Metric Spaces Notes

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Contents

1 Introduction to Metric Spaces

1.1 Defining a Metric

Metric is another name for distance. A **Metric Space** is a set equipped with a metric. A standard example is $\mathbb{R}$ with the standard metric

$$d(x, y) = |x - y|$$

We will now formally define what it means to have a metric

Theorem 1.1.1: Definition of a Metric

Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** iff for all $x,\ y,\ z \in X$,

- $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

A non-empty set X equipped with a metric d is called a **metric space**

1.2 Examples of Metric Spaces

We can construct a metric space using the **Absolute value** equipped with the standard triangle inequality

Example 1.2.1: The Real Line

Let $X = \mathbb{R}$. Define our metric $x : X \times X \to \mathbb{R}$ by

$$d(x, y) = |x - y|$$

The first two properties are fairly trivial. The third property follows using the regular triangle inequality

$$d(x, y) = |x - y| = |(x - z) + (z - y)| \ \leq\ |x - z| + |z - y| = d(x, z) + d(z, y)$$

Remark: This can be extended not just in $\mathbb{R}^2$, but to all $\mathbb{R}^n$. By induction,

$$|x_1 + \cdots + x_N| \leq |x_1| + \cdots + |x_N|$$

If $\displaystyle\sum_{n=1}^{\infty} x_n$ converges absolutely, let $N \to +\infty$ to see that

$$\left|\sum_{n=1}^{\infty} x_n\right| \leq \sum_{n=1}^{\infty} |x_n|$$

A second example is the **Euclidean Plane**. The metric is defined using the **inner product** and the **norm**.

Definition 1.2.2: Inner Product

The **inner product** is defined as

$$\langle x, y\rangle = x_1y_1 + x_2y_2$$

Properties of the inner product: For all vectors x, y, $z \in \mathbb{R}^2$ and all real scalars a, b,

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

Remark: This is basically a formalisation of the dot product

Definition 1.2.3: Norm

The **norm** is defined as:

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \sqrt{x_1^2 + x_2^2}$$

Properties of the norm: For all x, $y \in \mathbb{R}^2$, $a \in \mathbb{R}$

- $\|x\|_2 \geq 0$ and $\|x\|_2 = 0 \iff x = 0$
- $\|ax\|_2 = |a|\|x\|_2$
- $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$ (triangle inequality)

Remark: This is a formalisation of the "length of a vector"
With these two properties, we can now define the **Euclidean Metric**

Example 1.2.4: Euclidean Metric

For all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, define

$$d_2(x, y) = \|x - y\|_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Remark: Derivation of the triangle inequality is basically the same as Example 1.2.1.

$$d_2(x, y) = \|x - y\|_2 = \|(x - z) + (z - y)\|_2 \leq \|x - z\|_2 + \|z - y\|_2 = d_2(x, z) + d_2(z, y)$$

1.2.5 Proof of the euclidean triangle inequality

W.T.S:

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$$

Proof: Square both sides

$$\begin{aligned} \text{LHS}^2 &= \langle x + y, x + y\rangle \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ &= \|x\|_2^2 + 2\langle x, y\rangle + \|y\|_2^2 \end{aligned}$$

$$\text{RHS}^2 = \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2$$

Discarding the equal terms, we get

$$\|x\|_2^2 + 2\langle x, y\rangle + \|y\|_2^2 \leq \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2$$
$$\langle x, y\rangle \leq \|x\|_2\|y\|_2$$
$$\text{i.e. } x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

This is the **Cauchy-Schwarz Inequality**. Various ways to prove this (watch lecture 1)

Example 1.2.6: Complex Plane

Let $X = \mathbb{C}$, $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$

$$d(z, w) = |z - w|$$

If $z = a + ib, w = c + id$, $a, b, c, d \in \mathbb{R}$, then

$$z - w = (a - c) + i(b - d)$$

therefore,

$$d(z, w) = \sqrt{(a - c)^2 + (b - d)^2}$$

Definition 1.2.7: n-dimensional Euclidean space

Let $X = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$
For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \text{ (inner product)}$$

Properties of n-inner product: For all vectors $x, y, z \in \mathbb{R}^n$ and all real scalars a, b,

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

For $x = (x_1, x_2, \ldots, .x_n) \in \mathbb{R}^n$ define

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \sqrt{x_1^2 + x_2^2 + \cdots + c_n^2}\text{(norm)}$$

Properties of n-norm: For $x, y \in \mathbb{R}^n$, $a \in \mathbb{R}$,

- $\|x\|_2 \geq 0$ and $\|x\|_2 = 0 \iff x = 0$
- $\|ax\|_2 = |a|\|x\|_2$
- $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$ (triangle inequality)

Example 1.2.8: Metric in n-dim euclidean space

For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$\begin{aligned} d_2(x, y) &= \|x - y\|_2 \\ &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \end{aligned}$$

Triangle inequality, cauchy schwarz, yadda yadda same as 2-dim case

1.2.9 L space

For two sequences $x = (x_1, \ldots, x_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots)$ of real numbers we wish to define

$$d_1(x, y) = \sum_{n=0}^{\infty} |x_n - y_n|$$

We need this series to converge - in particular when $y = (0, \ldots, 0, \ldots)$, we need the series $\sum_{n=1}^{\infty} |x_n|$ to converge

Definition 1.2.10: l space

We denote by ℓ^1 the set of real sequences $(x_n)_{n \in \mathbb{N}}$ for which the series $\sum_{n=1}^{\infty} |x_n|$ converges.

