

# Algebraic Topology Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

## 1 Introduction

### Recall 1.1.1: Topology

An **(open) topology** on  $X$  is a collection of subsets  $\tau \subset P(X)$  such that

- $\emptyset \in \tau$  and  $X \in \tau$
- $\tau$  is closed under finite intersections: If  $\{U_1, \dots, U_n\} \subset \tau$  then
- $\tau$  is closed under arbitrary unions: If  $\{U_1, \dots, U_n\} \subset \tau$  is a family of open subsets then

$$\bigcap_{i=1, \dots, n} U_i \in \tau$$

$$\bigcup_{i=1, \dots, n} U_i \in \tau$$

The subsets  $U \in \tau$  are called **open** and their complements in  $X$  define **closed subsets**.

Two examples of a topology on a set  $X$  are the following:

- The **Trivial Topology**:  $\tau_{\text{triv}} = \{\emptyset, X\}$
- The **Discrete Topology**:  $\tau_{\text{dis}} = P(X)$

A subset  $A \subset X$  is **clopen** if it is both closed and open

### Definition 1: Connected Spaces

A topological space  $X$  is **connected** if  $X = A \amalg B$  with  $A, B \subset X$  open implies that  $A = \emptyset$  or  $A = X$ .

### Proposition 1: Connectedness and Clopens

A topological space  $X$  is *connected* iff the only clopens are  $\emptyset$  and  $X$ .

### Example 1: Examples of Connected Topologies

- Every  $X$  with the trivial topology is connected.
- Every  $X$  with the discrete topology isn't connected unless  $X = \emptyset$  or  $X = \{*\}$  (in which it coincides with the trivial topology).
- The real line  $\mathbb{R}$  with the standard topology is connected.

### Proposition 2: Continuous Maps

Let  $f : X \rightarrow Y$  be a continuous map of topological spaces and let  $X$  be connected. Then  $f(X)$  is connected.

### Proposition 3: Connected Equivalence Relation

For a topological space  $X$ , define  $x \sim y$  if there exists some connected subset that contains both. The relation  $x \sim y$  is an equivalence relation.

### Definition 2: Connected Components

The equivalence classes of this relation are called **connected components**. In particular, a space  $X$  is connected iff it only has a single connected component.

### Definition 3: Path

Let  $I$  denote the closed unit interval  $[0, 1]$ . A **path** in  $X$  is a continuous map  $\alpha : I \rightarrow X$ . The points  $\alpha(0) \in X$  and  $\alpha(1) \in X$  will be called **start** and **end** points respectively.

We define a path relation between points in  $X$  by declaring  $x \sim y$  if there exists some path  $\alpha : I \rightarrow X$  that starts at  $x$  and ends in  $y$ , i.e.  $\alpha(0) = x$  and  $\alpha(1) = y$ . This is an equivalence relation from the following properties:

1. **Constant Path**: For all  $x \in X$  there exists the constant path  $c_x : I \rightarrow X$  defined by  $c_x(t) = x$  for all  $t \in I$
2. **Path reversal**: Let  $\alpha : I \rightarrow X$  be a path in  $X$ . Define its reversed path by

$$\bar{\alpha} : I \rightarrow X, \quad t \mapsto \alpha(1 - t) \quad (1)$$

3. **Path Concatenation**: Let  $\alpha, \beta : I \rightarrow X$  be two paths in  $X$  s.t.  $\alpha(1) = \beta(0)$ . Their concatenated path is defined by:

$$\alpha * \beta(t) := \begin{cases} \alpha(2t), & 0 \leq t \leq 1/2 \\ \beta(2t - 1) & 1/2 \leq t \leq 1 \end{cases} \quad (2)$$

### Definition 4: Path-Connected Components

The equivalence classes are called **path-connected components** and their set is denoted by  $\pi_0(X)$ . A space  $X$  is called **path-connected** if  $\pi_0(X)$  is a one-point set, i.e. any two points  $x, y$  can be related by a path in  $X$ .

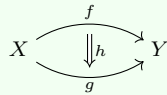
### Remark 1: Random examples

The following statements are true:

- A homeomorphism  $X \cong Y$  induces a bijection  $\pi_0(X) \cong \pi_0(Y)$ .
- If  $X$  is path-connected, it is also connected.
- The *topologist's sine curve* defined by  $X = \{0\} \times [-1, 1] \cup \{(x, \sin(1/x)) \mid 0 < x\}$  is connected but not path-connected.

### Definition 5: Homotopy

A **homotopy** of maps  $f, g : X \rightarrow Y$  is a continuous map  $h : X \times I \rightarrow Y$  such that  $h(-, 0) = f$  and  $h(-, 1) = g$ .



If such a homotopy exists,  $f$  is **homotopic** to  $g$ . This defines an equivalence relation  $f \simeq g$  on the space of maps  $\text{Map}(X, Y)$ .

### Example 2: Paths as Homotopies

Points in  $X$  are the same as maps  $*$   $\rightarrow$   $X$  from the one-point set  $*$  to  $X$ . A path  $\alpha : I \rightarrow X$  corresponds to a homotopy  $*$   $\times$   $I \rightarrow X$ .

### Remark 1.5: Composition of Homotopies

- **Vertical Composition**: Let  $h, h' : X \times I \rightarrow Y$  be two homotopies in  $X$  such that  $h(-, 1) = h'(-, 0) : X \rightarrow Y$ . Their concatenated homotopy is defined by

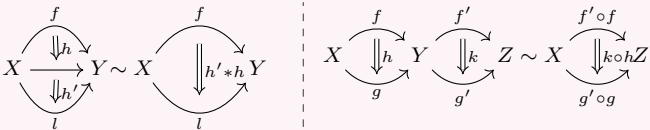
$$h * h'(-, t) := \begin{cases} h(-, 2t) & 0 \leq t \leq 1/2 \\ h'(-, 2t - 1) & 1/2 \leq t \leq 1 \end{cases} \quad (4)$$

- **Horizontal Composition**: Let  $h : X \times I \rightarrow Y, k : Y \times I \rightarrow Z$  be two homotopies on maps from  $X$  to  $Y$ , and  $Y$  to  $Z$ . Then

$$k \circ h := [X \times I \xrightarrow{\text{id} \times \Delta} X \times I^2 \xrightarrow{h \times \text{id}} Y \times I \xrightarrow{k} Z] \quad (5)$$

where  $\Delta : I \rightarrow I^2, t \mapsto (t, t)$  is the diagonal map, or explicitly,

$$k \circ h(x, t) = k(h(x, t), t)$$



### Lemma 1: Concatenation Relation

Let  $f, f' : X \rightarrow Y$  and  $g, g' : Y \rightarrow Z$  be maps such that  $f \simeq f'$  and  $g \simeq g'$ . Then  $g \circ f \simeq g' \circ f'$  as maps from  $X$  to  $Z$ . In particular,  $g' \circ f \sim g \circ f$  and  $g \circ f' \sim g \circ f$ .

### Definition 6: Homotopy Equivalence

A map  $f : X \rightarrow Y$  is called a **homotopy equivalence** if there exists a map  $g : Y \rightarrow X$  and homotopies  $f \circ g \simeq \text{id}_Y, g \circ f \simeq \text{id}_X$ . In other words,  $g$  satisfies the properties of an inverse up to homotopy. It is called a **homotopy inverse** of  $f$ .

### Example 3: Circle to $\mathbb{R}^2$

The inclusion map  $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2$  is not a homotopy equivalence, but the inclusion  $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$  is a homotopy equivalence.

### Proposition 4: Unique Inverses of Homotopy

Homotopy inverses are unique up to homotopy.

### Definition 7: Homotopic Spaces

Two spaces  $X$  and  $Y$  are called **homotopy equivalent**, or of the **same homotopy type**, and denoted by  $X \simeq Y$  if there exists a homotopy equivalence  $f : X \rightarrow Y$ .

