

# General Topology Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

## 1 Topological Spaces and Examples

### Definition 1.1: Topological Space

A **topological space** is a pair  $(X, \mathcal{T})$ , where  $X$  is a nonempty set, and  $\mathcal{T}$  is a collection of subsets of  $X$  which satisfies:

- $\emptyset \in \mathcal{T}$  and  $X \in \mathcal{T}$
- if  $U_\lambda \in \mathcal{T}$  for each  $\lambda \in \Lambda$  (where  $\Lambda$  is some indexing set), then  $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$
- if  $U_1, U_2 \in \mathcal{T}$ , then  $U_1 \cap U_2 \in \mathcal{T}$

The collection  $\mathcal{T}$  is called the **topology** of the topological space, and the members of  $\mathcal{T}$  are called the **open sets** of the topology

### Example 1.7: Euclidean Spaces

Let  $\mathbb{R}^n$  denote the  $n$ -dimensional Euclidean vector space with elements  $x = (x_1, x_2, \dots, x_n)$  and  $x_i \in \mathbb{R}$ , and let

$$|x| = \sqrt{\sum_{i=1}^n x_i^2} \geq 0$$

be the length of  $x$ . ( $\mathbb{R}^1 = \mathbb{R}$  is the real line). A subset  $U$  of  $\mathbb{R}^n$  is **open (for the usual topology)** iff for each  $a \in U$  there exists an  $r > 0$  such that

$$|x - a| < r \implies x \in U.$$

The collection of open sets thus defined is called the **usual topology** on  $\mathbb{R}^n$ . Note that open balls  $B(y, \rho) = \{x \in \mathbb{R}^n : |x - y| < \rho\}$  are open sets under this definition.

### Example 1.8: Metric Spaces

A **metric space**  $(X, d)$  is a nonempty set  $X$  together with a function  $d : X \times X \rightarrow \mathbb{R}$  with the following properties:

- $d(x, y) \geq 0$  and  $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$  (Triangle Inequality)

The function  $d$  is called the **metric**.

Let  $(X, d)$  be a metric space,  $x$  be a point in  $X$ , and  $r > 0$ . The **open ball** with center  $x$  and radius  $r$  is defined by

$$B(x, r) = \{y, \in X : d(x, y) < r\}.$$

A subset  $U$  of  $X$  is **open (in the metric topology given by  $d$ )** iff for each  $a \in U$  there is an  $r > 0$  such that  $B(a, r) \subseteq U$ . Just like euclidean spaces, open balls are open in this sense.

### Proposition 1.11: Topology Equality

Let  $d, d'$  be metrics on the same set  $X$ , and let  $\mathcal{T}, \mathcal{T}'$  be the corresponding metric topologies. If for real numbers  $A, B > 0$  we have

$$d(x, y) \leq Ad'(x, y), d'(x, y) \leq Bd(x, y) \text{ for all } x, y \in X,$$

then  $\mathcal{T} = \mathcal{T}'$ .

### Definition 1.16: Subspace Topology

Let  $(X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$  be any subset. Then the **subspace topology** on  $A$  consists of all sets of the form  $U \cap A$  where  $U \in \mathcal{T}$ .

### Definition 1.17: Closed Set

Let  $(X, \mathcal{T})$  be a topological space. A subset  $A \subseteq X$  is **closed** iff its complement  $X \setminus A := \{x \in X \mid x \notin A\}$  is open in  $X$ . Note that a set being *closed* does not mean it isn't *open*. Sets that are both *closed* and *open* are called **clopen**.

### Definition 1.20: Properties of Topological Spaces

For a subset  $A \subseteq X$ ,

- The **closure** of  $A$  is

$$\bar{A} := \bigcap_{\substack{C \subseteq X \text{ closed;} \\ A \subseteq C}} C.$$

- The **interior** of  $A$  is

$$\text{int } A = A^\circ := \bigcap_{\substack{C \subseteq X \text{ open;} \\ A \subseteq C}} C.$$

- The **boundary** (or **frontier**) of  $A$  is

$$\partial A := \bar{A} \setminus A^\circ.$$

- $A$  is **dense** in  $X$  iff  $\bar{A} = X$ .
- A **limit point** of  $A$  is a point  $x \in X$  s.t. for every open subset  $U \subseteq X$  with  $x \in U$  there exists an element  $a \in A \cup U$  with  $a \neq x$ . Let  $A'$  be the set of limit points of  $A$ . Note that this has nothing to do with limits of sequences.

### Proposition 1.22: Relating Topological Properties

- $\bar{A}$  is closed, and contains  $A$  and is the smallest set with this property. So  $A$  is closed iff  $\bar{A} = A$ .
- $A^\circ$  is open, and is contained in  $A$ , and is the largest set with this property. So  $A$  is open iff  $A^\circ = A$ .
- The closure of the complement is the complement of the interior:
$$\overline{X \setminus A} = X \setminus (A^\circ).$$
- The interior of the complement is the complement of the closure:
$$(X \setminus A)^\circ = X \setminus \bar{A}.$$

### Proposition 1.26: Union of Limit Points

Let  $(X, \mathcal{T})$  be a topological space, and suppose  $A \subseteq X$ . Then

$$\bar{A} = A \cup A'$$

### Corollary 1.27

A subset  $A \subseteq X$  is closed iff it contains all its limit points.

### Theorem 1.19: Properties of open and closed sets

Let  $(X, \mathcal{T})$  be a topological space.

- $\emptyset$  and  $X$  are **closed**.
- The union of **finitely many** closed sets is a closed set.
- The intersection of **any collection** of closed sets is a closed set.

### Lemma 1.24: Limit Points and Open Balls

An element  $x \in X$  in a metric space  $(X, d)$  is a **limit point** of a subset  $A \subseteq X$  iff for every  $\epsilon > 0$  there exists  $a \in A$  with  $0 < d(x, a) < \epsilon$ , or iff there exists a sequence  $a_1, a_2, a_3, \dots$  of elements  $a_i \in A$ , with  $a_i \neq x$  for all  $i$ , s.t.  $d(x_i, a_i) \rightarrow 0$  as  $i \rightarrow \infty$ . This interpretation does not extend to general topological spaces.