If $x, y \in \ell^1$ i.e. if $\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ converge, then $\sum_{n=1}^{\infty} |x_n - y_n|$ converges, because for all n,

$$|x_n - y_n| \leq |x_n| + y_n$$

For $x = (x_1, \ldots, x_n, \ldots)$ in ℓ^1, we may now define

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$$

For $x = (x_1, \ldots, x_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots)$ in ℓ^1 we may now define

$$d_1(x, y) = \|x - y\|_1 = \sum_{n=1}^{\infty} |x_n - y_n|$$

1.3 Real Vector Spaces

Definition 1.3.1: Real Vector Spaces

A *real vector space* is a set X with two operations, addition(+) and scalar multiplication $\cdot$, with the following properties: for all $x, y, z \in X$, $a, b \in \mathbb{R}$, we have $x + y, a \cdot x \in X$, and

- $x + y = y + x$
- $x + (y + z) = (x + y) + z$
- There is an element of X denoted by 0 such that, for all x, $0 + x = x + 0 = x$
- For every $x \in X$ there exists an element of X denoted by $-x$ such that $x + (-x) = (-x) + x = 0$
- $a \cdot (x + y) = a \cdot x + a \cdot y$
- $(a + b) \cdot x = a \cdot x + b \cdot x$
- $a \cdot (b \cdot x) = (ab) \cdot X$
- $1 \cdot x = x$

(we usually write ax instead of x)

1.3.2 Normalising l 1

Properties: For all sequences $x, y \in \ell^1$ and all real scalars a,

- $\|x\|_1 \geq 0$ and $\|x\|_1 = 0 \iff x = 0$
- $\|ax\|_1 = |a|\|x\|_1$
- $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$

1.3.3 Space l-2

We denote by ℓ^2 the set of real sequences $(x_1, \dots, x_n, \dots)$ such that the seriese $\sum\limits_{n=1}^{\infty} |x_n|^2$ converges

For $x = (x_1, \dots, x_n, \dots) \in \ell^2$, $y = (y_1, \dots, y_n, \dots) \in \ell^2$ we define

- $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n$ (inner product)
- $\|x\|_2 = \left(\sum\limits_{n=1}^{\infty} |x_n|^2 \right)^{1/2}$ (norm)
- $d_2(x, y) = \|x - y\|_2 = \left(\sum\limits_{n=1}^{\infty} |x_n - y_n|^2 \right)^{1/2}$ (Metric)

Theorem 1.3.4: 4

ℓ^2 is a real vector space proof icba

more stuff on ℓ^2 - typical properties watch video 1

1.4 Generalising metric space features

Definition 1.4.1: Normed Vector Spaces

A *normed vector space* (or *normed linear space* or *normed space*) is a real vector space X equipped with a *norm*, i.e. a function that assigns to every vector $x \in X$ a real number $\|x\|$ so that, for all vectors x and y in X and all real scalars a,

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- $\|ax\| = |a|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

If $(X, \|\cdot\|)$ is a normed vector space then

$$d(x, y) = \|x - y\|$$

defines a metric in X

Definition 1.4.2: Inner Product Spaces

Let X be a real vector space. An *inner product* on X is a function that assigns to every pair $(x, y) \in X \times X$ a real number denoted by $\langle x, y\rangle$ and has the following properties

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $ax + by, z = a\langle x, z\rangle + b\langle y, z\rangle$

A *real inner product space* is a real vector space equipped with an inner product. If $\|\cdot, \cdot\|$ is an inner product on X, then

$$\|x\| = \sqrt{\langle x, x\rangle}$$

defines a norm and

$$d(x, y) = \|x - y\|$$

defines a metric

Example 1.4.3: Discrete metric

Let X be a non-empty set. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, x \neq y \end{cases}$$

Example of metric space without norm or inner prod. Another example is post office metric

theres lots of examples, i kinda cba

1.5 Open Balls

Definition 1.5.1: Open Ball

Let (X, d) be a metric space, c be a point in X, and $r > 0$. The **open ball** with center c and radius r is defined by

$$B(c, r) = \{x \in X : d(c, x) < r\}$$

Note: there are lots of different notations for this, e.g. calling it a sphere
Example: on the real line with the standard metric

$$b(c, r) = \{x \in \mathbb{R} : |x - c| < r\} = (c - r, c + r)$$

Example: on the real plane with the Euclidean metric, $X = \mathbb{R}^2$m

$$d_2(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|}^2$$

$B(c, r)$ is the open disc with center c and radius r

Watch lecture recording for examples of open balls on:

- Discrete metric
- $\mathbb{R}^2$ with the d_1 metric
- $\mathbb{R}^2$ with the d_∞ metric

2 Convergence

2.1 Convergent Sequences in Metric Spaces

On the real line, $x_n \to x$ iff for every positive ϵ, there exists an index N such that for all indices n where $n \geq N$, we have $|x_n - x| < \epsilon$.

