Group Theory Notes

Leon Lee

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1 Recapping from previous courses

1.1 Groups, Subgroups, Cosets, oh my!

Definition 1.1.1: Group

A **group** consists of a set G together with a function $G \times G \to G$ which maps an ordered pair $(g, h) \in G \times G$ to an element $g * h \in G$. The following axioms must be satisfied:

1. **Associativity**: $(g * h) * k = g * (h * k)$ for each triple $(g, h, k) \in G \times G \times G$
2. **Identity**: There is an element $e \in G$ s.t. $e * g = g = g * e$ for each element $g \in G$
3. **Inverse**: To each element $g \in G$ there is an element $h \in G$ s.t. $gh = e = hg$

Every single course seems to have its own definition for a group, this one is a bit more compact than others. FPM had the **closure** axiom, but that is satisfied by the definition of the function $G \times G \to G$

Note on notation: Usually just write gh instead of $g * h$. Additionally g^{-1} is the inverse of g

Definition 1.3.1: Subgroups

If H is a nonempty subset of G, then H is a **subgroup** provided that

1. $hk \in H$ for all $h, k \in H$
2. $h^{-1} \in H$ for each $h \in H$

Alternatively, we can say "H is closed under the group operation"

Notation

- $H \leq G$ means H is a subgroup of G, whereas $H \subseteq G$ means H is a subset of G.
- $H < G$ means that H is a subgroup of G and also $H \neq G$.
- A subgroup is **proper** if $H \neq G$
- A subgroup is **non-trivial** if $H \neq \{e\}$

Note: $e \in H$ follows from the definition, and associativity follows from the fact that G is a group. Any subgroup H of G is a group using the same product as G

Definition 1.3.6: Cosets

Let $H \leq G$ and let $g \in G$. Then the **left coset of H determined by** g is the set $gH := \{gh : h \in H\}$. $Hg := \{hg : h \in H\}$ is the **right coset of H determined by** g

Notation

- The set of left cosets of H is denoted G/H, the set of right cosets is denoted $H\backslash G$.
- The number of elements in a group G is denoted by $\#G$ or $|G|$, and is known as the **order** of G. We will use $|G|$ in this course.
- The number of left cosets of a subgroup H of G is the **index** of H in G and is denoted by $|G : H|$ or $[G : H]$ (That is, $[G : H] = |G/H|$). We will use $[G : H]$ in this course.

Theorem 1.1.1: Coset Lemmas

If H if finite, $|gH| = |H|$
If $g_1H \cap g_2H \neq \emptyset$, then $g_1H = g_2H$

Theorem 1.3.8: Lagrange's Theorem

Let H be a subgroup of a finite group G. Then

$$|G| = [G : H] \cdot |H|$$

Consequences and Results

- The order of a subgroup must divide the order of the group, e.g. A group of order 12 cannot have a subgroup of order 8
- The converse of Lagrange's Theorem is false, e.g. there is a group of order 12 that doesn't have a subgroup of order 6

Example: If $G = S_3$ and $H = \{e, (12)\}$, what are the left cosets of H?

$$H = eH = \{e, (12)\} \quad \{(23), (132)\} \quad \{(13), (123)\}$$

Example: If $H \triangle G$ then the left cosets are right cosets

Proof.

$$gH = \{gh : h \in H\} = \{(ghg^{-1})g : h \in H\} \subseteq Hg$$

□

Theorem 1.3.9: Cauchy's Theorem

If G is a finite group and p is a prime that divides the order of G, then G has a subgroup of order p

Definition 1.3.10: Order of an element

Let $g \in G$. The **order** of g is the least positive integer such that $g^n = g$ or ∞ if such n does not exist. We write the order of g as $o(g)$. Note that $o(g) = |\langle g \rangle|$.
It thus follows from Lagrange's Theorem that the order of an element of G must divide $|G|$, since if $o(g) = n$ then $\langle g \rangle = \{g, g^2, \ldots, g^n = e\}$ is a subgroup of G. We also have:

Corollary 1.3.11: If $|G|$ is prime, then G is cyclic

Example A: Examples of Groups and Subgroups

- $\mathbb{Z}/n$ under addition, where $a * b = a + b \mod n$
- $(\mathbb{R}\backslash\{0\}, \times)$, or $K\backslash\{0\}$ for any field K
- Alternating group: $A_n \subset S_n$ - permutations from an even number of transpositions?

