General Topology Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Topological Spaces and Examples

Definition 1.1: Topological Space

A **topological space** is a pair $(X, \mathcal{T})$, where X is a nonempty set, and $\mathcal{T}$ is a collection of subsets of X which satisfies:

a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

b) if $U_\lambda \in \mathcal{T}$ for each $\lambda \in \Lambda$ (where Λ is some indexing set), then $\bigcup_{\lambda\in\Lambda} U_\lambda \in \mathcal{T}$

c) if $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

The collection $\mathcal{T}$ is called the **topology** of the topological space, and the members of $\mathcal{T}$ are called the **open sets** of the topology

Example 1.7: Euclidean Spaces

Let $\mathbb{R}^n$ enote the n-dimensional Euclidean vector space with elements $x = (x_1, x_2, \ldots, x_n)$ and $x_i \in \mathbb{R}$, and let

$$|x| = \sqrt{\sum_{i=1}^{n} x_i^2} \geq 0$$

be the length of x. ($\mathbb{R}^1 = \mathbb{R}$ is the real line). A subset U of $\mathbb{R}^n$ is **open (for the usual topology)** iff for each $a \in U$ there exists an $r > 0$ such that

$$|x - a| < r \implies x \in U.$$

The collection of open sets thus defined is called the **usual topology** on $\mathbb{R}^n$. Note that open balls $B(y, \rho) = \{x \in \mathbb{R}^n : |x - y| < \rho\}$ are open sets under this definition.

Example 1.8: Metric Spaces

A **metric space** (X, d) is a nonempty set X together with a function $d : X \times X \to \mathbb{R}$ with the following properties:

a) $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$

b) $d(x, y) = d(y, x)$

c) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

The function d is called the **metric**.

Let (X, d) be a metric space, x be a point in X, and $r > 0$. The **open ball** with center x and radius r is defined by

$$B(x, r) = \{y, , \in X : d(x, y) < r\}.$$

A subset U of X is **open (in the metric topology given by d)** iff for each $a \in U$ there is an $r > 0$ such that $B(a, r) \subseteq U$. Just like euclidean spaces, open balls are open in this sense.

Example 1.0.1: Other Topologies and Metrics

If $(X, \mathcal{T})$ is a topological space, and if X admits a metric whose metric topology is precisely $\mathcal{T}$, then we say that $(X, \mathcal{T})$ is **metrisable**

- Euclidean spaces with their usual topologies are metrisable.

1.9) The **Discrete Topology** is the topology of all subsets of a set X. We can define the **discrete metric** of X to be

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}.$$

1.10) The **Trivial** or **Indiscrete Topology** is the topology $\mathcal{T} := \{\emptyset, X\}$ for a set X. This is a non-metrisable topology when X has more than one member.

1.14) Let $X = \{a, b, c\}$, where a, b, c are distinct. Then

$$\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$$

is a topology on X

1.15) Give $\mathbb{R}$ the topolgoy whose open subsets $U \subseteq \mathbb{R}$ are precisesly the subsets with finite complement $\mathbb{R}\backslash U$, or $U = \emptyset$. Then $\mathbb{R}$ with this topology is not metrisable. This is an example of a **Zariski Topology**

Proposition 1.11: Topology Equality

Let d, d' be metrics on the same set X, and let $\mathcal{T}$, $\mathcal{T}'$ be the corresponding metric topologies. If for real numbers A, $B > 0$ we have

$$d(x, y) \leq Ad'(x, y),\ d'(x, y) \leq Bd(x, y) \text{ for all } x, y \in X,$$

then $\mathcal{T} = \mathcal{T}'$.

Example 1.12: Example of Topology Equality

- The **Euclidean metric** on $\mathbb{R}^n$ is defined as:

$$d\big((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\big) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

- The **Box metric** on $\mathbb{R}^n$ is defined as:

$$d(x, y) \leq \sqrt{n} d'(x, y),\ d'(x, y) \leq d(x, y)$$

By 1, these have the same topology.

Definition 1.16: Subspace Topology

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be any subset. Then teh **subspace topology** on A consists of all sets of the form $U \cap A$ where $U \in \mathcal{T}$.

Definition 1.17: Closed Set

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is **closed** iff its complement $X\backslash A := \{x \in X \mid x \notin A\}$ is open in X. Note that a set being *closed* does not mean it isn't *open*. Sets that are both *closed* and *open* are called **clopen**.

Theorem 1.19: Properties of open and closed sets

Let $(X, \mathcal{T})$ be a topological space.

