General Topology Math Notes

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Contents

1 Intro to Topology

1.1 Why Topology?

Topology can appear where we least expect it...

- Algebraic Number Theory - Next to Euclidean topology, can define other topologies on $\mathbb{Q}$ (related to how often primes divide a number). Extends to Adeles, Langlands programme, etc

- Arithmetic Progressions in the Integers - An arithmetic progression of length k is a set $\{a, a+d, \ldots, a+(k-1)d\}$ Finding subsets of $\mathbb{N}$ that contain arbitrarily long APs:

  - $2\mathbb{N}$ or $\mathbb{N}$
  - Primes (Green-Tao Theorem, 2007). Green-Tao theorem relies on **Szemeredi's Theorem**: Any dense enough subset of $\mathbb{N}$ contains arbitrarily long APs

    Furstenburg's idea: Get from $A \subseteq \mathbb{N}$ to $(a_i \in \{0,1\}^{\mathbb{N}})$ with $a_i \begin{cases} 1 & i \in A \\ 0 & \text{else} \end{cases}$

    Use topological dynamics to study this: A topological dynamical system is a triple of X cpt, $T : X \to X$ continuous, and a probability measure μ preserved by T (what)

1.2 Topological Spaces and Examples

> **Definition 1.2.1: Topological Space**
>
> A **topological space** is a pair $(X, \mathcal{T})$, where X is a nonempty set, and $\mathcal{T}$ is a collection of subsets of X which satisfies:
>
> 1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
>
> 2. if $U_\lambda \in \mathcal{T}$ for each $\lambda \in A$ (where A is some indexing set), then $\bigcup_{\lambda \in A} U_\lambda \in \mathcal{T}$
>
> 3. If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

1.2.2 Examples of Topological Spaces

1. $\mathbb{R}^n$ with the Euclidean Topology - induced by the Euclidean Metric

2. For any set X, $\mathcal{T} = \mathcal{P}(X)$ (discrete topology)

3. For any set X, $\mathcal{T} = \{\emptyset, X\}$ (indiscrete topology)

4. $X = \{0, 1, 2\}$ with $\mathcal{T} = \{\emptyset, X, \{0\}, \{0, 1\}, \{0, 2\}\}$

5. $X = \mathbb{R}$ and U open (aka, in $\mathcal{T}$) if $R \backslash U$ is finite or $U = \emptyset$

Proof for 5:

1. $\emptyset \in \mathcal{T}$, $\emptyset$ is finite $\implies X \in \mathcal{T}$

2. Intersections of finite sets are finite

3. Unions of finite sets are finite

Definition 1.2.3: Neighbourhood of a point

A **neighbourhood** of a point $x \in X$ is a subset $N \subseteq X$ s.t. $x \in U \subseteq N$ for some open subset $U \subseteq X$

Definition 1.2.4: Metric Space

A **metric space** (X, d) is a nonempty set X together with a function

$$d : X \times X \to \mathbb{R}$$

with the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$

2. $d(x, y) = d(y, x)$ for all $x, y \in X$

3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

The function d is called the metric. Point 3 is called the *triangle inequality*

For any $x \in X$ and any positive real number r the **open ball** in X with centre x and radius r is defined by
$$B(x, r) = \{y \in X | d(x, y) < r\}$$

We declare a subset U of X to be *open in the metric topology given by d* iff for each $a \in U$ there is an $r > 0$ such that $B(a, r) \subseteq U$

If $(X, \mathcal{T})$ is a topological space, and if X admits a metric whose metric topology is precisely $\mathcal{T}$ we say that $(X, \mathcal{T})$ is **metrisable**. Thus the euclidean spaces with their usual topologies are metrisable

1.2.5 Examples of Metric Spaces

1. Any set X with $d(x, y) = \begin{cases} 0, & x = y \\ 1, & \text{else} \end{cases}$

2. $\mathbb{R}^n$ with $d(x, y) = |x - y| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$

3. $C([0, 1])$ with $d(f, g) = \max_{t \in [0,1]} |f(t) - g(t)|$

4. $C([0, 1])$ with $d(f, g) = \sqrt{\int_0^n |f(t) - g(t)|^2 dt}$

1.2.6 Topologies on Metric spaces

We want to define a topology on (X, d). For this, we want open balls to be open in the topology

Definition 1.2.7: Base

For a set X, a basis $\mathcal{B}$ is a collection of subsets such that

1. $\bigcup_{B \in \mathcal{B}} B = X$

2. $B_1 \cap B_2 \in \mathcal{B}$ for all $B_1, B_2 \in \mathcal{B}$

The **topology generated by** $\mathcal{B}$ is

$$\mathcal{T} := \{\bigcup_{i \in I} B_i, I \text{ index set}, B_i \in \mathcal{B}\}$$