1.2.1 S_n, the n**-th symmetric group** is the group of permutations of $\{1, 2, \ldots, n\}$. The group operation is composition of fucntions

1.2.6 A group $(G, *)$ is **abelian** if $g * h = h * g$ for all $g, h \in G$

- Let F be a field
  - The **general linear group** $GL(n, F)$ is the set of all invertible $n \times n$ matrices
  - The **special linear group** $SL(n, F)$ is the set of all invertible $n \times n$ matrices with determinant equal to 1

1.3.5 Let G be a group and let $g \in G$. Then $\langle g \rangle := \{g^n : n \in \mathbb{Z}\}$ is a subgroup of G. It is called the **subgroup generated by** g. If $G = \langle g \rangle$ for some $g \in G$, then G is referred to as **cyclic**

1.3.7 A *subgroup* $H \leq G$ is **normal** if $gH = Hg$ for all $g \in G$. In this case we write $H \trianglelefteq G$

1.2 Group Homomorphisms

Definition 1.4.1: Group Homomorphism

Let G, H be groups. A function $\phi : G \to H$ such that

$$\phi(ab) = \phi(a)\phi(b)$$

for all $a, b \in G$ is a **group homomorphism**

Example: If ϕ is a group homomorphism then $\phi(e) = e$

Proof.

$$\begin{aligned} \phi(e \cdot e) &= \phi(e)\phi(e) \\ \implies \phi(e) &= \phi(e)\phi(e) \\ \text{multiply by } \phi(e)^{-1} \quad e &= \phi(e)^{-1}\phi(e)\phi(e) = \phi(e) \end{aligned}$$

□

Example: Show $\phi(g^{-1}) = \phi(g)^{-1}$

Proof.

$$\begin{aligned} \phi(g \cdot g^{-1}) &= \phi(g)\phi(g^{-1}) \\ \phi(e) &= \phi(g)\phi(g^{-1}) \\ \text{Multiply by } \phi(g)^{-1} \quad \phi(g)^{-1}\phi(e) &= \phi(g)^{-1}\phi(g)\phi(g^{-1}) \\ \phi(g)^{-1} &= \phi(g^{-1}) \end{aligned}$$

□

Example 1.4.2: Cyclic Group Homomorphisms

Let C_n be the **cyclic group of order** n. We can think of C_n as the set of rotations of an equilaterial n-gon. If g is a rotation of $2\pi/n$ radians, then $C_n = \{g, g^2, \ldots, g^n = e\}$. The group C_n is cyclic since all elements are powers of a single element g. Then

$$\begin{aligned} \phi : \mathbb{Z} &\to C_n \\ a &\mapsto g^a \end{aligned}$$

is a group homomorphism. (proof in lecture notes)

Definition 1.4.3: Group Isomorphism

If G and H are groups and $\psi : G \to H$ is a bijective *group homomorphism*, we say that ψ is a **group isomorphism** and that G and H are **isomorphic**

Definition 1.4.5: Kernel of a Homomorphism

Let $\phi : G \to H$ be a group homomorphism. The **kernel** of ϕ is $\{g \to G : \phi(g) = e\}$

Definition 1.4.6: Automorphisms

Let G be a group. The st of all isomorphisms $\phi : G \to G$ is also a group. It is called the **automorphism group of** G, and is written $\mathrm{Aut}(G)$. The group operation is composition of functions

Example: What is $\mathrm{Aut}(C_3)$?

Proof.

$$C_3 = \{e, r, r^{-1}\}$$

□

Definition 1.4.8: Direct Product

Let G, H be groups. The **product** (or **direct product**) $G \times H$ is a group, with group operation $*$ given by

$$(g, h) * (g', h') = (g *_G g', h *_G h')$$

Note: we usually just say that $(g, h) * (g', h') = (gg', hh')$

1.3 something...

Let $H \leq G$ (H a subgroup of G). TFAE

1. $\forall g \in G, h \in H$, $ghg^{-1} \in H$
2. $gHg^{-1} = H$, $\forall g \in G$
3. $gH = Hg$, $\forall g \in G$

Proof. Show conditions imply each other

- (2) $\implies$ (1) immediately
- (1) says that $gHg^{-1} \subseteq H$, $\forall g \in G$
  WTS: $gHg^{-1} \supseteq H$
  $$H = g^{-1}gHg^{-1}g \subseteq g^{-1}Hg, \forall g \in G$$
  replacing g with g^{-1}:
  $$H \subseteq gHg^{-1}, \forall g \in G$$
- (2) $\implies$ (3): Multiply by g on right
  (3) $\implies$ (2): Multiply by g^{-1} on left

□

Theorem 1.3.1: lma

If $\phi : G \to H$ is a group homomorphism, then $\ker \phi \triangle G$

Proof. If $\phi(x) = e$, then

$$\phi(gxg^{-1}) = \phi(g)\phi(x)\phi(g) = \phi(g)e\phi(g)^{-1} = \phi(g)\phi(g)^{-1} = e$$

□

Theorem 1.3.2

If $N \leq G$, then $N \triangleleft G$ iff $\exists \phi : G \to H$ s.t. $N = \ker \phi$

Proof. $\ker \phi$ is normal by the above lemma
Conversely, given $N \triangleleft G$, we can form **factor group** G/N
G/N is the set of left cosets, with:

- Identity N
- Inverses $(gN)^{-1} : g^{-1}N$
- Multiplication: $(g_1N) \times (g_2N) := g_1g_2N$