### Theorem 1.30: Open and Closed sets in $\mathbb{R}$

Consider  $\mathbb{R}$  with the usual topology.

- A nonempty set  $U$  is open iff it can be written as a countable union of disjoint nonempty open intervals  $I_j$  (shown left):
- A set  $F$  is closed iff it can be written as a countable intersection where each  $F_j$  is a finite union of closed intervals (shown right).

$$U = \bigcup_{j=1}^{\infty} I_j, \quad F = \bigcap_{j=1}^{\infty} F_j.$$

### Definition 1.32: Hausdorff Spaces

A topological space  $(X, \mathcal{T})$  is **Hausdorff** if for each  $x, y \in X$  with  $x \neq y$  there exist **disjoint** open sets  $U$  and  $V$  such that  $x \in U$  and  $y \in V$ .

Any metrisable space is Hausdorff, The trivial topology on a set with more than one element is not Hausdorff.

### Definition 1.33: Convergence of a Topological space

A sequence  $(x_n)$  of members of a topological space  $X$  converges to  $x \in X$  if for every open set  $U$  containing  $x$ , there exists an  $N$  such that  $n \geq N \implies x_n \in U$

### Proposition 1.34: Convergence of Hausdorff Spaces

Suppose  $(X, \mathcal{T})$  is Hausdorff. Then a sequence  $(x_n)$  can converge to at most one limit.

### Definition 1.36: Cauchy and Completeness

Let  $(X, d)$  be a metric space.

- A **Cauchy sequence** is a sequence  $(x_n)$  with each  $x_n \in X$  with the property that for each  $\epsilon > 0$ , there exists an  $N$  such that  $m, n \geq N \implies d(x_m, x_n) < \epsilon$
- $(X, d)$  is **complete** if every Cauchy sequence converges.

### Definition 1.37: Topology Basis

A **basis for a topology** on a set  $X$  is a collection  $\mathcal{B}$  of subsets  $B \subseteq X$  such that:

1.  $X = \bigcup_{B \in \mathcal{B}} B$
2. The intersection of sets  $B_1, B_2 \in \mathcal{B}$  is a set  $B_1 \cap B_2 \in \mathcal{B}$ .

The **topology  $\mathcal{T}$  generated by a basis  $\mathcal{B}$**  has open sets the arbitrary unions of basis elements  $B_\lambda \in \mathcal{B}$ :

$$U = \bigcup_{\lambda \in \Lambda} B_\lambda$$

(Don't forget to check that this really is a topology)

### Example 1.38: Finite Intersections of open balls

For any metric space  $(X, \mathcal{T})$  the finite intersections of open balls

$$B(x, r) = \{y \in X \mid d(x, y) < r\} \subseteq X \quad (r > 0, x \in X)$$

constitute a basis for the metric topology on  $X$

$$\mathcal{B} = \{B(x_1, r_1) \cap B(x_2, r_2) \cap \cdots \cap B(x_k, r_k) \mid x_1, x_2, \dots, x_k \in X, r_1, r_2, \dots, r_k > 0\}$$

## 2 Continuous functions and Homeomorphisms

### Definition 2.1: Continuity

Let  $(X, \mathcal{T})$ ,  $(Y, \mathcal{U})$  be topological spaces. A function  $f : X \rightarrow Y$  is **continuous** iff

$$U \in \mathcal{U} \text{ implies } f^{-1}(U) \in \mathcal{T}.$$

That is, **inverse** images of open sets are open. Continuous functions are often called **maps** or **mappings** of topological spaces.

### Proposition 2.6: Topological and Analytic Continuity

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces with their induced topologies  $\mathcal{T}$  and  $\mathcal{U}$  respectively. A function  $f : X \rightarrow Y$  is continuous (topologically) iff it is continuous analytically: for every  $a \in X$  and every  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$d(x, a) < \delta \implies \rho(f(x), f(a)) < \epsilon$$

### Definition 2.7: Homeomorphism

Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{U})$  be topological spaces. A **homeomorphism** is a bijection  $f : X \rightarrow Y$  such that both  $f$  and  $f^{-1}$  are continuous. Two topological spaces are **homeomorphic** if there is a homeomorphism between them.

### Proposition 2.18: The Punctured Sphere

Consider the  $n$ -dimensional sphere

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

with the metric topology inherited from  $\mathbb{R}^{n+1}$ . Let  $x_0 \in \mathbb{S}^n$ . Then  $\mathbb{S}^n \setminus \{x_0\}$  is homeomorphic to  $\mathbb{R}^n$ .

## 3 Subspaces Revisited

### Definition 3.65: Disjoint Unions

Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{U})$  be topological spaces. Their **disjoint union**  $X + Y$  is the set  $(X \times \{0\}) \cup (Y \times \{1\})$  with the topology consisting of all sets of the form

$$(T \times \{0\}) \cup (U \times \{1\}) \text{ such that } T \in \mathcal{T}, U \in \mathcal{U}$$

### Definition 3.8: Product Topology

Let  $(X, \mathcal{T})$ ,  $(Y, \mathcal{U})$  be topological spaces. The **product topology** on their product  $X \times Y$  consists of all sets of the form

$$T = \bigcup_{\alpha \in A} (U_\alpha \times V_\alpha)$$

where  $A$  is an arbitrary indexing set, and  $U_\alpha \in \mathcal{U}$  and  $V_\alpha \in \mathcal{V}$ .

### Lemma 3.10

The product topology is indeed a topology. (lol)

### Lemma 3.9: Openness in Product Topologies

Let  $(X, \mathcal{T})$ ,  $(Y, \mathcal{U})$  be topological spaces. Then  $T \subseteq X \times Y$  is open in the product topology if and only if for all  $t \in T$  there exists  $U \in \mathcal{U}$  and  $V \in \mathcal{V}$  such that  $t \in U \times V$  and  $U \times V \subseteq T$ .

