Algebraic Topology Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Introduction

Recall 1.1.1: Topology

An **(open) topology** on X is a collection of subsets $\tau \subset P(X)$ such that

- $\emptyset \in \tau$ and $X \in \tau$
- τ is closed under finite intersections: If $\{U_1, \dots, U_n\} \subset \tau$ then
$$\bigcap_{i=1,\dots,n} U_i \in \tau$$
- τ is closed under arbitrary unions: If $\{U_1, .., U_n\} \subset \tau$ is a family of open subsets then
$$\bigcup_{i=1,\dots,n} U_i \in \tau$$

The subsets $U \in \mathcal{T}$ are called **open** and their complements in X define **closed subsets**.

Two examples of a topology on a set X are the following:

- The **Trivial Topology**: $\tau_{\text{triv}} = \{\emptyset, X\}$
- The **Discrete Topology**: $\tau_{\text{dis}} = P(X)$

A subset $A \subset X$ is **clopen if it is both closed and open**

Definition 1: Connected Spaces

A topological space X is **connected** if $X = A \amalg B$ with A, $B \subset X$ open implies that $A = \emptyset$ or $A = X$.

Proposition 1: Connectedness and Clopens

A topological space X is *connected* iff the only clopens are $\emptyset$ and X.

Example 1: Examples of Connected Topologies

- Every X with the trivial topology is connected.
- Every X with the discrete topology isn't connected unless $X = \emptyset$ or $X = \{*\}$ (in which it coincides with the trivial topology).
- The real line $\mathbb{R}$ with the standard topology is connected.

Proposition 2: Continuous Maps

Let $f : X \to Y$ be a continuous map of topological spaces and let X be connected. Then $f(X)$ is connected.

Proposition 3: Connected Equivalence Relation

For a topological space X, define $x \sim y$ if there exists some connected subset that contains both. The relation $x \sim y$ is an equivalence relation.

Definition 2: Connected Components

The equivalence classes of this relation are called **connected components**. In particular, a space X is connected iff it only has a single connected component.

Definition 3: Path

Let I denote the closed unit interval $[0, 1]$. A **path** in X is a continuous map $\alpha : I \to X$. The points $\alpha(0) \in X$ and $\alpha(1) \in X$ will be called **start** and **end** points respectively.
We define a path relation between points in X by declaring $x \sim y$ if there exists some path $\alpha : I \to X$ that starts at x and ends in y, i.e. $\alpha(0) = x$ and $\alpha(1) = y$. This is an equivalence relation from the following properties:

1. **Constant Path**: For all $x \in X$ there exists the constant path $c_x : I \to X$ defined by $c_x(t) = x$ for all $t \in I$
2. **Path reversal**: Let $\alpha : I \to X$ be a path in X. Define its reversed path by
$$\overline{\alpha} : I \to X, \quad t \mapsto \alpha(1-t) \tag{1}$$
3. **Path Concatenation**: Let α, $\beta : I \to X$ be two paths in X s.t. $\alpha(1) = \beta(0)$. Their concatenated path is defined by:
$$\alpha * \beta(t) := \begin{cases} \alpha(2t), & 0 \le t \le 1/2 \\ \beta(2t-1) & 1/2 \le t \le 1 \end{cases} \tag{2}$$

Definition 4: Path-Connected Components

The equivalence classes are called **path-connected components** and their set is denoted by $\pi_0(X)$. A space X is called **path-connected** if $\pi_0(X)$ is a one-point set, i.e. any two points x, y can be related by a path in X.

Remark 1: Random examples

The following statements are true:

- A homeomorphism $X \cong Y$ induces a bijection $\pi_0(X) \cong \pi_0(Y)$.
- If X is path-connected, it is also connected.
- The *topologist's sine curve* defined by $X = \{0\} \times [-1, 1] \times \{(x, \sin(1/x)) \mid 0 < x\}$ is connected but not path-connected.

Definition 5: Homotopy

A **homotopy** of maps f, $g : X \to Y$ is a continuous map $h : X \times I \to Y$ such that $h(-, 0) = f$ and $h(-, 1) = g$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow h"] & Y \end{tikzcd}$$

If such a homotopy exists, f is **homotopic** to g. This defines an equivalence relation $f \simeq g$ on the space of maps $\mathrm{Map}(X, Y)$.

Example 2: Paths as Homotopies

Points in X are the same as maps $* \to X$ from the one-point set $*$ to X. A path $\alpha : I \to K$ corresponds to a homotopy $* \times I \to X$.

Remark 1.5: Composition of Homotopies

- **Vertical Composition**: Let $h, h' : X \times I \to Y$ be two homotopies in X such that $h(-, 1) = h'(-, 0) : X \to Y$. Their concatenated homotopy is defined by
$$h * h'(-, t) := \begin{cases} h(-, 2t) & 0 \le t \le 1/2 \\ h'(-, 2t-1) & 1/2 \le t \le 1 \end{cases} \tag{4}$$
- **Horizontal Composition**: Let $h : X \times I \to Y$, $k : Y \times I \to Z$ be two homotopies on maps from X to Y, and Y to Z. Then
$$k \circ h := [X \times I \xrightarrow{\text{id} \times \Delta} X \times I^2 \xrightarrow{h \times \text{id}} Y \times I \xrightarrow{k} Z] \tag{5}$$
where $\Delta : I \to I^2$, $t \mapsto (t, t)$ is the diagonal map, or explicitly,
$$k \circ h(x, t) = k(h(x, t), t)$$

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"] \arrow[r] \arrow[r, bend right=60, "l"'] \arrow[r, phantom, shift left=2, "\Downarrow h"] \arrow[r, phantom, shift right=2, "\Downarrow h'"] & Y \end{tikzcd} \sim \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "l"'] \arrow[r, phantom, "\Downarrow h' * h"] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow h"] & Y \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow k"] & Z \end{tikzcd} \sim \begin{tikzcd} X \arrow[r, bend left, "f' \circ f"] \arrow[r, bend right, "g' \circ g"'] \arrow[r, phantom, "\Downarrow k \circ h"] & Z \end{tikzcd}$$

Lemma 1: Concatenation Relation

Let f, $f' : X \to Y$ and $g, g' : Y \to Z$ be maps such that $f \simeq f'$ and $g \simeq g'$. Then $g \circ f \simeq g' \circ f'$ as maps from X to Z. In particular, $g' \circ f \sim g \circ f$ and $g \circ f' \sim g \circ f$.

