

# Metric Spaces Exam Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

## 1 Introduction to Metric Spaces

### Definition 1: Definition of a Metric

Let  $X$  be a non-empty set. A function  $d : X \times X \rightarrow \mathbb{R}$  is called a **metric** iff for all  $x, y, z \in X$ ,

- $d(x, y) \geq 0$  and  $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$  (Triangle Inequality)

A non-empty set  $X$  equipped with a metric  $d$  is a **metric space**

### Definition A: Real Vector Spaces

A **real vector space**  $V$  is a set with two operations  $(X, +, \cdot)$ , where:

- $+$  is addition, and  $\cdot$  is scalar multiplication
- $(X, +)$  is an abelian group - i.e. for all (vectors)  $x, y, z \in X$ :
  - Closure:**  $x + y \in X$
  - Commutativity:**  $x + y = y + x$
  - Associativity:**  $x + (y + z) = (x + y) + z$
  - Identity:**  $\exists 0 \in X$  s.t. for all  $x \in X$  we have  $0 + x = x + 0 = x$
  - Inverse:**  $\forall x \in X$  we have  $-x$  s.t.  $x + (-x) = (-x) + x = 0$
- Vector space axioms: for all  $x, y, z \in X$  and  $\mu, \lambda \in \mathbb{R}$  we have:
  - Closure-ish thing:**  $\lambda x \in X$
  - Distributivity 1:**  $\lambda(x + y) = \lambda x + \lambda y$
  - Distributivity 2:**  $(\lambda + \mu)x = \lambda x + \mu x$
  - Associativity:**  $\lambda(\mu x) = (\lambda\mu)x$
  - Identity:**  $1x = x$

### Definition B: Normed and Inner Product Spaces

#### Def 5 (Normed Vector Spaces)

A **normed vector space** is a real vector space  $X$  equipped with a **norm**, i.e. a function that assigns to every vector  $x \in X$  a real number  $\|x\|$  so that, for all vectors  $x$  and  $y$  in  $X$  and all real scalars  $a$ :

- $\|x\| \geq 0$  and  $\|x\| = 0 \iff x = 0$
- $\|ax\| = |a|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

**Remark:** If  $(X, \|\cdot\|)$  is a normed vector space then

$$d(x, y) = \|x - y\|$$

defines a metric in  $X$

#### Def 6 (Inner Product Spaces)

Let  $X$  be a real vector space. An **inner product** on  $X$  is a function that assigns to every pair  $(x, y) \in X \times X$  a real number denoted by  $\langle x, y \rangle$  and has the following properties:

- $\langle x, x \rangle \geq 0$  and  $\langle x, x \rangle = 0 \iff x = 0$
- $\langle x, y \rangle = \langle y, x \rangle$
- $ax + by, z = a\langle x, z \rangle + b\langle y, z \rangle$

A **real inner product space** is a real vector space equipped with an inner product. If  $\langle \cdot, \cdot \rangle$  is an inner product on  $X$ , then

- $\|x\| = \sqrt{\langle x, x \rangle}$  defines a norm in  $X$
- $d(x, y) = \|x - y\|$  defines a metric in  $X$

### Definition C: $n$ -dimensional Euclidean space

Let  $X = \mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1, x_2, \dots, x_n \in \mathbb{R}\}$

For  $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$  in  $\mathbb{R}^n$ , define




$$\langle x, y \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n \text{ (inner product)}$$

$$\|x\|_2 = \langle x, x \rangle^{1/2} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \text{ (norm)}$$

### Example D: Examples of Metric Spaces

Unless stated otherwise let  $X = \mathbb{R}^n$ . The case  $X = \mathbb{R}^2$  is listed in **red**

Name	Norm and Metric
Standard	$X = \mathbb{R}$ and $ x $ = Absolute Value $d(x, y) =  x - y $
Taxicab	$\ x\ _1 =  x_1  +  x_2  + \dots +  x_n $ $d_1(x, y) =  x_1 - y_1  +  x_2 - y_2  + \dots +  x_n - y_n $
Euclidean	$\ x\ _2 = \sqrt{ x_1 ^2 +  x_2 ^2 + \dots +  x_n ^2}$ $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$
$p$ -metric	$\ x\ _p = \left( \sum_{k=1}^n  x_k ^p \right)^{1/p}$ $d_p(x, y) = \left( \sum_{k=1}^n  x_k - y_k ^p \right)^{1/p}$
Chebyshev	$\ x\ _\infty = \max\{ x_1 ,  x_2 , \dots,  x_n \}$ $d(x, y) = \max\{ x_1 - y_1 ,  x_2 - y_2 , \dots,  x_n - y_n \}$
Discrete	$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$
Post Office	$d(x, y) = \begin{cases} \ x\ _2 + \ y\ _2 & x \neq y \\ 0 & x = y \end{cases}$

1	1	1	$\sqrt{2}$	1	$\sqrt{2}$	2	1	2
1		1	1		1	1		1
1	1	1	$\sqrt{2}$	1	$\sqrt{2}$	2	1	2
Chebyshev			Euclidean			Taxicab		

### The complex plane

Let  $X = \mathbb{C}, d : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$

$$d(z, w) = |z - w|$$

If  $z = a + ib, w = c + id, a, b, c, d \in \mathbb{R}$ , then

$$z - w = (a - c) + i(b - d)$$

therefore,

$$d(z, w) = \sqrt{(a - c)^2 + (b - d)^2}$$

### Example E: Sequence Spaces

#### The space $\ell^1$

$\ell^1$  is the set of real sequences  $(x_n)_{n \in \mathbb{N}}$  where  $\sum_{n=1}^\infty |x_n|$  converges. For  $x = (x_1, \dots, x_n, \dots) \in \ell^1, y = (y_1, \dots, y_n, \dots) \in \ell^1$  we define

- Norm:**  $\|x\|_1 = \sum_{n=1}^\infty |x_n|$
- Metric:**  $d_1(x, y) = \|x - y\|_1 = \sum_{n=1}^\infty |x_n - y_n|$

#### The space $\ell^2$

$\ell^2$  is the set of real seqs  $(x_n)_{n \in \mathbb{N}}$  where  $\sum_{n=1}^\infty |x_n|^2$  converges. For  $x = (x_1, \dots, x_n, \dots) \in \ell^2, y = (y_1, \dots, y_n, \dots) \in \ell^2$  we define

- Inner product:**  $\langle x, y \rangle = \sum_{n=1}^\infty x_n y_n$
- Norm:**  $\|x\|_2 = \left( \sum_{n=1}^\infty |x_n|^2 \right)^{1/2}$
- Metric:**  $d_2(x, y) = \|x - y\|_2 = \left( \sum_{n=1}^\infty |x_n - y_n|^2 \right)^{1/2}$

**Thm:**  $\ell^2$  is a real vector space

#### The space $\ell^\infty$

$\ell^\infty$  is the set of all bounded sequences of real numbers. For  $x = (x_1, \dots, x_n, \dots), y = (y_1, \dots, y_n, \dots) \in \ell^\infty$

- Norm:**  $\|x\|_\infty = \sup\{|x_1|, \dots, |x_n|, \dots\}$
- Metric:**  $\|x - y\|_\infty = \sup\{|x_1 - y_1|, \dots, |x_n - y_n|, \dots\}$

