General Topology Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Topological Spaces and Examples

Definition 1.1: Topological Space

A **topological space** is a pair $(X, \mathcal{T})$, where X is a nonempty set, and $\mathcal{T}$ is a collection of subsets of X which satisfies:

a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

b) if $U_\lambda \in \mathcal{T}$ for each $\lambda \in \Lambda$ (where Λ is some indexing set), then $\bigcup_{\lambda\in\Lambda} U_\lambda \in \mathcal{T}$

c) if $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

The collection $\mathcal{T}$ is called the **topology** of the topological space, and the members of $\mathcal{T}$ are called the **open sets** of the topology

Example 1.7: Euclidean Spaces

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean vector space with elements $x = (x_1, x_2, \ldots, x_n)$ and $x_i \in \mathbb{R}$, and let

$$|x| = \sqrt{\sum_{i=1}^n x_i^2} \geq 0$$

be the length of x. ($\mathbb{R}^1 = \mathbb{R}$ is the real line). A subset U of $\mathbb{R}^n$ is **open (for the usual topology)** iff for each $a \in U$ there exists an $r > 0$ such that

$$|x - a| < r \implies x \in U.$$

The collection of open sets thus defined is called the **usual topology** on $\mathbb{R}^n$. Note that open balls $B(y, \rho) = \{x \in \mathbb{R}^n : |x - y| < \rho\}$ are open sets under this definition.

Example 1.8: Metric Spaces

A **metric space** (X, d) is a nonempty set X together with a function $d : X \times X \to \mathbb{R}$ with the following properties:

a) $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$

b) $d(x, y) = d(y, x)$

c) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

The function d is called the **metric**.

Let (X, d) be a metric space, x be a point in X, and $r > 0$. The **open ball** with center x and radius r is defined by

$$B(x, r) = \{y, , \in X : d(x, y) < r\}.$$

A subset U of X is **open (in the metric topology given by d)** iff for each $a \in U$ there is an $r > 0$ such that $B(a, r) \subseteq U$. Just like euclidean spaces, open balls are open in this sense.

Proposition 1.11: Topology Equality

Let d, d' be metrics on the same set X, and let $\mathcal{T}$, $\mathcal{T}'$ be the corresponding metric topologies. If for real numbers A, $B > 0$ we have

$$d(x, y) \leq Ad'(x, y),\ d'(x, y) \leq Bd(x, y) \text{ for all } x, y \in X,$$

then $\mathcal{T} = \mathcal{T}'$.

Definition 1.16: Subspace Topology

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be any subset. Then the **subspace topology** on A consists of all sets of the form $U \cap A$ where $U \in \mathcal{T}$.

Definition 1.17: Closed Set

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is **closed** iff its complement $X \backslash A := \{x \in X \mid x \notin A\}$ is open in X. Note that a set being *closed* does not mean it isn't *open*. Sets that are both *closed* and *open* are called **clopen**.

Definition 1.20: Properties of Topological Spaces

For a subset $A \subseteq X$,

- The **closure** of A is
$$\overline{A} := \bigcap_{\substack{C \subseteq X \text{ closed;} \\ A \subseteq C}} C.$$
- The **interior** of A is
$$\operatorname{int} A = A^\circ := \bigcap_{\substack{C \subseteq X \text{ open;} \\ A \subseteq C}} C.$$
- The **boundary** (or **frontier**) of A is
$$\partial A := \overline{A} \backslash A^\circ.$$
- A is **dense** in X iff $\overline{A} = X$.
- A **limit point** of A is a point $x \in X$ s.t. for every open subset $U \subseteq X$ with $x \in U$ there exists an element $a \in A \cup U$ with $a \neq x$. Let A' be the set of limit points of A.
  Note that this has nothing to do with limits of sequences.

Proposition 1.22: Relating Toplogical Properties

- $\overline{A}$ is closed, and contains A and is the smallest set with this property. So A is closed iff $\overline{A} = A$.
- A° is open, and is contained in A, and is the largest set with this property. So A is open iff $A^\circ = A$.
- The closure of the complement is the complement of the interior:
$$\overline{X \backslash A} = X \backslash (A^\circ).$$
- The interior of the complement is the complement of the closure:
$$(X \backslash A)^\circ = X \backslash \overline{A}.$$

Proposition 1.26: Union of Limit Points

Let $(X, \mathcal{T})$ be a topological space, and suppose $A \subseteq X$. Then

$$\overline{A} = A \cup A'$$

Corollary 1.27

A subset $A \subseteq X$ is closed iff it contains all its limit points.

Theorem 1.19: Properties of open and closed sets

Let $(X, \mathcal{T})$ be a topological space.

1. $\emptyset$ and X are **closed**.
2. The union of **finitely many** closed sets is a closed set.
3. The intersection of **any collection** of closed sets is a closed set.

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1. The union of **any collection** of open sets is an open set.
2. The intersection of **finitely many** open sets is an open set

Lemma 1.24: Limit Points and Open Balls

An element $x \in X$ in a metric space (X, d) is a limit point of a subset $A \subseteq X$ iff for every $\epsilon > 0$ there exists $a \in A$ with $0 < d(x, a) < \epsilon$, or iff there exists a sequence $a_1, a_2, a_3, \cdots$ of elements $a_i \in A$, with $a_i \neq x$ for all i, s.t. $d(x_i, a_i) \to 0$ as $i \to \infty$. This interpretation does not extend to general topological spaces.

