Metric Spaces Exam Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Introduction to Metric Spaces

Theorem 1.0.1: Definition of a Metric

Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** iff for all $x,\ y,\ z \in X$,

- $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

A non-empty set X equipped with a metric d is a **metric space**

Definition 1.0.2: Real Vector Spaces

A **real vector space** V is a set with two operations $(X, +, \cdot)$, where:

- $+$ is addition, and $\cdot$ is scalar multiplication
- $(X, +)$ is an abelian group - i.e. for all (vectors) $x, y, z \in X$:
  - **Closure**: $x + y \in X$
  - **Commutativity**: $x + y = y + x$
  - **Associativity**: $x + (y + z) = (x + y) + z$
  - **Identity**: $\exists 0 \in X$ s.t. for all $x \in X$ we have $0 + x = x + 0 = x$
  - **Inverse**: $\forall x \in X$ we have $-x$ s.t. $x + (-x) = (-x) + x = 0$
- Vector space axioms: for all $x, y, z \in X$ and $\mu, \lambda \in \mathbb{R}$ we have:
  - **Closure-ish thing**: $\lambda x \in X$
  - **Distributivity 1**: $\lambda(x + y) = \lambda x + \lambda y$
  - **Distributivity 2**:$(\lambda + \mu)x = \lambda y + \mu x$
  - **Associativity**: $\lambda(\mu x) = (\lambda\mu)x$
  - **Identity**: $1x = x$

Definition 1.0.3: Normed and Inner Product Spaces

Normed Vector Spaces

A **normed vector space** is a real vector space X equipped with a **norm**, i.e. a function that assigns to every vector $x \in X$ a real number $\|x\|$ so that, for all vectors x and y in X and all real scalars a:

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- $\|ax\| = |a|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

Remark: If $(X, \|\cdot\|)$ is a normed vector space then

$$d(x, y) = \|x - y\|$$

defines a metric in X

Remark: This is a generalisation of the "length of a vector"

Inner Product Spaces

Let X be a real vector space. An **inner product** on X is a function that assigns to every pair $(x, y) \in X \times X$ a real number denoted by $\langle x, y\rangle$ and has the following properties:

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $ax + by, z = a\langle x, z\rangle + b\langle y, z\rangle$

A **real inner product space** is a real vector space equipped with an inner product. If $\langle\cdot,\cdot\rangle$ is an inner product on X, then

- $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm in X
- $d(x, y) = \|x - y\|$ defines a metric in X

Remark: This is a generalisation of the dot product

Definition 1.1.4: n-dimensional Euclidean space

Let $X = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$
For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \text{ (inner product)}$$

Properties of n-inner product: For all vectors $x, y, z \in \mathbb{R}^n$ and all real scalars a, b,

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

For $x = (x_1, x_2, \ldots, .x_n) \in \mathbb{R}^n$ define

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \sqrt{x_1^2 + x_2^2 + \cdots + c_n^2} \text{(norm)}$$

Properties of n-norm: For $x, y \in \mathbb{R}^n$, $a \in \mathbb{R}$,

- $\|x\|_2 \geq 0$ and $\|x\|_2 = 0 \iff x = 0$
- $\|ax\|_2 = |a|\|x\|_2$
- $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$ (triangle inequality)

Example 1.1.5: Examples of Metric Spaces

Unless stated otherwise let $X = \mathbb{R}^n$. The case $X = \mathbb{R}^2$ is listed in red

Name	Norm and Metric
Standard	$\lvert x\rvert = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$
Taxicab	$\lVert x\rVert_1 = \lvert x_1\rvert + \lvert x_2\rvert + \cdots \lvert x_n\rvert$ $d_1(x, y) = \lvert x_1 - y_1\rvert + \lvert x_2 - y_2\rvert + \cdots + \lvert x_n - y_n\rvert$
Euclidean	$\lVert x\rVert_2 = \sqrt{\lvert x_1\rvert^2 + \lvert x_2\rvert^2 + \cdots \lvert x_n\rvert^2}$ $d_2(x, y) = \sqrt{\lvert x_1 - y_1\rvert^2 + \lvert x_2 - y_2\rvert^2 + \cdots + \lvert x_n - y_n\rvert^2}$
p-metric	$\lVert x\rVert_p = \left(\sum_{k=1}^{n} \lvert x_k\rvert^p\right)^{1/p}$ $d_p(x, y) = \left(\sum_{k=1}^{n} \lvert x_k - y_k\rvert^p\right)^{1/p}$
Chebyshev	$\lVert x\rVert_\infty = \max\{\lvert x_1\rvert, \lvert x_2\rvert, \ldots, \lvert x_n\rvert\}$ $d(x, y) = \max\{\lvert x_1 - y_1\rvert, \lvert x_2 - y_2\rvert, \ldots, \lvert x_n - y_n\rvert\}$
Discrete	Not induced by a metric $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$
Post Office	Not induced by a metric $d(x, y) = \begin{cases} \lVert x\rVert_2 + \lVert y\rVert_2 & x = y \\ 1 & x \neq y \end{cases}$

The complex plane

Let $X = \mathbb{C}$, $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$

$$d(z, w) = |z - w|$$

If $z = a + ib, w = c + id,\ a, b, c, d \in \mathbb{R}$, then

$$z - w = (a - c) + i(b - d)$$

therefore,

$$d(z, w) = \sqrt{(a - c)^2 + (b - d)^2}$$

Example 1.1.6: Sequence Spaces

The space ℓ^1

ℓ^1 is the set of real sequences $(x_n)_{n\in\mathbb{N}}$ where $\sum_{n=1}^{\infty}|x_n|$ converges.
For $x=(x_1,\dots,x_n,\dots)\in\ell^1$, $y=(y_1,\dots,y_n,\dots)\in\ell^1$ we define

- **Norm**: $\|x\|_1=\sum_{n=1}^{\infty}|x_n|$
- **Metric**: $d_1(x,y)=\|x-y\|_1=\sum_{n=1}^{\infty}|x_n-y_n|$

The space ℓ^2

ℓ^2 is the set of real seqs $(x_n)_{n\in N}$ where $\sum_{n=1}^{\infty}|x_n|^2$ converges
For $x=(x_1,\dots,x_n,\dots)\in\ell^2$, $y=(y_1,\dots,y_n,\dots)\in\ell^2$ we define