Definition 6: Homotopy Equivalence

A map $f : X \to Y$ is called a **homotopy equivalence** if there exists a map $g : Y \to X$ and homotopies $f \circ g \simeq \text{id}_Y$, $g \circ f \simeq \text{id}_X$. In other words, g satisfies the properties of an inverse up to homotopy. It is called a **homotopy inverse** of f.

Example 3: Circle to $\mathbb{R}^2$

The inclusion map $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2$ is not a homotopy equivalence, but the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2 \backslash \{0\}$ is a homotopy equivalence.

Proposition 4: Unique Inverses of Homotopy

Homotopy inverses are unique up to homotopy.

Definition 7: Homotopic Spaces

Two spaces X and Y are called **homotopy equivalent**, or **of the same homotopy type**, and denoted by $X \simeq Y$ if there exists a homotopy equivalence $f : X \to Y$.

Note: $\cong$ for homeomorphisms and $\simeq$ for homotopy equivalence.

Lemma 2: Composition of Inverses

Let $f : X \to y$, $g : Y \to Z$ with homotopy inverses $\overline{f} : Y \to X$ and $\overline{g} : Z \to Y$ respectively. Then $\overline{f} \circ \overline{g} : Z \to X$ is a homotopy inverse of $g \circ f : X \to Z$. In particular, $X \simeq Y$, $Y \simeq Z$ implies $X \simeq Z$.

2 Contractible Spaces

Definition 8: Contractible Space

A space X is called **contractible** if it is homotopy equivalent to a point, i.e. $X \simeq *$.

The **terminal map** is the unique map $X \to *$. Contractibility requires that there is a homotopy inverse of that map, i.e. a map $* \to x$ along with homotopies

$$h : [* \to X \to *] \simeq \mathrm{id}_*, \quad k : [X \to * \to X] \simeq \mathrm{id}_X \tag{6}$$

Example 4: Examples of Contractible Spaces

1. $\mathbb{R}^n$ is contractible. Let x_0 be a fixed point in $\mathbb{R}^n$ and define the (straight line) homotopy $h : c_{x_0} \simeq \mathrm{id}_{\mathbb{R}^n}$ by
$$h(x,t) = (1-t)x_0 + tx.$$
2. $\mathbb{S}^{n-1} \simeq \mathbb{R}^n \backslash \{0\}$. The inclusion $\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n\backslash\{0\}$ and the shrinking map
$$\mathbb{R}^n\backslash\{0\} \to \mathbb{S}^{n-1}, \quad x \mapsto \frac{x}{|x|}$$
are homotopy inverses.

Remark 3: Remarks about Contractible Spaces

1. Contractible spaces are path-connected. Let x_0 be the point where the space X contracts to. In particular, we are given with a homotopy $h : c_{x_0} \simeq \mathrm{id}_X$. For any $x \in X$, the map $h(x,-) : I \to X$ defines a path from x_0 to x and thus every element $x \in X$ is path-connected to x_0.
2. The converse does not hold, for example $X = \mathbb{S}^1$.
3. A contractible space X is contractible at any point x_0. X is path-connected, so a path x to x' defines a homotopy $c_x \simeq c_{x'}$.
4. Any two maps $f, g : X \to Y$ are homotopic if Y is contractible.

Definition 9: Retracts and Deformation Retracts

- A **retract** of X onto a subspace $A \subset X$ is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$. Equivalently, this is a map $r : X \to X$ such that $r^2 = r$ and $r(X) = A$.
- A **deformation retract** of X onto A is the additional datum of a homotopy $h : \mathrm{id}_X \simeq i \circ r$.

In other words, a deformation retract is a homotopy $h : X \times I \to X$ such that $h(x,0) = x$ and $h(x,1) \in A$ for all $x \in X$ and $h(a,1) = a$ for all $a \in A$. Not all retracts can form deformation retracts. For instance, the retract X onto a point $\{x_0\}$ can be a deformation retract iff X is contractible.

Remark 4: Strong vs Weak Deformation Retracts

This notion is called **weak** deformation retract. A **strong** deformation retract has the condition $h(a,t) = a$ for all $t \in I$, $a \in A$. Our notion of a (weak) deformation retract deforms X into A while allowing to deform A to do so, while a strong deformation retract deforms X into A while keeping A fixed at all times

Proposition 5: Deformation Retracts and Homotopies

A deformation retract of X onto A induces a homotopy equivalence $X \simeq A$.

Recall 2: Quotient Space

Let X be a topological space and let $\sim$ be an equivalence relation on X. Then, $X/\sim$ is equipped with the quotient topology and called a **quotient space**. If Z is a closed subset in X, then we can also define the quotient space X/Z.

Another form of quotient spaces: Let $f : Z \to Y$ be a continuous map between a closed subset $Z \subset X$ and Y. Then

$$X \amalg_f Y = X \amalg Y/z \sim f(z).$$

Example 5: Examples of Quotient Spaces

- The quotient of the n-dimensional closed disk by its boundary is the n-sphere, i.e. $\mathbb{D}^n/\partial\mathbb{D}^n \cong \mathbb{S}^n$.
- The 2-torus: $\mathbb{R}^2/\mathbb{Z}^2$.
- The projective space: $\mathbb{RP}^n = \mathbb{R}^{n+1}\backslash\{0\}/\sim$ by the relation $x \sim y$ iff there exists some $\lambda \in \mathbb{R}^\times$ such that $x = \lambda y$. This corresponds to the space of lines through the origin in $\mathbb{R}^{n+1}$.

Definition 10: Mapping Quotients

Let $f : X \to Y$ be a continuous map.

- Its **mapping cylinder** is defined as the topological space
$$M_f := (X \times I) \amalg Y/\sim$$
where the quotient identifies $(x,0) \sim f(x)$ for any $x \in X$.
- Its **cone** is the further quotient:
$$C_f = M_f/X \times \{1\}.$$
- The **cone** of a topological space X is
$$C_X := C_{\mathrm{id}_X} = X \times I/X \times \{1\}.$$

In other words, the mapping cylinder of $f : X \times Y$ is the pushout of the diagram:

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \times I \arrow[r] & M_f \end{tikzcd}$$

Example 5.5: Spheres

For $\mathbb{S}^n$ with the standard embedding $\mathbb{R}^{n+1}\backslash\{0\}$, the following map is a retract, because if x has norm $|x| = 1$, then $r(x) = x$.

$$r : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{S}^n, \quad x \mapsto \frac{x}{|x|}$$

For a deformation retract one needs to find a homotopy $h : i \circ r \simeq \mathrm{id}_X$. We use the following straight-line homotopy:

$$h : \mathbb{R}^{n+1}\backslash\{0\} \times I \to \mathbb{R}^{n+1}\backslash\{0\}, \quad (x,t) \mapsto (1-t)\frac{x}{|x|} + tx.$$

Indeed, $h(x,0) = r(x)$ and $h(x,1) = x$ for all x.

