# Metric Spaces Exam Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

## 1 Introduction to Metric Spaces

### Definition 1: Definition of a Metric

Let $X$ be a non-empty set. A function $d: X \times X \to \mathbb{R}$ is called a **metric** iff for all $x, y, z \in X$,

- $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$ (Triangle Inequality)

A non-empty set $X$ equipped with a metric $d$ is a **metric space**

### Definition A: Real Vector Spaces

A **real vector space** $V$ is a set with two operations $(X, +, \cdot)$, where:

- $+$ is addition, and $\cdot$ is scalar multiplication
- $(X, +)$ is an abelian group - i.e. for all (vectors) $x, y, z \in X$:
  - **Closure**: $x + y \in X$
  - **Commutativity**: $x + y = y + x$
  - **Associativity**: $x + (y + z) = (x + y) + z$
  - **Identity**: $\exists 0 \in X$ s.t. for all $x \in X$ we have $0 + x = x + 0 = x$
  - **Inverse**: $\forall x \in X$ we have $-x$ s.t. $x + (-x) = (-x) + x = 0$
- Vector space axioms: for all $x, y, z \in X$ and $\mu, \lambda \in \mathbb{R}$ we have:
  - **Closure-ish thing**: $\lambda x \in X$
  - **Distributivity 1**: $\lambda(x + y) = \lambda x + \lambda y$
  - **Distributivity 2**:$(\lambda + \mu)x = \lambda y + \mu x$
  - **Associativity**: $\lambda(\mu x) = (\lambda\mu)x$
  - **Identity**: $1x = x$

### Definition B: Normed and Inner Product Spaces

**Def 5: Normed Vector Spaces**

A **normed vector space** is a real vector space $X$ equipped with a **norm**, i.e. a function that assigns to every vector $x \in X$ a real number $\|x\|$ so that, for all vectors $x$ and $y$ in $X$ and all real scalars $a$:

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- $\|ax\| = |a|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

**Remark**: If $(X, \|\cdot\|)$ is a normed vector space then

$$d(x,y) = \|x - y\|$$

defines a metric in $X$

**Def 6: Inner Product Spaces**

Let $X$ be a real vector space. An **inner product** on $X$ is a function that assigns to every pair $(x, y) \in X \times X$ a real number denoted by $\langle x, y\rangle$ and has the following properties:

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $ax + by, z = a\langle x, z\rangle + b\langle y, z\rangle$

A **real inner product space** is a real vector space equipped with an inner product. If $\langle \cdot, \cdot\rangle$ is an inner product on $X$, then

- $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm in $X$
- $d(x,y) = \|x - y\|$ defines a metric in $X$

### Definition C: $n$-dimensional Euclidean space

Let $X = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$
For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

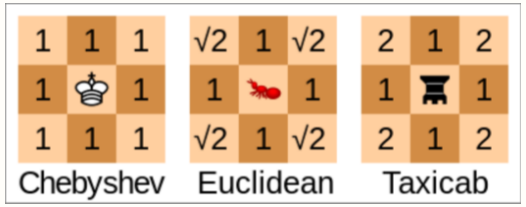
$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \text{ (inner product)}$$

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \sqrt{x_1^2 + x_2^2 + \cdots + c_n^2} \text{(norm)}$$

### Example D: Examples of Metric Spaces

Unless stated otherwise let $X = \mathbb{R}^n$. The case $X = \mathbb{R}^2$ is listed in red

| Name | Norm and Metric |
|---|---|
| Standard | $X = \mathbb{R}$ and $\lvert x\rvert =$ Absolute Value<br>$d(x,y) = \lvert x - y\rvert$ |
| Taxicab | $\\|x\\|_1 = \lvert x_1\rvert + \lvert x_2\rvert + \cdots \lvert x_n\rvert$<br>$d_1(x,y) = \lvert x_1 - y_1\rvert + \lvert x_2 - y_2\rvert + \cdots + \lvert x_n - y_n\rvert$ |
| Euclidean | $\\|x\\|_2 = \sqrt{\lvert x_1\rvert^2 + \lvert x_2\rvert^2 + \cdots \lvert x_n\rvert^2}$<br>$d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$ |
| $p$-metric | $\\|x\\|_p = \left(\sum_{k=1}^{n} \lvert x_k\rvert^p\right)^{1/p}$<br>$d_p(x,y) = \left(\sum_{k=1}^{n} \lvert x_k - y_k\rvert^p\right)^{1/p}$ |
| Chebyshev | $\\|x\\|_\infty = \max\{\lvert x_1\rvert, \lvert x_2\rvert, \ldots, \lvert x_n\rvert\}$<br>$d(x,y) = \max\{\lvert x_1 - y_1\rvert, \lvert x_2 - y_2\rvert, \ldots, \lvert x_n - y_n\rvert\}$ |
| Discrete | $d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$ |
| Post Office | $d(x,y) = \begin{cases} \\|x\\|_2 + \\|y\\|_2 & x = y \\ 1 & x \neq y \end{cases}$ |




**The complex plane**

Let $X = \mathbb{C}$, $d: \mathbb{C} \times \mathbb{C} \to \mathbb{R}$

$$d(z, w) = |z - w|$$

If $z = a + ib, w = c + id$, $a, b, c, d \in \mathbb{R}$, then

$$z - w = (a - c) + i(b - d)$$

therefore,

$$d(z, w) = \sqrt{(a - c)^2 + (b - d)^2}$$

### Example E: Sequence Spaces

**The space $\ell^1$**

$\ell^1$ is the set of real sequences $(x_n)_{n\in\mathbb{N}}$ where $\sum_{n=1}^{\infty}|x_n|$ converges.
For $x = (x_1, \ldots, x_n, \ldots) \in \ell^1$, $y = (y_1, \ldots, y_n, \ldots) \in \ell^1$ we define

- **Norm**: $\|x\|_1 = \sum_{n=1}^{\infty}|x_n|$
- **Metric**: $d_1(x,y) = \|x - y\|_1 = \sum_{n=1}^{\infty}|x_n - y_n|$