- **Inner product**: $\langle x,y\rangle=\sum_{n=1}^{\infty}x_ny_n$
- **Norm**: $\|x\|_2=\left(\sum_{n=1}^{\infty}|x_n|^2\right)^{1/2}$
- **Metric**: $d_2(x,y)=\|x-y\|_2=\left(\sum_{n=1}^{\infty}|x_n-y_n|^2\right)^{1/2}$

Thm: ℓ^2 is a real vector space

The space ℓ^∞

ℓ^∞ is the set of all bounded sequences of real numbers For
$x=(x_1,\dots,x_n,\dots)$, $y=(y_1,\dots,y_n,\dots)\in\ell^\infty$

- **Norm**: $\|x\|_\infty=\sup\{|x_1|,\dots,|x_n|,\dots\}$
- **Metric**: $\|x-y\|_\infty=\sup\{|x_1-y_1|,\dots,|x_n-y_n|,\dots\}$

The space $C([a,b])$

$X=C([a,b])$ is the set of all continuous functions $f:[a,b]\to\mathbb{R}$

- **Norm**: $\|f\|_\infty=\max\{|f(x)|:a\le x\le b\}$
- **Metric**: $d_\infty(f,g)=\|f-g\|=\max\{|f(x)-g(x)|:a\le x\le b\}$

The L^1 metric

X is the set of all continuous functions $f:[a,b]\to\mathbb{R}$

- **Norm**: $\|f\|_1=\int_a^b|f(x)|dx$
- **Metric**: $d_2(f,g)=\|f-g\|_1=\int_a^b|f(x)-g(x)|dx$

The L^2 metric

X is the set of all continuous functions $f:[a,b]\to\mathbb{R}$

- **Inner Product**: $\langle f,g\rangle=\int_a^b f(x)g(x)dx$
- **Norm**: $\|f\|_2=\langle f,f\rangle^{1/2}=\left(\int_a^b|f(x)|^2dx\right)^{1/2}$
- **Metric**: $d_1(f,g)=\left(\int_a^b|f(x)-g(x)|^2dx\right)^{1/2}$

Definition 1.1.7: Metric Subspaces

Let (X,d) be a metric space and Y a non-empty subset of X. Define

- $d_Y:Y\times Y\to\mathbb{R}$
- $d_y(y,y')=d(y,y')$

Then d_Y is a metric on Y. d_Y is called the **induced** or **inherited** metric, and (Y,d_Y) is said to be a metric subspace of the metric space (X,d)

Definition 1.1.8: Open Ball

Let (X, d) be a metric space, c be a point in X, and $r > 0$. The **open ball** with center c and radius r is defined by

$$B(c, r) = \{x \in X : d(c, x) < r\}$$

2 Convergence

2.1 Convergent Sequences in Metric Spaces

On the real line, $x_n \to x$ iff for every positive ϵ, there exists an index N such that for all indices n where $n \geq N$, we have $|x_n - x| < \epsilon$.

Definition 2.1.1: Convergent Sequence

Let (X, d) be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in X, and $x \in X$. We say that $(x_n)_{n=1}^{\infty}$ converges to x iff for every positive ϵ, there exists an index N s.t. for all indices n with $n \geq N$a we have $d(x_n, x) < \epsilon$.
Observe that:

- $d(x_n, x) < \epsilon$ is equivalent to $x_n \in B(x, \epsilon)$.
- $x_n \to x$ in (X, d) iff $d(x_n, x) \to 0$ on the real line

Theorem 2.1.2: Uniqueness of metric limit

- Let (X, d) be a metric space, and $x, x' \in X$, $x \neq x'$. Then there exists a positive radius r s.t. $B(x, r) \cap B(x', r) = \emptyset$
- A sequence in a metric space can have at most one limit

Example 2.1.3: convergence in $(\mathbb{R}^N, d_2)$

A sequence

$$x_1 = (x_{11}, \dots, x_{1j}, \dots x_{1N})$$
$$x_2 = (x_{21}, \dots, x_{2j}, \dots x_{2N})$$
$$\vdots$$
$$x_n = (x_{n1}, \dots, x_{nj}, \dots x_{nN})$$
$$\vdots$$
$$\downarrow$$
$$x = (x_1, \dots, x_j, \dots, x_N)$$

in $\mathbb{R}^N, d_2$ converges to $x = (x_1, \dots, x_j, \dots, x_N)$ iff for each j,

$$x_{nj} \xrightarrow[j \to +\infty]{} x_j$$

Definition 2.1.4: Bounded Sequence

A sequence in a metric space is said to be **bounded** iff there exists an open ball that contains all of its terms

---

Note: this is the same definition as "sequence is bounded if there is upper and lower bound", as open ball implies the same thing

Theorem 2.1.5

Every convergence is bounded

2.2 Cauchy Sequences

Convergence: For every ϵ, there is an N such that for $n \geq N$, $d(x_n, x) < \epsilon$

$$x_1 \quad x_2 \quad \cdots \quad x_N \quad \cdots x_n \quad \cdots \quad \to x$$

Replace x by any x_m with $m \geq N$

$$x_1 \quad x_2 \quad \cdots \quad x_N \quad \cdots x_n \quad \cdots \quad x_m \quad \cdots$$

'$d(x_n, x) < \epsilon$' becomes '$\forall m \geq N$, $d(x_n, x_m) < \epsilon$'

Definition 2.2.1: Cauchy Sequence

A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) is said to be a **Cauchy sequence** iff for every positive ϵ, there exists an index N, s.t. for all indices n, m with $n, m \geq N$,

$$d(x_n, x_m) < \epsilon$$

Theorem 2.2.2

If a sequence in a metric space converges, then it is a Cauchy sequence

Definition 2.2.3: Complete Metric Spaces

A metric space is said to be **complete** if and only if every Cauchy Sequence is convergent

Examples:

- $\mathbb{R}$ with the standard metric is complete
- $\mathbb{Q}$ with the standard metric is not complete
- $(0, 1)$ with the standard metric is not complete
- $[0, 1]$ with the standard metric is complete
- $\mathbb{R}^n$, ℓ^p, $C([a, b])$ is complete (proof later)

2.3 Open sets and closed sets

Definition 2.3.1: Open Sets and Closed Sets

Let (X, d) be a metric space.