Theorem 1.30: Open and Closed sets in $\mathbb{R}$

Consider $\mathbb{R}$ with the usual topology.

1. A nonempty set U is open iff it can be written as a countable union of disjoint nonempty open intervals I_j (shown left):
2. A set F is closed iff it can be written as a countable intersection where each F_j is a finite union of closed intervals (shown right).

$$U = \bigcup_{j=1}^\infty I_j, \qquad F = \bigcap_{j=1}^\infty F_j.$$

Definition 1.32: Hausdorff Spaces

A topological space $(X, \mathcal{T})$ is **Hausdorff** if for each $x, y \in X$ with $x \neq y$ there exist **disjoint** open sets U and V such that $x \in U$ and $y \in V$.

Any metrisable space is Hausdorff, The trivial topology on a set with more than one element is not Hausdorff.

Definition 1.33: Convergence of a Topological space

A sequence (x_n) of members of a topological space X converges to $x \in X$ if for every open set U containing x, there exists an N such that $n \geq N \implies x_n \in U$

Proposition 1.34: Convergence of Hausdorff Spaces

Suppose $(X, \mathcal{T})$ is Hausdorff. Then a sequence (x_n) can converge to at most one limit.

Definition 1.36: Cauchy and Completeness

Let (X, d) be a metric space.

1. A **Cauchy sequence** is a sequence (x_n) with each $x_n \in X$ with the property that for each $\epsilon > 0$, there exists an N such that $m, n \geq N \implies d(x_m, x_n) < \epsilon$
2. (X, d) is **complete** if every Cauchy sequence converges.

Definition 1.37: Topology Basis

A **basis for a topology** on a set X is a collection $\mathcal{B}$ of subsets $B \subseteq X$ such that:

1. $X = \bigcup_{B\in\mathcal{B}} B$
2. The intersection of sets $B_1,\, B_2 \in \mathcal{B}$ is a set $B_1 \cap B_2 \in \mathcal{B}$.

The **topology $\mathcal{T}$ generated by a basis** $\mathcal{B}$ has open sets the arbitrary unions of basis elements $B_\lambda \in \mathcal{B}$:

$$U = \bigcup_{\lambda\in\Lambda} B_\lambda$$

(Don't forget to check that this really is a topology)

Example 1.38: Finite Intersections of open balls

For any metric space $(X, \mathcal{T})$ the finite intersections of open balls

$$B(x,r) = \{y \in X \mid d(x,y) < r\} \subseteq X \quad (r > 0,\, x \in X)$$

constitute a basis for the metric topology on X

$$\mathcal{B} = \{B(x_1,r_1) \cap B(x_2,r_2) \cap \cdots \cap B(x_k,r_k) \mid \\ x_1,x_2,\ldots,x_k \in X,\, r_1,r_2,\ldots,r_k > 0\}$$

2 Continuous functions and Homeomorphisms

Definition 2.1: Continuity

Let $(X,\mathcal{T})$, $(Y,\mathcal{U})$ be topological spaces. A function $f : X \to Y$ is **continuous** iff

$$U \in \mathcal{U} \text{ implies } f^{-1}(U) \in \mathcal{T}.$$

That is, **inverse** images of open sets are open. Continuous functions are often called **maps** or **mappings** of topological spaces.

Proposition 2.6: Topological and Analytic Continuity

Let (X,d) and (Y,ρ) be metric spaces with their induced topologies $\mathcal{T}$ and $\mathcal{U}$ respectively. A function $f : X \to Y$ is continuous (topologically) iff it is continuous analytically: for every $a \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(x,a) < \delta \implies \rho(f(x), f(a)) < \epsilon$$

Definition 2.7: Homeomorphism

Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be topological spaces. A **homeomorphism** is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous. Two topological spaces are **homeomorphic** if there is a homeomorphism between them.

Proposition 2.18: The Punctured Sphere

Consider the n-dimensional sphere

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

with the metric topology inherited from $\mathbb{R}^{n+1}$. Let $x_0 \in \mathbb{S}^n$. Then $\mathbb{S}^n \backslash \{x_0\}$ is homeomorphic to $\mathbb{R}^n$.

3 Subspaces Revisited

Definition 3.65: Disjoint Unions

Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be topological spaces. Their **disjoint union** $X + Y$ is the set $(X \times \{0\}) \cup (Y \times \{1\})$ with the topology consisting of all sets of the form

$$(T \times \{0\}) \cup (U \times \{1\}) \text{ such that } T \in \mathcal{T},\, U \in \mathcal{U}$$

Definition 3.8: Product Topology

Let $(X,\mathcal{T})$, $(Y,\mathcal{U})$ be topological spaces. The **product topology** on their product $X \times Y$ consists of all sets of the form

$$T = \bigcup_{\alpha\in A} (U_\alpha \times V_\alpha)$$

where $\mathcal{A}$ is an arbitrary indexing set, and $U_\alpha \in \mathcal{U}$ and $V_\alpha \in \mathcal{V}$.

Lemma 3.10

The product topology is indeed a topology. (lol)

Lemma 3.9: Openness in Product Topologies

Let $(X,\mathcal{T})$ $(Y,\mathcal{U})$ be topological spaces. Then $T \subseteq X \times Y$ is open in the product topology if and only if for all $t \in T$ there exists $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $t \in U \times V$ and $U \times V \subseteq T$.