#### The space $C([a, b])$

$X = C([a, b])$  is the set of all continuous functions  $f : [a, b] \rightarrow \mathbb{R}$

- Norm:**  $\|f\|_\infty = \max\{|f(x)| : a \leq x \leq b\}$
- Metric:**  $d_\infty(f, g) = \|f - g\|_\infty = \max\{|f(x) - g(x)| : a \leq x \leq b\}$

#### The $L^1$ metric

$X$  is the set of all continuous functions  $f : [a, b] \rightarrow \mathbb{R}$

- Norm:**  $\|f\|_1 = \int_a^b |f(x)| dx$
- Metric:**  $d_2(f, g) = \|f - g\|_1 = \int_a^b |f(x) - g(x)| dx$

#### The $L^2$ metric

$X$  is the set of all continuous functions  $f : [a, b] \rightarrow \mathbb{R}$

- Inner Product:**  $\langle f, g \rangle = \int_a^b f(x)g(x) dx$
- Norm:**  $\|f\|_2 = \langle f, f \rangle^{1/2} = \left( \int_a^b |f(x)|^2 dx \right)^{1/2}$
- Metric:**  $d_1(f, g) = \left( \int_a^b |f(x) - g(x)|^2 dx \right)^{1/2}$

### Definition F: Metric Subspaces

**Ex 7:** Let  $(X, d)$  be a metric space and  $Y$  a non-empty subset of  $X$ . Define

- $d_Y : Y \times Y \rightarrow \mathbb{R}$
- $d_Y(y, y') = d(y, y')$

Then  $d_Y$  is a metric on  $Y$ .  $d_Y$  is called the **induced** or **inherited** metric, and  $(Y, d_Y)$  is said to be a metric subspace of the metric space  $(X, d)$

#### Interior

Let  $(X, d)$  be a metric space. The **interior**  $\mathring{A}$  of a subset  $A$  of  $X$  is the largest open set contained in  $A$ . e.g.  $A = [0, 1], \mathring{A} = (0, 1)$

### Theorem G: a lack of equality or fair treatment in t...

#### Good old fashioned Triangle Inequality

If it ain't broke...

$$|x + y| \leq |x| + |y|, \quad |x - y| \geq \left| |x| - |y| \right|, \quad |x - y| \leq |x - z| + |z - y|$$

#### Cauchy-Schwarz Inequality

For all  $x$  and  $y$  of an inner product space:

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

#### Minkowski's Inequality

Let  $p \geq 1$ , and real numbers  $x_i, y_i$ , ( $i = 1, \dots, n$ ). Then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$
$$\left( \sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}}$$

#### Ex 56 (Young's Inequality)

Let  $1 \leq p, q \leq \infty$  s.t.  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $a, b \leq 0$ . Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

#### Thm 169 (Hölder Inequality)

Let  $1 \leq p, q \leq \infty$  s.t.  $\frac{1}{p} + \frac{1}{q} = 1$  and  $x, y \in \mathbb{R}^n$ . Then

$$\sum_{i=1}^n |x_i y_i| \leq \|x\|_p \|y\|_q$$

### Definition 166: Equivalent Norms

Two norms on the same real vector space are said to be **equivalent** iff their corresponding metrics are equivalent

**Thm 167:** If  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are norms on the same real vector space  $X$ , then they are equivalent if there exist positive constants  $C$  and  $C'$  such that, for all  $x \in X$ ,

$$D\|x\|_1 \leq \|x\|_2 \leq C'\|x\|_1$$

#### Equivalence Theorems of $p$ -metrics

**171:** Any of the following norms are equivalent:

$$\|x\|_p = (|x_1|^p + \dots + |x_n|^p)^{1/p}, \quad x \in \mathbb{R}^n, \quad 1 \leq p < \infty$$

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}, \quad x \in \mathbb{R}^n$$

**172:** Let  $1 \leq p \leq q < \infty$ . For all  $x \in \mathbb{R}^n$ :

$$\|x\|_q \leq \|x\|_p \quad \text{and} \quad \|x\|_\infty \leq \|x\|_q \leq \|x\|_p \leq \|x\|_1$$

**173:** All norms in  $\mathbb{R}^n$  are equivalent

### Definition 8: Open Ball

Let  $(X, d)$  be a metric space,  $c$  be a point in  $X$ , and  $r > 0$ . The **open ball** with center  $c$  and radius  $r$  is defined by

$$B(c, r) = \{x \in X : d(c, x) < r\}$$

#### Boundedness and Distance Set

Let  $A \subseteq X$ .  $A$  is **bounded** iff there exists a  $c \in X$  and radius  $r$  s.t.  $A \subseteq B(c, r)$ . The **distance set** of  $D$ ,  $D(A)$  is defined by

$$D(A) = \{d(x, y) : x, y \in A\}$$

#### Boundary Points

Let  $(X, d)$  metric space and  $A \subseteq X$ .  $x \in X$  is a **boundary point** of  $A$  iff every open ball centered at  $x$  intersects both  $A$  and  $A^c$ , i.e.

$$B(x, r) \cap A \neq \emptyset \quad \text{and} \quad B(x, r) \cap A^c \neq \emptyset$$

**Boundary** of  $A$ : The set of all boundary points of  $A$ , denoted  $\partial A$

## 2 Convergence

### Definition 15: Convergent Sequence

Let  $(X, d)$  be a metric space,  $(x_n)_{n=1}^\infty$  be a sequence in  $X$ , and  $x \in X$ . We say that  $(x_n)_{n=1}^\infty$  converges to  $x$  iff for every  $\epsilon > 0$ , there exists an index  $N$  s.t. for all  $n \geq N$  we have  $d(x_n, x) < \epsilon$ .

Observe that:

- $d(x_n, x) < \epsilon$  is equivalent to  $x_n \in B(x, \epsilon)$ .
- $x_n \rightarrow x$  in  $(X, d)$  iff  $d(x_n, x) \rightarrow 0$  on the real line

### Theorem 16: Uniqueness of metric limit

- Let  $(X, d)$  be a metric space, and  $x, x' \in X$ ,  $x \neq x'$ . Then there exists a positive radius  $r$  s.t.  $B(x, r) \cap B(x', r) = \emptyset$
- A sequence in a metric space can have at most one limit

### Definition 19: Bounded Sequence

A sequence in a metric space is said to be **bounded** iff there exists an open ball that contains all of its terms

**Note:** this is the same definition as “sequence is bounded if there is upper and lower bound”, as open ball implies the same thing

**Thm 20:** Every convergent sequence is bounded

### Definition 21: Cauchy Sequence

A sequence  $(x_n)_{n=1}^\infty$  in a metric space  $(X, d)$  is said to be a **Cauchy sequence** iff for every positive  $\epsilon$ , there exists an index  $N$ , s.t. for all indices  $n, m$  with  $n, m \geq N$ ,

$$d(x_n, x_m) < \epsilon$$

**Thm 22:** If a sequence in a metric space converges, then it is a Cauchy sequence. **Note:** the converse is not true

### Definition 24: Complete Metric Spaces

A metric space is said to be **complete** if and only if every Cauchy Sequence is convergent

### Example 25: Examples of Complete Metric Spaces

- $\mathbb{R}$  with the standard metric is complete
- $\mathbb{Q}$  with the standard metric is not complete
- $(0, 1)$  with the standard metric is not complete
- $[0, 1]$  with the standard metric is complete
- $\mathbb{R}^n$ ,  $\ell^p$ ,  $C([a, b])$  is complete (proof later)

### Definition 26: Open Sets and Closed Sets

Let  $(X, d)$  be a metric space.