Check that the group is well defined

1. If $gN = g'N$, then $g' = gx$ for $x \in N$

$$(g'N)^{-1} = (g')^{-1}N = (gx)^{-1}N = x^{-1}g^{-1}N$$

   As N is normal, $gx^{-1}g^{-1} \in N$

$$\implies x^{-1}g^{-1}N = g^{-1}(gx^{-1}g^{-1})N = g^{-1}N, \text{ as } gx^{-1}g^{-1} \in N$$

2. If $g_1N = g_1'N$ and $g_2N = g_2'N$, then $g_1' = g_1x$ and $g_2' = g_2y$ for $x, y \in N$

$$(g_1'N) \times (g_2'N) = g_1'g_2'N = g_1xg_2yN$$

$$yN = N, \quad \text{so} \quad g_1xg_2y_1N = g_1xg_2N$$

   N normal, so $g_2^{-1}xg_2 \in N \implies g_1g_2(g_2^{-1}xg_2)N = g_1g_2N$

then prove the group axioms lol

Define $\text{can} : G \to G/N$, $g \mapsto gN$. This is a group homomorphism

$$\text{can}(g_1g_2) = g_1g_2N = (g_1N) * (g_2N) = \text{can}(g_1) * \text{can}(g_2)$$

Kernel of can

$$\ker(\text{can}) = \{g \in G : \text{can}(g) = N\} = \{g \in G : gN = N\} = N$$

□

Example: If $G = \mathbb{Z}$, (normal) subgroups are $n\mathbb{Z} = \{ni : i \in \mathbb{Z}\}$. What is $\mathbb{Z}/n\mathbb{Z}$?
Elements of $\mathbb{Z}/n\mathbb{Z}$ are cosets, $i + n\mathbb{Z}$ (fixed i), or $\{x \in \mathbb{Z} : x \equiv i \mod n\}$
Group operation: $(i + n\mathbb{Z}) * (j + n\mathbb{Z}) = i + j + n\mathbb{Z} = i + j \mod n$
soooo... $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n$, where elements are $n\mathbb{Z}, 1 + n\mathbb{Z}, \ldots, n - 1 + n\mathbb{Z}$
lol !

1.4 First Isomorphism Theorem and stuff

Theorem 1.4.1: First Isomorphism Theorem

If $\theta : G \to H$ a group homomorphism, then:

- $\mathrm{im}(\theta)$ is a subgroup of H
- $\ker(\theta) \triangleleft G$
- $\exists$ a group homomorphism $\overline{\theta} : \theta / \ker\theta \tilde{\to} \mathrm{im}(\theta)$

Proof. Prove all 3

- If $\theta(a), \theta(b) \in \mathrm{im}(\theta)$, then $\theta(a)\theta(b) = \theta(ab) \in \mathrm{im}(\theta)$
  $\theta(a)^{-1} = \theta(a^{-1}) \in \mathrm{im}(\theta)$ thererfore $\mathrm{im}(\theta) \leq H$
- Already $\ker(\theta) \triangleleft G$
- Let $N = \ker(\theta)$. Then $gN \in G/N$. Define $\overline{\theta}(gN) := \theta(g)$.
  Well defined: If $gN = g'N$, then $g' = gx$ for some $x \in N$. Then $\overline{\theta}(g'N) = \theta(g') = \theta(g)\theta(x) = \theta(g)e$ as $x \in \ker(\theta) = \theta(g)$

□

Ex 1: $\theta : \mathbb{C} \to \mathbb{C}\ \{0\}$

Theorem 1.4.2: Property of Finite Groups

Lf $N \triangleleft$, then for any homomorphism $\psi : G \to H$ with $N \subseteq \ker\psi$. $\exists$ a group homomorphism $\overline{\psi} : G/N \to H$ s.t. $\psi = \overline{\psi} \circ \mathrm{can}$

If $\psi : G \to K$ surjective...? $\psi : G \to H$ with $\ker\phi \subseteq \ker\psi$, then @$\exists$ $\overline{\psi} : K \to H$ s.t. $\psi = \overline{\psi} \circ \psi$

Theorem 1.4.3

Let $N \triangleleft G$, can $G \to G/N$ and $K \leq G/N$

1. $\mathrm{can}^{-1}(K) \leq G$ with $\mathrm{can}^{-1}(K) \geq N$
2. $\mathrm{can}^{-1}(K) \triangleleft G \iff K \triangleleft G/N$

Theorem 1.4.4: Correspondence Theorem

If we have $N \triangleleft G$, $\mathrm{can} : G \to G/N$, then:

- $H \to \mathrm{can}(H)$ gives a bijection between subgroups of G/N and subgroups of G containing N
- Normal subgroups of G containing $N \iff$ normal subgroups of G/N
- If $A, B \leq G$ with $N \subseteq A, N \subseteq B$, then: $A \subseteq B$ iff $\mathrm{can}(A) \subseteq \mathrm{can}(B)$