### Definition 3.11.5: Projection Maps

Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{U})$  be topological spaces, and consider their product  $X \times Y$  with the product topology. There are two natural maps  $\Pi_X$  and  $\Pi_Y$ , the projections of  $X \times Y$  onto  $X$  and  $Y$  respectively, given by

$$\Pi_X : X \times Y \rightarrow X, \quad (x, y) \mapsto x$$

$$\Pi_Y : X \times Y \rightarrow Y, \quad (x, y) \mapsto y.$$

### Definition 3.14: Weak Topology

Suppose that  $X$  is a set.  $(X_\lambda, \mathcal{T}_\lambda)$  is a family of topological spaces, and that  $f_\lambda : X \rightarrow X_\lambda$  are functions. The **weak topology generated by  $\{f_\lambda\}$**  is the smallest topology on  $X$  making all the  $f_\lambda$  continuous.

Thus, the product topology on  $X \times Y$  is the weak topology generated by the two maps  $\Pi_X$  and  $\Pi_Y$ .

### Definition 3.15: Cartesian Product Topology

If  $X_\lambda$  is a topological space, (with  $\lambda$  in some arbitrary indexing set  $\Lambda$ ), the product topology on the cartesian product  $\prod_{\lambda \in \Lambda} X_\lambda$  is defined to be the weak topology generated by the projections

$$\Pi_\lambda : \prod_{\mu \in \Lambda} X_\mu \rightarrow X_\lambda$$

### Definition 3.5.0: Equivalence Relation

An **equivalence** relation on a set  $X$  is a binary operation  $\sim$  on  $X$  which is:

1. **Reflexive**:  $x \sim x$  for all  $x \in X$ .
2. **Symmetric**: if  $x \sim y$  then  $y \sim x$ .
3. **Transitive**: if  $x \sim y$  and  $y \sim z$  then  $x \sim z$ .

The **equivalence class** of any element  $x \in X$  is the set

$$[x] = \{y \in X \mid x \sim y\},$$

and the set of equivalence classes is denoted by  $X/\sim$ . The function which assigns to each  $x \in X$  the equivalence class  $[x] \in X/\sim$  is a surjection

$$p : X \rightarrow X/\sim; \quad x \mapsto [x]$$

### Definition 3.17: Quotient Space

Given a topological space  $(X, \mathcal{T})$ , and an equivalence relation  $\sim$  on  $X$ , the **quotient space** or **identification space** is the set of equivalence classes  $X/\sim$  together with the topology

$$\{U \subseteq X/\sim : p^{-1}(U) \in \mathcal{T}\}$$

### Definition 3.25: Generated Topological Spaces

Let  $X$  be a topological space, and let  $Y_0, Y_1 \subseteq X$  be subspaces related by a continuous function  $f : Y_0 \rightarrow Y_1$ . Let  $\sim_f$  be the equivalence relation on  $X$  **generated by  $f$** , the intersection of all the equivalence relations on  $X$  (regarded as subsets of  $X \times X$ ) containing the pairs  $(y_0, f(y_0))$  with  $y_0 \in Y_0$ . The identification space

$$X/\sim_f = X/\{y_1 = f(y_0)\}$$

is obtained by identifying each  $y_0 \in Y_0 \subseteq X$  with  $y_1 = f(y_0) \in Y_1 \subseteq X$ .

### Proposition 3.34: Homeomorphisms of Relations

Given a continuous function  $f : X \rightarrow Y$  let  $\sim$  be the equivalence relation defined on  $X$  by  $x \sim x'$  if  $f(x) = f(x') \in Y$ . The function

$$g : X/\sim \rightarrow Y; [x] \mapsto f(x)$$

is continuous and injective, with

$$f = g \circ p : X \xrightarrow{p} X/\sim \xrightarrow{g} Y.$$

If  $f$  is onto, and such that  $f(U) \subseteq Y$  is open for every open subset  $U \subseteq X$  then  $g$  is a homeomorphism.

## 4 Compact Spaces

### Definition 4.1: Open Covers and Compact Spaces

1. An **open cover** of a topological space  $X$  is a collection  $\{U_\lambda \mid \lambda \in \Lambda\}$  of open subsets  $U_\lambda$  of  $X$  such that

$$\bigcup_{\lambda \in \Lambda} U_\lambda = X$$

2. A topological space  $X$  is **compact** if every open cover  $\{U_\lambda \mid \lambda \in \Lambda\}$  of  $X$  has a finite subcover, i.e. there exists  $\lambda_1, \dots, \lambda_n \in \Lambda$  such that

$$X = \bigcup_{j=1}^n U_{\lambda_j}.$$

### Definition 4.2: Open Covers as Collections

1. If  $A \subseteq X$  is a subset of a topological space  $X$ , an **open cover** of  $A$  is a collection  $\{V_\lambda \mid \lambda \in \Lambda\}$  of subsets  $V_\lambda$  which are open in  $X$  such that

$$X = \bigcup_{\lambda \in \Lambda} V_\lambda$$

2. A subset  $A$  of a topological space  $X$  is **compact** if it is compact as a subspace of  $X$ .

### Proposition 4.7: Boundedness of Compact Spaces

A compact metric space  $(X, d)$  is bounded, i.e. there exists a number  $K \geq 0$  such that  $d(x, y) \leq K$  for all  $x, y \in X$ .

### Proposition 4.8: Compactness of Products

A product of closed bounded intervals  $R = [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$  is compact in the usual topology. A collection of subsets of a set  $X$  has the **finite intersection property** if every finite intersection of their members is nonempty.

### Corollary 4.12: Limit Property of Compactness

Suppose that  $f : X \rightarrow \mathbb{R}^n$  is a continuous map and that  $X$  is compact. Then there exists an  $M$  such that

$$|f(x)| \leq M \text{ for all } x \in X.$$

Moreover, there exists an  $x \in X$  such that

$$|f(x)| = \sup_{y \in X} |f(y)|.$$

If  $n = 1$  there are  $x_0$  and  $x_1 \in X$  such that

$$f(x_0) = \min_{x \in X} f(x) \quad \text{and} \quad f(x_1) = \max_{x \in X} f(x).$$

### Theorem 4.13: Compact, Hausdorff, Bijection, Oh My!

Suppose  $X$  is compact,  $Y$  is Hausdorff, and that  $f : X \rightarrow Y$  is a continuous bijection. Then it is a homeomorphism.