- A subset  $G$  of  $X$  is said to be **open** iff for every point  $x$  in  $G$  there exists a positive radius  $r$  such that  $B(x, r) \subseteq G$ .
- A subset  $F$  of  $X$  is said to be **closed** iff  $F^c$  is open

### Definition 31: Discrete Spaces and Clopens

A metric space is called **discrete** iff all its subsets are open (equiv. all subsets are closed)

**Example:**  $[0, 1] \cap (2, 3)$

**Def 33:** A set that is both open and closed is called **clopen**

### Theorem 34: Properties of open and closed sets

Let  $(X, d)$  be a metric space

- The union of **any family** of open sets is an open set
- The intersection of **finitely many** open sets is an open set
- The intersection of **any family** of closed sets is an closed set
- The union of **finitely many** closed sets is an closed set

### Remark 35: Infinite open sets

The intersection of infinitely many open sets isn't always an open set e.g., let  $G_n = (-\frac{1}{n}, \frac{1}{n})$ ,  $n = 1, 2, \dots$  on  $\mathbb{R}$  with the standard metric. Each  $G_n$  is open but

$$\bigcap_{n=1}^{\infty} G_n = \{0\}$$

is not open

### Remark 38 (Infinite Closed Sets)

The union of infinitely many closed sets is not always a closed set. For example, let  $F_n = [\frac{1}{n}, 1]$ ,  $n = 1, 2, \dots$ , on the real line with the standard metric. Each  $F_n$  is closed but

$$\bigcup_{n=1}^{\infty} F_n = (0, 1]$$

is not closed.

### Theorem 18: Relatively open sets

Let  $(X, d)$  be a metric space and  $A$  a nonempty subset ocomplementf  $X$  equipped with the induced metric  $d_A$ . Let  $G \subseteq A$ . Then  $G$  is open in  $(A, d_A)$  iff there exists a subset  $O$  of  $X$ , open in  $(X, d)$ , s.t.  $G = A \cap O$

The open sets of  $(A, d_A)$  are referred to as **relatively open**

### Theorem 36

Let  $(X, d)$  be a metric space,  $(x_n)_{n=1}^\infty$  be a sequence in  $X$  and  $x$  be a point in  $X$ .

$x_n \rightarrow x$  iff every open set that contains  $x$  contains eventually all terms of the sequence

### Definition H: Neighbourhoods of points

An **open neighbourhood** of a point  $x$  is any open set that has  $x$ .  $x_n \rightarrow x$  iff every open neighbourhood of  $x$  contains eventually all terms of the sequence.

A **neighbourhood** of a point  $x$  is a set that contains an open neighbourhood of  $x$ .

$x_n \rightarrow x$  iff every neighbourhood of  $x$  contains eventually all terms of the sequence.

#### Theorem 41

A subset  $F$  of a metric space is closed iff the limit of every convergent sequence of elements of  $F$  belongs to  $F$

- In any metric space  $(X, d)$ , singletons  $F = \{x\}$  are closed.
- In any metric space, any finite set is closed because

$$\{x_1, \dots, x_n\} = \{x_1\} \cup \dots \cup \{x_n\}$$

#### Definition 43: Closure

Let  $(X, d)$  be a metric space and  $A \subseteq X$ . The **closure** of  $A$ , denoted by  $\bar{A}$ , is the smallest closed subset of  $X$  that contains  $A$

There exists at least one closed subset of  $X$  that contains  $A$ , namely  $X$  itself. The smallest closed subset of  $X$  that contains  $A$  is

$$\bigcap_{\substack{A \subseteq F \subseteq X \\ F \text{ closed}}} F$$

#### Theorem 44: Properties of Closure

Let  $(X, d)$  be a metric space and  $A, B \subseteq X$ .

1.  $\bar{\emptyset} = \emptyset$  and  $\bar{X} = X$
2.  $A \subseteq \bar{A}$  and  $\bar{A}$  is closed
3.  $A$  is closed iff  $A = \bar{A}$
4.  $\overline{\bar{A}} = \bar{A}$
5. If  $A \subseteq B$ , then  $\bar{A} \subseteq \bar{B}$
6.  $\overline{A \cup B} = \bar{A} \cup \bar{B}$

#### Definition 49: Dense Subset of a Metric Space

Let  $(X, d)$  be a metric space. A subset  $D \subseteq X$  is **dense** iff  $\bar{D} = X$

**Random Fact:** In  $\mathbb{R}^n$  with the Euclidean metric  $d_2$ ,  $\mathbb{Q}^n$  is dense.

#### Theorem 50: Adherent Points

Let  $(X, d)$  be a metric space,  $A \subseteq X, x \in X$ . The following are equiv.

1.  $x \in \bar{A}$
2. For every positive  $r$ ,  $B(x, r) \cap A \neq \emptyset$
3. There exists a sequence  $(a_n)_{n \in \mathbb{N}}$  with  $a_n \in A$  for all  $n$ , such that  $a_n \rightarrow x$

A point  $x$  with any of these properties is called an **adherent point** of  $A$ . So,  $\bar{A}$  is the set of all adherent points of  $A$ .

#### Definition 52: Limit points of sets

Let  $(X, d)$  be a metric space,  $A \subseteq X$  and  $x \in X$ . We say that  $x$  is a **limit point** or an **accumulation point** of  $A$  iff every open ball centered at  $x$  contains an element of  $A$  distinct from  $x$ , i.e.

$$\forall r > 0 \quad (B(x, r) \setminus \{x\}) \cap A \neq \emptyset$$

The set of all limit points of  $A$  is called the **derived set** of  $A$  and is denoted by  $A'$  or  $\bar{A}$ .

**Thm 78:** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces,  $x_0$  be a limit point of  $X$ ,  $y_0 \in Y$  and  $f : X \rightarrow Y$  be a function.

We say that  $\lim_{x \rightarrow x_0} f(x) = y_0$  iff for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $x \in B_X(x_0, \delta) \setminus \{x_0\}$  we have

$$f(x) \in B_Y(y_0, \epsilon)$$

#### Definition 54: Continuity at a point

Let  $(X, d_X), (Y, d_Y)$  be metric spaces and  $f : X \rightarrow Y$  be a function. We say that  $f$  is **continuous at a point**  $x_0$  in  $X$  iff...