Proof. Given $K < G/N$, $can^{-1}K \leq G$ and $N \leq \text{can}^{-1} K$ since $\text{can}^{-1}\{e\} = N$
Last prop says: $\text{can}^{-1}\text{can}(H) = H$ when $N \subseteq H$

$$\text{can}(\text{can}^{-1} K) \subseteq K$$

Since can is surjective, $\forall x \in K$, $\exists y \in G$ s.t. $\text{can}(y) = x$. Then $y \in can^{-1}K$ so $x \in \text{can}(\text{can}^{-1} K)$
So, $\text{can}(\text{can}^{-1} K) = K$ since can is surjective. Therefore can & can^{-1} give a bijection

$$\{\text{subgroups of } G \text{ containing } N\} \leftrightsquigarrow \{\text{subgroups of } G/N\}$$

□

1.4.5 Recap of last time (which is not on the notes)

- $\text{can}(H) \triangleleft G/N \iff H \triangleleft G$
- If $A \subseteq B$ then $\text{can}(K) \subseteq \text{can}(B)$
  Conversely, if $\text{can}(A) \subseteq \text{can}(B)$ then $\text{can}^{-1} \underbrace{\text{can}}_{=A}(A) \subseteq can^{-1} \underbrace{\text{can}}_{=B}(B)$

Definition 1.4.6: Random notation

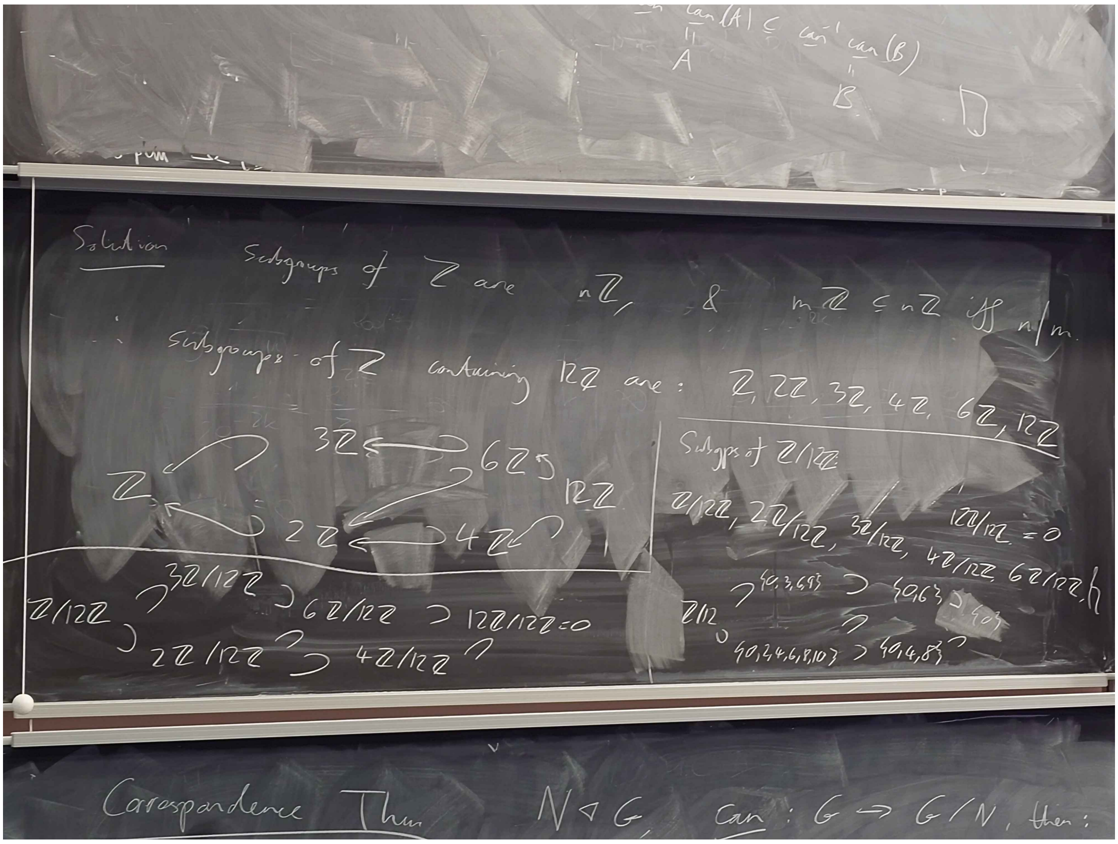
- $\exists$: There exists
- $\exists!$: There exists unique
- $\nexists$: there does not exist

Example: Let $G = \mathbb{Z}$, $N = 12\mathbb{Z}$.

