General Topology Math Notes

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1 Intro to Topology

1.1 Why Topology?

Topology can appear where we least expect it...

- Algebraic Number Theory - Next to Euclidean topology, can define other topologies on $\mathbb{Q}$ (related to how often primes divide a number). Extends to Adeles, Langlands programme, etc

- Arithmetic Progressions in the Integers - An arithmetic progression of length k is a set $\{a, a+d, \ldots, a+(k-1)d\}$ Finding subsets of $\mathbb{N}$ that contain arbitrarily long APs:
  - $2\mathbb{N}$ or $\mathbb{N}$
  - Primes (Green-Tao Theorem, 2007). Green-Tao theorem relies on **Szemeredi's Theorem**: Any dense enough subset of $\mathbb{N}$ contains arbitrarily long APs

    Furstenburg's idea: Get from $A \subseteq \mathbb{N}$ to $(a_i \in \{0,1\}^{\mathbb{N}})$ with $a_i \begin{cases} 1 & i \in A \\ 0 & \text{else} \end{cases}$

    Use topological dynamics to study this: A topological dynamical system is a triple of X cpt, $T : X \to X$ continuous, and a probability measure μ preserved by T (what)

1.2 Topological Spaces and Examples

Definition 1.2.1: Topological Space

A **topological space** is a pair $(X, \mathcal{T})$, where X is a nonempty set, and $\mathcal{T}$ is a collection of subsets of X which satisfies:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
2. if $U_\lambda \in \mathcal{T}$ for each $\lambda \in A$ (where A is some indexing set), then $\bigcup_{\lambda \in A} U_\lambda \in \mathcal{T}$
3. If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

1.2.2 Examples of Topological Spaces

1. $\mathbb{R}^n$ with the Euclidean Topology - induced by the Euclidean Metric
2. For any set X, $\mathcal{T} = \mathcal{P}(X)$ (discrete topology)
3. For any set X, $\mathcal{T} = \{\emptyset, X\}$ (indiscrete topology)
4. $X = \{0,1,2\}$ with $\mathcal{T} = \{\emptyset, X, \{0\}, \{0,1\}, \{0,2\}\}$
5. $X = \mathbb{R}$ and U open (aka, in $\mathcal{T}$) if $R \backslash U$ is finite or $U = \emptyset$

Proof for 5:

1. $\emptyset \in \mathcal{T}$, $\emptyset$ is finite $\implies X \in \mathcal{T}$
2. Intersections of finite sets are finite
3. Unions of finite sets are finite

Definition 1.2.3: Neighbourhood of a point

A **neighbourhood** of a point $x \in X$ is a subset $N \subseteq X$ s.t. $x \in U \subseteq N$ for some open subset $U \subseteq X$

Definition 1.2.4: Metric Space

A **metric space** (X, d) is a nonempty set X together with a function

$$d : X \times X \to \mathbb{R}$$

with the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

The function d is called the metric. Point 3 is called the *triangle inequality*

For any $x \in X$ and any positive real number r the **open ball** in X with centre x and radius r is defined by

$$B(x, r) = \{y \in X | d(x, y) < r\}$$

We declare a subset U of X to be *open in the metric topology given by d* iff for each $a \in U$ there is an $r > 0$ such that $B(a, r) \subseteq U$

If $(X, \mathcal{T})$ is a topological space, and if X admits a metric whose metric topology is precisely $\mathcal{T}$ we say that $(X, \mathcal{T})$ is **metrisable**. Thus the euclidean spaces with their usual topologies are metrisable

1.2.5 Examples of Metric Spaces

1. Any set X with $d(x, y) = \begin{cases} 0, & x = y \\ 1, & \text{else} \end{cases}$
2. $\mathbb{R}^n$ with $d(x, y) = |x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
3. $C([0, 1])$ with $d(f, g) = \max_{t \in [0,1]} |f(t) - g(t)|$
4. $C([0, 1])$ with $d(f, g) = \sqrt{\int_0^n |f(t) - g(t)|^2 dt}$