1. $\emptyset$ and X are closed.
2. The union of **finitely many** closed sets is an closed set.
3. The intersection of **any collection** of closed sets is a closed set.
4. The union of **any collection** of open sets is an open set.
5. The intersection of **finitely many** open sets is an open set

Definition 1.20: Properties of Topological Spaces

1. The **closure** of a set $A \subseteq X$ is
$$\overline{A} := \bigcap_{C \subseteq X \text{ closed};\ A \subseteq C} C.$$
2. The **interior** of a set $A \subseteq X$ is
$$\operatorname{int} A = A^\circ := \bigcap_{C \subseteq X \text{ open};\ A \subseteq C} C.$$
3. The **boundary** (or **frontier**) of a subset $A \subseteq X$ is
$$\partial A := \overline{A} \backslash A^\circ.$$
4. A subset A of X is **dense** in X iff $\overline{A} = X$.

$\overline{A}$ is closed, and contains A and is the smallest set with this property. So A is closed iff $\overline{A} = A$.
A° is open, and is contained in A, and is the largest set with this proprety. So A is open iff $A^\circ = A$.

Proposition 1.22: Relating Topological Properties

The closure of the complement is the complement of the interior:

$$\overline{X\backslash A} = X\backslash(A^\circ).$$

The interior of the complement is the complement of the closure:

$$(X\backslash A)^\circ = X\backslash\overline{A}.$$

Definition 1.23: Limit Points

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be a subset. A **limit point** of A is a point $x \in X$ s.t. for every open subset $U \subseteq X$ with $x \in U$ there exists an element $a \in A \cup U$ with $a \neq x$.
Let A' be the set of limit points of A.
Note that this has nothing to do with limits of sequences.

Lemma 1.24: Limit Points and Open Balls

An element $x \in X$ in a metric space (X, d) is a limit point of a subset $A \subseteq X$ iff for every $\epsilon > 0$ there exists $a \in A$ with $0 < d(x, a) < \epsilon$, or iff there exists a sequence $a_1, a_2, a_3, \cdots$ of elements $a_i \in A$, with $a_i \neq x$ for all i, such that $d(x_i, a_i) \to 0$ as $i \to \infty$. This interpretation does not extend to general topological spaces.

Example 1.0.2: Examples of limit points

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Proposition 1.26: Union of Limit points

Let $(X, \mathcal{T})$ be a topological space, and suppose $A \subseteq X$. Then

$$\overline{A} = A \cup A'$$

Corollary 1.27

A subset $A \subseteq X$ is closed iff it contains all its limit points.

Theorem 1.30: Open and Closed sets in $\mathbb{R}$

Consider $\mathbb{R}$ with the usual topology.

1. A nonempty set U is open iff it can be written as a countable union of disjoint nonempty open intervals I_j:
$$U = \bigcup_{j=1}^{\infty} I_j.$$
2. A set F is closed iff it can be written as a countable intersection
$$F = \bigcap_{j=1}^{\infty} F_j$$
where each F_j is a finite union of closed intervals.

Definition 1.32: Hausdorff Spaces

A topological space $(X, \mathcal{T})$ is **Hausdorff** if for each $x, y \in X$ with $x \neq y$ there exist **disjoint** open sets U and V such that $x \in U$ and $y \in V$.

Any metrisable space is Hausdorff, The trivial topology n a set with more than one element is not Hausdorff.

Definition 1.33: Convergence of a Topological space

A sequence (x_n) of members of a topological space X converges to $x \in X$ if for every open set U containing x, there exists an N such that $n \geq N \implies x_n \in U$

Proposition 1.34: Convergence of Hausdorff Spaces

Suppose $(X, \mathcal{T})$ is Hausdorff. Then a sequence (x_n) can converge to at most one limit.

Definition 1.36: Cauchy and Completeness

Let (X, d) be a metric space.

1. A **Cauchy sequence** is a sequence (x_n) with each $x_n \in X$ with the property that for each $\epsilon > 0$, there exists an N such that $m, n \in N \implies d(x_m, x_n) < \epsilon$
2. (X, d) is **complete** if every Cauchy sequence converges.