Definition 11: Star-Shaped Spaces

A subset $S \subset \mathbb{R}^n$ is called **star-shaped** at a point $x_0 \in S$, if for any $x \in S$ the line segment from x_0 to x is contained in S, i.e.

$$\{(1-t)x_0 + tx \mid t \in [0,1]\} \subset S$$

If S is star-shaped at every point, then it is called **convex**.

Example 5.6: Star-Shaped Spaces are Contractible

Let S be star-shaped at x_0 and $i : \{x_0\} \leftrightarrow S : r$ be the inclusion and constant maps. Define the straight line homotopy

$$h : S \times I \to S, \quad (x,t) \mapsto (1-t)x_0 + tx$$

which is well-defined by the star-shaped condition. Moreover, $h(x,0) = x_0 = r(x)$ and $h(x,1) = x$ for all x. Hence, star-shaped, and in particular convex spaces, are contractible.

Example 5.7: Möbius band

The Möbius band M can be defined as

$$M = I^2/\sim$$

where $\sim$ identifies the two vertical edges of I^2 by flipping one, i.e. $(0,b) \sim (1, 1-b)$ for $b \in I$. Its core $C \subset M$ is the line $\{[a, 1/2] \mid a \in I\}$. Thus, the core is homeomorphic to $\mathbb{S}^1$. The Möbius band deformation retracts onto its core, e.g. the retract $r : M \to C$ defined by $r([a,b]) := [(a, 1/2)]$ and the homotopy

$$h : M \times I \to M, \quad ([(a,b)], t) \mapsto \left[\left(a, (1-t)\frac{1}{2} + \right)\right].$$

In particular, $M \simeq \mathbb{S}^1$.

Proposition 6: Retracts of the Mapping Cylinder

Via Definition 10, the mapping cylinder is formed by the cylinder of X by gluing Y onto the bottom with the map f.
The mapping cylinder M_f strongly deformation retracts onto Y.

Proof. Consider the retract:

$$r : M_f \to Y$$

defined by $r([x,s]) := [(x,0)] = [f(x)]$ on the class of $(x,s) \in X \times I$ and $r([y]) = y$ for $y \in Y$. This is well-defined and by definition a retract on Y. Define the homotopy

$$h : M_f \times I \to M_f$$

by $h([(x,s)],t) := [(x,st)]$ for $(x,s) \in X \times I$ and $t \in I$, and by $h([y],t) := y$ for $y \in Y$. In particular, $h(-,0)i \circ r$ and $h(-,1) = \mathrm{id}_{M_f}$. This forms a strong deformation retract. $\square$

Remark 6: Continuous Maps are Homotopic

Any continuous $f : X \to Y$ can be replaced up to homotopy equivalence by the closed inclusion $X \hookrightarrow M_f$, $x \mapsto [(x,1)]$. More precisely, it factorises through an inclusion and a homotopy equivalence as the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & M_f \arrow[ur, "\simeq"'] & \end{tikzcd}$$

Definition 12: Relative Homotopy

Let X, Y be topological spaces and $A \subset X$ a subset in X. A homotopy $h : X \times I \to y$ is called **relative to** A if $h(a,t)$ is independent of t for all $a \in A$. In particular, this defines homotopies between maps $f, g : X \to Y$ such that $f|_A = g|_A$.

This definition generalises ordinary homotopies, as an ordinary homotopy is the same as a homotopy relative to $\emptyset$.

Example 6: Relative Homotopies and Retracts

A strong deformation retract of X onto A is a deformation retract such that the homotopy $h : i \circ r \simeq \mathrm{id}_X$ is relative to A.

Definition 13: Homotopic Path

Let $\alpha, \beta : I \to X$ be paths in X such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. A relative homotopy from α to β is a homotopy $h : I \times I \to x$ relative to $\partial I = \{0, 1\}$, i.e.

$$h(-,0) = \alpha, \quad h(-,1) = \beta \tag{7}$$

and

$$h(0,t) = \alpha(0) = \beta(0), \quad h(1,t) = \alpha(1) = \beta(1), \quad \forall t \in I. \tag{8}$$

In particular, at any point $t \in I$ a relative homotopy h defines a path $h_t := h(-,t) : I \to X$ with start $\alpha(0) = \beta(0)$ and end $\alpha(1) = \beta(1)$. If one omits the relative condition, the start and end points of h_t would be allowed to vary.

Remark 7: Ordinary Homotopies and Paths

Ordinary homotopies are not well suited for paths: Any path $\alpha : I \to X$ is homotopic (rel. $\emptyset$) to a constant - as the homotopy

$$h : I \times I \to X, \quad (s,t) \mapsto \alpha(st)$$

defines a homotopy from the constant path $c_{\alpha(0)}$ on $\alpha(0)$ to α, i.e. $c_{\alpha(0)} \simeq \alpha$. Hence, (ordinary) homotopy classes of paths in X are in 1-to-1 correspondence with path-connected components of X.

Proposition 7: Homotopic Properties of Paths

Path concatenation is **unital**, **associative**, and **invertible** up to homotopy in the following sense: Let $\alpha, \beta, \gamma : I \to x$ be paths such that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. Then there exists homotopies relative to $\{0, 1\}$:

1. **Left Unitality**: $c_{\alpha(0)} * \alpha \simeq \alpha$
2. **Right Unitality**: $\alpha \simeq c_{\alpha(0)} * \alpha$
3. **Associativity**: $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$
4. **Right Inverse**: $\alpha * \overline{\alpha} \simeq c_{\alpha(0)}$
5. **Left Inverse**: $\overline{\alpha} * \alpha \simeq c_{\alpha(1)}$

where c_x for some $x \in X$ denotes the constant path on x and $\overline{\alpha}$ is the reversed path.