1.2.6 Topologies on Metric spaces

We want to define a topology on (X, d). For this, we want open balls to be open in the topology

Definition 1.2.7: Base

For a set X, a basis $\mathcal{B}$ is a collection of subsets such that

1. $\bigcup_{B \in \mathcal{B}} B = X$
2. $B_1 \cap B_2 \in \mathcal{B}$ for all $B_1, B_2 \in \mathcal{B}$

The **topology generated by** $\mathcal{B}$ is

$$\mathcal{T} := \{\bigcup_{i \in I} B_i, I \text{ index set}, B_i \in \mathcal{B}\}$$

Note: This is a topology because

$$(\cup_{i \in I} B_i) \cap (\cup_{j \in J} B_j) = \bigcup_{i \in I, j \in J} \underbrace{B_i \cap B_j}_{\in \mathcal{B}} \in \sqcup$$

Definition 1.2.8: Metric Topology

$$\text{Let } \mathcal{B} = \{\bigcap_{i=1}^{n} Br_1(x_1), n \in \mathbb{N},\, r_i > 0,\, x_i \in X,\, \forall i\}$$

The **metric topology** is the topology generated by this basis

Observation A set U is open in the metric topology $\iff \forall x \in U, \exists r > 0$ s.t. $Br(x) \subseteq U$

- $\impliedby$: For each $x \in U$, let r_x s.t. $B_{r_x}(x) \subseteq U$. Then $U = \bigcup_{x \subseteq U} B_{r_x}(x)$ is open
- $\implies$: Let $x \in U$ be given. Knwo that $x \in B_{r_1}(x_1) \cup \cdots \cup B_{r_n}(x_n)$ for some n, r_1, x_1. For each i, there is $\delta_i > 0$ s.t. $B_{\delta_i}(x) \leq B_{r_1}(x_1)$.

huh?

Theorem 1.2.9: random ms prop

If X carries metrics $d, \tilde{d}$ such that $ad(x,y) \leq \tilde{d}(x,y) \leq Ad(x,y)$ for some $a, A > 0$, then the induced topologies agree

Definition 1.2.10: Subspace Topology

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be any subset. Then the **subspace topology** on A consists of all sets of the form $U \cap A$ where $U \in \mathcal{T}$

Example: $(-1,1) \subseteq \mathbb{R}$ with euclidean topology. The subspace topology is

$$\{(-1,1) \cap U,\, U \subseteq \mathbb{R} \text{ open}\}$$

$(-1,1)$ is closed in the subspace topology

Theorem 1.2.11: Topology Lemmas

1.3 If $(X, \mathcal{T})$ is a topological space and $U_1, \ldots, U_n$ are open sets, then the intersection $\bigcap_{i=1}^{n} u_i$ is also open

1.6 In order to show that a set $U \subseteq X$ is open, it is enough to show that for every $x \in U$ there is an open set V with $x \in V \subseteq U$

1.6 A subset U of $\mathbb{R}^n$ is *open for the usual topology* iff for each $a \in U$ there exists an $r > 0$ s.t.

$$|x - a| < r \implies x \in U$$

The collection of open sets thus defined is called the **usual topology on** $\mathbb{R}^n$. Note that open balls are open sets under this definition

Definition 1.2.12: Topology Small Definitions

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1.3 Closed sets, Closure, Interior, and Boundary

Definition 1.3.1: Closed Subsets

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is **closed** iff its complement $X \backslash A = A^C := \{x \in X \; x \notin A\}$ is open in X

Note: A set being "closed" has no connection with "not being open"

1.3.2 Examples of open and closed sets

- A set that is neither open nor closed: $[0, 1) \subseteq \mathbb{R}$ under Euclidean topology
- A set that is both closed and open: $\emptyset$ or X

Theorem 1.3.3

Let $(X, \mathcal{T})$ be a topological space. Then

1. $\emptyset$ and X are closed.
2. The union of finitely many closed sets is a closed set
3. The intersection of any collection of closed sets is a closed set