Definition 1.37: Topology Basis

A **basis for a topology** on a set X is a collection $\mathcal{B}$ of subsets $B \subseteq X$ such that:

1. $X = \bigcup_{B \in \mathcal{B}} B$
2. The intersection of sets $B_1, B_2 \in \mathcal{B}$ in a set $B_1 \cap B_2 \in \mathcal{B}$

The **topology $\mathcal{T}$ generated by a basis $\mathcal{B}$** has open sets the arbitrary unions of basis elements $B_\lambda \in \mathcal{B}$:

$$U = \bigcup_{\lambda \in \Lambda} B_\lambda$$

(Don't forget to check that this really is a topology)

Example 1.38: Finite Intersections of open balls

For any metric space $(X, \mathcal{T})$ the finite intersections of open balls

$$B(x, r) = \{y \in X \mid d(x, y) < r\} \subseteq X \quad (r > 0, x \in X)$$

constitute a basis for the metric topology on X

$$\mathcal{B} = \{B(x_1, r_1) \cap B(x_2, r_2) \cap \cdots \cap B(x_k, r_k) \mid x_1, x_2, \ldots, x_k \in X, r_1, r_2, \ldots, r_k > 0\}$$

2 Continuous functions and Homeomorphisms

Definition 2.1: Continuity

Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be topological spaces. A function $f : X \to Y$ is **continuous** iff

$$U \in \mathcal{U} \text{ implies } f^{-1}(U) \in \mathcal{T}.$$

That is, **inverse** images of open sets are open. Continuous functions are often called **maps** or **mappings** of topological spaces.

Proposition 2.6: Topological and Analytic Continuity

Let (X, d) and (Y, ρ) be metric spaces with their induced topologies $\mathcal{T}$ and $\mathcal{U}$ respectively. A function $f : X \to Y$ is continuous (topologically) iff it is continuous analytically: for every $a \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(x, a) < \delta \implies \rho(f(x), f(a)) < \epsilon$$

Definition 2.7: Homeomorphism

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. A **homeomorphism** is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous. Two topological spaces are **homeomorphic** if there is a homeomorphism between them.

Proposition 2.8: Open Homeomorphisms

Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a homeomorphism. Then U is open in Y iff $f^{-1}(U)$ is open in X.

Example 2.10: Examples of homeomorphisms

1. Let $(X, \mathcal{T})$ be an arbitrary topological space. Then the identity map
$$\iota : X \to X; \quad x \mapsto x$$
is continuous, and indeed a homeomorphism.
2. Suppose $(X, \mathcal{T})$, $(Y, \mathcal{U})$, and $(Z, \mathcal{W})$ are topological spaces, and that $f : X \to Y$ and $g : Y \to Z$ are continuous functions. Then their composition
$$g \circ f : X \to Z; \quad x \mapsto g(f(x))$$
is continuous.
3. For any topological spaces X, Y, and any element $y_0 \in Y$ the constant function
$$f_0 : X \to Y; \quad x \mapsto y_0$$
is continuous.

Proposition 2.14: Continuity and Closed sets

- Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces and that $f : X \to Y$. Then f is continuous iff for every closed subset $F \subseteq Y$ its inverse image $f^{-1}(F)$ is closed in X.
- f is continuous iff the image of the closure of every subset $A \subseteq X$ is contained in the closure of the image, i.e., $\forall A \subseteq X$,
$$f(\overline{A}) \subseteq \overline{f(A)}$$

Proposition 2.18: The Punctured Sphere

Consider the n-dimensional sphere

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

with the metric topology inherited from $\mathbb{R}^{n+1}$. Let $x_0 \in \mathbb{S}^n$. Then $\mathbb{S}^n \backslash \{x_0\}$ is homeomorphic to $\mathbb{R}^n$.

3 Subspaces Revisited

Proposition 3.4: Hausdorff and Subspaces

Suppose $(X, \mathcal{T})$ is a Hausdorff topological space and suppose A is a subspace. Then A is Hausdorff.

Proposition 3.5: Continuity and Subspaces

Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces and suppose A is a subspace of X. Let $f : X \to Y$ be continuous. Then $f|_A : A \to Y$ is continuous.

Corollary 3.6: Homeomorphisms and Exclusions

Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are homeomorphic via f. Then $X \backslash \{x_0\}$ is homeomorphic to $Y \backslash \{f(x_0)\}$

Definition 3.65: Disjoint Unions

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Their **disjoint union** $X + Y$ is the set $(X \times \{0\}) \cup (Y \times \{1\})$ with the topology consisting of all sets of the form

$$(T \times \{0\}) \cup (U \times \{1\}) \text{ such that } T \in \mathcal{T}, U \in \mathcal{U}$$

Definition 3.8: Product Topology

Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be topological spaces. The **product topology** on their product $X \times Y$ consists of all sets of the form

$$T = \bigcup_{\alpha \in A} (U_\alpha \times V_\alpha)$$

where $\mathcal{A}$ is an arbitrary indexing set, and $U_\alpha \in \mathcal{U}$ and $V_\alpha \in \mathcal{V}$.

Lemma 3.9: Openness in Product Topologies

Let $(X, \mathcal{T})$ $(Y, \mathcal{U})$ be topological spaces. Then $T \subseteq X \times Y$ is open in the product topology if and only if for all $t \in T$ there exists $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $t \in U \times V$ and $U \times V \subseteq T$.