Note: This is a topology because

$$(\cup_{i \in I} B_i) \cap (\cup_{j \in J} B_j) = \bigcup_{i \in I, j \in J} \underbrace{B_i \cap B_j}_{\in \mathcal{B}} \in \sqcup$$

Definition 1.2.8: Metric Topology

$$\text{Let } \mathcal{B} = \{\bigcap_{i=1}^{n} Br_1(x_1), n \in \mathbb{N}, r_i > 0, x_i \in X, \forall i\}$$

The **metric topology** is the topology generated by this basis

Observation A set U is open in the metric topology $\iff \forall x \in U, \exists r > 0$ s.t. $Br(x) \subseteq U$

- $\impliedby$: For each $x \in U$, let r_x s.t. $B_{r_x}(x) \subseteq U$. Then $U = \bigcup_{x \subseteq U} B_{r_x}(x)$ is open

- $\implies$: Let $x \in U$ be given. Knwo that $x \in B_{r_1}(x_1) \cup \cdots \cup B_{r_n}(x_n)$ for some n, r_1, x_1. For each i, there is $\delta_i > 0$ s.t. $B_{\delta_i}(x) \le B_{r_1}(x_1)$.

huh?

Theorem 1.2.9: random ms prop

If X carries metrics $d, \tilde{d}$ such that $ad(x, y) \le \tilde{d}(x, y) \le Ad(x, y)$ for some $a, A > 0$, then the induced topologies agree

Definition 1.2.10: Subspace Topology

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be any subset. Then the **subspace topology** on A consists of all sets of the form $U \cap A$ where $U \in \mathcal{T}$

Example: $(-1, 1) \subseteq \mathbb{R}$ with euclidean topology. The subspace topology is

$$\{(-1, 1) \cap U, U \subseteq \mathbb{R} \text{ open}\}$$

$(-1, 1)$ is closed in the subspace topology

Theorem 1.2.11: Topology Lemmas

1.3 If $(X, \mathcal{T})$ is a topological space and $U_1, \ldots, U_n$ are open sets, then the intersection $\bigcap_{i=1}^{n} u_i$ is also open

1.6 In order to show that a set $U \subseteq X$ is open, it is enough to show that for every $x \in U$ there is an open set V with $x \in V \subseteq U$

1.6 A subset U of $\mathbb{R}^n$ is *open for the usual topology* iff for each $a \in U$ there exists an $r > 0$ s.t.
$$|x - a| < r \implies x \in U$$

The collection of open sets thus defined is called the **usual topology on** $\mathbb{R}^n$. Note

that open balls are open sets under this definition

Definition 1.2.12: Topology Small Definitions

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1.3   Closed sets, Closure, Interior, and Boundary

Definition 1.3.1: Closed Subsets

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is **closed** iff its complement $X \backslash A = A^C := \{x \in X \ x \notin A\}$ is open in X

Note: A set being "closed" has no connection with "not being open"

1.3.2   Examples of open and closed sets

- A set that is neither open nor closed: $[0, 1) \subseteq \mathbb{R}$ under Euclidean topology

- A set that is both closed and open: $\emptyset$ or X

Theorem 1.3.3

Let $(X, \mathcal{T})$ be a topological space. Then

1. $\emptyset$ and X are closed.

2. The union of finitely many closed sets is a closed set

3. The intersection of any collection of closed sets is a closed set

$\bigcup_{i \in I} A_i$ is not necessarily closed.

$$A_n = \left[-2 + \frac{1}{n}, 2 - \frac{1}{n}\right] \text{ in } \mathbb{R}, \bigcup_{i=1}^{n} A_i = (-2, 2)$$

1.3.4   Topological proof that there are infinitely many primes (Furstenberg)

Proof. Look at $\mathbb{Z}$ with

$$\mathcal{B} := \{S(a, b),\ a \neq 0,\ b \in \mathbb{Z}\} \quad \text{and} \quad S(a, b) = \{an + b,\ n \in \mathbb{Z}\}$$

Let the open sets be the one generated by this basis. We can show

1. $S(a, b)$ is both open and closed.