$\bigcup_{i \in I} A_i$ is not necessarily closed.

$$A_n = \left[-2 + \frac{1}{n}, 2 - \frac{1}{n}\right] \text{ in } \mathbb{R}, \bigcup_{i=1}^{n} A_i = (-2, 2)$$

1.3.4 Topological proof that there are infinitely many primes (Furstenberg)

Proof. Look at $\mathbb{Z}$ with

$$\mathcal{B} := \{S(a, b), \, a \neq 0, \, b \in \mathbb{Z}\} \quad \text{and} \quad S(a, b) = \{an + b, \, n \in \mathbb{Z}\}$$

Let the open sets be the one generated by this basis. We can show

1. $S(a, b)$ is both open and closed.
2. All open sets are infinite.

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1. $S(a, b) = \mathbb{Z} \backslash \bigcup_{i=1}^{a-1} S(a, b-1)$
2. Clear

Thus:

$$\underbrace{\mathbb{Z} \backslash \{-1, 1\}}_{\text{not closed}} = \bigcup_{p \text{ primes}} \overbrace{S(p, 0)}^{\text{closed}}$$

If there were only finitely many primes, right hand side would be closed □

Definition 1.3.5: Closure, Interior, Boundary

Let $(X, \mathcal{T})$ be a topological space.

1. The **closure** of a subset $A \subseteq X$ is

$$\overline{A} := \bigcap_{C \subseteq X \text{closed};\ A \subseteq C} C$$

2. The **interior** of a subset $A \subseteq X$ is

$$\operatorname{int} A = A^\circ := \bigcap_{U \subseteq X \text{open};\ U \subseteq A} C$$

3. The **boundary** or **frontier** of a subset $A \subseteq X$ is

$$\partial A := \overline{A} \backslash A^\circ$$

4. A subset A of X is **dense** in X iff $\overline{A} = X$

Theorem 1.3.6: Closure and Interior of Complement

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then

1. The closure of the complement is the complement of the interior:

$$\overline{X \backslash A} = X \backslash (A^\circ)$$

2. the interior of the complement is the complement of the closure:

$$(X \backslash A)^\circ = X \backslash \overline{A}$$

1.4 Open and closed sets in $\mathbb{R}$ with the usual topology

1.5 Hausdorff Spaces

Definition 1.5.1: Hausdorff Space

A topological space $(X, \mathcal{T})$ is **Hausdorff** if for each $x, y \in X$ with $x \neq y$ there exist *disjoint* open sets U and V s.t. $x \in U$ and $y \in V$

If (X, d) is a metric space then it is automatically Hausdorff, so any metrisable space is Hausdorff. The trivial topology on a set with more than one element is not Hausdorff. Not every Hausdorff space is metrisable
Non-Hausdorff spaces are a lot more annoying to work with - for example you can have multiple limits in non-Hausdorff spaces

Definition 1.5.2: Convergence of Hausdorff Spaces

A sequence (x_n) of members of a topological space X converges to $x \in X$ if for every open set U containing x, there exists an N such that $n \geq N \implies x_n \in U$

Theorem 1.5.3: Haussdorf Convergence Uniqueness

Suppose $(X, \mathcal{T})$ is Hausdorff. Then a sequence (x_n) can converge to at most one limit.

Being Hausdorff is what's called a *topological property*, which means whether or not it is true in a particular case depends only on the open sets of the space in question.
In contrast, the property of *completeness* of a metric space is not a topological property as there exist sets upon which one can put two distinct metrics, one complete and one not, yet for which the metric topologies coincide

Definition 1.5.4: Cauchy Sequences

Let (X, d) be a metric space

1. A **Cauchy Sequence** is a sequence (x_n) with each $x_n \in X$ with the property that for each $\epsilon > 0$, there exists an N s.t. $m, n \geq N \implies d(x_m, x_n) < \epsilon$

2. (X, d) is **complete** if every Cauchy Sequence converges