Definition 2.1.1: Convergent Sequence

Let (X, d) be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in X, and $x \in X$. We say that $(x_n)_{n=1}^{\infty}$ converges to x iff for every positive ϵ, there exists an index N s.t. for all indices n with $n \geq N$a we have $d(x_n, x) < \epsilon$.
Observe that:

- $d(x_n, x) < \epsilon$ is equivalent to $x_n \in B(x, \epsilon)$.
- $x_n \to x$ in (X, d) iff $d(x_n, x) \to 0$ on the real line

Theorem 2.1.2: Uniqueness of metric limit

- Let (X, d) be a metric space, and $x, x' \in X$, $x \neq x'$. Then there exists a positive radius r s.t. $B(x, r) \cap B(x', r) = \emptyset$
- A sequence in a metric space can have at most one limit

Proof of first: $d(x, x') > 0$ because $x \neq x'$. Choose any r with $0 < r \leq \frac{d(x,x')}{2}$. If $y \in B(x, r)$, then $d(y, x) < r$, therefore

$$d(y, x' \geq d(x, x') - d(y, x) > d(x, x') - r)$$

and $d(x, x') - r \geq r$, therefore

$$d(y, x') > r$$

Therefore, $y \notin B(x', r)$
Proof of second: Let $x_n \to x$ and $x_n \to x'$ in a metric space (X, d). We claim that $x = x'$. Assume $x \neq x'$. Let $r > 0$ be s.t.

$$B(x, r) \cap B(x', r) = \emptyset$$

Since $x_n \to x$, there exists N s.t. for all n with $n \geq N$ we have

$$x_n \in B(x, r)$$

Since $x_n \to x$, there exists N' s.t. for all n with $n \geq N'$ we have

$$x_n \in B(x', r)$$

For any n with $n \geq \max\{N, N'\}$, the term x_n belongs to both balls - contradiction

Example 2.1.3: convergence in $(\mathbb{R}^N, d_2)$

A sequence

$$\begin{aligned} x_1 &= (x_{11}, \dots, x_{1j}, \dots x_{1N}) \\ x_2 &= (x_{21}, \dots, x_{2j}, \dots x_{2N}) \\ &\vdots \\ x_n &= (x_{n1}, \dots, x_{nj}, \dots x_{nN}) \\ &\vdots \\ &\downarrow \\ x &= (x_1, \dots, x_j, \dots, x_N) \end{aligned}$$

in $\mathbb{R}^N, d_2$ converges to $x = (x_1, \dots, x_j, \dots, x_N)$ iff for each j,

$$x_{nj} \xrightarrow[j \to +\infty]{} x_j$$

Watch lecture recording 23/01 for examples of:

- Convergence in ℓ^2
- Convergecnce in $C([a, b])$

Definition 2.1.4: Bounded Sequence

A sequence in a metric space is said to be **bounded** iff there exists an open ball that contains all of its terms

Note: this is the same definition as "sequence is bounded if there is upper and lower bound", as open ball implies the same thing

Theorem 2.1.5

Every convergence is bounded

Proof: Let $x_n \to x$ in a metric space (X, d). There exists an index N s.t. for all n with $n \geq N$,

$$x_n \in B(x, 1)$$

Let r be any positive number such that

$$r > 1,\ r > d(x, x_1), \dots, r > d(x, x_{N-1})$$

Then, for all n,

$$d(x_n, x) < r$$

therefore

$$x_n \in B(x, r)$$

2.2 Cauchy Sequences

Convergence: For every ϵ, there is an N such that for $n \geq N$, $d(x_n, x) < \epsilon$

$$x_1 \quad x_2 \quad \cdots \quad x_N \quad \cdots x_n \quad \cdots \quad \to x$$

Replace x by any x_m with $m \geq N$

$$x_1 \quad x_2 \quad \cdots \quad x_N \quad \cdots x_n \quad \cdots \quad x_m \quad \cdots$$

'$d(x_n, x) < \epsilon$' becomes '$\forall m \geq N$, $d(x_n, x_m) < \epsilon$'

Definition 2.2.1: Cauchy Sequence

A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) is said to be a **Cauchy sequence** iff for every positive ϵ, there exists an index N, s.t. for all indices n, m with $n, m \geq N$,

$$d(x_n, x_m) < \epsilon$$

Theorem 2.2.2

If a sequence in a metric space converges, then it is a Cauchy sequence

Proof: If $x_n \to L$ in a metric space (X, d), then for every positive ϵ, there exists an index N, such that for all indices n with $n \geq N, d(x_n, L) < \frac{\epsilon}{2}$. Therefore for all $n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, L) + d(x_m, L) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Note: The converse is not true.
Counterexample:

$$X = (0,1), d(x,y) = |x - y|,\ x_n = \frac{1}{n},\ (n \geq 2)$$

This sequence is Cauchy but not convergent
Cauchy: Let ϵ be positive. Pick N s.t. $\frac{1}{N} < \frac{\epsilon}{2}$. For $n, m \geq N$ we have

$$d(x_n, x_m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \epsilon$$

Not convergent: Let $x \in (0,1)$. Find N s.t. $\frac{1}{N} < x$. For $n \geq N$ we have $x_n = \frac{1}{n} \leq \frac{1}{N}$, so the open interval $(\frac{1}{N}, 1)$ contains x and only finitely many terms of the sequence. Therefore $x_n \not\to x$

Watch Lecture 23/01 for example of counterexample

- Metric spaces $(\mathbb{R}, d_{\mathbb{R}})$ and $(\mathbb{Q}, d_{\mathbb{Q}})$

Definition 2.2.3: Complete Metric Spaces

A metric space is said to be **complete** if and only if every Cauchy Sequence is convergent

Examples:

- $\mathbb{R}$ with the standard metric is complete
- $\mathbb{Q}$ with the standard metric is not complete

- $(0, 1)$ with the standard metric is not complete
- $[0, 1]$ with the standard metric is complete
- $\mathbb{R}^n$, ℓ^p, $C([a, b])$ is complete (proof later)

2.3 Open sets and closed sets

Definition 2.3.1: Open Sets and Closed Sets

Let (X, d) be a metric space.