Definition 3.11.5: Projection Maps

Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be topological spaces, and consider their product $X \times Y$ with the product topology. There are two natural maps Π_X and Π_Y, the projections of $X \times Y$ onto X and Y respectively, given by

$$\Pi_X : X \times Y \to X, \quad (x,y) \mapsto x$$
$$\Pi_Y : X \times Y \to Y, \quad (x,y) \mapsto y.$$

Definition 3.14: Weak Topology

Suppose that X is a set. $(X_\lambda, \mathcal{T}_\lambda)$ is a family of topological spaces, and that $f_\lambda : X \to X_\lambda$ are functions. The **weak topology generated by** $\{f_\lambda\}$ is the smallest topology on X making all the f_λ continuous.
Thus, the product topology on $X \times Y$ is the weak topology generated by the two maps Π_X and Π_Y

Definition 3.15: Cartesian Product Topology

If X_λ is a topological space, (with λ in some arbitrary indexing set Λ), the product topology on the cartesian product $\Pi_{\lambda\in\Lambda} X_{Ll}$ is defined to be the weak topology generated by the projections

$$\Pi_\lambda : \prod_{\mu\in\Lambda} X_\mu \to X_\lambda$$

Definition 3.5.0: Equivalence Relation

An **equivalence** relation on a set X is a binary operation $\sim$ on X which is:

1. **Reflexive**: $x \sim x$ for all $x \in X$.
2. **Symmetric**: if $x \sim y$ then $y \sim x$.
3. **Transitive**: if $x \sim y$ and $y \sim z$ then $x \sim z$.

The **equivalence class** of any element $x \in X$ is the set

$$[x] = \{y \in X \mid x \sim y\},$$

and the set of equivalence classes is denoted by $X/\sim$. The function which assigns to each $x \in X$ the equivalence class $[x] \in X/\sim$ is a surjection

$$p : X \to X/\sim; \quad x \to [x]$$

Definition 3.17: Quotient Space

Given a topological space $(X,\mathcal{T})$, and an equivalence relation $\sim$ on X, the **quotient space** or **identification space** is the set of equivalence classes $X/\sim$ together with the topology

$$\{U \subseteq X/\sim : p^{-1}(U) \in \mathcal{T}\}$$

Definition 3.25: Generated Topological Spaces

Let X be a topological space, and let $Y_0,\, Y_1 \subseteq X$ be subspaces related by a continuous function $f : Y_0 \to Y_1$. Let $\sim_f$ be the equivalence relation on X **generated by** f, the intersection of all the equivalence relations on X (regarded as subsets of $X \times X$) containing the pairs $(y_0, f(y_0))$ with $y_0 \in Y_0$. The identification space

$$X/\sim_f = X/\{y_1 = f(y_0)\}$$

is obtained by identifying each $y_0 \in Y_0 \subseteq X$ with $y_1 = f(y_0) \in Y_1 \subseteq X$.

Proposition 3.34: Homeomorphisms of Relations

Given a continuous function $f : X \to Y$ let $\sim$ be the equivalence relation defined on X by $x \sim x'$ if $f(x) = f(x') \in Y$. The function

$$g : X/\sim \to Y\,; [x] \to f(x)$$

is continuous and injective, with

$$f = g \circ p : X \xrightarrow{p} X/\sim \xrightarrow{g} Y.$$

If f is onto, and such that $f(U) \subseteq Y$ is open for every open subset $U \subseteq X$ then g is a homeomorphism.

4 Compact Spaces

Definition 4.1: Open Covers and Compact Spaces

1. An **open cover** of a topological space X is a collection $\{U_\lambda \mid \lambda \in \Lambda\}$ of open subsets U_λ of X such that
$$\bigcup_{\lambda\in\Lambda} U_\lambda = X$$
2. A topological space X is **compact** if every open cover $\{U_\lambda \mid \lambda \in \Lambda\}$ of X has a finite subcover, i.e. there exists $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that
$$X = \bigcup_{j=1}^{n} U_{\lambda_j}.$$

Definition 4.2: Open Covers as Collections

1. If $A \subseteq X$ is a subset of a topological space X, an **open cover** of A is a collection $\{V_\lambda \mid \lambda \in \Lambda\}$ of subsets V_λ which are open in X such that
$$X = \bigcup_{\lambda\in\Lambda} V_\lambda$$
2. A subset A of a toplogical space X is **compact** if it is compact as a subspace of X.

Proposition 4.7: Boundedness of Compact Spaces

A compact metric space (X, d) is bounded, i.e. there exists a number $K \geq 0$ such that $d(x, y) \leq K$ for all $x, y \in X$.

Proposition 4.8: Compactness of Products

A product of closed bounded intervals $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is compact in the usual topology. A collection of subsets of a set X has the **finite intersection property** if every finite intersection of their members is nonempty.

Corollary 4.12: Limit Property of Compactness

Suppose that $f : X \to \mathbb{R}^n$ is a continuous map and that X is compact. Then there exists an M such that
$$|f(x)| \leq M \text{ for all } x \in X.$$
Moreover, there exists an $x \in X$ such that
$$|f(x)| = \sup_{y\in X} |f(y)|.$$
If $n = 1$ there are x_0 and $x_1 \in X$ such that
$$f(x_0) = \min_{x\in X} f(x) \quad \text{and} \quad f(x_1) = \max_{x\in X} f(x).$$

Theorem 4.13: Compact,Hausdorff,Bijection, Oh My!

