

General Topology Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Topological Spaces and Examples

Definition 1.1: Topological Space

A **topological space** is a pair (X, \mathcal{T}) , where X is a nonempty set, and \mathcal{T} is a collection of subsets of X which satisfies:

- $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
- if $U_\lambda \in \mathcal{T}$ for each $\lambda \in \Lambda$ (where Λ is some indexing set), then $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$
- if $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

The collection \mathcal{T} is called the **topology** of the topological space, and the members of \mathcal{T} are called the **open sets** of the topology

Example 1.7: Euclidean Spaces

Let \mathbb{R}^n denote the n -dimensional Euclidean vector space with elements $x = (x_1, x_2, \dots, x_n)$ and $x_i \in \mathbb{R}$, and let

$$|x| = \sqrt{\sum_{i=1}^n x_i^2} \geq 0$$

be the length of x . ($\mathbb{R}^1 = \mathbb{R}$ is the real line). A subset U of \mathbb{R}^n is **open (for the usual topology)** iff for each $a \in U$ there exists an $r > 0$ such that

$$|x - a| < r \implies x \in U.$$

The collection of open sets thus defined is called the **usual topology** on \mathbb{R}^n . Note that open balls $B(y, \rho) = \{x \in \mathbb{R}^n : |x - y| < \rho\}$ are open sets under this definition.

Example 1.8: Metric Spaces

A **metric space** (X, d) is a nonempty set X together with a function $d : X \times X \rightarrow \mathbb{R}$ with the following properties:

- $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

The function d is called the **metric**.

Let (X, d) be a metric space, x be a point in X , and $r > 0$. The **open ball** with center x and radius r is defined by

$$B(x, r) = \{y, \in X : d(x, y) < r\}.$$

A subset U of X is **open (in the metric topology given by d)** iff for each $a \in U$ there is an $r > 0$ such that $B(a, r) \subseteq U$. Just like euclidean spaces, open balls are open in this sense.

Proposition 1.11: Topology Equality

Let d, d' be metrics on the same set X , and let $\mathcal{T}, \mathcal{T}'$ be the corresponding metric topologies. If for real numbers $A, B > 0$ we have

$$d(x, y) \leq Ad'(x, y), d'(x, y) \leq Bd(x, y) \text{ for all } x, y \in X,$$

then $\mathcal{T} = \mathcal{T}'$.

Definition 1.16: Subspace Topology

Let (X, \mathcal{T}) be a topological space, and let $A \subseteq X$ be any subset. Then the **subspace topology** on A consists of all sets of the form $U \cap A$ where $U \in \mathcal{T}$.

Definition 1.17: Closed Set

Let (X, \mathcal{T}) be a topological space. A subset $A \subseteq X$ is **closed** iff its complement $X \setminus A := \{x \in X \mid x \notin A\}$ is open in X . Note that a set being *closed* does not mean it isn't *open*. Sets that are both *closed* and *open* are called **clopen**.

Definition 1.20: Properties of Topological Spaces

For a subset $A \subseteq X$,

- The **closure** of A is $\bar{A} := \bigcap_{\substack{C \subseteq X \text{ closed;} \\ A \subseteq C}} C.$
- The **interior** of A is $\text{int } A = A^\circ := \bigcap_{\substack{C \subseteq X \text{ open;} \\ A \subseteq C}} C.$
- The **boundary** (or **frontier**) of A is $\partial A := \bar{A} \setminus A^\circ.$

- A is **dense** in X iff $\bar{A} = X$.
- A **limit point** of A is a point $x \in X$ s.t. for every open subset $U \subseteq X$ with $x \in U$ there exists an element $a \in A \cup U$ with $a \neq x$. Let A' be the set of limit points of A . Note that this has nothing to do with limits of sequences.

Proposition 1.22: Relating Topological Properties

- \bar{A} is closed, and contains A and is the smallest set with this property. So A is closed iff $\bar{A} = A$.
- A° is open, and is contained in A , and is the largest set with this property. So A is open iff $A^\circ = A$.
- The closure of the complement is the complement of the interior: $\overline{X \setminus A} = X \setminus (A^\circ).$
- The interior of the complement is the complement of the closure: $(X \setminus A)^\circ = X \setminus \bar{A}.$

Proposition 1.26: Union of Limit Points

Let (X, \mathcal{T}) be a topological space, and suppose $A \subseteq X$. Then

$$\bar{A} = A \cup A'$$

Corollary 1.27

A subset $A \subseteq X$ is closed iff it contains all its limit points.

Theorem 1.19: Properties of open and closed sets

Let (X, \mathcal{T}) be a topological space.