Lemma 3.10: Product Topology is a topology

The product topology is indeed a topology. (lol)

Definition 3.11.5: Projection Maps

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and consider their product $X \times Y$ with the product topology. There are two natural maps Π_X and Π_Y, the projections of $X \times Y$ onto X and Y respectively, given by

$$\Pi_X : X \times Y \to X, \quad (x, y) \mapsto x$$

and

$$\Pi_Y : X \times Y \to Y, \quad (x, y) \mapsto y.$$

Theorem 3.12: Continuity of Projection Maps

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and $\mathcal{T}$ the product topology on $X \times Y$. Then the projection maps Π_X and Π_Y are continuous. Moreover, $\mathcal{T}$ is the smallest topology on $X \times Y$ such that the projection maps are continuous.

Proposition 3.13: Continuity of compositions

Let X, Y, Z be topological spaces. Endow $X \times Y$ with the product topology. A function $f : Z \to X \times Y$ is continuous iff the functions $\Pi_X \circ f : Z \to X$ and $\Pi_Y \circ f : Z \to Y$ are both continuous.

Definition 3.14: Weak Topology

Suppose that X is a set. $(X_\lambda, \mathcal{T}_\lambda)$ is a family of topological spaces, and that $f_\lambda : X \to X_\lambda$ are functions. The **weak topology generated by** $\{f_\lambda\}$ is the smallest topology on X making all the f_λ continuous.
Thus the product topology on $X \times Y$ is the weak topology generated by the two maps Π_X and Π_Y

Definition 3.15: Cartesian Product Topology

If X_λ is a topological space, (with λ in some arbitrary indexing set Λ), the product topology on the cartesian product $\Pi_{\lambda \in \Lambda} X_{Ll}$ is defined to be the weak topology generated by the projections

$$\Pi_\lambda : \prod_{\mu \in \Lambda} X_\mu \to X_\lambda$$

Definition 3.5.0: Equivalence Relation

An **equivalence** relation on a set X is a binary operation $\sim$ on X which is:

1. **Reflexive**: $x \sim x$ for all $x \in X$.
2. **Symmetric**: if $x \sim y$ then $y \sim x$.
3. **Transitive**: if $x \sim y$ and $y \sim z$ then $x \sim z$.

The **equivalence class** of any element $x \in X$ is the set

$$[x] = \{y \in X \mid x \sim y\},$$

and the set of equivalence classes is denoted by $X/\sim$. The function which assigns to each $x \in X$ the equivalence class $[x] \in X/\sim$ is a surjection

$$p : X \to X/\sim; \quad x \to [x]$$

Definition 3.17: Quotient Space

Given a topological space $(X, \mathcal{T})$, and an equivalence relation $\sim$ on X, the **quotient space** or **identification space** is the set of equivalence classes $X/\sim$ together with the topology

$$\{U \subseteq X/\sim : p^{-1}(U) \in \mathcal{T}\}$$

Example 3.18: Circle as an Interval

The circle S^1 is homeomorphic to an identification space of the unit interval $I = [0, 1]$. The topology on I is defined by regarding I as a subspace of $\mathbb{R}$: a subset $Y \subseteq I$ is open iff $Y = I \cup U$ for an open subset $U \subseteq \mathbb{R}$. Define an equivalence relation $\sim$ on I by

$$x \sim y \text{ if } x = y \text{ or if } (x, y) = (1, 0) \text{ or if } (x, y) = (0, 1)$$

The identification space $I/\sim$ tying the two endpoints of I together is homeomorphic to S^1, with a homeomorphism

$$I/\sim \to S^1; \quad t \mapsto (\cos 2\pi t, \sin 2\pi t).$$

The subset $Y = [0, 1/2) \subseteq I$ is open, since $Y = I \cup (-\infty, 1/2)$ with $(-\infty, 1/2)$ is open in $\mathbb{R}$. The image $p(Y) \subseteq I/\sim$ is not open. In fact, the open subsets $Y \subseteq I$ such that $p(Y) \subseteq I/\sim$ is open are those for which $\{0, 1\} \subseteq Y$ or $\{0, 1\} \cap Y = \emptyset$.

Definition 3.25: Generated Topological Spaces

Let X be a topological space, and let Y_0, $Y_1 \subseteq X$ be subspaces, which are related by a continuous function $f : Y_0 \to Y_1$. Let $\sim_f$ be the equivalence relation on X **generated by** f, the intersection of all the equivalence relations on X (regarded as subsets of $X \times X$) containing the pairs $(y_0, f(y_0))$ with $y_0 \in Y_0$. The identification space

$$X/\sim_f = X/\{y_1 = f(y_0)\}$$

is obtained by identifying each $y_0 \in Y_0 \subseteq X$ with $y_1 = f(y_0) \in Y_1 \subseteq X$.

