Galois Theory Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Galois Groups

Definition 1.1.1: Conjugate Numbers

Two complex numbers z and z' are **conjugate over** $\mathbb{R}$ if for all polynomials p with coefficients in $\mathbb{R}$,

$$p(z) = 0 \iff p(z') = 0$$

Lemma 1.1.2: Characterising Conjugates

$z, z' \in \mathbb{C}$ are conjugate over $\mathbb{R}$ iff either $z = z'$ or $\overline{z} = z'$

Definition 1.1.9: Conjugacy in $\mathbb{Q}$

$z, z' \in \mathbb{C}$ are **conjugate over** $\mathbb{Q}$ if $\forall p(t) \in \mathbb{Q}[t]$

$$p(z) = 0 \iff p(z') = 0$$

Definition 1.1.9: Conjugacy for sets

$(z_1, \ldots, z_n), z_i, z_i' \in \mathbb{C}$ is conjugate over $\mathbb{Q}$ to $(z_1', \ldots, z_n')$ if $\forall p(t_1, \ldots, t_n) \in \mathbb{Q}[t_1, \ldots, t_n]$

Additionally, if $(z_1, \ldots, z_n)$ conjugate to $(z_1', \ldots, z_n')$, then z_i is conjugate to z_i' for all i

Definition 1.2.1: Galois Group

Let f be a polynomia lwith coefficients in $\mathbb{Q}$. Write $\alpha_1, \ldots, \alpha_k$ for its distinct roots in $\mathbb{C}$. The **Galois group** of f is

$$\mathrm{Gal}(g) = \{\sigma \in S_n \mid (\alpha_1, \ldots, \alpha_n) \text{ conjugate to } (\alpha_{S(1)}, \ldots, \alpha_{\sigma(n)})\}$$

Note: distinct roots mean that we ignore any repetition of roots.

Definition 1.3.0: Solvability (Simple Definition)

A complex number is **radical** if it can be obtained from the rationals using only the usual arithmetic operations and kth roots. A polynomial over $\mathbb{Q}$ is **solvable (or soluble) by radicals** if all of its complex roots are radical.

Theorem 1.3.5: Galois

Let f be a polynomial over $\mathbb{Q}$. Then

$$f \text{ is solvable by radicals} \iff \mathrm{Gal}(f) \text{ is a solvable group.}$$

2 Groups, Rings, and Fields

Definition 2.1.1: Group Action

Let G be a group and X a set. An **action** of G on X is a function $G \times X \to X$, written as $(g, x) \mapsto gx$ such that

$$(gh)x = g(hx)$$

for all $g, h \in G$ and $x \in X$ and

$$1x = x$$

for all $x \in X$, where 1 is the identity of G

Definition 2.1.7: Faithful Actions

An action of a group G on a set X is **faithful** if for $g, h \in G$,

$$gx = hx \text{ for all } x \in X \implies g = h$$

Faithfulness means that if two elements of the group *do* the same, they *are* the same.

Lemma 2.1.8: Faithful Properties

For an action of a group G on a set X, the following are equivalent:

1. The action is faithful
2. For $g \in G$, if $gx = x$ for all $x \in X$ then $g = 1$
3. The homomorphism $\Sigma : G \to \mathrm{Sym}(X)$ is injective
4. $\ker \Sigma$ is trivial.

Lemma 2.1.11: Isomorphisms of Faithful Groups

Let G be a group acting faithfully on a set X. then G is isomorphic to the subgroup

$$\operatorname{im} \Sigma = \{\overline{g} \mid g \in G\}$$

of $\mathrm{Sym}(X)$, where $\Sigma : G \to \mathrm{Sym}(X)$ and $\overline{g}$ are defined as above.

Definition 2.1.1: Fixed Set

Let G be a group acting on a set X. Let $S \subseteq G$. The **fixed set** of S is

$$\mathrm{Fix}(S) = \{x \in X \mid sx = x \text{ for all } s \in S\}$$

Lemma 2.1.15: Normal Fixed Sets

Let G be a group acting on a set X, let $S \subseteq G$, and let $g \in G$.
Then $\mathrm{Fix}(gSg^{-1}) = g\,\mathrm{Fix}(S)$.
Here, $gSg^{-1} = \{gsg^{-1} \mid s \in S\}$ and $g\,\mathrm{Fix}(S) = \{gx \mid x \in \mathrm{Fix}(S)\}$

Definition 2.2.1: Ring Homomorphism

Given rings R and S, a **homomorphism** from R to S is a function $\varphi : R \to S$ satisfying the following equations for all $r, r' \in R$:

- $\varphi(r + r') = \varphi(r) + \varphi(r')$
- $\varphi(rr') = \varphi(r)\varphi(r')$
- $\varphi(0) = 0$, $\varphi(1) = 1$
- $\varphi(-r) = -\varphi(r)$

A **subring** of a ring R is a subset $S \subseteq R$ that contains 0 and 1 and is closed under addition, multiplication, and negatives. Whenever S is a subring of R, the inclusion $\iota : S \to R$ (defined by $\iota(s) = s$) is a homomorphism.

