Algebraic Topology Notes

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Contents

1 Introduction to Algebraic Topology

1.1 Topologies to Algebra

We want to turn topological spaces into algebraic objects through operations called Invariants. An example is that if two topological spaces X and Y are isomorphic, the translated algebraic object should also be isomorphic

$$\begin{aligned} \text{TOP} &\rightsquigarrow \text{ALG} \\ X &\mapsto A(X) \quad \text{“algebraic objects”} \\ X \cong Y &\mapsto A(X) \cong A(Y) \end{aligned}$$

Example 1.1.1: Examples of Algebraic Objects

Some examples of algebraic objects:

- The set of Connected Components $\pi_0(X)$
- The Fundamental Group $\pi_1(X)$
- Higher homotopy groups $\pi_n(X)$

Note: the more involved the algebraic invariant is, the more topology it sees.
Computability problem leads to Homology Theory (this is non-examinable)

1.2 Connected Spaces

Recall 1.2.1: Topologies

A topology on X, $\mathcal{T}$, is a family of subsets s.t.

- $\emptyset, X \in \mathcal{T}$
- Closed under finite intersection, $U_1, U_2 \in \mathcal{T} \implies U_1 \cap U_2 \in \mathcal{T}$
- Closed under arbitrary unions

Examples of topological spaces:

- Trivial topology $\mathcal{T} = \{\emptyset, X\}$
- Discrete Topology $\mathcal{T} = \mathcal{P}(X)$
- $\mathbb{R}$ or anything made from a metric space

Definition 1.2.2: Connected Spaces

A topological space X is **connected** if $X = A \uplus B$ (A and B are open) means that $A = \emptyset$ or $A = X$

Prop 1.2.3: Connected Spaces and Clopens

X is connected iff the only clopens are $\emptyset, X$

Proof.
($\implies$): A clopen then $X = A \uplus A^C \implies A = \emptyset, X$ (both A and A^C open)
($\impliedby$): $A \uplus B \implies A = B^C \implies A$ is clopen □

Examples:

- $\mathbb{R}$ is connected. Opens are generated by intervals like $(-\infty, a)$, (a, b), (a, ∞).
- The trivial topology is connected. (by definition since there are only two sets).
- The discrete topology is *not* connected, unless $X = \emptyset$ or $X = \{*\}$ in which case it coincides with the trivial topology.

Prop 1.2.4: Connectedness of Maps

For a continuous map $f : X \to Y$, and X connected, we have that $f(X)$ is connected.

Proof. $f(X) = U \uplus V \implies f^{-1}(U) \uplus f^{-1}(V) = X \implies f^{-1}(U) = \emptyset, X$ □

Corollary 1.2.5

If $X \cong Y$ are homeomorphic, then X is connected iff Y is connected

Prop 1.2.6

The relation ($x \sim y$ if $\exists$ connected subset $A \subseteq X$ s.t. $x, y \in A$) is an equivalence relation.

Proof. We show the relation fulfils all requirements for an equivalence relation:

- **Reflexivity**: $x \sim x$: $x \in \{x\} \subseteq X$
- **Symmetry**: $x \sim y \iff y \sim x$ tautological (we don't specify between x and y so just take $y = x$ and $x = y$)
- **Transitivity**: $x \sim y \land y \sim z \implies x \sim z$, $x, y \in A$, $y, z \in B$. Claim: $A \cup B$ is connected. Proof in workshop

□

Definition 1.2.7: Components

The equivalence classes of the above proposition are called **components**

1.3 Path-Connectedness

Definition 1.3.1: Path

A **path** in X is a continuous map $\alpha : I \to X$ for $I = \mathcal{T}(0, 1)$.
$x \sim y \iff \exists \alpha : I \xrightarrow{\text{path}} X$ s.t. $\alpha(0) = x, \alpha(1) = y$

$x \sim y$ is an equivalence relation due to the following operations on paths:

1. Constant path. If $x \in X$, $c_X : I \to X$, $c_x(t) := X$
2. Path reversal. Let $\alpha : I \to X$ be a path. Then $\overline{\alpha} : I \to X, t \mapsto \alpha(1-t)$
3. Path concatenation: $\alpha : I \to X$, $\beta : I \to X$ s.t. $\alpha(1) = \beta(0)$. Then

$$(a * b)(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Definition 1.3.2: Connected Components

The set of path-connected components (equivalence classes) is denoted by $\pi_0(X)$

Remarks:

- We have that $X \cong Y \implies \pi_0(X) \cong \pi_0(Y)$
- Path-connected $\implies$ Connected (but not vice-versa). Counterexample: Pick

$$X = \{(x, \sin(\frac{1}{x})) \mid 0 < x < 1\}$$

is connected but not path connected