Lemma 3:

Let $\alpha : I \to X$ be a path and $\lambda : I \to I$ a boundary preserving map, i.e. $\lambda(0) = 0$ and $\lambda(1) = 1$. Then,

$$\alpha \circ \lambda \simeq \alpha, \quad \text{rel.}\, \partial I.$$

Definition 14: Fundamental Group

Let X be a topological space and $x_0 \in X$ some fixed point. The **fundamental group** of X at x_0 is the group of homotopy classes of paths in X that start and end on x_0. i.e. $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x_0$, i.e.

$$\pi_1(X, x) = \{\alpha : I \to X \mid \alpha(0) = \alpha(1)\} / \sim.$$

Theorem 1: Defining the Fundamental Group

The fundamental group $\pi_1(X, x_0)$ is a well-defined group with:

- **Multiplication**: $[\alpha] \cdot [\beta] := [\alpha * \beta]$
- **Unit**: $1 = [c_{x_0}]$
- **Inverse**: $[\alpha]^{-1} = [\overline{\alpha}]$

Lemma 4: Relative Concated Homotopic Paths

Let $\alpha \simeq \alpha' : I \to X$ and $\beta \simeq \beta' : I \to X$ be two pairs of relative homotopic paths such that $\alpha(1) = \alpha'(1) = \beta(0) = \beta'(0)$. Then the concatenations are relative homotopic, i.e.

$$\alpha * \beta \simeq \alpha' * \beta', \quad \text{rel.}\{0, 1\}.$$

Proposition 8: Fundamental Group is Point Independent

Let $\gamma : I \to X$ be a path from $\gamma(0) = x$ to $\gamma(1) = x'$. Then it induces a group isomorphism:

$$(\gamma)_\# : \pi(X, x) \to \pi(X, x'), \quad [\alpha] \mapsto [\overline{\gamma} * \alpha * \gamma].$$

We abuse notation to say that for a path-connected space X, $\pi_1(X)$ is the fundamental group omitting the choice of base point.

Example 7: Examples of Fundamental Groups

- **Euclidean**: $\pi_1(\mathbb{R}^n) \cong 1$.
- **n-Sphere**, $n \geq 2$: $\pi_1(\mathbb{S}^n) \cong 1$.
- **Circle**: $\pi_1(\mathbb{S}_1) \cong \mathbb{Z}$.
- **Torus**: $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- **Projective Spaces**: $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ for $n \geq 2$.

Definition 15: Pointed Space and Loop Space

- A **pointed space** is a pair (X, x) consisting of a topological space X and a point $x \in X$.
- A **map of pointed spaces** $f : (X, x) \to (Y, y)$ is a continuous map $f : X \to Y$ such that $f(x) = y$.
- The **space of pointed maps** from (X, x) to (Y, y) is denoted

$$\mathrm{Map}_*((X, x), (Y, y)) \subset \mathrm{Map}(X, Y).$$

With the (pointed) homeomorphism $(\mathbb{S}^1, 1) \cong (I/\partial I, [0])$, closed paths (where $\alpha(0) = \alpha(1) = x$) are the same as pointed maps

$$(\mathbb{S}^1, 1) \to (X, x)$$

The space of such **loops** based at x is called the **loop space at** x.

$$\Omega X := \mathrm{Map}_*((\mathbb{S}^1, 1), (X, x))$$

It is itself a pointed space with the compact-open topology, and the constant map c_x as the base point. Paht concatenation is the operation $* : \Omega X \times \Omega X \to \Omega X$ which is associative, unital, invertible up to path-connectedness, which gives a group structure

$$\pi_0(\Omega X).$$

Proposition 9: Loop Space Isormophism

We have a group isomorphism: $\pi_1(X, x) \cong \pi_0(\Omega X)$.
Iteratively defining the n-fold loop space:
$\Omega^n X := \Omega\Omega^{n-1} X = \Omega \cdot \cdot \Omega X$
There is a homeomorphism: $\Omega^n X \cong \mathrm{Map}_*((\mathbb{S}^\kappa, 1), (X, x))$

Definition 16: n-th Homotopy Group

The n-th homotopy group $\pi_n(X, x)$ is defined by:

$$\pi_n(X, x) := \pi_0(\Omega^n X) \cong \pi_0(\mathrm{Map}_*(\mathbb{S}^n, (X, x))).$$

Definition 17: Simply Connected Space

A path-connected space X is **simply connected** if its fundamental group is trivial, i.e. $\pi_1(X) = 1$.

Some examples are $\mathbb{R}^n$, $\mathbb{S}^n$ for $n > 1$, and some non-examples are $\mathbb{S}^1$, $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{RP}^2$.

Theorem 2: Fundamental Group Isomorphism

Let $f : X \to Y$ be a homotopy equivalence and $x \in X$ an arbitrary base point. Then, the following map is a group isomorphism:

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x)), \quad [\alpha] \mapsto [f \circ \alpha]$$

In particular, for homotopy equivalent spaces $X \simeq Y$ which are path-connected, we get $\pi_1(X) \cong \pi_1(Y)$.

A map of pointed spaces $f : (X, x) \to (Y, y)$ is a **homotopy equivalence of pointed spaces** or **homotopy equivalence relative** $\{x\}$ if there exists a map of pointed spaces $g : (Y, y) \to (X, x)$ along with relative homotopies

$$h : f \circ g \simeq \mathrm{id}_Y \quad \text{rel. } \{y\} \quad \text{and} \quad k : g \circ f \simeq \mathrm{id}_X \quad \text{rel. } \{x\}$$

Example 9: Strong Deformation Retracts Homotopies

A strong deformation retract of X onto a subspace A gives a homotopy equivalence of pointed spaces $(x, a) \to (A, a)$ for any choice of $a \in A$. In particular, a contractible space $X \simeq *$ determines a homotopy equivalence of pointed spaces $(X, x) \to *$ for any choice of base point x.

Lemma 5: Pointed Space Isomorphism

Let $f : (X, x) \to (Y, y)$ be a homotopy equivalence of pointed spaces. Then the map

$$f_* : \pi_1(X, x) \to \pi_1(Y, y)$$

is a group isomorphism.

Corollary 1: Let $r : X \to A$ be a strong deformation retract of X onto $A \subset X$. Then for any $a \in A$,

$$\pi_1(X, a) \cong \pi_1(A, a)$$

In particular, contractible spaces are simply connected.

Lemma 6: Identity Homomorphic Isormorphism

Let $f : X \to X$ be a cts. map homotopic to id_X. Then, the map

$$f_* : \pi_1(X, x_0) \to \pi_1(X, f(x_0))$$

is a group isomorphism for any choice of base point $x_0 \in X$.