**Note**:  $\cong$  for homeomorphisms and  $\simeq$  for homotopy equivalence.

### Lemma 2: Composition of Inverses

Let  $f : X \rightarrow y, g : Y \rightarrow Z$  with homotopy inverses  $\bar{f} : Y \rightarrow X$  and  $\bar{g} : Z \rightarrow Y$  respectively. Then  $\bar{f} \circ \bar{g} : Z \rightarrow X$  is a homotopy inverse of  $g \circ f : X \rightarrow Z$ . In particular,  $X \simeq Y, Y \simeq Z$  implies  $X \simeq Z$ .

## 2 Contractible Spaces

### Definition 8: Contractible Space

A space  $X$  is called **contractible** if it is homotopy equivalent to a point, i.e.  $X \simeq *$ .

The **terminal map** is the unique map  $X \rightarrow *$ . Contractibility requires that there is a homotopy inverse of that map, i.e. a map  $*$   $\rightarrow$   $x$  along with homotopies

$$h : [* \rightarrow X \rightarrow *] \simeq \text{id}_*, \quad k : [X \rightarrow * \rightarrow X] \simeq \text{id}_X \quad (6)$$

### Example 4: Examples of Contractible Spaces

- $\mathbb{R}^n$  is contractible. Let  $x_0$  be a fixed point in  $\mathbb{R}^n$  and define the (straight line) homotopy  $h : c_{x_0} \simeq \text{id}_{\mathbb{R}^n}$  by

$$h(x, t) = (1 - t)x_0 + tx.$$

- $\mathbb{S}^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$ . The inclusion  $\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$  and the shrinking map

$$\mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{S}^{n-1}, \quad x \mapsto \frac{x}{|x|}$$

are homotopy inverses.

### Remark 3: Remarks about Contractible Spaces

- Contractible spaces are path-connected. Let  $x_0$  be the point where the space  $X$  contracts to. In particular, we are given with a homotopy  $h : c_{x_0} \simeq \text{id}_X$ . For any  $x \in X$ , the map  $h(x, -) : I \rightarrow X$  defines a path from  $x_0$  to  $x$  and thus every element  $x \in X$  is path-connected to  $x_0$ .
- The converse does not hold, for example  $X = \mathbb{S}^1$ .
- A contractible space  $X$  is contractible at any point  $x_0$ .  $X$  is path-connected, so a path  $x$  to  $x'$  defines a homotopy  $c_x \simeq c_{x'}$ .
- Any two maps  $f, g : X \rightarrow Y$  are homotopic if  $Y$  is contractible.

### Definition 9: Retracts and Deformation Retracts

- A **retract** of  $X$  onto a subspace  $A \subset X$  is a map  $r : X \rightarrow A$  such that  $r|_A = \text{id}_A$ . Equivalently, this is a map  $r : X \rightarrow X$  such that  $r^2 = r$  and  $r(X) = A$ .
- A **deformation retract** of  $X$  onto  $A$  is the additional datum of a homotopy  $h : \text{id}_X \simeq i \circ r$ .

In other words, a deformation retract is a homotopy  $h : X \times I \rightarrow X$  such that  $h(x, 0) = x$  and  $h(x, 1) \in A$  for all  $x \in X$  and  $h(a, 1) = a$  for all  $a \in A$ . Not all retracts can form deformation retracts. For instance, the retract  $X$  onto a point  $\{x_0\}$  can be a deformation retract iff  $X$  is contractible.

### Remark 4: Strong vs Weak Deformation Retracts

This notion is called **weak** deformation retract. A **strong** deformation retract has the condition  $h(a, t) = a$  for all  $t \in I$ ,  $a \in A$ . Our notion of a (weak) deformation retract deforms  $X$  into  $A$  while allowing to deform  $A$  to do so, while a strong deformation retract deforms  $X$  into  $A$  while keeping  $A$  fixed at all times

### Proposition 5: Deformation Retracts and Homotopies

A deformation retract of  $X$  onto  $A$  induces a homotopy equivalence  $X \simeq A$ .

### Recall 2: Quotient Space

Let  $X$  be a topological space and let  $\sim$  be an equivalence relation on  $X$ . Then,  $X/\sim$  is equipped with the quotient topology and called a **quotient space**. If  $Z$  is a closed subset in  $X$ , then we can also define the quotient space  $X/Z$ .

Another form of quotient spaces: Let  $f : Z \rightarrow Y$  be a continuous map between a closed subset  $Z \subset X$  and  $Y$ . Then

$$X \amalg_f Y = X \amalg Y / z \sim f(z).$$

### Example 5: Examples of Quotient Spaces

- The quotient of the  $n$ -dimensional closed disk by its boundary is the  $n$ -sphere, i.e.  $\mathbb{D}^n / \partial \mathbb{D}^n \cong \mathbb{S}^n$ .
- The 2-torus:  $\mathbb{R}^2 / \mathbb{Z}^2$ .
- The projective space:  $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$  by the relation  $x \sim y$  iff there exists some  $\lambda \in \mathbb{R}^\times$  such that  $x = \lambda y$ . This corresponds to the space of lines through the origin in  $\mathbb{R}^{n+1}$ .

### Definition 10: Mapping Quotients

Let  $f : X \rightarrow Y$  be a continuous map.

- Its **mapping cylinder** is defined as the topological space

$$M_f := (X \times I) \amalg Y / \sim$$

where the quotient identifies  $(x, 0) \sim f(x)$  for any  $x \in X$ .

- Its **cone** is the further quotient:  $C_f = M_f / X \times \{1\}$ .
- The **cone** of a topological space  $X$  is  $C_X := C_{\text{id}_X} = X \times I / X \times \{1\}$ .

In other words, the mapping cylinder of  $f : X \rightarrow Y$  is the pushout of the diagram:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & M_f \end{array}$$

### Example 5.5: Spheres

For  $\mathbb{S}^n$  with the standard embedding  $\mathbb{R}^{n+1} \setminus \{0\}$ , the following map is a retract, because if  $x$  has norm  $|x| = 1$ , then  $r(x) = x$ .

$$r : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{S}^n, \quad x \mapsto \frac{x}{|x|}$$

For a deformation retract one needs to find a homotopy  $h : i \circ r \simeq \text{id}_X$ . We use the following straight-line homotopy:

$$h : \mathbb{R}^{n+1} \setminus \{0\} \times I \rightarrow \mathbb{R}^{n+1} \setminus \{0\}, \quad (x, t) \mapsto (1 - t) \frac{x}{|x|} + tx.$$

Indeed,  $h(x, 0) = r(x)$  and  $h(x, 1) = x$  for all  $x$ .

### Definition 11: Star-Shaped Spaces

A subset  $S \subset \mathbb{R}^n$  is called **star-shaped** at a point  $x_0 \in S$ , if for any  $x \in S$  the line segment from  $x_0$  to  $x$  is contained in  $S$ , i.e.

$$\{(1 - t)x_0 + tx \mid t \in [0, 1]\} \subset S$$

If  $S$  is star-shaped at every point, then it is called **convex**.

### Example 5.6: Star-Shaped Spaces are Contractible

Let  $S$  be star-shaped at  $x_0$  and  $i : \{x_0\} \hookrightarrow S : r$  be the inclusion and constant maps. Define the straight line homotopy

$$h : S \times I \rightarrow S, \quad (x, t) \mapsto (1 - t)x_0 + tx$$

which is well-defined by the star-shaped condition. Moreover,  $h(x, 0) = x_0 = r(x)$  and  $h(x, 1) = x$  for all  $x$ . Hence, star-shaped, and in particular convex spaces, are contractible.