### Theorem 4.14: Lebesgue Numbers

Let  $X$  be a compact metric space and  $\{U_\lambda \mid \lambda \in \Lambda\}$  an open cover of  $X$ . Then there exists a positive number  $\delta > 0$  (the **Lebesgue number** of the cover) such that for all  $x \in X$ ,  $B(x, \delta)$  lies *entirely inside some single*  $U_\lambda$ .

### Corollary 4.17: Compactness of Identification Spaces

1. An identification space  $X/\sim$  of a compact space  $X$  is compact.
2. If  $f : X \rightarrow Y$  is a map from a compact space  $X$  to a Hausdorff space  $Y$  and  $\sim$  is the equivalence relation on  $X$  defined by  $x \sim x'$  if  $f(x) = f(x') \in Y$ , then the continuous bijection 
$$g : X/\sim \rightarrow f(X); \quad [x] \mapsto f(x)$$
 is a homeomorphism.

### Lemma 4.20: Open sets in Product spaces

Let  $X$  be a topological space,  $Y$  a compact space,  $x \in X$ ,  $N$  an open set in  $X \times Y$  such that  $\{x\} \times Y \subseteq N$ . Then there is an open set  $W \subseteq X$  such that  $x \in W$  and  $W \times Y \subseteq N$ .

### Lemma 4.22 - 4.23: Collections and Intersections

- 4.22) Let  $X$  be a set, and suppose  $\mathcal{C}$  is a collection of subsets of  $X$  which has the finite intersection property. Then there is a collection  $\mathcal{B}$  of subsets of  $X$ , with  $\mathcal{C} \subseteq \mathcal{B}$ , such that  $\mathcal{B}$  has the finite intersection property, and such that  $\mathcal{B}$  is maximal with respect to this property: i.e. no collection containing  $\mathcal{B}$  as a proper subcollection has the finite intersection property.
- 4.23) Let  $X$  be a set, and suppose that  $\mathcal{B}$  is a collection of subsets of  $X$  which is maximal with respect to the finite intersection property. Then  $\mathcal{B}$  is closed under finite intersections, and any set which meets all members of  $\mathcal{B}$  is also in  $\mathcal{B}$ .

### Definition 4.24: Compactifications

1. A **compactification** of a topological space  $X$  is a compact space  $Y$  which contains a homeomorphic copy of  $X$  as a subspace, i.e. such that there is a one-one map  $f : X \rightarrow Y$  such that  $X \rightarrow f(X); \quad x \mapsto f(x)$  is a homeomorphism.
2. A compactification  $Y$  is **dense** if  $X$  is dense in  $Y$ , i.e.  $\overline{X} = Y$ .

### Definition 4.27: One-point compactification

The **one-point compactification** of a topological space  $X$  is the set

$$X^\infty = X \cup \{\infty\}$$

obtained by adjoining a “point at infinity”  $\infty$ , where  $\infty$  is a symbol *not* in  $X$ , with open sets of the form either

1.  $U$ , where  $U \subseteq X$  is open, or
2.  $X^\infty \setminus K$ , where  $K \subseteq X$  is compact and closed.

### Lemma 4.28

1. The collection of open sets just defined does form a topology
2. The subspace topology on  $X$  induced by this topology coincides with its original topology.

### Definition 4.32: Local Compactness

A topological space  $X$  is **locally compact** if for each  $x \in X$ , there exists an open subset  $U \subseteq X$  and a compact  $C$  such that  $x \in U \subseteq C$ .

### Remark 4.33

When  $X$  is Hausdorff, it is locally compact iff for each  $x \in X$  there exists an open subset  $U \subseteq X$  and a compact  $x \in U$  and the closure  $\overline{U}$  is compact.

### Definition 4.35: Normal Space

A topological space  $(X, \mathcal{T})$  is **normal** if for every pair of disjoint closed subsets  $C$  and  $D \subseteq X$ , there are disjoint open subsets  $U, V \subseteq X$  such that  $C \subseteq U$  and  $D \subseteq V$ .

### Lemma 4.37: Normal Complements

A space  $X$  is normal iff for every closed  $F \subseteq X$  and open  $G \subseteq X$  with  $F \subseteq G$ , there exist open  $G'$  and closed  $F'$  such that

$$F \subseteq G' \subseteq F' \subseteq G.$$

### Theorem 4.38: Urysohn's Lemma

Suppose that  $X$  is a normal topological space, and that  $C, D$  are disjoint closed subsets of  $X$ . Then there is a continuous function  $f : X \rightarrow \mathbb{R}$  such that

- $f(x) = 0$  for all  $x \in C$
- $f(x) = 1$  for all  $x \in D$
- $0 \leq f(x) \leq 1$  for all  $x \in X$

### Theorem 4.39: Tietze extension theorem

Suppose that  $X$  is a normal topological space, and that  $C$  is a closed subset of  $X$ . Suppose that  $f : C \rightarrow \mathbb{R}$  is continuous. Then there is a continuous function  $\bar{f} : X \rightarrow \mathbb{R}$  such that

- $\bar{f}(x) = f(x)$  for all  $x \in C$
- If  $a \leq f(x) \leq b$  for all  $x \in C$ , then  $a \leq \bar{f}(x) \leq b$  for all  $x \in X$ .

### Theorem 4.40: Stone-Weierstrass Theorem

The algebra  $A$  is dense in the normed space  $C(X)$ , i.e.  $\overline{A} = C(X)$ , i.e. for all  $f \in C(X)$  and for all  $\epsilon > 0$  there is  $g \in A$  such that  $\sup_{x \in X} |f(x) - g(x)| < \epsilon$ .