Suppose X is compact, Y is Hausdorff, and that $f : X \to Y$ is a continuous bijection. Then it is a homeomorphism.

Theorem 4.14: Lebesgue Numbers

Let X be a compact metric space and $\{U_\lambda \mid \lambda \in \Lambda\}$ an open cover of X. Then there exists a positive number $\delta > 0$ (the **Lebesgue number** of the cover) such that for all $x \in X$, $B(x, \delta)$ lies *entirely inside some single* U_λ.

Corollary 4.17: Compactness of Identification Spaces

1. An identification space $X/\sim$ of a compact space X is compact.
2. If $f : X \to Y$ is a map from a compact space X to a Hausdorff space Y and $\sim$ is the equivalence relation on X defined by $x \sim x'$ if $f(x) = f(x') \in Y$, then the continuous bijection
$$g : X/\sim \to f(X)\,; \quad [x] \mapsto f(x)$$
is a homeomorphism.

Lemma 4.20: Open sets in Product spaces

Let X be a topological space, Y a compact space, $x \in X$, N an open set in $X \times Y$ such that $\{x\} \times Y \subseteq N$. Then there is an open set $W \subseteq X$ such that $x \in W$ and $W \times Y \subseteq N$.

Lemma 4.22 - 4.23: Collections and Intersections

4.22) Let X be a set, and suppose $\mathcal{C}$ is a collection of subsets of X which has the finite intersection property. Then there is a collection $\mathcal{B}$ of subsets of X, with $\mathcal{C} \subseteq \mathcal{B}$, such that $\mathcal{B}$ has the finite intersection property, and such that $\mathcal{B}$ is maximal with respect to this property: i.e. no collection containing $\mathcal{B}$ as a proper subcollection has the finite intersection property.

4.23) Let X be a set, and suppose that $\mathcal{B}$ is a collection of subsets of X which is maximal with respect to the finite intersection property. Then $\mathcal{B}$ is closed under finite intersections, and any set which meets all members of $\mathcal{B}$ is also in $\mathcal{B}$.

Definition 4.24: Compactifications

1. A **compactification** of a topological space X is a compact space Y which contains a homeomorphic copy of X as a subspace, i.e. such that there is a one-one map $f : X \to Y$ such that $X \to f(X)\,; \quad x \mapsto f(x)$ is a homeomorphism.
2. A compactification Y is **dense** if X is dense in Y, i.e. $\overline{X} = Y$.

Definition 4.27: One-point compactification

The **one-point compactification** of a topological space X is the set
$$X^\infty = X \cup \{\infty\}$$
obtained by adjoining a "point at infinity" ∞, where ∞ is a symbol *not* in X, with open sets of the form either

1. U, where $U \subseteq X$ is open, or
2. $X^\infty \backslash K$, where $K \subseteq X$ is compact and closed.

Lemma 4.28

1. The collection of open sets just defined does form a topology
2. The subspace topology on X induced by this topology coincides with its original topology.

Definition 4.32: Local Compactness

A topological space X is **locally compact** if for each $x \in X$, there exists an open subset $U \subseteq X$ and a compact C such that $x \in U \subseteq C$.

Remark 4.33

When X is Hausdorff, it is locally compact iff for each $x \in X$ there exists an open subset $U \subseteq X$ and a compact $x \in U$ and the closure $\overline{U}$ is compact.

Definition 4.35: Normal Space

A topological space $(X, \mathcal{T})$ is **normal** if for every pair of disjoint closed subsets C and $D \subseteq X$, there are disjoint open subsets $U, V \subseteq X$ such that $C \subseteq U$ and $D \subseteq V$

Lemma 4.37: Normal Complements

A space X is normal iff for every closed $F \subseteq X$ and open $G \subseteq X$ with $F \subseteq G$, there exist open G' and closed F' such that
$$F \subseteq G' \subseteq F' \subseteq G.$$

Theorem 4.38: Urysohn's Lemma

Suppose that X is a normal topological space, and that C, D are disjoint closed subsets of X. Then there is a continuous function $f : X \to \mathbb{R}$ such that

- $f(x) = 0$ for all $x \in C$
- $f(x) = 1$ for all $x \in D$
- $0 \leq f(x) \leq 1$ for all $x \in X$

Theorem 4.39: Tietze extension theorem

Suppose that X is a normal topological space, and that C is a closed subset of X. Suppose that $f : C \to \mathbb{R}$ is continuous. Then there is a continuous function $\overline{f} : X \to \mathbb{R}$ such that

- $\overline{f}(x) = f(x)$ for all $x \in C$
- If $a \leq f(x) \leq b$ for all $x \in C$, then $a \leq \overline{f}(x) \leq b$ for all $x \in X$.

Theorem 4.40: Stone-Weierstrass Theorem

The algebra A is dense in the normed space $C(X)$, i.e. $\overline{A} = C(X)$, i.e. for all $f \in C(X)$ and for all $\epsilon > 0$ there is $g \in A$ such that $\sup_{x\in X} |f(x) - g(x)| < \epsilon$

5 Connected Spaces

Definition 5.1: Connected Spaces

1. A topological space X is **connected** if it *cannot* be written as a union
$$X = A \cup B$$
where A and B are disjoint nonempty open subsets of X
2. A topological space X is **disconnected** if it is not connected, i.e. if it *can* be expressed as a union
$$X = A \cup B$$
where A and B are disjoint nonempty open subsets of X

Connectedness is a **Topological Property** (See P6).

