# General Topology Math Notes

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# 1 Intro to Topology

## 1.1 Why Topology?

Topology can appear where we least expect it...

- Algebraic Number Theory - Next to Euclidean topology, can define other topologies on $\mathbb{Q}$ (related to how often primes divide a number). Extends to Adeles, Langlands programme, etc

- Arithmetic Progressions in the Integers - An arithmetic progression of length $k$ is a set $\{a, a+d, \ldots, a+(k-1)d\}$ Finding subsets of $\mathbb{N}$ that contain arbitrarily long APs:

  - $2\mathbb{N}$ or $\mathbb{N}$

  - Primes (Green-Tao Theorem, 2007). Green-Tao theorem relies on **Szemeredi's Theorem**: Any dense enough subset of $\mathbb{N}$ contains arbitrarily long APs

    Furstenburg's idea: Get from $A \subseteq \mathbb{N}$ to $(a_i \in \{0,1\}^{\mathbb{N}})$ with $a_i \begin{cases} 1 & i \in A \\ 0 & \text{else} \end{cases}$

    Use topological dynamics to study this: A topological dynamical system is a triple of $X$ cpt, $T : X \to X$ continuous, and a probability measure $\mu$ preserved by $T$ (what)

## 1.2 Topological Spaces and Examples

**Definition 1.2.1: Topological Space**

A **topological space** is a pair $(X, \mathcal{T})$, where $X$ is a nonempty set, and $\mathcal{T}$ is a collection of subsets of $X$ which satisfies:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
2. if $U_\lambda \in \mathcal{T}$ for each $\lambda \in A$ (where $A$ is some indexing set), then $\bigcup_{\lambda \in A} U_\lambda \in \mathcal{T}$
3. If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

### 1.2.2 Examples of Topological Spaces

1. $\mathbb{R}^n$ with the Euclidean Topology - induced by the Euclidean Metric
2. For any set $X$, $\mathcal{T} = \mathcal{P}(X)$ (discrete topology)
3. For any set $X$, $\mathcal{T} = \{\emptyset, X\}$ (indiscrete topology)
4. $X = \{0,1,2\}$ with $\mathcal{T} = \{\emptyset, X, \{0\}, \{0,1\}, \{0,2\}\}$
5. $X = \mathbb{R}$ and $U$ open (aka, in $\mathcal{T}$) if $R \backslash U$ is finite or $U = \emptyset$

Proof for 5:

1. $\emptyset \in \mathcal{T}$, $\emptyset$ is finite $\implies X \in \mathcal{T}$
2. Intersections of finite sets are finite
3. Unions of finite sets are finite

**Definition 1.2.3: Neighbourhood of a point**

A **neighbourhood** of a point $x \in X$ is a subset $N \subseteq X$ s.t. $x \in U \subseteq N$ for some open subset $U \subseteq X$

**Definition 1.2.4: Metric Space**

A **metric space** $(X, d)$ is a nonempty set $X$ together with a function

$$d : X \times X \to \mathbb{R}$$

with the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

The function $d$ is called the metric. Point 3 is called the *triangle inequality*

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For any $x \in X$ and any positive real number $r$ the **open ball** in $X$ with centre $x$ and radius $r$ is defined by

$$B(x, r) = \{y \in X | d(x, y) < r\}$$

We declare a subset $U$ of $X$ to be *open in the metric topology given by $d$* iff for each $a \in U$ there is an $r > 0$ such that $B(a, r) \subseteq U$

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If $(X, \mathcal{T})$ is a topological space, and if $X$ admits a metric whose metric topology is precisely $\mathcal{T}$ we say that $(X, \mathcal{T})$ is **metrisable**. Thus the euclidean spaces with their usual topologies are metrisable