## 5 Connected Spaces

### Definition 5.1: Connected Spaces

1. A topological space  $X$  is **connected** if it *cannot* be written as a union

$$X = A \cup B$$

where  $A$  and  $B$  are disjoint nonempty open subsets of  $X$

2. A topological space  $X$  is **disconnected** if it is not connected, i.e. if it *can* be expressed as a union

$$X = A \cup B$$

where  $A$  and  $B$  are disjoint nonempty open subsets of  $X$

Connectedness is a **Topological Property** (See P6).

### Remark 5.8: Connected Homeomorphisms

- If  $X$  is a compact connected metric space with exactly two points  $x$  such that  $X \setminus \{x\}$  is connected, then  $X$  is homeomorphic to  $[0, 1]$
- If  $X$  is a compact connected space, where for every pair of distinct points  $x, y \in X$  the complement  $X \setminus \{x, y\}$  is disconnected, then  $X$  is homeomorphic to the circle  $\mathbb{S}_1$

### Proposition 5.11: Connectedness other properties

The following statements about a nonempty subset  $A \subseteq \mathbb{R}$  are equivalent:

1.  $A$  is connected
2.  $A$  has the interval property
3.  $A$  is an interval

### Theorem 5.12: Intermediate Value Theorem

Let  $I$  be a closed bounded interval and suppose  $f : I \rightarrow \mathbb{R}$  is continuous. Then the image  $f(I)$  is a closed bounded interval

$$f(I) = [a, b] \subseteq \mathbb{R} (a \leq b).$$

### Definition 5.13: Fixed Points of Maps

A **fixed point** of a map  $f : X \rightarrow X$  is an  $x \in X$  s.t.  $f(x) = x$ .

### Theorem 5.15: 1-D Brouwer Fixed Point Theorem

Every continuous map  $f : [0, 1] \rightarrow [0, 1]$  has a fixed point, i.e. there exists  $x \in [0, 1]$  such that  $f(x) = x$ .

General Case: Every continuous map  $f : \mathbb{D}^n \rightarrow \mathbb{D}^n$  has a fixed point

### Definition 5.16: Path

A **path** in a topological space  $X$  is a continuous map  $\alpha : I = [0, 1] \rightarrow X$ . Its **initial point** is  $\alpha(0) \in X$  and its **terminal point** is  $\alpha(1) \in X$ .

### Definition 5.18: Path Connectedness

A topological space  $X$  is **path-connected** if for any two points  $x_0, x_1 \in X$  there exists a path  $\alpha : I \rightarrow X$  from  $\alpha(0) = x_0$  to  $\alpha(1) = x_1$ .

### Theorem 5.24: Homeomorphisms of Real Spaces

If  $n \geq 2$ , the spaces  $\mathbb{R}^n$  and  $\mathbb{R}$  are not homeomorphic. Additionally, there is no bijection  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  which is continuous.

### Definition 5.35: Connected Components

We define an equivalence relation  $\sim$  on a topological space  $x$  by  $x \sim y$  iff there is a connected subset of  $X$  which contains both  $x$  and  $y$ . The resulting equivalence classes are called the **components** or **connected components** of  $X$ . For two homeomorphic topological spaces there will be a bijection between the sets of their components. So spaces with differing numbers of components cannot be homeomorphic. The components are the maximal connected subsets of a topological space and by Prop 5.9 are always closed in  $X$ . If we take  $U \subseteq \mathbb{R}$  an open set, its connected component decomposition is its canonical representation as in Thm 1.30 as a countable disjoint union of open intervals.

### Lemma 5.31.5: Path Components

Define a path (equivalence) relation

$$x_0 \sim x_1 \text{ if there exists a path } \alpha : I \rightarrow X \\ \text{from } \alpha(0) = x_0 \in X \text{ to } \alpha(1) = x_1 \in X.$$

- 5.32) The **constant path** at  $x \in X$  is the path

$$\alpha_x : I \rightarrow X; \quad t \mapsto x$$

from  $\alpha_x(0) = x \in X$  to  $\alpha_x(1) = x \in X$

- 5.33) The **reverse** of a path  $\alpha : I \rightarrow X$  is the path

$$-\alpha : I \rightarrow X; \quad t \mapsto \alpha(1 - t)$$

retracting  $\alpha$  backwards, with

$$\begin{array}{ccc} -\alpha(0) = \alpha(1) & & -\alpha(1) = \alpha(0) \\ \downarrow & \xrightarrow{-\alpha} & \downarrow \end{array}$$

- 5.34) The **concatenation** of paths  $\alpha : I \rightarrow X, \beta : I \rightarrow X$  with  $\alpha(1) = \beta(0) \in X$

is the path

$$\alpha \bullet \beta : I \rightarrow X, \quad t \mapsto \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq 1/2 \\ \beta(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

which starts at  $\alpha(0)$ , follows along  $\alpha$  at twice the speed in the first half, switching at  $\alpha(1) = \beta(0)$  to follow  $\beta$  at twice the speed in the second half.

$$\begin{array}{ccccc} \alpha \bullet \beta(0) = \alpha(0) & & \alpha(1) = \beta(0) & & \beta(1) = \alpha \bullet \beta(1) \\ \downarrow & \xrightarrow{\alpha} & \downarrow & \xrightarrow{\beta} & \downarrow \end{array}$$

### Lemma 5.31: Connected Components and Openness

Let  $X$  be a topological space and  $C$  a connected component of  $X$ . Then  $C$  is open iff for all  $x \in C$  there is an open connected  $V$  such that  $x \in V \subseteq C$ .

### Proposition 5.35: Equivalence of Path Relations

The path relation defined on a space  $X$  by  $x_0 \sim x_1$  if there exists a path  $\alpha : I \rightarrow X$  from  $\alpha(0) = x_0$  to  $\alpha(1) = x_1$  is an equivalence relation.

### Definition 5.36: Path Components Formally

Let  $X$  be a topological space.