Remark 5.8: Connected Homeomorphisms

- If X is a compact connected metric space with exactly two points x such that $X \backslash \{x\}$ is connected, then X is homeomorphic to $[0, 1]$
- If X is a compact connected space, where for every pair of distinct points $x, y \in X$ the complement $X \backslash \{x, y\}$ is disconnected, then X is homeomorphic to the circle $\mathbb{S}_1$

Proposition 5.11: Connectedness other properties

The following statements about a nonempty subset $A \subseteq \mathbb{R}$ are equivalent:

1. A is connected
2. A has the interval property
3. A is an interval

Theorem 5.12: Intermediate Value Theorem

Let I be a closed bounded interval and suppose $f : I \to \mathbb{R}$ is continuous. Then the image $f(I)$ is a closed bounded interval
$$f(I) = [a, b] \subseteq \mathbb{R}(a \leq b).$$

Definition 5.13: Fixed Points of Maps

A **fixed point** of a map $f : X \to X$ is an $x \in X$ s.t. $f(x) = x$.

Theorem 5.15: 1-D Brouwer Fixed Point Theorem

Every continuous map $f : [0, 1] \to [0, 1]$ has a fixed point, i.e. there exists $x \in [0, 1]$ such that $f(x) = x$.
General Case: Every continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point

Definition 5.16: Path

A **path** in a topological space X is a continuous map $\alpha : I = [0, 1] \to X$. Its **initial point** is $\alpha(0) \in X$ and its **terminal point** is $\alpha(1) \in X$.

Definition 5.18: Path Connectedness

A topological space X is **path-connected** if for any two points $x_0, x_1 \in X$ there exists a path $\alpha : I \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$.

Theorem 5.24: Homeomorphisms of Real Spaces

If $n \geq 2$, the spaces $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic.
Additionally, there is no bijection $f : \mathbb{R} \to \mathbb{R}^n$ which is continuous.

Definition 5.35: Connected Components

We define an equivalence relation $\sim$ on a topological space x by $x \sim y$ iff there is a connected subset of X which contains both x and y. The resulting equivalence classes are called the **components** or **connected components** of X. For two homeomorphic topologial spaces there will be a bijection between the sets of their components. So spaces with differing numbers of components cannot be homeomorphic. The components are the maximal connected subsets of a topological space and by Prop 5.9 are always closed in X. If we take $U \subseteq \mathbb{R}$ an open set, its connected component decomposition is its canonical representation as in Thm 1.30 as a countable disjoint union of open intervals.

Lemma 5.31.5: Path Components

Define a path (equivalence) relation

$$x_0 \sim x_1 \text{ if there exists a path } \alpha : I \to X \text{ from } \alpha(0) = x_0 \in X \text{ to } \alpha(1) = x_1 \in X.$$

5.32) The **constant path** at $x \in X$ is the path
$$\alpha_x : I \to X; \quad t \mapsto x$$
from $\alpha_x(0) = x \in X$ to $\alpha_x(1) = x \in X$

5.33) The **reverse** of a path $\alpha : I \to X$ is the path
$$-\alpha : I \to X; \quad t \mapsto \alpha(1 - t)$$
retracting α backwards, with
$$\underset{-\alpha(0) = \alpha(1)}{\bullet} \xrightarrow{-\alpha} \underset{-\alpha(1) = \alpha(0)}{\bullet}$$

5.34) The **concatenation** of paths $\alpha : I \to X$, $\beta : I \to X$ with
$$\alpha(1) = \beta(0) \in X$$
is the path
$$\alpha \bullet \beta : I \to X, \quad t \mapsto \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq 1/2 \\ \beta(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$
which starts at $\alpha(0)$, follows along α at twice the speed in the first half, switching at $\alpha(1) = \beta(0)$ to follow β at twice the speed in the second half.
$$\underset{\alpha \bullet \beta(0) = \alpha(0)}{\bullet} \xrightarrow{\alpha} \underset{\alpha(1) = \beta(0)}{\bullet} \xrightarrow{\beta} \underset{\beta(1) = \alpha \bullet \beta(1)}{\bullet}$$

Lemma 5.31: Connected Compomennts and Openness

Let X be a topological space and C a connected component of X. Then C is open iff for all $x \in C$ there is an open connected V such that $x \in V \subseteq C$.

Proposition 5.35: Equivalence of Path Relations

The path relation defined on a space X by $x_0 \sim x_1$ if there exists a path $\alpha : I \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_0$ is an equivalence relation.

Definition 5.36: Path Compoments Formally

Let X be a topological space.

1. The **path components** of X are the equivalence classes of the path equivalence relation $\sim$, i.e. the subspaces
$$\begin{aligned}[x] &= \{y \in X \mid y \sim x\} \\ &= \{y \in X \mid \exists \alpha : I \to X \text{ from } a(0) = x \text{ to } \alpha(1) = y\}\end{aligned}$$
2. The **set of path components** (which may be infinite) is denoted by
$$X/\sim = \pi_0(X)$$
3. The function
$$X \to \pi_0(X), \quad x \mapsto [x] = \{\text{equivalence class of } x\}$$
is surjective.