- \emptyset and X are **closed**.
- The union of **finitely many** closed sets is a closed set.
- The intersection of **any collection** of closed sets is a closed set.

Lemma 1.24: Limit Points and Open Balls

An element $x \in X$ in a metric space (X, d) is a limit point of a subset $A \subseteq X$ iff for every $\epsilon > 0$ there exists $a \in A$ with $0 < d(x, a) < \epsilon$, or iff there exists a sequence a_1, a_2, a_3, \dots of elements $a_i \in A$, with $a_i \neq x$ for all i , s.t. $d(x_i, a_i) \rightarrow 0$ as $i \rightarrow \infty$. This interpretation does not extend to general topological spaces.

Theorem 1.30: Open and Closed sets in \mathbb{R}

Consider \mathbb{R} with the usual topology.

- A nonempty set U is open iff it can be written as a countable union of disjoint nonempty open intervals I_j (shown left):
- A set F is closed iff it can be written as a countable intersection where each F_j is a finite union of closed intervals (shown right).

$$U = \bigcup_{j=1}^{\infty} I_j, \quad F = \bigcap_{j=1}^{\infty} F_j.$$

Definition 1.32: Hausdorff Spaces

A topological space (X, \mathcal{T}) is **Hausdorff** if for each $x, y \in X$ with $x \neq y$ there exist **disjoint** open sets U and V such that $x \in U$ and $y \in V$.

Any metrisable space is Hausdorff, The trivial topology on a set with more than one element is not Hausdorff.

Definition 1.33: Convergence of a Topological space

A sequence (x_n) of members of a topological space X converges to $x \in X$ if for every open set U containing x , there exists an N such that $n \geq N \implies x_n \in U$

Proposition 1.34: Convergence of Hausdorff Spaces

Suppose (X, \mathcal{T}) is Hausdorff. Then a sequence (x_n) can converge to at most one limit.

Definition 1.36: Cauchy and Completeness

Let (X, d) be a metric space.

- A **Cauchy sequence** is a sequence (x_n) with each $x_n \in X$ with the property that for each $\epsilon > 0$, there exists an N such that $m, n \geq N \implies d(x_m, x_n) < \epsilon$
- (X, d) is **complete** if every Cauchy sequence converges.

Definition 1.37: Topology Basis

A **basis for a topology** on a set X is a collection \mathcal{B} of subsets $B \subseteq X$ such that:

1. $X = \bigcup_{B \in \mathcal{B}} B$
2. The intersection of sets $B_1, B_2 \in \mathcal{B}$ is a set $B_1 \cap B_2 \in \mathcal{B}$.

The **topology \mathcal{T} generated by a basis \mathcal{B}** has open sets the arbitrary unions of basis elements $B_\lambda \in \mathcal{B}$:

$$U = \bigcup_{\lambda \in \Lambda} B_\lambda$$

(Don't forget to check that this really is a topology)

Example 1.38: Finite Intersections of open balls

For any metric space (X, \mathcal{T}) the finite intersections of open balls

$$B(x, r) = \{y \in X \mid d(x, y) < r\} \subseteq X \quad (r > 0, x \in X)$$

constitute a basis for the metric topology on X

$$\mathcal{B} = \{B(x_1, r_1) \cap B(x_2, r_2) \cap \cdots \cap B(x_k, r_k) \mid x_1, x_2, \dots, x_k \in X, r_1, r_2, \dots, r_k > 0\}$$

2 Continuous functions and Homeomorphisms

Definition 2.1: Continuity

Let (X, \mathcal{T}) , (Y, \mathcal{U}) be topological spaces. A function $f : X \rightarrow Y$ is **continuous** iff

$$U \in \mathcal{U} \text{ implies } f^{-1}(U) \in \mathcal{T}.$$

That is, **inverse** images of open sets are open. Continuous functions are often called **maps** or **mappings** of topological spaces.

Proposition 2.6: Topological and Analytic Continuity

Let (X, d) and (Y, ρ) be metric spaces with their induced topologies \mathcal{T} and \mathcal{U} respectively. A function $f : X \rightarrow Y$ is continuous (topologically) iff it is continuous analytically: for every $a \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(x, a) < \delta \implies \rho(f(x), f(a)) < \epsilon$$

Definition 2.7: Homeomorphism

Let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces. A **homeomorphism** is a bijection $f : X \rightarrow Y$ such that both f and f^{-1} are continuous. Two topological spaces are **homeomorphic** if there is a homeomorphism between them.

