General Topology Math Notes

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1 Intro to Topology

1.1 Why Topology?

Topology can appear where we least expect it...

- Algebraic Number Theory - Next to Euclidean topology, can define other topologies on $\mathbb{Q}$ (related to how often primes divide a number). Extends to Adeles, Langlands programme, etc
- Arithmetic Progressions in the Integers - An arithmetic progression of length k is a set $\{a, a+d, \ldots, a+(k-1)d\}$ Finding subsets of $\mathbb{N}$ that contain arbitrarily long APs:
  - $2\mathbb{N}$ or $\mathbb{N}$
  - Primes (Green-Tao Theorem, 2007). Green-Tao theorem relies on **Szemeredi's Theorem**: Any dense enough subset of $\mathbb{N}$ contains arbitrarily long APs

    Furstenburg's idea: Get from $A \subseteq \mathbb{N}$ to $(a_i \in \{0,1\}^{\mathbb{N}})$ with $a_i \begin{cases} 1 & i \in A \\ 0 & \text{else} \end{cases}$

    Use topological dynamics to study this: A topological dynamical system is a triple of X cpt, $T : X \to X$ continuous, and a probability measure μ preserved by T (what)

1.2 Topological Spaces and Examples

Definition 1.2.1: Topological Space

A **topological space** is a pair $(X, \mathcal{T})$, where X is a nonempty set, and $\mathcal{T}$ is a collection of subsets of X which satisfies:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
2. if $U_\lambda \in \mathcal{T}$ for each $\lambda \in A$ (where A is some indexing set), then $\bigcup_{\lambda \in A} U_\lambda \in \mathcal{T}$
3. If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

1.2.2 Examples of Topological Spaces

1. $\mathbb{R}^n$ with the Euclidean Topology - induced by the Euclidean Metric
2. For any set X, $\mathcal{T} = \mathcal{P}(X)$ (discrete topology)
3. For any set X, $\mathcal{T} = \{\emptyset, X\}$ (indiscrete topology)
4. $X = \{0,1,2\}$ with $\mathcal{T} = \{\emptyset, X, \{0\}, \{0,1\}, \{0,2\}\}$
5. $X = \mathbb{R}$ and U open (aka, in $\mathcal{T}$) if $R \backslash U$ is finite or $U = \emptyset$

Proof for 5:

1. $\emptyset \in \mathcal{T}$, $\emptyset$ is finite $\implies X \in \mathcal{T}$
2. Intersections of finite sets are finite
3. Unions of finite sets are finite

Definition 1.2.3: Neighbourhood of a point

A **neighbourhood** of a point $x \in X$ is a subset $N \subseteq X$ s.t. $x \in U \subseteq N$ for some open subset $U \subseteq X$

Definition 1.2.4: Metric Space

A **metric space** (X, d) is a nonempty set X together with a function

$$d : X \times X \to \mathbb{R}$$

with the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

The function d is called the metric. Point 3 is called the *triangle inequality*

For any $x \in X$ and any positive real number r the **open ball** in X with centre x and radius r is defined by

$$B(x, r) = \{y \in X | d(x, y) < r\}$$

We declare a subset U of X to be *open in the metric topology given by d* iff for each $a \in U$ there is an $r > 0$ such that $B(a, r) \subseteq U$

If $(X, \mathcal{T})$ is a topological space, and if X admits a metric whose metric topology is precisely $\mathcal{T}$ we say that $(X, \mathcal{T})$ is **metrisable**. Thus the euclidean spaces with their usual topologies are metrisable

1.2.5 Examples of Metric Spaces

1. Any set X with $d(x, y) = \begin{cases} 0, & x = y \\ 1, & \text{else} \end{cases}$
2. $\mathbb{R}^n$ with $d(x, y) = |x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
3. $C([0, 1])$ with $d(f, g) = \max_{t \in [0,1]} |f(t) - g(t)|$
4. $C([0, 1])$ with $d(f, g) = \sqrt{\int_0^n |f(t) - g(t)|^2 dt}$

1.2.6 Topologies on Metric spaces

We want to define a topology on (X, d). For this, we want open balls to be open in the topology

Definition 1.2.7: Base

For a set X, a basis $\mathcal{B}$ is a collection of subsets such that

1. $\bigcup_{B \in \mathcal{B}} B = X$
2. $B_1 \cap B_2 \in \mathcal{B}$ for all $B_1, B_2 \in \mathcal{B}$

The **topology generated by** $\mathcal{B}$ is

$$\mathcal{T} := \{\bigcup_{i\in I} B_i, I \text{ index set}, B_i \in \mathcal{B}\}$$