Lemma 2.2.3: Intersection of Subrings

Let R be a ring and let $\mathcal{S}$ be any set (perhaps infinite) of subrings of R. Then their intersection $\bigcap_{S \in \mathcal{S}} S$ is also a subring of R.

Recall 2.0.1: Ideals and Quotient Rings

Let R be a ring. $I \subseteq R$ is an **ideal**, $I \trianglelefteq R$, if the following hold:

1. $I \neq \emptyset$
2. I is closed under subtraction
3. for all $i \in I$ and $r \in R$ we have $ri, ir \in I$

Every ring homomorphism $\varphi : R \to S$ has an image $\operatorname{im} \varphi$, which is a subring of S, and a kernel $\ker \varphi$, which is an ideal of R.

Given an ideal $I \trianglelefteq R$, we obtain the quotient ring R/I and the canonical homomorphism $\pi_I : R \to R/I$ which is surjective and hs kernel I.
Universal Prop: Given any ring S and any homomorphism $\varphi : R \to S$ satisfying $\ker \varphi \supseteq I$, there is exactly one homomorphism $\overline{\varphi} : R/I \to S$ such that this diagram communutes.

$$\begin{tikzcd}
R \arrow[d, "\pi_I"'] \arrow[dr, "\varphi"] & \\
R/I \arrow[r, "\overline{\varphi}"'] & S
\end{tikzcd}$$

Recall 2.0.2: Integral Domain

An **integral domain** is a ring R such that $0_R \neq 1_R$ and for $r, r' \in R$,

$$rr' = 0 \implies r = 0 \text{ or } r' = 0$$

Recall 2.0.3: Generated Ideal

Let Y be a subset of a ring R. The **ideal** $\langle Y \rangle$ **generated by** Y is defined as the intersection of all the ideals of R containing Y.

- Ideals of the form $\langle r \rangle$ are called **principal ideals**. A **principle ideal domain** is an integral domain where every ideal is principal.
- Let r and s be elements of a ring R. We say that r **divides** s, and write $r \mid s$ if there exists $a \in R$ such that $s = ar$. This condition is equivalent to $s \in \langle r \rangle$, and to $\langle s \rangle \supseteq \langle r \rangle$.
- An element $u \in R$ is a **unit** if it has a multiplicative inverse, or equivalently, if $\langle u \rangle = R$. The units form a group $R^\times$ under multiplication.
- Elements r and s of a ring are **coprime** if for $a \in R$,
$$a \mid r \text{ and } a \mid s \implies a \text{ is a unit}$$

Lemma 2.2.11: Characterisation of Generated Ideals

Let R be a ring and let $Y = \{r_1, \ldots, r_n\}$ be a finite subset. Then
$$\langle Y \rangle = \{a_1 r_1 + \cdots + a_n r_n : a_1, \ldots, a_n \in R\}$$

Proposition 2.2.16: Coprime and PIDs

Let R be a principal ideal domain and $r, s \in R$. Then
$$r \text{ and } s \text{ are coprime} \iff ar + bs = 1 \text{ for some } a, b \in R$$

Recall 2.3.0: Field

A **field** is a ring K in which $0 \neq 1$ and every nonzero element is a unit. Equivalently, it is a ring such that $K^\times = K \backslash \{0\}$. Every field is an integral domain.
A field K has exactly two ideals: $\{0\}$ and K.
A **subfield** of a field K is a subring that is a field

Example 2.3.2: Rational Expressions

Let K be a field. A **rational expression** over K is a ratio of two polynomials
$$\frac{f(t)}{g(t)}$$
where $f(t), g(t) \in K[t]$ with $g \neq 0$. Two such expressions, f_1/g_1 and f_2/g_2 are regarded as equal if $f_1 g_2 = f_2 g_1$ in $K[t]$. i.e. equivalence class. The set of rational expressions over K is denoted by $K(t)$

Lemma 2.3.3: Homomorphisms between fields

Every (ring) homomorphism between fields is injective.

Lemma 2.3.6: Images of Subfields

Let $\varphi : K \to L$ be a homomorphism between fields.

1. For any subfield K' of K, the image $\varphi K'$ is a subfield of L
2. For any subfield L' of L, the preimage $\varphi^{-1} L'$ is a subfield of K

Definition 2.3.7: Equaliser

Let X and Y be sets, and let $S \subseteq \{ \text{ functions } X \to Y\}$. The **equalizer** of S is
$$\mathrm{Eq}(S) = \{x \in X \mid f(x) = g(x) \text{ for all } f, g \in S\}$$
i.e., it is the part of X where all the functions in S are equal.

Lemma 2.3.8: Equalisers are Subfields

Let K and L be fields, and let $S \subseteq \{\text{homomorphisms} K \to L\}$. Then $\mathrm{Eq}(S)$ is a subfield of K.