- A subset G of X is said to be **open** iff for every point x in G there exists a positive radius r such that $B(x, r) \subseteq G$.
- A subset F of X is said to be **closed** iff F^c is open

Definition 2.3.2: Discrete Metric Space

A metric space is called **discrete** iff all its subsets are open (equiv. all subsets are closed)
Example: $[0, 1] \cap (2, 3)$

Theorem 2.3.3: Properties of open sets

Let (X, d) be a metric space

1. The union of any family of open sets is an open set
2. The intersection of finitely many open sets is an open set

Theorem 2.3.4: Infinite open sets

The intersection of infinitely many open sets is not always an open set
For example, let $G_n = (-\frac{1}{n}, \frac{1}{n}), n = 1, 2, \dots$ on the real line with the standard metric.
Each G_n is open but

$$\bigcap_{n=1}^{\infty} G_n = \{0\}$$

Theorem 2.3.5: Relatively open sets

Let (X, d) be a metric space and A be a non-empty subset of X equipped with the induced metric d_A. Let $G \subseteq A$. G is open in (A, d_A) iff there exists a subset O of X, open in (X, d), such that $G = A \cap O$
The open sets of (A, d_A) are sometimes referred to as **relatively open**

Theorem 2.3.6

Let (X, d) be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in X and x be a point in X.
$x_n \to x$ iff every open set that contains x contains eventually all terms of the sequence

Definition 2.3.7: Neighbourhoods of points

An **open neighbourhood** of a point x is any open set that contains x. $x_n \to x$ iff every open neighbourhood of x contains eventually all terms of the sequence.

A **neighbourhood** of a point x is a set that contains an open neighbourhood of x. $x_n \to x$ iff every neighbourhood of x contains eventually all terms of the sequence.

Theorem 2.3.8: Properties of Closed sets

Let (X, d) be a metric space.

1. The intersection of any family of closed sets is a closed set
2. The union of finitely many closed sets is a closed set.

Theorem 2.3.9: Infinite closed sets

The union of infinitely many closed sets is not always a closed set.
For example, let $F_n = [\frac{1}{n}, 1], n = 1, 2, \dots,$ on the real line with the standard metric. Each F_n is closed but

$$\bigcup_{n=1}^{\infty} F_n = (0, 1]$$

is not closed.

Theorem 2.3.10

A subset F of a metric space is closed iff the limit of every convergent sequence of elements of F belongs to F

- In any metric space (X, d), singletons $F = \{x\}$ are closed.
- In any metric space, any finite set is closed because

$$\{x_1, \ldots, x_n\} = \{x_1\} \cup \cdots \cup \{x_n\}$$

2.4 Closure

Definition 2.4.1: Closure

Let (X, d) be a metric space and $A \subseteq X$. The **closure** of A, deonted by $\overline{A}$, is the smallest closed subset of X that contains A

There exists at least one closed subset of X that contains A, namely X itself. The smallest closed subset of X that contains A is

$$\bigcap_{A \subseteq F \subseteq X,\ F \text{closed}} F$$

Theorem 2.4.2: Properties of Closure

Let (X, d) be a metric space and A, $B \subseteq X$.

1. $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$
2. $A \subseteq \overline{A}$ and $\overline{A}$ is closed
3. A is closed iff $A = \overline{A}$
4. $\overline{\overline{A}} = \overline{A}$
5. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$
6. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

- On the real line with the standard metric, $\overline{(a, b)} = [a, b]$
- In $\mathbb{R}^n$ with the Euclidean metric d_2, the closure of the open ball $B(c, r)$ is the closed ball $\{x \in \mathbb{R}^n : d_2(x, c) \leq r\}$
- On the complex plane with its standard metric, the closure of an open disc is the corresponding closed disc
- Let X be a non-empty set with the discrete metric, $c \in X$ and $r = 1$. Then $B(c, 1) = \{c\}$, therefore $\overline{B(c, 1)} - \overline{\{c\}} = \{c\}$, while

$$\{x \in X : d(x, c) \leq 1\} = X$$

  The closure of an open ball is not always equal to the corresponding closed ball
- $X = \mathbb{R}$, $d(x, y) = |x - y|$. $\overline{\mathbb{Q}} = \mathbb{R}$

Definition 2.4.3: Dense Subset of a Metric Space

Let (X, d) be a metric space. A subset D of X is said to be **dense** iff $\overline{D} = X$

Random fact: In $\mathbb{R}^n$ with the Euclidean metric d_2, $\mathbb{Q}^n$ is dense.

Theorem 2.4.4: Closure Equivalence

Let (X, d) be a metric space, $A \subseteq X, x \in X$. The following are equivalent

1. $x \in \overline{A}$
2. For every positive r, $B(x, r) \cap A \neq \emptyset$
3. There exists a sequence $(a_n)_{n \in \mathbb{N}}$ with $a_n \in A$ for all n, such that $a_n \to x$

A point x with any of these properties is called an **adherent point** of A. So, $\overline{A}$ is the set of all adherent points of A.

Definition 2.4.5: Limit points of sets

Let (X, d) be a metric space, $A \subseteq X$ and $x \in X$. We say that x is a **limit point** or an **accumulation point** of A iff every open ball centered at x contains an element of A distinct from x, i.e.

$$\forall r > 0 \quad (B(x, r) \backslash \{x\}) \cap A \neq \emptyset$$

The set of all limit points of A is called the **derived set** of A and is denoted by A' or $\tilde{A}$.

2.5 Continuous functions between metric spaces

Definition 2.5.1: Continuity at a point

Let (X, d_X), (Y, d_Y) be metric spaces and $f : X \to Y$ be a function. We say that f is **continuous at a point** x_0 in X iff for for every positive ϵ, there exists a positive δ, s.t., for all $x \in X$ with $d_X(x, x_0) < \delta$ we have $d_Y(f(x), f(x_0)) < \epsilon$

Alternatively, f is **continuous at a point** $x_0 \in X$ iff, for every positive ϵ, there exists a positive δ, such that, for all $x \in B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$

Definition 2.5.2: Continuity of a function

Let $(X, d_X), (Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is said to be **continuous** iff it is continuous at every point in X

Theorem 2.5.3

Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f : X \to Y$ be a function and x_0 be a point in X. Then f is continuous at x_0 iff for every open neighbourhood G of $f(x_0)$ there exists an open neighbourhood O of x_0 such that, for all $x \in O$, we have $f(x) \in G$