Note: This is a topology because

$$(\cup_{i\in I}B_i)\cap(\cup_{j\in J}B_j) = \bigcup_{i\in I, j\in J} \underbrace{B_i \cap B_j}_{\in\mathcal{B}} \in \sqcup$$

Definition 1.2.8: Metric Topology

$$\text{Let } \mathcal{B} = \{\bigcap_{i=1}^{n} Br_1(x_1), n\in\mathbb{N},\, r_i > 0,\, x_i \in X,\, \forall i\}$$

The **metric topology** is the topology generated by this basis

Observation A set U is open in the metric topology $\iff \forall x\in U, \exists r>0$ s.t. $Br(x)\subseteq U$

- $\Longleftarrow$: For each $x\in U$, let r_x s.t. $B_{r_x}(x)\subseteq U$. Then $U = \bigcup_{x\subseteq U} B_{r_x}(x)$ is open
- $\Longrightarrow$: Let $x\in U$ be given. Knwo that $x\in B_{r_1}(x_1)\cup\cdots\cup B_{r_n}(x_n)$ for some n, r_1, x_1. For each i, there is $\delta_i > 0$ s.t. $B_{\delta_i}(x)\leq B_{r_1}(x_1)$.

huh?

Theorem 1.2.9: random ms prop

If X carries metrics $d, \tilde{d}$ such that $ad(x,y)\leq\tilde{d}(x,y)\leq Ad(x,y)$ for some $a, A > 0$, then the induced topologies agree

Definition 1.2.10: Subspace Topology

Let $(X,\mathcal{T})$ be a topological space, and let $A\subseteq X$ be any subset. Then the **subspace topology** on A consists of all sets of the form $U\cap A$ where $U\in\mathcal{T}$

Example: $(-1,1)\subseteq\mathbb{R}$ with euclidean topology. The subspace topology is

$$\{(-1,1)\cap U,\, U\subseteq\mathbb{R} \text{ open}\}$$

$(-1,1)$ is closed in the subspace topology

Theorem 1.2.11: Topology Lemmas

1.3 If $(X,\mathcal{T})$ is a topological space and $U_1,\ldots,U_n$ are open sets, then the intersection $\bigcap_{i=1}^n u_i$ is also open

1.6 In order to show that a set $U\subseteq X$ is open, it is enough to show that for every $x\in U$ there is an open set V with $x\in V\subseteq U$

1.6 A subset U of $\mathbb{R}^n$ is *open for the usual topology* iff for each $a\in U$ there exists an $r>0$ s.t.

$$|x-a|<r \implies x\in U$$

The collection of open sets thus defined is called the **usual topology on** $\mathbb{R}^n$. Note that open balls are open sets under this definition

Definition 1.2.12: Topology Small Definitions

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1.3 Closed sets, Closure, Interior, and Boundary

Definition 1.3.1: Closed Subsets

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is **closed** iff its complement $X \backslash A = A^C := \{x \in X \; x \notin A\}$ is open in X

Note: A set being "closed" has no connection with "not being open"

1.3.2 Examples of open and closed sets

- A set that is neither open nor closed: $[0, 1) \subseteq \mathbb{R}$ under Euclidean topology
- A set that is both closed and open: $\emptyset$ or X

Theorem 1.3.3

Let $(X, \mathcal{T})$ be a topological space. Then

1. $\emptyset$ and X are closed.
2. The union of finitely many closed sets is a closed set
3. The intersection of any collection of closed sets is a closed set

$\bigcup_{i \in I} A_i$ is not necessarily closed.

$$A_n = \left[-2 + \frac{1}{n}, 2 - \frac{1}{n}\right] \text{ in } \mathbb{R}, \bigcup_{i=1}^{n} A_i = (-2, 2)$$

1.3.4 Topological proof that there are infinitely many primes (Furstenberg)

Proof. Look at $\mathbb{Z}$ with

$$\mathcal{B} := \{S(a, b),\, a \neq 0,\, b \in \mathbb{Z}\} \quad \text{and} \quad S(a, b) = \{an + b,\, n \in \mathbb{Z}\}$$

Let the open sets be the one generated by this basis. We can show

1. $S(a, b)$ is both open and closed.
2. All open sets are infinite.

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1. $S(a, b) = \mathbb{Z} \backslash \bigcup_{i=1}^{a-1} S(a, b - 1)$
2. Clear

Thus:

$$\underbrace{\mathbb{Z} \backslash \{-1, 1\}}_{\text{not closed}} = \bigcup_{p \text{ primes}} \overbrace{S(p, 0)}^{\text{closed}}$$

If there were only finitely many primes, right hand side would be closed $\square$

1.4 Closure and stuff

Definition 1.4.1: Closure, Interior, Boundary

Let $(X, \mathcal{T})$ be a topological space.