Recall 2.3.9: Characteristic

Let R be any ring. There is a unique homomorphism $\chi : \mathbb{Z} \to R$. Its kernel is an ideal of the principal ideal domain $\mathbb{Z}$. Hence $\ker \chi = \langle n \rangle$ for a unique integer $n \geq 0$. This n is called the **characteristic** of R, and written as $\operatorname{char} R$. So for $m \in \mathbb{Z}$, we have that $m \cdot 1_R = 0$ iff m is a multiple of $\operatorname{char} R$. Or equivalently,
$$\operatorname{char} R = \begin{cases} \text{the least } n > 0 \text{ s.t. } n \cdot 1_R = 0_R, & \text{if such an } n \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2.3.11: Characteristic of Integral Domains

The characteristic of an integral domain is 0 or a prime number.

Lemma 2.3.12: Characteristics of Homomorphisms

Let $\varphi : K \to L$ be a homomorphism of fields. Then $\operatorname{char} K = \operatorname{char} L$.

Recall 2.3.C: Prime Subfield

The **prime subfield** of K is the inersection of all the subfields of K. Any intersection of subfields is a subfield, and is the smallest subfield of K, in teh sense that any other subfield of K contains it. Concretely, the prime subfield of K is
$$\left\{ \frac{m \cdot 1_K}{n \cdot 1_K} \mid m, n \in \mathbb{Z} \text{ with } n \cdot 1_K \neq 0 \right\}$$

Lemma 2.3.16: Prime Subfields

Let K be a field.

- If $\operatorname{char} K = 0$ then the prime subfield of K is (iso to) $\mathbb{Q}$.
- If $\operatorname{char} K = p > 0$ then the prime subfield of K is (iso to) $\mathbb{F}_p$

Lemma 2.3.17: Characteristic of Finite Fields

Every finite field has positive characteristic.

Lemma 2.3.19: Prime Division

Let p be a prime and $0 < i < p$. Then $p \mid \binom{p}{i}$

Proposition 2.3.20: Characteristics and Primes

Let p be a prime number and R a ring of characteristic p.

1. The function
$$\theta : R \to R \quad r \mapsto r^p$$
is a homomorphism.
2. If R is a field then θ is injective.
3. If R is a finite field then θ is an automorphism of R

The homomorphism $\theta : r \mapsto r^p$ is called the **Frobenius map**, or, in the case of finite fields, the **Frobenius Automorphism**.

Corollary 2.3.22: Roots by Characteristic

Let p be a prime number.

1. In a field of characteristic p, every element has *at most* one pth root.
2. In a finite field of characteristic p, every element has *exactly* one pth root.

Recall 2.3.D: Reducible Elements

An element r of a ring R is **irreducible** if r is not 0 or a unit, and if for $a, b \in R$.
$$r = ab \implies a \text{ or } b \text{ is a tu}$$
For example, the irreducibles in $\mathbb{Z}$ are $\pm 2, \pm 3, \pm 5, \ldots$. An element of a ring is **reducible** if it is not 0, a unit, or irreducible.

Warning: The 0 and units of a ring are neither reducible nor irreducible, in much the same way that the integers 0 and 1 are neither prime nor composite.

Proposition 2.3.26

Let R be a principal ideal domain and $0 \neq r \in R$. Then

$$r \text{ is irreducible} \iff R/\langle r\rangle \text{ is a field}$$

This lets us construct fields from irreducible elements of a PID.

3 Polynomials

Definition 3.1.1: Polynomial Ring

Let R be a ring. A **polynomial over** R is an infinite sequence $(a_0, a_1, a_2, \dots)$ of elements of R s.t. $\{i \mid a_i \neq 0\}$ is finite.
The set of polynomials over R forms a ring as follows:

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (a_0 + b_0, a_1 + b_1, \dots),$$
$$(a_0, a_1, \dots) \cdot (b_0, b_1, \dots) = (c_0, c_1, \dots),$$
$$\text{where } c_k = \sum_{i,j: i+j=k} a_i b_j$$

The zero is $(0, 0, \dots)$ and the mult. identity is $(1, 0, 0, \dots)$.
The set of polynomials over R is written as $R[t]$. Since $R[t]$ is itself a ring S, we can consider the ring $S[u] = (R[t, u])[v]$, etc. Polynomials are typically written as f or $f(t)$, interchangeable. A polynomial $f = (a_0, a_1, \dots)$ over R gives rise to a function

$$R \to R$$
$$r \mapsto a_0 + a_1 r + a_2 r^2 + \cdots.$$

Remark 3.1.5: Rational Functions vs Expressions

$K(t)$ is the field of *rational expressions* over a field K. These are **not** functions, e.g. $1/(t-1)$ is a totally respectable element of $K(t)$, and you don't need to worry about $t = 1$.