Proposition 3.33: Continuity of Relations

Let X be a topological space with an equivalence relation $\sim$.

1. The function $p : X \to X/\sim\,; \quad x \mapsto [x]$ is continuous.
2. A continuous function $f : X \to Y$ such that $f(x) = f(x') \in Y$ for all x, $x \in X$ with $x \sim x'$ determines a continuous function
$$g : X/\sim \to Y\,; \quad [x] \mapsto f(x)$$
such that
$$f = g \circ p : X \xrightarrow{p} X/\sim \xrightarrow{g} Y$$

$f = g \circ p$ is best described by a commutative triangle:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & X/\sim \arrow[ur, "g"'] & \end{tikzcd}$$

In fact, every continuous function on X determines an equivalence relation.

Proposition 3.34: Homeomorphisms of Relations

Given a continuous function $f : X \to Y$ let $\sim$ be the equivalence relation defined on X by $x \sim x'$ if $f(x) = f(x') \in Y$. The function
$$g : X/\sim \to Y\,; [x] \to f(x)$$
is continuous and injective, with
$$f = g \circ p : X \xrightarrow{p} X/\sim \xrightarrow{g} Y.$$
If f is onto, and such that $f(U) \subseteq Y$ is open for every oopen subset $U \subseteq X$ then g is a homeomorphism.

4 Compact Spaces

Definition 4.1: Open Covers and Compct Spaces

1. An **open cover** of a topological space X is a collection $\{U_\lambda \mid \lambda \in \Lambda\}$ of open subsets U_λ of X such that
$$\bigcup_{\lambda \in \Lambda} U_\lambda = X$$
2. A topological space X is **compact** if every open cover $\{U_\lambda \mid \lambda \in \Lambda\}$ of X has a finite subcover, i.e. there exists $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that
$$X = \bigcup_{j=1}^{n} U_{\lambda_j}.$$

Definition 4.2: Open Covers as Collections

1. If $A \subseteq X$ is a subset of a topological space X, an **open cover** of A is a collection $\{V_\lambda \mid \lambda \in \Lambda\}$ of subsets V_λ which are open in X such that
$$X = \bigcup_{\lambda \in \Lambda} V_\lambda$$
2. A subset A of a toplogical space X is **compact** if it is compact as a subspace of X.

Proposition 4.3: Compactness and Subcoverings

Let X be a topological space and $A \subseteq X$. Then A is compact iff every open cover of A has a finite subcover.

Theorem 4.5: Heine-Borel Theorem

A subset $F \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.

Proposition 4.6: Properties of Compact Spaces

Let X be a topological space.

1. If X is compact and $A \subseteq X$ is closed, then A is compact
2. If X is Hausdorff and $A \subseteq X$ is compact, then A is closed.

Proposition 4.7: Boundedness of Compact Spaces

A compact metric space (X, d) is bounded, i.e. there exists a number $K \geq 0$ such that $d(x, y) \leq K$ for all x, $y \in X$.

Proposition 4.8: Compactness of Products

A product of closed bounded intervals $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is compact in the usual topology. A collection of subsets of a set X has the **finite intersection property** if every finite intersection of their members is nonempty.

Theorem 4.10: Compactness of Functions

Let $f : X \to Y$ be a continuous map between topological spaces. If X is compact, so is $f(X)$.

Corollary 4.11

Compactness is a topological invariant. For example, $\mathbb{S}$ and $\mathbb{R}^n$ are not homeomorphic as the former is compact while the latter is not.

Corollary 4.12: Limit Property of Compactness

Suppose that $f : X \to \mathbb{R}^n$ is a continuous map and that X is compact. Then there exists an M such that
$$|f(x)| \leq M \text{ for all } x \in X.$$
Moreover, there exists an $x \in X$ such that
$$|f(x)| = \sup_{y \in X} |f(y)|.$$
If $n = 1$ there are x_0 and $x_1 \in X$ such that
$$f(x_0) = \min_{x \in X} f(x) \quad \text{ and } \quad f(x_1) = \max_{x \in X} f(x).$$

Theorem 4.13: Compact,Hausdorff,Bijection, Oh My!

Suppose X is compact, Y is Hausdorff, and that $f : X \to Y$ is a continuous bijection. Then it is a homeomorphism.