- Find all subgroups of G containing N and all inclusions between them
- Find all subgroups of $\mathbb{Z}/12$

Solution: Subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$. $m\mathbb{Z} \subseteq n\mathbb{Z}$ iff n/m
Therefore, subgroups of $\mathbb{Z}$ containing $12\mathbb{Z}$ are:

$$\mathbb{Z},\ 2\mathbb{Z},\ 3\mathbb{Z},\ 4\mathbb{Z},\ 6\mathbb{Z},\ 12\mathbb{Z}$$



Subgroups of $\mathbb{Z}/12\mathbb{Z}$:

$$12\mathbb{Z}/12\mathbb{Z},\ \mathbb{Z}/12\mathbb{Z},\ 2\mathbb{Z}/12\mathbb{Z},\ 3\mathbb{Z}/12\mathbb{Z},\ 4\mathbb{Z}/12\mathbb{Z},\ 6\mathbb{Z}/12\mathbb{Z}$$

some working out

Theorem 1.4.7: Third Isomorphism Theorem

If $N, H \triangleleft G$, with $N \leq H$, then

$$(G/N)/(H/N) \cong G/H$$

Proof. $N \leq \ker(\text{can}_H) = H$, so $\exists!\pi$ by universal property of finite groups
π is surjective, because can_H is isomorphic
Explicitly,

$$\pi(gN) = gH = \pi(\text{can}_N(g)) = \text{can}_H(g)$$

$\ker(\pi) = \{gN : g \in H\} = H/N$
By the first isomorphism theorem,

$$G/H \equiv (G/N)/\ker\pi = (G/N)/(H/N)$$

□

Theorem 1.4.8: Second Isormorphism Theorem

Let $N \triangleleft G$ and $H \leq G$. Then:

1. $HN \leq G$
2. $N \triangleleft HN$
3. $H \cap N \triangleleft H$
4. $HN/N \equiv H/H \cap N$

Proof. Let $h_1 h_2 \in H$, $n_1 n_2 \in N$

1. $$h_1 n_1 h_2 n_2 = \underbrace{h_1 h_2}_{\in H} \underbrace{(h_2^{-1} n_1 h_2) n_2}_{\in N}$$
$$(hn)^{-1} = n^{-1} h^{-1} = \underbrace{h^{-1}}_{\in H} \underbrace{(hn^{-1} h^{-1})}_{\in N}$$

2. If $g \in HN$ and $n \in N$, then $g \cap g^{-1} \in n$ since $g \in G$

3. If $x \in H \cap N$ and $h \in H$, then $\underbrace{hxh^{-1}}_{N \triangleleft G} \in N$ and $\underbrace{hxh^{-1}}_{x \in H} \in H$

4. Need $\theta : H \to HN/N$ surjective with kernel $H \cap N$

   Let $\theta(h) = hN$ i.e. $\theta = \mathrm{can}_N \big|_H$, ($\mathrm{can}_N \, G \to G/N$)

   Surjective: cosets of HN/N are cosets xN for $x \in HN$ but $x = hn$, $h \in H$, $n \in N$ and $xN = hN = \theta(n)$ (wtf?)

   Kernel: If $\theta(h) = e$, $kN = N$, so $h \in N$, so $\ker \theta = H \cap N$, so by the correspondence theorem,
$$H/H \cap N \subseteq HN/N$$

□

2 Group Actions

Definition 2.0.1: Free Group

The **free group on generators** $x_1, \dots, x_m$ is the group whose elements are words in the symbols $x_1, \dots, x_m, x_1^{-1}, \dots, x_m^{-1}$, subject to the group axioms and all logical consequences. The group operation is concatenation. The free group is written

$$\langle x_1, \dots, x_m \rangle$$

Example: Find presentations for:

- $\mathbb{Z}^2 = \langle x, y \mid xy = yx \rangle = \langle x, y \mid xyx^{-1}y^{-1} = e \rangle \cong \{x^i y^i = i, j \in \mathbb{Z}\}$

Example 2.0.2: Random group action E

Let
$$E = \langle a, b \mid a^2, b^5, (ab)^2 \rangle$$

Lemma 2.0.3

Any element $x \in E$ can be written $x = a^i b^j$, where $i \in \{0, 1\}$ and $j \in \{0, 1, 2, 3, 4\}$

Corollary 2.0.4

Group homomorphisms

$$\phi : \langle x_1, \dots, x_n \mid r_1(\underline{x}), \dots, r_n(\underline{x}) \rangle \to G$$

correspond to multiples $(g_1, \dots, g_m) \in G^m$ s.t. $r_1(\underline{g}) = e, \dots, r_n(\underline{g})$

Example: For $E = \langle a, b \mid a^2, b^5, (ab)^2 \rangle$
Group homomorphism - $Q : E \to G$ correspond to:

$$(g, h) \in G \times G \quad \text{s.t.} \quad g^2 = e,\ h^5 = e,\ (gh)^2 = e$$

In particular, we have:

$$\begin{aligned} \phi : E &\to D_5 \\ b &\mapsto \text{rotation} \\ a &\mapsto \text{reflection} \end{aligned}$$

We also have that $\operatorname{im}(Q) = D_5$, and Q surjective

Definition 2.0.5: Reduced Word

A word $x^{m_1} y^{n_1} x^{m_2} y^{n_2} \dots x^{m_k} y^{n_k}$ is **reduced** if no $m_i, n_j = 0$ except possibly for m_1 or n_k (That is, a word doesn't need to start with a power of x or end with a power of y)