- for every  $\epsilon > 0$ , there exists a  $\delta > 0$ , such that, for all  $x \in X$  with  $d_X(x, x_0) < \delta$  we have

$$d_Y(f(x), f(x_0)) < \epsilon$$

- for every  $\epsilon > 0$ , there exists a  $\delta > 0$ , such that, for all  $x \in B_X(x_0, \delta)$  we have

$$f(x) \in B_Y(f(x_0), \epsilon)$$

- **Thm 57:** for every open nbhd  $G$  of  $f(x_0)$ , there exists an open nbhd  $O$  of  $x_0$  such that, for all  $x \in O$ , we have  $f(x) \in G$

#### Def 55 (Continuity of a Function)

Let  $(X, d_X), (Y, d_Y)$  be metric spaces. A function  $f : X \rightarrow Y$  is said to be **continuous** iff it is continuous at every point in  $X$

#### Theorem 58: Continuity and Convergence

Let  $(X, d_X), (Y, d_Y)$  be metric spaces,  $x_0$  be a point in  $X$ , and  $f : X \rightarrow Y$  be a function. The following are equivalent:

1.  $f$  is continuous at  $x_0$
2. For every sequence  $(x_n)_{n=1}^\infty$  in  $X$ , if  $x_n \xrightarrow{n \rightarrow +\infty}$  in  $(X, d_X)$ , then  $f(x_n) \xrightarrow{n \rightarrow +\infty} f(x_0)$  in  $(Y, d_Y)$

#### Theorem 59: Continuity for Open and Closed Sets

Let  $(X, d_X), (Y, d_Y)$  be metric spaces. A function  $f : X \rightarrow Y$  is continuous iff...

- the inverse image  $f^{-1}(G)$  of any open subset  $G$  of  $Y$  is an open subset of  $X$
- the inverse image  $f^{-1}(G)$  of any closed subset  $G$  of  $Y$  is a closed subset of  $X$

#### Definition 60: Topological Space

A **topological space** is a set  $X$  together with a family  $\mathcal{T}$  of subsets of  $X$  that has the following properties:

- $\emptyset, X \in \mathcal{T}$
- Any union of elements of  $\mathcal{T}$  is an element of  $\mathcal{T}$
- Any finite intersection of elements of  $\mathcal{T}$  is an element of  $\mathcal{T}$

$\mathcal{T}$  is called a **topology** and the elements of  $\mathcal{T}$  are called **open sets**

#### Definition 61: Continuity of Topological Spaces

- Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be two topological spaces. A function  $f : X \rightarrow Y$  is said to be **continuous** iff for every  $G$  in  $\mathcal{T}_Y$  the pre-image  $f^{-1}(G)$  is an element of  $\mathcal{T}_X$ .
- $f$  is said to be a **homeomorphism** iff it is a continuous bijection and its inverse is continuous.
- If such a homeomorphism exists then  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  are said to be **homeomorphic**

#### Theorem 66: $d : X \times X \rightarrow \mathbb{R}$ is continuous

Let  $(X, d)$  be a metric space.  $f : X \times X \rightarrow \mathbb{R}$  is continuous, where

- $\mathbb{R}$  is equipped with the standard metric.
- $X \times X$  is equipped with the product metric

#### Definition 67: Bounded Linear Operators

A linear operator  $T : X \rightarrow Y$  is said to be **bounded** iff there exists a positive constant  $C$  such that, for all  $x \in X$ ,

$$\|T(x)\|_Y \leq C\|x\|_X$$

**Thm 68:** Let  $T : X \rightarrow Y$  be a linear operator. The following are equivalent:

1.  $T$  is continuous
2.  $T$  is continu. at 0
3.  $T$  is bounded

#### Definition 70: Lipschitz Functions

Let  $(X, d_X), (Y, d_Y)$  be metric spaces. A function  $f : X \rightarrow Y$  is said to be a **Lipschitz** function iff there exists a constant  $L$  such that for all  $x, x' \in X$ ,

$$d_Y(f(x), f(x')) \leq L d_X(x, x')$$

If  $L < 1$ ,  $f$  is said to be a **contraction**

If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz function and  $x$  is any point in  $\mathbb{R}$ , then for any  $x \in \mathbb{R}$  we have

$$|f(x) - f(x')| \leq L|x - x'|$$

For  $x \geq x'$  this can be expanded to

$$f(x') - L(x - x') \leq f(x) \leq f(x') + L(x - x')$$

#### Lipschitz Theorem Bank

**71:** Every Lipschitz function is continuous

**175:** Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces, and  $f : X \rightarrow Y$  be a Lipschitz function. Then there exists a smallest Lipschitz constant of  $f$

**176:** Let  $I$  be a non-degenerate open interval on the real line and let  $f : I \rightarrow \mathbb{R}$  be a differentiable function. Then  $f$  is Lipschitz iff  $f'$  is bounded. When that is the case,

$$|f|_{\text{Lip}} = \sup\{|f'(x)| : x \in I\}$$

#### Definition 72: Fixed Points

A **fixed point** of a function  $f : S \rightarrow S$  where  $S$  is a non-empty set, is any element  $x$  of  $S$  such that  $f(x) = x$

Solving equations can sometimes be reduced to finding fixed points

#### Theorem 75: Banach's Fixed Point Theorem

Let  $(X, d)$  be a complete metric space and let  $f : X \rightarrow X$  be a contraction. Then  $f$  has a unique fixed point

#### Definition 76: Equivalent Metrics

Two metrics on the same non-empty set  $X$  are said to be **equivalent** iff they have the same open sets

**Thm 77:** Let  $d_1$  and  $d_2$  be metrics on the same non-empty set  $X$ . If there exist positive constants  $C$  and  $C'$  such that for all  $x, y$  in  $X$ ,

$$C d_1(x, y) \leq d_2(x, y) \leq C' d_1(x, y)$$

then  $d_1$  and  $d_2$  are equivalent

3 Completeness

Theorem I: Completeness of the Classical Spaces

Some examples of complete metric spaces:

<b>79:</b> $(\mathbb{R}^n, d_2)$	<b>80:</b> $\ell^2$	<b>81:</b> $\ell^p$	<b>82:</b> $C([a, b])$	<b>83:</b> $\ell^\infty$
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Exercise 31

- Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces and assume that  $(Y, d_Y)$  is complete.
- Let  $C(X, Y)$  be the set of all continuous and bounded functions from  $X$  to  $Y$ . For  $f, g \in C(X, Y)$  define
$$D(f, g) = \sup\{d_Y(f(x), g(x)) : x \in X\}$$
- Then  $D$  is a metric and the metric space  $(C(X, Y), D)$  is complete

Definition 83: The product space  $X^\mathbb{N}$

Let  $(X, d)$  be a metric space and  $n \in \mathbb{N}$ . Define  $D : X^\mathbb{N} \rightarrow \mathbb{R}$  by
$$D(x_1, x_2) = d(x_{11}, x_{21}) + d(x_{12}, x_{22}) + \dots + d(x_{1n}, x_{2n})$$

Lemma Bank

- Ex.33:**  $D$  is a metric and a sequence converges in  $(X^\mathbb{N}, D)$  iff it converges componentwise
- Ex.34:** If  $(X, d)$  is complete then  $(X^\mathbb{N}, D)$  is complete

Definition 84: The product space  $X^\mathbb{N}$

Let  $B^A$ , where  $A, B$  are sets, be the set of all functions from  $A$  to  $B$

**Def 85:** Let  $(X, d)$  be a metric space. Define a metric  $D : X^\mathbb{N} \times X^\mathbb{N} \rightarrow \mathbb{R}$  by

$$D(x_1, x_2) = \sum_{n=1}^\infty \frac{1}{2^n} \frac{d(x_{1n}, x_{2n})}{1 + d(x_{1n}, x_{2n})}$$

- $x_1 = (x_{11}, \dots, x_{1n}, \dots)$ ,  $x_2 = (x_{21}, \dots, x_{2n}, \dots)$
- $(X^\mathbb{N}, D)$  is called a **product space**

Theorem J: Product space Convergence & Completeness

Thm 86 (Convergence)

Let  $(X, d)$  be a metric space, let  $(x_k)_{k=1}^\infty$  be a sequence in  $X^\mathbb{N}$  and let  $x \in X^\mathbb{N}$ . Write  $x_k = (x_{k1}, \dots, x_{kn}, \dots)$  and  $x = (l_1, \dots, l_n, \dots)$ .

