Algebraic Topology Notes

Leon Lee

February 3, 2025

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1 Introduction to Algebraic Topology

1.1 Topologies to Algebra

We want to turn topological spaces into algebraic objects through operations called Invariants. An example is that if two topological spaces X and Y are isomorphic, the translated algebraic object should also be isomorphic

$$\begin{aligned} \text{TOP} &\rightsquigarrow \text{ALG} \\ X &\mapsto A(X) \quad \text{“algebraic objects”} \\ X \cong Y &\mapsto A(X) \cong A(Y) \end{aligned}$$

Example 1.1.1: Examples of Algebraic Objects

Some examples of algebraic objects:

- The set of Connected Components $\pi_0(X)$
- The Fundamental Group $\pi_1(X)$
- Higher homotopy groups $\pi_n(X)$

Note: the more involved the algebraic invariant is, the more topology it sees.
Computability problem leads to Homology Theory (this is non-examinable)

1.2 Connected Spaces

Recall 1.2.1: Topologies

A topology on X, $\mathcal{T}$, is a family of subsets s.t.

- $\emptyset, X \in \mathcal{T}$
- Closed under finite intersection, $U_1, U_2 \in \mathcal{T} \implies U_1 \cap U_2 \in \mathcal{T}$
- Closed under arbitrary unions

Examples of topological spaces:

- Trivial topology $\mathcal{T} = \{\emptyset, X\}$
- Discrete Topology $\mathcal{T} = \mathcal{P}(X)$
- $\mathbb{R}$ or anything made from a metric space

Definition 1.2.2: Connected Spaces

A topological space X is **connected** if $X = A \uplus B$ (A and B are open) means that $A = \emptyset$ or $A = X$

Prop 1.2.3: Connected Spaces and Clopens

X is connected iff the only clopens are $\emptyset, X$

Proof.
($\implies$): A clopen then $X = A \uplus A^C \implies A = \emptyset, X$ (both A and A^C open)
($\impliedby$): $A \uplus B \implies A = B^C \implies A$ is clopen $\square$

Examples:

- $\mathbb{R}$ is connected. Opens are generated by intervals like $(-\infty, a)$, (a, b), (a, ∞).
- The trivial topology is connected. (by definition since there are only two sets).
- The discrete topology is *not* connected, unless $X = \emptyset$ or $X = \{*\}$ in which case it coincides with the trivial topology.

Prop 1.2.4: Connectedness of Maps

For a continuous map $f : X \to Y$, and X connected, we have that $f(X)$ is connected.

Proof. $f(X) = U \uplus V \implies f^{-1}(U) \uplus f^{-1}(V) = X \implies f^{-1}(U) = \emptyset, X$ $\square$

Corollary 1.2.5

If $X \cong Y$ are homeomorphic, then X is connected iff Y is connected

Prop 1.2.6

The relation ($x \sim y$ if $\exists$ connected subset $A \subseteq X$ s.t. $x, y \in A$) is an equivalence relation.

Proof. We show the relation fulfils all requirements for an equivalence relation:

- **Reflexivity**: $x \sim x$: $x \in \{x\} \subseteq X$
- **Symmetry**: $x \sim y \iff y \sim x$ tautological (we don't specify between x and y so just take $y = x$ and $x = y$)
- **Transitivity**: $x \sim y \wedge y \sim z \implies x \sim z$, $x, y \in A$, $y, z \in B$. Claim: $A \cup B$ is connected. Proof in workshop

$\square$

Definition 1.2.7: Components

The equivalence classes of the above proposition are called **components**

1.3 Path-Connectedness

Definition 1.3.1: Path

A **path** in X is a continuous map $\alpha : I \to X$ for $I = \mathcal{T}(0, 1)$.
$x \sim y \iff \exists \alpha : I \xrightarrow{\text{path}} X$ s.t. $\alpha(0) = x, \alpha(1) = y$

$x \sim y$ is an equivalence relation due to the following operations on paths:

