Metric Spaces Exam Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Introduction to Metric Spaces

Theorem 1.0.1: Definition of a Metric

Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** iff for all $x,\ y,\ z \in X$,

- $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

A non-empty set X equipped with a metric d is a **metric space**

Definition 1.0.2: Real Vector Spaces

A **real vector space** V is a set with two operations $(X, +, \cdot)$, where:

- $+$ is addition, and $\cdot$ is scalar multiplication
- $(X, +)$ is an abelian group - i.e. for all (vectors) $x, y, z \in X$:
  - **Closure**: $x + y \in X$
  - **Commutativity**: $x + y = y + x$
  - **Associativity**: $x + (y + z) = (x + y) + z$
  - **Identity**: $\exists 0 \in X$ s.t. for all $x \in X$ we have $0 + x = x + 0 = x$
  - **Inverse**: $\forall x \in X$ we have $-x$ s.t. $x + (-x) = (-x) + x = 0$
- Vector space axioms: for all $x, y, z \in X$ and $\mu, \lambda \in \mathbb{R}$ we have:
  - **Closure-ish thing**: $\lambda x \in X$
  - **Distributivity 1**: $\lambda(x + y) = \lambda x + \lambda y$
  - **Distributivity 2**:$(\lambda + \mu)x = \lambda y + \mu x$
  - **Associativity**: $\lambda(\mu x) = (\lambda\mu)x$
  - **Identity**: $1x = x$

Definition 1.0.3: Normed and Inner Product Spaces

Normed Vector Spaces

A **normed vector space** is a real vector space X equipped with a **norm**, i.e. a function that assigns to every vector $x \in X$ a real number $\|x\|$ so that, for all vectors x and y in X and all real scalars a:

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- $\|ax\| = |a|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

Remark: If $(X, \|\cdot\|)$ is a normed vector space then

$$d(x, y) = \|x - y\|$$

defines a metric in X

Remark: This is a generalisation of the "length of a vector"

Inner Product Spaces

Let X be a real vector space. An **inner product** on X is a function that assigns to every pair $(x, y) \in X \times X$ a real number denoted by $\langle x, y\rangle$ and has the following properties:

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $ax + by, z = a\langle x, z\rangle + b\langle y, z\rangle$

A **real inner product space** is a real vector space equipped with an inner product. If $\langle\cdot,\cdot\rangle$ is an inner product on X, then

- $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm in X
- $d(x, y) = \|x - y\|$ defines a metric in X

Remark: This is a generalisation of the dot product

Definition 1.1.4: n-dimensional Euclidean space

Let $X = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$
For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \text{ (inner product)}$$

Properties of n-inner product: For all vectors $x, y, z \in \mathbb{R}^n$ and all real scalars a, b,

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

For $x = (x_1, x_2, \ldots, .x_n) \in \mathbb{R}^n$ define

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \sqrt{x_1^2 + x_2^2 + \cdots + c_n^2} \text{(norm)}$$

Properties of n-norm: For $x, y \in \mathbb{R}^n$, $a \in \mathbb{R}$,

- $\|x\|_2 \geq 0$ and $\|x\|_2 = 0 \iff x = 0$
- $\|ax\|_2 = |a|\|x\|_2$
- $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$ (triangle inequality)

Example 1.1.5: Examples of Metric Spaces

Unless stated otherwise let $X = \mathbb{R}^n$. The case $X = \mathbb{R}^2$ is listed in red

Name	Norm and Metric
Standard	$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$
Taxicab	$\|x\|_1 = \|x_1\| + \|x_2\| + \cdots \|x_n\|$ $d_1(x, y) = \|x_1 - y_1\| + \|x_2 - y_2\| + \cdots + \|x_n - y_n\|$
Euclidean	$\|x\|_2 = \sqrt{\|x_1\|^2 + \|x_2\|^2 + \cdots \|x_n\|^2}$ $d_2(x, y) = \sqrt{\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \cdots + \|x_n - y_n\|^2}$
p-metric	$\|x\|_p = \left(\sum_{k=1}^{n} \|x_k\|^p\right)^{1/p}$ $d_p(x, y) = \left(\sum_{k=1}^{n} \|x_k - y_k\|^p\right)^{1/p}$
Chebyshev	$\|x\|_\infty = \max\{\|x_1\|, \|x_2\|, \ldots, \|x_n\|\}$ $d(x, y) = \max\{\|x_1 - y_1\|, \|x_2 - y_2\|, \ldots, \|x_n - y_n\|\}$
Discrete	Not induced by a metric $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$
Post Office	Not induced by a metric $d(x, y) = \begin{cases} \|x\|_2 + \|y\|_2 & x = y \\ 1 & x \neq y \end{cases}$