Proposition 2.18: The Punctured Sphere

Consider the n -dimensional sphere

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

with the metric topology inherited from \mathbb{R}^{n+1} . Let $x_0 \in \mathbb{S}^n$. Then $\mathbb{S}^n \setminus \{x_0\}$ is homeomorphic to \mathbb{R}^n .

3 Subspaces Revisited

Definition 3.65: Disjoint Unions

Let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces. Their **disjoint union** $X + Y$ is the set $(X \times \{0\}) \cup (Y \times \{1\})$ with the topology consisting of all sets of the form

$$(T \times \{0\}) \cup (U \times \{1\}) \text{ such that } T \in \mathcal{T}, U \in \mathcal{U}$$

Definition 3.8: Product Topology

Let (X, \mathcal{T}) , (Y, \mathcal{U}) be topological spaces. The **product topology** on their product $X \times Y$ consists of all sets of the form

$$T = \bigcup_{\alpha \in A} (U_\alpha \times V_\alpha)$$

where A is an arbitrary indexing set, and $U_\alpha \in \mathcal{U}$ and $V_\alpha \in \mathcal{V}$.

Lemma 3.10

The product topology is indeed a topology. (lol)

Lemma 3.9: Openness in Product Topologies

Let (X, \mathcal{T}) , (Y, \mathcal{U}) be topological spaces. Then $T \subseteq X \times Y$ is open in the product topology if and only if for all $t \in T$ there exists $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $t \in U \times V$ and $U \times V \subseteq T$.

Definition 3.11.5: Projection Maps

Let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces, and consider their product $X \times Y$ with the product topology. There are two natural maps Π_X and Π_Y , the projections of $X \times Y$ onto X and Y respectively, given by

$$\Pi_X : X \times Y \rightarrow X, \quad (x, y) \mapsto x$$

$$\Pi_Y : X \times Y \rightarrow Y, \quad (x, y) \mapsto y.$$

Definition 3.14: Weak Topology

Suppose that X is a set. $(X_\lambda, \mathcal{T}_\lambda)$ is a family of topological spaces, and that $f_\lambda : X \rightarrow X_\lambda$ are functions. The **weak topology generated by $\{f_\lambda\}$** is the smallest topology on X making all the f_λ continuous.

Thus, the product topology on $X \times Y$ is the weak topology generated by the two maps Π_X and Π_Y .

Definition 3.15: Cartesian Product Topology

If X_λ is a topological space, (with λ in some arbitrary indexing set Λ), the product topology on the cartesian product $\prod_{\lambda \in \Lambda} X_\lambda$ is defined to be the weak topology generated by the projections

$$\Pi_\lambda : \prod_{\mu \in \Lambda} X_\mu \rightarrow X_\lambda$$

Definition 3.5.0: Equivalence Relation

An **equivalence** relation on a set X is a binary operation \sim on X which is:

1. **Reflexive:** $x \sim x$ for all $x \in X$.
2. **Symmetric:** if $x \sim y$ then $y \sim x$.
3. **Transitive:** if $x \sim y$ and $y \sim z$ then $x \sim z$.

The **equivalence class** of any element $x \in X$ is the set

$$[x] = \{y \in X \mid x \sim y\},$$

and the set of equivalence classes is denoted by X/\sim . The function which assigns to each $x \in X$ the equivalence class $[x] \in X/\sim$ is a surjection

$$p : X \rightarrow X/\sim; \quad x \mapsto [x]$$

Definition 3.17: Quotient Space

Given a topological space (X, \mathcal{T}) , and an equivalence relation \sim on X , the **quotient space** or **identification space** is the set of equivalence classes X/\sim together with the topology

$$\{U \subseteq X/\sim : p^{-1}(U) \in \mathcal{T}\}$$

Definition 3.25: Generated Topological Spaces

Let X be a topological space, and let $Y_0, Y_1 \subseteq X$ be subspaces related by a continuous function $f : Y_0 \rightarrow Y_1$. Let \sim_f be the equivalence relation on X **generated by f** , the intersection of all the equivalence relations on X (regarded as subsets of $X \times X$) containing the pairs $(y_0, f(y_0))$ with $y_0 \in Y_0$. The identification space

$$X/\sim_f = X/\{y_1 = f(y_0)\}$$

is obtained by identifying each $y_0 \in Y_0 \subseteq X$ with $y_1 = f(y_0) \in Y_1 \subseteq X$.

Proposition 3.34: Homeomorphisms of Relations

Given a continuous function $f : X \rightarrow Y$ let \sim be the equivalence relation defined on X by $x \sim x'$ if $f(x) = f(x') \in Y$. The function

$$g : X/\sim \rightarrow Y; [x] \mapsto f(x)$$

is continuous and injective, with

$$f = g \circ p : X \xrightarrow{p} X/\sim \xrightarrow{g} Y.$$

If f is onto, and such that $f(U) \subseteq Y$ is open for every open subset $U \subseteq X$ then g is a homeomorphism.

