# Algebraic Topology Notes

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# 1 Introduction to Algebraic Topology

# 1.1 Topologies to Algebra

We want to turn topological spaces into algebraic objects through operations called Invariants. An example is that if two topological spaces X and Y are isomorphic, the translated algebraic object should also be isomorphic

TOP 
$$\leadsto$$
 ALG 
$$X \mapsto A(X) \quad \text{``algebraic objects''}$$
 
$$X \cong Y \mapsto A(X) \cong A(Y)$$

#### Example 1.1.1: Examples of Algebraic Objects

Some examples of algebraic objects:

- The set of Connected Components  $\pi_0(X)$
- The Fundamental Group  $\pi_1(X)$
- Higher homotopy groups  $\pi_n(X)$

Note: the more involved the algebraic invariant is, the more topology it sees. Computability problem leads to Homology Theory (this is non-examinable)

# 1.2 Connected Spaces

#### Recall 1.2.1: Topologies

A topology on X,  $\mathcal{T}$ , is a family of subsets s.t.

- $\emptyset, X \in \mathcal{T}$
- Closed under finite intersection,  $U_1, U_2 \in \mathcal{T} \implies U_1 \cap U_2 \in \mathcal{T}$
- Closed under arbitrary unions

Examples of topological spaces:

- Trivial topology  $\mathcal{T} = \{\emptyset, X\}$
- Discrete Topology  $\mathcal{T} = \mathcal{P}(X)$
- $\mathbb{R}$  or anything made from a metric space

#### Definition 1.2.2: Connected Spaces

A topological space X is **connected** if  $X = A \uplus B$  (A and B are open) means that  $A = \emptyset$  or A = X

#### Prop 1.2.3: Connected Spaces and Clopens

X is connected iff the only clopens are  $\emptyset$ , X

Proof.

$$(\Longrightarrow)$$
: A clopen then  $X = A \uplus A^C \Longrightarrow A = \emptyset, X$  (both  $A$  and  $A^C$  open)  
 $(\Longleftrightarrow)$ :  $A \uplus B \Longrightarrow A = B^C \Longrightarrow A$  is clopen

#### Examples:

- $\mathbb{R}$  is connected. Opens are generated by intervals like  $(-\infty, a)$ , (a, b),  $(a, \infty)$ .
- The trivial topology is connected. (by definition since there are only two sets).
- The discrete topology is *not* connected, unless  $X = \emptyset$  or  $X = \{*\}$  in which case it coincides with the trivial topology.

#### Prop 1.2.4: Connectedness of Maps

For a continuous map  $f: X \to Y$ , and X connected, we have that f(X) is connected.

Proof. 
$$f(X) = U \uplus V \implies f^{-1}(U) \uplus f^{-1}(V) = X \implies f^{-1}(U) = \emptyset, X$$

#### Corollary 1.2.5

If  $X \cong Y$  are homeomorphic, then X is connected iff Y is connected

## Prop 1.2.6

The relation  $(x \sim y \text{ if } \exists \text{ connected subset } A \subseteq X \text{ s.t. } x, y \in A)$  is an equivalence relation.

*Proof.* We show the relation fulfils all requirements for an equivalence relation:

- Reflexivity:  $x \sim x$ :  $x \in \{x\} \subseteq X$
- Symmetry:  $x \sim y \iff y \sim x$  tautological (we don't specify between x and y so just take y = x and x = y)
- Transitivity:  $x \sim y \land y \sim z \implies x \sim z, \, x,y \in A, \, y,z \in B$ . Claim:  $A \cup B$  is connected. Proof in workshop

## Definition 1.2.7: Components

The equivalence classes of the above proposition are called **components** 

#### 1.3 Path-Connectedness

#### Definition 1.3.1: Path

A path in X is a continuous map  $\alpha: I \to X$  for  $I = \mathcal{T}(0,1)$ .  $x \sim y \iff \exists \alpha: I \xrightarrow{\text{path}} X \text{ s.t. } \alpha(0) = x, \alpha(1) = y$ 

 $x \sim y$  is an equivalence relation due to the following operations on paths:

- 1. Constant path. If  $x \in X$ ,  $c_X : I \to X$ ,  $c_x(t) := X$
- 2. Path reversal. Let  $\alpha: I \to X$  be a path. Then  $\overline{\alpha}: I \to X, t \mapsto \alpha(1-t)$
- 3. Path concatenation:  $\alpha: I \to X$ ,  $\beta: I \to X$  s.t.  $\alpha(1) = \beta(0)$ . Then

$$(a*b)(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2} \\ \beta(2t-1), \frac{1}{2} \le t \le 1 \end{cases}$$