The complex plane

Let $X = \mathbb{C}$, $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$

$$d(z, w) = |z - w|$$

If $z = a + ib, w = c + id,\ a, b, c, d \in \mathbb{R}$, then

$$z - w = (a - c) + i(b - d)$$

therefore,

$$d(z, w) = \sqrt{(a - c)^2 + (b - d)^2}$$

Example 1.1.6: Sequence Spaces

The space ℓ^1

ℓ^1 is the set of real sequences $(x_n)_{n\in\mathbb{N}}$ where $\sum_{n=1}^{\infty} |x_n|$ converges.
For $x = (x_1, \ldots, x_n, \ldots) \in \ell^1$, $y = (y_1, \ldots, y_n, \ldots) \in \ell^1$ we define

- **Norm**: $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$
- **Metric**: $d_1(x, y) = \|x - y\|_1 = \sum_{n=1}^{\infty} |x_n - y_n|$

The space ℓ^2

ℓ^2 is the set of real seqs $(x_n)_{n\in N}$ where $\sum_{n=1}^{\infty} |x_n|^2$ converges
For $x = (x_1, \ldots, x_n, \ldots) \in \ell^2$, $y = (y_1, \ldots, y_n, \ldots) \in \ell^2$ we define

- **Inner product**: $\langle x, y\rangle = \sum_{n=1}^{\infty} x_ny_n$
- **Norm**: $\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$
- **Metric**: $d_2(x, y) = \|x - y\|_2 = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2\right)^{1/2}$

Thm: ℓ^2 is a real vector space

The space ℓ^∞

ℓ^∞ is the set of all bounded sequences of real numbers For $x = (x_1, \ldots, x_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots) \in \ell^\infty$

- **Norm**: $\|x\|_\infty = \sup\{|x_1|, \ldots, |x_n|, \ldots\}$
- **Metric**: $\|x - y\|_\infty = \sup\{|x_1 - y_1|, \ldots, |x_n - y_n|, \ldots\}$

The space $C([a, b])$

$X = C([a, b])$ is the set of all continuous functions $f : [a, b] \to \mathbb{R}$

- **Norm**: $\|f\|_\infty = \max\{|f(x)| : a \leq x \leq b\}$
- **Metric**: $d_\infty(f, g) = \|f - g\| = \max\{|f(x) - g(x)| : a \leq x \leq b\}$

The L^1 metric

X is the set of all continuous functions $f : [a, b] \to \mathbb{R}$

- **Norm**: $\|f\|_1 = \int_a^b |f(x)|dx$
- **Metric**: $d_2(f, g) = \|f - g\|_1 = \int_a^b |f(x) - g(x)|dx$

The L^2 metric

X is the set of all continuous functions $f : [a, b] \to \mathbb{R}$

- **Inner Product**: $\langle f, g\rangle = \int_a^b f(x)g(x)dx$
- **Norm**: $\|f\|_2 = \langle f, f\rangle^{1/2} = \left(\int_a^b |f(x)|^2dx\right)^{1/2}$
- **Metric**: $d_1(f, g) = \left(\int_a^b |f(x) - g(x)|^2dx\right)^{1/2}$

Definition 1.1.7: Metric Subspaces

Let (X, d) be a metric space and Y a non-empty subset of X. Define

- $d_Y : Y \times Y \to \mathbb{R}$
- $d_y(y, y') = d(y, y')$

Then d_Y is a metric on Y. d_Y is called the **induced** or **inherited** metric, and (Y, d_Y) is said to be a metric subspace of the metric space (X, d)

Definition 1.1.8: Open Ball

Let (X, d) be a metric space, c be a point in X, and $r > 0$. The **open ball** with center c and radius r is defined by

$$B(c, r) = \{x \in X : d(c, x) < r\}$$

2 Convergence

2.1 Convergent Sequences in Metric Spaces

On the real line, $x_n \to x$ iff for every positive ϵ, there exists an index N such that for all indices n where $n \geq N$, we have $|x_n - x| < \epsilon$.