4 Compact Spaces

Definition 4.1: Open Covers and Compact Spaces

1. An **open cover** of a topological space X is a collection $\{U_\lambda \mid \lambda \in \Lambda\}$ of open subsets U_λ of X such that

$$\bigcup_{\lambda \in \Lambda} U_\lambda = X$$

2. A topological space X is **compact** if every open cover $\{U_\lambda \mid \lambda \in \Lambda\}$ of X has a finite subcover, i.e. there exists $\lambda_1, \dots, \lambda_n \in \Lambda$ such that

$$X = \bigcup_{j=1}^n U_{\lambda_j}.$$

Definition 4.2: Open Covers as Collections

1. If $A \subseteq X$ is a subset of a topological space X , an **open cover** of A is a collection $\{V_\lambda \mid \lambda \in \Lambda\}$ of subsets V_λ which are open in X such that

$$X = \bigcup_{\lambda \in \Lambda} V_\lambda$$

2. A subset A of a topological space X is **compact** if it is compact as a subspace of X .

Proposition 4.7: Boundedness of Compact Spaces

A compact metric space (X, d) is bounded, i.e. there exists a number $K \geq 0$ such that $d(x, y) \leq K$ for all $x, y \in X$.

Proposition 4.8: Compactness of Products

A product of closed bounded intervals $R = [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is compact in the usual topology. A collection of subsets of a set X has the **finite intersection property** if every finite intersection of their members is nonempty.

Corollary 4.12: Limit Property of Compactness

Suppose that $f : X \rightarrow \mathbb{R}^n$ is a continuous map and that X is compact. Then there exists an M such that

$$|f(x)| \leq M \text{ for all } x \in X.$$

Moreover, there exists an $x \in X$ such that

$$|f(x)| = \sup_{y \in X} |f(y)|.$$

If $n = 1$ there are x_0 and $x_1 \in X$ such that

$$f(x_0) = \min_{x \in X} f(x) \quad \text{and} \quad f(x_1) = \max_{x \in X} f(x).$$

Theorem 4.13: Compact, Hausdorff, Bijection, Oh My!

Suppose X is compact, Y is Hausdorff, and that $f : X \rightarrow Y$ is a continuous bijection. Then it is a homeomorphism.

Theorem 4.14: Lebesgue Numbers

Let X be a compact metric space and $\{U_\lambda \mid \lambda \in \Lambda\}$ an open cover of X . Then there exists a positive number $\delta > 0$ (the **Lebesgue number** of the cover) such that for all $x \in X$, $B(x, \delta)$ lies *entirely inside some single* U_λ .

Corollary 4.17: Compactness of Identification Spaces

1. An identification space X/\sim of a compact space X is compact.
2. If $f : X \rightarrow Y$ is a map from a compact space X to a Hausdorff space Y and \sim is the equivalence relation on X defined by $x \sim x'$ if $f(x) = f(x') \in Y$, then the continuous bijection
$$g : X/\sim \rightarrow f(X); \quad [x] \mapsto f(x)$$
 is a homeomorphism.

Lemma 4.20: Open sets in Product spaces

Let X be a topological space, Y a compact space, $x \in X$, N an open set in $X \times Y$ such that $\{x\} \times Y \subseteq N$. Then there is an open set $W \subseteq X$ such that $x \in W$ and $W \times Y \subseteq N$.

Lemma 4.22 - 4.23: Collections and Intersections

- 4.22) Let X be a set, and suppose \mathcal{C} is a collection of subsets of X which has the finite intersection property. Then there is a collection \mathcal{B} of subsets of X , with $\mathcal{C} \subseteq \mathcal{B}$, such that \mathcal{B} has the finite intersection property, and such that \mathcal{B} is maximal with respect to this property: i.e. no collection containing \mathcal{B} as a proper subcollection has the finite intersection property.
- 4.23) Let X be a set, and suppose that \mathcal{B} is a collection of subsets of X which is maximal with respect to the finite intersection property. Then \mathcal{B} is closed under finite intersections, and any set which meets all members of \mathcal{B} is also in \mathcal{B} .

Definition 4.24: Compactifications

1. A **compactification** of a topological space X is a compact space Y which contains a homeomorphic copy of X as a subspace, i.e. such that there is a one-one map $f : X \rightarrow Y$ such that $X \rightarrow f(X); \quad x \mapsto f(x)$ is a homeomorphism.
2. A compactification Y is **dense** if X is dense in Y , i.e. $\overline{X} = Y$.

