Algebraic Topology Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Introduction

Recall 1.1.1: Topology

An **(open) topology** on X is a collection of subsets $\tau \subset P(X)$ such that

- $\emptyset \in \tau$ and $X \in \tau$
- τ is closed under finite intersections: If $\{U_1, \dots, U_n\} \subset \tau$ then
$$\bigcap_{i=1,\dots,n} U_i \in \tau$$
- τ is closed under arbitrary unions: If $\{U_1, \dots, U_n\} \subset \tau$ is a family of open subsets then
$$\bigcup_{i=1,\dots,n} U_i \in \tau$$

The subsets $U \in \mathcal{T}$ are called **open** and their complements in X define **closed subsets**.

Two examples of a topology on a set X are the following:

- The **Trivial Topology**: $\tau_{\text{triv}} = \{\emptyset, X\}$
- The **Discrete Topology**: $\tau_{\text{dis}} = P(X)$

A subset $A \subset X$ is **clopen if it is both closed and open**

Definition 1: Connected Spaces

A topological space X is **connected** if $X = A \cup B$ with $A,\, B \subset X$ open implies that $A = \emptyset$ or $A = X$.

Proposition 1: Connectedness and Clopens

A topological space X is *connected* iff the only clopens are $\emptyset$ and X.

Example 1: Examples of Connected Topologies

- Every X with the trivial topology is connected.
- Every X with the discrete topology is not connected unless $X = \emptyset$ or $X = \{*\}$ (in which it coincides with the trivial topology).
- The real line $\mathbb{R}$ with the standard topology is connected.

Proposition 2: Continuous Maps

Let $f : X \to Y$ be a continuous map of topological spaces and let X be connected. Then $f(X)$ is connected.

Proposition 3: Connected Equivalence Relation

For a topological space X, define $x \sim y$ if there exists some connected subset that contains both. The relation $x \sim y$ is an equivalence relation.

Definition 2: Connected Components

The equivalence classes of this relation are called **connected components**. In particular, a space X is connected iff it only has a single connected component.

Definition 3: Path

Let I denote the closed unit interval $[0, 1]$. A **path** in X is a continuous map $\alpha : I \to X$. The points $\alpha(0) \in X$ and $\alpha(1) \in X$ will be called **start** and **end** points respectively.
We define a path relation between points in X be declaring $x \sim y$ if there exists some path $\alpha : I \to X$ that starts at x and ends in y, i.e. $\alpha(0) = x$ and $\alpha(1) = y$. This is an equivalence relation from the following properties:

1. **Constant Path**: For all $x \in X$ there exists the constant path $c_x : I \to X$ defined by $c_x(t) = x$ for all $t \in I$
2. **Path reversal**: Let $\alpha : I \to X$ be a pth in X. Define its reversed path by
$$\overline{\alpha} : I \to X, \quad t \mapsto \alpha(1-t) \tag{1}$$
3. **Path Concatenation**: Let $\alpha,\, \beta : I \to X$ be two paths in X s.t. $\alpha(1) = \beta(0)$. Their concatenated path is defined by:
$$\alpha * \beta(t) := \begin{cases} \alpha(2t), & 0 \leq t \leq 1/2 \\ \beta(2t-1) & 1/2 \leq t \leq 1 \end{cases} \tag{2}$$

Definition 4: Path-Connected Components

The equivalence clases are called **path-connected components** and their set is denoted by $\pi_0(X)$. A space X is called **path-connected** if $\pi_0(X)$ is a one-point set, i.e. any two points x, y can be related by a path in X.

Remark 1: Random examples

The following statements are true:

- A homeomorphism $X \cong Y$ induces a bijection $\pi_0(X) \cong \pi_0(Y)$.
- If X is path-connected, it is also connected.
- The *topologist's sine curve* defined by $X = \{0\} \times [-1, 1] \times \{(x, \sin(1/x)) \mid 0 < x\}$ is connected but not path-connected.

Definition 5: Homotopy

A **homotopy** of maps $f, g : X \to Y$ is a continuous map $h : X \times I \to Y$ such that $h(-, 0) = f$ and $h(-, 1) = g$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & Y \arrow[Rightarrow, from=U, to=D, "h"] \end{tikzcd} \tag{3}$$

If such a homotopy exists, f is called **homotopic** to g. This defines an equivalence relation $f \simeq g$ on the space of maps $\mathrm{Map}(X, Y)$.