Lemma 5.39: Open Condition of Path Compoments

Let X be a topological space and P a path component of X. Then P is open iff for all $x \in P$ there is an open path connected V such that $x \in V \subseteq P$.

Lemma 5.40: Openness and Singular Components

Let C be a connected component of a topological space X. If every path component $P \subseteq C$ is open, then C consists of a single path component. Note that the converse of this is not true.

6 Relations between Top Props

Proposition A: Topological Invariants

A **topological property** of a topological space is one which is **invariant** under homeomorphism. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a homeomorphism. The following properties are true:

2.8) $\mathcal{U}$ is open in Y iff $f^{-1}(\mathcal{U})$ is open in X.

- X is Hausdorff iff Y is Hausdorff.

3.6) $X \backslash \{x_0\}$ is homeomorphic to $Y \backslash \{f(x_0)\}$.

4.11) X is compact, iff Y is compact.

5.6) X is connected iff Y is connected.

5.21) X is path-connected iff Y is path-connected.

5.37) There exists a bijection between the set of path components $\pi_0(X)$ and $\pi_0(Y)$. However, existence of a bijection between $\pi_0(X)$ and $\pi_0(Y)$ does *not* necessarily imply that X and Y are homeomorphic.

Proposition B: Hausdorff if...

3.4) Suppose $(X, \mathcal{T})$ is a Hausdorff topological space and suppose A is a subspace. Then A is Hausdorff.

4.34) The one-point compactification X^∞ of a space X is Hausdorff iff X is Hausdorff and locally compact.

Proposition C: Compact if...

4.3) Let X be a topological space and $A \subseteq X$. Then A is compact iff every open cover of A has a finite subcover.

4.5) **Heine-Borel Theorem**: A subset $F \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.

4.6) Let X be a topological space and $A \subseteq X$.

1. If X is compact and A is closed, then A is compact
2. If X is Hausdorff and A is compact, then A is closed.

4.10) Let $f : X \to Y$ be a continuous map between topological spaces. If X is compact, so is $f(X)$.

4.18) **Tychonoff's Theorem**: Suppose X and Y are compact spaces. Then their product $X \times Y$ is compact. The converse is also true.

4.21) **Tychonoff's Theorem (General)**: Suppose that $\mathcal{A}$ is an indexing set and that for each $\alpha \in \mathcal{A}$, X_α is a compact topological space. Then the product $\prod_{\alpha \in \mathcal{A}} X_\alpha$ is compact.

4.30) Suppose $X^\infty = X \cup \{\infty\}$ is the *one-point compactification* of X. Then either X^∞ is compact, or X is dense in X^∞

Proposition D: Continuous if..

2.14) Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces and that $f : X \to Y$. Then f is continuous iff for every closed subset $F \subseteq Y$ its inverse image $f^{-1}(F)$ is closed in X.

2.14) f is continuous iff the image of the closure of every subset $A \subseteq X$ is contained in the closure of the image, i.e., $\forall A \subseteq X$,

$$f(\overline{A}) \subseteq \overline{f(A)}$$

3.5) Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces and suppose A is a subspace of X. Let $f : X \to Y$ be continuous. Then $f|_A : A \to Y$ is continuous.

3.12) Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and $\mathcal{T}$ the product topology on $X \times Y$. Then the projection maps Π_X and Π_Y are continuous. Moreover, $\mathcal{T}$ is the smallest topology on $X \times Y$ such that the projection maps are continuous.

3.13) Let X, Y, Z be topological spaces. Endow $X \times Y$ with the product topology. A function $f : Z \to X \times Y$ is continuous iff the functions $\Pi_X \circ f : Z \to X$ and $\Pi_Y \circ f : Z \to Y$ are both continuous.

Let X be a topological space with an equivalence relation $\sim$.

1. The function $p : X \to X/\sim; \quad x \mapsto [x]$ is continuous.
2. A continuous function $f : X \to Y$ such that $f(x) = f(x') \in Y$ for all x, $x \in X$ with $x \sim x'$ determines a continuous function
$$g : X/\sim \to Y; \quad [x] \mapsto f(x)$$
such that
$$f = g \circ p : X \xrightarrow{p} X/\sim \xrightarrow{g} Y$$

$f = g \circ p$ is best described by a commutative triangle:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & X/\sim \arrow[ur, "g"'] & \end{tikzcd}$$

In fact, every continuous function on X determines an equivalence relation.

Proposition E: Connected if...

5.2) X is connected iff the only subsets of X which are clopen are $\emptyset$ and X

5.4) $\mathbb{R}$ with the usual topology is connected.

5.5) If $f : X \to Y$ is continuous and X is connected, then $f(X)$ (with the subspace topology) is connected.

5.9) Let A be a connected subset of a topological space X and suppose $A \subseteq B \subseteq \overline{A}$. Then B is connected.

5.10) Every nonempty interval $I \subseteq \mathbb{R}$ is connected.

5.25) If a topological space X is path-connected, then it is also connected. Note that the converse need not be true.

5.30) Let $A_\lambda \subseteq X$, $(\lambda \in \Lambda)$ be a family of connected subsets of a topological space X. Suppose $\bigcup_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected.