Definition 4.27: One-point compactification

The **one-point compactification** of a topological space X is the set

$$X^\infty = X \cup \{\infty\}$$

obtained by adjoining a “point at infinity” ∞ , where ∞ is a symbol *not* in X , with open sets of the form either

1. U , where $U \subseteq X$ is open, or
2. $X^\infty \setminus K$, where $K \subseteq X$ is compact and closed.

Lemma 4.28

1. The collection of open sets just defined does form a topology
2. The subspace topology on X induced by this topology coincides with its original topology.

Definition 4.32: Local Compactness

A topological space X is **locally compact** if for each $x \in X$, there exists an open subset $U \subseteq X$ and a compact C such that $x \in U \subseteq C$.

Remark 4.33

When X is Hausdorff, it is locally compact iff for each $x \in X$ there exists an open subset $U \subseteq X$ and a compact $x \in U$ and the closure \overline{U} is compact.

Definition 4.35: Normal Space

A topological space (X, \mathcal{T}) is **normal** if for every pair of disjoint closed subsets C and $D \subseteq X$, there are disjoint open subsets $U, V \subseteq X$ such that $C \subseteq U$ and $D \subseteq V$.

Lemma 4.37: Normal Complements

A space X is normal iff for every closed $F \subseteq X$ and open $G \subseteq X$ with $F \subseteq G$, there exist open G' and closed F' such that

$$F \subseteq G' \subseteq F' \subseteq G.$$

Theorem 4.38: Urysohn's Lemma

Suppose that X is a normal topological space, and that C, D are disjoint closed subsets of X . Then there is a continuous function $f : X \rightarrow \mathbb{R}$ such that

- $f(x) = 0$ for all $x \in C$
- $f(x) = 1$ for all $x \in D$
- $0 \leq f(x) \leq 1$ for all $x \in X$

Theorem 4.39: Tietze extension theorem

Suppose that X is a normal topological space, and that C is a closed subset of X . Suppose that $f : C \rightarrow \mathbb{R}$ is continuous. Then there is a continuous function $\bar{f} : X \rightarrow \mathbb{R}$ such that

- $\bar{f}(x) = f(x)$ for all $x \in C$
- If $a \leq f(x) \leq b$ for all $x \in C$, then $a \leq \bar{f}(x) \leq b$ for all $x \in X$.

Theorem 4.40: Stone-Weierstrass Theorem

The algebra A is dense in the normed space $C(X)$, i.e. $\overline{A} = C(X)$, i.e. for all $f \in C(X)$ and for all $\epsilon > 0$ there is $g \in A$ such that $\sup_{x \in X} |f(x) - g(x)| < \epsilon$.

5 Connected Spaces

Definition 5.1: Connected Spaces

1. A topological space X is **connected** if it *cannot* be written as a union

$$X = A \cup B$$

where A and B are disjoint nonempty open subsets of X

2. A topological space X is **disconnected** if it is not connected, i.e. if it *can* be expressed as a union

$$X = A \cup B$$

where A and B are disjoint nonempty open subsets of X

Connectedness is a **Topological Property** (See P6).

Remark 5.8: Connected Homeomorphisms

- If X is a compact connected metric space with exactly two points x such that $X \setminus \{x\}$ is connected, then X is homeomorphic to $[0, 1]$
- If X is a compact connected space, where for every pair of distinct points $x, y \in X$ the complement $X \setminus \{x, y\}$ is disconnected, then X is homeomorphic to the circle \mathbb{S}_1

Proposition 5.11: Connectedness other properties

The following statements about a nonempty subset $A \subseteq \mathbb{R}$ are equivalent:

1. A is connected
2. A has the interval property
3. A is an interval

Theorem 5.12: Intermediate Value Theorem

Let I be a closed bounded interval and suppose $f : I \rightarrow \mathbb{R}$ is continuous. Then the image $f(I)$ is a closed bounded interval

$$f(I) = [a, b] \subseteq \mathbb{R} (a \leq b).$$

Definition 5.13: Fixed Points of Maps

A **fixed point** of a map $f : X \rightarrow X$ is an $x \in X$ s.t. $f(x) = x$.

Theorem 5.15: 1-D Brouwer Fixed Point Theorem

Every continuous map $f : [0, 1] \rightarrow [0, 1]$ has a fixed point, i.e. there exists $x \in [0, 1]$ such that $f(x) = x$.