**The space $\ell^2$**

$\ell^2$ is the set of real seqs $(x_n)_{n\in N}$ where $\sum_{n=1}^{\infty}|x_n|^2$ converges
For $x = (x_1, \ldots, x_n, \ldots) \in \ell^2$, $y = (y_1, \ldots, y_n, \ldots) \in \ell^2$ we define

- **Inner product**: $\langle x, y\rangle = \sum_{n=1}^{\infty} x_ny_n$
- **Norm**: $\|x\|_2 = \left(\sum_{n=1}^{\infty}|x_n|^2\right)^{1/2}$
- **Metric**: $d_2(x,y) = \|x - y\|_2 = \left(\sum_{n=1}^{\infty}|x_n - y_n|^2\right)^{1/2}$

**Thm**: $\ell^2$ is a real vector space

**The space $\ell^\infty$**

$\ell^\infty$ is the set of all bounded sequences of real numbers For $x = (x_1, \ldots, x_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots) \in \ell^\infty$

- **Norm**: $\|x\|_\infty = \sup\{|x_1|, \ldots, |x_n|, \ldots\}$
- **Metric**: $\|x - y\|_\infty = \sup\{|x_1 - y_1|, \ldots, |x_n - y_n|, \ldots\}$

**The space $C([a,b])$**

$X = C([a,b])$ is the set of all continuous functions $f: [a,b] \to \mathbb{R}$

- **Norm**: $\|f\|_\infty = \max\{|f(x)| : a \leq x \leq b\}$
- **Metric**: $d_\infty(f,g) = \|f - g\| = \max\{|f(x) - g(x)| : a \leq x \leq b\}$

**The $L^1$ metric**

$X$ is the set of all continuous functions $f: [a,b] \to \mathbb{R}$

- **Norm**: $\|f\|_1 = \int_a^b |f(x)|dx$
- **Metric**: $d_2(f,g) = \|f - g\|_1 = \int_a^b |f(x) - g(x)|dx$

**The $L^2$ metric**

$X$ is the set of all continuous functions $f: [a,b] \to \mathbb{R}$

- **Inner Product**: $\langle f, g\rangle = \int_a^b f(x)g(x)dx$
- **Norm**: $\|f\|_2 = \langle f, f\rangle^{1/2} = \left(\int_a^b |f(x)|^2dx\right)^{1/2}$
- **Metric**: $d_1(f,g) = \left(\int_a^b |f(x) - g(x)|^2dx\right)^{1/2}$

### Definition F: Metric Subspaces

**Ex 7**: Let $(X, d)$ be a metric space and $Y$ a non-empty subset of $X$. Define

- $d_Y : Y \times Y \to \mathbb{R}$
- $d_y(y, y') = d(y, y')$

Then $d_Y$ is a metric on $Y$. $d_Y$ is called the **induced** or **inherited** metric, and $(Y, d_Y)$ is said to be a metric subspace of the metric space $(X, d)$

### Theorem G: a lack of equality or fair treatment in t...

**Good old fashioned Triangle Inequality**

If it ain't broke...

$$|x+y| \le |x| + |y| \quad \text{and} \quad |x-y| \ge ||x| - |y||$$

**Cauchy-Schwarz Inequality**

For all $x$ and $y$ of an inner product space:

$$|\langle x, y\rangle| \le \|x\|\|y\|$$

**Minkowski's Inequality**

Let $p \ge 1$, and real numbers $x_i, y_i$, $(i = 1, \dots, n)$. Then

$$\|x+y\|_p \le \|x\|_p + \|y\|_p$$

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

**Ex 56 (Young's Inequality)**

Let $1 \le p, q \le \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$, and $a, b \le 0$. Then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

**Thm 169 (Hölder Inequality)**

Let $1 \le p, q \le \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$ and $x, y \in \mathbb{R}^n$. Then

$$\sum_{i=1}^{n} |x_i y_i| \le \|x\|_p \|y\|_q$$

### Definition 166: Equivalent Norms

Two norms on the same real vector space are said to be equivalent iff their corresponding metrics are equivalent

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**Thm 167**: If $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on the same real vector space $X$ and there exist positive constants $C$ and $C'$ s.t., for all $x \in X$,

$$D\|x\|_1 \le \|x\|_2 \le C'\|x\|_1$$

then they are equivalent

**Equivalence Theorems of $p$-metrics**

**171**: Any of the following norms are equivalent:

$$\|x\|_p = (|x_1|^p + \dots + |x_n|^p)^{1/p},\ x \in \mathbb{R}^n,\ 1 \le p < \infty$$
$$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\},\ x \in \mathbb{R}^n$$

**172**: Let $1 \le p \le q < \infty$. For all $x \in \mathbb{R}^n$:

$$\|x\|_q \le \|x\|_p$$

As a consequence,

$$\|x\|_\infty \le \|x\|_q \le \|x\|_p \le \|x\|_1$$

**173**: All norms in $\mathbb{R}^n$ are equivalent

### Definition 8: Open Ball

Let $(X, d)$ be a metric space, $c$ be a point in $X$, and $r > 0$. The **open ball** with center $c$ and radius $r$ is defined by

$$B(c, r) = \{x \in X : d(c, x) < r\}$$

## 2 Convergence

### Definition 15: Convergent Sequence

Let $(X, d)$ be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in $X$, and $x \in X$. We say that $(x_n)_{n=1}^{\infty}$ converges to $x$ iff for every $\epsilon > 0$, there exists an index $N$ s.t. for all $n \ge N$ we have $d(x_n, x) < \epsilon$.