Then,  $x_k \xrightarrow[k \rightarrow +\infty]{(X^\mathbb{N}, D)} x$  if and only if, for all  $n$ ,  $x_{kn} \xrightarrow[k \rightarrow +\infty]{(X, d)} l_n$

Thm 87 (Completeness)

Let  $(X, d)$  be a complete metric space. Then the product space  $(X^\mathbb{N}, D)$  is complete.

Theorem K: Completeness of  $\mathbb{R}$

- Thm (Least Upper Bound Principle):** Every non-empty bounded above subset of  $\mathbb{R}$  has a least upper bound
- Thm 88 (Monotone Convergence):** Every bounded monotone sequence of real numbers has a limit
- Thm/Ex. 36 ( $\epsilon$ -convergence):** Let  $A$  be a non-empty bounded subset of  $\mathbb{R}$  and let  $\epsilon$  be positive. If the distance between any two elements of  $A$  is  $< \epsilon$ , then

$$\sup(A) - \inf(A) \leq \epsilon$$

- Thm 89:** Every Cauchy sequence of real numbers is convergent

Definition L: Limit Superior and Inferior

Let  $(x_n)_{n=1}^\infty$  be a Cauchy sequence in  $\mathbb{R}$ . Then  $(x_n)_{n=1}^\infty$  is bounded. Define:

$$I_n = \inf\{x_n, x_{n+1}, \dots\} \quad S_n = \sup\{x_n, x_{n+1}, \dots\}$$

**Thm:**  $(S_n)_{n=1}^\infty$  and  $(I_n)_{n=1}^\infty$  are monotone and bounded

$$I_1 \leq I_n \leq S_n \leq S_1, \quad n = 1, 2, \dots$$

Therefore  $I_n \rightarrow I$  and  $S_n \rightarrow S$  for some reals  $I$  and  $S$ . Since  $S_n - I_n \rightarrow 0$  we have  $S = I$ . We also have  $x_n \rightarrow S = I$

Def 90: Limsup and Liminf

- The limit of the sequence  $(I_n)_{n=1}^\infty$  is called the **limit inferior** of  $(x_n)_{n=1}^\infty$  and is denoted by  $\liminf x_n$ 
$$\liminf x_n = \lim_{n \rightarrow +\infty} I_n = \lim_{n \rightarrow +\infty} \inf\{x_n, x_{n+1}, \dots\}$$
- The limit of the sequence  $(S_n)_{n=1}^\infty$  is called the **limit superior** of  $(x_n)_{n=1}^\infty$  and is denoted by  $\limsup x_n$ 
$$\limsup x_n = \lim_{n \rightarrow +\infty} S_n = \lim_{n \rightarrow +\infty} \sup\{x_n, x_{n+1}, \dots\}$$

- $\liminf x_n$  is the smallest subsequential limit of  $(x_n)_{n=1}^\infty$
- $\limsup x_n$  is the largest subsequential limit of  $(x_n)_{n=1}^\infty$
- $(x_n)_{n=1}^\infty$  converges iff  $\liminf x_n = \limsup x_n$

4 Compactness

Definition 96: Open Covers and Subcovers

An **open cover** of a set  $S$  in a metric space is a family  $(G_i)_{i \in I}$  of open sets such that  $S \subseteq \bigcup_{i \in I} G_i$ . A **subcover** of an open cover

$(G_i)_{i \in I}$  is a sub-family  $(G_i)_{i \in I'}$  where  $I' \subset I$ , such that  $S \subseteq \bigcup_{i \in I'} G_i$

Definition M: Compacting Compactness

Def 102 (Compactness and Sequential Compactness)

Let  $(X, d)$  be a metric space and  $K \subseteq X$

- $K$  is **compact** iff every open cover of  $K$  has a finite subcover
- $K$  is **sequentially compact** iff every sequence in  $K$  has a subsequence that converges to an element of  $K$   
For the case  $K = X$ ,  $X$  compact iff every sequence in  $X$  has a convergent subsequence

Def 111 (Uniform Continuity)

- Let  $(X, d_X), (Y, d_Y)$  be metric spaces. A function  $f : X \rightarrow Y$  is said to be **uniformly continuous** iff for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that, for all  $x, x' \in X$  with  $d_X(x, x') < \delta$  we have

$$d_Y(f(x), f(x')) < \epsilon$$

Def 117 (Totally bounded Spaces)

- 117:** A metric space  $(X, d)$  is said to be **totally bounded** iff for every positive  $\delta$ ,  $X$  can be covered by a finite number of open balls of radius  $\delta$ .
- 118:** If  $(X, d)$  is totally bounded then it is bounded, but the converse is not necessarily true

Example N: Examples of compactness

Compact sets

- $[a, b]$  is compact
- $\emptyset$  is compact
- $\mathbb{R} \cup \{-\infty, +\infty\}$  is compact!

Not Compact sets

- $(0, 1)$  is not compact
- $\mathbb{R}$  is not compact

Theorem 116: Lebesgue's Lemma

Let  $(X, d)$  be a sequentially compact metric space and  $X = \bigcup_{i \in I} G_i$  be an open cover of  $X$ . There exists a  $\delta > 0$  such that for any two points  $x, y \in X$  with  $d(x, y) < \delta$  there exists an  $i$  such that  $x, y \in G_i$ . Any such  $\delta$  is called a **Lebesgue number** of the open cover

**Ex.44:** Let  $(X, d)$  be a sequentially compact m.s. and  $X = \bigcup_{i \in I} G_i$  be an open cover of  $X$ . Then there exists a  $\delta > 0$  s.t. any nonempty subset of  $X$  of diameter  $< \delta$  can be covered by a single  $G_i$

Theorem O: big theorem bank of obvious shit

Regular Compactness

- For a set  $K$  in  $\mathbb{R}$  with the standard metric, or  $\mathbb{R}^n$  with the Euclidean metric:
  - 93:**  $K$  is compact  $\iff K$  is closed and bounded
- 99:** Every open cover of the interval  $[a, b]$ , where  $a, b \in \mathbb{R}$ ,  $a \leq b$  has a finite subcover

Continuous Functions

- Let  $K \subseteq \mathbb{R}$  be compact, and  $f : K \rightarrow \mathbb{R}$  continuous:
  - 94:**  $f$  is bounded
  - 95:**  $f$  has a maximum and minimum (EVT)
- Let  $(X, d)$  be a metric space,  $K$  be a sequentially compact subset of  $X$  and  $f : K \rightarrow \mathbb{R}$  be a continuous function:
  - 110:**  $f$  has a maximum and a minimum. In particular,  $f$  is bounded. (EVT ..again)