Theorem 4.14: Lebesgue Numbers

Let X be a compact metric space and $\{U_\lambda \mid \lambda \in \Lambda\}$ an open cover of X. Then there exists a positive number $\delta > 0$ (the **Lebesgue number** of the cover) such that for all $x \in X$, $B(x, \delta)$ lies *entirely inside some single* U_λ.

Corollary 4.17: Compactness of Identification Spaces

1. An identification space $X/\sim$ of a compact space X is compact.
2. If $f : X \to Y$ is a map from a compact space X to a Hausdorff space Y and $\sim$ is the equivalence relation on X defined by $x \sim x'$ if $f(x) = f(x') \in Y$, then the continuous bijection
$$g : X/\sim \to f(X)\,; \quad [x] \mapsto f(x)$$
is a homeomorphism.

Theorem 4.18: Tychonoff's Theorem - Two Products

Suppose X and Y are compact spaces. Then their product $X \times Y$ is compact. The converse is also true.

Lemma 4.20: Open sets in Product spaces

Let X be a topological space, Y a compact space, $x \in X$, N an open set in $X \times Y$ such that $\{x\} \times Y \subseteq N$. Then there is an open set $W \subseteq X$ such that $x \in W$ and $W \times Y \subseteq N$.

Theorem 4.21: Tychonoff's Theorem

Suppose that $\mathcal{A}$ is an indexing set and that for ech $\alpha \in \mathcal{A}$, X_α is a compact topological space. Then the product $\prod_{\alpha \in \mathcal{A}} X_\alpha$ is compact.

Lemma 4.22 - 4.23: Collections and Intersections

4.22) Let X be a set, and suppose that $\mathcal{C}$ is a collection of subsets of X which has the finite intersection property. Then there is a collection $\mathcal{B}$ of subsets of X, with $\mathcal{C} \subseteq \mathcal{B}$, such that $\mathcal{B}$ has the finite intersection property, and such that $\mathcal{B}$ is maximal with respect to this property: i.e. no collection containing $\mathcal{B}$ as a propert subcollection has the finite intersection property.

4.23) Let X be a set, and suppose that $\mathcal{B}$ is a collection of subsets of X which is maximal with respect to the finite intersection property. Then $\mathcal{B}$ is closed under finite intersections, and any set which meets all members of $\mathcal{B}$ is also in $\mathcal{B}$.

Definition 4.24: Compactifications

1. A **compactification** of a topological space X is a compact space Y which contains a homeomorphic copy of X as a subspace, i.e. such that there is a one-one map $f : X \to Y$ such that $X \to f(X);\quad x \mapsto f(x)$ is a homeomorphism.
2. A compactification Y is **dense** if X is dense in Y, i.e. $\overline{X} = Y$.

Definition 4.27: One-point compactification

the **one-point compactification** of a topological space X is the set

$$X^\infty = X \cup \{\infty\}$$

obtained by adjoining a "point at infinity" ∞, where ∞ is a symbol *not* in X, with open sets of the form either

1. U, where $U \subseteq X$ is open, or
2. $X^\infty \backslash K$, where $K \subseteq X$ is compact and closed.

Lemma 4.28

1. The collection of open sets just defined does form a topology
2. The subspace topology on X induced by this topology coincides with its original topology.

Proposition 4.30: Compactness of OPC

1. X^∞ is compact
2. If X is not compact, then X is dense in X^∞

Definition 4.32: Local Compactness

A topological space X is **locally compact** if for each

$$x \in X \subseteq (X, \mathcal{T})\frac{1}{2}$$

there exists an open subset $U \subseteq X$ and a compact C such that $x \in U \subseteq C$.

Remark 4.33

When X is Hausdorff, it is locally compact iff for each $x \in X$ there exists an open subset $U \subseteq X$ and a compact $x \in U$ and the closure $\overline{U}$ is compact.

Proposition 4.34: Hausdorff OPC

The one-point compactification X^∞ of a space X is Hausdorff iff X is Hausdorff and locally compact.

Definition 4.35: Normal Space

A topological space $(X, \mathcal{T})$ is **normal** if for every pair of disjoint closed subsets C and $D \subseteq X$, there are disjoint open subsets $U, V \subseteq X$ such that $C \subseteq U$ and $D \subseteq V$

Lemma 4.37: Normal Complements

Show that a space X is normal iff for every closed $F \subseteq X$ and open $G \subseteq X$ with $F \subseteq G$, there exist open G' and closed F' such that

$$F \subseteq G' \subseteq F' \subseteq G.$$

Theorem 4.38: Urysohn's Lemma

Suppose that X is a normal topological space, and that C, D are disjoint closed subsets of X. Then there is a continuous function $f : X \to \mathbb{R}$ such that

- $f(x) = 0$ for all $x \in C$
- $f(x) = 1$ for all $x \in D$
- $0 \leq f(x) \leq 1$ for all $x \in X$

Theorem 4.39: Tietze extension theorem

Suppose that X is a normal topological space, and that C is a closed subset of X. Suppose that $f : C \to \mathbb{R}$ is continuous. Then there is a continuous function $\overline{f} : X \to \mathbb{R}$ such that

- $\overline{f}(x) = f(x)$ for all $x \in C$
- If $a \leq f(x) \leq b$ for all $x \in C$, then $a \leq \overline{f}(x) \leq b$ for all $x \in X$.