General Case: Every continuous map $f : \mathbb{D}^n \rightarrow \mathbb{D}^n$ has a fixed point

Definition 5.16: Path

A **path** in a topological space X is a continuous map $\alpha : I = [0, 1] \rightarrow X$. Its **initial point** is $\alpha(0) \in X$ and its **terminal point** is $\alpha(1) \in X$.

Definition 5.18: Path Connectedness

A topological space X is **path-connected** if for any two points $x_0, x_1 \in X$ there exists a path $\alpha : I \rightarrow X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$.

Theorem 5.24: Homeomorphisms of Real Spaces

If $n \geq 2$, the spaces \mathbb{R}^n and \mathbb{R} are not homeomorphic. Additionally, there is no bijection $f : \mathbb{R} \rightarrow \mathbb{R}^n$ which is continuous.

Definition 5.35: Connected Components

We define an equivalence relation \sim on a topological space X by $x \sim y$ iff there is a connected subset of X which contains both x and y . The resulting equivalence classes are called the **components** or **connected components** of X . For two homeomorphic topological spaces there will be a bijection between the sets of their components. So spaces with differing numbers of components cannot be homeomorphic. The components are the maximal connected subsets of a topological space and by Prop 5.9 are always closed in X . If we take $U \subseteq \mathbb{R}$ an open set, its connected component decomposition is its canonical representation as in Thm 1.30 as a countable disjoint union of open intervals.

Lemma 5.31.5: Path Components

Define a path (equivalence) relation

$$x_0 \sim x_1 \text{ if there exists a path } \alpha : I \rightarrow X \text{ from } \alpha(0) = x_0 \in X \text{ to } \alpha(1) = x_1 \in X.$$

- 5.32) The **constant path** at $x \in X$ is the path

$$\alpha_x : I \rightarrow X; \quad t \mapsto x$$

from $\alpha_x(0) = x \in X$ to $\alpha_x(1) = x \in X$

- 5.33) The **reverse** of a path $\alpha : I \rightarrow X$ is the path

$$-\alpha : I \rightarrow X; \quad t \mapsto \alpha(1 - t)$$

retracting α backwards, with

$$\begin{array}{ccc} -\alpha(0) = \alpha(1) & & -\alpha(1) = \alpha(0) \\ \downarrow & \xrightarrow{-\alpha} & \downarrow \end{array}$$

- 5.34) The **concatenation** of paths $\alpha : I \rightarrow X, \beta : I \rightarrow X$ with $\alpha(1) = \beta(0) \in X$

is the path

$$\alpha \bullet \beta : I \rightarrow X, \quad t \mapsto \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq 1/2 \\ \beta(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

which starts at $\alpha(0)$, follows along α at twice the speed in the first half, switching at $\alpha(1) = \beta(0)$ to follow β at twice the speed in the second half.

$$\begin{array}{ccccc} \alpha \bullet \beta(0) = \alpha(0) & & \alpha(1) = \beta(0) & & \beta(1) = \alpha \bullet \beta(1) \\ \downarrow & \xrightarrow{\alpha} & \downarrow & \xrightarrow{\beta} & \downarrow \end{array}$$

Lemma 5.31: Connected Components and Openness

Let X be a topological space and C a connected component of X . Then C is open iff for all $x \in C$ there is an open connected V such that $x \in V \subseteq C$.

Proposition 5.35: Equivalence of Path Relations

The path relation defined on a space X by $x_0 \sim x_1$ if there exists a path $\alpha : I \rightarrow X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$ is an equivalence relation.

Definition 5.36: Path Components Formally

Let X be a topological space.

1. The **path components** of X are the equivalence classes of the path equivalence relation \sim , i.e. the subspaces

$$\begin{aligned} [x] &= \{y \in X \mid y \sim x\} \\ &= \{y \in X \mid \exists \alpha : I \rightarrow X \text{ from } \alpha(0) = x \text{ to } \alpha(1) = y\} \end{aligned}$$

2. The **set of path components** (which may be infinite) is denoted by

$$X / \sim = \pi_0(X)$$

3. The function

$$X \rightarrow \pi_0(X), \quad x \mapsto [x] = \{\text{equivalence class of } x\}$$

is surjective.

Lemma 5.39: Open Condition of Path Components

Let X be a topological space and P a path component of X . Then P is open iff for all $x \in P$ there is an open path connected V such that $x \in V \subseteq P$.

Lemma 5.40: Openness and Singular Components

Let C be a connected component of a topological space X . If every path component $P \subseteq C$ is open, then C consists of a single path component. Note that the converse of this is not true.