Definition 2.1.1: Convergent Sequence

Let (X, d) be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in X, and $x \in X$. We say that $(x_n)_{n=1}^{\infty}$ converges to x iff for every positive ϵ, there exists an index N s.t. for all indices n with $n \geq N$a we have $d(x_n, x) < \epsilon$.
Observe that:

- $d(x_n, x) < \epsilon$ is equivalent to $x_n \in B(x, \epsilon)$.
- $x_n \to x$ in (X, d) iff $d(x_n, x) \to 0$ on the real line

Theorem 2.1.2: Uniqueness of metric limit

- Let (X, d) be a metric space, and $x, x' \in X$, $x \neq x'$. Then there exists a positive radius r s.t. $B(x, r) \cap B(x', r) = \emptyset$
- A sequence in a metric space can have at most one limit

Example 2.1.3: convergence in $(\mathbb{R}^N, d_2)$

A sequence

$$\begin{aligned} x_1 &= (x_{11}, \dots, x_{1j}, \dots x_{1N}) \\ x_2 &= (x_{21}, \dots, x_{2j}, \dots x_{2N}) \\ &\vdots \\ x_n &= (x_{n1}, \dots, x_{nj}, \dots x_{nN}) \\ &\vdots \\ &\downarrow \\ x &= (x_1, \dots, x_j, \dots, x_N) \end{aligned}$$

in $\mathbb{R}^N, d_2$ converges to $x = (x_1, \dots, x_j, \dots, x_N)$ iff for each j,

$$x_{nj} \xrightarrow[j \to +\infty]{} x_j$$

Definition 2.1.4: Bounded Sequence

A sequence in a metric space is said to be **bounded** iff there exists an open ball that contains all of its terms

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Note: this is the same definition as "sequence is bounded if there is upper and lower bound", as open ball implies the same thing

Theorem 2.1.5

Every convergence is bounded

2.2 Cauchy Sequences

Convergence: For every ϵ, there is an N such that for $n \geq N$, $d(x_n, x) < \epsilon$

$$x_1 \quad x_2 \quad \cdots \quad x_N \quad \cdots x_n \quad \cdots \quad \to x$$

Replace x by any x_m with $m \geq N$

$$x_1 \quad x_2 \quad \cdots \quad x_N \quad \cdots x_n \quad \cdots \quad x_m \quad \cdots$$

'$d(x_n, x) < \epsilon$' becomes '$\forall m \geq N,\ d(x_n, x_m) < \epsilon$'

Definition 2.2.1: Cauchy Sequence

A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) is said to be a **Cauchy sequence** iff for every positive ϵ, there exists an index N, s.t. for all indices n, m with $n, m \geq N$,

$$d(x_n, x_m) < \epsilon$$

Theorem 2.2.2

If a sequence in a metric space converges, then it is a Cauchy sequence

Definition 2.2.3: Complete Metric Spaces

A metric space is said to be **complete** if and only if every Cauchy Sequence is convergent

Examples:

- $\mathbb{R}$ with the standard metric is complete
- $\mathbb{Q}$ with the standard metric is not complete
- $(0, 1)$ with the standard metric is not complete
- $[0, 1]$ with the standard metric is complete
- $\mathbb{R}^n$, ℓ^p, $C([a, b])$ is complete (proof later)

2.3 Open sets and closed sets

Definition 2.3.1: Open Sets and Closed Sets

Let (X, d) be a metric space.