#### **Definition 1.3.2: Connected Components**

The set of path-connected components (equivalence classes) is denoted by  $\pi_0(X)$ 

#### Remarks:

- We have that  $X \cong Y \implies \pi_0(X) \cong \pi_0(Y)$
- ullet Path-connected  $\Longrightarrow$  Connected (but not vice-versa). Counterexample: Pick

$$X = \{(x, \sin(\frac{1}{x})) \mid 0 < x < 1\}$$

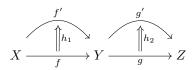
is connected but not path connected

#### **Definition 1.3.3: Homotopy**

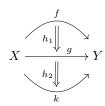
Let  $f, g: X \to Y$  continuous maps. A **homotopy** from f to g is a continuous map  $h: X \times I \to Y$  s.t.

$$h(-,0) = f \iff h(x,0) = f(x), \forall x$$
  
 $h(-,1) = g$ 

**Terminology**: f is homotopy equivalent to g if there exists a homotopy h homotopies on homotopies - horizontal composition



Vertical composition



#### 1.4 Homotopy Equivalence

#### Definition 1.4.1: Homotopy Equivalence

Two spaces X, Y are called **homotopy equivalent** or **of the same homotopy type**, and denoted by  $X \simeq Y$ , if there exists a homotopy equivalence  $f: X \to Y$ 

**Note**: We use  $\cong$  for homeomorphisms and  $\simeq$  for homotopy equivalences.

#### Lemma 1.4.2: Homotopy inverses

Let  $f: X \to Y$  and  $g: Y \to Z$  with homotopy inverses  $\tilde{f}: Y \to X$  and  $\tilde{g}: Z \to Y$  respectively. Then,  $\tilde{f} \circ \tilde{g}: Z \to X$  is a homotopy inverse of  $g \circ f: X \to Z$ . In particular,  $X \simeq Y$  and  $Y \simeq Z$  implies  $X \simeq Z$ .

#### Definition 1.4.3: Contractible Spaces

A space X is called **contractible** if it is homotopy equivalent to a point, i.e.  $X \simeq *$ 

**Example**:  $\mathbb{R}^n$  is contractible. Let  $x_0$  be a fixed point in  $\mathbb{R}^n$  and define the (straight line) homotopy  $h: c_{x_0} \simeq \mathrm{id}_{\mathbb{R}^n}$  by

$$h(x,t) = (1-t)x_0 + tx$$

#### Remark 1.4.4

- 1. Contractible spaces are path-connected
- 2. The converse does not hold. For example  $X = \mathbb{S}^1$  will lead to a counterexample.
- 3. A contractible space X is contractible at any point  $x_0$ . Since X is path-connected a path from x to x' defines a homotopy  $c_x \simeq c_{x'}$
- 4. Any two maps  $f, g: X \to Y$  are homotopic if Y is contractible.

#### Definition 1.4.5: Retractions and Detractions

- A retract of X onto a subspace  $A \subset X$  is a map  $r: X \to A$  such that  $r|_A = \mathrm{id}_A$ . Equivalently, this is a map  $r: X \to X$  such that  $r^2 = r$  and r(X) = A
- A deformation retract of X onto A is the additional datum of a homotopy  $h: \mathrm{id}_X \simeq i \circ r$ , where  $i: A \hookrightarrow X$  denotes the inclusion

In other words, a deformation retract is a homotopy  $h: X \times I \to X$  such that h(x,0) = x and  $h(x,1) \in A$  for all  $x \in X$  and h(a,1) = a for all  $a \in A$ 

Not all retracts can form deformation retracts. For instance, notice that the retract X onto a point  $\{x_0\}$  can be a deformation retract if and only if X is contractible.

# Remark 1.4.6

Ordinary tomotopy are not interesting for paths, e.g.  $\alpha:I\to X$  is homotopic to a constant path

# Prop 1.4.7

Path concatenation is unital and associative up to relative union

# Lemma 1.4.8

Let  $\alpha: I \to X$  be a path, and  $\lambda: I \to I$  continuous s.t.  $\lambda(0) = 0$  and  $\lambda(1) = 1$ . Then,  $\alpha \circ \lambda \cong \alpha$  (relative to  $\{0,1\}$ )

# Definition 1.4.9: Fundamental Group

The fundamental group of X at  $x_0 \in X$  is the homotophy equivalence class of "loops" at  $x_0$ . i.e. paths in X s.t.  $\alpha(0) = \alpha(0) = x$