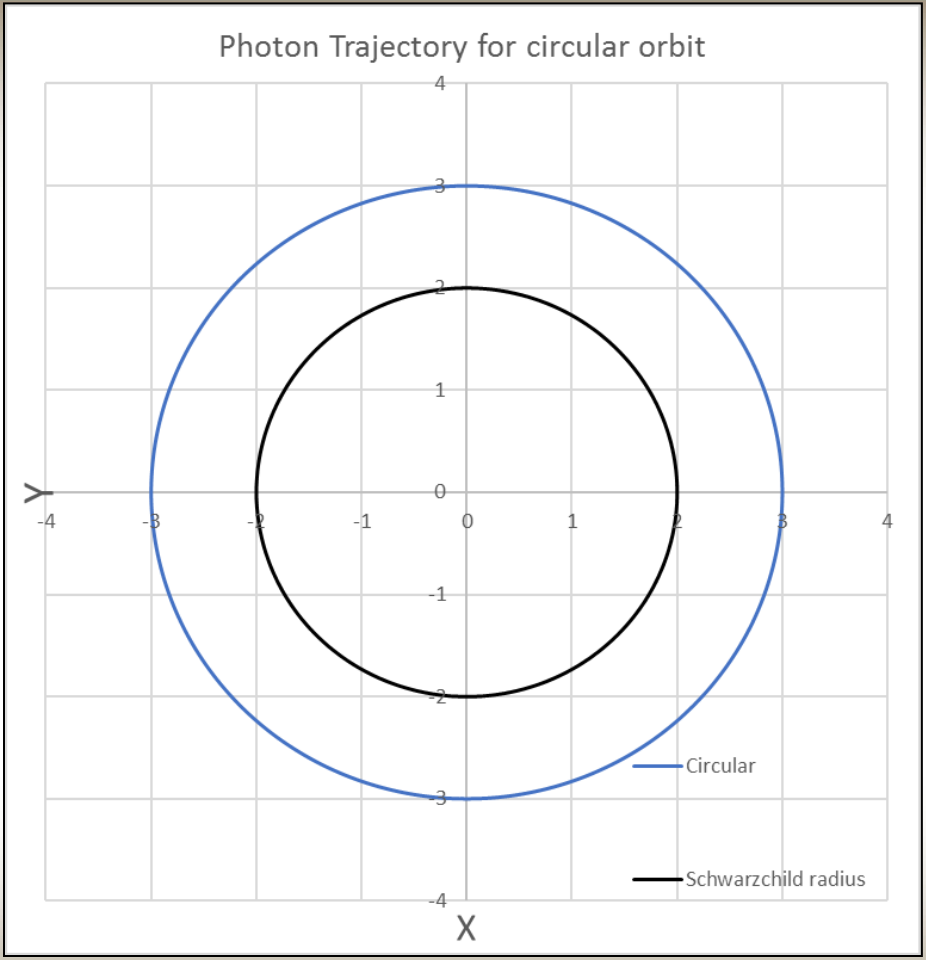
Einstein ring effect



Circular orbit

- When $\frac{1}{b^2} = \frac{1}{27M^2}$ and initial radial position $r = 3M$, there will be a circular orbit
- The circular orbit is independent of the initial angular position
- All photon emitted at $r = 3M$ with $\frac{1}{b^2} = \frac{1}{27M^2}$ will go into circular orbit
- Usually named as Photon sphere