### Example 5.7: Möbius band

The Möbius band  $M$  can be defined as

$$M = I^2 / \sim$$

where  $\sim$  identifies the two vertical edges of  $I^2$  by flipping one, i.e.  $(0, b) \sim (1, 1 - b)$  for  $b \in I$ . Its core  $C \subset M$  is the line  $\{[a, 1/2] \mid a \in I\}$ . Thus, the core is homeomorphic to  $\mathbb{S}^1$ . The Möbius band deformation retracts onto its core, e.g. the retract  $r : M \rightarrow C$  defined by  $r([a, b]) := [(a, 1/2)]$  and the homotopy

$$h : M \times I \rightarrow M, \quad ([a, b], t) \mapsto \left[ \left( a, (1 - t) \frac{1}{2} + \right) \right].$$

In particular,  $M \simeq \mathbb{S}^1$ .

### Proposition 6: Retracts of the Mapping Cylinder

Via Definition 10, the mapping cylinder is formed by the cylinder of  $X$  by gluing  $Y$  onto the bottom with the map  $f$ . The mapping cylinder  $M_f$  strongly deformation retracts onto  $Y$ .

*Proof.* Consider the retract:

$$r : M_f \rightarrow Y$$

defined by  $r([x, s]) := [(x, 0)] = [f(x)]$  on the class of  $(x, s) \in X \times I$  and  $r([y]) = y$  for  $y \in Y$ . This is well-defined and by definition a retract on  $Y$ . Define the homotopy

$$h : M_f \times I \rightarrow M_f$$

by  $h([x, s], t) := [(x, st)]$  for  $(x, s) \in X \times I$  and  $t \in I$ , and by  $h([y], t) := y$  for  $y \in Y$ . In particular,  $h(-, 0) = i \circ r$  and  $h(-, 1) = \text{id}_{M_f}$ . This forms a strong deformation retract.  $\square$

### Remark 6: Continuous Maps are Homotopic

Any continuous  $f : X \rightarrow Y$  can be replaced up to homotopy equivalence by the closed inclusion  $X \hookrightarrow M_f$ ,  $x \mapsto [(x, 1)]$ . More precisely, it factorises through an inclusion and a homotopy equivalence as the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \nearrow \simeq \\ & M_f & \end{array}$$

### Definition 12: Relative Homotopy

Let  $X, Y$  be topological spaces and  $A \subset X$  a subset in  $X$ . A homotopy  $h : X \times I \rightarrow Y$  is called **relative to  $A$**  if  $h(a, t)$  is independent of  $t$  for all  $a \in A$ . In particular, this defines homotopies between maps  $f, g : X \rightarrow Y$  such that  $f|_A = g|_A$ .

This definition generalises ordinary homotopies, as an ordinary homotopy is the same as a homotopy relative to  $\emptyset$ .

### Example 6: Relative Homotopies and Retracts

A strong deformation retract of  $X$  onto  $A$  is a deformation retract such that the homotopy  $h : i \circ r \simeq \text{id}_X$  is relative to  $A$ .

### Definition 13: Homotopic Path

Let  $\alpha, \beta : I \rightarrow X$  be paths in  $X$  such that  $\alpha(0) = \beta(0)$  and  $\alpha(1) = \beta(1)$ . A relative homotopy from  $\alpha$  to  $\beta$  is a homotopy  $h : I \times I \rightarrow X$  relative to  $\partial I = \{0, 1\}$ , i.e.

$$h(-, 0) = \alpha, \quad h(-, 1) = \beta \quad (7)$$

and

$$h(0, t) = \alpha(0) = \beta(0), \quad h(1, t) = \alpha(1) = \beta(1), \quad \forall t \in I. \quad (8)$$

In particular, at any point  $t \in I$  a relative homotopy  $h$  defines a path  $h_t := h(-, t) : I \rightarrow X$  with start  $\alpha(0) = \beta(0)$  and end  $\alpha(1) = \beta(1)$ . If one omits the relative condition, the start and end points of  $h_t$  would be allowed to vary.

### Remark 7: Ordinary Homotopies and Paths

Ordinary homotopies are not well suited for paths: Any path  $\alpha : I \rightarrow X$  is homotopic (rel.  $\emptyset$ ) to a constant - as the homotopy

$$h : I \times I \rightarrow X, \quad (s, t) \mapsto \alpha(st)$$

defines a homotopy from the constant path  $c_{\alpha(0)}$  on  $\alpha(0)$  to  $\alpha$ , i.e.  $c_{\alpha(0)} \simeq \alpha$ . Hence, (ordinary) homotopy classes of paths in  $X$  are in 1-to-1 correspondence with path-connected components of  $X$ .

### Proposition 7: Homotopic Properties of Paths

Path concatenation is **unital**, **associative**, and **invertible** up to homotopy in the following sense: Let  $\alpha, \beta, \gamma : I \rightarrow X$  be paths such that  $\alpha(1) = \beta(0)$  and  $\beta(1) = \gamma(0)$ . Then there exists homotopies relative to  $\{0, 1\}$ :

1. **Left Unitality:**  $c_{\alpha(0)} * \alpha \simeq \alpha$
2. **Right Unitality:**  $\alpha \simeq c_{\alpha(0)} * \alpha$
3. **Associativity:**  $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$
4. **Right Inverse:**  $\alpha * \bar{\alpha} \simeq c_{\alpha(0)}$
5. **Left Inverse:**  $\bar{\alpha} * \alpha \simeq c_{\alpha(1)}$

where  $c_x$  for some  $x \in X$  denotes the constant path on  $x$  and  $\bar{\alpha}$  is the reversed path.

### Lemma 3:

Let  $\alpha : I \rightarrow X$  be a path and  $\lambda : I \rightarrow I$  a boundary preserving map, i.e.  $\lambda(0) = 0$  and  $\lambda(1) = 1$ . Then,

$$\alpha \circ \lambda \simeq \alpha, \quad \text{rel. } \partial I.$$

### Definition 14: Fundamental Group

Let  $X$  be a topological space and  $x_0 \in X$  some fixed point. The **fundamental group** of  $X$  at  $x_0$  is the group of homotopy classes of paths in  $X$  that start and end on  $x_0$ . i.e.  $\alpha : I \rightarrow X$  such that  $\alpha(0) = \alpha(1) = x_0$ , i.e.

$$\pi_1(X, x) = \{\alpha : I \rightarrow X \mid \alpha(0) = \alpha(1)\} / \sim.$$

### Theorem 1: Defining the Fundamental Group

The fundamental group  $\pi_1(X, x_0)$  is a well-defined group with:

- **Multiplication:**  $[\alpha] \cdot [\beta] := [\alpha * \beta]$
- **Unit:**  $1 = [c_{x_0}]$
- **Inverse:**  $[\alpha]^{-1} = [\bar{\alpha}]$

### Lemma 4: Relative Concatenated Homotopic Paths

Let  $\alpha \simeq \alpha' : I \rightarrow X$  and  $\beta \simeq \beta' : I \rightarrow X$  be two pairs of relative homotopic paths such that  $\alpha(1) = \alpha'(1) = \beta(0) = \beta'(0)$ . Then the concatenations are relative homotopic, i.e.

$$\alpha * \beta \simeq \alpha' * \beta', \quad \text{rel. } \{0, 1\}.$$

### Proposition 8: Fundamental Group is Point Independent

Let  $\gamma : I \rightarrow X$  be a path from  $\gamma(0) = x$  to  $\gamma(1) = x'$ . Then it induces a group isomorphism:

$$(\gamma)_\# : \pi_1(X, x) \rightarrow \pi_1(X, x'), \quad [\alpha] \mapsto [\bar{\gamma} * \alpha * \gamma].$$

We abuse notation to say that for a path-connected space  $X$ ,  $\pi_1(X)$  is the fundamental group omitting the choice of base point.

### Example 7: Examples of Fundamental Groups

- **Euclidean:**  $\pi_1(\mathbb{R}^n) \cong 1$ .
- **$n$ -Sphere,**  $n \geq 2$ :  $\pi_1(\mathbb{S}^n) \cong 1$ .
- **Circle:**  $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ .
- **Torus:**  $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ .
- **Projective Spaces:**  $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$  for  $n \geq 2$ .