Definition 18: Homotopy Lifting Property

A continuous map $p : E \to X$ satisfies the **homotopy lifting property** (HLP) with respect to a topological space Y if for any commuting diagram:

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "H_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"] \arrow[ur, dashed, "\exists"] & X \end{tikzcd}$$

There exists a map $H : Y \times I \to E$ s.t. both triangles commute, i.e. $H|_{Y\times\{0\}} = H_0$ and $p \circ H = h$.

The map $p : E \to X$ has the HLP if for any homotopy $h : Y \times I \to X$ of maps $h(-,0) := f_0$ and $h(-,1) := f_1$ of maps $Y \to X$ and a choice of lift H_0 of f_0, then the homotopy h lifts to a homotopy $H : Y \times I \to E$. In particular, if $f_0 \simeq f_1 : Y \to X$ and H_0 is a lift of f_0, we find $H_0 \simeq H_1$ where H_1 lifts f_1.

Ex. 10: The identity map $\mathrm{id}_X : X \to X$ has the HLP with respect to any space Y.

Definition 19: Covering Space

A **covering space** of X is a topological space $\overline{X}$ along with a continuous map $p : X \to x$ s.t. for any point $x \in X$ there exists an open nbhd $U \subset X$ whose preimage $p^{-1}(U) = \bigcup_{j\in J} V_j$ and the opens $V_j \subset \overline{X}$ map homeomorphically to U under p. A covering space of X looks locally like a product of X with a discrete space.

Example 11: Example of a Covering Space

1. The projection map $p : X \times Z \to X$ is a covering map if Z is a discrete topological space. If Z is not discrete, then this is not a covering map in general.
2. The identity map $\mathrm{id}_X : X \to X$ is trivially a covering map.
3. While the projection of $p : X \times I \to X$ from the cylinder is not a covering map, its restriction to the boundary $\partial(X \times I) = X \times \{0,1\} =: \overline{X}$ gives a trivial (2-fold) cover of X.
4. Recall that the Möbius band M deformation retracts onto its core $\mathbb{S}^1$. Restricting to the boundary $\partial M = \mathbb{S}^1$, one obtains a (non-trivial) covering map $\mathbb{S}^1 \to \mathbb{S}^1$. This map coincides with $z \mapsto z^2$ if we identify S^1 as the unit circle in $\mathbb{C}$.

Theorem 3: Unique HLPs from Covering Maps

Let $p : \tilde{X} \to X$ be a covering map and Y any topological space. Then p satisfies the HLP uniquely: i.e. the lift H not only exists, but it is also unique.

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "H_0"] \arrow[d, hook] & \tilde{X} \arrow[d, "p"] \\ Y \times I \arrow[r, "h"] \arrow[ur, dashed, "\exists!"] & X \end{tikzcd}$$

Corollary 2:

1. Let $\gamma : I \to X$ be a path and fix a point $\tilde{x_0} \in \tilde{X}$ such that $p(\tilde{x_0}) = \gamma(0)$. Then, there exists a unique path $\tilde{\gamma} : I \to \tilde{X}$ which starts at $\tilde{x_0}$ and lifts γ i.e. $p \circ \tilde{\gamma} = \gamma$
2. Let $h : I \times I \to X$ be a (relative) homotopy of paths $h(-,0) =: \gamma_0$ and $h(-,1) =: \gamma_1$, and fix a point $\tilde{x_0}$ such that $p(\tilde{x_0}) = h(0,t) = \gamma_0(0) = \gamma_1(0)$. Suppose $\tilde{\gamma_0} : I \to X$ is a lift of γ starting at $\tilde{\gamma_0}(0) = \tilde{x_0}$. Then, there exists a unique homotopy of paths $\tilde{h} : I \times I \to \tilde{X}$ which lifts h and $\tilde{h}(-,0) = \tilde{\gamma_0}$

Theorem 4-7: Fundamental Groups

- **Theorem 4**: The fundamental group of the circle is $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. It is generated by the class of
$$\alpha : I \to \mathbb{S}^1, \quad t \mapsto e^{2\pi i t}.$$
- **Theorem 5 (Brouwer's Fixed Point Theorem)**: Any continuous map $f : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point, i.e. there exists $x \in \mathbb{D}^2$ such that $f(x) = x$.
- **Theorem 6 (Fundamental Theorem of Algebra)**: Every non-constant complex polynomial $p \in \mathbb{C}[z]$ has at least one root, i.e. $p(z_0) = 0$ for some z_0.
- **Theorem 7**: The fundamental group of $\mathbb{S}^n$ is trivial for $n \geq 2$, i.e. $\pi_1(\mathbb{S}^2) \cong 1$ for $n \geq 2$

Lemma 7: Closed Paths Homotopic to Loops

Let (X, x_0) be a topological space with an open cover $\{U_j\}_{j\in J}$ such that U_j are path-connected neighbourhoods of x_0 and $U_j \cap U_{j'}$ is path-connected for any $j, j' \in J$. Then, any closed path γ based at x_0 is homotopic to a concatenation $\gamma_1 * \gamma_2 * \cdots * \gamma_n$ of loops at x_0 each of them contained in a single U_j.

Corollary 3: Homemorphisms between $\mathbb{R}^2$ and $\mathbb{R}^n$

There is no homeomorphism between $\mathbb{R}^2$ and $\mathbb{R}^n$ for $n \neq 2$.

Recall 4: Defining the Real Projective Space

1. The space $\mathbb{RP}^2$ is the quotient space:
$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \backslash \{0\}) / \sim$$
where $x \sim y$ if there exists $\lambda \in \mathbb{R}$ s.t. $x = \lambda y$. i.e., the real projective n-space represents the lines in $\mathbb{R}^{n+1}$ through the origin.
2. Picking representatives that lie in the unit n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1} \backslash \{0\}$, we obtain $\mathbb{RP}^n \cong \mathbb{S}^n / \sim$ where $x \sim -x$ for all $x \in \mathbb{S}^n$, i.e. identifying antipodal points on the n-sphere.
3. Further restricting to the upper half $\mathbb{D}^n \subset \mathbb{S}^n$ we obtain:
$$\mathbb{RP}^n \cong \mathbb{D}^n / \sim$$
where $x \sim -x$ for any boundary points $x \in \partial\mathbb{D}^n \cong \mathbb{S}^{n-1}$

For example, $\mathbb{RP}^0$ is a one point space, $\mathbb{RP}^1 \cong \mathbb{S}^1$, while $\mathbb{RP}^n$ are different than spheres for larger n.