- A subset G of X is said to be **open** iff for every point x in G there exists a positive radius r such that $B(x, r) \subseteq G$.
- A subset F of X is said to be **closed** iff F^c is open

Definition 2.3.2: Discrete Metric Space

A metric space is called **discrete** iff all its subsets are open (equiv. all subsets are closed)
Example: $[0, 1] \cap (2, 3)$

Theorem 2.3.3: Properties of open sets

Let (X, d) be a metric space

1. The union of any family of open sets is an open set
2. The intersection of finitely many open sets is an open set

Theorem 2.3.4: Infinite open sets

The intersection of infinitely many open sets is not always an open set
For example, let $G_n = (-\frac{1}{n}, \frac{1}{n}), n = 1, 2, \dots$ on the real line with the standard metric.
Each G_n is open but

$$\bigcap_{n=1}^{\infty} G_n = \{0\}$$

Theorem 2.3.5: Relatively open sets

Let (X, d) be a metric space and A be a non-empty subset of X equipped with the induced metric d_A. Let $G \subseteq A$. G is open in (A, d_A) iff there exists a subset O of X, open in (X, d), such that $G = A \cap O$
The open sets of (A, d_A) are sometimes referred to as **relatively open**

Theorem 2.3.6

Let (X, d) be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in X and x be a point in X.
$x_n \to x$ iff every open set that contains x contains eventually all terms of the sequence

Definition 2.3.7: Neighbourhoods of points

An **open neighbourhood** of a point x is any open set that contains x. $x_n \to x$ iff every open neighbourhood of x contains eventually all terms of the sequence.

A **neighbourhood** of a point x is a set that contains an open neighbourhood of x. $x_n \to x$ iff every neighbourhood of x contains eventually all terms of the sequence.

Theorem 2.3.8: Properties of Closed sets

Let (X, d) be a metric space.

1. The intersection of any family of closed sets is a closed set
2. The union of finitely many closed sets is a closed set.

Theorem 2.3.9: Infinite closed sets

The union of infinitely many closed sets is not always a closed set.
For example, let $F_n = [\frac{1}{n}, 1], n = 1, 2, \dots,$ on the real line with the standard metric. Each F_n is closed but

$$\bigcup_{n=1}^{\infty} F_n = (0, 1]$$

is not closed.

Theorem 2.3.10

A subset F of a metric space is closed iff the limit of every convergent sequence of elements of F belongs to F

- In any metric space (X,d), singletons $F=\{x\}$ are closed.
- In any metric space, any finite set is closed because

$$\{x_1,\dots,x_n\}=\{x_1\}\cup\cdots\cup\{x_n\}$$

2.4 Closure

Definition 2.4.1: Closure

Let (X,d) be a metric space and $A\subseteq X$. The **closure** of A, deonted by $\overline{A}$, is the smallest closed subset of X that contains A
There exists at least one closed subset of X that contains A, namely X itself. The smallest closed subset of X that contains A is

$$\bigcap_{A\subseteq F\subseteq X,\ F\text{closed}} F$$

Theorem 2.4.2: Properties of Closure

Let (X,d) be a metric space and $A,\ B\subseteq X$.

1. $\overline{\emptyset}=\emptyset$ and $\overline{X}=X$
2. $A\subseteq\overline{A}$ and $\overline{A}$ is closed
3. A is closed iff $A=\overline{A}$
4. $\overline{\overline{A}}=\overline{A}$
5. If $A\subseteq B$, then $\overline{A}\subseteq\overline{B}$
6. $\overline{A\cup B}=\overline{A}\cup\overline{B}$

- On the real line with the standard metric, $\overline{(a,b)}=[a,b]$
- In $\mathbb{R}^n$ with the Euclidean metric d_2, the closure of the open ball $B(c,r)$ is the closed ball $\{x\in\mathbb{R}^n : d_2(x,c)\le r\}$
- On the complex plane with its standard metric, the closure of an open disc is the corresponding closed disc
- Let X be a non-empty set with the discrete metric, $c\in X$ and $r=1$. Then $B(c,1)=\{c\}$, therefore $\overline{B(c,1)}-\overline{\{c\}}=\{c\}$, while

  $$\{x\in X : d(x,c)\le 1\}=X$$

  The closure of an open ball is not always equal to the corresponding closed ball
- $X=\mathbb{R}$, $d(x,y)=|x-y|$. $\overline{\mathbb{Q}}=\mathbb{R}$

Definition 2.4.3: Dense Subset of a Metric Space

Let (X,d) be a metric space. A subset D of X is said to be **dense** iff $\overline{D}=X$

Random fact: In $\mathbb{R}^n$ with the Euclidean metric d_2, $\mathbb{Q}^n$ is dense.