Theorem 2.5.4: Continuity and Convergence

Let (X, d_X), (Y, d_Y) be metric spaces, x_0 be a point in X, and $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous at x_0
2. For every sequence $(x_n)_{n=1}^{\infty}$ in X, if $x_n \xrightarrow[n \to +\infty]{}$ in (X, d_X), then $f(x_n) \xrightarrow[n \to +\infty]{} f(x_0)$ in (Y, d_Y)

Theorem 2.5.5: Continuity and Open Sets

Let (X, d_X), (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous iff the inverse image $f^{-1}(G)$ of any open subset G of Y is an open subset of X

3 Topology!!!

3.1 Homeomorphisms and Topological Properties

Definition 3.1.1: Topological Space

A **topological space** is a set X together with a family $\mathcal{T}$ of subsets of X that has the following properties:

- $\emptyset, X \in \mathcal{T}$
- Any union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$
- Any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$

$\mathcal{T}$ is called a **topology** and the elements of $\mathcal{T}$ are called **open sets**

Definition 3.1.2: Continuity of Topological Spaces

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. A function $f : X \to Y$ is said to be **continuous** iff for every G in $\mathcal{T}_Y$ the pre-image $f^{-1}(G)$ is an element of $\mathcal{T}_X$.

f is said to be a **homeomorphism** iff it is a continuous bijection and its inverse is continuous.

If such a homeomorphism exists then $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are said to be **homeomorphic**

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic, and one of them is compact or connected or separable etc etc, then so is the other

Properties that are preserved by homeomorphisms are called topological properties

Theorem 3.1.3: $d : X \times X \to \mathbb{R}$ is continuous

Let (X, d) be a metric space. The function $f : X \times X \to \mathbb{R}$ is continuous.

$\mathbb{R}$ is equipped with the standard metric. $X \times X$ is equipped with the product metric

3.1.4 Continuity of linear operators between normed vector spaces

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed vector spaces. Recall that $d_X : X \times X \to \mathbb{R}$, $d(x, x') = \|x - x'\|_X$, and $d_Y : Y \times Y \to \mathbb{R}$, $d_Y(y, y') = \|y - y'\|_Y$ are metrics

Definition 3.1.5: Bounded Linear Operators

A linear operator $T : X \to Y$ is said to be **bounded** iff there exists a positive constant C such that, for all $x \in X$,

$$\|T(x)\|_Y \leq C\|x\|_X$$

Theorem 3.1.6: Linear Operator Equivalence

Let $T : X \to Y$ be a linear operator. The following are equivalent:

1. T is continuous
2. T is continuous at 0
3. T is bounded

- Let $(X, \|\cdot\|)$ be a normed vector space and define $f : \mathbb{R} \times X \to X$ by $f(\lambda, x) = \lambda x$. Define $g : X \times X \to X$ by $g(x, y) = x + y$. f and g are continuous

3.2 Fixed Points and Lipschitz

Definition 3.2.1: Lipschitz Functions

Let (X, d_X), (Y, d_Y) be metric spaces. A function $f : X \to Y$ is said to be a **Lipschitz** function iff there exists a constant L such that for all $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq L d_X(x, x')$$

If $L < 1$, f is said to be a **contraction**

Note: Magnus uses non-standard terminology here:

- When the equation is satisifed and $L < 1$, Magnus calls f a **strict contraction**
- He uses **contraction** for a functino f that satisfies the weaker condition: for all $x, x' \in X$ with $x \neq x'$

$$d_Y(f(x), f(x')) < d_X(x, x')$$

Theorem 3.2.2: Lipschitz Continuity

Every Lipschitz function is continuous

Definition 3.2.3: Fixed Points

A **fixed point** of a function $f : S \to S$ where S is a non-empty set, is any element x of S such that $f(x) = x$

Solving equations can sometimes be reduced to finding fixed points

Watch lecture recording 06/03 for more in-depth examples

- Newton's Method for solving $f(x) = 0$
- Picard's Method for solving the Initial Value Problem

Theorem 3.2.4: Banach's Fixed Point Theorem

Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction. Then f has a unique fixed point

3.3 Equivalent Metrics

Definition 3.3.1: Equivalent Metrics

Two metrics on the same non-empty set X are said to be **equivalent** iff they have the same open sets

Theorem 3.3.2: Equivalent Metrics Theorem

Let d_1 and d_2 be metrics on the same non-empty set X. If there exist positive constants C and C' such that for all x, y in X,

$$C d_1(x, y) \leq d_2(x, y) \leq C' d_1(x, y)$$

then d_1 and d_2 are equivalent

Definition 3.3.3: Limits of functions between metric spaces

Let (X, d_X) and (Y, d_Y) be metric spaces, x_0 be a limit point of X, $y_0 \in Y$ and $f : X \to Y$ be a function. We say that $\lim_{x \to x_0} f(x) = y_0$ iff

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in B_X(x_0, \delta) \backslash \{x_0\} \quad f(x) \in B_Y(y_0, \epsilon)$$

Theorem 3.3.4: Completeness of the Classical Spaces

Some examples of complete metric spaces:

- $(\mathbb{R}^n, d_2)$
- ℓ^2
- $C([a, b])$
- ℓ^∞

Let (X, d_X) and (Y, d_Y) be two metric spaces and assume that (Y, d_Y) is complete.

Let $C(X, Y)$ be the set of all continuous and bounded functions from X to Y. For $f, g \in C(X, Y)$ define

$$D(f, g) = \sup\{d_Y(f(x), g(x)) : x \in X\}$$

D is a metric and the metric space $(C(X, Y), D)$ is complete

Definition 3.3.5: The product space X^n

Let (X, d) be a metric space and $n \in \mathbb{N}$. Define $D : X^n \to \mathbb{R}$ by

$$D(x_1, x_2) = d(x_{11}, x_{21}) + d(x_{12}, x_{22}) + \cdots + d(x_{1n}, x_{2n})$$

Lemma: D is a metric and a sequence converges in (X^n, D) iff it converges componentwise

Lemma: If (X, d) is complete then (X^n, D) is complete

Definition 3.3.6: The product space $X^{\mathbb{N}}$

Let B^A, where A, B are sets, be the set of all functions from A to B

Def: Let (X, d) be a metric space. Define a metric $D : X^{\mathbb{N}} \times X^{\mathbb{N}} \to \mathbb{R}$ by

$$D(x_1, x_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d(x_{1n}, x_{2n})}{1 + d(x_{1n}, x_{2n})}$$

where $x_1 = (x_{11}, \ldots, x_{1n}, \ldots)$, $x_2 = (x_{21}, \ldots, x_{2n}, \ldots)$

$(X^{\mathbb{N}}, D)$ is called a **product space**

Theorem 3.3.7: Convergence of Product spaces

Let (X, d) be a metric space, let $(x_k)_{k=1}^\infty$ be a sequence in $X^{\mathbb{N}}$ and let $x \in X^{\mathbb{N}}$. Write $x_k = (x_{k1}, \ldots, x_{kn}, \ldots)$ and $x = (l_1, \ldots, l_n, \ldots)$.