Proposition F: Path-Connected if...

- Suppose $f : X \to Y$ is a continuous map between topological spaces and that X is path-connected. Then $f(X)$ is path-connected as a subspace of Y.
- For any equivalence relation $\sim$ on a path-connected space X the identification space $Y = X/\sim$ is path-connected.
- Any connected open subset $\Omega \subseteq \mathbb{R}^n$ is also path-connected.
- Let X be a topological space. Then X is path connected iff X is connected *and* for all $x \in X$ there is an open path connected V such that $x \in V$.

Example G: Topological Invariancy Proofs

- **Compactness**: Let U_λ be open subsets of Y which cover $f(X)$. Then $f^{-1}(U_\lambda)$ are open sets in X which cover X. Hence there is a finite subcover $\{f^{-1}(U_{\lambda_1}), \ldots, f^{-1}(U_{\lambda_1})\}$, and so $\{U_{\lambda_1}, \ldots, U_{\lambda_1}\}$ covers $f(X)$.
- **Connectedness**: If $f(X)$ is disconnected then we can write it as a disjoint union $f(X) = (A \cap f(X)) \cup (B \cap f(X))$ for some open subsets $A, B \subseteq Y$. The inverse images $f^{-1}(A \cup f(X)) = f^{-1}(A)$ and $f^{-1}(B \cap f(X)) = f^{-1}(B)$ are disjoint open subsets of X s.t. $X = f^{-1}(A) \cup f^{-1}(B)$, in contradiction to the connectedness of X. Hence $f(X)$ is connected.
- **Path-Connectedness**: Pick y_0 and y_1 in $f(X)$. So there are $x_0, x_1 \in X$ such that $f(x_0) = y_0$ and $f(x_1) = y_1$. Let $\alpha : [0,1] \to X$ be a cts map with $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Then $\beta = f \circ a$ is a path in $f(X)$ joining y_0 to y_1.

7 Examples

Example a: Other Topologies and Metrics

If $(X, \mathcal{T})$ is a topological space, and X admits a metric whose metric topology is precisely $\mathcal{T}$, then we say that $(X, \mathcal{T})$ is **metrisable**.

- Euclidean spaces with their usual topologies are metrisable.

1.9) The **Discrete Topology** is the topology of all subsets of a set X. We can define the **discrete metric** of X to be
$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}.$$

1.10) The **Trivial** or **Indiscrete Topology** is the topology $\mathcal{T} := \{\emptyset, X\}$ for a set X. This is a non-metrisable topology when X has more than one member.

1.14) Let $X = \{a, b, c\}$, where a, b, c are distinct. Then
$$\mathcal{T} = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$$
is a topology on X

1.15) Give $\mathbb{R}$ the topology whose open subsets $U \subseteq \mathbb{R}$ are precisely the subsets with finite complement $\mathbb{R}\backslash U$, or $U = \emptyset$. Then $\mathbb{R}$ with this topology is not metrisable. This is an example of a **Zariski Topology**

- The **Co-finite** topology is the subsets of K whose complements are finite, along with $\emptyset$. Every subset of the co-finite topology is compact.
- The **Co-countable** topology is the subsets of K whose complements are countable, along with $\emptyset$. Every compact subset of the co-countable topology is finite.
- The **Hawaiian Earring** space is the subspace of $\mathbb{R}^2$ with the usual topology given by $H = \bigcup_{n=1}^{\infty} C_n$, where $C_n \subseteq \mathbb{R}^2$ is given by
$$C_n = \left\{ (x,y) \in \mathbb{R}^2 \mid (x - \frac{1}{n}^2) + y^2 = \frac{1}{n^2} \right\}$$
- $\mathbb{S}^n$ and an eq. relation $\sim$ where $x \sim y$ iff $x = y$ or $x = -y$ is the **Real Projective Space** $\mathbb{RP}^n$, or "the lines in $\mathbb{R}^{n+1}$ which pass through the origin".
- The **Particular Point Topology** is a topology where a set is open if it contains a particular point of the space, i.e. $T = \{S \subseteq X \mid p \in S\} \cup \{\emptyset\}$. The closure of any open set other than $\emptyset$ is X, so the interior of every closed set other than X is $\emptyset$. $X\backslash\{p\}$ is totally disconnected. $\{p\}$ is compact, but the closure is X therefore not compact if X is infinite. If $Y \subseteq X$ doesn't contain p, Y has no limit point, and if it does then every point is a limit point of Y.

Example B: Compact Sets

- $\mathbb{R}$ is not compact. Take $\{[0,n) \mid n = 1, 2, \ldots\}$. This covers $\mathbb{R}$ but has no finite subcover.
- $\mathbb{R}^n$ is not compact. Take the same argument, but with open balls of dimension n.
- $\mathbb{S}^n$ is compact, as it is a closed (under the euclidean norm), bounded (by 1) subspace of $\mathbb{R}^n$.
- $[0,1]$ is closed and bounded, therefore compact via Heine-Borel.
- The cantor space $\{0,1\}^w$ is bounded by $[0,1]$, and as thirds C_n are closed, and $\{0,1\}^w$ is an intersection of such sets, it is closed and therefore compact via Heine-Borel.
- The quotient space of a Topological space $K/\sim$ is compact. The quotient map $p : K \to K/\sim$ is continuous, therefore since K is compact, so is $K/\sim$ via Theorem 4.10.