### 1.2.5 Examples of Metric Spaces

1. Any set $X$ with $d(x, y) = \begin{cases} 0, & x = y \\ 1, & \text{else} \end{cases}$
2. $\mathbb{R}^n$ with $d(x, y) = |x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
3. $C([0, 1])$ with $d(f, g) = \max_{t \in [0,1]} |f(t) - g(t)|$
4. $C([0, 1])$ with $d(f, g) = \sqrt{\int_0^n |f(t) - g(t)|^2 dt}$

### 1.2.6 Topologies on Metric spaces

We want to define a topology on $(X, d)$. For this, we want open balls to be open in the topology

**Definition 1.2.7: Base**

For a set $X$, a basis $\mathcal{B}$ is a collection of subsets such that

1. $\bigcup_{B \in \mathcal{B}} B = X$
2. $B_1 \cap B_2 \in \mathcal{B}$ for all $B_1, B_2 \in \mathcal{B}$

The **topology generated by** $\mathcal{B}$ is

$$\mathcal{T} := \{\bigcup_{i\in I} B_i, I \text{ index set}, B_i \in \mathcal{B}\}$$

**Note**: This is a topology because

$$(\cup_{i\in I} B_i) \cap (\cup_{j\in J} B_j) = \bigcup_{i\in I, j\in J} \underbrace{B_i \cap B_j}_{\in\mathcal{B}} \in \sqcup$$

**Definition 1.2.8: Metric Topology**

$$\text{Let } \mathcal{B} = \{\bigcap_{i=1}^{n} Br_1(x_1), n \in \mathbb{N},\ r_i > 0,\ x_i \in X,\ \forall i\}$$

The **metric topology** is the topology generated by this basis

**Observation** A set $U$ is open in the metric topology $\iff \forall x \in U, \exists r > 0$ s.t. $Br(x) \subseteq U$

- $\impliedby$ : For each $x \in U$, let $r_x$ s.t. $B_{r_x}(x) \subseteq U$. Then $U = \bigcup_{x\subseteq U} B_{r_x}(x)$ is open
- $\implies$ : Let $x \in U$ be given. Knwo that $x \in B_{r_1}(x_1) \cup \cdots \cup B_{r_n}(x_n)$ for some $n, r_1, x_1$. For each $i$, there is $\delta_i > 0$ s.t. $B_{\delta_i}(x) \leq B_{r_1}(x_1)$.

huh?

**Theorem 1.2.9: random ms prop**

If $X$ carries metrics $d, \tilde{d}$ such that $ad(x,y) \leq \tilde{d}(x,y) \leq Ad(x,y)$ for some $a, A > 0$, then the induced topologies agree

**Definition 1.2.10: Subspace Topology**

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be any subset. Then the **subspace topology** on $A$ consists of all sets of the form $U \cap A$ where $U \in \mathcal{T}$

**Example**: $(-1, 1) \subseteq \mathbb{R}$ with euclidean topology. The subspace topology is

$$\{(-1, 1) \cap U,\ U \subseteq \mathbb{R} \text{ open}\}$$

$(-1, 1)$ is closed in the subspace topology

**Theorem 1.2.11: Topology Lemmas**

**1.3** If $(X, \mathcal{T})$ is a topological space and $U_1, \ldots, U_n$ are open sets, then the intersection $\bigcap_{i=1}^{n} u_i$ is also open

1.6 In order to show that a set $U \subseteq X$ is open, it is enough to show that for every $x \in U$ there is an open set $V$ with $x \in V \subseteq U$

1.6 A subset $U$ of $\mathbb{R}^n$ is *open for the usual topology* iff for each $a \in U$ there exists an $r > 0$ s.t.

$$|x - a| < r \implies x \in U$$

The collection of open sets thus defined is called the **usual topology on** $\mathbb{R}^n$. Note that open balls are open sets under this definition

**Definition 1.2.12: Topology Small Definitions**

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## 1.3 Closed sets, Closure, Interior, and Boundary

**Definition 1.3.1: Closed Subsets**

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is **closed** iff its complement $X \backslash A = A^C := \{x \in X \; x \notin A\}$ is open in $X$