Theorem 2.4.4: Closure Equivalence

Let (X,d) be a metric space, $A\subseteq X, x\in X$. The following are equivalent

1. $x\in\overline{A}$
2. For every positive r, $B(x,r)\cap A\neq\emptyset$
3. There exists a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n\in A$ for all n, such that $a_n\to x$

A point x with any of these properties is called an **adherent point** of A. So, $\overline{A}$ is the set of all adherent points of A.

Definition 2.4.5: Limit points of sets

Let (X,d) be a metric space, $A\subseteq X$ and $x\in X$. We say that x is a **limit point** or an **accumulation point** of A iff every open ball centered at x contains an element of A distinct from x, i.e.

$$\forall r>0\quad (B(x,r)\backslash\{x\})\cap A\neq\emptyset$$

The set of all limit points of A is called the **derived set** of A and is denoted by A' or $\tilde{A}$.

2.5 Continuous functions between metric spaces

Definition 2.5.1: Continuity at a point

Let (X,d_X), (Y,d_Y) be metric spaces and $f:X\to Y$ be a function. We say that f is **continuous at a point** x_0 in X iff for for every positive ϵ, there exists a positive δ, s.t., for all $x\in X$ with $d_X(x,x_0)<\delta$ we have $d_Y(f(x),f(x_0))<\epsilon$

Alternatively, f is **continuous at a point** $x_0\in X$ iff, for every positive ϵ, there exists a positive δ, such that, for all $x\in B_X(x_0,\delta)$ we have $f(x)\in B_Y(f(x_0),\epsilon)$

Definition 2.5.2: Continuity of a function

Let $(X,d_X),(Y,d_Y)$ be metric spaces. A function $f:X\to Y$ is said to be **continuous** iff it is continuous at every point in X

Theorem 2.5.3

Let $(X,d_X),(Y,d_Y)$ be metric spaces, $f:X\to Y$ be a function and x_0 be a point in X. Then f is continuous at x_0 iff for every open neighbourhood G of $f(x_0)$ there exists an open neighbourhood O of x_0 such that, for all $x\in O$, we have $f(x)\in G$

Theorem 2.5.4: Continuity and Convergence

Let (X,d_X), (Y,d_Y) be metric spaces, x_0 be a point in X, and $f:X\to Y$ be a function. The following are equivalent:

1. f is continuous at x_0
2. For every sequence $(x_n)_{n=1}^{\infty}$ in X, if $x_n\xrightarrow[n\to+\infty]{}$ in (X,d_X), then $f(x_n)\xrightarrow[n\to+\infty]{} f(x_0)$ in (Y,d_Y)

Theorem 2.5.5: Continuity and Open Sets

Let (X,d_X), (Y,d_Y) be metric spaces. A function $f:X\to Y$ is continuous iff the inverse image $f^{-1}(G)$ of any open subset G of Y is an open subset of X

3 Topology!!!

3.1 Homeomorphisms and Topological Properties

Definition 3.1.1: Topological Space

A **topological space** is a set X together with a family $\mathcal{T}$ of subsets of X that has the following properties:

- $\emptyset, X\in\mathcal{T}$
- Any union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$
- Any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$

$\mathcal{T}$ is called a **topology** and the elements of $\mathcal{T}$ are called **open sets**