1. The **closure** of a subset $A \subseteq X$ is the smallest closed set such that $A \subseteq \overline{A}$.

$$\overline{A} := \bigcap_{\substack{C \subseteq X \text{closed} \\ A \subseteq C}} C$$

2. The (open) **interior** of a subset $A \subseteq X$ is the biggest open set U contained in A

$$\operatorname{int} A = A^\circ := \bigcap_{\substack{U \subseteq X \text{open} \\ U \subseteq A}} C$$

3. The **boundary** or **frontier** of a subset $A \subseteq X$ is

$$\partial A := \overline{A} \backslash A^\circ$$

4. A subset A of X is **dense** in X iff $\overline{A} = X$

   E.g.: $\mathbb{Q} \subseteq \mathbb{R}$ with the Euclidean topology

Theorem 1.4.2: Closure and Interior of Complement

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then

1. The closure of the complement is the complement of the interior:

$$\overline{X \backslash A} = X \backslash (A^\circ)$$

2. the interior of the complement is the complement of the closure:

$$(X \backslash A)^\circ = X \backslash \overline{A}$$

Definition 1.4.3: Limits in Topological spaces

A sequence (x_n) converges to $x \in X$ if $\forall U$ open with $x \in U$, $\exists N$ s.t. $x_n \in U$ for all $n \geq N$

Definition 1.4.4: Limit Set

$\overline{A}$ can be thought as the set of limits. Formally define the **Limit Set** as

$$A' = \{x \in X : \forall U \subseteq X, x \in U, \text{ open } \exists a \in A, \text{ s.t. } a \in U\}$$

But, limits in general are much worse behaved in topological spaces, e.g. no unique limit point
Example: a topological space X and a sequence (x_n) which does not have a unique limit (i.e. $\exists x \neq y$ s.t. $x_n \to x$ and $x_n \to y$ in the sense defined): Nontrivial X with the indiscrete topology $\{\emptyset, X\}$

1.5 Hausdorff Spaces

Problem: Non-unique limits are nasty :(

Definition 1.5.1: Hausdorff Space

A topological space $(X, \mathcal{T})$ is **Hausdorff** if for each $x, y \in X$ with $x \neq y$ there exist *disjoint* open sets U and V s.t. $x \in U$ and $y \in V$
This space has *unique limits*!

If (X, d) is a metric space then it is automatically Hausdorff, so any metrisable space is Hausdorff. The trivial topology on a set with more than one element is not Hausdorff. Not every Hausdorff space is metrisable
Non-Hausdorff spaces are a lot more annoying to work with - for example you can have multiple limits in non-Hausdorff spaces

Theorem 1.5.2: Open sets on $\mathbb{R}$ with Euclidean Topology

- A set U is open iff there are open intervals I_j s.t.

$$U = \bigcup_{j=1}^{\infty} I_j$$

- A set A is closed iff there are F_j (union of two closed intervals) s.t.

$$A = \bigcap_{j=1}^{\infty} F_j$$

Definition 1.5.3: Convergence of Hausdorff Spaces

A sequence (x_n) of members of a topological space X converges to $x \in X$ if for every open set U containing x, there exists an N such that $n \geq N \implies x_n \in U$

Theorem 1.5.4: Haussdorf Convergence Uniqueness

Suppose $(X, \mathcal{T})$ is Hausdorff. Then a sequence (x_n) can converge to at most one limit.

Being Hausdorff is what's called a *topological property*, which means whether or not it is true in a particular case depends only on the open sets of the space in question.
In contrast, the property of *completeness* of a metric space is not a topological property as there exist sets upon which one can put two distinct metrics, one complete and one not, yet for which the metric topologies coincide

Definition 1.5.5: Cauchy Sequences

Let (X, d) be a metric space

1. A **Cauchy Sequence** is a sequence (x_n) with each $x_n \in X$ with the property that for each $\epsilon > 0$, there exists an N s.t. $m, n \geq N \implies d(x_m, x_n) < \epsilon$

2. (X, d) is **complete** if every Cauchy Sequence converges

Caveat: In general, this does not have to converge to an $x \in X$
Example: $\mathbb{Q}$ with the Euclidean metric.

Definition 1.5.6: Complete Space

A metric space is called **complete** if all Cauchy sequences converge to a point in the space

Definition 1.5.7: Closure in Metric Spaces

Let (X, d) be a complete metric space and $A \subseteq X$. A point x is in the **closure** of $A \iff \exists x_i \to x$ with $x \in A$