Example C: Homeomorphisms

- For the sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, the punctured sphere $\mathbb{S}\backslash\{x_0\}$ for some x_0 is homeomorphic to $\mathbb{R}^n$

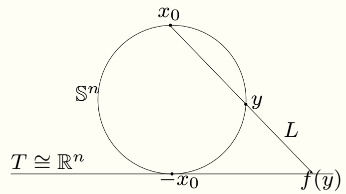

Figure 1: Homeomorphism of $\mathbb{S}^2$ to $\mathbb{R}$

- $(0,1)$ is homeomorphic to $\mathbb{R}$. Take $f(x) = \tan(\pi x - \frac{\pi}{2})$ or $f(x) = \frac{x}{\sqrt{1+x^2}}$
- $[0,1]$ is not homeomorphic to $(0,1)$. $[0,1]$ is closed and bounded $\implies$ compact via Heine-Borel, while $\mathbb{R}$ is not compact.
- $[0,1)$ is not homeomorphic to $(0,1)$. Let $f : [0,1) \to (0,1)$. Then there is $f(0) \in (0,1)$. Now take $[0,1]\backslash\{0\}$. This is still connected, but $(0,1)\backslash\{f(0)\}$ is disconnected.
- $Y = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$ is not homeomorphic to $\mathbb{R}$. There is a point $(0,0)$ where $Y\backslash(0,0)$ has 4 connected components but this does not follow for $\mathbb{R}$.
- $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$. For $\mathbb{R}$ vs $\mathbb{R}^2$ consider a hole and exclusion on $\mathbb{R}$ not being path-connected via IVT.
- $\mathbb{R} + \mathbb{R}$ (disjoint union) is homeomorphic to $\mathbb{R}\backslash\{0\}$
- $\mathbb{S}^1$ is homeomorphic to the identification space of $I = [0,1]$ under a equivalence relation that glues both ends together
$$x \sim y \text{ if } x = y \text{ or if } (x,y) = (1,0) \text{ or if } (x,y) = (0,1)$$
- $\mathbb{S}_1$ is not homeomorphic to $[0,1]$, if there was $f : [0,1] \to \mathbb{S}^1$ then the spaces $[0,1]\backslash\{1/2\} = [0,1/2) \cup (1/2,1]$ disconnected, while $\mathbb{S}^1\backslash\{f(1/2)\}$ is homeomorphic to an open interval and therefore connected.

Example D: Random counterexample

- The **topologist's sine curve** is connected but not path-connected
$$X = \{(0,y) \mid -1 \le y \le 1\} \cup \{(x, \sin(\frac{\pi}{x})) \mid 0 < x \le 1\} \subseteq \mathbb{R}^2$$

Example E: Compactification

- The open interval $X = (0,1)$ has dense compactification the closed interval $Y = [0,1]$.
- Let $\sim$ be the equivalencer relation on $[0,1]$ generated by $0 \sim 1$. Then $Z = [0,1]/\sim = \mathbb{S}_1$ is a dense compactification of $X = (0,1)$.
- $\mathbb{R}^n$ has dense compactification $\mathbb{S}^n$ since $\mathbb{S}^n\backslash\{x\} \subseteq \mathbb{S}^n$ is a dense subspace homeomorphic to $\mathbb{R}^n$.
- $\mathbb{R}^n$ has dense compactification $\mathbb{D}^n$ since the open unit ball
$$\mathbb{B}^n = B(0,1) = \{x \in \mathbb{R}^n \mid |x| < 1\} \subset \mathbb{D}^n$$
is a dense subspace homeomorphic to $\mathbb{R}^n$

One Point compactification

- $(0,1)^\infty = \mathbb{S}_1$
- $(\mathbb{R}^n)^\infty = \mathbb{S}^n$

Example F: Topological Objects

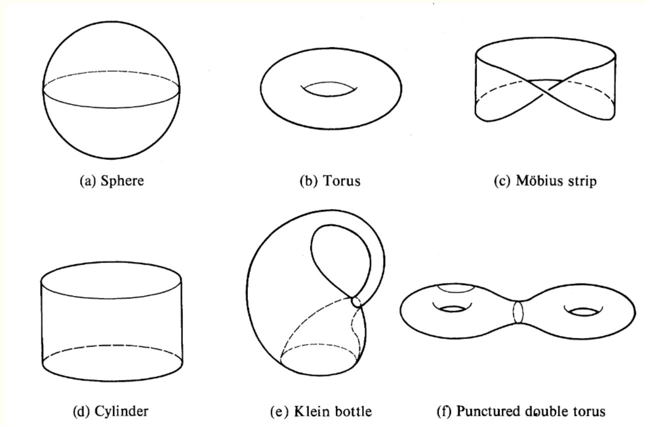

Figure 2: Standard Topological Objects

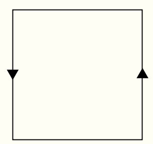

Fig 3: Möbius Strip

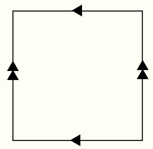

Fig 4: Torus

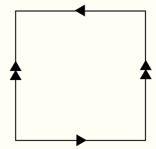

Fig 5: Klein Bottle

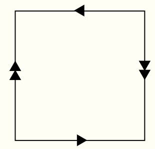

Fig 6: $\mathbb{RP}^2$