Observe that:

- $d(x_n, x) < \epsilon$ is equivalent to $x_n \in B(x, \epsilon)$.
- $x_n \to x$ in $(X, d)$ iff $d(x_n, x) \to 0$ on the real line

### Theorem 16: Uniqueness of metric limit

- Let $(X, d)$ be a metric space, and $x, x' \in X$, $x \ne x'$. Then there exists a positive radius $r$ s.t. $B(x, r) \cap B(x', r) = \emptyset$
- A sequence in a metric space can have at most one limit

### Definition 19: Bounded Sequence

A sequence in a metric space is said to be **bounded** iff there exists an open ball that contains all of its terms

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**Note**: this is the same definition as "sequence is bounded if there is upper and lower bound", as open ball implies the same thing

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**Thm 20**: Every convergent sequence is bounded

### Definition 21: Cauchy Sequence

A sequence $(x_n)_{n=1}^{\infty}$ in a metric space $(X, d)$ is said to be a **Cauchy sequence** iff for every positive $\epsilon$, there exists an index $N$, s.t. for all indices $n, m$ with $n, m \ge N$,

$$d(x_n, x_m) < \epsilon$$

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**Thm 22**: If a sequence in a metric space converges, then it is a Cauchy sequence. **Note**: the converse is not true

### Definition 24: Complete Metric Spaces

A metric space is said to be **complete** if and only if every Cauchy Sequence is convergent

### Example 25: Examples of Complete Metric Spaces

- $\mathbb{R}$ with the standard metric is complete
- $\mathbb{Q}$ with the standard metric is not complete
- $(0, 1)$ with the standard metric is not complete
- $[0, 1]$ with the standard metric is complete
- $\mathbb{R}^n$, $\ell^p$, $C([a, b])$ is complete (proof later)

### Definition 26: Open Sets and Closed Sets

Let $(X, d)$ be a metric space.

- A subset $G$ of $X$ is said to be **open** iff for every point $x$ in $G$ there exists a positive radius $r$ such that $B(x, r) \subseteq G$.
- A subset $F$ of $X$ is said to be **closed** iff $F^c$ is open

### Definition 31: Discrete Spaces and Clopens

A metric space is called **discrete** iff all its subsets are open (equiv. all subsets are closed)

**Example:** $[0, 1] \cap (2, 3)$

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**Def 33**: A set that is both open and closed is called **clopen**

### Theorem 34: Properties of open and closed sets

Let $(X, d)$ be a metric space

1. The union of **any family** of open sets is an open set
2. The intersection of **finitely many** open sets is an open set
3. The intersection of **any family** of closed sets is an closed set
4. The union of **finitely many** closed sets is an closed set

### Remark 35: Infinite open sets

The intersection of infinitely many open sets isn't always an open set e.g., let $G_n = (-\frac{1}{n}, \frac{1}{n}), n = 1, 2, \dots$ on $\mathbb{R}$ with the standard metric. Each $G_n$ is open but

$$\bigcap_{n=1}^{\infty} G_n = \{0\}$$

### Theorem 18: Relatively open sets

Let $(X, d)$ be a metric space and $A$ a nonempty subset of $X$ equipped with the induced metric $d_A$. Let $G \subseteq A$. Then $G$ is open in $(A, d_A)$ iff there exists a subset $O$ of $X$, open in $(X, d)$, s.t. $G = A \cap O$

The open sets of $(A, d_A)$ are referred to as **relatively open**

### Theorem 36

Let $(X, d)$ be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in $X$ and $x$ be a point in $X$.

$x_n \to x$ iff every open set that contains $x$ contains eventually all terms of the sequence

### Definition H: Neighbourhoods of points

An **open neighbourhood** of a point $x$ is any open set that has $x$.

$x_n \to x$ iff every open neighbourhood of $x$ contains eventually all terms of the sequence.

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A **neighbourhood** of a point $x$ is a set that contains an open neighbourhood of $x$.

$x_n \to x$ iff every neighbourhood of $x$ contains eventually all terms of the sequence.

### Remark 38: Infinite closed sets

The union of infinitely many closed sets is not always a closed set. For example, let $F_n = [\frac{1}{n}, 1], n = 1, 2, \dots,$ on the real line with the standard metric. Each $F_n$ is closed but

$$\bigcup_{n=1}^{\infty} F_n = (0, 1]$$

is not closed.

### Theorem 41

A subset $F$ of a metric space is closed iff the limit of every convergent sequence of elements of $F$ belongs to $F$

- In any metric space $(X,d)$, singletons $F=\{x\}$ are closed.
- In any metric space, any finite set is closed because

$$\{x_1,\ldots,x_n\}=\{x_1\}\cup\cdots\cup\{x_n\}$$

### Definition 43: Closure

Let $(X,d)$ be a metric space and $A\subseteq X$. The **closure** of $A$, deonted by $\overline{A}$, is the smallest closed subset of $X$ that contains $A$

There exists at least one closed subset of $X$ that contains $A$, namely $X$ itself. The smallest closed subset of $X$ that contains $A$ is

$$\bigcap_{\substack{A\subseteq F\subseteq X\\ F\,\text{closed}}} F$$

### Theorem 44: Properties of Closure

Let $(X,d)$ be a metric space and $A,\ B\subseteq X$.

1. $\overline{\emptyset}=\emptyset$ and $\overline{X}=X$
2. $A\subseteq\overline{A}$ and $\overline{A}$ is closed
3. $A$ is closed iff $A=\overline{A}$
4. $\overline{\overline{A}}=\overline{A}$
5. If $A\subseteq B$, then $\overline{A}\subseteq\overline{B}$
6. $\overline{A\cup B}=\overline{A}\cup\overline{B}$

### Definition 49: Dense Subset of a Metric Space

Let $(X,d)$ be a metric space. A subset $D\subseteq X$ is **dense** iff $\overline{D}=X$

**Random Fact**: In $\mathbb{R}^n$ with the Euclidean metric $d_2$, $\mathbb{Q}^n$ is dense.

### Theorem 50: Adherent Points

Let $(X,d)$ be a metric space, $A\subseteq X, x\in X$. The following are equiv.

1. $x\in\overline{A}$
2. For every positive $r$, $B(x,r)\cap A\neq\emptyset$
3. There exists a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n\in A$ for all $n$, such that $a_n\to x$

A point $x$ with any of these properties is called an **adherent point** of $A$. So, $\overline{A}$ is the set of all adherent points of $A$.

### Definition 52: Limit points of sets

Let $(X,d)$ be a metric space, $A\subseteq X$ and $x\in X$. We say that $x$ is a **limit point** or an **accumulation point** of $A$ iff every open ball centered at $x$ contains an element of $A$ distinct from $x$, i.e.

$$\forall r>0\quad (B(x,r)\backslash\{x\})\cap A\neq\emptyset$$

The set of all limit points of $A$ is called the **derived set** of $A$ and is denoted by $A'$ or $\tilde{A}$.