Proposition 3.1.6: Universal Property of the Polyring

Let R and B be rings. For every homomorphism $\varphi : R \to B$ and every $b \in B$, there is exactly one homomorphism $\theta : R[t] \to B$ such that

$$\theta(a) = \varphi(a) \text{ for all } a \in R$$
$$\theta(t) = b$$

Definition 3.1.7: Induced Homomorphism

Let $\varphi : R \to S$ be a ring homomorphism. The **induced homomorphism**

$$\varphi_* : R[t] \to S[t]$$

is the unique homomorphism $R[t] \to S[t]$ s.t. $\varphi_* = \varphi(a)$ for all $a \in R$ and $\varphi_*(t) = t$

Definition 3.1.9: Degree

The **degree**, $\deg(f)$, of a nonzero polynomial $f(t) = \sum a_i t^i$ is the largest $n \geq 0$ s.t. $a_n \neq 0$. By convention, $\deg(0) = -\infty$, where $-\infty$ is a formal symbol which we give the properties

$$-\infty < n, \quad (-\infty) + n = -\infty, \quad (-\infty) + (-\infty) = -\infty$$

for all integers n

Lemma 3.1.11: Degree and Integral Domains

Let R be an integral domain. Then:

1. $\deg(fg) = \deg(f) + \deg(g)$ for all $f, g \in R[t]$
2. $R[t]$ is an integral domain.

The one and only polynomial of degree $-\infty$ is the zero polynomial. The polynomials of degree 0 are the nonzero constants. The polynomials of degree > 0 are therefore the nonconstant polynomials.

Lemma 3.1.14

Let K be a field. Then

1. The units in $K[t]$ are the nonzero constants
2. $f \in K[t]$ is irreducible iff f is nonconstant and cannot be expressed as a product of two nonconstant polynomials.

Proposition 3.2.1: Uniqueness of Poly Division

Let K be a field and $f, g \in K[t]$ with $g \neq 0$. Then there is exactly one pair of polynomials $q, r \in K[t]$ such that $f = qg + r$ and $\deg(r) < \deg(g)$

Proposition 3.2.2: Polynomial PIDs

Let K be a field. Then $K[t]$ is a principal ideal domain.

Corollary 3.2.5: Irreducibility and Fields

Let K be a field and let $0 \neq f \in K[t]$. Then

$$f \text{ is irreducible} \iff K[t]/\langle f\rangle \text{ is a field.}$$

Lemma 3.2.6: Divisibility by Irreducibles

Let K be a field and let $f(t) \in K[t]$ be a nonconstant polynomial. Then $f(t)$ is divisible by some irreducible in $K[t]$

Lemma 3.2.7: Divisibility of Products

Let K be a field and $f, g, h \in K[t]$. Suppose that f is irreducible and $f \mid gh$. Then $f \mid g$ or $f \mid h$

Theorem 3.2.8: Unique Determination of Polys

Let K be a field and $0 \neq f \in K[t]$. Then

$$f = af_1 f_2 \cdots f_n$$

for some $n \geq 0$, $a \in K$, and monic irreducibles $f_1, \dots, f_n \in K[t]$. Moreover, n and a are uniquely determined by f, and $f_1, \dots, f_n$ are uniquely determind up to reordering.

Monic means that the leading coefficient is 1

Lemma 3.2.9: Root Finding

One way to find an irreducible factor of a polynomial $f(t) \in K[t]$ is to find a **root**. Let K be a field, $f(t) \in K[t]$, and $a \in K$. Then

$$f(a) = 0 \iff (t - a) \mid f(t).$$

A field is **algebraically closed** if every nonconstant polynomial has at least one root.

Lemma 3.2.10: Algebraically Closed Field

Let K be an algebraically closed field and $0 \neq f \in K[t]$. then

$$f(t) = c(t - a_1)^{m_1} \cdots (t - a_k)^{m_k},$$

where c is the leading coefficient of f, and $a_1, \dots, a_k$ are the distinct roots of f in K, and $m_1, \dots, m_k \geq 1$

Lemma 3.3.1: Degrees and Irreducibility

Let K be a field and $f \in K[t]$.

1. If f is constant then f is not irreducible.
2. If $\deg(f) = 1$ then f is irreducible.
3. If $\deg(f) \geq 2$ and f has a root then f is reducible.
4. If $\deg(f) \in \{2, 3\}$ and f has no root then f is irreducible.

Warning: To show a polynomial is irreducible, it's generally *not* enough to show it has no root. The converse of 3 is false!

Definition 3.3.6: Primitive Polynomial

A polynomial over $\mathbb{Z}$ is **primitive** if its coefficients have no common divisor except for ± 1.

Lemma 3.3.7: Existence of Primitive Polynomials

Let $f(t) \in \mathbb{Q}[t]$. Then there exists a primitive polynomial $F(t) \in \mathbb{Z}[t]$ and $\alpha \in \mathbb{Q}$ such that $f = \alpha F$.

