General Topology Math Notes

Leon Lee

September 17, 2024

Contents

1 Intro to Topology

1.1 Why Topology?

Topology can appear where we least expect it...

- Algebraic Number Theory - Next to Euclidean topology, can define other topologies on $\mathbb{Q}$ (related to how often primes divide a number). Extends to Adeles, Langlands programme, etc
- Arithmetic Progressions in the Integers - An arithmetic progression of length k is a set $\{a, a+d, \ldots, a+(k-1)d\}$ Finding subsets of $\mathbb{N}$ that contain arbitrarily long APs:
  - $2\mathbb{N}$ or $\mathbb{N}$
  - Primes (Green-Tao Theorem, 2007). Green-Tao theorem relies on **Szemeredi's Theorem**: Any dense enough subset of $\mathbb{N}$ contains arbitrarily long APs

    Furstenburg's idea: Get from $A \subseteq \mathbb{N}$ to $(a_i \in \{0,1\}^{\mathbb{N}})$ with $a_i \begin{cases} 1 & i \in A \\ 0 & \text{else} \end{cases}$

    Use topological dynamics to study this: A topological dynamical system is a triple of X cpt, $T : X \to X$ continuous, and a probability measure μ preserved by T (what)

1.2 Topological Spaces and Examples

Definition 1.1: Topological Space

A **topological space** is a pair $(X, \mathcal{T})$, where X is a nonempty set, and $\mathcal{T}$ is a collection of subsets of X which satisfies:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
2. if $U_\lambda \in \mathcal{T}$ for each $\lambda \in A$ (where A is some indexing set), then $\bigcup_{\lambda \in A} U_\lambda \in \mathcal{T}$
3. If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

1.2.1 Examples of Topological Spaces

1. $\mathbb{R}^n$ with the Euclidean Topology - induced by the Euclidean Metric
2. For any set X, $\mathcal{T} = \mathcal{P}(X)$ (discrete topology)
3. For any set X, $\mathcal{T} = \{\emptyset, X\}$ (indiscrete topology)
4. $X = \{0,1,2\}$ with $\mathcal{T} = \{\emptyset, X, \{0\}, \{0,1\}, \{0,2\}\}$
5. $X = \mathbb{R}$ and U open (aka, in $\mathcal{T}$) if $R \backslash U$ is finite or $U = \emptyset$

Proof for 5:

1. $\emptyset \in \mathcal{T}$, $\emptyset$ is finite $\implies X \in \mathcal{T}$
2. Intersections of finite sets are finite
3. Unions of finite sets are finite

reading for next time: section 1.2 def 1.17 - 1.19 section 1.4 section 1.1