**Thm 78**: Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces, $x_0$ be a limit point of $X$, $y_0\in Y$ and $f:X\to Y$ be a function.

We say that $\lim_{x\to x_0}f(x)=y_0$ iff for all $\epsilon>0$, there exists $\delta>0$ such that for all $x\in B_X(x_0,\delta)\backslash\{x_0\}$ we have

$$f(x)\in B_Y(y_0,\epsilon)$$

### Definition 54: Continuity at a point

Let $(X,d_X)$, $(Y,d_Y)$ be metric spaces and $f:X\to Y$ be a function. We say that $f$ is **continuous at a point** $x_0$ in $X$ iff...

- for every $\epsilon>0$, there exists a $\delta>0$, such that, for all $x\in X$ with $d_X(x,x_0)<\delta$ we have

$$d_Y(f(x),f(x_0))<\epsilon$$

- for every $\epsilon>0$, there exists a $\delta>0$, such that, for all $x\in B_X(x_0,\delta)$ we have

$$f(x)\in B_Y(f(x_0),\epsilon)$$

- **Thm 57**: for every open nbhd $G$ of $f(x_0)$, there exists an open nbhd $O$ of $x_0$ such that, for all $x\in O$, we have $f(x)\in G$

**Def 55: Continuity of a Function**

Let $(X,d_X),(Y,d_Y)$ be metric spaces. A function $f:X\to Y$ is said to be **continuous** iff it is continuous at every point in $X$

### Theorem 58: Continuity and Convergence

Let $(X,d_X)$, $(Y,d_Y)$ be metric spaces, $x_0$ be a point in $X$, and $f:X\to Y$ be a function. The following are equivalent:

1. $f$ is continuous at $x_0$
2. For every sequence $(x_n)_{n=1}^{\infty}$ in $X$, if $x_n \xrightarrow[n\to+\infty]{} $ in $(X,d_X)$, then $f(x_n)\xrightarrow[n\to+\infty]{} f(x_0)$ in $(Y,d_Y)$

### Theorem 59: Continuity and Open Sets

Let $(X,d_X)$, $(Y,d_Y)$ be metric spaces. A function $f:X\to Y$ is continuous iff the inverse image $f^{-1}(G)$ of any open subset $G$ of $Y$ is an open subset of $X$

### Definition 60: Topological Space

A **topological space** is a set $X$ together with a family $\mathcal{T}$ of subsets of $X$ that has the following properties:

- $\emptyset, X\in\mathcal{T}$
- Any union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$
- Any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$

$\mathcal{T}$ is called a **topology** and the elements of $\mathcal{T}$ are called **open sets**

### Definition 61: Continuity of Topological Spaces

- Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be two topological spaces. A function $f:X\to Y$ is said to be **continuous** iff for every $G$ in $\mathcal{T}_Y$ the pre-image $f^{-1}(G)$ is an element of $\mathcal{T}_X$.
- $f$ is said to be a **homeomorphism** iff it is a continuous bijection and its inverse is continuous.
- If such a homeomorphism exists then $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ are said to be **homeomorphic**

### Theorem 66: $d:X\times X\to\mathbb{R}$ is continuous

Let $(X,d)$ be a metric space. $f:X\times X\to\mathbb{R}$ is continuous, where

- $\mathbb{R}$ is equipped with the standard metric.
- $X\times X$ is equipped with the product metric

### Definition 67: Bounded Linear Operators

A linear operator $T:X\to Y$ is said to be **bounded** iff there exists a positive constant $C$ such that, for all $x\in X$,

$$\|T(x)\|_Y\leq C\|x\|_X$$

**Thm 68**: Let $T:X\to Y$ be a linear operator. The following are equivalent:

1. $T$ is continuous
2. $T$ is continu. at 0
3. $T$ is bounded

### Definition 70: Lipschitz Functions

Let $(X,d_X)$, $(Y,d_Y)$ be metric spaces. A function $f:X\to Y$ is said to be a **Lipschitz** function iff there exists a constant $L$ such that for all $x,x'\in X$,

$$d_Y(f(x),f(x'))\leq Ld_X(x,x')$$

If $L<1$, $f$ is said to be a **contraction**

If $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function and $x$ is any point in $\mathbb{R}$, then for any $x\in\mathbb{R}$ we have

$$|f(x)-f(x')|\leq L|x-x'|$$

For $x\geq x'$ this can be expanded to

$$f(x')-L(x-x')\leq f(x)\leq f(x')+L(x-x')$$

**Lipschitz Theorem Bank**

**71**: Every Lipschitz function is continuous

**175**: Let $(X,d_X)$ and $(Y,d_Y)$ be two metric spaces, and $f:X\to Y$ be a Lipschitz function. Then there exists a smallest Lipschitz constant of $f$

**176**: Let $I$ be a non-degenerate open interval on the real line and let $f:I\to\mathbb{R}$ be a differentiable function. Then $f$ is Lipschitz iff $f'$ is bounded. When that is the case,

$$|f|_{\text{Lip}}=\sup\{|f'(x)|:x\in I\}$$

### Definition 72: Fixed Points

A **fixed point** of a function $f:S\to S$ where $S$ is a non-empty set, is any element $x$ of $S$ such that $f(x)=x$

Solving equations can sometimes be reduced to finding fixed points

### Theorem 75: Banach's Fixed Point Theorem

Let $(X,d)$ be a complete metric space and let $f:X\to X$ be a contraction. Then $f$ has a unique fixed point

### Definition 76: Equivalent Metrics

Two metrics on the same non-empty set $X$ are said to be **equivalent** iff they have the same open sets

