Algebraic Topology Notes

Made by Leon:) Note: Any reference numbers are to the lecture notes

1 Introduction

Recall 1.1.1: Topology

An (open) topology on X is a collection of subsets $\tau \subset P(X)$ such that

- $\emptyset \in \tau$ and $X \in \tau$
- τ is closed under finite intersections: If $\{U_1,\ldots,U_n\}\subset \tau$ then

$$\bigcap_{i=1,\ldots,n} U_i \in \tau$$

• τ is closed under arbitrary unions: If $\{U_1,\ldots,U_n\}\subset \tau$ is a family of open subsets then

$$\bigcup_{i=1,\ldots,n} U_i \in \tau$$

The subsets $U \in \mathcal{T}$ are called **open** and their complements in X define **closed subsets**.

Two examples of a topology on a set X are the following:

- The Trivial Topology: $\tau_{\mathrm{triv}} = \{\emptyset, X\}$
- The Discrete Topology: $\tau_{dis} = P(X)$

A subset $A \subset X$ is clopen if it is both closed and open

Definition 1: Connected Spaces

A topological space X is **connected** if $X = A \cup B$ with $A, B \subset X$ open implies that $A = \emptyset$ or A = X.

Proposition 1: Connectedness and Clopens

A topological space X is connected iff the only clopens are \emptyset and X.

Example 1: Examples of Connected Topologies

- \bullet Every X with the trivial topology is connected.
- Every X with the discrete topology is not connected unless
 X = Ø or X = {*} (in which it coincides with the trivial topology).
- The real line \mathbb{R} with the standard topology is connected.

Proposition 2: Continuous Maps

Let $f: X \to Y$ be a continuous map of topological spaces and let X be connected. Then f(X) is connected.

Proposition 3: Connected Equivalence Relation

For a topological space X, define $x\sim y$ if there exists some connected subset that contains both. The relation $x\sim y$ is an equivalence relation.

Definition 2: Connected Components

The equivalence classes of this relation are called **connected components**. In particular, a space X is connected iff it only has a single connected component.

Definition 3: Path

Let I denote the closed unit interval [0,1]. A **path** in X is a continuous map $\alpha: I \to X$. The points $\alpha(0) \in X$ and $\alpha(1) \in X$ will be called **start** and **end** points respectively.

We define a path relation between points in X be declaring $x \sim y$ if there exists some path $\alpha: I \to X$ that starts at x and ends in y, i.e. $\alpha(0) = x$ and $\alpha(1) = y$. This is an equivalence relation from the following properties:

- 1. Constant Path: For all $x \in X$ there exists the constant path $c_x : I \to X$ defined by $c_x(t) = x$ for all $t \in I$
- 2. Path reversal: Let $\alpha: I \to X$ be a path in X. Define its reversed path by

$$\overline{\alpha}: I \to X, \quad t \mapsto \alpha(1-t)$$
 (1)

3. Path Concatenation: Let α , $\beta: I \to X$ be two paths in X s.t. $\alpha(1) = \beta(0)$. Their concatenated path is defined by:

$$\alpha * \beta(t) := \begin{cases} \alpha(2t), & 0 \le t \le 1/2\\ \beta(2t-1) & 1/2 \le t \le 1 \end{cases}$$
 (2)

Definition 4: Path-Connected Components

The equivalence classes are called **path-connected components** and their set is denoted by $\pi_0(X)$. A space X is called **path-connected** if $\pi_0(X)$ is a one-point set, i.e. any two points x, y can be related by a path in X.

Remark 1: Random examples

The following statements are true:

- A homeomorphism $X \cong Y$ induces a bijection $\pi_0(X) \cong \pi_0(Y)$.
- If X is path-connected, it is also connected.
- The topologist's sine curve defined by $X=\{0\}\times[-1,1]\times\{(x,\sin(1/x))\mid 0< x\}$ is connected but not path-connected.

Definition 5: Homotopy

A **homotopy** of maps $f, g: X \to Y$ is a continuous map $h: X \times I \to Y$ such that h(-,0) = f and h(-,1) = g.



If such a homotopy exists, f is called **homotopic** to g. This defines an equivalence relation $f \simeq g$ on the space of maps $\operatorname{Map}(X,Y)$.

Example 2: Paths as Homotopies

Points in X are the same as maps $* \to X$ from the one-point set * to X. A path $\alpha: I \to K$ corresponds to a homotopy $* \times I \to X$.