### Definition 15: Pointed Space and Loop Space

- A **pointed space** is a pair  $(X, x)$  consisting of a topological space  $X$  and a point  $x \in X$ .
- A **map of pointed spaces**  $f : (X, x) \rightarrow (Y, y)$  is a continuous map  $f : X \rightarrow Y$  such that  $f(x) = y$ .
- The **space of pointed maps** from  $(X, x)$  to  $(Y, y)$  is denoted

$$\text{Map}_*((X, x), (Y, y)) \subset \text{Map}(X, Y).$$

With the (pointed) homeomorphism  $(\mathbb{S}^1, 1) \cong (I/\partial I, [0])$ , closed paths (where  $\alpha(0) = \alpha(1) = x$ ) are the same as pointed maps

$$(\mathbb{S}^1, 1) \rightarrow (X, x)$$

The space of such **loops** based at  $x$  is called the **loop space at  $x$** .

$$\Omega X := \text{Map}_*((\mathbb{S}^1, 1), (X, x))$$

It is itself a pointed space with the compact-open topology, and the constant map  $c_x$  as the base point. Path concatenation is the operation  $*$ :  $\Omega X \times \Omega X \rightarrow \Omega X$  which is associative, unital, invertible up to path-connectedness, which gives a group structure

$$\pi_0(\Omega X).$$

### Proposition 9: Loop Space Isomorphism

We have a group isomorphism:  $\pi_1(X, x) \cong \pi_0(\Omega X)$ .

Iteratively defining the  $n$ -fold loop space:

$$\Omega^n X := \Omega \Omega^{n-1} X = \Omega \cdot \Omega X$$

There is a homeomorphism:  $\Omega^n X \cong \text{Map}_*((\mathbb{S}^n, 1), (X, x))$

### Definition 16: $n$ -th Homotopy Group

The  $n$ -th homotopy group  $\pi_n(X, x)$  is defined by:

$$\pi_n(X, x) := \pi_0(\Omega^n X) \cong \pi_0(\text{Map}_*(\mathbb{S}^n, (X, x))).$$

### Definition 17: Simply Connected Space

A path-connected space  $X$  is **simply connected** if its fundamental group is trivial, i.e.  $\pi_1(X) = 1$ .

Some examples are  $\mathbb{R}^n$ ,  $\mathbb{S}^n$  for  $n > 1$ , and some non-examples are  $\mathbb{S}^1$ ,  $\mathbb{S}^1 \times \mathbb{S}^1$  and  $\mathbb{RP}^2$ .

### Theorem 2: Fundamental Group Isomorphism

Let  $f : X \rightarrow Y$  be a homotopy equivalence and  $x \in X$  an arbitrary base point. Then, the following map is a group isomorphism:

$$f_* : \pi_1(X, x) \rightarrow \pi_1(Y, f(x)), \quad [\alpha] \mapsto [f \circ \alpha]$$

In particular, for homotopy equivalent spaces  $X \simeq Y$  which are path-connected, we get  $\pi_1(X) \cong \pi_1(Y)$ .

A map of pointed spaces  $f : (X, x) \rightarrow (Y, y)$  is a **homotopy equivalence of pointed spaces** or **homotopy equivalence relative  $\{x\}$**  if there exists a map of pointed spaces  $g : (Y, y) \rightarrow (X, x)$  along with relative homotopies

$$h : f \circ g \simeq \text{id}_Y \quad \text{rel. } \{y\} \quad \text{and} \quad k : g \circ f \simeq \text{id}_X \quad \text{rel. } \{x\}$$

### Example 9: Strong Deformation Retracts Homotopies

A strong deformation retract of  $X$  onto a subspace  $A$  gives a homotopy equivalence of pointed spaces  $(x, a) \rightarrow (A, a)$  for any choice of  $a \in A$ . In particular, a contractible space  $X \simeq *$  determines a homotopy equivalence of pointed spaces  $(X, x) \rightarrow *$  for any choice of base point  $x$ .

### Lemma 5: Pointed Space Isomorphism

Let  $f : (X, x) \rightarrow (Y, y)$  be a homotopy equivalence of pointed spaces. Then the map

$$f_* : \pi_1(X, x) \rightarrow \pi_1(Y, y)$$

is a group isomorphism.

**Corollary 1:** Let  $r : X \rightarrow A$  be a strong deformation retract of  $X$  onto  $A \subset X$ . Then for any  $a \in A$ ,

$$\pi_1(X, a) \cong \pi_1(A, a)$$

In particular, contractible spaces are simply connected.

### Lemma 6: Identity Homomorphic Isomorphism

Let  $f : X \rightarrow X$  be a cts. map homotopic to  $\text{id}_X$ . Then, the map

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(X, f(x_0))$$

is a group isomorphism for any choice of base point  $x_0 \in X$ .



### Definition 18: Homotopy Lifting Property

A continuous map  $p : E \rightarrow X$  satisfies the **homotopy lifting property** (HLP) with respect to a topological space  $Y$  if for any commuting diagram:

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{H_0} & E \\ \downarrow & \searrow \exists & \downarrow p \\ Y \times I & \xrightarrow{h} & X \end{array}$$

There exists a map  $H : Y \times I \rightarrow E$  s.t. both triangles commute, i.e.  $H|_{Y \times \{0\}} = H_0$  and  $p \circ H = h$ .

The map  $p : E \rightarrow X$  has the HLP if for any homotopy  $h : Y \times I \rightarrow X$  of maps  $h(-, 0) := f_0$  and  $h(-, 1) := f_1$  of maps  $Y \rightarrow X$  and a choice of lift  $H_0$  of  $f_0$ , then the homotopy  $h$  lifts to a homotopy  $H : Y \times I \rightarrow E$ . In particular, if  $f_0 \simeq f_1 : Y \rightarrow X$  and  $H_0$  is a lift of  $f_0$ , we find  $H_0 \simeq H_1$  where  $H_1$  lifts  $f_1$ .

**Ex. 10:** The identity map  $\text{id}_X : X \rightarrow X$  has the HLP with respect to any space  $Y$ .

### Definition 19: Covering Space

A **covering space** of  $X$  is a topological space  $\bar{X}$  along with a continuous map  $p : \bar{X} \rightarrow X$  s.t. for any point  $x \in X$  there exists an open nbhd  $U \subset X$  whose preimage  $p^{-1}(U) = \bigcup_{j \in J} V_j$  and the opens  $V_j \subset \bar{X}$  map homeomorphically to  $U$  under  $p$ . A covering space of  $X$  looks locally like a product of  $X$  with a discrete space.

### Example 11: Example of a Covering Space

1. The projection map  $p : X \times Z \rightarrow X$  is a covering map if  $Z$  is a discrete topological space. If  $Z$  is not discrete, then this is not a covering map in general.
2. The identity map  $\text{id}_X : X \rightarrow X$  is trivially a covering map.
3. While the projection of  $p : X \times I \rightarrow X$  from the cylinder is not a covering map, its restriction to the boundary  $\partial(X \times I) = X \times \{0, 1\} =: \bar{X}$  gives a trivial (2-fold) cover of  $X$ .
4. Recall that the Möbius band  $M$  deformation retracts onto its core  $S^1$ . Restricting to the boundary  $\partial M = S^1$ , one obtains a (non-trivial) covering map  $S^1 \rightarrow S^1$ . This map coincides with  $z \mapsto z^2$  if we identify  $S^1$  as the unit circle in  $\mathbb{C}$ .

### Theorem 3: Unique HLPs from Covering Maps

Let  $p : \bar{X} \rightarrow X$  be a covering map and  $Y$  any topological space. Then  $p$  satisfies the HLP uniquely: i.e. the lift  $H$  not only exists, but it is also unique.