Definition 20: Lift of a Path

- A lift of a path $\alpha : I \to \mathbb{RP}^n$ is a path $\tilde{\alpha} : I \to \mathbb{S}^n$ s.t. $p \circ \tilde{\alpha} = \alpha$
- If α is a closed path, then $[\tilde{\alpha}(0)] = [\tilde{\alpha}(1)]$ which implies $\tilde{\alpha}(0) = \pm\tilde{\alpha}(1)$. The **sign** of α is defined by
$$\mathrm{sgn}(\alpha) \mapsto \begin{cases} +1 & \tilde{\alpha}(0) = \tilde{\alpha}(1) \\ -1 & \tilde{\alpha}(0) = -\tilde{\alpha}(1) \end{cases}$$

Theorem 8: Group Homomorphism of the Sign

The sign induces a surjective group homomorphism
$$\mathrm{sgn} : \pi_1(\mathbb{RP}^n) \to \mathbb{Z}_2, \quad [\alpha] \mapsto \mathrm{sgn}(\alpha)$$
which is an isomorphism for $n \geq 2$.

3 Covering Theory

Definition 21: Right Lifting Property

A map $p : X \to Y$ satisfies the **right lifting property** (RLP) w.r.t. a map $i : A \to B$ if any commutative square has a solution to the lifting problem making both triangles commute.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

Explicitly, if $f : B \to Y$ and $g : A \to X$ such that $f \circ i = p \circ g$, then there exists a map $l : B \to X$ satisfying $l \circ i = g$ and $p \circ l = f$. Dually, the map $i : A \to B$ is said to satisfy the **left lefting property** (LLP) with respect to $p : X \to Y$.

Example 13: Homotopy Lifting Property WRT Spaces

1. A map $p : X \to Y$ satisfies the **homotopy lifting property** w.r.t. a space Z iff it has the RLP with respect to the inclusion map $i : Z \times \{0\} \hookrightarrow Z \times I$, i.e. solves the following lifting problem:
$$\begin{tikzcd} Z \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Z \times I \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$
In other words, given a homotopy $h : Z \times I \to Y$ and a lift $\tilde{f} : Z \to X$ of $h(-,0) =: f$, there is a homotopy lift $\tilde{h} : Z \times I \to X$ with $\tilde{h}(-,0) = \tilde{f}$.

2. Dually, a map $i : A \to b$ satisfies the **homotopy extension property** (HEP) with w.r.t. a space Z iff it has the LLP w.r.t. the map
$$p : Z^I \to Z, \quad \gamma \mapsto \gamma(0)$$
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Z^I \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "\exists"] & Z \end{tikzcd}$$
Where $Z^I := \mathrm{Map}(I, Z)$ is the space of paths in Z. In other words, one can solve the following lifting problem.

Note that a map $A \to Z^I$ is the same datum as a homotopy $h : A \times I \to Z$. Given an extension $\tilde{f} : B \to Z$ of $h(-,0)$ along i, the existence of a map $B \to Z^I$ which makes both triangles commute provides an extension of the homotopy h to a homotopy $\tilde{h} : B \times I \to Z$ along i.

Example 15: Covering Spaces

1. The projection map $p : X \times D \to X$ where D is a discrete space. Note that $X \times D$ cannot be path-connected unless D is a one-point set.
2. The covering map $\mathbb{R} \to \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$ which we can use to compute the fundamental group of $\mathbb{S}^1$.
3. The degree-n map $F_n : \mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto z^n$ provides an n-fold covering of $\mathbb{S}^1$ by itself.
4. The product of two covering maps $p_i : \tilde{X}_i \to X_i$ is also a covering map $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$
5. The product of F_n and F_m in the third example also provides a self covering of the torus:
$$T^2 = \mathbb{S}^1 \times \mathbb{S}^1 \to T^2, \quad (z, w) \mapsto (z^n, w^m)$$
6. Similarly, there is a covering $\mathbb{R}^2 \to T^2$.
7. The 2-fold covering $\mathbb{S}^n \to \mathbb{RP}^n$ which was used to compute the fundamental group of $\mathbb{RP}^n$

Theorem 10: Homomorphism of Covering Maps

Let $p : \tilde{X} \to X$ be a covering map. The induced group homomorphism

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$$

is injective for any $\tilde{x}_0 \in p^{-1}(x_0)$. Its image consists of (classes of) loops in X based at x_0 that lift to loops in $\tilde{X}$ based at $\tilde{x}_0$

Remark 5: Notation for Covers

Fix a covering map $p : \tilde{X} \to X$ and $x_0 \in X$ a fixed point. Write $G := \pi_1(X, x_0)$ for the fundamental group of X at x_0 and $H := p_*\pi_1(\tilde{X}, \tilde{x}_0) \subset G$ for the subgroup determined by the covering map.
The subgroup H depends on the choice of fiber point $\tilde{x}_0 \in p^{-1}(x_0)$ and we shall see that it subgroups for different fiber points are conjugate to each other. Finally, the fiber over x_0 will be denoted by

$$F_{x_0} := p^{-1}(x_0)$$

Lemma 8: Transitive Actions

If $\tilde{X}$ is path-connected, then the G-action on F_{x_0} is transitive, i.e. for any $\tilde{x}, \tilde{x}' \in F_{x_0}$, there exists a $\alpha \in G$ such that $\tilde{x}.\alpha = \tilde{x}'$.

Theorem 11: Path-Connected Correspondence

If $\tilde{X}$ is path-connected, then there is a one-to-one correspondence between right cosets and fiber points, i.e. a bijection

$$G_{\tilde{x}} \backslash G \to F_{x_0}, \qquad G_{\tilde{x}} \cdot g \mapsto \tilde{x}.g$$

Thus the index of $G_{\tilde{x}}$ in G coincides with the cardinality of the fiber F_{x_0}:

$$[G : G_{\tilde{x}}] = |F_{x_0}| \tag{20}$$

Corollary 4: If $\tilde{X}$ is simply-connected, then there is a bijection

$$G \to F_{x_0}$$

Equation (20) becomes

$$|G| = |F_{x_0}| \tag{21}$$

4 Deck Transformations, Further Cover Theory

Definition 23: Deck Transformation

A **deck transformation** of a covering map $p : \tilde{X} \to X$ is a self-homeomorphism $D : \tilde{X} \xrightarrow{\cong} \tilde{X}$ such that $p \circ D = p$.