Example 2: Paths as Homotopies

Points in X are the same as maps $* \to X$ from the one-point set $*$ to X. A path $\alpha : I \to K$ corresponds to a homotopy $* \times I \to X$.

Remark 1.5: Composition of Homotopies

- **Horizontal Composition**: Let $h, h' : X \times I \to Y$ be two homotopies in X such that $h(-, 1) = h'(-, 0) : X \to Y$. Their concatenated homotopy is defined by
$$h * h'(-, t) := \begin{cases} h(-, 2t) & 0 \leq t \leq 1/2 \\ h'(-, 2t-1) & 1/2 \leq t \leq 1 \end{cases} \tag{5}$$
- **Vertical Composition**: Let $h : X \times I \to Y$ and $k : Y \times I \to Z$ be two homotopies on maps from X to Y, and Y to Z. Then
$$k \circ h := [X \times I \xrightarrow{\mathrm{id} \times \Delta} X \times I^2 \xrightarrow{h \times \mathrm{id}} Y \times I \xrightarrow{k} Z] \tag{6}$$
where $\Delta : I \to I^2$, $t \mapsto (t, t)$ is the diagonal map, or explicitly,
$$k \circ h(x, t) = k(h(x, t), t)$$

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"{name=A}] \arrow[r, ""{name=B}] \arrow[r, bend right=60, "l"'{name=C}] & Y \arrow[Rightarrow, from=A, to=B, "h"'] \arrow[Rightarrow, from=B, to=C, "h'"'] \end{tikzcd} \sim \begin{tikzcd} X \arrow[r, bend left=60, "f"{name=A}] \arrow[r, bend right=60, "l"'{name=C}] & Y \arrow[Rightarrow, from=A, to=C, "k' * h"] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, bend left, "f"{name=A}] \arrow[r, bend right, "g"'{name=B}] & Y \arrow[r, bend left, "f'"{name=C}] \arrow[r, bend right, "g'"'{name=D}] & Z \arrow[Rightarrow, from=A, to=B, "h"] \arrow[Rightarrow, from=C, to=D, "k"] \end{tikzcd} \sim \begin{tikzcd} X \arrow[r, bend left, "f' \circ f"{name=A}] \arrow[r, bend right, "g' \circ g"'{name=B}] & Z \arrow[Rightarrow, from=A, to=B, "k \circ h"] \end{tikzcd}$$

Lemma 1: Concatenation Relation

Let $f,\, f' : X \to Y$ and $g, g' : Y \to Z$ be maps such that $f \simeq f'$ and $g \simeq g'$. Then $f' \circ f \simeq g' \circ g$ as maps from X to Z. In particular, $g' \circ f \sim g \circ f$ and $g \circ f' \sim g \circ f$.

Definition 6: Homotopy Equivalence

A map $f : X \to Y$ is called a **homotopy equivalence** if there exists a map $g : Y \to X$ and homotopies $f \circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$. In other words, g satisfies the properties of an inverse up to homotopy. It is called a **homotopy inverse** of f.

Example 3: Circle to $\mathbb{R}^2$

The inclusion map $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2$ is not a homotopy equivalence, but the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2 \backslash \{0\}$ is a homotopy equivalence.

Proposition 4: Unique Inverses of Homotopy

Homotopy inverses are unique up to homotopy.

Definition 7: Homotopic Spaces

Two spaces X and Y are called **homotopy equivalent**, or **of the same homotopy type**, and denoted by $X \simeq Y$ if there exists a homotopy equivalence $f : X \to Y$.

Notation: We use $\cong$ for homeomorphisms and $\simeq$ for homotopy equivalence.

Lemma 2: Composition of Inverses

Let $f : X \to y$ and $g : Y \to Z$ with homotopy inverses $\overline{f} : Y \to X$ and $\overline{g} : Z \to Y$ respectively. Then $\overline{f} \circ \overline{g} : Z \to X$ is a homotopy inverse of $g \circ f : X \to Z$. In particular, $X \simeq Y$ and $Y \simeq Z$ implies $X \simeq Z$.

Definition 8: Contractible Space

A space X is called **contractible** if it is homotopy equivalent to a point, i.e. $X \simeq *$.