**Note**: A set being "closed" has no connection with "not being open"

### 1.3.2 Examples of open and closed sets

- A set that is neither open nor closed: $[0, 1) \subseteq \mathbb{R}$ under Euclidean topology
- A set that is both closed and open: $\emptyset$ or $X$

**Theorem 1.3.3**

Let $(X, \mathcal{T})$ be a topological space. Then

1. $\emptyset$ and $X$ are closed.
2. The union of finitely many closed sets is a closed set
3. The intersection of any collection of closed sets is a closed set

$\bigcup_{i \in I} A_i$ is not necessarily closed.

$$A_n = \left[-2 + \frac{1}{n}, 2 - \frac{1}{n}\right] \text{ in } \mathbb{R}, \bigcup_{i=1}^{n} A_i = (-2, 2)$$

### 1.3.4 Topological proof that there are infinitely many primes (Furstenberg)

*Proof.* Look at $\mathbb{Z}$ with

$$\mathcal{B} := \{S(a,b),\, a \neq 0,\, b \in \mathbb{Z}\} \quad \text{and} \quad S(a,b) = \{an + b,\, n \in \mathbb{Z}\}$$

Let the open sets be the one generated by this basis. We can show

1. $S(a, b)$ is both open and closed.
2. All open sets are infinite.

---

1. $S(a, b) = \mathbb{Z} \backslash \bigcup_{i=1}^{a-1} S(a, b-1)$
2. Clear

Thus:

$$\underbrace{\mathbb{Z} \backslash \{-1, 1\}}_{\text{not closed}} = \bigcup_{p \text{ primes}} \overbrace{S(p, 0)}^{\text{closed}}$$

If there were only finitely many primes, right hand side would be closed ☐

## 1.4 Closure and stuff

**Definition 1.4.1: Closure, Interior, Boundary**

Let $(X, \mathcal{T})$ be a topological space.

1. The **closure** of a subset $A \subseteq X$ is the smallest closed set such that $A \subseteq \overline{A}$.

$$\overline{A} := \bigcap_{\substack{C \subseteq X \text{closed} \\ A \subseteq C}} C$$

2. The (open) **interior** of a subset $A \subseteq X$ is the biggest open set $U$ contained in $A$

$$\operatorname{int} A = A^\circ := \bigcap_{\substack{U \subseteq X \text{open} \\ U \subseteq A}} C$$

3. The **boundary** or **frontier** of a subset $A \subseteq X$ is

$$\partial A := \overline{A} \backslash A^\circ$$

4. A subset $A$ of $X$ is **dense** in $X$ iff $\overline{A} = X$

   E.g.: $\mathbb{Q} \subseteq \mathbb{R}$ with the Euclidean topology

**Theorem 1.4.2: Closure and Interior of Complement**

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then

1. The closure of the complement is the complement of the interior:

$$\overline{X \backslash A} = X \backslash (A^\circ)$$

2. the interior of the complement is the complement of the closure:

$$(X \backslash A)^\circ = X \backslash \overline{A}$$

**Definition 1.4.3: Limits in Topological spaces**

A sequence $(x_n)$ converges to $x \in X$ if $\forall U$ open with $x \in U$, $\exists N$ s.t. $x_n \in U$ for all $n \geq N$

**Definition 1.4.4: Limit Set**

$\overline{A}$ can be thought as the set of limits. Formally define the **Limit Set** as

$$A' = \{x \in X : \forall U \subseteq X, x \in U, \text{ open } \exists a \in A, \text{ s.t. } a \in U\}$$

But, limits in general are much worse behaved in topological spaces, e.g. no unique limit point
**Example**: a topological space $X$ and a sequence $(x_n)$ which does not have a unique limit (i.e. $\exists x \neq y$ s.t. $x_n \to x$ and $x_n \to y$ in the sense defined): Nontrivial $X$ with the indiscrete topology $\{\emptyset, X\}$

## 1.5 Hausdorff Spaces

**Problem**: Non-unique limits are nasty :(

**Definition 1.5.1: Hausdorff Space**

A topological space $(X, \mathcal{T})$ is **Hausdorff** if for each $x, y \in X$ with $x \neq y$ there exist *disjoint* open sets $U$ and $V$ s.t. $x \in U$ and $y \in V$
This space has *unique limits*!