Definition 3.1.2: Continuity of Topological Spaces

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be two topological spaces. A function $f:X\to Y$ is said to be **continuous** iff for every G in $\mathcal{T}_Y$ the pre-image $f^{-1}(G)$ is an element of $\mathcal{T}_X$.
f is said to be a **homeomorphism** iff it is a continuous bijection and its inverse is continuous.
If such a homeomorphism exists then $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ are said to be **homeomorphic**

If $(X,\mathcal{T}_X)$ and (Y,T_Y) are homeomorphic, and one of them is compact or connected or separable etc etc, then so is the other
Properties that are preserved by homeomorphisms are called topological properties

Theorem 3.1.3: $d:X\times X\to\mathbb{R}$ is continuous

Let (X,d) be a metric space. The function $f:X\times X\to\mathbb{R}$ is continuous.
$\mathbb{R}$ is equipped with the standard metric. $X\times X$ is equipped with the product metric

3.1.4 Continuity of linear operators between normed vector spaces

Let $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$ be normed vector spaces. Recall that $d_X:X\times X\to\mathbb{R}$, $d(x,x')=\|x-x'\|_X$, and $d_Y:Y\times Y\to\mathbb{R}$, $d_Y(y,y')=\|y-y'\|_Y$ are metrics

Definition 3.1.5: Bounded Linear Operators

A linear operator $T:X\to Y$ is said to be **bounded** iff there exists a positive constant C such that, for all $x\in X$,

$$\|T(x)\|_Y\le C\|x\|_X$$

Theorem 3.1.6: Linear Operator Equivalence

Let $T:X\to Y$ be a linear operator. The following are equivalent:

1. T is continuous
2. T is continuous at 0
3. T is bounded

- Let $(X,\|\cdot\|)$ be a normed vector space and define $f:\mathbb{R}\times X\to X$ by $f(\lambda,x)=\lambda x$. Define $g:X\times X\to X$ by $g(x,y)=x+y$. f and g are continuous

3.2 Fixed Points and Lipschitz

Definition 3.2.1: Lipschitz Functions

Let (X, d_X), (Y, d_Y) be metric spaces. A function $f : X \to Y$ is said to be a **Lipschitz** function iff there exists a constant L such that for all $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq L d_X(x, x')$$

If $L < 1$, f is said to be a **contraction**

Note: Magnus uses non-standard terminology here:

- When the equation is satisifed and $L < 1$, Magnus calls f a **strict contraction**
- He uses **contraction** for a functino f that satisfies the weaker condition: for all $x, x' \in X$ with $x \neq x'$

$$d_Y(f(x), f(x')) < d_X(x, x')$$

Theorem 3.2.2: Lipschitz Continuity

Every Lipschitz function is continuous

Definition 3.2.3: Fixed Points

A **fixed point** of a function $f : S \to S$ where S is a non-empty set, is any element x of S such that $f(x) = x$
Solving equations can sometimes be reduced to finding fixed points

Watch lecture recording 06/03 for more in-depth examples

- Newton's Method for solving $f(x) = 0$
- Picard's Method for solving the Initial Value Problem

Theorem 3.2.4: Metric Space Unique Fixed Points

Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction. Then f has a unique fixed point

Proof. Let $x_1 \in X$ and define $x_{n+1} = f(x_n)$, $n = 1, 2, \ldots$
$(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Observe first that, for all n,

$$d(x_{n+1}, d + n) = d(f(x_n, f(x_{n-1})) \leq Ld(x_n, x_{n-1}))$$

Therefore, for all n,

$$d(x_{n+1}, x_n) \leq Ld(x_n, x_{n-1}) \leq L^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq L^{n-1} d(x_2, x_1)$$

This goes on for like 10 more lines, watch 09/06 42 min $\square$

3.3 Equivalent Metrics

Definition 3.3.1: Equivalent Metrics

Two metrics on the same non-empty set X are said to be **equivalent** iff they have teh same open sets

Exercise: Let X be a non-empty set and d_1, d_2 be two metrics on X. Prove that d_1 and d_2 are equivalent iff the identity function

$$i : (X, d_1) \to (X, d_2)$$

is a homeomorphism (i.e. i is continuous and its inverse $i^{-1} = i : (X, d_2) \to (X, d_1)$ is continuous)
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