6 Relations between Top Props

Proposition A: Topological Invariants

A **topological property** of a topological space is one which is **invariant** under homeomorphism. Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be a homeomorphism. The following properties are true:

- 2.8) \mathcal{U} is open in Y iff $f^{-1}(\mathcal{U})$ is open in X .
 - X is Hausdorff iff Y is Hausdorff.
- 3.6) $X \setminus \{x_0\}$ is homeomorphic to $Y \setminus \{f(x_0)\}$.
- 4.11) X is compact, iff Y is compact.
- 5.6) X is connected iff Y is connected.
- 5.21) X is path-connected iff Y is path-connected.
- 5.37) There exists a bijection between the set of path components $\pi_0(X)$ and $\pi_0(Y)$. However, existence of a bijection between $\pi_0(X)$ and $\pi_0(Y)$ does *not* necessarily imply that X and Y are homeomorphic.

Proposition B: Hausdorff if...

- 3.4) Suppose (X, \mathcal{T}) is a Hausdorff topological space and suppose A is a subspace. Then A is Hausdorff.
- 4.34) The one-point compactification X^∞ of a space X is Hausdorff iff X is Hausdorff and locally compact.

Proposition C: Compact if...

- 4.3) Let X be a topological space and $A \subseteq X$. Then A is compact iff every open cover of A has a finite subcover.
- 4.5) **Heine-Borel Theorem:** A subset $F \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.
- 4.6) Let X be a topological space and $A \subseteq X$.
 - 1. If X is compact and A is closed, then A is compact
 - 2. If X is Hausdorff and A is compact, then A is closed.
- 4.10) Let $f : X \rightarrow Y$ be a continuous map between topological spaces. If X is compact, so is $f(X)$.
- 4.18) **Tychonoff's Theorem:** Suppose X and Y are compact spaces. Then their product $X \times Y$ is compact. The converse is also true.
- 4.21) **Tychonoff's Theorem (General):** Suppose that \mathcal{A} is an indexing set and that for each $\alpha \in \mathcal{A}$, X_α is a compact topological space. Then the product $\prod_{\alpha \in \mathcal{A}} X_\alpha$ is compact.
- 4.30) Suppose $X^\infty = X \cup \{\infty\}$ is the *one-point compactification* of X . Then either X^∞ is compact, or X is dense in X^∞

Proposition D: Continuous if..

- 2.14) Suppose (X, \mathcal{T}) and (Y, \mathcal{U}) are topological spaces and that $f : X \rightarrow Y$. Then f is continuous iff for every closed subset $F \subseteq Y$ its inverse image $f^{-1}(F)$ is closed in X .
- 2.14) f is continuous iff the image of the closure of every subset $A \subseteq X$ is contained in the closure of the image, i.e., $\forall A \subseteq X$,

$$f(\overline{A}) \subseteq \overline{f(A)}$$
- 3.5) Suppose (X, \mathcal{T}) and (Y, \mathcal{U}) are topological spaces and suppose A is a subspace of X . Let $f : X \rightarrow Y$ be continuous. Then $f|_A : A \rightarrow Y$ is continuous.
- 3.12) Let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces and \mathcal{T} the product topology on $X \times Y$. Then the projection maps Π_X and Π_Y are continuous. Moreover, \mathcal{T} is the smallest topology on $X \times Y$ such that the projection maps are continuous.
- 3.13) Let X, Y, Z be topological spaces. Endow $X \times Y$ with the product topology. A function $f : Z \rightarrow X \times Y$ is continuous iff the functions $\Pi_X \circ f : Z \rightarrow X$ and $\Pi_Y \circ f : Z \rightarrow Y$ are both continuous.

Let X be a topological space with an equivalence relation \sim .

1. The function $p : X \rightarrow X/\sim; \quad x \mapsto [x]$ is continuous.
2. A continuous function $f : X \rightarrow Y$ such that $f(x) = f(x') \in Y$ for all $x, x' \in X$ with $x \sim x'$ determines a continuous function

$$g : X/\sim \rightarrow Y; \quad [x] \mapsto f(x)$$

such that

$$f = g \circ p : X \xrightarrow{p} X/\sim \xrightarrow{g} Y$$

$f = g \circ p$ is best described by a commutative triangle:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p \quad \nearrow g & \\ & X/\sim & \end{array}$$

In fact, every continuous function on X determines an equivalence relation.

Proposition E: Connected if...