Then, $x_k \xrightarrow[k \to +\infty]{(X^{\mathbb{N}}, D)} x$ if and only if, for all n, $x_{kn} \xrightarrow[k \to +\infty]{(X^{\mathbb{N}} l_n}$

Theorem 3.3.8: Completeness of product spaces

Let (X, d) be a complete metric space. Then the product space $(X^{\mathbb{N}}, D)$ is complete.

Theorem 3.3.9: Completeness of $\mathbb{R}$

- **Thm (Least Upper Bound Principle)**: Every non-empty bounded above subset of $\mathbb{R}$ has a least upper bound
- **Thm (Monotone Convergence)**: Every bounded monotone sequence of real numbers has a limit
- **Thm (ϵ-convergence)**: Let A be a non-empty boudned subset of $\mathbb{R}$ and let ϵ be positive. If the distance between any two elements of A is $< \epsilon$, then

$$\sup(A) - \inf(A) \leq \epsilon$$

- **Thm**: Every Cauchy sequence of real numbers is convergent

Definition 3.3.10: Limit Superior and Inferior

Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathbb{R}$. Then $(x_n)_{n=1}^\infty$ is bounded. Define:

$$I_n = \inf\{x_n, x_{n+1}, \ldots\} \quad S_n = \sup\{x_n, x_{n+1}, \ldots\}$$

Thm: $(S_n)_{n=1}^\infty$ and $(I_n)_{n=1}^\infty$ are monotone and bounded

$$I_1 \leq I_n \leq S_n \leq S_1, \quad n = 1, 2, \ldots$$

Therefore $I_n \to I$ and $S_n \to S$ for some reals I and S. Since $S_n - I_n \to 0$ we have $S = I$. We also have $x_n \to S = I$

Limsup and Liminf

- The limit of the sequence $(I_n)_{n=1}^\infty$ is called the **limit inferior** of $(x_n)_{n=1}^\infty$ and is denoted by $\liminf x_n$

$$\liminf x_n = \lim_{n \to +\infty} I_n = \lim_{n \to +\infty} \inf\{x_n, x_{n+1}, \ldots\}$$

- The limit of the sequence $(S_n)_{n=1}^\infty$ is called the **limit superior** of $(x_n)_{n=1}^\infty$ and is denoted by $\limsup x_n$

$$\limsup x_n = \lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \sup\{x_n, x_{n+1}, \ldots\}$$

- $\liminf x_n$ is the smallest subsequential limit of $(x_n)_{n=1}^\infty$
- $\limsup x_n$ is the largest subsequential limit of $(x_n)_{n=1}^\infty$
- $(x_n)_{n=1}^\infty$ converges iff $\liminf x_n = \limsup x_n$

4 Compactness

Definition 4.0.1: Compactness

Let $X = \mathbb{R}$ and d be the standard metric. A subset K of $\mathbb{R}$ is said to be **compact** iff every sequence of elements of K has a subsequence that converges to an element of K

Example 4.0.2: Examples of compactness

Compact sets

- $[a, b]$ is compact
- $\emptyset$ is compact
- $\mathbb{R} \cup \{-\infty, +\infty\}$ is compact!

Not Compact sets

- $(0, 1)$ is not compact
- $\mathbb{R}$ is not compact

Theorem 4.0.3: Heine-Borel Theorem

On the real line with the standard metric, a set is compact if and only if it is closed and bounded

---

In $\mathbb{R}^n$ with the Euclidean metric, a set is compact if and only if it is closed and bounded

Theorem 4.0.4: Continuous Functions on Compact Sets

Let $K \subseteq \mathbb{R}$ be compact, and $f : K \to \mathbb{R}$ be a continuous function. Then f is bounded

Theorem 4.0.5: Extreme Value Theorem

Let $K \subseteq \mathbb{R}$ be compact and $f : K \to \mathbb{R}$ be a continuous function. Then f has a maximum and a minimum

Theorem 4.0.6: Open Covers

An **open cover** of a set S in a metric space is a family $(G_i)_{i\in I}$ of open sets such that $S \subset \bigcup_{i \in I} G_i$. A **subcover** of an open cover $(G_i)_{i\in I}$ is a sub-family $(G_i)_{i\in I'}$ where $I' \subset I$, such that $S \subseteq \bigcup_{i\in I'} G_i$

Thm: On the real line with the standard metric, a set K is compact iff every open cover of K has a finite subcover

---

Lemma: Every open cover of the interval $[a, b]$, where $a, b \in \mathbb{R}$, $a \leq b$ has a finite cover

---

Thm: Let $K \subseteq \mathbb{R}$ and assume that every open cover of K has a finite subcover. Then K is closed and bounded, hence compact

Definition 4.0.7: Sequentially compact sets

Let (X, d) be a metric space and $K \subseteq X$

- We say that K is **sequentially compact** iff every sequence in K has a subsequence that converges to an element of K

  For $K = X$ this becomes: X is compact iff every sequence in X has a convergent subsequence
- We say that K is **compact** iff every open cover of K has a finite subcover

These two notions of compactness are equivalent

Theorem 4.0.8: idk more thms

Thm: Let (X, d) be a metric space and $K \subseteq X, K \neq \emptyset$, d_K be the induced metric on K. K is a (sequentially) compact subset of X iff the metric space (K, d_K) is (sequentially compact)

Thm: Let (X, d) be a metric space and $K \subseteq X$ be sequentially compact. Then K is closed and bounded

---

Thm: Let (X, d) be a sequentially compact metric space and $K \subseteq X$. The set K is sequentially compact iff it is closed

---

Thm: If a metric space (X, d) is sequentially compact, then it is complete

Theorem 4.0.9: Extreme Value Theorem again

Let (X, d) be a metric space, K be a sequentially compact subset of X and $f : K \to \mathbb{R}$ be a continous function. Then f has a maximum and a minimum. In particular, f is bounded.