**Thm 77**: Let $d_1$ and $d_2$ be metrics on the same non-empty set $X$. If there exist positive constants $C$ and $C'$ such that for all $x,y$ in $X$,

$$Cd_1(x,y)\leq d_2(x,y)\leq C'd_1(x,y)$$

then $d_1$ and $d_2$ are equivalent

# 3 Completeness

**Theorem I: Completeness of the Classical Spaces**

Some examples of complete metric spaces:

| | | | | |
|---|---|---|---|---|
| **79**: $(\mathbb{R}^n, d_2)$ | **80**: $\ell^2$ | **81**: $\ell^p$ | **82**: $C([a,b])$ | **83**: $\ell^\infty$ |

**Exercise 31**

- Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces and assume that $(Y, d_Y)$ is complete.
- Let $C(X,Y)$ be the set of all continuous and bounded functions from $X$ to $Y$. For $f, g \in C(X,Y)$ define

$$D(f,g) = \sup\{d_Y(f(x), g(x)) : x \in X\}$$

- Then $D$ is a metric and the metric space $(C(X,Y), D)$ is complete

**Definition 83: The product space $X^n$**

Let $(X,d)$ be a metric space and $n \in \mathbb{N}$. Define $D : X^n \to \mathbb{R}$ by

$$D(x_1, x_2) = d(x_{11}, x_{21}) + d(x_{12}, x_{22}) + \cdots + d(x_{1n}, x_{2n})$$

**Lemma Bank**

**Ex.33**: $D$ is a metric and a sequence converges in $(X^n, D)$ iff it converges componentwise

**Ex.34**: If $(X,d)$ is complete then $(X^n, D)$ is complete

**Definition 84: The product space $X^{\mathbb{N}}$**

Let $B^A$, where $A, B$ are sets, be the set of all functions from $A$ to $B$

**Def 85:** Let $(X,d)$ be a metric space. Define a metric $D : X^{\mathbb{N}} \times X^{\mathbb{N}} \to \mathbb{R}$ by

$$D(x_1, x_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d(x_{1n}, x_{2n})}{1 + d(x_{1n}, x_{2n})}$$

- $x_1 = (x_{11}, \ldots, x_{1n}, \ldots)$, $x_2 = (x_{21}, \ldots, x_{2n}, \ldots)$
- $(X^{\mathbb{N}}, D)$ is called a **product space**

**Theorem J: Product space Convergence & Completeness**

**Thm 86 (Convergence)**

Let $(X,d)$ be a metric space, let $(x_k)_{k=1}^\infty$ be a sequence in $X^{\mathbb{N}}$ and let $x \in X^{\mathbb{N}}$. Write $x_k = (x_{k1}, \ldots, x_{kn}, \ldots)$ and $x = (l_1, \ldots, l_n, \ldots)$.

Then, $x_k \xrightarrow[k \to +\infty]{(X^{\mathbb{N}}, D)} x$ if and only if, for all $n$, $x_{kn} \xrightarrow[k \to +\infty]{(X^{\mathbb{N}}} l_n$

**Thm 87 (Completeness)**

Let $(X,d)$ be a complete metric space. Then the product space $(X^{\mathbb{N}}, D)$ is complete.

**Theorem K: Completeness of $\mathbb{R}$**

- **Thm (Least Upper Bound Principle)**: Every non-empty bounded above subset of $\mathbb{R}$ has a least upper bound
- **Thm 88 (Monotone Convergence)**: Every bounded monotone sequence of real numbers has a limit
- **Thm/Ex. 36 ($\epsilon$-convergence)**: Let $A$ be a non-empty boudned subset of $\mathbb{R}$ and let $\epsilon$ be positive. If the distance between any two elements of $A$ is $< \epsilon$, then

$$\sup(A) - \inf(A) \le \epsilon$$

- **Thm 89**: Every Cauchy sequence of real numbers is convergent

**Definition L: Limit Superior and Inferior**

Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathbb{R}$. Then $(x_n)_{n=1}^\infty$ is bounded. Define:

$$I_n = \inf\{x_n, x_{n+1}, \ldots\} \quad S_n = \sup\{x_n, x_{n+1}, \ldots\}$$

**Thm**: $(S_n)_{n=1}^\infty$ and $(I_n)_{n=1}^\infty$ are monotone and bounded

$$I_1 \le I_n \le S_n \le S_1, \quad n = 1, 2, \ldots$$

Therefore $I_n \to I$ and $S_n \to S$ for some reals $I$ and $S$. Since $S_n - I_n \to 0$ we have $S = I$. We also have $x_n \to S = I$

**Def 90: Limsup and Liminf**

- The limit of the sequence $(I_n)_{n=1}^\infty$ is called the **limit inferior** of $(x_n)_{n=1}^\infty$ and is denoted by $\liminf x_n$

$$\liminf x_n = \lim_{n \to +\infty} I_n = \lim_{n \to +\infty} \inf\{x_n, x_{n+1}, \ldots\}$$

- The limit of the sequence $(S_n)_{n=1}^\infty$ is called the **limit superior** of $(x_n)_{n=1}^\infty$ and is denoted by $\limsup x_n$

$$\limsup x_n = \lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \sup\{x_n, x_{n+1}, \ldots\}$$

- $\liminf x_n$ is the smallest subsequential limit of $(x_n)_{n=1}^\infty$
- $\limsup x_n$ is the largest subsequential limit of $(x_n)_{n=1}^\infty$
- $(x_n)_{n=1}^\infty$ converges iff $\liminf x_n = \limsup x_n$

# 4 Compactness

**Definition 96: Open Covers and Subcovers**

An **open cover** of a set $S$ in a metric space is a family $(G_i)_{i \in I}$ of open sets such that $S \subset \bigcup_{i \in I} G_i$. A **subcover** of an open cover $(G_i)_{i \in I}$ is a sub-family $(G_i)_{i \in I'}$ where $I' \subset I$, such that $S \subseteq \bigcup_{i \in I'} G_i$

**Definition 4.0.1: Compacting Compactness**

**Def 91 (Compactness)**

Let $X = \mathbb{R}$ and $d$ be the standard metric. A subset $K$ of $\mathbb{R}$ is said to be **compact** iff every sequence of elements of $K$ has a subsequence that converges to an element of $K$

**Def 102 (Sequential Compactness)**

1. $K$ is **sequentially compact** iff every sequence in $K$ has a subsequence that converges to an element of $K$

   For the case $K = X$ it's just the definition (1) defined above
2. $K$ is **compact** iff every open cover of $K$ has a finite subcover

**Def 111 (Uniform Continuity)**

3. Let $(X, d_X), (Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is said to be **uniformly continuous** iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that, for all $x, x'$ in $X$ with $d_X(x, x') < \delta$ we have

$$d_Y(f(x), f(x')) < \epsilon$$

**Def 117 (Totally bounded Spaces)**

**117**: A metric space $(X,d)$ is said to be **totally bounded** iff for every positive $\delta$, $X$ can be covered by a finite number of open balls of radius $\delta$.