Lemma 2.0.6

Every element of $\langle x, y \rangle$ has a unique expansion as a reduced word

3 Sylow Theorems

The converse of Lagrange's Theorem doesn't hold - i.e. if G is a group and $s|\langle G\rangle$ then there is no guarantee that G contains a subgroup of order s. The closest thing we have is Cauchy's Theorem

Theorem 3.0.1: Cauchy's Theorem

If p is a prime that divides the order of G, then G has a (cyclic) subgroup of order p

3.1 Sylow Theorems - Statements

Definition 3.1.1: Sylow Subgroups

Let G be a finite group and let p be a prime. A subgroup H of G is a **p-subgroup of** G if it is a p-group, that is it has order p^n for some n, and it is a **Sylow p-subgroup of** G if its order is the highest power of p that divides the order of G. We say that H is a **Sylow subgroup of** G if it is s Sylow p-subgroup for some prime p

If p does not divide $|G|$ then the trivial subgroup $\{e\}$ is the Sylow p-subgroup of G. When we wish to consider only Sylow p-subgroups of G for primes p that divide $|G|$ then we refer to nontrivial Sylow p-subgroups

Theorem 3.1.2: Sylow I

Let $|G| = n$ and suppose that p is a prime that divides n. Write $n = p^m r$ with p not dividing r.
Then there exists at least one subgroup of order p^m. i.e., there is at least one Sylow p-subgroup

Theorem 3.1.3: Sylow II

Let $|G| = n$ and suppose that p is a prime that divides n. Write $n = p^m r$ with p not dividing r. Suppose that P is a Sylow p-subgroup and that $H \leq G$ is any p-subgroup of G. Then there exists $x \in G$ with $H \subseteq xPx^{-1}$. In particular, any two Sylow p-subgroups of G are conjugate in G

Theorem 3.1.4: Sylow III

Let $|G| = n$ and suppose that p is a prime that divides n. Write $n = p^m r$ with p not dividing r. Let n_p be the number of distinct Sylow p-subgroups of G. Then $n_p \mid r$ and $n_p \equiv 1 \mod p$

Lemma 3.1.5

If $n_p = 1$, then the Sylow p-subgroup P is normal in G.

Prop 3.1.6

Every group G with $|G| = 30$ has a normal subgroup

3.2 Group Actions

Definition 3.2.1: Group Action

An **action** of a group G on a set X is a function

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

such that

- $e \cdot x = x$ for all $x \in X$
- $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in X$ and $x \in X$

Examples of actions

- D_n acting on an n-gon
- S_n acting on $\{1, 2, \ldots, n\}$
- $\mathrm{GL}_n(F)$ acting on F^n

Definition 3.2.2: Orbits

Given a G acting on X, and $x \in X$, define

- The **Orbit** $G \cdot x$ or $\mathrm{Orb}_G(x)$ is $\{g \cdot x : g \in X\} \subseteq X$
- The **Stabiliser** $\mathrm{Stab}_G(x)$ is $\{h \in G : h \cdot x = x\} \subseteq G$

Lemma 3.2.3

$\mathrm{Stab}_G(x)$ is a subgroup of G

Proof. If $g_1 h \in \mathrm{Stab}_G(x)$, then $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$
$g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1} g) \cdot x = e \cdot x = x$ $\square$

Theorem 3.2.4: Orbit-Stabiliser Theorem

Let G be a finite group acting on a set X, and let $x \in x$. then

$$|G| = |\mathrm{Stab}_G(x)||G \cdot x|$$

Or more cleanly,

$$G \cdot x \cong G / \mathrm{Stab}_G(x)$$

Lemma 3.2.5

Let G act on X

1. An action defines an equivalence relation $X : x \sim y \iff \exists g \in G$ s.t. $g \cdot x = y$
2. Equivalence relations are orbits
3. The orbits partition X

[diagram of D3]

Theorem 3.2.6: Conjugacy Class

If $|G| = p^n$ for some n, then $Z(G) \neq \{e\}$

$$Z(G) = \{x \in G : xg = gx, \quad \forall g \in G\}$$

Proof. Conjugacy classes partition G and $x \in |(G) \iff Cl(x) = \{x\}$

$$G = Z(G) \sqcup Cl(g_1) \sqcup \cdots \sqcup Cl(g_n) \text{ for conjugacy classes } |Cl(g_i)| < 1$$

□

3.3 Proofs of Sylow theorems

Proof. Sylow 1: Subgroups exist
Something about permutations. QED □

Corollary 3.3.1

A Sylow p-subgroup P is **normal** $\iff$ $n_p = 1$ i.e. P is the unique Sylow p-subgroup

Definition 3.3.2: Normalizer

Let G be a group and $H \leq G$. The **normalizer** of H is

$$N_G(H) = \{g \in G : gHg^{-1} = H\}$$

Example: Let $G = S_4$ and $H = \langle(123)\rangle$. ??
random properties

- $H \leq N_G(H)$ since $hHh^{-1} = h$ and $H \lhd N_G(H)$
- $N_G(H)$ is the largest subgroup in which H is normal
- G acts by conjugation on its set of subgroups
  - The orbit of H: $\{gHg^{-1} : g \in G\}$ is a conjugation of H
  - The stabiliser of H: $\{g \in G : gHg^{-1} = H\} = N_G(H)$
  - $\implies |G| = |N_G(H)| \cdot (\text{no. of conjugations of } H)$

Lemma 3.3.3

Let G be a finite group.