Definition 4.0.10: Uniform Continuity

Let $(X, d_X), (Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is said to be **uniformly continuous** iff for every positive ϵ there exists a positive δ such that, for all x, x' in X with $d_X(x, x') < \delta$ we have $d_Y(f(x), f(x')) < \epsilon$

Thm: Let (X, d_X) be a sequentially compact metric space, (Y, d_Y) be a metric space and $f : X \to Y$ be a continuous function. Then f is uniformly continuous

Theorem 4.0.11: yet another compactness thm

If a metric spasce (X, d) is compact, then it is sequentially compact

If a subset K of X is compact, prove that K is sequentially compact

Let (X, d) be a compact metric space and A be an infinite subset of X. Then A has at least one limit point

Theorem 4.0.12: Lebesgue's Lemma

Let (X, d) be a sequentially compact metric space and $X = \bigcup_{i\in I} G_i$ be an open cover of X. There exists a positive δ such that for any two points $x, y \in X$ with $d(x, y) < \delta$ there exists an i such that $x, y \in G_i$. Any such δ is called a **Lebesgue number** of the open cover

Lemma: Let (D, d) be a sequentially compact metric space and $X = \bigcup_{i\in I} G_i$ be an open cover of X. Then there exists a $\delta > 0$ s.t. any nonempty subset of X of diameter $< \delta$ can be covered by a single G_i

Definition 4.0.13: Totally bounded spaces

A metric space (X, d) is said to be **totally bounded** iff for every positive δ, X can be covered by a finite number of open balls of radius δ.

Note: If (X, d) is totally bounded then it is bounded, but the converse is not necessarily true

Theorem 4.0.14: Sequentially compactness boundedness

If a metric space is sequentially compact, then it is totally bounded

---

Thm: Every sequentially compact metric space is compact. (From now on, refer to sequentially compact spaces as compact)

Thm: A metric space is compact iff it is complete and totally bounded

Definition 4.0.15: Countable and Uncountable Sets

A set S is said to be:

- **Infinitely coutnable** iff there is a bijection $f : \mathbb{N} \to S$
- **Countable** if it is finite or infinitely countable
- **Uncountable** iff it isn't countable

Examples

- $\{1, 2, 3\}$ and $\mathbb{R}$ are countable sets
- $\mathbb{Q}$ is infinitely countable
- $\mathbb{R}$ is uncountable

Theorem 4.0.16: Dense Subset equivalence

Let (X, d) be a metric space and $D \subseteq X$. The following are equivalent:

1. D is dense
2. For every $x \in X$ and $\epsilon > 0$ there exists $y \in D$ s.t. $d(x, y) < \epsilon$
3. For every $x \in X$ there is a sequence $(y_n)_{n=1}^{\infty}$ of elements of D s.t. $y_n \to x$
4. For every element $x \in X$ and every open nbhd G of x, $G \cap D \neq \emptyset$
5. D intersects every non-empty open set

Definition 4.0.17: Separable spaces

A metric space is said to be **separable** iff it has a countable dense subset

Examples

- $\mathbb{R}$ with the standard metric is a separable metric because $\mathbb{Q}$ is dense and countable
- $\mathbb{R}^n$ with the Euclidean metric is a separable metric space because $\mathbb{Q}^n$ is dense and countable
- $\mathbb{C}$ with its standard metric is a separable metric space because $\{z \in \mathbb{C} : \mathrm{Re}(z), \mathrm{Im}(z) \in \mathbb{Q}\}$
- ℓ^2 is separable, and ℓ^p is separable for $1 \leq p < \infty$

Theorem 4.0.18: Weierstrass Approximation Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\epsilon > 0$. There exists a polynomial p with *real* coefficients s.t. for all $x \in [a, b]$

$$|f(x) - p(x)| < \epsilon$$

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\epsilon > 0$. There exists a polynomial p with *rational* coefficients s.t. for all $x \in [a, b]$

$$|f(x) - p(x)| < \epsilon$$

Theorem 4.0.19: more theorems

The set of all polynomials with rational coefficients is countable

Thm: $C([a, b])$ is separable

Theorem 4.0.20: Separability of subspaces

Let (X, d) be a separable metric space, $A \subseteq X$, $A \neq \emptyset$, and d_A be the induced metric on A. Then the metric space (A, d_A) is separable

---

Thm: Every compact metric space is separable (compact $\implies$ separable)

Theorem 4.0.21: Open Ball countability

Let (X, d) be a separable metric space and let D be a countable dense subset of X. Let

$$\mathcal{B} = \{B(c, r) : c \in D, r \in \mathbb{Q}^+\}$$

be the set of all open balls with centers in D and rational radii. Then $\mathcal{B}$ is countable and every open set in X can be written as a union of elements of $\mathcal{B}$

Definition 4.0.22: Open Bases and Second Countability

Open Bases

Let $(X, \mathcal{T})$ be a topological space. An **open base** (or **base**) for the topology $\mathcal{T}$ is a family $\mathcal{B}$ of open sets such that every open set in $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$

Second Countability

A topological space $(X, \mathcal{T})$ is said to satisfy the **second Axiom of Countability**, or is **second countable** iff it has a countable open base

---

Thm: In a separable metric space, every family of pairwise disjoint non-empty open sets is countable

---

Thm: On the real line with the standard metric, every open set can be written as a countable union of disjoint open intervals

Definition 4.0.23: Continuous Extensions

Let $(X, d_X), (Y, d_Y)$ be metric spaces, D be a dense subset of X, $f, g : X \to Y$ continuous functions s.t. $f(x) = g(x)$ for all $x \in D$. Then $f = g$

Let $(X, d_X), (Y, d_Y)$ be metric spaces, $D \subseteq X$ be dense, $f : D \to Y$ be uniformly continuous, and assume that (Y, d_Y) is complete. Then f has a unique continuous extension $F : X \to Y$

Theorem 4.0.24: complete ms props

Let (X, d) be a metric space, F be a nonempty subset of X and d_F be the induced metric on F. If the metric space (F, d_F) is complete then F is a closed subset of X

Thm: Let (X, d) be a complete metric space, F be a nonempty subset of X, and d_F be the induced metric on F. If F is a closed subset of X, then the metric space (F, d_F) is complete

---

Thm: Let (X, d) be a complete metric space, $A \subseteq X$, $A \neq \emptyset$. Then

1. The metric space $(\overline{A}, d_{\overline{A}})$ is complete
2. If $A \subseteq B \subseteq X$ and (B, d_B) is complete, then $\overline{A} \subseteq B$