Sequential compactness stuff

Let  $(X, d)$  be a metric space, and  $K \subseteq X$ :

- Let  $K \neq \emptyset$ , and let  $d_K$  be the induced metric on  $K$ .
  - Ex.39:**  $K$  (seq.) compact  $\iff$  the M.S.  $(K, d_K)$  is (seq.) compact
- 105:**  $K$  sequentially compact  $\implies K$  is closed and bounded
- 107:**  $(X, d)$  and  $K$  are both sequentially compact  $\iff K$  is closed
- 108:**  $(X, d)$  is sequentially compact  $\implies (X, d)$  is complete
- 115:**  $(X, d)$  is compact  $\implies (X, d)$  is sequentially compact
- x43:**  $(X, d)$  is compact, and let  $A$  be an infinite subset of  $X \implies A$  has at least one limit point

Thm 114 (Uniform Continuity)

Let  $(X, d_X)$  be a sequentially compact metric space,  $(Y, d_Y)$  be a metric space and  $f : X \rightarrow Y$  be a continuous function. Then  $f$  is uniformly continuous

Totally Bounded Spaces

Let  $(X, d)$  be a metric space:

- 120:**  $(X, d)$  is sequentially compact  $\implies (X, d)$  is totally bounded
- 122:**  $(X, d)$  is compact  $\iff (X, d)$  complete and totally bounded
- 121:** Every sequentially compact metric space is compact.



### Definition 123: Countable and Uncountable Sets

A set  $S$  is said to be:

- **Infinitely countable** iff there is a bijection  $f : \mathbb{N} \rightarrow S$
- **Countable** if it is finite or infinitely countable
- **Uncountable** iff it isn't countable

#### Example 124

- $\{1, 2, 3\}$  and  $\mathbb{R}$  are countable sets
- $\mathbb{Q}$  is infinitely countable
- $\mathbb{R}$  is uncountable

### Theorem or rather Ex 45: Dense Subset equivalence

Let  $(X, d)$  be a metric space,  $D \subseteq X$ . The following are equivalent:

1.  $D$  is dense
2. For every  $x \in X$  and  $\epsilon > 0$  there exists  $y \in D$  s.t.  $d(x, y) < \epsilon$
3. For every  $x \in X$  there is a sequence  $(y_n)_{n=1}^{\infty}$  of elements of  $D$  s.t.  $y_n \rightarrow x$
4. For every element  $x \in X$  and every open nbhd  $G$  of  $x$ ,  $G \cap D \neq \emptyset$
5.  $D$  intersects every non-empty open set

### Definition 125: Separable spaces

A metric space is **separable** iff it has a countable dense subset

#### Examples

- $\mathbb{R}$  with the standard metric is a separable metric because  $\mathbb{Q}$  is dense and countable
- $\mathbb{R}^n$  with the Euclidean metric is a separable metric space because  $\mathbb{Q}^n$  is dense and countable
- $\mathbb{C}$  with its standard metric is a separable metric space because  $\{z \in \mathbb{C} : \text{Re}(z), \text{Im}(z) \in \mathbb{Q}\}$
- $\ell^2$  is separable, and  $\ell^p$  is separable for  $1 \leq p < \infty$

### Theorem P: Polynomials

#### Thm 130 (Weierstrass Approximation Theorem)

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function and  $\epsilon > 0$ . There exists a polynomial  $p$  with *real* coefficients s.t. for all  $x \in [a, b]$

$$|f(x) - p(x)| < \epsilon$$

#### Thm 131 (literally same thing but with $\mathbb{Q}$ )

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function and  $\epsilon > 0$ . There exists a polynomial  $p$  with *rational* coefficients s.t. for all  $x \in [a, b]$

$$|f(x) - p(x)| < \epsilon$$

#### More Theorems

- **Ex 47:** The set of all polynomials (of one variable and any degree) with rational coefficients is countable
- **Thm 132:**  $C([a, b])$  is separable

### Theorem 133: Separability of subspaces

Let  $(X, d)$  be a separable metric space,  $A \subseteq X$ ,  $A \neq \emptyset$ , and  $d_A$  be the induced metric on  $A$ . Then the metric space  $(A, d_A)$  is separable

**Thm 135:** Every compact metric space is separable (compact  $\implies$  separable)

### Theorem 136: Open Ball countability

Let  $(X, d)$  be a separable metric space and let  $D$  be a countable dense subset of  $X$ . Let

$$\mathcal{B} = \{B(c, r) : c \in D, r \in \mathbb{Q}^+\}$$

be the set of all open balls with centers in  $D$  and rational radii. Then  $\mathcal{B}$  is countable and every open set in  $X$  can be written as a union of elements of  $\mathcal{B}$

### Definition Q: Open Bases and Second Countability

#### Def 137 (Open Bases)

Let  $(X, \mathcal{T})$  be a topological space. An **open base** (or **base**) for the topology  $\mathcal{T}$ , is a family  $\mathcal{B}$  of open sets such that every open set in  $\mathcal{T}$  can be written as a union of elements of  $\mathcal{B}$

#### Def 139 (Second Countability)

A topological space  $(X, \mathcal{T})$  satisfies the **second Axiom of Countability**, or is **second countable** iff it has a countable open base

#### Other theorems

- **Thm 140:** In a separable metric space, every family of pairwise disjoint non-empty open sets is countable
- **Thm 141:** On the real line with the standard metric, every open set can be written as a countable union of disjoint open intervals

### Theorem 142: Continuous Extensions

Let  $(X, d_X), (Y, d_Y)$  be metric spaces,  $D$  be a dense subset of  $X$ ,  $f, g : X \rightarrow Y$  continuous functions s.t.  $f(x) = g(x)$  for all  $x \in D$ . Then  $f = g$

**Thm 143:** Let  $(X, d_X), (Y, d_Y)$  be metric spaces,  $D \subseteq X$  be dense,  $f : D \rightarrow Y$  be uniformly continuous, and assume that  $(Y, d_Y)$  is complete. Then  $f$  has a unique continuous extension  $F : X \rightarrow Y$

### Theorem R: Properties of Complete Metric Spaces

- **144:** Let  $(X, d)$  be a metric space,  $F$  be a nonempty subset of  $X$  and  $d_F$  be the induced metric on  $F$ . If the metric space  $(F, d_F)$  is complete then  $F$  is a closed subset of  $X$
- **145:** Let  $(X, d)$  be a complete metric space,  $F$  be a nonempty subset of  $X$ , and  $d_F$  be the induced metric on  $F$ . If  $F$  is a closed subset of  $X$ , then the metric space  $(F, d_F)$  is complete
- **146:** Let  $(X, d)$  be a complete metric space,  $A \subseteq X$ ,  $A \neq \emptyset$ . Then
  1. The metric space  $(\bar{A}, d_{\bar{A}})$  is complete
  2. If  $A \subseteq B \subseteq X$  and  $(B, d_B)$  is complete, then  $\bar{A} \subseteq B$

### Definition 147: Isometries

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. A function  $f : X \rightarrow Y$  is called a **isometry** iff for all  $x_1, x_2 \in X$ ,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

The metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  are said to be **isometric** iff there exists an isometry  $f$  from  $X$  onto  $Y$