- A subset G of X is said to be **open** iff for every point x in G there exists a positive radius r such that $B(x, r) \subseteq G$.
- A subset F of X is said to be **closed** iff F^c is open

Example: In any metric space (X, d), the sets $\emptyset$ and X are both open and closed.
$\emptyset$ is open because the following statement is true:

$$\forall x(x \in \emptyset \implies \exists r \dots)$$

X is open because, for every x in X we can take $r = 1234$ to have $B(x, r) \subseteq X$
$\emptyset^c = X$ and $X^c = \emptyset$ are closed

Watch lecture recording 26/01 for details on examples

- Every open ball is an open set
- If d is the discrete metric on a non-empty set X, then every subset of X is both open and closed
- $X = \mathbb{Z}$, $d(x, y) = |x - y|$, all subsets of X are both open and closed

Definition 2.3.2: Discrete Metric Space

A metric space is called **discrete** iff all its subsets are open (equiv. all subsets are closed)

Example: $[0, 1] \cap (2, 3)$

Theorem 2.3.3: Properties of open sets

Let (X, d) be a metric space

1. The union of any family of open sets is an open set
2. The intersection of finitely many open sets is an open set

Proof for 1: Let $(G_i)_{i \in I}$ be a family of open sets and define $G = \bigcup_{i \in I} G_i$. If $x \in G$, then $x \in G_i$ for some i. Since G_i is open, there exists a positive r such that $B(x, r) \subseteq G_i$. Then $B(x, r) \subseteq G$
Proof for 2: Let $G_1, \dots, G_n$ be open sets. Define $G = G_1 \cap \dots \cap G_n$. If $x \in G$, then $x \in G_i$ for all i. Since each G_i is open, there exists a positive r_i such that $B(x, r_i) \subseteq G_i$. Let $r = \min\{r_1, \dots, r_n\}$. For each i,

$$B(x, r) \subseteq B(x, r_i) \subseteq G_i$$

Therefore, $B(x, r) \subseteq G_1 \cap \dots \cap G_n = G$

Theorem 2.3.4: Infinite open sets

The intersection of infinitely many open sets is not always an open set
For example, let $G_n = (-\frac{1}{n}, \frac{1}{n}), n = 1, 2, \ldots$ on the real line with the standard metric.
Each G_n is open but

$$\bigcap_{n=1}^{\infty} G_n = \{0\}$$

Theorem 2.3.5: Relatively open sets

Let (X, d) be a metric space and A be a non-empty subset of X equipped with the induced metric d_A. Let $G \subseteq A$. G is open in (A, d_A) iff there exists a subset O of X, open in (X, d), such that $G = A \cap O$
The open sets of (A, d_A) are sometimes referred to as **relatively open**

Theorem 2.3.6

Let (X, d) be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in X and x be a point in X.
$x_n \to x$ iff every open set that contains x contains eventually all terms of the sequence

Proof: Assume $x_n \to x$. Let G be any open set with $x \in G$. There is a positive r such that $B(x, r) \subseteq G$. There is an N such that for all n with $n \geq N$ we have $x_n \in B(x, r)$, hence, $x_n \in G$. Conversely, assume that every open set containing x contains eventually all terms of the sequence. Every open ball centered at x is an open set, therefore it contains eventually all terms of the sequence. It follows that $x_n \to x$.

Definition 2.3.7: Neighbourhoods of points

An **open neighbourhood** of a point x is any open set that contains x. $x_n \to x$ iff every open neighbourhood of x contains eventually all terms of the sequence.

A **neighbourhood** of a point x is a set that contains an open neighbourhood of x.
$x_n \to x$ iff every neighbourhood of x contains eventually all terms of the sequence.

Theorem 2.3.8: Properties of Closed sets

Let (X, d) be a metric space.

1. The intersection of any family of closed sets is a closed set
2. The union of finitely many closed sets is a closed set.

Proof for 1: Let $(F_i)_{i \in I}$ be a family of closed sets. Then each F_i^c is open, therefore, $\bigcup_{i \in I} F_i^c$ is open, therefore $\left(\bigcup_{i \in I} F_i^c\right)$ is closed. By De Morgan's rule, $\left(\bigcup_{i \in I} F_i^c\right)^c = \bigcap_{i \in I} F_i$. Therefore, $\bigcap_{i \in I} F_i$ is closed.
Proof for 2: Let $F_1, \ldots, F_n$ be closed sets. Then $F_1^c, .., F_n^c$ are open, therefore $F_1^c \cap \cdots \cap F_n^c$ is open, therefore $(F_1^c \cap \cdots \cap F_n^c)^c$ is closed. By de Morgan's rule, $(F_1^c \cap \cdots \cap F_n^c)^c = F \cup \cdots \cup F_n$. Therefore, $F \cup \cdots \cup F_n$ is closed

Theorem 2.3.9: Infinite closed sets

The union of infinitely many closed sets is not always a closed set.
For example, let $F_n = [\frac{1}{n}, 1], n = 1, 2, \ldots$, on the real line with the standard metric. Each F_n is closed but

$$\bigcup_{n=1}^{\infty} F_n = (0, 1]$$

is not closed.