Deck transformations form a group $\mathrm{Deck}(p)$. For any two deck transformations D, D' their composite is also a deck transformation since $p \circ D \circ D' = p \circ D' = p$. If D is a deck transformation then so is its inverse D^{-1} as $p \circ D^{-1} = p \circ D \circ D' = p$.
For example, deck transformations of the covering map $\mathbb{R} \to \mathbb{S}^1$ are precisely translations by integers:

$$D_n : \mathbb{R} \to \mathbb{R}, \qquad t \mapsto t + n$$

In particular, the group of deck transformations is $\mathbb{Z}$. More generally, the group of deck transformations $\mathrm{Deck}(p)$ of a universal covering p is isomorphic to the fundamental group G. From now on $p : \tilde{X} \to x$ will be a covering with $\tilde{X}$ path-connected and X path-connected and locally path-connected.

Example 16: Topologist's Sine Curve

Recall that the topologist's sine curve

$$X = \{0\} \times [-1, 1] \cup \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) \mid 0 < x \le \frac{1}{2\pi} \right\} \subset \mathbb{R}^2$$

is an example of a connected, but not path-connected space. Let Z be the quotient of X by identifying the points $(0,0) \sim (\frac{1}{2\pi}, 0)$. Z is a path-connected space but not locally path-connected.

Theorem 12: Solutions to the Lifting Problem

Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering with X path-connected and locally path-connected, and let $g : (Z, z_0) \to (X, x_0)$ be a pointed map. Then, there exists a solution to the lifting problem

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Z, z_0) \arrow[ur, dashed, "\exists \tilde{g}"] \arrow[r, "g"] & (X, x_0) \end{tikzcd} \quad \text{iff } g_*\pi_1(Z, z_0) \subset p_*\pi_1(\tilde{X}, \tilde{x}_0)$$

Moreover, if a solution to the lifting problem exists, then it is also unique.

Remark 10: If g is a covering map, then so is its lift $\tilde{g}$. In particular, homomorphisms of covering maps are also covering maps.

Corollary 5: Commuting Covering Maps

Let $p : \tilde{X} \to X$ be a covering with $\tilde{X}$ simply connected. Then, for any covering $p' : X' \to X$ there exists a covering $\tilde{p} : \tilde{X} \to X'$ such that the following diagram commutes:

$$\begin{tikzcd} & X' \arrow[d, "p'"] \\ \tilde{X} \arrow[ur, "\tilde{p}"] \arrow[r, "p"] & X \end{tikzcd}$$

In other words, if a simply connected covering $\tilde{X}$ of X exists, then it covers all other possible coverings. This is why such a covering is called the **universal covering of** X. For example, we have seen $\mathbb{R}$ as the universal covering of $\mathbb{S}^1$ of $\mathbb{S}^n$ as the universal covering of $\mathbb{RP}^n$.

Theorem 24: Covering Isomorphism

Let $p : \tilde{X} \to X$ be a covering with X path-connected and locally path-connected. Let $H \subset \pi_1(X, x_0)$ denote the subgroup determined by the covering map. Then, there exists a group isomorphism:

$$\mathrm{Deck}(p) \cong N(H)/H$$

where $N(H)$ denotes the normalizer.

Definition 24: Normal Coverings

A covering $p : \tilde{X} \to X$ is **normal** if the subgroup H is normal

Trivially, universal coverings are always normal. All the examples so far were normal since all the fundamental groups we have seen so far were abelian.

Corollary 6: Let $\tilde{X}$ be simply-connected. Then

$$\mathrm{Deck}(p) \cong \pi_1(X, x_0).$$

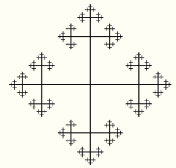
Example 19: The Figure Eight Space

Let X be the figure eight space, $X = \mathbb{S}^1 \vee \mathbb{S}^1$.
Consider an oriented bicolored graph $\tilde{X}$ whose vertices are all 4-valent with one incoming edge of each color and one outgoing edge of each color. Bicolored means each edge is labelled by a or b.

Such a graph determines a covering map

$$p : \tilde{X} \to X$$

by sending all vertices to the unique vertex of the figure-eight graph and the edges are sent to one of the loops. A universal covering is obtained by the following graph:



Vertical edges are oriented upwards and labelled by b, horizontal edges are oriented to the right and labelled by a. Deck transformations are freely generated by either D_a or D_b, where D_a (resp. D_b) acts on the graph by shifting all edges once to the right, rescaling them appropriately. In other words..

Theorem 14: Fundamental Group of $\mathbb{S}^1 \vee \mathbb{S}^1$

The fundamental group of $\mathbb{S}^1 \vee \mathbb{S}^1$ is the free group generated by two elements, i.e.

$$\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \langle a, b \rangle$$

Example 20.1: Covering The Möbius Band

Consider $M : \mathbb{R} \times I / \sim$ where $(x, y) \sim (x + 1, 1 - y)$. We obtain the homotopy equivalence using covering theory. The quotient map

$$q : \mathbb{R} \times I \to M$$

is the universal covering, since $\mathbb{R} \times I$ is simply-connected. For some $n \in \mathbb{Z}$, let D_n be the deck transformation:

$$D_n : \mathbb{R} \times I \to \mathbb{R} \times I, \qquad (x, y) \mapsto (x + n, y_n)$$

where $y_n = y$ if n is even and $y_n = 1 - y$ for n odd. These are all deck transformations and $\mathrm{Deck}(p)$ is generated by D_1 since

$$D_n = (D_1)^n$$

For odd n, there are n-fold self-coverings $M \to M$. For even n, there are n-fold coverings by the cylinder $S^I \to I$.

Example 20.2: Covering the Klein Bottle

Consider $K = \mathbb{R}^2 / \sim$, where $(x, y) \sim (x + 1, 1 - y) \sim (x, y + 1)$ for all $(x, y) \in \mathbb{R}^2$. The quotient map of the Klein bottle

$$q : \mathbb{R}^2 \to K$$

is the universal covering map. Consider the deck transformation

$$D_a : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto (x, y + 1)$$

and

$$D_b : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto (x + 1, 1 - y)$$

These two deck transformations generate the deck transformation group $\mathrm{Deck}(q)$ and satisfy the relation:

$$D_b \circ D_a \circ D_b^{-1} \circ D_a = \mathrm{id}\,.$$

Proposition 10: Fundamental Group of the Klein Bottle

The fundamental group of the Klein Bottle is:

$$\pi_1(K) = \langle a, b \rangle / \langle aba^{-1}b \rangle$$

5 Free Space and Examples

6 Unexaminable Material

Definition 22: Fibration

1. A map $p : X \to Y$ is a **fibration** if it satisfies the HLP w.r.t. all spaces Z. i.e., it has the RLP w.r.t. the set of maps $\{i : Z \times \{0\} \hookrightarrow Z \times I\}_Z$ where Z runs over all topo. spaces.
2. Dually, a map $i : A \to B$ is a **cofibration** if it satisfies the HEP with respect to all spaces Z.