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{H_0} & \bar{X} \\ \downarrow & \searrow \exists! & \downarrow p \\ Y \times I & \xrightarrow{h} & X \end{array}$$

### Corollary 2:

1. Let  $\gamma : I \rightarrow X$  be a path and fix a point  $\tilde{x}_0 \in \bar{X}$  such that  $p(\tilde{x}_0) = \gamma(0)$ . Then, there exists a unique path  $\tilde{\gamma} : I \rightarrow \bar{X}$  which starts at  $\tilde{x}_0$  and lifts  $\gamma$  i.e.  $p \circ \tilde{\gamma} = \gamma$
2. Let  $h : I \times I \rightarrow X$  be a (relative) homotopy of paths  $h(-, 0) =: \gamma_0$  and  $h(-, 1) =: \gamma_1$ , and fix a point  $\tilde{x}_0$  such that  $p(\tilde{x}_0) = h(0, t) = \gamma_0(0) = \gamma_1(0)$ . Suppose  $\tilde{\gamma}_0 : I \rightarrow \bar{X}$  is a lift of  $\gamma_0$  starting at  $\tilde{\gamma}_0(0) = \tilde{x}_0$ . Then, there exists a unique homotopy of paths  $\tilde{h} : I \times I \rightarrow \bar{X}$  which lifts  $h$  and  $\tilde{h}(-, 0) = \tilde{\gamma}_0$

### Theorem 4-7: Fundamental Groups

- **Theorem 4:** The fundamental group of the circle is  $\pi_1(S^1) \cong \mathbb{Z}$ . It is generated by the class of

$$\alpha : I \rightarrow S^1, \quad t \mapsto e^{2\pi i t}.$$

- **Theorem 5 (Brouwer's Fixed Point Theorem):** Any continuous map  $f : \mathbb{D}^2 \rightarrow \mathbb{D}^2$  has a fixed point, i.e. there exists  $x \in \mathbb{D}^2$  such that  $f(x) = x$ .
- **Theorem 6 (Fundamental Theorem of Algebra):** Every non-constant complex polynomial  $p \in \mathbb{C}[z]$  has at least one root, i.e.  $p(z_0) = 0$  for some  $z_0$ .
- **Theorem 7:** The fundamental group of  $S^n$  is trivial for  $n \geq 2$ , i.e.  $\pi_1(S^2) \cong 1$  for  $n \geq 2$

### Lemma 7: Closed Paths Homotopic to Loops

Let  $(X, x_0)$  be a topological space with an open cover  $\{U_j\}_{j \in J}$  such that  $U_j$  are path-connected neighbourhoods of  $x_0$  and  $U_j \cap U_{j'}$  is path-connected for any  $j, j' \in J$ . Then, any closed path  $\gamma$  based at  $x_0$  is homotopic to a concatenation  $\gamma_1 * \gamma_2 * \dots * \gamma_n$  of loops at  $x_0$  each of them contained in a single  $U_j$ .

### Corollary 3: Homomorphisms between $\mathbb{R}^2$ and $\mathbb{R}^n$

There is no homeomorphism between  $\mathbb{R}^2$  and  $\mathbb{R}^n$  for  $n \neq 2$ .

### Recall 4: Defining the Real Projective Space

1. The space  $\mathbb{RP}^2$  is the quotient space:

$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$$

where  $x \sim y$  if there exists  $\lambda \in \mathbb{R}$  s.t.  $x = \lambda y$ . i.e., the real projective  $n$ -space represents the lines in  $\mathbb{R}^{n+1}$  through the origin.

2. Picking representatives that lie in the unit  $n$ -sphere  $S^n \subset \mathbb{R}^{n+1} \setminus \{0\}$ , we obtain  $\mathbb{RP}^n \cong S^n / \sim$  where  $x \sim -x$  for all  $x \in S^n$ , i.e. identifying antipodal points on the  $n$ -sphere.
3. Further restricting to the upper half  $\mathbb{D}^n \subset S^n$  we obtain:

$$\mathbb{RP}^n \cong \mathbb{D}^n / \sim$$

where  $x \sim -x$  for any boundary points  $x \in \partial \mathbb{D}^n \cong S^{n-1}$

For example,  $\mathbb{RP}^0$  is a one point space,  $\mathbb{RP}^1 \cong S^1$ , while  $\mathbb{RP}^n$  are different than spheres for larger  $n$ .

### Definition 20: Lift of a Path

- A lift of a path  $\alpha : I \rightarrow \mathbb{RP}^n$  is a path  $\tilde{\alpha} : I \rightarrow S^n$  s.t.  $p \circ \tilde{\alpha} = \alpha$
- If  $\alpha$  is a closed path, then  $[\tilde{\alpha}(0)] = [\tilde{\alpha}(1)]$  which implies  $\tilde{\alpha}(0) = \pm \tilde{\alpha}(1)$ . The **sign** of  $\alpha$  is defined by

$$\text{sgn}(\alpha) \mapsto \begin{cases} +1 & \tilde{\alpha}(0) = \tilde{\alpha}(1) \\ -1 & \tilde{\alpha}(0) = -\tilde{\alpha}(1) \end{cases}$$

### Theorem 8: The Fundamental Group of $\mathbb{RP}^2$

The sign induces a surjective group homomorphism  $\text{sgn} : \pi_1(\mathbb{RP}^n) \rightarrow \mathbb{Z}_2$ ,  $[\alpha] \mapsto \text{sgn}(\alpha)$  which is an isomorphism for  $n \geq 2$ .

## 3 Covering Theory

### Definition 21: Right Lifting Property

A map  $p : X \rightarrow Y$  satisfies the **right lifting property** (RLP) w.r.t. a map  $i : A \rightarrow B$  if any commutative square has a solution to the lifting problem making both triangles commute.

$$\begin{array}{ccc} A & \xrightarrow{\quad} & X \\ \downarrow i & \nearrow \exists! & \downarrow p \\ B & \xrightarrow{\quad} & Y \end{array}$$

Explicitly, if  $f : B \rightarrow Y$  and  $g : A \rightarrow X$  such that  $f \circ i = p \circ g$ , then there exists a map  $l : B \rightarrow X$  satisfying  $l \circ i = g$  and  $p \circ l = f$ . Dually, the map  $i : A \rightarrow B$  is said to satisfy the **left lifting property** (LLP) with respect to  $p : X \rightarrow Y$ .

### Example 13: Homotopy Lifting Property WRT Spaces

1. A map  $p : X \rightarrow Y$  satisfies the **homotopy lifting property** w.r.t. a space  $Z$  iff it has the RLP with respect to the inclusion map  $i : Z \times \{0\} \hookrightarrow Z \times I$ , i.e. solves the following lifting problem:

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \exists & \downarrow p \\ Z \times I & \xrightarrow{\quad} & Y \end{array}$$

In other words, given a homotopy  $h : Z \times I \rightarrow Y$  and a lift  $\tilde{f} : Z \rightarrow X$  of  $h(-, 0) =: f$ , there is a homotopy lift  $\tilde{h} : Z \times I \rightarrow X$  with  $\tilde{h}(-, 0) = \tilde{f}$ .

2. Dually, a map  $i : A \rightarrow b$  satisfies the **homotopy extension property** (HEP) with w.r.t. a space  $Z$  iff it has the LLP w.r.t. the map

$$p : Z^I \rightarrow Z, \quad \gamma \mapsto \gamma(0)$$

$$\begin{array}{ccc} A & \xrightarrow{\quad} & Z^I \\ \downarrow i & \nearrow \exists & \downarrow p \\ B & \xrightarrow{\quad} & Z \end{array}$$

Where  $Z^I := \text{Map}(I, Z)$  is the space of paths in  $Z$ . In other words, one can solve the following lifting problem.