The **terminal map** is the unique map $X \to *$. Contractibility requires that there is a homotopy inverse of that map, i.e. a map $* \to x$ along with homotopies

$$h : [* \to X \to *] \simeq \mathrm{id}_*, \quad k : [X \to * \to X] \simeq \mathrm{id}_X \tag{7}$$

Example 4: Examples of Contractible Spaces

1. $\mathbb{R}^n$ is contractible. Let x_0 be a fixed point in $\mathbb{R}^n$ and define the (straight line) homotopy $h : c_{x_0} \simeq \mathrm{id}_{\mathbb{R}^n}$ by
$$h(x,t) = (1-t)x_0 + tx.$$
2. $\mathbb{S}^{n-1} \simeq \mathbb{R}^n \backslash \{0\}$. The inclusion $\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n \backslash \{0\}$ and the shrinking map
$$\mathbb{R}^n \backslash \{0\} \to \mathbb{S}^{n-1}, \quad x \mapsto \frac{x}{|x|}$$
are homotopy inverses.

Remark 3: Remarks about Contractible Spaces

1. Contractible spaces are path-connected. Let x_0 be the point where the space X contracts to. In particular, we are given with a homotopy $h : c_{x_0} \simeq \mathrm{id}_X$. For any $x \in X$, the map $h(x, -) : I \to X$ defines a path from x_0 to x and thus every element $x \in X$ is path-connected to x_0.
2. The converse does not hold, for example $X = \mathbb{S}^1$.
3. A contractible space X is contractible at any point x_0. Since X is path-connected, a path from x to x' defines a homotopy $c_x \simeq c_{x'}$.
4. Any two maps $f, g : X \to Y$ are homotopic if Y is contractible.

Definition 9: Retracts and Deformation Retracts

- A **retract** of X onto a subspace $A \subset X$ is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$. Equivalently, this is a map $r : X \to X$ such that $r^2 = r$ and $r(X) = A$.
- A **deformation retract** of X onto A is the additional datum of a homotopy $h : \mathrm{id}_X \simeq i \circ r$.

In other words, a deformation retract is a homotopy $h : X \times I \to X$ such that $h(x, 0) = x$ and $h(x, 1) \in A$ for all $x \in X$ and $h(a, 1) = a$ for all $a \in A$. Not all retracts can form deformation retracts. For instance, the retract X onto a point $\{x_0\}$ can be a deformation retract iff X is contractible.

This notion is called **weak** deformation retract. A **strong** deformation retract has the condition $h(a, t) = a$ for all $t \in I$, $a \in A$. i.e. Our notion of a (weak) deformation retract deforms X into A while allowing to deform A to do so, while a strong deformation retract deforms X into A while keeping A fixed at all times

Proposition 5: Deformation Retracts and Homotopies

A deformation retract of X onto A induces a homotopy equivalence $X \simeq A$.

Recall 2: Quotient Space

Let X be a topological space and let $\sim$ be an equivalence relation on X. Then, $X/\sim$ is equipped with the quotient topology and called a **quotient space**. If Z is a closed subset in X, then we can also define the quotient space X/Z.

Another form of quotient spaces: Let $f : Z \to Y$ be a continuous map between a closed subset $Z \subset X$ and Y. Then

$$X \cup_f Y = X \cup Y/z \sim f(z).$$

Example 5: Examples of Quotient Spaces

- The quotient of the n-dimensional closed disk by its boundary is the n-sphere, i.e. $\mathbb{D}^n/\partial\mathbb{D}^n \cong \mathbb{S}^n$.
- The 2-torus: $\mathbb{R}^2/\mathbb{Z}^2$. The projective space:
$\mathbb{RP}^n = \mathbb{R}^{n+1} \backslash \{0\}/\sim$ by the relation $x \sim y$ iff there exists some $\lambda \in \mathbb{R}^\times$ such that $x = \lambda y$. This corresponds to the space of lines through the origin in $\mathbb{R}^{n+1}$.

Definition 10: Mapping Quotients

Let $f : X \to Y$ be a continuous map.

- Its **mapping cylinder** is defined as the topological space
$$M_f := (X \times I) \cup Y/\sim$$
where the quotient identifies $(x, 0) \sim f(x)$ for any $x \in X$.
- Its **cone** is the further quotient:
$$C_f = M_f/X \times \{1\}.$$
- The **cone** of a topological space X is
$$C_X := C_{\mathrm{id}_X} = X \times I/X \times \{1\}.$$

In other words, the mapping cylinder of $f : X \times Y$ is the pushout of the diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \times I \arrow[r] & M_f \end{tikzcd}$$

Example 5.5: Spheres

Consider the n-sphere $\mathbb{S}^n$ with the standard embedding $\mathbb{R}^{n+1} \backslash \{0\}$. Then the map

$$r : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{S}^n, \quad x \mapsto \frac{x}{|x|}$$

is a retract. Indeed, if x has norm $|x| = 1$, then $r(x) = x$. for a deformation retract one needs to find a homotopy $h : i \circ r \simeq \mathrm{id}_X$. This can easily be realised by following straight-line homotopy:

$$h : \mathbb{R}^{n+1} \backslash \{0\} \times I \to \mathbb{R}^{n+1} \backslash \{0\}, \quad (x, t) \mapsto (1-t)\frac{x}{|x|} + tx.$$

Indeed, $h(x, 0) = r(x)$ and $h(x, 1) = x$ for all x.