Remark 3.3.7A: Irreducibility over

If the coefficients of a polynomial $f(t) \in \mathbb{Q}[t]$ happen to all be integers, the word "irreducible" could mean two things: irreducibility in the ring $\mathbb{Q}[t]$ or in the ring $\mathbb{Z}[t]$. We say that f is irreducible **over** $\mathbb{Q}$ or $\mathbb{Z}$ to distinguish between the two.

Lemma 3.3.8: Gauss' Lemma

1. The product of two primitive polynomials over $\mathbb{Z}$ is primitive.
2. If a nonconstant polynomial over $\mathbb{Z}$ is irreducible over $\mathbb{Z}$, it is irreducible over $\mathbb{Q}$

Proposition 3.3.9: Mod p method

Let $f(t) = a_0 + a_1 t + \cdots + a_n t^n \in \mathbb{Z}[t]$. If there is some prime p such that $p \nmid a_n$ and $\overline{f} \in \mathbb{F}_p[t]$ is irreducible, then f is irreducible over $\mathbb{Q}$.
Warning: This only tells you that a polynomial is *irreducible* over $\mathbb{Q}$ and says nothing about whether it is *reducible*.

Proposition 3.3.12: Eisenstein's Criterion

Let $f(t) = a_0 + \cdots + a_n t^n \in \mathbb{Z}[t]$, with $n \geq 1$. Suppose there exists a prime p such that

- $p \nmid a_n$
- $p \mid a_i$ for all $i \in \{0, \ldots, n-1\}$
- $p^2 \nmid a_0$

Then f is irreducible over $\mathbb{Q}$.

Example 3.3.16: Cyclotomic Polynomial

Let p be a prime. The p**th cyclotomic polynomial** is

$$\Phi_p(t) = 1 + t + \cdots + t^{p-1} = \frac{t^p - 1}{t - 1}$$

Φ_p is irreducible.

4 Field Extensions

Remark 4.1.A: Inclusion Funtion

Given a set A and a subset $B \subseteq A$, there is an **inclusion** function $\iota : B \to A$ defined by $\iota(b) = b$ for all $b \in B$.
On the other hand, given any injective funtion between sets, say $\varphi : X \to A$, the image im A is a subset of A, and there is a bijection $\varphi' : X \to \operatorname{im}\varphi$ given by $\varphi'(x) = \varphi(x)$ $(x \in X)$. Hence the set X is isomorphic to (in bijection with) the subset $\operatorname{im}\varphi$ of A.
So given any subset of A, we get an injection into A, and vice versa. These two back-and-forth processes are mutually inverse (up to iso), so subsets and injections are more or less the same thing. (wtf?)

Definition 4.1.1: Field Extension

Let K be a field. An **extension** of K is a field M together with a homomorphism $\iota : K \to M$.
We can write $M : K$ to mean that M is an extension of K, not bothering to mention ι.

Definition 4.1.4: Generated Subfield

Let K be a field and X a subset of K. The subfield of K **generated by** X is the intersection of all the subfields of K containing X.
Let F be the subfield of K generated by X. F contains X, and F is also the *smallest* subfield of K containing X (in the sense that any subfield of K containing X contains F)

Definition 4.1.8: Adjoined Subfields

Let $M : K$ be a field extension and $Y \subseteq M$. We write $K(Y)$ for the subfield of M generated by $K \cup Y$. We call it K with Y **adjoined**, or the subfield of M **generated by** Y **over** K

So, $K(Y)$ is the smallest subfield of M containing both K and Y. When Y is a finite set $\{\alpha_1, \ldots, \alpha_n\}$, we write $K(\{\alpha_1, \ldots, \alpha_n\})$ as $K(\alpha_1, \ldots, \alpha_n)$

Remark 4.2.A: Algebraic Number

A complex number α is said to be "algebraic" if

$$a_0 + a_1\alpha + \cdots + a_n a^n = 0$$

for some rational numbers a_i, not all zero. This concept generalises to arbitrary field extensions:

Definition 4.2.1: Algebraic Numbers for Extensions

Let $M : K$ be a field extension and $\alpha \in M$. Then α is **algebraic** over K if there exists $f \in K[t]$ s.t. $f(\alpha) = 0$ but $f \neq 0$, and **transcendental** otherwise.

Let $M : K$ be a field extension and $\alpha \in M$. An **annihilating polynomial** of α is a polynomial $f \in K[t]$ such that $f(\alpha) = 0$.
So, α is algebraic iff it has some nonzero annihilating polynomial.

Lemma 4.2.6: Annihilaters

Let $M : K$ be a field extension and $\alpha \in M$. Then there is a polynomial $m(t) \in K[t]$ such that

$$\langle m \rangle = \{\text{annihilating polynomials of } \alpha \text{ over } K\}. \tag{1}$$

If α is transcendental over K then $m = 0$. If α is algebraic over K then there is a unique monic polynomial m satisfying (1).