2. All open sets are infinite.

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1. $S(a, b) = \mathbb{Z} \backslash \bigcup_{i=1}^{a-1} S(a, b-1)$

2. Clear

Thus:

$$\underbrace{\mathbb{Z} \backslash \{-1, 1\}}_{\text{not closed}} = \bigcup_{p \text{ primes}} \overbrace{S(p, 0)}^{\text{closed}}$$

If there were only finitely many primes, right hand side would be closed $\qquad \square$

1.4 Closure and stuff

> **Definition 1.4.1: Closure, Interior, Boundary**
>
> Let $(X, \mathcal{T})$ be a topological space.
>
> 1. The **closure** of a subset $A \subseteq X$ is the smallest closed set such that $A \subseteq \overline{A}$.
> $$\overline{A} := \bigcap_{\substack{C \subseteq X \,\text{closed} \\ A \subseteq C}} C$$
>
> 2. The (open) **interior** of a subset $A \subseteq X$ is the biggest open set U contained in A
> $$\text{int}\, A = A^{\circ} := \bigcap_{\substack{U \subseteq X \,\text{open} \\ U \subseteq A}} C$$
>
> 3. The **boundary** or **frontier** of a subset $A \subseteq X$ is
> $$\partial A := \overline{A} \backslash A^{\circ}$$
>
> 4. A subset A of X is **dense** in X iff $\overline{A} = X$
>
>   E.g.: $\mathbb{Q} \subseteq \mathbb{R}$ with the Euclidean topology

> **Theorem 1.4.2: Closure and Interior of Complement**
>
> Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then
>
> 1. The closure of the complement is the complement of the interior:
> $$\overline{X \backslash A} = X \backslash (A^{\circ})$$
>
> 2. the interior of the complement is the complement of the closure:
> $$(X \backslash A)^{\circ} = X \backslash \overline{A}$$

> **Definition 1.4.3: Limits in Topological spaces**
>
> A sequence (x_n) converges to $x \in X$ if $\forall U$ open with $x \in U$, $\exists N$ s.t. $x_n \in U$ for all $n \geq N$

> **Definition 1.4.4: Limit Set**
>
> $\overline{A}$ can be thought as the set of limits. Formally define the **Limit Set** as
> $$A' = \{x \in X : \forall U \subseteq X, x \in U, \text{ open } \exists a \in A, \text{ s.t. } a \in U\}$$

But, limits in general are much worse behaved in topological spaces, e.g. no unique limit point
Example: a topological space X and a sequence (x_n) which does not have a unique limit (i.e. $\exists x \neq y$ s.t. $x_n \to x$ and $x_n \to y$ in the sense defined): Nontrivial X with the indiscrete topology $\{\emptyset, X\}$

1.5   Hausdorff Spaces

Problem: Non-unique limits are nasty :(

> **Definition 1.5.1: Hausdorff Space**
>
> A topological space $(X, \mathcal{T})$ is **Hausdorff** if for each $x, y \in X$ with $x \neq y$ there exist
> *disjoint* open sets U and V s.t. $x \in U$ and $y \in V$
> This space has *unique limits*!

If (X, d) is a metric space then it is automatically Hausdorff, so any metrisable space is Hausdorff.
The trivial topology on a set with more than one element is not Hausdorff. Not every Hausdorff
space is metrisable
Non-Hausdorff spaces are a lot more annoying to work with - for example you can have multiple
limits in non-Hausdorff spaces

> **Theorem 1.5.2: Open sets on $\mathbb{R}$ with Euclidean Topology**
>
> - A set U is open iff there are open intervals I_j s.t.
>
> $$U = \bigcup_{j=1}^{\infty} I_j$$
>
> - A set A is closed iff there are F_j (union of two closed intervals) s.t.
>
> $$A = \bigcap_{j=1}^{\infty} F_j$$

> **Definition 1.5.3: Convergence of Hausdorff Spaces**
>
> A sequence (x_n) of members of a topological space X converges to $x \in X$ if for every open
> set U containing x, there exists an N such that $n \geq N \implies x_n \in U$

> **Theorem 1.5.4: Haussdorf Convergence Uniqueness**
>
> Suppose $(X, \mathcal{T})$ is Hausdorff. Then a sequence (x_n) can converge to at most one limit.

Being Hausdorff is what's called a *topological property*, which means whether or not it is true in
a particular case depends only on the open sets of the space in question.
In contrast, the property of *completeness* of a metric space is not a topological property as there
exist sets upon which one can put two distinct metrics, one complete and one not, yet for which
the metric topologies coincide

> **Definition 1.5.5: Cauchy Sequences**
>
> Let (X, d) be a metric space
>
> 1. A **Cauchy Sequence** is a sequence (x_n) with each $x_n \in X$ with the property that for each $\epsilon > 0$, there exists an N s.t. $m, n \geq N \implies d(x_m, x_n) < \epsilon$
>
> 2. (X, d) is **complete** if every Cauchy Sequence converges

Caveat: In general, this does not have to converge to an $x \in X$
Example: $\mathbb{Q}$ with the Euclidean metric.