Watch lecture recording 30/01 for examples

Theorem 2.3.10

A subset F of a metric space is closed iff the limit of every convergent sequence of elements of F belongs to F

Proof $\Longrightarrow$: Assume F is closed, and let $(x_n)_{n=1}^{\infty}$ be a convergent sequence of elements of F. Let x be its limit. We wish to show that $x \in F$. We argue by contradiction. Suppose $x \notin F$. Then $x \in F^c$, and since F^c is open, there exists a positive r such that $B(x, r) \subseteq F^c$. Then $B(x, r)$ contains no terms of the sequence - contradiction
Proof $\Longleftarrow$: assume that the limit of every convergent sequence of elements of F belongs to F. We wish to show that F is closed.
We show that F^c is open. Let $x \in F^c$. We need to show that there exists a postive r such that $B(x, r) \subseteq F^c$. If not, then for every r there exists a point in $B(x, r)$ that belongs to F.
Using this with $r = \frac{1}{n}, n = 1, 2, 3, \ldots$, we find points x_n with $x_n \in B(x, 1/n)$ and $x_n \in F$. Then $x_n \to x$ but $x \notin F$a. Contradiction

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Watch lecture recording 30/01 for examples

- In any metric space (X, d), singletons $F = \{x\}$ are closed.
- In any metric space, any finite set is closed because

$$\{x_1, \ldots, x_n\} = \{x_1\} \cup \cdots \cup \{x_n\}$$

2.4 Closure

Definition 2.4.1: Closure

Let (X, d) be a metric space and $A \subseteq X$. The **closure** of A, deonted by $\overline{A}$, is the smallest closed subset of X that contains A
There exists at least one closed subset of X that contains A, namely X itself. The smallest closed subset of X that contains A is

$$\bigcap_{A \subseteq F \subseteq X,\, F \text{closed}} F$$

Theorem 2.4.2: Properties of Closure

Let (X, d) be a metric space and A, $B \subseteq X$.

1. $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$
2. $A \subseteq \overline{A}$ and $\overline{A}$ is closed
3. A is closed iff $A = \overline{A}$
4. $\overline{\overline{A}} = \overline{A}$
5. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$
6. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Lecture 30/01 45m for proofs

Example: $X = \mathbb{R}$, $d(x, y) = |x - y|$, $A = (0, 1)$. We claim that $\overline{A} = [0, 1]$
$A \subseteq [0, 1]$ and $[0, 1]$ is a closed set. The smallest such set is $\overline{A}$. Therefore $\overline{A} \subseteq [0, 1]$.
Next we show that $[0, 1] \subseteq \overline{A}$. clearly, $(0, 1) = A \subseteq \overline{A}$
$(1/2, 1/3, \dots, 1/n \dots) \to 0$, each term belongs to $\overline{A}$, and $\overline{A}$ is closed, therefore $0 \in \overline{A}$. Similarly, $1 \in \overline{A}$

Watch lecture recording 02/02 10m for more in-depth examples of closure things

- On the real line with the standard metric, $\overline{(a, b)} = [a, b]$
- In $\mathbb{R}^n$ with the Euclidean metric d_2, the closure of the open ball $B(c, r)$ is the closed ball $\{x \in \mathbb{R}^n : d_2(x, c) \leq r\}$
- On the complex plane with its standard metric, the closure of an open disc is the corresponding closed disc
- Let X be a non-empty set with the discrete metric, $c \in X$ and $r = 1$. Then $B(c, 1) = \{c\}$, therefore $\overline{B(c, 1)} - \overline{\{c\}} = \{c\}$, while

$$\{x \in X : d(x, c) \leq 1\} = X$$

  The closure of an open ball is not always equal to the corresponding closed ball
- $X = \mathbb{R}$, $d(x, y) = |x - y|$. $\overline{\mathbb{Q}} = \mathbb{R}$

Definition 2.4.3: Dense Subset of a Metric Space

Let (X, d) be a metric space. A subset D of X is said to be **dense** iff $\overline{D} = X$

Random fact: In $\mathbb{R}^n$ with the Euclidean metric d_2, $\mathbb{Q}^n$ is dense.

Theorem 2.4.4: Closure Equivalence

Let (X, d) be a metric space, $A \subseteq X, x \in X$. The following are equivalent

1. $x \in \overline{A}$
2. For every positive r, $B(x, r) \cap A \neq \emptyset$
3. There exists a sequence $(a_n)_{n \in \mathbb{N}}$ with $a_n \in A$ for all n, such that $a_n \to x$

A point x with any of these properties is called an **adherent point** of A. So, $\overline{A}$ is the set of all adherent points of A.

Example: $X = \mathbb{R}$, $d(x, y) = |x - y|, A = (0, 1) \cup \{2\}$, $\overline{A} = [0, 1] \cup \{2\}$
2 is an adherent point of A. 0 is an adherent point of A.
Observe: $2 \in A, 0 \notin A$