1. The **path components** of  $X$  are the equivalence classes of the path equivalence relation  $\sim$ , i.e. the subspaces

$$\begin{aligned} [x] &= \{y \in X \mid y \sim x\} \\ &= \{y \in X \mid \exists \alpha : I \rightarrow X \text{ from } \alpha(0) = x \text{ to } \alpha(1) = y\} \end{aligned}$$

2. The **set of path components** (which may be infinite) is denoted by

$$X / \sim = \pi_0(X)$$

3. The function

$$X \rightarrow \pi_0(X), \quad x \mapsto [x] = \{\text{equivalence class of } x\}$$

is surjective.

### Lemma 5.39: Open Condition of Path Components

Let  $X$  be a topological space and  $P$  a path component of  $X$ . Then  $P$  is open iff for all  $x \in P$  there is an open path connected  $V$  such that  $x \in V \subseteq P$ .

### Lemma 5.40: Openness and Singular Components

Let  $C$  be a connected component of a topological space  $X$ . If every path component  $P \subseteq C$  is open, then  $C$  consists of a single path component. Note that the converse of this is not true.



## 6 Relations between Top Props

### Proposition A: Topological Invariants

A **topological property** of a topological space is one which is **invariant** under homeomorphism. Let  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$  be a homeomorphism. The following properties are true:

- 2.8)  $\mathcal{U}$  is open in  $Y$  iff  $f^{-1}(\mathcal{U})$  is open in  $X$ .
  - $X$  is Hausdorff iff  $Y$  is Hausdorff.
- 3.6)  $X \setminus \{x_0\}$  is homeomorphic to  $Y \setminus \{f(x_0)\}$ .
- 4.11)  $X$  is compact, iff  $Y$  is compact.
- 5.6)  $X$  is connected iff  $Y$  is connected.
- 5.21)  $X$  is path-connected iff  $Y$  is path-connected.
- 5.37) There exists a bijection between the set of path components  $\pi_0(X)$  and  $\pi_0(Y)$ . However, existence of a bijection between  $\pi_0(X)$  and  $\pi_0(Y)$  does *not* necessarily imply that  $X$  and  $Y$  are homeomorphic.

### Proposition B: Hausdorff if...

- 3.4) Suppose  $(X, \mathcal{T})$  is a Hausdorff topological space and suppose  $A$  is a subspace. Then  $A$  is Hausdorff.
- 4.34) The one-point compactification  $X^\infty$  of a space  $X$  is Hausdorff iff  $X$  is Hausdorff and locally compact.

### Proposition C: Compact if...

- 4.3) Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is compact iff every open cover of  $A$  has a finite subcover.
- 4.5) **Heine-Borel Theorem:** A subset  $F \subseteq \mathbb{R}^n$  is compact iff it is closed and bounded.
- 4.6) Let  $X$  be a topological space and  $A \subseteq X$ .
  - 1. If  $X$  is compact and  $A$  is closed, then  $A$  is compact
  - 2. If  $X$  is Hausdorff and  $A$  is compact, then  $A$  is closed.
- 4.10) Let  $f : X \rightarrow Y$  be a continuous map between topological spaces. If  $X$  is compact, so is  $f(X)$ .
- 4.18) **Tychonoff's Theorem:** Suppose  $X$  and  $Y$  are compact spaces. Then their product  $X \times Y$  is compact. The converse is also true.
- 4.21) **Tychonoff's Theorem (General):** Suppose that  $\mathcal{A}$  is an indexing set and that for each  $\alpha \in \mathcal{A}$ ,  $X_\alpha$  is a compact topological space. Then the product  $\prod_{\alpha \in \mathcal{A}} X_\alpha$  is compact.
- 4.30) Suppose  $X^\infty = X \cup \{\infty\}$  is the *one-point compactification* of  $X$ . Then either  $X^\infty$  is compact, or  $X$  is dense in  $X^\infty$

### Proposition D: Continuous if..

- 2.14) Suppose  $(X, \mathcal{T})$  and  $(Y, \mathcal{U})$  are topological spaces and that  $f : X \rightarrow Y$ . Then  $f$  is continuous iff for every closed subset  $F \subseteq Y$  its inverse image  $f^{-1}(F)$  is closed in  $X$ .
- 2.14)  $f$  is continuous iff the image of the closure of every subset  $A \subseteq X$  is contained in the closure of the image, i.e.,  $\forall A \subseteq X$ ,
 
$$f(\overline{A}) \subseteq \overline{f(A)}$$
- 3.5) Suppose  $(X, \mathcal{T})$  and  $(Y, \mathcal{U})$  are topological spaces and suppose  $A$  is a subspace of  $X$ . Let  $f : X \rightarrow Y$  be continuous. Then  $f|_A : A \rightarrow Y$  is continuous.
- 3.12) Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{U})$  be topological spaces and  $\mathcal{T}$  the product topology on  $X \times Y$ . Then the projection maps  $\Pi_X$  and  $\Pi_Y$  are continuous. Moreover,  $\mathcal{T}$  is the smallest topology on  $X \times Y$  such that the projection maps are continuous.
- 3.13) Let  $X, Y, Z$  be topological spaces. Endow  $X \times Y$  with the product topology. A function  $f : Z \rightarrow X \times Y$  is continuous iff the functions  $\Pi_X \circ f : Z \rightarrow X$  and  $\Pi_Y \circ f : Z \rightarrow Y$  are both continuous.

Let  $X$  be a topological space with an equivalence relation  $\sim$ .

1. The function  $p : X \rightarrow X/\sim; \quad x \mapsto [x]$  is continuous.
2. A continuous function  $f : X \rightarrow Y$  such that  $f(x) = f(x') \in Y$  for all  $x, x' \in X$  with  $x \sim x'$  determines a continuous function

$$g : X/\sim \rightarrow Y; \quad [x] \mapsto f(x)$$

such that

$$f = g \circ p : X \xrightarrow{p} X/\sim \xrightarrow{g} Y$$

$f = g \circ p$  is best described by a commutative triangle:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p \quad \nearrow g & \\ & X/\sim & \end{array}$$

In fact, every continuous function on  $X$  determines an equivalence relation.