> **Definition 1.5.6: Complete Space**
>
> A metric space is called **complete** if all Cauchy sequences converge to a point in the space

> **Definition 1.5.7: Closure in Metric Spaces**
>
> Let (X, d) be a complete metric space and $A \subseteq X$. A point x is in the **closure** of $A \iff \exists x_i \to x$ with $x \in A$

1.6 Continuity

> **Definition 1.6.1: Continuity**
>
> Let $(X, \mathcal{T}), (Y, \tilde{\mathcal{T}})$ be topological spaces and $f : X \to Y$. f is **continuous** if for all $U \in \tilde{\mathcal{T}}$, $f^{-1}(U) \in \mathcal{T}$
>
> ---
>
> Equivalently:
>
> - $U \subseteq Y$ open $\implies f^{-1}(U)$ open
>
> - $A \subseteq Y$ closed $\implies f^{-1}(A)$ closed

1.6.2 Why take f^{-1}

Properties: For U, V sets in Y,

- $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(U)$

- $f^{-1}(U^C) = f^{-1}(U)^C$

- $f^{-1}(U \cup U) = f^{-1}(U) \cup f^{-1}(U)$

Example: $\mathbb{R}$ with Euclidean Topology

Proof. "Proof" that $[-1, 1]$ is open:
Take $[-1, 1]$ with the subspace topology $\mathcal{T} := \{[-1, 1] \cap U, U \subseteq \mathbb{R} \text{ open}\}$
Embedding $i : [-1, 1] \to \mathbb{R}, x \mapsto x$ is continuous
$[-1, 1]$ open in subspace topology
i cont $\implies i([-1, 1])$ is open   this is actually wrong! U open $\not\implies f(U)$ open
But $i([-1, 1]) = [-1, 1] \subseteq \mathbb{R}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

> **Definition 1.6.3: Formal Definition of Continuity**
>
> Let $(X, d), (Y, d)$ be metric spaces with the metric topology. $f : X \to Y$ is continuous as above iff $\forall x \in X$, $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.
>
> $$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$$

Proof. $\implies$ **Direction**
Recall: U open in metric topology if $\forall x \in U$, $\exists r > 0$ s.t. $B_r(x) \subseteq U$, where $B_r(x) = \{y \in X : d(x, y) < r\}$
$\implies$ Let $x \in X$ be given, $\epsilon > 0$. Let $y = f(x) \in Y$, $U = B_\epsilon(y) = \{y' \in Y : \tilde{d}(y, y') < \epsilon\}$.
f cont $\implies f^{-1}(U)$ is open. $x \in f^{-1}(U) \implies \exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(U)$
$\implies \forall x' \in X$ s.t. $d(x, x') < \delta$, $x' \in B_\delta(x) \subseteq f^{-1}(U)$.
$\implies f(x') \in B_\delta(f(x)) \implies \tilde{d}(f(x), f(x')) < \epsilon$
$\impliedby$ **Direction**
Let U be open in Y. WTS: $f^{-1}(U)$ is open.
So it is enough to show for all $x \in f^{-1}(U)$, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(U)$.
Let x be given, $y := f(x) \in U$. U open $\implies \exists \epsilon > 0$ s.t. $B_\epsilon(y) \subseteq U$.
By assumption $\exists \delta > 0$ s.t.

$$d(x', x) < \delta \implies \tilde{d}(f(x'), f(x)) < \epsilon$$

But, $\{y' : d(y', f(x)) < \epsilon\} \subseteq U$ by choice of ϵ.
$\implies B_\delta(x) \subseteq f^{-1}(U)$ $\qquad \square$

1.7 Homeomorphisms

> **Definition 1.7.1: Homeomorphism**
>
> Let $(X, \mathcal{T}), (Y, \tilde{\mathcal{T}})$ be topological spaces. A function $f : X \to Y$ is a **homeomorphism** (or **bi-continuous**) if f is bijective, f is continuous, and $f^{-1} Y \to X$ is continuous

A "Great goal of Topology": Understand topological spaces up to homeomorphisms.
Say that a property of a topological space is a **topological invariant** if it is preserved by homeomorphism. Example: Being Hausdorff

> **Example 1.7.2: Examples of Homeomorphisms**
>
> - $(X, \mathcal{T})$ topological space, id $: X \to X$, $x \mapsto x$
>
> - $X = \mathbb{R}^n$, $A : \mathbb{R}^n \to \mathbb{R}^n$ Linear + Invertible
>
> - Example which is **not** a homeomorphism:
>
> $$f : \quad \underbrace{\mathbb{R}}_{\text{metric topology}} \quad \to \quad \underbrace{\mathbb{R}}_{\text{indiscrete topology } \{\emptyset, \mathbb{R}\}} \quad , x \mapsto x$$
>
> Problem: f^{-1} is not continuous