Proof: $1 \implies 2$
Assume $x \in \overline{A}$. Fix a positive r. We show: $B(x, r) \cap A \neq \emptyset$.
The set $\overline{A} \backslash B(x, r)$ is closed and $\overline{A} \backslash B(x, r) \subsetneq \overline{A}$
Therefore, $A \nsubseteq \overline{A} \backslash B(x, r)$
Therefore there exists an element $a \in A$ s.t. $a \notin \overline{A} \backslash B(x, r)$. But $a \in \overline{A}$. Therefore $a \in B(x, r)$
Proof: $2 \implies 3$
If A intersects every open ball centered at x, then for every n there is a point a_n that belongs to A and to $B(x, 1/n)$. Then $d(a_n, x) < 1/n$, therefore $a_n \to x$
Proof: $3 \implies 1$ Assume that there is a sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \in A$ for all n, and $a_n \to x$. We show that $x \in \overline{A}$.
For each n we have $a_n \in \overline{A}$. Also, $a_n \to x$ and $\overline{A}$ is closed. Therefore $x \in \overline{A}$

Definition 2.4.5: Limit points of sets

Let (X, d) be a metric space, $A \subseteq X$ and $x \in X$. We say that x is a **limit point** or an **accumulation point** of A iff every open ball centered at x contains an element of A distinct from x, i.e.

$$\forall r > 0 \quad (B(x, r) \backslash \{x\}) \cap A \neq \emptyset$$

The set of all limit points of A is called the **derived set** of A and is denoted by A' or $\tilde{A}$.

Note w/o proof: x is a limit point of A iff there exists a sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \in A, a_n \neq x$ for all n, and $a_n \to x$
Note w/o proof: Let (X, d) be a metric space and $A \subseteq X$. Then $\overline{A} = A \cup A'$

Example: On the real line with the standard metric, let $A = (0, 1) \cup \{2\}$. Then $\overline{A} = [0, 1] \cup \{2\}$, so $0, 2 \in \overline{A}$ 0 is a limit point of A 2 isn't a limit point of A

2.5 Continuous functions between metric spaces

Definition 2.5.1: Continuity at a point

Let (X, d_X), (Y, d_Y) be metric spaces and $f : X \to Y$ be a function. We say that f is **continuous at a point** x_0 in X iff for for every positive ϵ, there exists a positive δ, s.t., for all $x \in X$ with $d_X(x, x_0) < \delta$ we have $d_Y(f(x), f(x_0)) < \epsilon$

Alternatively, f is **continuous at a point** $x_0 \in X$ iff, for every positive ϵ, there exists a positive δ, such that, for all $x \in B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$

Definition 2.5.2: Continuity of a function

Let $(X, d_X), (Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is said to be **continuous** iff it is continuous at every point in X

Example: Let (X, d) be a metric space and p be a point in X. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, p)$. f is continuous.
Watch lecture recording 02/02 40m for proof

Theorem 2.5.3

Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f : X \to Y$ be a function and x_0 be a point in X. Then f is continuous at x_0 iff for every open neighbourhood G of $f(x_0)$ there exists an open neighbourhood O of x_0 such that, for all $x \in O$, we have $f(x) \in G$

Proof. Assume f is continuous at x_0. Let G be an open set in Y with $f(x_0) \in G$. There exists a positive ϵ such that $B_Y(f(x_0), \epsilon) \subseteq G$. By continuity, there exists a positive δ such that for all $x \in B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$. Let $O = B_X(x_0, \delta)$. For all $x \in O$ we have $f(x) \in G$ □

Conversely, assume that for every open neighbourhood G of $f(x_0)$ there exists an open neighbourhood O of x_0 s.t. for all $x \in O$, we have $f(x) \in G$. We wish to show that f is continuous at x_0
Let ϵ be positive. Apply our hypothesis with $G = B_Y(f(x_0), \epsilon)$ to see that there exists an open set O in X with $x_0 \in O$, s.t. for all $x \in O$ we have $f(x) \in G$.
Since O is open, there exists a positive δ such that $B_X(x_0, \delta) \subseteq O$.
For all x in $B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$

Theorem 2.5.4: Continuity and Convergence

Let (X, d_X), (Y, d_Y) be metric spaces, x_0 be a point in X, and $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous at x_0
2. For every sequence $(x_n)_{n=1}^{\infty}$ in X, if $x_n \xrightarrow[n \to +\infty]{}$ in (X, d_X), then $f(x_n) \xrightarrow[n \to +\infty]{} f(x_0)$ in (Y, d_Y)

Proof. 1 $\implies$ 2: Assume f is continuous at x_0 and let $x_n \to x_0 \in X$
Let ϵ be positive. There exists a positive δ such that, for all $x \in B_X(x_0, \delta)$, $f(x) \in B_Y(f(x_0), \epsilon)$.
Eventually all x_n belong to $B_X(x_0, \delta)$. Therefore eventually all $f(x_n)$ belong to $B_Y(f(x_0), \epsilon)$

---

2 $\implies$ 1: Contrapositive - not 1 $\implies$ not 2
Assume that f is not continuous at x_0. Then

$$\text{not} \, (\forall \epsilon, \, \exists \delta, \, \forall x \in B_X(x_0, \delta) \quad f(x) \in B_Y(f(x_0), \epsilon))$$

i.e.

$$\exists \epsilon, \, \forall \delta, \, \exists x \in B_X(x_0, \delta) \quad f(x) \notin B_Y(f(x_0), \epsilon)$$

Apply this with $\delta = 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots$ to see that there exists $x_1, x_2, \ldots, x_n,$ such that

$$x_n \in B_X(x_0, 1/n) \text{ and } f(x_n) \notin B_Y(f(x_0), \epsilon)$$

Then $x_n \to x_0$ in X and $f(x_n) \not\to f(x_0)$ in Y, so, not 2 □

Theorem 2.5.5: Continuity and Open Sets

Let (X, d_X), (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous iff the inverse image $f^{-1}(G)$ of any open subset G of Y is an open subset of X

Proof. Assume f is continuous and let G be an open subset of Y. Let $x_0 \in f^{-1}(G)$. Then $f(x_0) \in G$, therefore there exists a positive ϵ such that $B_Y(f(x_0), \epsilon) \subseteq G$. Since f is continuous at x_0, there exists a positive δ such that, for all $x \in B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$, therefore $f(x) \in G$, therefore $x \in f^{-1}(G)$. This shows that $B_X(x_0, \delta) \subseteq f^{-1}(G)$.