Note that a map  $A \rightarrow Z^I$  is the same datum as a homotopy  $h : A \times I \rightarrow Z$ . Given an extension  $\tilde{f} : B \rightarrow Z$  of  $h(-, 0)$  along  $i$ , the existence of a map  $B \rightarrow Z^I$  which makes both triangles commute provides an extension of the homotopy  $h$  to a homotopy  $\tilde{h} : B \times I \rightarrow Z$  along  $i$ .

### Example 15: Covering Spaces

1. The projection map  $p : X \times D \rightarrow X$  where  $D$  is a discrete space. Note that  $X \times D$  cannot be path-connected unless  $D$  is a one-point set.
2. The covering map  $\mathbb{R} \rightarrow S^1, t \mapsto e^{2\pi i t}$  which we can use to compute the fundamental group of  $S^1$ .
3. The degree- $n$  map  $F_n : S^1 \rightarrow S^1, z \mapsto z^n$  provides an  $n$ -fold covering of  $S^1$  by itself.
4. The product of two covering maps  $p_i : \tilde{X}_i \rightarrow X_i$  is also a covering map  $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \rightarrow X_1 \times X_2$
5. The product of  $F_n$  and  $F_m$  in the third example also provides a self covering of the torus:
$$T^2 = S^1 \times S^1 \rightarrow T^2, \quad (z, w) \mapsto (z^n, w^m)$$
6. Similarly, there is a covering  $\mathbb{R}^2 \rightarrow T^2$ .
7. The 2-fold covering  $S^n \rightarrow \mathbb{RP}^n$  which was used to compute the fundamental group of  $\mathbb{RP}^n$

### Theorem 10: Homomorphism of Covering Maps

Let  $p : \tilde{X} \rightarrow X$  be a covering map. The induced group homomorphism

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$$

is injective for any  $\tilde{x}_0 \in p^{-1}(x_0)$ . Its image consists of (classes of) loops in  $X$  based at  $x_0$  that lift to loops in  $\tilde{X}$  based at  $\tilde{x}_0$

### Remark 5: Notation for Covers

Fix a covering map  $p : \tilde{X} \rightarrow X$  and  $x_0 \in X$  a fixed point. Write  $G := \pi_1(X, x_0)$  for the fundamental group of  $X$  at  $x_0$  and  $H := p_*\pi_1(\tilde{X}, \tilde{x}_0) \subset G$  for the subgroup determined by the covering map.

The subgroup  $H$  depends on the choice of fiber point  $\tilde{x}_0 \in p^{-1}(x_0)$ , and we shall see that its subgroups for different fiber points are conjugate to each other. Finally, the fiber over  $x_0$  will be denoted by

$$F_{x_0} := p^{-1}(x_0)$$

### Lemma 8: Transitive Actions

If  $\tilde{X}$  is path-connected, then the  $G$ -action on  $F_{x_0}$  is transitive, i.e. for any  $\tilde{x}, \tilde{x}' \in F_{x_0}$ , there exists an  $\alpha \in G$  such that  $\tilde{x} \cdot \alpha = \tilde{x}'$ .

### Theorem 11: Path-Connected Correspondence

If  $\tilde{X}$  is path-connected, then there is a one-to-one correspondence between right cosets and fiber points, i.e. a bijection

$$G\tilde{x} \backslash G \rightarrow F_{x_0}, \quad G\tilde{x} \cdot g \mapsto \tilde{x} \cdot g$$

Thus the index of  $G_{\tilde{x}}$  in  $G$  coincides with the cardinality of the fiber  $F_{x_0}$ :

$$[G : G_{\tilde{x}}] = |F_{x_0}| \quad (20)$$

**Corollary 4:** If  $\tilde{X}$  is simply-connected, then there is a bijection

$$G \rightarrow F_{x_0}$$

Equation (20) becomes

$$|G| = |F_{x_0}| \quad (21)$$

## 4 Deck Transformations, Further Cover Theory

### Definition 23: Deck Transformation

A **deck transformation** of a covering map  $p : \tilde{X} \rightarrow X$  is a self-homeomorphism  $D : \tilde{X} \xrightarrow{\cong} \tilde{X}$  such that  $p \circ D = p$ .

Deck transformations form a group  $\text{Deck}(p)$ . For any two deck transformations  $D, D'$  their composite is also a deck transformation since  $p \circ D \circ D' = p \circ D' = p$ . If  $D$  is a deck transformation then so is its inverse  $D^{-1}$  as  $p \circ D^{-1} = p \circ D \circ D' = p$ . For example, deck transformations of the covering map  $\mathbb{R} \rightarrow \mathbb{S}^1$  are precisely translations by integers:

$$D_n : \mathbb{R} \rightarrow \mathbb{R}, \quad t \mapsto t + n$$

In particular, the group of deck transformations is  $\mathbb{Z}$ . More generally, the group of deck transformations  $\text{Deck}(p)$  of a universal covering  $p$  is isomorphic to the fundamental group  $G$ . From now on  $p : \tilde{X} \rightarrow X$  will be a covering with  $\tilde{X}$  path-connected and  $X$  path-connected and locally path-connected.

### Example 16: Topologist's Sine Curve

Recall that the topologist's sine curve

$$X = \{0\} \times [-1, 1] \cup \left\{ \left( x, \sin\left(\frac{1}{x}\right) \right) \mid 0 < x \leq \frac{1}{2\pi} \right\} \subset \mathbb{R}^2$$

is an example of a connected, but not path-connected space. Let  $Z$  be the quotient of  $X$  by identifying the points  $(0, 0) \sim (\frac{1}{2\pi}, 0)$ .  $Z$  is a path-connected space but not locally path-connected.

### Theorem 12: Solutions to the Lifting Problem

Let  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  be a covering with  $X$  path-connected and locally path-connected, and let  $g : (Z, z_0) \rightarrow (X, x_0)$  be a pointed map. Then, there exists a solution to the lifting problem

$$\begin{array}{ccc} & (\tilde{X}, \tilde{x}_0) & \\ \nearrow \exists \tilde{g} & \downarrow p & \\ (Z, z_0) & \xrightarrow{g} & (X, x_0) \end{array} \quad \text{iff } g_*\pi_1(Z, z_0) \subset p_*\pi_1(\tilde{X}, \tilde{x}_0)$$

Moreover, if a solution to the lifting problem exists, then it is also unique.

**Remark 10:** If  $g$  is a covering map, then so is its lift  $\tilde{g}$ . In particular, homomorphisms of covering maps are also covering maps.

### Corollary 5: Universal Coverings

Let  $p : \tilde{X} \rightarrow X$  be a covering with  $\tilde{X}$  simply connected. Then, for any covering  $p' : X' \rightarrow X$  there exists a covering  $\tilde{p} : \tilde{X} \rightarrow X'$  such that the following diagram commutes:

$$\begin{array}{ccc} & X' & \\ \tilde{p} \nearrow & \downarrow p' & \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

In other words, if a simply connected covering  $\tilde{X}$  of  $X$  exists, then it covers all other possible coverings. This is why such a covering is called the **universal covering** of  $X$ . For example, we have seen  $\mathbb{R}$  as the universal covering of  $\mathbb{S}^1$  or  $\mathbb{S}^n$  as the universal covering of  $\mathbb{R}P^n$ .

### Theorem 24: Covering Isomorphism

Let  $p : \tilde{X} \rightarrow X$  be a covering with  $X$  path-connected and locally path-connected. Let  $H \subset \pi_1(X, x_0)$  denote the subgroup determined by the covering map. Then, there exists a group isomorphism:

$$\text{Deck}(p) \cong N(H)/H$$

where  $N(H)$  denotes the normalizer.

### Definition 24: Normal Coverings

A covering  $p : \tilde{X} \rightarrow X$  is **normal** if the subgroup  $H$  is normal

Trivially, universal coverings are always normal. All the examples so far were normal since all the fundamental groups we have seen so far were abelian.