- 5.2) X is connected iff the only subsets of X which are clopen are \emptyset and X
- 5.4) \mathbb{R} with the usual topology is connected.
- 5.5) If $f : X \rightarrow Y$ is continuous and X is connected, then $f(X)$ (with the subspace topology) is connected.
- 5.9) Let A be a connected subset of a topological space X and suppose $A \subseteq B \subseteq \overline{A}$. Then B is connected.
- 5.10) Every nonempty interval $I \subseteq \mathbb{R}$ is connected.
- 5.25) If a topological space X is path-connected, then it is also connected. Note that the converse need not be true.
- 5.30) Let $A_\lambda \subseteq X$, $(\lambda \in \Lambda)$ be a family of connected subsets of a topological space X . Suppose $\bigcup_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected.

Proposition F: Path-Connected if...

- Suppose $f : X \rightarrow Y$ is a continuous map between topological spaces and that X is path-connected. Then $f(X)$ is path-connected as a subspace of Y .
- For any equivalence relation \sim on a path-connected space X the identification space $Y = X/\sim$ is path-connected.
- Any connected open subset $\Omega \subseteq \mathbb{R}^n$ is also path-connected.
- Let X be a topological space. Then X is path connected iff X is connected *and* for all $x \in X$ there is an open path connected V such that $x \in V$.

7 Examples

Example 7.0.1: Other Topologies and Metrics

If (X, \mathcal{T}) is a topological space, and X admits a metric whose metric topology is precisely \mathcal{T} , then we say that (X, \mathcal{T}) is **metrisable**.

- Euclidean spaces with their usual topologies are metrisable.
- 1.9) The **Discrete Topology** is the topology of all subsets of a set X . We can define the **discrete metric** of X to be

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}.$$

- 1.10) The **Trivial** or **Indiscrete Topology** is the topology $\mathcal{T} := \{\emptyset, X\}$ for a set X . This is a non-metrisable topology when X has more than one member.
- 1.14) Let $X = \{a, b, c\}$, where a, b, c are distinct. Then $\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ is a topology on X
- 1.15) Give \mathbb{R} the topology whose open subsets $U \subseteq \mathbb{R}$ are precisely the subsets with finite complement $\mathbb{R} \setminus U$, or $U = \emptyset$. Then \mathbb{R} with this topology is not metrisable. This is an example of a **Zariski Topology**

Example B: The Punctured Sphere

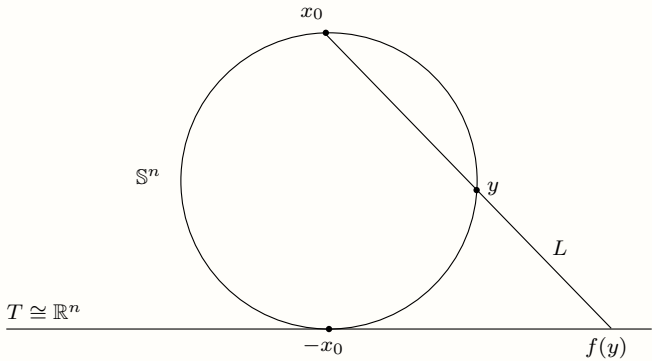


Figure 1: Homeomorphism of \mathbb{S}^2 to \mathbb{R}

Example C: Topological Objects

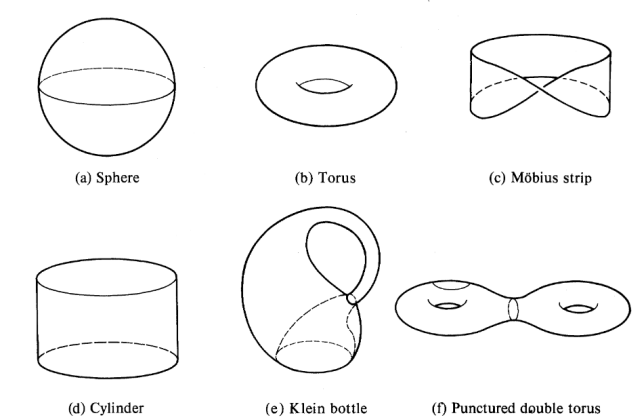


Figure 2: Standard Topological Objects

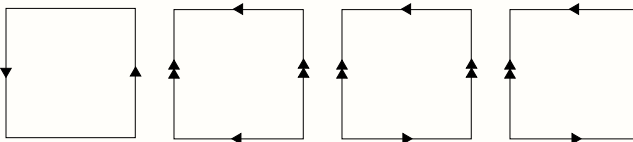


Fig 3: Möbius Strip

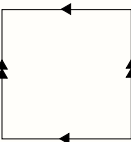


Fig 4: Torus

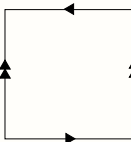


Fig 5: Klein Bottle

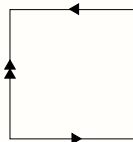


Fig 6: \mathbb{RP}^2