Definition 11: Star-Shaped Spaces

A subset $S \subset \mathbb{R}^n$ is called **star-shaped** at a point $x_0 \in S$, if for any $x \in S$ the line segment from x_0 to x is contained in S, i.e.

$$\{(1-t)x_0 + tx \mid t \in [0, 1]\} \subset S$$

If S is star-shaped at every point, then it is called **convex**.

Example 5.6: Star-Shaped Spaces are Contractible

Let S be star-shaped at x_0 and $i : \{x_0\} \leftrightarrow S : r$ be the inclusion and constant maps. Define the straight line homotopy

$$h : S \times I \to S, \quad (x, t) \mapsto (1-t)x_0 + tx$$

which is well defined by the star-shaped condition. Moreover, $h(x,0) = x_0 = r(x)$ and $h(x,1) = x$ for all x. Hence, star-shaped, and in particular convex spaces, are contractible.

Example 5.7: Möbius band

The Möbius band M can be be defined as

$$M = I^2 / \sim$$

where the equivalence relation $\sim$ identifies the two vertical edges of I^2 by flipping one, i.e. $(0,b) \sim (1, 1-b)$ for $b \in I$. Its core $C \subset M$ is the line $\{[a, 1/2] \mid a \in I\}$. Thus the core is homeomorphic to $\mathbb{S}^1$. The Möbius band deformation retracts onto its core. Indeed, consider the retract $r : M \to C$ defined by $r([a,b]) := [(a, 1/2)]$ and the homotopy

$$h : M \times I \to M, \quad ([(a,b)], t) \mapsto \left[\left(a, (1-t)\frac{1}{2} + \right)\right].$$

In particular, $M \simeq \mathbb{S}^1$.

Proposition 6: Retracts of the Mapping Cylinder

Via Definition 1.1.10, the mapping cylinder is formed by the cylinder of X by gluing Y onto the bottom with the map f.
The mapping cylinder M_f strongly deformation retracts onto Y.

Proof. Consider the retract:

$$r : M_f \to Y$$

defined by $r([x,s]) := [(x,0)] = [f(x)]$ on the class of $(x,s) \in X \times I$ and $r([y]) = y$ for $y \in Y$. This is well-defined and by definition a retract on Y. Define the homotopy

$$h : M_f \times I \to M_f$$

by $h([(x,s)], t) := [(x, st)]$ for $(x,s) \in X \times I$ and $t \in I$, and by $h([y], t) := y$ for $y \in Y$. In particular, $h(-,0) i \circ r$ and $h(-,1) = \mathrm{id}_{M_f}$. This forms a strong deformation retract. □

Remark 6: Continuous Maps are Homotopic

Any continuous $f : X \to Y$ can be replaced up to homotopy equivalence by the closed inclusion $X \hookrightarrow M_f$, $x \mapsto [(x,1)]$. More precisely, it factorises through an inclusion and a homotopy equivalence as the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook] & & Y \\ & M_f \arrow[ur, "\simeq"'] & \end{tikzcd}$$

Definition 12: Relative Homotopy

Let X, Y be topological spaces and $A \subset X$ be a subset in X. A homotopy $h : X \times I \to y$ is called **relative to** A if $h(a,t)$ is independent of t for all $a \in A$. In particular, this defines homotopies between maps $f, g : X \to Y$ such that $f|_A = g|_A$.

This definition generalises ordinary homotopies, as an ordinary homotopy is the same as a homotopy relative to $\emptyset$.

Example 6: Relative Homotopies and Retracts

A strong deformation retarct of X onto A is a deformation retract such that the homotopy $h : i \circ r \simeq \mathrm{id}_X$ is relative to A.

