Algebraic Topology Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Introduction

Recall 1.1.1: Topology

An **(open) topology** on X is a collection of subsets $\tau \subset P(X)$ such that

- $\emptyset \in \tau$ and $X \in \tau$
- τ is closed under finite intersections: If $\{U_1, \dots, U_n\} \subset \tau$ then
$$\bigcap_{i=1,\dots,n} U_i \in \tau$$
- τ is closed under arbitrary unions: If $\{U_1, \dots, U_n\} \subset \tau$ is a family of open subsets then
$$\bigcup_{i=1,\dots,n} U_i \in \tau$$

The subsets $U \in \mathcal{T}$ are called **open** and their complements in X define **closed subsets**.

Two examples of a topology on a set X are the following:

- The **Trivial Topology**: $\tau_{\text{triv}} = \{\emptyset, X\}$
- The **Discrete Topology**: $\tau_{\text{dis}} = P(X)$

A subset $A \subset X$ is **clopen if it is both closed and open**

Definition 1: Connected Spaces

A topological space X is **connected** if $X = A \cup B$ with $A,\, B \subset X$ open implies that $A = \emptyset$ or $A = X$.

Proposition 1: Connectedness and Clopens

A topological space X is *connected* iff the only clopens are $\emptyset$ and X.

Example 1: Examples of Connected Topologies

- Every X with the trivial topology is connected.
- Every X with the discrete topology is not connected unless $X = \emptyset$ or $X = \{*\}$ (in which it coincides with the trivial topology).
- The real line $\mathbb{R}$ with the standard topology is connected.

Proposition 2: Continuous Maps

Let $f : X \to Y$ be a continuous map of topological spaces and let X be connected. Then $f(X)$ is connected.

Proposition 3: Connected Equivalence Relation

For a topological space X, define $x \sim y$ if there exists some connected subset that contains both. The relation $x \sim y$ is an equivalence relation.

Definition 2: Connected Components

The equivalence classes of this relation are called **connected components**. In particular, a space X is connected iff it only has a single connected component.

Definition 3: Path

Let I denote the closed unit interval $[0, 1]$. A **path** in X is a continuous map $\alpha : I \to X$. The points $\alpha(0) \in X$ and $\alpha(1) \in X$ will be called **start** and **end** points respectively.
We define a path relation between points in X be declaring $x \sim y$ if there exists some path $\alpha : I \to X$ that starts at x and ends in y, i.e. $\alpha(0) = x$ and $\alpha(1) = y$. This is an equivalence relation from the following properties:

1. **Constant Path**: For all $x \in X$ there exists the constant path $c_x : I \to X$ defined by $c_x(t) = x$ for all $t \in I$
2. **Path reversal**: Let $\alpha : I \to X$ be a pth in X. Define its reversed path by
$$\overline{\alpha} : I \to X, \quad t \mapsto \alpha(1-t) \tag{1}$$
3. **Path Concatenation**: Let $\alpha,\, \beta : I \to X$ be two paths in X s.t. $\alpha(1) = \beta(0)$. Their concatenated path is defined by:
$$\alpha * \beta(t) := \begin{cases} \alpha(2t), & 0 \le t \le 1/2 \\ \beta(2t-1) & 1/2 \le t \le 1 \end{cases} \tag{2}$$

Definition 4: Path-Connected Components

The equivalence clases are called **path-connected components** and their set is denoted by $\pi_0(X)$. A space X is called **path-connected** if $\pi_0(X)$ is a one-point set, i.e. any two points $x,\, y$ can be related by a path in X.

Remark 1: Random examples

The following statements are true:

- A homeomorphism $X \cong Y$ induces a bijection $\pi_0(X) \cong \pi_0(Y)$.
- If X is path-connected, it is also connected.
- The *topologist's sine curve* defined by $X = \{0\} \times [-1, 1] \times \{(x, \sin(1/x)) \mid 0 < x\}$ is connected but not path-connected.

Definition 5: Homotopy

A **homotopy** of maps $f,\, g : X \to Y$ is a continuous map $h : X \times I \to Y$ such that $h(-, 0) = f$ and $h(-, 1) = g$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & Y \arrow[Rightarrow, from=U, to=D, "h"] \end{tikzcd} \tag{3}$$

If such a homotopy exists, f is called **homotopic** to g. This defines an equivalence relation $f \simeq g$ on the space of maps $\mathrm{Map}(X, Y)$.