If $(X, d)$ is a metric space then it is automatically Hausdorff, so any metrisable space is Hausdorff. The trivial topology on a set with more than one element is not Hausdorff. Not every Hausdorff space is metrisable
Non-Hausdorff spaces are a lot more annoying to work with - for example you can have multiple limits in non-Hausdorff spaces

**Theorem 1.5.2: Open sets on $\mathbb{R}$ with Euclidean Topology**

- A set $U$ is open iff there are open intervals $I_j$ s.t.

$$U = \bigcup_{j=1}^{\infty} I_j$$

- A set $A$ is closed iff there are $F_j$ (union of two closed intervals) s.t.

$$A = \bigcap_{j=1}^{\infty} F_j$$

**Definition 1.5.3: Convergence of Hausdorff Spaces**

A sequence $(x_n)$ of members of a topological space $X$ converges to $x \in X$ if for every open set $U$ containing $x$, there exists an $N$ such that $n \geq N \implies x_n \in U$

**Theorem 1.5.4: Haussdorf Convergence Uniqueness**

Suppose $(X, \mathcal{T})$ is Hausdorff. Then a sequence $(x_n)$ can converge to at most one limit.

Being Hausdorff is what's called a *topological property*, which means whether it is true in a particular case depends only on the open sets of the space in question.
In contrast, the property of *completeness* of a metric space is not a topological property as there exist sets upon which one can put two distinct metrics, one complete and one not, yet for which the metric topologies coincide

**Definition 1.5.5: Cauchy Sequences**

Let $(X, d)$ be a metric space

1. A **Cauchy Sequence** is a sequence $(x_n)$ with each $x_n \in X$ with the property that for each $\epsilon > 0$, there exists an $N$ s.t. $m, n \geq N \implies d(x_m, x_n) < \epsilon$
2. $(X, d)$ is **complete** if every Cauchy Sequence converges

**Caveat**: In general, this does not have to converge to an $x \in X$
**Example**: $\mathbb{Q}$ with the Euclidean metric.

**Definition 1.5.6: Complete Space**

A metric space is called **complete** if all Cauchy sequences converge to a point in the space

**Definition 1.5.7: Closure in Metric Spaces**

Let $(X, d)$ be a complete metric space and $A \subseteq X$. A point $x$ is in the **closure** of $A \iff \exists x_i \to x$ with $x \in A$

# 2 Continuity

## 2.1 Continuity

**Definition 2.1.1: Continuity**

Let $(X, \mathcal{T}), (Y, \tilde{\mathcal{T}})$ be topological spaces and $f : X \to Y$. $f$ is **continuous** if for all $U \in \tilde{\mathcal{T}}$, $f^{-1}(U) \in \mathcal{T}$

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Equivalently:

- $U \subseteq Y$ open $\implies f^{-1}(U)$ open
- $A \subseteq Y$ closed $\implies f^{-1}(A)$ closed

### 2.1.2 Why take $f^{-1}$

**Properties**: For $U, V$ sets in $Y$,

- $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(U)$
- $f^{-1}(U^C) = f^{-1}(U)^C$
- $f^{-1}(U \cup U) = f^{-1}(U) \cup f^{-1}(U)$

**Example**: $\mathbb{R}$ with Euclidean Topology

*Proof.* "Proof" that $[-1, 1]$ is open:
Take $[-1, 1]$ with the subspace topology $\mathcal{T} := \{[-1, 1] \cap U, U \subseteq \mathbb{R} \text{ open}\}$
Embedding $i : [-1, 1] \to \mathbb{R}$, $x \mapsto x$ is continuous
$[-1, 1]$ open in subspace topology
$i$ cont $\implies i([-1, 1])$ is open   this is actually wrong! $U$ open $\not\Longrightarrow f(U)$ open
But $i([-1, 1]) = [-1, 1] \subseteq \mathbb{R}$ ∎