Theorem 4.40: Stone-Weierstrass Theorem

The algebra A is dense in the normed space $C(X)$, i.e. $\overline{A} = C(X)$, i.e. for all $f \in C(X)$ and for all $\epsilon > 0$ there is $g \in A$ such that $\sup_{x \in X} |f(x) - g(x)| < \epsilon$

5 Connected Spaces

Definition 5.1: Connected Spaces

1. A topological space X is **connected** if it *cannot* be written as a union
$$X = A \cup B$$
where A and B are disjoint nonempty open subsets of X
2. A topological space X is **disconnected** if it is not connected, i.e. if it *can* be expressed as a union
$$X = A \cup B$$
where A and B are disjoint nonempty open subsets of X

Remark 5.2

X is connected iff the only subsets of X which are clopen are $\emptyset$ and X

Theorem 5.4: Connectedness of $\mathbb{R}$

$\mathbb{R}$ with the usual topology is connected.

Theorem 5.5: Connectedness of Functions

If $f : X \to Y$ is continuous and X is connected, then $f(X)$ (with the subspace topology) is connected

Corollary 5.6: Connectedness is a Prop

Connectedness if a topological property: If X, Y are homeomorphic spaces, then X is connected iff Y is connected.

Remark 5.8: Connected Homeomorphisms

- If X is a compact connected metric space with exactly two points x such that $X \backslash \{x\}$ is connected, then X is homeomorphic to $[0, 1]$
- If X is a compact conneted space, where for every pair of distinct points x, $y \in X$ the complement $X \backslash \{x, y\}$ is disconnected, then X is homeomorphic to the circle $\mathbb{S}_1$

Proposition 5.9: Connected Squeeze theorem

Let A be a connected subsets of a topological space X and suppose $A \subseteq B \subseteq \overline{A}$. Then B is connected.

Corollary 5.10: Connected Intervals

Every nonempty interval $I \subseteq \mathbb{R}$ is connected.

Proposition 5.11: Connectedness other properties

The following statements about a nonempty subset $A \subseteq \mathbb{R}$ are equivalent:

1. A is connected
2. A has the interval property
3. A is an interval

Theorem 5.12: Intermediate Value Theorem

Let I be a closed bounded interval and suppose $f : I \to \mathbb{R}$ is continuous. Then the image $f(I)$ is a closed bounded interval

$$f(I) = [a, b] \subseteq \mathbb{R}(a \leq b).$$

Definition 5.13: Fixed Points of Maps

A **fixed point** of a map $f : X \to X$ is an $x \in X$ such that $f(x) = x$.

Theorem 5.15: 1-D Brouwer Fixed Point Theorem

Every continuous map $f : [0, 1] \to [0, 1]$ has a fixed point, i.e. there exists $x \in [0, 1]$ such that $f(x) = x$.
General Case: Every continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point

Definition 5.16: Path

A **path** in a topological space X is a continuous map $\alpha : I = [0, 1] \to X$. Its **initial point** is $\alpha(0) \in X$ and its **terminal point** is $\alpha(1) \in X$.

Definition 5.18: Path Connectedness

A topological psace X is **path-connected** if for any two points $x_0, x_1 \in X$ there exists a path $\alpha : I \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$.

Theorem 5.18: Path Connectedness and Continuity

Suppose $f : X \to Y$ is a continuous map between topological spaces and that X is path-connected. Then $f(X)$ is path-connected as a subspace of Y.

Corollary 5.21: Path-Connectedness Property

Path-connectedness is a topological property: If X, Y are homeomorphic spaces then X is path-connected iff Y is path-connected.

Proposition 5.22: Path-Connectedness and ID Spaces

For any equivalence relation $\sim$ on a path-connected space X the identification space $Y = X/\sim$ is path-connected.

Theorem 5.24: Homeomorphisms of Real Spaces

If $n \geq 2$, the spaces $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic.
Additionally, there is no bijection $f : \mathbb{R} \to \mathbb{R}^n$ which is continuous.