### Proposition E: Connected if...

- 5.2)  $X$  is connected iff the only subsets of  $X$  which are clopen are  $\emptyset$  and  $X$
- 5.4)  $\mathbb{R}$  with the usual topology is connected.
- 5.5) If  $f : X \rightarrow Y$  is continuous and  $X$  is connected, then  $f(X)$  (with the subspace topology) is connected.
- 5.9) Let  $A$  be a connected subset of a topological space  $X$  and suppose  $A \subseteq B \subseteq \overline{A}$ . Then  $B$  is connected.
- 5.10) Every nonempty interval  $I \subseteq \mathbb{R}$  is connected.
- 5.25) If a topological space  $X$  is path-connected, then it is also connected. Note that the converse need not be true.
- 5.30) Let  $A_\lambda \subseteq X$ ,  $(\lambda \in \Lambda)$  be a family of connected subsets of a topological space  $X$ . Suppose  $\bigcup_{\lambda \in \Lambda} A_\lambda \neq \emptyset$ . Then  $\bigcup_{\lambda \in \Lambda} A_\lambda$  is connected.

### Proposition F: Path-Connected if...

- Suppose  $f : X \rightarrow Y$  is a continuous map between topological spaces and that  $X$  is path-connected. Then  $f(X)$  is path-connected as a subspace of  $Y$ .
- For any equivalence relation  $\sim$  on a path-connected space  $X$  the identification space  $Y = X/\sim$  is path-connected.
- Any connected open subset  $\Omega \subseteq \mathbb{R}^n$  is also path-connected.
- Let  $X$  be a topological space. Then  $X$  is path connected iff  $X$  is connected *and* for all  $x \in X$  there is an open path connected  $V$  such that  $x \in V$ .

### Example G: Topological Invariancy Proofs

- **Compactness:** Let  $U_\lambda$  be open subsets of  $Y$  which cover  $f(X)$ . Then  $f^{-1}(U_\lambda)$  are open sets in  $X$  which cover  $X$ . Hence there is a finite subcover  $\{f^{-1}(U_{\lambda_1}), \dots, f^{-1}(U_{\lambda_n})\}$ , and so  $\{U_{\lambda_1}, \dots, U_{\lambda_n}\}$  covers  $f(X)$ .
- **Connectedness:** If  $f(X)$  is disconnected then we can write it as a disjoint union  $f(X) = (A \cap f(X)) \cup (B \cap f(X))$  for some open subsets  $A, B \subseteq Y$ . The inverse images  $f^{-1}(A \cap f(X)) = f^{-1}(A)$  and  $f^{-1}(B \cap f(X)) = f^{-1}(B)$  are disjoint open subsets of  $X$  s.t.  $X = f^{-1}(A) \cup f^{-1}(B)$ , in contradiction to the connectedness of  $X$ . Hence  $f(X)$  is connected.
- **Path-Connectedness:** Pick  $y_0$  and  $y_1$  in  $f(X)$ . So there are  $x_0, x_1 \in X$  such that  $f(x_0) = y_0$  and  $f(x_1) = y_1$ . Let  $\alpha : [0, 1] \rightarrow X$  be a cts map with  $\alpha(0) = x_0$  and  $\alpha(1) = x_1$ . Then  $\beta = f \circ \alpha$  is a path in  $f(X)$  joining  $y_0$  to  $y_1$ .

## 7 Examples

### Example a: Other Topologies and Metrics

If  $(X, \mathcal{T})$  is a topological space, and  $X$  admits a metric whose metric topology is precisely  $\mathcal{T}$ , then we say that  $(X, \mathcal{T})$  is **metrisable**.

- Euclidean spaces with their usual topologies are metrisable.

- 1.9) The **Discrete Topology** is the topology of all subsets of a set  $X$ . We can define the **discrete metric** of  $X$  to be

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

- 1.10) The **Trivial** or **Indiscrete Topology** is the topology  $\mathcal{T} := \{\emptyset, X\}$  for a set  $X$ . This is a non-metrisable topology when  $X$  has more than one member.

- 1.14) Let  $X = \{a, b, c\}$ , where  $a, b, c$  are distinct. Then

$$\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$$

is a topology on  $X$

- 1.15) Give  $\mathbb{R}$  the topology whose open subsets  $U \subseteq \mathbb{R}$  are precisely the subsets with finite complement  $\mathbb{R} \setminus U$ , or  $U = \emptyset$ . Then  $\mathbb{R}$  with this topology is not metrisable. This is an example of a **Zariski Topology**

- The **Co-finite** topology is the subsets of  $K$  whose complements are finite, along with  $\emptyset$ . Every subset of the co-finite topology is compact.
- The **Co-countable** topology is the subsets of  $K$  whose complements are countable, along with  $\emptyset$ . Every compact subset of the co-countable topology is finite.
- The **Hawaiian Earring** space is the subspace of  $\mathbb{R}^2$  with the usual topology given by  $H = \bigcup_{n=1}^{\infty} C_n$ , where  $C_n \subseteq \mathbb{R}^2$  is given by

$$C_n = \left\{ (x, y) \in \mathbb{R}^2 \mid \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2} \right\}$$

- $\mathbb{S}^n$  and an eq. relation  $\sim$  where  $x \sim y$  iff  $x = y$  or  $x = -y$  is the **Real Projective Space**  $\mathbb{RP}^n$ , or “the lines in  $\mathbb{R}^{n+1}$  which pass through the origin”.

### Example B: Compact Sets

- $\mathbb{R}$  is not compact. Take  $\{[0, n] \mid n = 1, 2, \dots\}$ . This covers  $\mathbb{R}$  but has no finite subcover.
- $\mathbb{R}^n$  is not compact. Take the same argument, but with open balls of dimension  $n$ .
- $\mathbb{S}^n$  is compact, as it is a closed (under the euclidean norm), bounded (by 1) subspace of  $\mathbb{R}^n$ .
- $[0, 1]$  is closed and bounded, therefore compact via Heine-Borel.
- The cantor space  $\{0, 1\}^{\omega}$  is bounded by  $[0, 1]$ , and as thirds  $C_n$  are closed, and  $\{0, 1\}^{\omega}$  is an intersection of such sets, it is closed and therefore compact via Heine-Borel.
- The quotient space of a Topological space  $K/\sim$  is compact. The quotient map  $p : K \rightarrow K/\sim$  is continuous, therefore since  $K$  is compact, so is  $K/\sim$  via Theorem 4.10.