Definition 4.0.25: Isometries

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called a **isometry** iff for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

Thm: Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be an isometry. Then f is an injection. If, moreover, f is a surjection (hence f bij.) then $f^{-1} : Y \to X$ is also an isometry

---

Thm: The metric spaces (X, d_X) and (Y, d_Y) are said to be **isometric** iff there exists an isometry f from X onto Y

Thm: if two metric spaces are isometric and one of them is complete/compact/connected/... then so is the other

Theorem 4.0.26: Isometry completion

Let (X, d) be a bounded metric space and let $C(X, \mathbb{R})$ be the set of all bounded continuous functions $f : X \to \mathbb{R}$ equipped with the metric

$$d_\infty(f_1, f_2) = \sup\{|f_1(t) - f_2(t)| : t \in X\}$$

For each $x \in X$, define $F_X : X \to \mathbb{R}$ be $F_X(x') = d(x, x')$. Then

1. $F_X \in C(X, \mathbb{R})$
2. The map $X \to C(X, \mathbb{R}), x \mapsto F_X$ is an isometry
3. $X^* = \{F_X : x \in X\}$, equipped with the induced metric, is a subspae of $C(X\mathbb{R})$ isometric to X
4. The closure $\overline{X^*}$ of X^* in $C(X, \mathbb{R})$, equipped with the induced metric, is a complete metric space
5. X^* is dense in $\overline{X^*}$

Definition 4.0.27: Completion of a Metric Space

Let (X, d) be a metric space. A **completion** of (X, d_X) is any metric space (Y, d_Y) with the following properties

1. (Y, d_Y) is complete
2. (Y, d_Y) has a subspace X^* isometric to (X, d_X)
3. X^* is dense in Y

It can be shown that any two completions of X are isometric to each other, i.e. a completion is unique up to isometries

Definition 4.0.28: Construction of Completion via Cauchy

Let (X, d) be a metric space and let $\mathcal{C}$ be the set of all Cauchy sequences of elements of X

We define an equivalence relation $\sim$ in $\mathcal{C}$ as follows: Let $x = (x_n)_{n \in \mathbb{N}}$, $y = (y_n)_{n \in \mathbb{N}} \in \mathcal{C}$. We say that $x \sim y$ iff $d(x_n, y_n) \to 0$

Distinct equivalence classes are disjoint and partition $\mathcal{C}$

The set of all equivalence classes is called the **quotient space**, denoted $\mathcal{C}/\sim$

Define a metric D on $\mathcal{C}/\sim$ as follows:

Let $\alpha, \beta \in \mathcal{C}/\sim$. Then

$$\alpha = [(x_1, \ldots, x_n, \ldots)] \text{ and } \beta = [(y_1, \ldots, y_n, \ldots)]$$

for some $(x_1, \ldots, x_n, \ldots), (y_1, \ldots, y_n, \ldots) \in \mathcal{C}$. Define

$$D(\alpha, \beta) = \lim_{n \to +\infty} d(x_n, y_n)$$

$(\mathcal{C}/\sim, D)$ is complete. Additionally, the following is an isometry:

$$X \to \mathcal{C}/\sim \qquad x \mapsto ([x, x, \ldots, x, \ldots])$$

Let X^* be its range. The metric space (X^*, D_{X^*}) is isometric to (X, d), $(\overline{X^*}, D_{\overline{X^*}})$ is a complete metric space, and X^* is dense in $\overline{X^*}$

Definition 4.0.29: Connected and Disconnected Spaces

A metric space (X, d) is said to be **disconnected** iff there exists non-empty disjoint open sets G_1 and G_2 such that

$$X = G_1 \cup G_2$$

Otherwise the metric space is called **connected**

A non-empty subset A of a metric space (X, d) is said to be disconnected iff the metric space (A, d_A), where d_A is the induced metric, is disconnected

Theorem 4.0.30: Connected Theorems

A subset O of A is open in (A, d_A) iff $O = A \cup G$ for some G that is open in X

Therefore, A is disconnected iff there exist open subsets G_1, G_2 of X s.t.

- $A = (A \cap G_1) \cup (A \cap G_2)$, which is equivalent to $A \subseteq G_1 \cup G_2$
- $A \cap G_1 \neq \emptyset$, $A \cap G_2 \neq \emptyset$
- $(A \cap G_1) \cap (A \cap G_2) = \emptyset$, which is equivalent to $A \cap G_1 \cap G_2 = \emptyset$

Thm: $\mathbb{R}$ with the standard metric is connected

Thm: On the real line with the standard metric, all intervals are connected sets

Thm: A non-empty subset of the real line is connected iff it is an interval

Thm: A metric space (X, d) is connected iff the only subsets of X with empty boundary are $\emptyset$ and X

Thm: Let (X, d_X) be a connected metric space, (Y, d_Y) be a metric space and $f : X \to Y$ be a continuous surjection. Then (Y, d_Y) is connected as well

Theorem 4.0.31: Intermediate Value Theorem

Let (X, d) be a connected metric space and $f : X \to \mathbb{R}$ be a continuous function. If $x_1, x_2 \in X$ with $f(x_1 \neq f(x_2))$ and y is a real number between $f(x_1)$ and $f(x_2)$, then there exists an $x \in X$ such that $f(x) = y$

Theorem 4.0.32: Clopen

A metric space (X, d) is connected iff the only clopen subsets are $\emptyset, X$

Definition 4.0.33: Connected Components

Let (X, d) be a metric space. We define an equivalence relation $\sim$ in X as follows: $x \sim x'$ iff there exists a connected subset C of X that contains both x and x'

Thm: If $(C_i)_{i\in I}$ is a family of connected subsets of X with nonempty intersection, then $\bigcup_{i\in I} C_i$ is connected

Theorem 4.0.34: Big equivalence classes

The equivalence class of any point in X is the largest connected subset of X that contains that point (what point?)