Theorem 9: Covering Maps are Fibrations

A covering map $p : \tilde{X} \to X$ is a fibration. Additionally, the homotopy lifts are unique:

$$\begin{tikzcd} Z \times \{0\} \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Z \times I \arrow[r] \arrow[ur, dashed, "\exists !"] & Y \end{tikzcd}$$

Example 14: Examples of Fibrations

1. By Theorem 9, fibrations include all covering maps
2. The projection map $p : X \times F \to X$ is always a fibration. However this map is a covering map iff F is a discrete space. Hence, this includes examples of fibrations that are not coming from covering maps.
3. An important example of a cofibration is the inclusion $i : X \to M_f$ where M_f is the mapping cylinder of $f : X \times Y$. We have seen that any continuous map $f : X \times Y$ factors through the mapping cylinder:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & M_f \arrow[ur, "r"'] & \end{tikzcd}$$

In particular, every map factors through a cofibration and a homotopy equivalence.

Theorem 15: Seifert-Vam Kampen Theorem

Let X be a topological space with a fixed point x_0. Let $\{U_\alpha\}_\alpha$ be an open cover of X consisting of path-connected open sets U_α containing the fixed point x_0. The inclusions $U_\alpha \subset X$ induce a group homomorphism:

$$\Phi : *_\alpha \pi_1(U_\alpha) \to \pi_1(X).$$

1. If $U_\alpha \cap U_\beta$ is path-connected for any α, β, then Φ is surjective.
2. If $U_\alpha \cap U_\beta \cap U_\gamma$ is path-connected for any α, β, γ, then the kernel of Φ is generated by elements $i_{\alpha\beta}(w) i_{\beta\alpha}(w)^{-1}$ where $w \in \pi_1(U_\alpha \cap U_\beta)$ and $i_{\alpha\beta} : \pi_1(U_\alpha \cap U_\beta) \to \pi_1(U_\alpha)$ is the induced homomorphism from the inclusion $U_\alpha \cap U_\beta \subset U_\alpha$.

The assumption $U_\alpha \cap U_\beta$ are path-connected ensures that words $\pi_1(U_\alpha)$ generate $\pi_1(X)$. The assumption $U_\alpha \cap U_\beta \cap U_\gamma$ is path-connected gives a presentation for the group $\pi_1(X)$.

Example 17: Sifert-Vam Kampen on $\mathbb{S}^1 \vee \mathbb{S}^1$

Consider the figure eight $\mathbb{S}^1 \vee \mathbb{S}^1 = \mathbb{S}^1 \times \{1\} \cup \{1\} \times \mathbb{S}^1$ and let

$$U_i := \mathbb{S}^1 \vee \mathbb{S}^1 \backslash \{x_i\}$$

be the complements of the points $x_1 = (-1, 1)$ and $x_2 = (1, -1)$. The sets U_1 and U_2 are open path-connected and cover $\mathbb{S}^1 \vee \mathbb{S}^1$. In fact, they are both homotopy equivalent to the circle $U_i \simeq \mathbb{S}^1$. Their intersection $U_1 \cap U_2$ is contractible, and applying SVK we find,

$$\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \mathbb{Z} * \mathbb{Z}.$$

Example 18: Fundamental Group of Wedged Circles

Let (X_α, x_α) be a fIYL of path-connected pointed spaces and consider theier wedge sum

$$X := \bigvee_\alpha X_\alpha.$$

suppose that each $x_a := X_\alpha \vee \bigvee_{\beta \neq \alpha} U_\beta \subset X$. By contractibility of the U_α's we have homotopy equivalences $A_\alpha \simeq X_\alpha$. Moreover, the intersection $A_\alpha \cap A_\beta$ is contractibel for any $\alpha \neq \beta$. Applying SVK we obtain

$$\pi_1(X) \cong *_\alpha \pi_1(X_\alpha).$$

In particular, the fundamental group of the n-th wedge sum of circles is the free group on n-generators:

$$\pi_1\left(\bigwedge^n \mathbb{S}^1\right) \cong \mathbb{Z} * \cdots * \mathbb{Z} \cong \langle \alpha_1, \ldots, \alpha_n \rangle. \tag{23}$$

Definition 25: CW Complexes

A special class of topological spaces which are constructed inducively attaching n-dimensional disks or n-cells are called **CW complexes**. They are described as follows:

1. A set X^0 of **vertices** or 0-cells
2. Inductively construct the n-skeleton X^n from X^{n-1} by attaching n-dimenaional disks $\mathbb{D}^n_\alpha$ by attaching maps $\phi_\alpha : \partial D^n_\alpha = \mathbb{S}^{n-1}_\alpha \to X^{n-1}$. In other words,

$$X^n = X^{n-1} \amalg_{\phi\alpha} \coprod_\alpha D^n_\alpha.$$

Equivalently, a **CW Complex** is a space X along with a filtration of subspaces

$$X^0 \subset \cdots X^n \subset X^{n+1} \subset \cdots \subset X$$

such that $X^n \backslash X^{n-1}$ is homeomorphic to a disjoint union of n-dimensional open disks, and X^0 is discrete.

Example 19: Examples of CW Complexes

1. The Torus $T^2 = I^2/\sim$ can be made into a CW complex with: $X^0 = \{[(0,0)]\}$, $X^1 = \{[(a,0)] \mid a \in I\} \cup \{[(0,b)] \mid B \in I\}$ and $X^2 = T^2$. In particular, it has one 0-cell, two 1-cell, and one 2-cell.
2. The real projective plane $\mathbb{RP}^2$ can be made into a CW complex with $X^0 = *$, $X^1 = \mathbb{RP}^1 = \mathbb{S}^1$ and X^2 obtained by attaching a 2-disk to $\mathbb{S}^1$ along the quotient map $\mathbb{S}^1 \to \mathbb{RP}^1$