**Definition 2.1.3: Formal Definition of Continuity**

Let $(X,d),(Y,d)$ be metric spaces with the metric topology. $f:X\to Y$ is continuous as above iff $\forall x\in X$, $\forall\epsilon>0$, $\exists\delta>0$ s.t.

$$d(x,y)<\delta \implies d(f(x),f(y))<\epsilon$$

*Proof.* $\implies$ **Direction**
Recall: $U$ open in metric topology if $\forall x\in U$, $\exists r>0$ s.t. $B_r(x)\subseteq U$, where $B_r(x)=\{y\in X: d(x,y)<r\}$
$\implies$ Let $x\in X$ be given, $\epsilon>0$. Let $y=f(x)\in Y$, $U=B_\epsilon(y)=\{y'\in Y:\tilde{d}(y,y')<\epsilon\}$.
$f$ cont $\implies f^{-1}(U)$ is open. $x\in f^{-1}(U)\implies\exists\delta>0$ s.t. $B_\delta(x)\subseteq f^{-1}(U)$
$\implies\forall x'\in X$ s.t. $d(x,x')<\delta$, $x'\in B_\delta(x)\subseteq f^{-1}(U)$.
$\implies f(x')\in B_\delta(f(x))\implies\tilde{d}(f(x),f(x'))<\epsilon$
$\impliedby$ **Direction**
Let $U$ be open in $Y$. WTS: $f^{-1}(U)$ is open.
So it is enough to show for all $x\in f^{-1}(U)$, $\exists\delta>0$ s.t. $B_\delta(x)\subseteq f^{-1}(U)$.
Let $x$ be given, $y:=f(x)\in U$. $U$ open $\implies\exists\epsilon>0$ s.t. $B_\epsilon(y)\subseteq U$.
By assumption $\exists\delta>0$ s.t.

$$d(x',x)<\delta\implies\tilde{d}(f(x'),f(x))<\epsilon$$

But, $\{y':d(y',f(x))<\epsilon\}\subseteq U$ by choice of $\epsilon$.
$\implies B_\delta(x)\subseteq f^{-1}(U)$ $\square$

## 2.2 Homeomorphisms

**Definition 2.2.1: Homeomorphism**

Let $(X,\mathcal{T}),(Y,\tilde{\mathcal{T}})$ be topological spaces. A function $f:X\to Y$ is a **homeomorphism** (or **bi-continuous**) if $f$ is bijective, $f$ is continuous, and $f^{-1}Y\to X$ is continuous

A "Great goal of Topology": Understand topological spaces up to homeomorphisms.
Say that a property of a topological space is a **topological invariant** if it is preserved by homeomorphism. Example: Being Hausdorff

**Example 2.2.2: Examples of Homeomorphisms**

- $(X,\mathcal{T})$ topological space, $\mathrm{id}:X\to X$, $x\mapsto x$
- $X=\mathbb{R}^n$, $A:\mathbb{R}^n\to\mathbb{R}^n$ Linear + Invertible
- Example which is **not** a homeomorphism:

$$f:\underbrace{\mathbb{R}}_{\text{metric topology}}\to\underbrace{\mathbb{R}}_{\text{indiscrete topology }\{\emptyset,\mathbb{R}\}},\ x\mapsto x$$

Problem: $f^{-1}$ is not continuous

**Definition 2.2.3: Another continuity definition**

Let $(X, d), (Y, \tilde{d})$ be metric spaces with the metric topology. $f : X \to Y$ is continuous iff

$$\forall x \in X, \forall \epsilon > 0, \exists \delta > 0 \ \text{ s.t. } \forall y \in X, d(x, y) < \delta \implies \tilde{d}(f(x), f(y)) < \epsilon$$