Example 2: Paths as Homotopies

Points in X are the same as maps $* \to X$ from the one-point set $*$ to X. A path $\alpha : I \to K$ corresponds to a homotopy $* \times I \to X$.

Remark 1.5: Composition of Homotopies

- **Horizontal Composition**: Let $h, h' : X \times I \to Y$ be two homotopies in X such that $h(-, 1) = h'(-, 0) : X \to Y$. Their concatenated homotopy is defined by
$$h * h'(-, t) := \begin{cases} h(-, 2t) & 0 \le t \le 1/2 \\ h'(-, 2t-1) & 1/2 \le t \le 1 \end{cases} \tag{5}$$
- **Vertical Composition**: Let $h : X \times I \to Y$ and $k : Y \times I \to Z$ be two homotopies on maps from X to Y, and Y to Z. Then
$$k \circ h := [X \times I \xrightarrow{\text{id} \times \Delta} X \times I^2 \xrightarrow{h \times \text{id}} Y \times I \xrightarrow{k} Z] \tag{6}$$
where $\Delta : I \to I^2,\, t \mapsto (t, t)$ is the diagonal map, or explicitly,
$$k \circ h(x, t) = k(h(x, t), t)$$

$$\begin{tikzcd} X \arrow[rr, bend left=50, "f"{name=F}] \arrow[rr, ""{name=M}] \arrow[rr, bend right=50, "l"'{name=L}] && Y \arrow[Rightarrow, from=F, to=M, "h"'] \arrow[Rightarrow, from=M, to=L, "h'"'] \end{tikzcd} \sim \begin{tikzcd} X \arrow[rr, bend left=50, "f"{name=F}] \arrow[rr, bend right=50, "l"'{name=L}] && Y \arrow[Rightarrow, from=F, to=L, "k' * h"] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, bend left, "f"{name=A}] \arrow[r, bend right, "g"'{name=B}] & Y \arrow[r, bend left, "f'"{name=C}] \arrow[r, bend right, "g'"'{name=D}] & Z \arrow[Rightarrow, from=A, to=B, "h"] \arrow[Rightarrow, from=C, to=D, "k"] \end{tikzcd} \sim \begin{tikzcd} X \arrow[r, bend left, "f' \circ f"{name=A}] \arrow[r, bend right, "g' \circ g"'{name=B}] & Z \arrow[Rightarrow, from=A, to=B, "k \circ h"] \end{tikzcd}$$

Lemma 1: Concatenation Relation

Let $f,\, f' : X \to Y$ and $g, g' : Y \to Z$ be maps such that $f \simeq f'$ and $g \simeq g'$. Then $f' \circ f \simeq g' \circ g$ as maps from X to Z. In particular, $g' \circ f \sim g \circ f$ and $g \circ f' \sim g \circ f$.

Definition 6: Homotopy Equivalence

A map $f : X \to Y$ is called a **homotopy equivalence** if there exists a map $g : Y \to X$ and homotopies $f \circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$. In other words, g satisfies the properties of an inverse up to homotopy. It is called a **homotopy inverse** of f.

Example 3: Circle to $\mathbb{R}^2$

The inclusion map $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2$ is not a homotopy equivalence, but the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2 \backslash \{0\}$ is a homotopy equivalence.

Proposition 4: Unique Inverses of Homotopy

Homotopy inverses are unique up to homotopy.

Definition 7: Homotopic Spaces

Two spaces X and Y are called **homotopy equivalent**, or **of the same homotopy type**, and denoted by $X \simeq Y$ if there exists a homotopy equivalence $f : X \to Y$.

Notation: We use $\cong$ for homeomorphisms and $\simeq$ for homotopy equivalence.

Lemma 2: Composition of Inverses

Let $f : X \to y$ and $g : Y \to Z$ with homotopy inverses $\overline{f} : Y \to X$ and $\overline{g} : Z \to Y$ respectively. Then $\overline{f} \circ \overline{g} : Z \to X$ is a homotopy inverse of $g \circ f : X \to Z$. In particular, $X \simeq Y$ and $Y \simeq Z$ implies $X \simeq Z$.

Definition 8: Contractible Space

A space X is called **contractible** if it is homotopy equivalent to a point, i.e. $X \simeq *$.