**Corollary 6:** Let  $\tilde{X}$  be simply-connected. Then

$$\text{Deck}(p) \cong \pi_1(X, x_0).$$

### Example 19: The Figure Eight Space

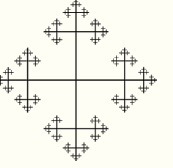
Let  $X$  be the figure eight space,  $X = \mathbb{S}^1 \vee \mathbb{S}^1$ .

Consider an oriented bicolored graph  $\tilde{X}$  whose vertices are all 4-valent with one incoming edge of each color and one outgoing edge of each color. Bicolored means each edge is labelled by  $a$  or  $b$ .

Such a graph determines a covering map

$$p : \tilde{X} \rightarrow X$$

by sending all vertices to the unique vertex of the figure-eight graph and the edges are sent to one of the loops. A universal covering is obtained by the following graph:



Vertical edges are oriented upwards and labelled by  $b$ , horizontal edges are oriented to the right and labelled by  $a$ . Deck transformations are freely generated by either  $D_a$  or  $D_b$ , where  $D_a$  (resp.  $D_b$ ) acts on the graph by shifting all edges once to the right, rescaling them appropriately. In other words..

### Theorem 14: Fundamental Group of $\mathbb{S}^1 \vee \mathbb{S}^1$

The fundamental group of  $\mathbb{S}^1 \vee \mathbb{S}^1$  is the free group generated by two elements, i.e.

$$\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \langle a, b \rangle$$

### Example 20.1: Covering The Möbius Band

Consider  $M : \mathbb{R} \times I / \sim$  where  $(x, y) \sim (x+1, 1-y)$ . We obtain the homotopy equivalence using covering theory. The quotient map

$$q : \mathbb{R} \times I \rightarrow M$$

is the universal covering, since  $\mathbb{R} \times I$  is simply-connected. For some  $n \in \mathbb{Z}$ , let  $D_n$  be the deck transformation:

$$D_n : \mathbb{R} \times I \rightarrow \mathbb{R} \times I, \quad (x, y) \mapsto (x+n, y_n)$$

where  $y_n = y$  if  $n$  is even and  $y_n = 1-y$  for  $n$  odd. These are all deck transformations and  $\text{Deck}(p)$  is generated by  $D_1$  since

$$D_n = (D_1)^n$$

For odd  $n$ , there are  $n$ -fold self-coverings  $M \rightarrow M$ . For even  $n$ , there are  $n$ -fold coverings by the cylinder  $S^1 \rightarrow I$ .

### Example 20.2: Covering the Klein Bottle

Consider  $K = \mathbb{R}^2 / \sim$ , where  $(x, y) \sim (x+1, 1-y) \sim (x, y+1)$  for all  $(x, y) \in \mathbb{R}^2$ . The quotient map of the Klein bottle

$$q : \mathbb{R}^2 \rightarrow K$$

is the universal covering map. Consider the deck transformation

$$D_a : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto (x, y+1)$$

and

$$D_b : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto (x+1, 1-y)$$

These two deck transformations generate the deck transformation group  $\text{Deck}(q)$  and satisfy the relation:

$$D_b \circ D_a \circ D_b^{-1} \circ D_a = \text{id}.$$

### Proposition 10: Fundamental Group of the Klein Bottle

The fundamental group of the Klein Bottle is:

$$\pi_1(K) = \langle a, b \rangle / \langle aba^{-1}b \rangle$$

## 5 Free Space and Examples

### 5.1 Fundamental Groups of Circles and Results

#### Example A: Indecipherable Proof of $\mathbb{S}^1$

**WTS:**  $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ , and generated by  $\alpha : I \rightarrow \mathbb{S}^1, t \mapsto e^{2\pi it}$ .

Let  $\alpha_n : I \rightarrow \mathbb{S}^1, t \mapsto e^{2\pi int}$  be the loop based at 1 and wraps  $n$  times around  $\mathbb{S}^1$ . The map

$$f : \mathbb{Z} \rightarrow \pi_1(\mathbb{S}^1, 1), \quad n \mapsto [\alpha_n]$$

is a group iso. It is a homo. as  $f(0) = [\alpha_0] = [c_1]$ ,  $c_1$  constant path at 1 and  $f(n+m) = f(n) \cdot f(m)$  follows from

$\alpha_{n+m} \simeq \alpha_n * \alpha_m$ . Let  $\gamma : I \rightarrow \mathbb{S}^1$  a loop based at  $1 \in \mathbb{S}^1$ .  $\exists n \in \mathbb{Z}$  s.t.  $\gamma \simeq \alpha_n$ , i.e.  $[\alpha] = [\alpha_n]$ . Consider the covering map

$$p : \mathbb{R} \rightarrow \mathbb{S}^1, \quad t \mapsto e^{2\pi it}$$

and note the fiber of  $1 \in \mathbb{S}^1$  is  $p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$ . By Corollary 2,  $\exists \tilde{\gamma} : I \rightarrow \mathbb{R}$  s.t.  $\tilde{\gamma}(0) = 0$  and  $p \circ \tilde{\gamma} = \gamma$ . In particular,  $p \circ \tilde{\gamma} = \gamma$  and  $p \circ \tilde{\gamma}(1) = \gamma(1) = 1$ , therefore  $\tilde{\gamma}(1) = n$  is some integer. Similarly

$$\tilde{\alpha}_n : I \rightarrow \mathbb{R}, \quad t \mapsto nt$$

is a lift of  $\alpha_n$  and also starts at 0, ends  $n$ . By contract. of  $\mathbb{R}$ , both  $\tilde{\gamma}$  and  $\tilde{\alpha}_n$  are homotopic as paths in  $\mathbb{R}$ . In particular,  $p \circ \tilde{g} = \gamma$  and  $p \circ \tilde{\alpha}_n = \alpha_n$  homotopic. Show that  $n$  is uniquely determined, i.e.  $[\alpha_n] = [\alpha_m]$  iff  $n = m$ . Suppose  $\exists$  homotopy  $h : I \times I \rightarrow \mathbb{S}^1$  paths  $h(-, 0) = \alpha_n$  and  $h(-, 1) = \alpha_m$ . By crl2  $\exists!$  rel. homotopy  $\tilde{h} : I \times I \rightarrow \mathbb{R}$  s.t.  $\tilde{h}(-, 0) = \tilde{\alpha}_n$ . In particular, we get a lift  $\tilde{h}(-, 1)$  of  $\alpha_m$  which by uniqueness must be  $\tilde{h}(-, 1) = \tilde{\alpha}_m$ . Comparing end points of  $\tilde{\alpha}_n$  and  $\tilde{\alpha}_m$  and noting  $\tilde{h}$  homotopy of paths,  $n = m$

We use: **Covering space**  $p : \mathbb{R} \rightarrow \mathbb{S}^1, t \mapsto e^{2\pi it}$  and its HLP to lift a closed path  $\alpha : I \rightarrow \mathbb{S}^1$  at 1 to a path  $\tilde{\alpha} : I \rightarrow \mathbb{R}$ . **Angle map** at  $\alpha$ :  $\alpha(t) = e^{2\pi i \tilde{\alpha}(t)}$ . If  $\tilde{\alpha}$  and  $\alpha'$  are both angle maps of  $\alpha$  s.t.  $\tilde{\alpha}(0) = \alpha'(0)$ , then  $\tilde{\alpha} = \alpha'$ , i.e. unique up to starting point. **Degree** of  $\alpha$ :  $\deg(\alpha) := \tilde{\alpha}(1) - \tilde{\alpha}(0)$  and is indep. of choice of angle map. Thm 4 states that  $\pi_1(\mathbb{S}^1) \rightarrow \mathbb{Z}, [\alpha] \mapsto \deg(\alpha)$  is a grp. iso.