**118**: If $(X,d)$ is totally bounded then it is bounded, but the converse is not necessarily true

**Example 4.0.2: Examples of compactness**

**Compact sets**
- $[a,b]$ is compact
- $\emptyset$ is compact
- $\mathbb{R} \cup \{-\infty, +\infty\}$ is compact!

**Not Compact sets**
- $(0,1)$ is not compact
- $\mathbb{R}$ is not compact

**Theorem 116: Lebesgue's Lemma**

Let $(X,d)$ be a sequentially compact metric space and $X = \bigcup_{i \in I} G_i$ be an open cover of $X$. There exists a $\delta > 0$ such that for any two points $x, y \in X$ with $d(x,y) < \delta$ there exists an $i$ such that $x, y \in G_i$. Any such $\delta$ is called a **Lebesgue number** of the open cover

**Ex.44**: Let $(X,d)$ be a sequentially compact m.s. and $X = \bigcup_{i \in I} G_i$ be an open cover of $X$. Then there exists a $\delta > 0$ s.t. any nonempty subset of $X$ of diameter $< \delta$ can be covered by a single $G_i$

**Theorem 4.0.3: big theorem bank of obvious shit**

**Regular Compactness**

- For a set $K$ in $\mathbb{R}$ with the standard metric:
  - **93**: $K$ is compact $\iff$ $K$ is closed and bounded
  - **100**: $K$ is compact $\iff$ every open cover of $K$ has a finite cover
- For a set $K$ in $\mathbb{R}^n$ with the Euclidean metric:

**Ex.38**: $K$ is compact $\iff$ $K$ is closed and bounded

- For a set $K$ in $\mathbb{R}$
  - **101**: Every open cover of $K$ has a finite subcover $\implies$ $K$ is closed and bounded $\implies$ $K$ is compact

**99**: Every open cover of the interval $[a,b]$, where $a, b \in \mathbb{R}$, $a \le b$ has a finite subcover

**Continuous Functions**

- Let $K \subseteq \mathbb{R}$ be compact, and $f : K \to \mathbb{R}$ continuous:
  - **94**: $f$ is bounded
  - **95**: $f$ has a maximum and minimum (EVT)
- Let $(X,d)$ be a metric space, $K$ be a sequentially compact subset of $X$ and $f : K \to \mathbb{R}$ be a continous function:
  - **110**: $f$ has a maximum and a minimum. In particular, $f$ is bounded. (EVT ..again)

**Sequential compactness stuff**

Let $(X,d)$ be a metric space, and $K \subseteq X$:

- Let $K \ne \emptyset$, and let $d_K$ be the induced metric on $K$.

**Ex.39**: $K$ (seq.) compact $\iff$ the M.S. $(K, d_K)$ is (seq.) compact

**105**: $K$ sequentially compact $\implies$ $K$ is closed and bounded

**107**: $(X,d)$ and $K$ are both sequentially compact $\iff$ $K$ is closed

**108**: $(X,d)$ is sequentially compact $\implies$ $(X,d)$ is complete

**115**: $K$ is compact $\iff$ $K$ is sequentially compact

**x42**: $(X,d)$ is compact $\implies$ $(X,d)$ is sequentially compact

**x43**: $(X,d)$ is compact, and let $A$ be an infinite subset of $X$ $\implies$ $A$ has at least one limit point

**Thm 114 (Uniform Continuity)**

Let $(X, d_X)$ be a sequentially compact metric space, $(Y, d_Y)$ be a metric space and $f : X \to Y$ be a continuous function. Then $f$ is uniformly continuous

**Totally Compact Spaces**

Let $(X,d)$ be a metric space:

**120**: $(X,d)$ is sequentially compact $\implies$ $(X,d)$ is totally bounded

**122**: $(X,d)$ is compact $\iff$ $(X,d)$ complete and totally bounded

**121**: Every sequentially compact metric space is compact.

### Definition 4.0.4: Countable and Uncountable Sets

A set $S$ is said to be:

- **Infinitely coutnable** iff there is a bijection $f : \mathbb{N} \to S$
- **Countable** if it is finite or infinitely countable
- **Uncountable** iff it isn't countable

**Examples**

- $\{1, 2, 3\}$ and $\mathbb{R}$ are countable sets
- $\mathbb{Q}$ is infinitely countable
- $\mathbb{R}$ is uncountable

### Theorem 4.0.5: Dense Subset equivalence

Let $(X, d)$ be a metric space and $D \subseteq X$. The following are equivalent:

1. $D$ is dense
2. For every $x \in X$ and $\epsilon > 0$ there exists $y \in D$ s.t. $d(x, y) < \epsilon$
3. For every $x \in X$ there is a sequence $(y_n)_{n=1}^{\infty}$ of elements of $D$ s.t. $y_n \to x$
4. For every element $x \in X$ and every open nbhd $G$ of $x$, $G \cap D \neq \emptyset$
5. $D$ intersects every non-empty open set

### Definition 4.0.6: Separable spaces

A metric space is said to be **separable** iff it has a countable dense subset

**Examples**

- $\mathbb{R}$ with the standard metric is a separable metric because $\mathbb{Q}$ is dense and countable
- $\mathbb{R}^n$ with the Euclidean metric is a separable metric space because $\mathbb{Q}^n$ is dense and countable
- $\mathbb{C}$ with its standard metric is a separable metric space because $\{z \in \mathbb{C} : \mathrm{Re}(z), \mathrm{Im}(z) \in \mathbb{Q}\}$
- $\ell^2$ is separable, and $\ell^p$ is separable for $1 \leq p < \infty$