The **terminal map** is the unique map $X \to *$. Contractibility requires that there is a homotopy inverse of that map, i.e. a map $* \to x$ along with homotopies

$$h : [* \to X \to *] \simeq \mathrm{id}_*, \quad k : [X \to * \to X] \simeq \mathrm{id}_X \tag{7}$$

Example 4: Examples of Contractible Spaces

1. $\mathbb{R}^n$ is contractible. Let x_0 be a fixed point in $\mathbb{R}^n$ and define the (straight line) homotopy $h : c_{x_0} \simeq \mathrm{id}_{\mathbb{R}^n}$ by
$$h(x,t) = (1-t)x_0 + tx.$$
2. $\mathbb{S}^{n-1} \simeq \mathbb{R}^n \backslash \{0\}$. The inclusion $\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n \backslash \{0\}$ and the shrinking map
$$\mathbb{R}^n \backslash \{0\} \to \mathbb{S}^{n-1}, \quad x \mapsto \frac{x}{|x|}$$
are homotopy inverses.

Remark 3: Remarks about Contractible Spaces

1. Contractible spaces are path-connected. Let x_0 be the point where the space X contracts to. In particular, we are given with a homotopy $h : c_{x_0} \simeq \mathrm{id}_X$. For any $x \in X$, the map $h(x, -) : I \to X$ defines a path from x_0 to x and thus every element $x \in X$ is path-connected to x_0.
2. The converse does not hold, for example $X = \mathbb{S}^1$.
3. A contractible space X is contractible at any point x_0. Since X is path-connected, a path from x to x' defines a homotopy $c_x \simeq c_{x'}$.
4. Any two maps $f, g : X \to Y$ are homotopic if Y is contractible.

Definition 9: Retracts and Deformation Retracts

- A **retract** of X onto a subspace $A \subset X$ is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$. Equivalently, this is a map $r : X \to X$ such that $r^2 = r$ and $r(X) = A$.
- A **deformation retract** of X onto A is the additional datum of a homotopy $h : \mathrm{id}_X \simeq i \circ r$.

In other words, a deformation retract is a homotopy $h : X \times I \to X$ such that $h(x,0) = x$ and $h(x,1) \in A$ for all $x \in X$ and $h(a,1) = a$ for all $a \in A$. Not all retracts can form deformation retracts. For instance, the retract X onto a point $\{x_0\}$ can be a deformation retract iff X is contractible.

This notion is called **weak** deformation retract. A **strong** deformation retract has the condition $h(a,t) = a$ for all $t \in I$, $a \in A$. i.e. Our notion of a (weak) deformation retract deforms X into A while allowing to deform A to do so, while a strong deformation retract deforms X into A while keeping A fixed at all times

Proposition 5: Deformation Retracts and Homotopies

A deformation retract of X onto A induces a homotopy equivalence $X \simeq A$.

Recall 2: Quotient Space

Let X be a topological space and let $\sim$ be an equivalence relation on X. Then, $X/\sim$ is equipped with the quotient topology and called a **quotient space**. If Z is a closed subset in X, then we can also define the quotient space X/Z.

Another form of quotient spaces: Let $f : Z \to Y$ be a continuous map between a closed subset $Z \subset X$ and Y. Then
$$X \cup_f Y = X \cup Y/z \sim f(z).$$

Example 5: Examples of Quotient Spaces

- The quotient of the n-dimensional closed disk by its boundary is the n-sphere, i.e. $\mathbb{D}^n/\partial\mathbb{D}^n \cong \mathbb{S}^n$.
- The 2-torus: $\mathbb{R}^2/\mathbb{Z}^2$. The projective space:

$\mathbb{RP}^n = \mathbb{R}^{n+1} \backslash \{0\}/\sim$ by the relation $x \sim y$ iff there exists some $\lambda \in \mathbb{R}^\times$ such that $x = \lambda y$. This corresponds to the space of lines through the origin in $\mathbb{R}^{n+1}$.

Definition 10: Mapping Quotients

Let $f : X \to Y$ be a continuous map.

- Its **mapping cylinder** is defined as the topological space
$$M_f := (X \times I) \cup Y/\sim$$
where the quotient identifies $(x,0) \sim f(x)$ for any $x \in X$.
- Its **cone** is the further quotient:
$$C_f = M_f / X \times \{1\}.$$
- The **cone** of a topological space X is
$$C_X := C_{\mathrm{id}_X} = X \times I / X \times \{1\}.$$

In other words, the mapping cylinder of $f : X \times Y$ is the pushout of the diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \times I \arrow[r] & M_f \end{tikzcd}$$

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