#### Example B: Proof of results of $\mathbb{S}^1$

**Brouwer:** Suppose  $\exists$  cts. map  $f : \mathbb{D}^2 \rightarrow \mathbb{D}^2$  s.t.  $f(x) \neq x$   $\forall x \in \mathbb{D}^2$ . Consider the half line

$$L_x = \{tx + (1-t)f(x) \mid t \in \mathbb{R}_{\geq 0}\}$$

starts at  $f(x)$ , passes thru  $x$  and defines  $r(x) \in \mathbb{S}^1$  by the intersection  $r(x) := (L_x \setminus \{f(x)\}) \cap \mathbb{S}^1$ . This defines a retract of  $\mathbb{D}^2$  onto  $\mathbb{S}^1$ ,  $r : \mathbb{D}^2 \rightarrow \mathbb{S}^1, x \mapsto r(x)$  cts by cts of  $f(x)$ . However, the existence of the retract implies we have a surj. grp.homo.  $r_* : \pi_1(\mathbb{D}^2) \rightarrow \pi_1(\mathbb{S}^1)$  which is a contradiction since  $\pi_1(\mathbb{D}^2)$  trivial while  $\pi_1(\mathbb{S}^1)$  non-trivial.

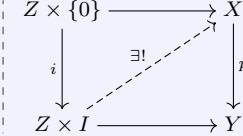
## 6 Unexamined Material

### Definition 22: Fibration

1. A map  $p : X \rightarrow Y$  is a **fibration** if it satisfies the HLP w.r.t. all spaces  $Z$ . i.e., it has the RLP w.r.t. the set of maps  $\{i : Z \times \{0\} \hookrightarrow Z \times I\}_Z$  where  $Z$  runs over all topo. spaces.
2. Dually, a map  $i : A \rightarrow B$  is a **cofibration** if it satisfies the HEP with respect to all spaces  $Z$ .

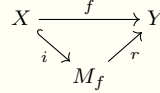
### Theorem 9: Covering Maps are Fibrations

A covering map  $p : \tilde{X} \rightarrow X$  is a fibration. Additionally, the homotopy lifts are unique:



### Example 14: Examples of Fibrations

1. By Theorem 9, fibrations include all covering maps
2. The projection map  $p : X \times F \rightarrow X$  is always a fibration. However this map is a covering map iff  $F$  is a discrete space. Hence, this includes examples of fibrations that are not coming from covering maps.
3. An important example of a cofibration is the inclusion  $i : X \rightarrow M_f$  where  $M_f$  is the mapping cylinder of  $f : X \rightarrow Y$ . We have seen that any continuous map  $f : X \times Y$  factors through the mapping cylinder:



In particular, every map factors through a cofibration and a homotopy equivalence.

### Theorem 15: Seifert-Van Kampen Theorem

Let  $X$  be a topological space with a fixed point  $x_0$ . Let  $\{U_\alpha\}_\alpha$  be an open cover of  $X$  consisting of path-connected open sets  $U_\alpha$  containing the fixed point  $x_0$ . The inclusions  $U_\alpha \subset X$  induce a group homomorphism:

$$\Phi : *_\alpha \pi_1(U_\alpha) \rightarrow \pi_1(X).$$

1. If  $U_\alpha \cap U_\beta$  is path-connected for any  $\alpha, \beta$ , then  $\Phi$  is surjective.
2. If  $U_\alpha \cap U_\beta \cap U_\gamma$  is path-connected for any  $\alpha, \beta, \gamma$ , then the kernel of  $\Phi$  is generated by elements  $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$  where  $w \in \pi_1(U_\alpha \cap U_\beta)$  and  $i_{\alpha\beta} : \pi_1(U_\alpha \cap U_\beta) \rightarrow \pi_1(U_\alpha)$  is the induced homomorphism from the inclusion  $U_\alpha \cap U_\beta \subset U_\alpha$ .

The assumption  $U_\alpha \cap U_\beta$  are path-connected ensures that words  $\pi_1(U_\alpha)$  generate  $\pi_1(X)$ . The assumption  $U_\alpha \cap U_\beta \cap U_\gamma$  is path-connected gives a presentation for the group  $\pi_1(X)$ .

### Example 17: Sifert-Van Kampen on $\mathbb{S}^1 \vee \mathbb{S}^1$

Consider the figure eight  $\mathbb{S}^1 \vee \mathbb{S}^1 = \mathbb{S}^1 \times \{1\} \cup \{1\} \times \mathbb{S}^1$  and let

$$U_i := \mathbb{S}^1 \vee \mathbb{S}^1 \setminus \{x_i\}$$

be the complements of the points  $x_1 = (-1, 1)$  and  $x_2 = (1, -1)$ . The sets  $U_1$  and  $U_2$  are open path-connected and cover  $\mathbb{S}^1 \vee \mathbb{S}^1$ . In fact, they are both homotopy equivalent to the circle  $U_i \simeq \mathbb{S}^1$ . Their intersection  $U_1 \cap U_2$  is contractible, and applying SVK we find,

$$\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \mathbb{Z} * \mathbb{Z}.$$

### Example 18: Fundamental Group of Wedged Circles

Let  $(X_\alpha, x_\alpha)$  be a fIYL of path-connected pointed spaces and consider their wedge sum

$$X := \bigvee_{\alpha} X_{\alpha}.$$

suppose that each  $x_\alpha := X_\alpha \vee \bigvee_{\beta \neq \alpha} U_\beta \subset X$ . By contractibility of the  $U_\alpha$ 's we have homotopy equivalences  $A_\alpha \simeq X_\alpha$ . Moreover, the intersection  $A_\alpha \cap A_\beta$  is contractible for any  $\alpha \neq \beta$ . Applying SVK we obtain

$$\pi_1(X) \cong *_\alpha \pi_1(X_\alpha).$$

In particular, the fundamental group of the  $n$ -th wedge sum of circles is the free group on  $n$ -generators:

$$\pi_1\left(\bigwedge^n \mathbb{S}^1\right) \cong \mathbb{Z} * \dots * \mathbb{Z} \cong \langle \alpha_1, \dots, \alpha_n \rangle. \quad (23)$$

### Definition 25: CW Complexes

A special class of topological spaces which are constructed inductively attaching  $n$ -dimensional disks or  $n$ -cells are called **CW complexes**. They are described as follows:

1. A set  $X^0$  of **vertices** or 0-cells
2. Inductively construct the  $n$ -skeleton  $X^n$  from  $X^{n-1}$  by attaching  $n$ -dimensional disks  $\mathbb{D}_\alpha^n$  by attaching maps  $\phi_\alpha : \partial \mathbb{D}_\alpha^n = \mathbb{S}_\alpha^{n-1} \rightarrow X^{n-1}$ . In other words,

$$X^n = X^{n-1} \amalg_{\phi_\alpha} \coprod_{\alpha} \mathbb{D}_\alpha^n.$$

Equivalently, a **CW Complex** is a space  $X$  along with a filtration of subspaces

$$X^0 \subset \dots \subset X^n \subset X^{n+1} \subset \dots \subset X$$

such that  $X^n \setminus X^{n-1}$  is homeomorphic to a disjoint union of  $n$ -dimensional open disks, and  $X^0$  is discrete.

### Example 19: Examples of CW Complexes

1. The Torus  $T^2 = I^2 / \sim$  can be made into a CW complex with:  $X^0 = \{[(0, 0)]\}$ ,  $X^1 = \{[(a, 0)] \mid a \in I\} \cup \{[(0, b)] \mid b \in I\}$  and  $X^2 = T^2$ . In particular, it has one 0-cell, two 1-cells, and one 2-cell.
2. The real projective plane  $\mathbb{RP}^2$  can be made into a CW complex with  $X^0 = *$ ,  $X^1 = \mathbb{RP}^1 = \mathbb{S}^1$  and  $X^2$  obtained by attaching a 2-disk to  $\mathbb{S}^1$  along the quotient map  $\mathbb{S}^1 \rightarrow \mathbb{RP}^1$