1. For any subgroup $H \leq G$, we have

$$[G : N_G(H)] = \text{the number of distinct conjugates of } H$$

2. Let $p||G|$ and let P be a Sylow p-subgroup of G. Then $n_p = [G : N_G(P)]$

Proof. **Proof of Sylow III**: $n_p = |X|$ is congruent to $1 \mod p$. QED □

Example: For S_4, Since $|S_4| = 24 = 2^3 \cdot 3$, we have

- Sylow 2-subgroups have order 8
- Sylow 3-subgroups have order 3

$$\begin{aligned} n_2 &\equiv 1 \mod 2 \quad \text{and } n_2|3 \\ n_2 &\equiv 1 \mod 3 \quad \text{and } n_2|8 \end{aligned}$$

2-subgroups: Copies of D_4 e.g. $\langle (1234), (12)(34) \rangle$ or $\langle (1324), (13)(24) \rangle$
$n_2 = 3$ therefore not normal in S_4

$$|N_{S_4}(D_4)| = \frac{|S_4|}{n_2} = \frac{24}{3} = 8$$

3-subgroups: possibilities for n_3 : 1 or 4: $\langle (123) \rangle$ not normal in $n_3 = 4$ Groups are:

- $\langle (123) \rangle$ $|N_{S_4}(\langle (123) \rangle)| = \frac{24}{4} = 6$
- $\langle (124) \rangle$
- $\langle (134) \rangle$
- $\langle (234) \rangle$

3.4 Finite Abelian Groups

Theorem 3.4.1

Every finite abelian group is isomorphic to the product of its Sylow Subgroups

If $|A| = \prod_{i=1}^{t} p_i^{s_i}$ and A_{p_i} the Sylow p_i-subgroup ($|A_{p_i} = p_i^{s_i}|$) then

$$\begin{aligned} A \cong A_{p_1} \times A_{p_2} \times \cdots \times A_{p_t} &\to A \\ (a_1, a_2, \dots, a_t) &\to a_1, a_2, a_3 \end{aligned}$$

Theorem 3.4.2

If A is an abelian p-group (i.e. $|A| = p^m$), then

$$A \cong C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots C_{p^{e_m}} \quad \text{s.t. } \sum_{i=1}^{n} e_i = m$$

Theorem 3.4.3: Chinese Remainder Theorem

Let m, n be nonzero coprime integers. then

$$C_{mn} \cong C_m \times C_n$$

Corollary 3.4.4: Fundamental Theorem of Finite Abelian Groups II

Every finite abelian group is isomorphic to

$$C_{n_1} \times C_{n_2} \times \cdots \times C_{n_s}$$

where n_i divides n_{i+1} for each $i = 1, 2, \ldots, s-1$ and $n_1 n_2 \ldots n_s = n$. This product is unique up to reordering the factors

Theorem 3.4.5

If A is a finite subgroup of multiplicative group $K \backslash \{0\}$, with K a field, then A is cyclic.

3.5 Linear Algebra over $\mathbb{Z}$

Definition 3.5.1: Module

Let R be a ring. An R-module is an abelian group $(M, +)$ together with a mapping

$$\begin{aligned} R \times M &\to M \\ (r, a) &\mapsto ra \end{aligned}$$

such that

- $1 \cdot m = m$
- $(r+s) \cdot m = r \cdot m + s \cdot m$
- $r \cdot (m+n) = r \cdot m + r \cdot n$
- $(rs) \cdot m = r \cdot (s \cdot m)$

Definition 3.5.2: Free Module

The free R-module R^m is the abelian group $(R^m, +)$ with

$$r \cdot (a_1, \dots, a_m) = (ra_1, \dots, ra_m)$$

Lemma 3.5.3: Abelian Groups and $\mathbb{Z}$ modules

An abelian group is the same as a $\mathbb{Z}$-module

Proof. Every $\mathbb{Z}$-module has an underlying abelian group
Conversely, given an abelian group $(A, +)$, there is a unique $\mathbb{Z}$-module structure:
We know $1 \cdot a = a$ for all $a \in A$, and so for $n > 0$

$$n \cdot a = (\underbrace{1 + 1 + \cdots + 1}_{n}) \cdot a = \underbrace{a + a + \cdots + a}_{n}$$