Conversely, assume that the inverse image of every open subset of Y is an open subset of X.
Fix a point $x_0 \in x$. We show that f is continuous at x_0.
Let ϵ be positive. The open ball $B_Y(f(x_0), \epsilon)$ is an open subset of Y, therefore $f^{-1}(B_Y(f(x_0), \epsilon))$ is an open subset of X that contains x_0.
Therefore, there exists a positive δ such that

$$B_X(x_0, \delta) \subseteq f^{-1}(B_Y(f(x_0), \epsilon))$$

For any $x \in B_X(x_0, \delta)$ we have $x \in f^{-1}(B_Y(f(x_0), \epsilon))$, therefore $f(x) \in B(f(x_0), \epsilon)$ □

Exercise: Let $(X, d_X), (Y, d_Y), Z_{d_Z}$ be three metric spaces. Let $f : X \to Y$ and $g : Y \to Z$ be two continuous functions. Then $g \circ f : X \to Z$ is continuous

3 Topology!!!

3.1 Homeomorphisms and Topological Properties

Definition 3.1.1: Topological Space

A **topological space** is a set X together with a family $\mathcal{T}$ of subsets of X that has the following properties:

- $\emptyset, X \in \mathcal{T}$
- Any union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$
- Any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$

$\mathcal{T}$ is called a **topology** and the elements of $\mathcal{T}$ are called **open sets**

Definition 3.1.2: Continuity of Topological Spaces

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. A function $f : X \to Y$ is said to be **continuous** iff for every G in $\mathcal{T}_Y$ the pre-image $f^{-1}(G)$ is an element of $\mathcal{T}_X$.
f is said to be a **homeomorphism** iff it is a continuous bijection and its inverse is continuous.
If such a homeomorphism exists then $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are said to be **homeomorphic**

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic, and one of them is compact or connected or separable etc etc, then so is the other
Properties that are preserved by homeomorphisms are called topological properties

3.2 Just kidding back to metric spaces

Example: Let (X, d_X) be a discrete metric space and (Y, d_Y) be any metric space. Show that every function $f : X \to Y$ is continuous.
Indeed, the inverse image $f^{-1}(G)$ of any open subset G of Y is an open subset of X (all subsets of X are open)

Example: Let $X = \mathbb{R}$ equipped with the standard metric d, and $Y = \mathbb{R}$ equipped with the discrete metric ρ. Show the function $f : X \to Y$, $f(x) = x$ is not continuous.

Proof. The set $\{0\}$ is open in Y, but the set $f^{-1}(\{0\}) = \{0\}$ is not open in X
Actually, for any point $x_0 \in X$, we have $x_0 + \frac{1}{n} \to x_0 \in X$, but

$$f\left(x_0 + \frac{1}{n}\right) = x_0 + \frac{1}{n} \not\to x_0 = f(x_0) \text{ in } Y$$

Therefore, f is not continuous at x_0 □

Watch lecture recording 06/02 for examples of continuous functions

Theorem 3.2.1: $d : X \times X \to \mathbb{R}$ is continuous

Let (X, d) be a metric space. The function $f : X \times X \to \mathbb{R}$ is continuous.
$\mathbb{R}$ is equipped with the standard metric. $X \times X$ is equipped with the product metric

Proof. Fix $(x, x') \in X \times X$. We'll show that d is continuous at (x, x').
Let $(x_n, x_n') \to (x, x')$ in $(X \times X, D)$. We'll show that

$$d(x_n, x_n') \to d(x, x') \text{ in } \mathbb{R}$$

By exercise 25, $x_n \to x$ and $x_n' \to x'$ in (X, d). By exercise 26,

$$|d(x_n, x_n') - d(x, x')| \leq d(x_n, x) + d(x_n', x') \to 0 + 0 = 0$$

$\square$

Let $X = Y = \mathbb{R}^n$, both equipped with the Euclidean metric d_2.
Let A be an $n \times n$ matrix, and define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = Ax$. Then T is continuous.

Proof. Fix $x_0 \in \mathbb{R}^n$. For all $x \in \mathbb{R}^n$ we have

$$\begin{aligned} d_2(T(x), T(x_0)) &= \|T(x) - T(x_0)\|_2 = \|T(x - x_0)\|_2 \\ &= \|A(x - x_0)\|_2 \leq C\|x - x_0\|_2 = Cd_2(x, x_0) \end{aligned}$$

Where C is a positive constant (independent of x, x_0).
Let $\epsilon > 0$. Define $\delta = \frac{\epsilon}{C}$. for all x with $d_2(x, x_0) < \delta$ we have

$$d_2(T(x), T(x_0)) \leq Cd_2(x, x_0) < C\delta = \epsilon$$

$\square$

We need: For every $n \times n$ matrix A there exists a constant C such that, for all vectors $x \in \mathbb{R}^n$

$$\|Ax\|_2 \leq C\|x\|_2$$

Proof. The i-th component of Ax is $(Ax)_i = \sum_{j=1}^n a_{ij}x_j$. By Cauchy-Schwarz,

$$|(Ax)_i|^2 \leq \left(\sum_{j=1}^n |a_{ij}|^2\right)\left(\sum_{j=1}^n |x_j|^2\right) = \left(\sum_{j=1}^n |a_{ij}|^2\right)\|x\|_2^2$$