Definition 4.2.7: Minimal Polynomial

Let $M : K$ be a field extension and let $\alpha \in M$ be algebraic over K. The **minimal polynomial** of α is the unique monic polynomial satisfying (1).
Warning: We do not define the minimal polynomial for a transcendental element. Therefore, some elements of M may have no minimal polynomial

Lemma 4.2.10: Minimal Polynomial Conditions

Let $M : K$ be a field extension, let $\alpha \in M$ be algebraic over K and let $m \in K[t]$ be a monic polynomial. The following are equivalent:

1. m is the minimal polynomial of α over K
2. $m(\alpha) = 0$ and $m \mid f$ for all annihilating polynomials f of α over K
3. $m(\alpha) = 0$ and $\deg(m) \leq \deg(f)$ for all nonzero annihilating polynomials.
4. $m(\alpha) = 0$ and m is irreducible over K.

Part 3 says the minimal polynomial is a monic annihilating polynomial of least degree.

Definition 4.3.1

Let K be a field.

1. Let $m \in K[t]$ be monic and irreducible. Write $\alpha \in K[t]/\langle m \rangle$ for the imge of t under the canonical homomorphism $K[t] \to K[t]/\langle m \rangle$. Then α has minimal polynomial m over K, and $K[t]/\langle m \rangle$ is generated by α over K.
2. The element t of the field $K(t)$ of rational expressions over K is transcendental over K, and $K(t)$ is generated by t over K

In part 1, we are viewing $K[t]/\langle m \rangle$ as an extension of K.

Definition 4.3.3: Homomorphism over Fields

Let K be a field, and let $\iota : K \to M$, and $\iota' : K \to M'$ be extensions of K. A homomorphism $\varphi : M \to M'$ is said to be a **homomorphism over** K if

$$\begin{tikzcd} M \arrow[rr, "\varphi"] & & M, \\ & K \arrow[ul, "\iota"] \arrow[ur, "\iota'"'] & \end{tikzcd}$$

commutes.

Lemma 4.3.6: Uniqueness of Field Homomorphisms

Let M and M' be extensions of a field K, and let $\varphi, \psi : M \to M'$ be homomorphisms over K. Let Y be a subset of M such that $M = K(Y)$. If $\varphi(\alpha) = \psi(\alpha)$ for all $\alpha \in Y$ then $\varphi = \psi$.

Proposition 4.3.7: Universal Props of $K[t]/\langle m\rangle$, $K(t)$

Let K be a field

1. Let $m \in K[t]$ be monic and irreducible, let $L : K$ be an extension of K, and let $\beta \in L$ with minimal polynomial m. Write α for the image of t under the canonical homomorphism $K[t] \to K[t]/\langle m\rangle$. Then there is exactly one homomorphism $\varphi : K[t]/\langle m\rangle \to L$ over K such that $\varphi(a) = \beta$
2. Let $L : K$ be an extension of K, and let $\beta \in L$ be transcendental. Then there is exactly one homomorphism $\varphi : K(t) \to L$ over K such that $\varphi(t) = \beta$.

$$\begin{tikzcd} & & \beta \\ \alpha \arrow[urr, dotted, mapsto] & & L \\ K[t]/\langle m\rangle \arrow[urr, dotted, "\varphi"] & & \\ & K \arrow[ul] \arrow[uur] & \end{tikzcd}$$

Figure 1: Diagram for 1

Remark 4.3.A: Isomorphism Over a Field

Let M and M' be extensions of a field K. A homomorphism $\varphi : M \to M'$ is an **isomorphism over** K if it is a homomorphism over K and an isomorphism of fields. If such a φ exists, we say that M and M' are **isomorphic over** K.

Corollary 4.3.11: Uniqueness of Isomorphisms

Let K be a field.

1. Let $m \in K[t]$ be monic and irreducible, let $L : K$ be an extension of K, and let $\beta \in L$ with minimal polynomical m and with $L = K(\beta)$. Write α for the image of t under the canonical homomorphism $K[t] \to K[t]/\langle m\rangle$. then there is exactly one isomorphism $\varphi : K[t]/\langle m\rangle \to L$ over K such that $\varphi(\alpha) = \beta$.
2. Let $L : K$ be an extension of K, and let $\beta \in L$ be transcendental with $L = K(\beta)$. Then there is exactly one isomorphism $\varphi : K(t) \to L$ over K such that $\varphi(t) = \beta$.

Definition 4.3.13: Simple Extension

A field extension $M : K$ is **simple** if there exists $\alpha \in M$ such that $M = K(\alpha)$.

Theorem 4.3.16: Classification of Simple Extensions

Let K be a field

1. Let $m \in K[t]$ be a monic irreducible polynomial. Then there exists an extension $M : K$ and an algebraic element $\alpha \in M$ such that $M = K(\alpha)$ and α has minimal polynomial m over K.

   Moreover, if (M, α) and (M', α') are two such pairs, there is exactly one isomorphism $\varphi : M \to M'$ over K such that $\varphi(\alpha) = \alpha'$
2. There exists an extension $M : K$ and a transcendental element $\alpha \in M$ such that $M = K(\alpha)$.