### Example C: Homeomorphisms

- For the sphere  $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ , the punctured sphere  $\mathbb{S}^n \setminus \{x_0\}$  for some  $x_0$  is homeomorphic to  $\mathbb{R}^n$

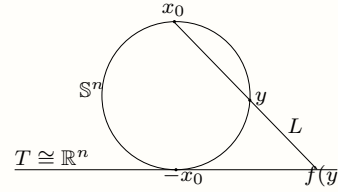


Figure 1: Homeomorphism of  $\mathbb{S}^2$  to  $\mathbb{R}$

- $(0, 1)$  is homeomorphic to  $\mathbb{R}$ . Take  $f(x) = \tan(\pi x - \frac{\pi}{2})$  or  $f(x) = \frac{x}{\sqrt{1+x^2}}$
- $[0, 1]$  is not homeomorphic to  $(0, 1)$ .  $[0, 1]$  is closed and bounded  $\Rightarrow$  compact via Heine-Borel, while  $\mathbb{R}$  is not compact.
- $[0, 1)$  is not homeomorphic to  $(0, 1)$ . Let  $f : [0, 1) \rightarrow (0, 1)$ . Then there is  $f(0) \in (0, 1)$ . Now take  $[0, 1] \setminus \{0\}$ . This is still connected, but  $(0, 1) \setminus \{f(0)\}$  is disconnected.
- $Y = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$  is not homeomorphic to  $\mathbb{R}$ . There is a point  $(0, 0)$  where  $Y \setminus \{(0, 0)\}$  has 4 connected components but this does not follow for  $\mathbb{R}$ .
- $\mathbb{R}^n$  is not homeomorphic to  $\mathbb{R}^m$ . For  $\mathbb{R}$  vs  $\mathbb{R}^2$  consider a hole and exclusion on  $\mathbb{R}$  not being path-connected via IVT.
- $\mathbb{R} + \mathbb{R}$  (disjoint union) is homeomorphic to  $\mathbb{R} \setminus \{0\}$
- $\mathbb{S}^1$  is homeomorphic to the identification space of  $I = [0, 1]$  under a equivalence relation that glues both ends together  
 $x \sim y$  if  $x = y$  or if  $(x, y) = (1, 0)$  or if  $(x, y) = (0, 1)$
- $\mathbb{S}^1$  is not homeomorphic to  $[0, 1]$ , if there was  $f : [0, 1] \rightarrow \mathbb{S}^1$  then the spaces  $[0, 1] \setminus \{1/2\} = [0, 1/2) \cup (1/2, 1]$  disconnected, while  $\mathbb{S}^1 \setminus \{f(1/2)\}$  is homeomorphic to an open interval and therefore connected.

### Example D: Random counterexample

- The **topologist's sine curve** is connected but not path-connected

$$X = \{(0, y) \mid -1 \leq y \leq 1\} \cup \{(x, \sin(\frac{\pi}{x})) \mid 0 < x \leq 1\} \subseteq \mathbb{R}^2$$

### Example E: Compactification

- The open interval  $X = (0, 1)$  has dense compactification the closed interval  $Y = [0, 1]$ .
- Let  $\sim$  be the equivalence relation on  $[0, 1]$  generated by  $0 \sim 1$ . Then  $Z = [0, 1]/\sim = \mathbb{S}^1$  is a dense compactification of  $X = (0, 1)$ .
- $\mathbb{R}^n$  has dense compactification  $\mathbb{S}^n$  since  $\mathbb{S}^n \setminus \{x\} \subseteq \mathbb{S}^n$  is a dense subspace homeomorphic to  $\mathbb{R}^n$ .
- $\mathbb{R}^n$  has dense compactification  $\mathbb{D}^n$  since the open unit ball  $\mathbb{B}^n = B(0, 1) = \{x \in \mathbb{R}^n \mid |x| < 1\} \subset \mathbb{D}^n$  is a dense subspace homeomorphic to  $\mathbb{R}^n$

#### One Point compactification

- $(0, 1)^{\infty} = \mathbb{S}_1$
- $(\mathbb{R}^n)^{\infty} = \mathbb{S}^n$

### Example F: Topological Objects

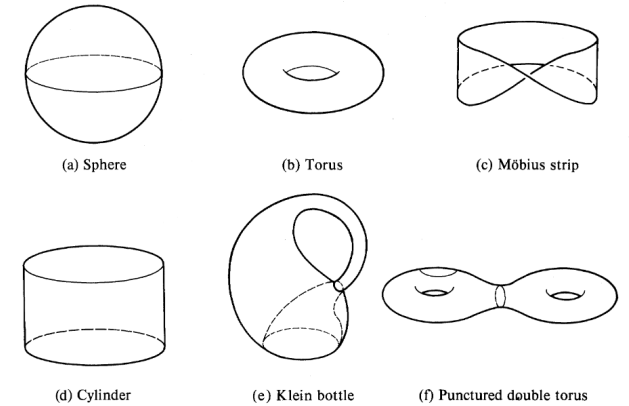


Figure 2: Standard Topological Objects

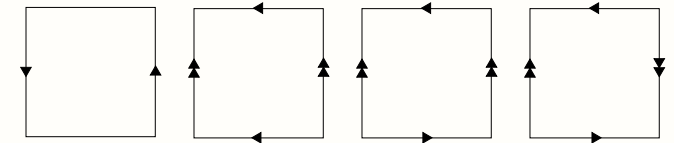


Fig 3: Möbius Strip

Fig 4: Torus

Fig 5: Klein Bottle

Fig 6:  $\mathbb{RP}^2$