Definition 4.0.35: Path Connected Metric Spaces

Let (X, d) be a metric space and $x_0, x_1 \in X$. A **path** in X from x_0 to x_1 is a continuous function $\gamma : [0, 1] \to X$ s.t. $\gamma(0) = x_0$, $\gamma(1) = x_1$

(X, d) is said to be **path-connected** iff for any two points x_0, x_1 in X there is a path in X from x_0 to x_1

A non-empty subset A of X is said to be **path-connected** iff the metric space (A, d_A), where d_A is the induced metric, is path connected

Theorem 4.0.36: Path connected theorem

Every path-connected metric space is connected

Not every connected metric space is necessarily path-connected

5 Applications

Definition 5.0.1: Equivalent Norms

Two norms on the same real vector space are said to be equivalent iff their corresponding metrics are equivalent

---

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on the same real vector space X and there exist positive constants C and C' such that, for all $x \in X$,

$$D\|x\|_1 \leq \|x\|_2 \leq C'\|x\|_1$$

then they are equivalent

Theorem 5.0.2: p metric again?

For any p with $1 \leq p < \infty$ and any $x \in \mathbb{R}^n$ we define

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$$

Young's Inequality

Let $1 \leq p, q \leq \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$, and $a, b \leq 0$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Holder Inequality

Let $1 \leq p, q \leq \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$ and $x, y \in \mathbb{R}^n$. Then

$$\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \|y\|_q$$

Equivalence of p-metrics

Thm: Any of the following norms are equivalent:

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p},\ x \in \mathbb{R}^n,\ 1 \leq p < \infty$$
$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\},\ x \in \mathbb{R}^n$$

Thm: Let $1 \leq p \leq q < \infty$. For all $x \in \mathbb{R}^n$:

$$\|x\|_q \leq \|x\|_p$$

As a consequence,

$$\|x\|_\infty \leq \|x\|_q \leq \|x\|_p \leq \|x\|_1$$

Thm: All norms in $\mathbb{R}^n$ are equivalent

Theorem 5.0.3: Picard's Theorem

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous and boudned function, and t_0, x_0 be real numbers. Assume that there exists a positive constant L s.t. for all real t, x_1, x_2 we have:

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$$

Then, there exists a positive δ and a unique differentiable function $x : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ s.t. for all $t \in [t_0 - \delta, t_0 + \delta]$,

$$x'(t) = f(t, x(t)) \quad \text{and} \quad x(t_0) = x_0$$

Definition 5.0.4: Lipschitz Functions again

Let (X, d_X), (Y, d_Y) be metric spaces. A function $f : X \to Y$ is said to be a **Lipschitz** function iff there exists a constant L such that for all $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq L d_X(x, x')$$

If $L < 1$, f is said to be a **contraction**

If $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function and x_0 is any point in $\mathbb{R}$, then for any $x \in \mathbb{R}$ we have

$$|f(x) - f(x_0)| \leq L|x - x_0|$$

For $x \geq x_0$ this can be expanded to

$$f(x_0) - L(x - x_0) \leq f(x) \leq f(x_0) + L(x - x_0)$$

Let (X, d_X) and (Y, d_Y) be two metric spaces, and $f : X \to Y$ be a Lipschitz function. Then there exists a smallest Lipschitz constant of f

---

Thm: Let I be a non-degenerate open interval on the real line and let $f : I \to \mathbb{R}$ be a differentiable function. Then f is Lipschitz iff f' is bounded. When that is the case,

$$|f|_{\text{Lip}} = \sup\{|f'(x)| : x \in I\}$$

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus.

Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.

Suspendisse vel felis. Ut lorem lorem, interdum eu, tincidunt sit amet, laoreet vitae, arcu. Aenean faucibus pede eu ante. Praesent enim elit, rutrum at, molestie non, nonummy vel, nisl. Ut lectus eros, malesuada sit amet, fermentum eu, sodales cursus, magna. Donec eu purus. Quisque vehicula, urna sed ultricies auctor, pede lorem egestas dui, et convallis elit erat sed nulla. Donec luctus. Curabitur et nunc. Aliquam dolor odio, commodo pretium, ultricies non, pharetra in, velit. Integer arcu est, nonummy in, fermentum faucibus, egestas vel, odio.

Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.

Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Donec odio elit, dictum in, hendrerit sit amet, egestas sed, leo. Praesent feugiat sapien aliquet odio. Integer vitae justo. Aliquam vestibulum fringilla lorem. Sed neque lectus, consectetuer at, consectetuer sed, eleifend ac, lectus. Nulla facilisi. Pellentesque eget lectus. Proin eu metus. Sed porttitor. In hac habitasse platea dictumst. Suspendisse eu lectus. Ut mi mi, lacinia sit amet, placerat et, mollis vitae, dui. Sed ante tellus, tristique ut, iaculis eu, malesuada ac, dui. Mauris nibh leo, facilisis non, adipiscing quis, ultrices a, dui.

Morbi luctus, wisi viverra faucibus pretium, nibh est placerat odio, nec commodo wisi enim eget quam. Quisque libero justo, consectetuer a, feugiat vitae, porttitor eu, libero. Suspendisse sed mauris vitae elit sollicitudin malesuada. Maecenas ultricies eros sit amet ante. Ut venenatis velit. Maecenas sed mi eget dui varius euismod. Phasellus aliquet volutpat odio. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Pellentesque sit amet pede ac sem eleifend consectetuer. Nullam elementum, urna vel imperdiet sodales, elit ipsum pharetra ligula, ac pretium ante justo a nulla. Curabitur tristique arcu eu metus. Vestibulum lectus. Proin mauris. Proin eu nunc eu urna hendrerit faucibus. Aliquam auctor, pede consequat laoreet varius, eros tellus scelerisque quam, pellentesque hendrerit ipsum dolor sed augue. Nulla nec lacus.

Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi.

Sed feugiat. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Ut pellentesque augue sed urna. Vestibulum diam eros, fringilla et, consectetuer eu, nonummy id, sapien. Nullam at lectus. In sagittis ultrices mauris. Curabitur malesuada erat sit amet massa. Fusce blandit. Aliquam erat volutpat. Aliquam euismod. Aenean vel lectus. Nunc imperdiet justo nec dolor.

Etiam euismod. Fusce facilisis lacinia dui. Suspendisse potenti. In mi erat, cursus id, nonummy sed, ullamcorper eget, sapien. Praesent pretium, magna in eleifend egestas, pede pede pretium lorem, quis consectetuer tortor sapien facilisis magna. Mauris quis magna varius nulla scelerisque imperdiet. Aliquam non quam. Aliquam porttitor quam a lacus. Praesent vel arcu ut tortor cursus volutpat. In vitae pede quis diam bibendum placerat. Fusce elementum convallis neque. Sed dolor orci, scelerisque ac, dapibus nec, ultricies ut, mi. Duis nec dui quis leo sagittis commodo.