   Moreover, if (M, α) and (M', α') are two such pairs, there is exactly one isomorphism $\varphi : M \to M'$ over K such that $\varphi(\alpha) = \alpha'$.

Remark 4.3.C: Field Extension Explanation

Given any field K and any monic irreducible $m(t) \in K[t]$, we can say the words "adjoin to K a root α of m", and this unambiguously defines an extension $K(\alpha) : K$. Similarly, we can unambiguously adjoin to K a transcendental element.

Remark 5.1.A: Field Extensions as Vector Spaces

Let $M : K$ be a field extension. Then M is a vector space over K in a natural way. Addition and subtraction in the vector space M are the same as in the field M. Scalar multiplication in the vector space is just multiplication of elements of M by elements of K, which makes sen because K is embedded as a subfield of M.

When we view M as a vector space over K rather than an extension, we forget how to multiply together elements of M that aren't in K.

Definition 5.1.1: Degree of a Field Extension

The **degree** $[M : K]$ of a field extension $M : K$ is the dimension of M as a vector space over K.
If M is an *infinite-dimensional* vector space over K, we write $[M : K] = \infty$, where ∞ is a formal symbol which we give the properties

$$n < \infty, \quad n \cdot \infty = \infty \ (n \geq 1), \quad \infty \cdot \infty = \infty$$

for integers n. An extension $M : K$ is **finite** if $[M : K] < \infty$.

Remark 5.1.4: Degree over itself

The degree $[K : K]$ of K over itself is 1, not 0. Degrees of extensions are never 0.

Theorem 5.1.5: Basis of Field Extensions

Let $K(\alpha) : K$ be a simple extension.

1. Suppose that α is algebraic over K. Write $m \in K[t]$ for the minimal polynomial of α and $n = \deg(m)$. Then
$$1, \alpha, \ldots, \alpha^{n-1}$$
is a basis of $K(\alpha)$ over K. In particular, $[K(\alpha) : K] = \deg(m)$
2. Suppose that α is transcendental over K. Then $1, \alpha, \alpha^2, \ldots$ are linearly independent over K. In particular, $[K(\alpha) : K] = \infty$

Corollary 5.1.10: Degree and Alegebraicness

Let $M : K$ be a field extension and $\alpha \in M$, the **degree** of α over K is $[K(\alpha) : K]$. We write it as $\deg_K(\alpha)$. Then

$$\deg_K(\alpha) < \infty \iff \alpha \text{ is algebraic over } K.$$

If α is algebraic over K then the degree of α over K is the degree of the minimal polynomial of α over K.

Corollary 5.1.12: Size of Nested Extensions

Let $M : L : K$ be a field extension and $\beta \in M$. Then

$$[L(\beta) : L] \leq [K(\beta) : K]$$

Corollary 5.1.14: Polynomial Form for Extensions

Let $M : K$ be a field extension. Let $\alpha_1, \ldots, \alpha_n \in M$, when α_i algebraic over K of degree d_i. Then every element $\alpha \in K(\alpha_1, \ldots, \alpha_n)$ can be expressed as a polynomial in $\alpha_1, \ldots, \alpha_n$ over K. More exactly,

$$\alpha = \sum_{r_1, \ldots, r_n} c_{r_1, \ldots, r_n} a_1^{r_1} \cdots a_n^{r_n}$$

for some $c_{r_1, \ldots, r_n} \in K$, where r_i ranges over $0, \ldots, d_i - 1$.

Theorem 5.1.17: Tower Law

Let $M : L : K$ be field extensions.

1. If $(\alpha_i)_{i\in I}$ is a basis of L over K and $(\beta_j)_{j\in J}$ is a basis of M over L, then $(\alpha_i\beta_j)_{(i,j)\in I\times J}$ is a basis of M over K.
2. $M : K$ is finite $\iff$ $M : L$ and $L : K$ are finite.
3. $[M : K] = [M : L][L : K]$

The sets I and J here could be infinite. A family $(\alpha_i)_{i\in I}$ of elements of a field is **finitely supported** if the set $\{i \in I \mid \alpha_i \neq 0\}$ is finite.

Corollary 5.1.19: Dividing Extensions

Let $M : L' : L : K$ be field extensions. If $M : K$ is finite, then $[L' : L]$ divides $[M : K]$

Corollary 5.1.21: Triangle Tower Inequality

Let $M : K$ be a field extension and $\alpha_1, \dots, \alpha_n \in M$. Then

$$[K(\alpha_1, \dots, \alpha_n) : K] \leq [K(\alpha_1) : K] \cdots [K(\alpha_n) : K].$$

Definition 5.2.1: Finitely Generated Extensions

A field extension $M : K$ is **finitely generated** if $M = K(Y)$ for some finite subset $Y \subseteq M$.