Definition 13: Homotopic Path

Let $\alpha, \beta : I \to X$ be paths in X such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. A relative homotopy from α to β is a homotopy $h : I \times I \to x$ relative to $\partial I = \{0,1\}$, i.e.

$$h(-,0) = \alpha, \quad h(-,1) = \beta \tag{8}$$

and

$$h(0,t) = \alpha(0) = \beta(0), \quad h(1,t) = \alpha(1) = \beta(1), \quad \forall t \in I. \tag{9}$$

In particular, at any point $t \in I$ a relative homotopy h defins a path $h_t := h(-,t) : I \to X$ with start $\alpha(0) = \beta(0)$ and end $\alpha(1) = \beta(1)$. If one omits the relative condition, the start and end points of h_t would be allowed to vary.

Remark 7: Ordinary Homotopies and Paths

Observe that ordinary homotopies are not well suited for paths: Any path $\alpha : I \to X$ is homotopic (relative $\emptyset$) to a constant. Indeed, the homotopy

$$h : I \times I \to X, \quad (s,t) \mapsto \alpha(st)$$

defines a homotopy from the constant path $c_{\alpha(0)}$ on $\alpha(0)$ to α, i.e. $c_{\alpha(0)} \simeq \alpha$. Hence, (ordinary) homotopy classes of paths in X are in one-to-one correspondence with path-connected components of X.

Proposition 7: Homotopic Properties of Paths

Path concatenation is **unital**, **associative**, and **invertible** up to homotopy in the following sense: Let $\alpha, \beta, \gamma : I \to x$ be paths such that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. Then there exists homotopies relative to $\{0,1\}$:

1. **Left Unitality**: $c_{\alpha(0)} * \alpha \simeq \alpha$
2. **Right Unitality**: $\alpha \simeq c_{\alpha(0)} * \alpha$
3. **Associativity**: $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$
4. **Right Inverse**: $\alpha * \overline{\alpha} \simeq c_{\alpha(0)}$
5. **Left Inverse**: $\overline{\alpha} * \alpha \simeq c_{\alpha(1)}$

where c_x for some $x \in X$ denotes the constant path on x and $\overline{\alpha}$ is the reversed path.

Lemma 3:

Let $\alpha : I \to X$ be a path and $\lambda : I \to I$ a boundary preserving map, i.e. $\lambda(0) = 0$ and $\lambda(1) = 1$. Then,

$$\alpha \circ \lambda \simeq \alpha, \quad \text{rel.}\, \partial I.$$

Definition 14: Fundamental Group

Let X be a topological space and $x_0 \in X$ some fixed point. The **fundamental group** of X at x_0 i the group of homotopy classes of paths in X that start and end on x_0. i.e. $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x_0$, i.e.

$$\pi_1(X, x) = \{\alpha : I \to X \mid \alpha(0) = \alpha(1)\} / \sim.$$

Theorem 1: The Fundamental Group is Well Defined

The fundamental group $\pi_1(X, x_0)$ is a well-defined group with:

- **Multiplication**: $[\alpha] \cdot [\beta] := [\alpha * \beta]$
- **Unit**: $1 = [c_{x_0}]$
- **Inverse**: $[\alpha]^{-1} = [\overline{\alpha}]$

Lemma 4: Relative Concated Homotopic Paths

Let $\alpha \simeq \alpha' : I \to X$ and $\beta \simeq \beta' : I \to X$ be two pairs of relative homotopic paths such that $\alpha(1) = \alpha'(1) = \beta(0) = \beta'(0)$. Then the concatenations are relative homotopic, i.e.

$$\alpha * \beta \simeq \alpha' * \beta', \quad \text{rel.}\{0,1\}.$$

Proposition 8: Fundamental Group is Point Independent

Let $\gamma : I \to X$ be a path from $\gamma(0) = x$ to $\gamma(1) = x'$. Then it induces a group isomorphism:

$$(\gamma)_\# : \pi(X, x) \to \pi(X, x'), \quad [\alpha] \mapsto [\overline{\gamma} * \alpha * \gamma].$$

We abuse notation to say that for a path-connected space X, $\pi_1(X)$ is the fundamental group omitting the choice of base point.

Example 7: Examples of Fundamental Groups

- **Euclidean**: $\pi_1(\mathbb{R}^n) \cong 1$.
- **Circle**: $\pi_1(\mathbb{S}_1) \cong \mathbb{Z}$.
- ***n*-Spheres**: $\pi_1(\mathbb{S}^n) \cong 1$ for $n \geq 2$.
- **Torus**: $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- **Projective Spaces**: $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ for $n \geq 2$.