1. Constant path. If $x \in X$, $c_X : I \to X$, $c_x(t) := X$
2. Path reversal. Let $\alpha : I \to X$ be a path. Then $\overline{\alpha} : I \to X, t \mapsto \alpha(1-t)$
3. Path concatenation: $\alpha : I \to X$, $\beta : I \to X$ s.t. $\alpha(1) = \beta(0)$. Then

$$(a * b)(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Definition 1.3.2: Connected Components

The set of path-connected components (equivalence classes) is denoted by $\pi_0(X)$

Remarks:

- We have that $X \cong Y \implies \pi_0(X) \cong \pi_0(Y)$
- Path-connected $\implies$ Connected (but not vice-versa). Counterexample: Pick

$$X = \{(x, \sin(\frac{1}{x})) \mid 0 < x < 1\}$$

is connected but not path connected

Definition 1.3.3: Homotopy

Let $f, g : X \to Y$ continuous maps. A **homotopy** from f to g is a continuous map $h : X \times I \to Y$ s.t.

$$\begin{aligned} h(-,0) &= f \iff h(x,0) = f(x), \forall x \\ h(-,1) &= g \end{aligned}$$

Terminology: f is homotopy equivalent to g if there exists a homotopy h
homotopies on homotopies - horizontal composition

$$\begin{tikzcd} X \arrow[r, bend left=50, "f'"] \arrow[r, "f"'] \arrow[r, phantom, "\Uparrow h_1"] & Y \arrow[r, bend left=50, "g'"] \arrow[r, "g"'] \arrow[r, phantom, "\Uparrow h_2"] & Z \end{tikzcd}$$

Vertical composition

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "k"'] \arrow[r, phantom, shift left=3ex, "h_1 \Downarrow"] \arrow[r, phantom, shift right=3ex, "h_2 \Downarrow"] & Y \end{tikzcd}$$

1.4 Homotopy Equivalence

Definition 1.4.1: Homotopy Equivalence

Two spaces X, Y are called **homotopy equivalent** or **of the same homotopy type**, and denoted by $X \simeq Y$, if there exists a homotopy equivalence $f : X \to Y$

Note: We use $\cong$ for homeomorphisms and $\simeq$ for homotopy equivalences.

Lemma 1.4.2: Homotopy inverses

Let $f : X \to Y$ and $g : Y \to Z$ with homotopy inverses $\tilde{f} : Y \to X$ and $\tilde{g} : Z \to Y$ respectively. Then, $\tilde{f} \circ \tilde{g} : Z \to X$ is a homotopy inverse of $g \circ f : X \to Z$.
In particular, $X \simeq Y$ and $Y \simeq Z$ implies $X \simeq Z$.

Definition 1.4.3: Contractible Spaces

A space X is called **contractible** if it is homotopy equivalent to a point, i.e. $X \simeq *$

Example: $\mathbb{R}^n$ is contractible. Let x_0 be a fixed point in $\mathbb{R}^n$ and define the (straight line) homotopy $h : c_{x_0} \simeq \mathrm{id}_{\mathbb{R}^n}$ by

$$h(x,t) = (1-t)x_0 + tx$$

Remark 1.4.4

1. Contractible spaces are path-connected
2. The converse does not hold. For example $X = \mathbb{S}^1$ will lead to a counterexample.
3. A contractible space X is contractible at any point x_0. Since X is path-connected a path from x to x' defines a homotopy $c_x \simeq c_{x'}$
4. Any two maps $f, g : X \to Y$ are homotopic if Y is contractible.

2 Retractions and Deformations

Definition 2.0.1: Retractions and Detractions

- A **retract** of X onto a subspace $A \subset X$ is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$. Equivalently, this is a map $r : X \to X$ such that $r^2 = r$ and $r(X) = A$
- A **deformation retract** of X onto A is the additional datum of a homotopy $h : \mathrm{id}_X \simeq i \circ r$, where $i : A \hookrightarrow X$ denotes the inclusion

In other words, a deformation retract is a homotopy $h : X \times I \to X$ such that $h(x, 0) = x$ and $h(x, 1) \in A$ for all $x \in X$ and $h(a, 1) = a$ for all $a \in A$
Not all retracts can form deformation retracts. For instance, notice that the retract X onto a point $\{x_0\}$ can be a deformation retract if and only if X is contractible.