Definition 5.2.2: Algebraic Extensions

A field extension $M : K$ is **algebraic** if every element of M is algebraic over K.

Proposition 5.2.4: Algebraic and Finiteness

The following conditions on a field extension $M : K$ are equivalent:

1. $M : K$ is finite
2. $M : K$ is finitely generated and algebraic
3. $M = K(\alpha_1, \dots, \alpha_n)$ for some finite set $\{\alpha_1, \dots, \alpha_n\}$ of elements of M algebraic over K.

Corollary 5.2.6: Algebraic and Finiteness (SEs)

Let $K(\alpha) : K$ be a simple extension. The following are equivalent:

1. $K(\alpha) : K$ is finite
2. $K(\alpha) : K$ is algebraic
3. α is algebraic over K.

Proposition 5.2.7

$\overline{\mathbb{Q}}$ is a subfield of $\mathbb{C}$.

Remark 5.3.A: Iterated Quadratic

For a subfield $K \subseteq \mathbb{R}$, an extension $K : \mathbb{Q}$ is **iterated quadratic** if there is some finite sequence of subfields

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$$

such that $[K_i : K_{i-1}] = 2$ for all $i \in \{1, \dots, n\}$

Definition 5.3.3: Compositum

Let L and L' be subfields of a field M. The **compositum** LL' of L and L' is the subfield of M generated by $L \cup L'$
That is, LL' is the smallest subfield of M containing both L and L'.

Lemma 5.3.6

Let $M : K$ be a field extension and let L, L' be subfields of M containing K. If $[L : K] = 2$ then $[LL' : L'] \in \{1, 2\}$.

Lemma 5.3.8

Let K and L be subfields of $\mathbb{R}$ such that the extensions $K : \mathbb{Q}$ and $L : \mathbb{Q}$ are iterated quadratic. Then there is some subfield M of $\mathbb{R}$ such that the extension $M : \mathbb{Q}$ is iterated quadratic and $K, L \subseteq M$.

Proposition 5.3.9: Iteratic Quadratics from Points

Let $(x, y) \in \mathbb{R}^2$. If (x, y) is constructable from $\{(0, 0), (1, 0)\}$ then there is an iterated quadratic extension of $\mathbb{Q}$ containing x and y.

Theorem 5.3.10: Quadratics and Constructability

Let $(x, y) \in \mathbb{R}^2$. If (x, y) is constructible from $\{(0, 0), (1, 0)\}$ then x and y are algebraic over $\mathbb{Q}$, and their degrees over $\mathbb{Q}$ are powers of 2.

Definition 6.1.1: Extending Homomorphism

Let $\iota : K \to M$ and $\iota : K' \to M'$ be field extensions. Let $\psi : K \to K'$ be a homomorphism of fields. A homomorphism $\varphi : M \to M'$ **extends** ψ if the square

$$\begin{tikzcd}
M \arrow[r, "\varphi"] & M' \\
K' \arrow[u, "\iota"] \arrow[r, "\psi"'] & K \arrow[u, "\iota'"']
\end{tikzcd}$$

commutes ($\varphi \circ \iota = \iota' \circ \psi$). Most of the time we view K as a subset of M, and K' as a subset of M', with ι and ι' be the inclusions. In this case, for φ to extend ψ just means that

$$\pi(a) = \psi(a) \text{ for all } a \in K$$

Lemma 6.1.3: Induced Homomorphism as sum

Let $M : K$ and $M' : K'$ be field extensions, let $\varphi : K \to K'$ be a homomorphism, and let $\varphi : M \to M'$ be a homomorphism extending ψ. Let $\alpha \in M$ and $f(t) \in K[t]$. Then

$$f(\alpha) = 0 \iff (\psi_* f)(\varphi(\alpha)) = 0.$$

Proposition 6.1.6: Unique Extending Isomorphisms

Let $\psi : K \to K'$ be an isomorphism of fields. Let $K(\alpha) : K$ be a simple extension where α has minimal polynomial m over K, and let $K'(\alpha') : K'$ be a simple extension where α' has minimal polynomial $\psi_* m$ over K'. Then there is exactly one isomorphism $\varphi : K(\alpha) \to K'(\alpha')$ that extends ψ and satisfies $\varphi(\alpha) = \alpha'$.

$$\begin{tikzcd}
K(\alpha) \arrow[r, dotted, "\varphi", "\cong"'] & K'(\alpha') \\
K' \arrow[u] \arrow[r, "\cong", "\psi"'] & K' \arrow[u]
\end{tikzcd}$$

A dotted arrow is used to denote a map whose existence is part of the conclusion of a theorem.

Definition 6.2.2: Splitting Field

Let f be a polynomial over a field M. Then f **splits** in M if

$$f(t) = \beta(t - \alpha_1) \cdots (t - a_n)$$

for some $n \neq 0$ and $\beta, \alpha_1, \dots, \alpha_n \in M$.
Equivalently, f splits in M if all its irreducible factors in $M[t]$ are linear.

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