Theorem 5.25: Connected means Path-Connected

If a topological space X is path-connected, then it is also connected.
Note that the converse is not ture, and a connected space need not be path-connected (Example 5.2.7).

Theorem 5.2.8

Any connected open subset $\Omega \subseteq \mathbb{R}^n$ is also path-connected.

Lemma 5.30: Connected Families

Let $A_\lambda \subseteq X$, $(\lambda \in \Lambda)$ be a family of connected subsets of a topological space X. Suppose $\bigcup_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected.

Definition 5.35: Connected Components

We define an equivalence relation $\sim$ on a topological space x by $x \sim y$ iff there is a connected subset of X which contains both x and y. The resulting equivalence classes are called the **components** or **connected components** of X. For two homeomorphic topologial spaces there will be a bijection between the sets of their components. So spaces with differing numbers of components cannot be homemorphic. The components are the maximal connected subsets of a topological space and by Prop 5.9 are always closed in X. If we take $U \subseteq \mathbb{R}$ an open set, its connected component decomposition is its canonical representation as in Thm 1.30 as a countable disjoint union of open intervals.

Lemma 5.31: Connected Compomennts and Openness

Let X be a topological space and C a connected component of X. Then C is open iff for all $x \in C$ there is an open connected V such that $x \in V \subseteq C$.

Lemma 5.31.5: Path Components

Define a path (equivalence) relation

$x_0 \sim x_1$ if there exists a path $\alpha : I \to X$

from $\alpha(0) = x_0 \in X$ to $\alpha(1) = x_1 \in X$.

5.32) The **constant path** at $x \in X$ is the path

$$\alpha_x : I \to X; \quad t \mapsto x$$

from $\alpha_x(0) = x \in X$ to $\alpha_x(1) = x \in X$

5.33) The **reverse** of a path $\alpha : I \to X$ is the path

$$-\alpha : I \to X; \quad t \mapsto \alpha(1 - t)$$

retracting α backwards, with

$$\underset{-\alpha(0) = \alpha(1)}{\bullet} \xrightarrow{-\alpha} \underset{-\alpha(1) = \alpha(0)}{\bullet}$$

5.34) The **concatenation** of paths $\alpha : I \to X$, $\beta : I \to X$ with

$$\alpha(1) = \beta(0) \in X$$

is the path

$$\alpha \bullet \beta : I \to X, \quad t \mapsto \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq 1/2 \\ \beta(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

which starts at $\alpha(0)$, follows along α at twice the speed in the first half, switching at $\alpha(1) = \beta(0)$ to follow β at twice the speed in the second half.

$$\underset{\alpha \bullet \beta(0) = \alpha(0)}{\bullet} \xrightarrow{\alpha} \underset{\alpha(1) = \beta(0)}{\bullet} \xrightarrow{\beta} \underset{\beta(1) = \alpha \bullet \beta(1)}{\bullet}$$

Proposition 5.35: Equivalence of Path Relations

The path relation defined on a space X by $x_0 \sim x_1$ if there exists a path $\alpha : I \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_0$ is an equivalence relation.

Definition 5.36: Path Compoments Formally

Let X be a topological space.

1. The **path compoments** of X are the equivalence classes of the path equivalence relation $\sim$, i.e. the subspaces
$$\begin{aligned}[x] &= \{y \in X \mid y \sim x\} \\ &= \{y \in X \mid \exists \alpha : I \to X \text{ from } a(0) = x \text{ to } \alpha(1) = y\}\end{aligned}$$
2. The **set of path components** (which may be infinite) is denoted by
$$X/\sim = \pi_0(X)$$
3. The function
$$X \to \pi_0(X), \quad x \mapsto [x] = \{\text{equivalence class of } x\}$$
is surjective.

Proposition 5.37: Invariance of Path Components

The cardinality of sets of of path compoments $\pi_0(X)$ of a topologicl space X is a topological invariant, i.e. if X and Y are homeomorphic, there is a bijection between $\pi_0(X)$ and $\pi_0(Y)$. However, existence of a bijection between $\pi_0(X)$ and $\pi_0(Y)$ does *not* necessarily imply that X and Y are homeomorphic.

Lemma 5.39: Open Condition of Path Compoments

Let X be a topological space and P a path compoment of X. Then P is open iff for all $x \in P$ there is an open path connected V such that $x \in V \subseteq P$.

Lemma 5.40: Openness and Singular Components

Let C be a connected component of a topological space X. If every path component $P \subseteq C$ is open, then C consists of a single path component. Note that the converse of this is not true.

Theorem 5.41: Path Connectedness and Openness

Let X be a topological space. Then X is path connected iff X is connected *and* for all $x \in X$ there is an open path connected V such that $x \in V$.

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