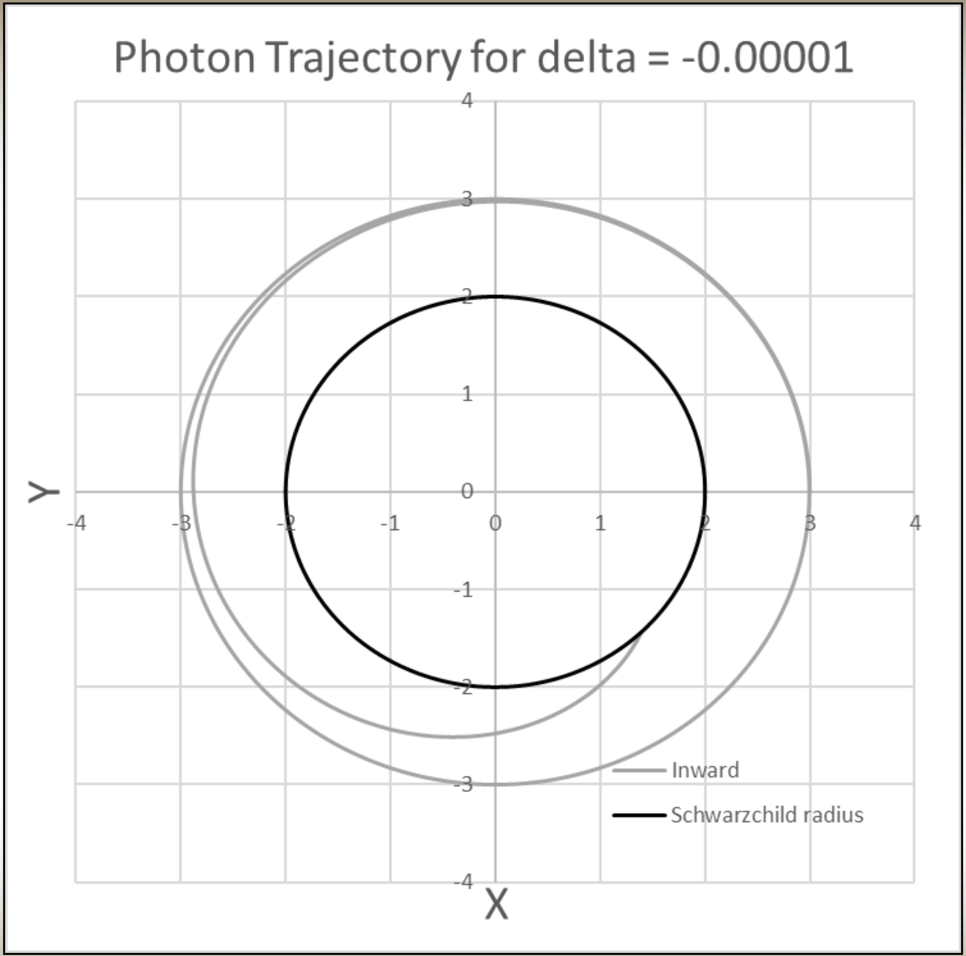
Circular orbit

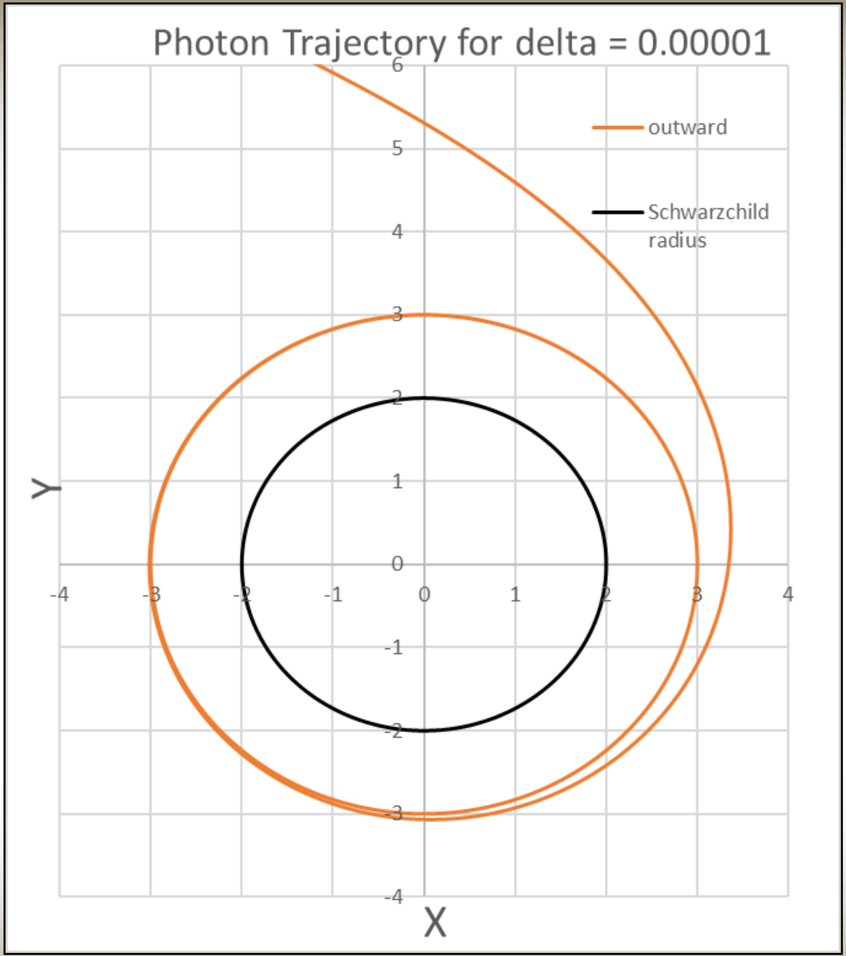


Circular orbit

- The circular orbit is unstable
- A small perturbation of $\delta = \pm 0.0001$ could be added
- The results would be very different
- The photon will first perform some circular orbit
- The unstable nature either push the photon into black hole, or escape

Unstable Circular orbit





Interesting fact

- The photon sphere will not be affected if the photon is emitted in a direction symmetric about x-axis
- If one could held stationary at r=3M, the light ray from its back could reach its eye by circular orbit
- One can see his/her back by looking in front!

Conclusion

- Successfully calculated the trajectory of photon near Schwarzschild spacetime
- For circular orbit, the photon sphere effect has been demonstrated. The instability of orbits around the turning point had also been verified.

Conclusion

- For outgoing photon and incoming photon
- The nature of the trajectory agrees with the prediction from the graph of effective potential.
- The Einstein ring effect and the partially dark ring region around black hole has been demonstrated.
- The instability of the turning point had also been verified.

Source of error?

- The numerical results of the point of transition deviates with prediction
- The error may due to the accumulation of global error of the RK4 method
- Moreover, error of finite digit of floating point number will also contribute.
- Can be reduced using smaller step size.

That’s the end. Thank you for your attention ☺