#### Isometry Theorems

- **Thm 148:** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and  $f : X \rightarrow Y$  be an isometry. Then  $f$  is an injection. If, moreover,  $f$  is a surjection (hence  $f$  bij.) then  $f^{-1} : Y \rightarrow X$  is also an isometry
- **Fun Fact:** if two metric spaces are isometric and one of them is complete/compact/connected/... then so is the other

### Theorem 150: Isometry completion

Let  $(X, d)$  be a bounded metric space and let  $C(X, \mathbb{R})$  be the set of all bounded continuous functions  $f : X \rightarrow \mathbb{R}$  equipped with the metric

$$d_{\infty}(f_1, f_2) = \sup\{|f_1(t) - f_2(t)| : t \in X\}$$

For each  $x \in X$ , define  $F_X : X \rightarrow \mathbb{R}$  be  $F_X(x') = d(x, x')$ . Then

1.  $F_X \in C(X, \mathbb{R})$
2. The map  $X \rightarrow C(X, \mathbb{R}), x \mapsto F_X$  is an isometry
3.  $X^* = \{F_X : x \in X\}$ , equipped with the induced metric, is a subspace of  $C(X, \mathbb{R})$  isometric to  $X$
4. The closure  $\overline{X^*}$  of  $X^*$  in  $C(X, \mathbb{R})$ , equipped with the induced metric, is a complete metric space
5.  $X^*$  is dense in  $\overline{X^*}$

### Definition 152: Completion of a Metric Space

Let  $(X, d)$  be a metric space. A **completion** of  $(X, d_X)$  is any metric space  $(Y, d_Y)$  with the following properties

1.  $(Y, d_Y)$  is complete
2.  $(Y, d_Y)$  has a subspace  $X^*$  isometric to  $(X, d_X)$
3.  $X^*$  is dense in  $Y$

It can be shown that any two completions of  $X$  are isometric to each other, i.e. a completion is unique up to isometries

### Definition S: Construction of Completion via Cauchy

Let  $(X, d)$  be a metric space and let  $\mathcal{C}$  be the set of all Cauchy sequences of elements of  $X$

We define an equivalence relation  $\sim$  in  $\mathcal{C}$  as follows: Let  $x = (x_n)_{n \in \mathbb{N}}, y = (y_n)_{n \in \mathbb{N}} \in \mathcal{C}$ . We say that  $x \sim y$  iff  $d(x_n, y_n) \rightarrow 0$

Distinct equivalence classes are disjoint and partition  $\mathcal{C}$

The set of all equivalence classes is called the **quotient space**, denoted  $\mathcal{C}/\sim$

Define a metric  $D$  on  $\mathcal{C}/\sim$  as follows:

Let  $\alpha, \beta \in \mathcal{C}/\sim$ . Then

$$\alpha = [(x_1, \dots, x_n, \dots)] \text{ and } \beta = [(y_1, \dots, y_n, \dots)]$$

for some  $(x_1, \dots, x_n, \dots), (y_1, \dots, y_n, \dots) \in \mathcal{C}$ . Define

$$D(\alpha, \beta) = \lim_{n \rightarrow +\infty} d(x_n, y_n)$$

$(\mathcal{C}/\sim, D)$  is complete. Additionally, the following is an isometry:

$$X \rightarrow \mathcal{C}/\sim \quad x \mapsto [(x, x, \dots, x, \dots)]$$

Let  $X^*$  be its range. The metric space  $(X^*, D_{X^*})$  is isometric to  $(X, d)$ ,  $(\overline{X^*}, D_{\overline{X^*}})$  is a complete metric space, and  $X^*$  is dense in  $\overline{X^*}$

### Definition 153: Connected and Disconnected Spaces

A metric space  $(X, d)$  is said to be **disconnected** iff there exists non-empty disjoint open sets  $G_1$  and  $G_2$  such that

$$X = G_1 \cup G_2$$

Otherwise the metric space is called **connected**

A non-empty subset  $A$  of a metric space  $(X, d)$  is said to be **disconnected** iff the metric space  $(A, d_A)$ , where  $d_A$  is the induced metric, is disconnected

### Theorem T: Connected Theorems

A subset  $O$  of  $A$  is open in  $(A, d_A)$  iff  $O = A \cup G$  for some  $G$  that is open in  $X$ . Therefore,  $A$  is disconnected iff there exist open subsets  $G_1, G_2$  of  $X$  s.t.

- $A = (A \cap G_1) \cup (A \cap G_2)$ , which is equivalent to  $A \subseteq G_1 \cup G_2$
- $A \cap G_1 \neq \emptyset, A \cap G_2 \neq \emptyset$
- $(A \cap G_1) \cap (A \cap G_2) = \emptyset$ , which is equivalent to  $A \cap G_1 \cap G_2 = \emptyset$

#### Connected Theorems

- **Thm 154:**  $\mathbb{R}$  with the standard metric is connected
- **Ex.53:** On the real line with the standard metric, all intervals are connected sets
- **Thm 155:** A non-empty subset of the real line is connected iff it is an interval
- **Thm 157:** A metric space  $(X, d)$  is connected iff the only subsets of  $X$  with empty boundary are  $\emptyset$  and  $X$
- **Thm 158:** Let  $(X, d_X)$  be a connected metric space,  $(Y, d_Y)$  be a metric space and  $f : X \rightarrow Y$  be a continuous surjection. Then  $(Y, d_Y)$  is connected as well
- **Thm 160:** A metric space  $(X, d)$  is connected iff the only clopen subsets are  $\emptyset, X$

### Theorem 159: Intermediate Value Theorem

Let  $(X, d)$  be a connected metric space and  $f : X \rightarrow \mathbb{R}$  be a continuous function. If  $x_1, x_2 \in X$  with  $f(x_1) \neq f(x_2)$  and  $y$  is a real number between  $f(x_1)$  and  $f(x_2)$ , then there exists an  $x \in X$  such that  $f(x) = y$

### Definition U: Connected Components

Let  $(X, d)$  be a metric space. We define an equivalence relation  $\sim$  in  $X$  as follows:  $x \sim x'$  iff there exists a connected subset  $C$  of  $X$  that contains both  $x$  and  $x'$

**Ex.55:** If  $(C_i)_{i \in I}$  is a family of connected subsets of  $X$  with nonempty intersection, then  $\bigcup_{i \in I} C_i$  is connected

### Theorem 161: Big equivalence classes

The equivalence class of any point in  $X$  is the largest connected subset of  $X$  that contains that point (what point?)

### Definition 162: Path Connected Metric Spaces

Let  $(X, d)$  be a metric space and  $x_0, x_1 \in X$ .