Prop 2.0.2: Deformation Retracts cause Homotopy Equivalence

A deformation retract of X onto A induces a homotopy equivalence $X \simeq A$.

2.1 Quotient spaces

Definition 2.1.1: Quotient Space

Let X be a topological space and let $\sim$ be an equivalence relation on X. Then $X/\sim$ is equipped with the quotient topology and called a **quotient space**. If Z is a closed subset in X, then we can also define the quotient space X/Z.

Examples of Quotient Spaces

- The quotient of the n-dimensional closed disk by its boundary is the n-sphere, i.e.

$$\mathbb{D}^n/\partial\mathbb{D}^n \cong \mathbb{S}^n$$

- The 2-torus: $\mathbb{R}^2/\mathbb{Z}^2$
- The projective space $\mathbb{R}P^n = \mathbb{R}^{n+1} - \{0\}/\sim$ by the relation $x \sim y$ iff there exists some $\lambda \in \mathbb{R}^\times$ such that $x = \lambda y$. This corresponds to the space of lines through the origin in $\mathbb{R}^{n+1}$.

Definition 2.1.2: Alternate Quotient Space

Let $f : Z \to Y$ be a continuous map between a closed subset $Z \subset X$ and Y. Then

$$X \amalg_f Y = X \amalg Y/f(z) \sim y$$

Additionally,

- Its **mapping cylinder** is defined as the topological space

$$M_f := (X \times I) \amalg Y/\sim$$

where the quotient identifies $(x, 0) \sim f(x)$ for any $x \in X$

- Its **cone** is the further quotient

$$C_f := M_f/X \times \{1\}$$

- The **cone** of a topological space X is:

$$C_X := C_{\mathrm{id}_X} = X \times I/X \times \{1\}$$

Remark 2.1.3: Commutative Diagram of the Mapping Cylinder

In other words, the mapping cylinder of $f : X \times Y$ is the pushout of the diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \times I \arrow[r] & M_f \end{tikzcd}$$

Lemma 2.1.4: Iclusion Map of the Mapping Cylinder

Let $f : X \to Y$ and M_f its mapping cylinder. The iclusion map $i : Y \hookrightarrow M_f$ is a strong deformation retract.

2.2 Examples of Deformation Retracts

Example 2.2.1: Shhere

Consider the n-sphere $\mathbb{S}^n$ with the standard embedding $\mathbb{R}^{n+1}\backslash\{0\}$. Then the map

$$r : \mathbb{R}^{n+1}\backslash_0\{0\} \to \mathbb{S}^n,\ x \mapsto \frac{x}{|x|}$$

is a retract. Indeed, if x has norm $|x| = 1$, then $r(x) = x$. For a deformation retract one needs to find a homotopy $h : i \circ r \simeq \mathrm{id}_X$. this can easily be realized by the following straight line homotopy:

$$h : \mathbb{R}^{n+1}\backslash\{0\} \times I \to \mathbb{R}^{n+1}\backslash\{0\}, \quad (x, t) \mapsto (1 - t)\frac{x}{|x|} + tx$$

Indeed $h(x, 0) = r(x)$ and $h(x, 1) = x$ for all x
In fact, one can easily check that the above forms a strong deformation retract as $h(x, t) = x$ for all $x \in \mathbb{S}^n$ and $t \in I$. Note that one could have also constructed a deformation retract that is not strong, for example by rotating in time.

Remark 2.2.2

Ordinary tomotopy are not interesting for paths, e.g. $\alpha : I \to X$ is homotopic to a constant path

Prop 2.2.3

Path concatenation is unital and asssociative up to relative union

Lemma 2.2.4

Let $\alpha : I \to X$ be a path, and $\lambda : I \to I$ continuous s.t. $\lambda(0) = 0$ and $\lambda(1) = 1$.
Then, $\alpha \circ \lambda \cong \alpha$ (relative to $\{0, 1\}$)

Definition 2.2.5: Fundamental Group

The fundamental group of X at $x_0 \in X$ is the homotophy equivalence class of "loops" at x_0. i.e. paths in X s.t. $\alpha(0) = \alpha(0) = x$