**Observe**: $\forall y \in X$ is equivalent to

$$B_\delta(x) \subseteq f^{-1}(\tilde{B}_\epsilon(f(x)))$$

Why? Let $A, B$ be things which can be true for $y \in X$. i.e.

$$A \implies B \text{ is equivalent to } \{y : A \text{ true} \subseteq \{y : B \text{ true}\}\}$$

Then: $B_\delta(x) = \{y, \underbrace{d(x, y) < \delta}_{A}\}$, $f^{-1}(\tilde{B}_\epsilon(f(x))) = \{y \in X : \underbrace{\tilde{d}(f(x), f(y)) < \epsilon}_{B}\}$
WTS: $U$ open $\iff \forall x \in U, \exists r > 0$ s.t. $B_\delta(x) \subseteq U$
$\implies$ "Let $x, \epsilon$ be given, WTS that $\exists \delta$ s.t. $B_\delta(x) \subseteq f^{-1}(\tilde{B}_\epsilon f(x))$
**Example**: $f$ cont + bijective but not a homeomorphism:

$$f : \underbrace{X}_{\text{discrete topology - every set is open}} \to \overbrace{X}^{\text{indiscrete topology: only } \emptyset \text{ and } X \text{ are open}} \text{ identity}$$

**Lemma 2.2.4: Homeomorphism-condition**

For a set $X$ with topologies $\mathcal{T}$, $\tilde{\mathcal{T}}$. The identity map $(X, \mathcal{T}) \to (X, \tilde{\mathcal{T}})$, $x \mapsto x$ is

- continuous $\iff \tilde{\mathcal{T}} \subseteq T$
- a homeo $\iff \tilde{\mathcal{T}} = \mathcal{T}$

**Theorem 2.2.5: Mapping prop**

- Let $f : X \to Y$, $g : Y \to Z$ continuous. The map $f \circ g$ is continuous

$$\text{As } (f \circ g)^{-1}(U) = f^{-1}(g^{-1}(U))$$

- If $f : X \to Y$ is constant, then $f$ is continuous
- In particular, $f, g$ homeo $\implies f \circ g$ is a homeo

# 3 The Clark Barwick Era

**Theorem 3.0.1: Clark Barwick Quotes List**

"Shadows are harshest when there is only one lamp" - 04/10/24

## 3.1 More top

### 3.1.1 Something weird

$$[0, 2\pi) \to S^1 = \{z \in \mathbb{Z} : \|z\| = 1\}$$
$$[0, 2\pi) \to [0, 1) \quad \text{is open, and is also creepy}$$

Not a homeomorphism
**Claim**: A continuous bijection in which the **image** of every open set is open is a homeomorphism

**Definition 3.1.2: Subspace Topology**

For $X$ a topological space, and $T \subseteq X, \mathcal{U} \subseteq T$ is open iff $\exists V \subseteq X$ open and $\mathcal{U} = V \cap T$

**Definition 3.1.3: Impromptu Set Theory - Products**

$\mathcal{F}$ is a family of sets. We can talk about a product

$$\prod_{x \in \mathcal{F}} X = \{(a_x)_{x \in \mathcal{F}} : a_X \in X\}$$

Example:

$$\mathbb{R}^\infty = \prod_{i=1}^{\infty} \mathbb{R} = \{(x_1, x_2, \dots) : x_i \in \mathbb{R}\}$$
$$\prod_{x \in \mathcal{F}} = \{\phi : \mathcal{F} \implies \bigcup_{x \in \mathcal{F}} X : \phi(X) \in X\}$$

Note: the $\mathcal{F}$ notation is pretty creepy - Clark

### 3.1.4 Topologising the above thing

Metrisable

$$\prod_{i \in I} X_i \to X_j$$

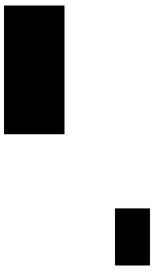

Figure 1: testt