Summing over i we have

$$\|Ax\|_2^2 = \sum_{i=1}^n |(Ax)_i|^2 \leq \underbrace{\left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2\right)}_{=C^2} \|x\|_2^2$$

$\square$

3.2.2 Continuity of linear operators between normed vector spaces

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed vector spaces. Recall that $d_X : X \times X \to \mathbb{R}$, $d(x, x') = \|x - x'\|_X$, and $d_Y : Y \times Y \to \mathbb{R}$, $d_Y(y, y') = \|y - y'\|_Y$ are metrics

Definition 3.2.3: Bounded Linear Operators

A linear operator $T : X \to Y$ is said to be **bounded** iff there exists a positive constant C such that, for all $x \in X$,

$$\|T(x)\|_Y \leq C\|x\|_X$$

Theorem 3.2.4: Linear Operator Equivalence

Let $T : X \to Y$ be a linear operator. The following are equivalent:

1. T is continuous
2. T is continuous at 0
3. T is bounded

Proof. $1 \implies 2$: Trivial
$2 \implies 3$: Assume that T is continuous at 0. We wish to show:

$$\exists C \forall x \|T(x)\|_Y \leq C\|x\|_X$$

If not, then

$$\forall C, \exists x \|T(x)\|_Y > C\|x\|_X$$

Observe that the x is $\neq 0$. Apply with $C = 1, 2, \ldots,$ to see that there exists a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that, for all n,

$$\|T(x_n)\|_Y > n\|x_n\|_X$$

Define $x'_n = \frac{1}{n}\frac{x_n}{\|x_n\|_X}$. Then $d_X(x'_n, 0) = \|x'_n\|_X = \frac{1}{n} \to 0$, therefore, $x'_n \to 0 \in X$, but $T(x'_n) \not\to 0 \in Y$ because $T(x'_n)$ is bigger than 1

$3 \implies 1$: Assume T is bounded. Fix $x_0 \in X$. Let $\epsilon > 0$. Define $\delta = \frac{\epsilon}{C}$. For all x with $d_X(x, x_0) < \delta$ we have

$$\begin{aligned} d_Y(T(x), T(x_0)) &= \|T(x) - T(x_0)\|_Y \\ &= \|T(x - x_0)\|_Y \\ &\leq C\|x - x_0\|_X \\ &= Cd_X(x, x_0) \\ &< C\delta \\ &= \epsilon \end{aligned}$$

□

Watch lecture recording 09/02 for proofs on examples:

- Let $(X, \|\cdot\|)$ be a normed vector space and define $f : \mathbb{R} \times X \to X$ by $f(\lambda, x) = \lambda x$. Define $g : X \times X \to X$ by $g(x, y) = x + y$. f and g are continuous

3.3 Fixed Points and Lipschitz

Definition 3.3.1: Lipschitz Functions

Let (X, d_X), (Y, d_Y) be metric spaces. A function $f : X \to Y$ is said to be a **Lipschitz** function iff there exists a constant L such that for all $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq Ld_X(x, x')$$

If $L < 1$, f is said to be a **contraction**

Note: Magnus uses non-standard terminology here:

- When the equation is satisifed and $L < 1$, Magnus calls f a **strict contraction**
- He uses **contraction** for a functino f that satisfies the weaker condition: for all $x, x' \in X$ with $x \neq x'$

$$d_Y(f(x), f(x')) < d_X(x, x')$$

Theorem 3.3.2: Lipschitz Continuity

Every Lipschitz function is continuous

Definition 3.3.3: Fixed Points

A **fixed point** of a function $f : S \to S$ where S is a non-empty set, is any element x of S such that $f(x) = x$
Solving equations can sometimes be reduced to finding fixed points

Watch lecture recording 06/03 for more in-depth examples

- Newton's Method for solving $f(x) = 0$
- Picard's Method for solving the Initial Value Problem

Theorem 3.3.4: Metric Space Unique Fixed Points

Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction. Then f has a unique fixed point

Proof. Let $x_1 \in X$ and define $x_{n+1} = f(x_n)$, $n = 1, 2, \ldots$
$(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Observe first that, for all n,

$$d(x_{n+1}, d + n) = d(f(x_n, f(x_{n-1})) \leq Ld(x_n, x_{n-1}))$$

Therefore, for all n,

$$d(x_{n+1}, x_n) \leq Ld(x_n, x_{n-1}) \leq L^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq L^{n-1} d(x_2, x_1)$$

This goes on for like 10 more lines, watch 09/06 42 min ☐

3.4 Equivalent Metrics

Definition 3.4.1: Equivalent Metrics

Two metrics on the same non-empty set X are said to be **equivalent** iff they have teh same open sets

Exercise: Let X be a non-empty set and d_1, d_2 be two metrics on X. Prove that d_1 and d_2 are equivalent iff the identity function

$$i : (X, d_1) \to (X, d_2)$$

is a homeomorphism (i.e. i is continuous and its inverse $i^{-1} = i : (X, d_2) \to (X, d_1)$ is continuous) test