### Theorem 4.0.7: Weierstrass Approximation Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\epsilon > 0$. There exists a polynomial $p$ with *real* coefficients s.t. for all $x \in [a, b]$

$$|f(x) - p(x)| < \epsilon$$

---

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\epsilon > 0$. There exists a polynomial $p$ with *rational* coefficients s.t. for all $x \in [a, b]$

$$|f(x) - p(x)| < \epsilon$$

### Theorem 4.0.8: more theorems

The set of all polynomials with rational coefficients is countable
**Thm**: $C([a, b])$ is separable

### Theorem 4.0.9: Separability of subspaces

Let $(X, d)$ be a separable metric space, $A \subseteq X$, $A \neq \emptyset$, and $d_A$ be the induced metric on $A$. Then the metric space $(A, d_A)$ is separable

---

**Thm**: Every compact metric space is separable (compact $\implies$ separable)

### Theorem 4.0.10: Open Ball countability

Let $(X, d)$ be a separable metric space and let $D$ be a countable dense subset of $X$. Let

$$\mathcal{B} = \{B(c, r) : c \in D, r \in \mathbb{Q}^+\}$$

be the set of all open balls with centers in $D$ and rational radii. Then $\mathcal{B}$ is countable and every open set in $X$ can be written as a union of elements of $\mathcal{B}$

### Definition 4.0.11: Open Bases and Second Countability

**Open Bases**

Let $(X, \mathcal{T})$ be a topological space. An **open base** (or **base**) for the topology $\mathcal{T}$ is a family $\mathcal{B}$ of open sets such that every open set in $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$

**Second Countability**

A topological space $(X, \mathcal{T})$ is said to satisfy the **second Axiom of Countability**, or is **second countable** iff it has a countable open base

---

**Thm**: In a separable metric space, every family of pairwise disjoint non-empty open sets is countable

---

**Thm**: On the real line with the standard metric, every open set can be written as a countable union of disjoint open intervals

### Definition 4.0.12: Continuous Extensions

Let $(X, d_X), (Y, d_Y)$ be metric spaces, $D$ be a dense subset of $X$, $f, g : X \to Y$ continuous functions s.t. $f(x) = g(x)$ for all $x \in D$. Then $f = g$

---

Let $(X, d_X), (Y, d_Y)$ be metric spaces, $D \subseteq X$ be dense, $f : D \to Y$ be uniformly continuous, and assume that $(Y, d_Y)$ is complete. Then $f$ has a unique continuous extension $F : X \to Y$

### Theorem 4.0.13: complete ms props

Let $(X, d)$ be a metric space, $F$ be a nonempty subset of $X$ and $d_F$ be the induced metric on $F$. If the metric space $(F, d_F)$ is complete then $F$ is a closed subset of $X$

---

**Thm**: Let $(X, d)$ be a complete metric space, $F$ be a nonempty subset of $X$, and $d_F$ be the induced metric on $F$. If $F$ is a closed subset of $X$, then the metric space $(F, d_F)$ is complete

---

**Thm**: Let $(X, d)$ be a complete metric space, $A \subseteq X$, $A \neq \emptyset$. Then

1. The metric space $(\overline{A}, d_{\overline{A}})$ is complete
2. If $A \subseteq B \subseteq X$ and $(B, d_B)$ is complete, then $\overline{A} \subseteq B$

### Definition 4.0.14: Isometries

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is called a **isometry** iff for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

---

**Thm**: Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $f : X \to Y$ be an isometry. Then $f$ is an injection. If, moreover, $f$ is a surjection (hence $f$ bij.) then $f^{-1} : Y \to X$ is also an isometry

---

**Thm**: The metric spaces $(X, d_X)$ and $(Y, d_Y)$ are said to be **isometric** iff there exists an isometry $f$ from $X$ onto $Y$

---

**Thm**: if two metric spaces are isometric and one of them is complete/compact/connected/... then so is the other

### Theorem 4.0.15: Isometry completion

Let $(X, d)$ be a bounded metric space and let $C(X, \mathbb{R})$ be the set of all bounded continuous functions $f : X \to \mathbb{R}$ equipped with the metric

$$d_\infty(f_1, f_2) = \sup\{|f_1(t) - f_2(t)| : t \in X\}$$

For each $x \in X$, define $F_X : X \to \mathbb{R}$ be $F_X(x') = d(x, x')$. Then

1. $F_X \in C(X, \mathbb{R})$
2. The map $X \to C(X, \mathbb{R}), x \mapsto F_X$ is an isometry
3. $X^* = \{F_X : x \in X\}$, equipped with the induced metric, is a subspae of $C(X\mathbb{R})$ isometric to $X$
4. The closure $\overline{X^*}$ of $X^*$ in $C(X, \mathbb{R})$, equipped with the induced metric, is a complete metric space
5. $X^*$ is dense in $\overline{X^*}$

### Definition 4.0.16: Completion of a Metric Space

Let $(X, d)$ be a metric space. A **completion** of $(X, d_X)$ is any metric space $(Y, d_Y)$ with the following properties

1. $(Y, d_Y)$ is complete
2. $(Y, d_Y)$ has a subspace $X^*$ isometric to $(X, d_X)$
3. $X^*$ is dense in $Y$

It can be shown that any two completions of $X$ are isometric to each other, i.e. a completion is unique up to isometries

**Definition 4.0.17: Construction of Completion via Cauchy**

Let $(X, d)$ be a metric space and let $\mathcal{C}$ be the set of all Cauchy sequences of elements of $X$

We define an equivalence relation $\sim$ in $\mathcal{C}$ as follows: Let $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}} \in \mathcal{C}$. We say that $x \sim y$ iff $d(x_n, y_n) \to 0$

Distinct equivalence classes are disjoint and partition $\mathcal{C}$

The set of all equivalence classes is called the **quotient space**, denoted $\mathcal{C}/\sim$

---

Define a metric $D$ on $\mathcal{C}/\sim$ as follows:

Let $\alpha, \beta \in \mathcal{C}/\sim$. Then

$$\alpha = [(x_1, \dots, x_n, \dots)] \text{ and } \beta = [(y_1, \dots, y_n, \dots)]$$

for some $(x_1, \dots, x_n, \dots), (y_1, \dots, y_n, \dots) \in \mathcal{C}$. Define

$$D(\alpha, \beta) = \lim_{n\to+\infty} d(x_n, y_n)$$

$(\mathcal{C}/\sim, D)$ is complete. Additionally, the following is an isometry:

$$X \to \mathcal{C}/\sim \qquad x \mapsto ([x, x, \dots, x, \dots])$$

Let $X^*$ be its range. The metric space $(X^*, D_{X^*})$ is isometric to $(X, d)$, $(\overline{X^*}, D_{\overline{X^*}})$ is a complete metric space, and $X^*$ is dense in $\overline{X^*}$

**Definition 4.0.18: Connected and Disconnected Spaces**

A metric space $(X, d)$ is said to be **disconnected** iff there exists non-empty disjoint open sets $G_1$ and $G_2$ such that

$$X = G_1 \cup G_2$$

Otherwise the metric space is called **connected**

---

A non-empty subset $A$ of a metric space $(X, d)$ is said to be disconnected iff the metric space $(A, d_A)$, where $d_A$ is the induced metric, is disconnected

**Theorem 4.0.19: Connected Theorems**

A subset $O$ of $A$ is open in $(A, d_A)$ iff $O = A \cup G$ for some $G$ that is open in $X$

Therefore, $A$ is disconnected iff there exist open subsets $G_1, G_2$ of $X$ s.t.

- $A = (A \cap G_1) \cup (A \cap G_2)$, which is equivalent to $A \subseteq G_1 \cup G_2$
- $A \cap G_1 \neq \emptyset$, $A \cap G_2 \neq \emptyset$
- $(A \cap G_1) \cap (A \cap G_2) = \emptyset$, which is equivalent to $A \cap G_1 \cap G_2 = \emptyset$

---

**Thm**: $\mathbb{R}$ with the standard metric is connected

**Thm**: On the real line with the standard metric, all intervals are connected sets

**Thm**: A non-empty subset of the real line is connected iff it is an interval

**Thm**: A metric space $(X, d)$ is connected iff the only subsets of $X$ with empty boundary are $\emptyset$ and $X$

**Thm**: Let $(X, d_X)$ be a connected metric space, $(Y, d_Y)$ be a metric space and $f : X \to Y$ be a continuous surjection. Then $(Y, d_Y)$ is connected as well

**Theorem 4.0.20: Intermediate Value Theorem**

Let $(X, d)$ be a connected metric space and $f : X \to \mathbb{R}$ be a continuous function. If $x_1, x_2 \in X$ with $f(x_1 \neq f(x_2))$ and $y$ is a real number between $f(x_1)$ and $f(x_2)$, then there exists an $x \in X$ such that $f(x) = y$

**Theorem 4.0.21: Clopen**

A metric space $(X, d)$ is connected iff the only clopen subsets are $\emptyset, X$

**Definition 4.0.22: Connected Components**

Let $(X, d)$ be a metric space. We define an equivalence relation $\sim$ in $X$ as follows: $x \sim x'$ iff there exists a connected subset $C$ of $X$ that contains both $x$ and $x'$

**Thm**: If $(C_i)_{i\in I}$ is a family of connected subsets of $X$ with nonempty intersection, then $\bigcup_{i\in I} C_i$ is connected

**Theorem 4.0.23: Big equivalence classes**

The equivalence class of any point in $X$ is the largest connected subset of $X$ that contains that point (what point?)

**Definition 4.0.24: Path Connected Metric Spaces**

Let $(X, d)$ be a metric space and $x_0, x_1 \in X$. A **path** in $X$ from $x_0$ to $x_1$ is a continuous function $\gamma : [0, 1] \to X$ s.t. $\gamma(0) = x_0$, $\gamma(1) = x_1$

$(X, d)$ is said to be **path-connected** iff for any two points $x_0, x_1$ in $X$ there is a path in $X$ from $x_0$ to $x_1$

A non-empty subset $A$ of $X$ is said to be **path-connected** iff the metric space $(A, d_A)$, where $d_A$ is the induced metric, is path connected

**Theorem 4.0.25: Path connected theorem**

Every path-connected metric space is connected

Not every connected metric space is necessarily path-connected

# 5 Applications

**Theorem 5.0.1: Picard's Theorem**

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous and boudned function, and $t_0, x_0$ be real numbers. Assume that there exists a positive constant $L$ s.t. for all real $t, x_1, x_2$ we have:

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$$

Then, there exists a positive $\delta$ and a unique differentiable function $x : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ s.t. for all $t \in [t_0 - \delta, t_0 + \delta]$,

$$x'(t) = f(t, x(t)) \quad \text{and} \quad x(t_0) = x_0$$

**Definition 5.0.2: Lipschitz Functions again**

Let $(X, d_X)$, $(Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is said to be a **Lipschitz** function iff there exists a constant $L$ such that for all $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq L d_X(x, x')$$

If $L < 1$, $f$ is said to be a **contraction**

**Example 5.0.3: Examples of closure**

- On the real line with the standard metric, $\overline{(a, b)} = [a, b]$
- In $\mathbb{R}^n$ with the Euclidean metric $d_2$, the closure of the open ball $B(c, r)$ is the closed ball $\{x \in \mathbb{R}^n : d_2(x, c) \leq r\}$
- On the complex plane with its standard metric, the closure of an open disc is the corresponding closed disc
- Let $X$ be a non-empty set with the discrete metric, $c \in X$ and $r = 1$. Then $B(c, 1) = \{c\}$, therefore $\overline{B(c, 1)} - \overline{\{c\}} = \{c\}$, while

  $$\{x \in X : d(x, c) \leq 1\} = X$$

  The closure of an open ball is not always equal to the corresponding closed ball
- $X = \mathbb{R}$, $d(x, y) = |x - y|$. $\overline{\mathbb{Q}} = \mathbb{R}$

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