Random axiom: $0 \cdot a = 0$

$0 \cdot a = (1 + (-1)) \cdot a = 1 \cdot a + (-1) \cdot a$, $\therefore (-1) \cdot a = -a$ and so for $n > 0$, $\quad (-n) \cdot a = -(\underbrace{a + \cdots + a}_{n})$

$$a = 1 \cdot a = (0+1) \cdot a = 0 \cdot a + 1 \cdot a = 0 \cdot a + a, \therefore (-a) \text{ gives } 0 = 0 \cdot a$$

□

Theorem 3.5.4: Fundamental Theorem of Finite Abelian Groups

Every finite abelian group A is of the form

$$A \cong \mathbb{Z}/r_1\mathbb{Z} \times \mathbb{Z}/r_2\mathbb{Z} \times \cdots \times \mathbb{Z}/r_k\mathbb{Z} \times \mathbb{Z}^\ell$$

for some $k, \ell \in \mathbb{N}$ and $r_1, \dots, r_k$ nonzero elements of $\mathbb{Z}$ with $r_1 | r_2 | \dots | r_k$

Important Fact: Every submodule of $\mathbb{Z}^s$ is finitely generated (no proof included)
A finitely generated abelian group A has generators $a_1, \dots, a_s$, say

$$A = \langle a_1, \dots, a_s \rangle$$

Which is equivalent to a surjective map θ

$$\begin{aligned}\mathbb{Z}^s &\xrightarrow{\theta} A \\ e_i &\mapsto a_i \\ \sum a_i s_i &\mapsto \sum a_i s_i\end{aligned}$$

Therefore, by FIT for modules, $A \cong \mathbb{Z}^s / \ker\theta$

4 Alternating Groups

4.1 Symmetric Groups

Permutations are written in cycle notation

$$\begin{pmatrix} 1 & 1 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

means "the permutation that sends $1 \mapsto 2$, $2 \mapsto 4$, $3 \mapsto 1$, and $4 \mapsto 3$

Definition 4.1.1: Disjoint Cycle

Two cycles are disjoint if no integer appears in both cycles. For example, (214)(35) is a product of disjoint cycles, and is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}$$

Lemma 4.1.2: Uniqueness of Disjoint permutations

Every permutation can be written as a product of disjoint cycles, and the product is unique up to reordering the factors

Example: The permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 6 & 2 & 5 & 3 \end{pmatrix}$$

is written as (142)(36)(5) but can also be written as (36)(5)(142)
bunch of other stuff

4.2 Alternating Groups

Definition 4.2.1: Even permutations

A permutation is **even** if

5 Solvable Groups

Definition 5.0.1: Subnormal series

A **subnormal series** is a sequence

$$\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_s = G$$

Definition 5.0.2: Solvable Group

A group is **solvable** if it has a subnormal series with all G_{i+1}/G_i abelian

Examples

- Abelian groups are solvable, $\{e\} \lhd G$
- D_n is solvable, $\{e\} \lhd C_n \lhd D_n$, $D_n/C_n \cong C_2$, $C_n/\{e\} \cong C_n$
- A_5 is not solvable because it is simple, so $\{e\}$ is the only solvable normal subgroup and $A_5/\{e\} \cong A_5$ not abelian
- S_4 is solvable
- Every p-group is solvable

Theorem 5.0.3: Solvable and Cyclic Groups

A group is solvable $\iff$ all its composition factors are cyclic

Proof.
($\Longleftarrow$) Composition series is a subnormal series with cyclic factors. Cyclic $\implies$ abelian, so G is solvable
($\implies$) The result that composition factors of G are those of N and G/N for $N \lhd G$
Since G is solvable, $\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd C_n = G$ and G_{i+1}/G_i abelian
Composition factors for G are those for G_n/G_{n+1} and those for G_{n-1} so by induction, composition factors for G are the disjoint union of all composition factors for G_i/G_{i-1} $\square$

Lemma 5.0.4

All composition factors of an abelian group are cyclic. Therefore all composition factors of G_i/G_{i-1} are cyclic, so all composition factors of G are cyclic

Example:

- Any group of order < 60 is solvable. This is because its composition factors have order of < 60 and are simple, and therefore cyclic (A_5 is the smallest non-cyclic simple group)
- $S_5, S_6, S_7, \ldots$ are not solvable as composition factors of S_n are A_n and C_2 for $n > 5$, and A_n is not cyclic

Theorem 5.0.5: Solvable subgroups

Every subgroup of a solvable group is solvable

Proof. Let G be solvabel and $G \leq G$. So we have

$$\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G$$

with G_{i+1}/G_i abelian. Let $H_i = H \cup G_i$. Note that $G_i \cup H \lhd G_{i+1} \cup H$ since $\forall n \in G_{iH} \cup H$ and all $x \in G_i \cup /H$ we have $hxh^{-1} \in G_i$ as $G_i \lhd G_{i+1}$ and $hxh^{-1} \in H$ as $h_1 x \in H$
$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$ is a subnormal series □

note: idk if this proof makees sense