- A **path** in  $X$  from  $x_0$  to  $x_1$  is a continuous function  $\gamma : [0, 1] \rightarrow X$  s.t.  $\gamma(0) = x_0, \gamma(1) = x_1$
- $(X, d)$  is **path-connected** iff for any two points  $x_0, x_1$  in  $X$  there is a path in  $X$  from  $x_0$  to  $x_1$
- A non-empty subset  $A$  of  $X$  is **path-connected** iff the metric space  $(A, d_A)$ , where  $d_A$  is the induced metric, is path connected

#### Thm 163 (Path Connected Theorem)

- Every path-connected metric space is connected
- Not every connected metric space is necessarily path-connected

## 5 Applications

**Newton's Method:** We wish to 'solve'  $f(x) = 0$ , where  $f : \mathbb{R} \rightarrow \mathbb{R}$  smooth. Assume  $f'(x) \neq 0 \forall x \in \mathbb{R}$  and  $f(x^*) = 0$  for some real  $x^*$ . Then  $x^*$  is unique. Define a sequence  $(x_n)_{n \in \mathbb{N}}$  as follows. Let  $x_0$  be any real number and set

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots$$

This is of the form  $x_{n+1} = F(x_n)$  where  $F(x) = x - \frac{f(x)}{f'(x)}$

Observe that  $f(x^*) = 0$  is equivalent to  $F(x^*) = x^*$

$$\text{The derivative of } F \text{ is } F'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

$$\text{Observe that } F'(x^*) = \frac{f(x^*)f''(x^*)}{f'(x^*)^2} = 0$$

Therefore there exists a  $\delta > 0$  s.t. for all  $x \in [x^* - \delta, x^* + \delta]$ ,  $|F'(x)| \leq \frac{1}{2}$   
 $F$  maps  $[x^* - \delta, x^* + \delta]$  into  $[x^* - \delta, x^* + \delta]$ . If  $x \in [x^* - \delta, x^* + \delta]$  then

$$|F(x) - x^*| = |F(x) - F(x^*)| \leq \frac{1}{2}|x - x^*| \leq \frac{1}{2}\delta \leq \delta$$

- $F : [x^* - \delta, x^* + \delta] \rightarrow [x^* - \delta, x^* + \delta]$  is a contraction
- $[x^* - \delta, x^* + \delta]$  is a complete metric space
- If  $x_0 \in [x^* - \delta, x^* + \delta]$  and  $x_{n+1} = F(x_n)$ ,  $n = 1, 2, \dots$  then all  $x_n$  are in  $[x^* - \delta, x^* + \delta]$
- The sequence  $(x_n)_{n \in \mathbb{N}}$  converges to the fixed point of  $F$  in  $[x^* - \delta, x^* + \delta]$ , namely  $x^*$

**Order of Convergence:** By Taylor's Formula,

$$\begin{aligned} F(x) &= F(x^*) + F'(x^*)(x - x^*) + \frac{1}{2}F''(c)(x - x^*)^2 \\ &= x^* + \frac{1}{2}F''(c)(x - x^*)^2 \end{aligned}$$

Therefore,  $|F(x) - x^*| = \frac{1}{2}|F''(c)||x - x^*|^2 \leq C|x - x^*|^2$ . Replace  $x$  by  $x_n$ ,

$$|x_{n+1} - x^*| \leq C|x_n - x^*|^2$$

If the  $n$ -th error is  $|x_n - x^*| = \frac{1}{100}$  then the  $(n+1)$ -th error is

$$|x_{n+1} - x^*| \leq C|x_n - x^*|^2 \leq \frac{C}{10000}$$

**Heron's Method:** Let  $a > 0$ . We wish to approximate  $\sqrt{a}$  which is one of the two roots of  $f(x) = x^2 - a$ . Let

$$F : (0, \infty) \rightarrow \mathbb{R}, F(x) = x - \frac{f(x)}{f'(x)} = \frac{1}{2} \left( x + \frac{a}{x} \right)$$

$$\text{Properties: } F(x) \geq f(\sqrt{a}) = \sqrt{a}, F'(x) = \frac{1}{2} \left( 1 - \frac{a}{x^2} \right) \leq \frac{1}{2}$$

So,  $F : [\sqrt{a}, +\infty) \rightarrow [\sqrt{a}, +\infty)$  is a contraction. The metric space  $[\sqrt{a}, +\infty)$  is complete. Fix  $x_0$  with  $x_0 \geq \sqrt{a}$  and define

$$x_{n+1} = F(x_n) = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right), \quad n = 0, 1, 2, \dots$$

then  $(x_n)_{n \in \mathbb{N}}$  converges to the unique fixed point of  $F$ , namely  $\sqrt{a}$

Start with  $0 < x_0 < \sqrt{a}$ , then  $x_1 = F(x_0) > \sqrt{a}$ . So,  $x_n \geq \sqrt{a}, n = 1, 2, \dots$

### Theorem V: Picard's Theorem

Let  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous and bounded function, and  $t_0, x_0$  be real numbers. Assume that there exists a positive constant  $L$  s.t. for all real  $t, x_1, x_2$  we have:

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$$

Then, there exists a positive  $\delta$  and a unique differentiable function  $x : [t_0 - \delta, t_0 + \delta] \rightarrow \mathbb{R}$  s.t. for all  $t \in [t_0 - \delta, t_0 + \delta]$ ,

$$x'(t) = f(t, x(t)) \quad \text{and} \quad x(t_0) = x_0$$

### Example X: Past Paper Questions

**Q:** Let  $(X, d_X), (Y, d_Y)$  be two metric spaces and  $f : X \rightarrow Y$  cont. Prove that for every subset of  $A$  of  $X$  we have  $f(\overline{X}) \subseteq \overline{f(A)}$

**A:**  $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$ . Moreover, the set  $\overline{f(A)}$  is closed in  $Y$ , and  $f$  cont. therefore  $f^{-1}(\overline{f(A)})$  is closed in  $X$ . It follows that  $\overline{A} \subseteq f^{-1}(\overline{f(A)})$ , therefore  $f(\overline{A}) \subseteq \overline{f(A)}$

**Q:** Give an example where  $f(\overline{A}) \neq \overline{f(A)}$ . Let  $X = (0, 1), Y = \mathbb{R}$  with std. meric, and  $A = (0, 1)$ . Define  $f : X \rightarrow Y$  by  $f(x) = x$ . Then  $\overline{A} = A$ , therefore  $f(\overline{A}) = f(A) = (0, 1)$ , while  $\overline{f(A)} = \overline{(0, 1)} = [0, 1]$

**Q:** Prove that if  $(X, d_X)$  compact, then we have equality in the eq

**A:** We know that  $f(\overline{A}) \subseteq \overline{f(A)}$ , so now prove  $\overline{f(A)} \subseteq f(\overline{A})$ .  $\overline{A}$  is closed in  $X$ , and  $X$  compact, therefore  $\overline{A}$  is compact in  $X$ .  $f$  cont. therefore  $f(\overline{A})$  is compact in  $Y$ , therefore  $\overline{f(A)}$  is closed in  $Y$ . Alos,  $f(A) \subseteq f(\overline{A})$ . It follows that  $f(\overline{A}) \subseteq \overline{f(A)}$

**Q:** Let  $(X, d)$  be a metric space,  $x \in X$  and  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$  be two seqs in  $X$ . If  $x_n \rightarrow x$  and  $d(x_n, y_n) \rightarrow 0$ , prove that  $y_n \rightarrow x$ .

**A:** By the triangle inequality,  $d(y_n, x) \leq d(y_n, x_n) + d(x_n, x) \forall n$ . We have  $d(x_n, x) \rightarrow 0$  coz  $x_n \rightarrow x$ , and we are assuming  $d(y_n, x_n) \rightarrow 0 \implies d(y_n, x_n) + d(x_n, x) \rightarrow 0 \implies d(y_n, x) \rightarrow 0 \implies y_n \rightarrow x$