Definition 15: Pointed Space

- A **pointed space** is a pair (X, x) consisting of a topological space X and a point $x \in X$.
- A **map of pointed spaces** $f : (X, x) \to (Y, y)$ is a continu-

ous map $f : X \to Y$ such that $f(x) = y$.

- The **space of pointed maps** from (X, x) to (Y, y) is denoted by

$$\mathrm{Map}_*((X, x), (Y, y)) \subset \mathrm{Map}(X, Y).$$

Proposition 9: Point and Path Space Isormophism

We have a group isomorphism:

$$\pi_1(X, x) \cong \pi_0(\Omega X).$$

Similarly, one can iteratively define the n-fold loop space

$$\Omega^n X := \Omega\Omega^{n-1} X = \Omega \cdots \Omega X$$

There is a homeomorphism

$$\Omega^n X \cong \mathrm{Map}_*((\mathbb{S}^{\ltimes}, 1), (X, x))$$

Definition 16: n-th Homotopy Group

The n-th homotopy group $\pi_n(X, x)$ is defined by:

$$\pi_n(X, x) := \pi_0(\Omega^n X) \cong \pi_0(\mathrm{Map}_*(\mathbb{S}^n, (X, x))).$$

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.

Suspendisse vel felis. Ut lorem lorem, interdum eu, tincidunt sit amet, laoreet vitae, arcu. Aenean faucibus pede eu ante. Praesent enim elit, rutrum at, molestie non, nonummy vel, nisl. Ut lectus eros, malesuada sit amet, fermentum eu, sodales cursus, magna. Donec eu purus. Quisque vehicula, urna sed ultricies auctor, pede lorem egestas dui, et convallis elit erat sed nulla. Donec luctus. Curabitur et nunc. Aliquam dolor odio, commodo pretium, ultricies non, pharetra in, velit. Integer arcu est, nonummy in, fermentum faucibus, egestas vel, odio.

Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.

Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Donec odio elit, dictum in, hendrerit sit amet, egestas sed, leo. Praesent feugiat sapien aliquet odio. Integer vitae justo. Aliquam vestibulum fringilla lorem. Sed neque lectus, consectetuer at, consectetuer sed, eleifend ac, lectus. Nulla facilisi. Pellentesque eget lectus. Proin eu metus. Sed porttitor. In hac habitasse platea dictumst. Suspendisse eu lectus. Ut mi mi, lacinia sit amet, placerat et, mollis vitae, dui. Sed ante tellus, tristique ut, iaculis eu, malesuada ac, dui. Mauris nibh leo, facilisis non, adipiscing quis, ultrices a, dui.

Morbi luctus, wisi viverra faucibus pretium, nibh est placerat odio, nec commodo wisi enim eget quam. Quisque libero justo, consectetuer a, feugiat vitae, porttitor eu, libero. Suspendisse sed mauris vitae elit sollicitudin malesuada. Maecenas ultricies eros sit amet ante. Ut venenatis velit. Maecenas sed mi eget dui varius euismod. Phasellus aliquet volutpat odio. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Pellentesque sit amet pede ac sem eleifend consectetuer. Nullam elementum, urna vel imperdiet sodales, elit ipsum pharetra ligula, ac pretium ante justo a nulla. Curabitur tristique arcu eu metus. Vestibulum lectus. Proin mauris. Proin eu nunc eu urna hendrerit faucibus. Aliquam auctor, pede consequat laoreet varius, eros tellus scelerisque quam, pellentesque hendrerit ipsum dolor sed augue. Nulla nec lacus.

Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi.

Sed feugiat. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Ut pellentesque augue sed urna. Vestibulum diam eros, fringilla et, consectetuer eu, nonummy id, sapien. Nullam at lectus. In sagittis ultrices mauris. Curabitur malesuada erat sit amet massa. Fusce blandit. Aliquam erat volutpat. Aliquam euismod. Aenean vel lectus. Nunc imperdiet justo nec dolor.

Etiam euismod. Fusce facilisis lacinia dui. Suspendisse potenti. In mi erat, cursus id, nonummy sed, ullamcorper eget, sapien. Praesent pretium, magna in eleifend egestas, pede pede pretium lorem, quis consectetuer tortor sapien facilisis magna. Mauris quis magna varius nulla scelerisque imperdiet. Aliquam non quam. Aliquam porttitor quam a lacus. Praesent vel arcu ut tortor cursus volutpat. In vitae pede quis diam bibendum placerat. Fusce elementum convallis neque. Sed dolor orci, scelerisque ac, dapibus nec, ultricies ut, mi. Duis nec dui quis leo sagittis commodo.