Remark 1.5: Composition of Homotopies

• Horizontal Composition: Let $h, h': X \times I \to Y$ be two homotopies in X such that $h(-,1) = h'(-,0): X \to Y$. Their concatenated homotopy is defined by

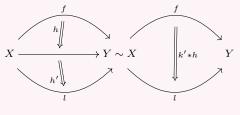
$$h * h'(-,t) := \begin{cases} h(-,2t) & 0 \le t \le 1/2 \\ h'(-,2t-1) & 1/2 \le t \le 1 \end{cases}$$
 (5)

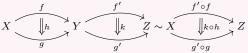
• Vertical Composition: Let $h: X \times I \to Y$ and $k: Y \times I \to Z$ be two homotopies on maps from X to Y, and Y to Z. Then

$$k \circ h := [X \times I \xrightarrow{\operatorname{id} \times \Delta} X \times I^2 \xrightarrow{h \times \operatorname{id}} Y \times I \xrightarrow{k} Z]$$
 (6)

where $\Delta:I\to I^2,\,t\mapsto(t,t)$ is the diagonal map, or explicitly,

$$k\circ h(x,t)=k(h(x,t),t)$$





Lemma 1: Concatenation Relation

Let $f, f': X \to Y$ and $g, g': Y \to Z$ be maps such that $f \simeq f'$ and $g \simeq g'$. Then $f' \circ f \simeq g' \circ g$ as maps from X to Z. In particular, $g' \circ f \sim g \circ f$ and $g \circ f' \sim g \circ f$.

Definition 6: Homotopy Equivalence

A map $f: X \to Y$ is called a **homotopy equivalence** if there exists a map $g: Y \to X$ and homotopies $f \circ g \simeq \operatorname{id}_Y$, $g \circ f \simeq \operatorname{id}_X$. In other words, g satisfies the properties of an inverse up to homotopy. It is called a **homotopy inverse** of f.

Example 3: Circle to \mathbb{R}^2

The inclusion map $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2$ is not a homotopy equivalence, but the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$ is a homotopy equivalence.

Proposition 4: Unique Inverses of Homotopy

Homotopy inverses are unique up to homotopy.

Definition 7: Homotopic Spaces

Two spaces X and Y are called **homotopy equivalent**, or **of the same homotopy type**, and denoted by $X \simeq Y$ if there exists a homotopy equivalence $f: X \to Y$.

Notation: We use \cong for homeomorphisms and \simeq for homotopy equivalence.

Lemma 2: Composition of Inverses

Let $f:X \to y$ and $g:Y \to Z$ with homotopy inverses $\overline{f}:Y \to X$ and $\overline{g}:Z \to Y$ respectively. Then $\overline{f}\circ \overline{g}:Z \to X$ is a homotopy inverse of $g\circ f:X \to Z$. In particular, $X\simeq Y$ and $Y\simeq Z$ implies $X\simeq Z$.

Definition 8: Contractible Space

A space X is called ${\bf contractible}$ if it is homotopy equivalent to a point, i.e. $X \simeq *.$

The **terminal map** is the unique map $X \to *$. Contractibility requires that there is a homotopy inverse of that map, i.e. a map $* \to x$ along with homotopies

$$h: [* \to X \to *] \simeq \mathrm{id}_*, \quad k: [X \to * \to X] \simeq \mathrm{id}_X$$
 (7)

Example 4: Examples of Contractible Spaces

1. \mathbb{R}^n is contractible. Let x_0 be a fixed point in \mathbb{R}^n and define the (straight line) homotopy $h:c_{x_0}\simeq \mathrm{id}_{\mathbb{R}^n}$ by

$$h(x,t) = (1-t)x_0 + tx$$
.

2. $\mathbb{S}^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$. The inclusion $\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ and the shrinking map

$$\mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}, \quad x \mapsto \frac{x}{|x|}$$

are homotopy inverses.

Remark 3: Remarks about Contractible Spaces

- 1. Contractible spaces are path-connected. Let x_0 be the point where the space X contracts to. In particular, we are given with a homotopy $h: c_{x_0} \simeq \operatorname{id}_X$. For any $x \in X$, the map $h(x,-): I \to X$ defines a path from x_0 to x and thus every element $x \in X$ is path-connected to x_0 .
- 2. The converse does not hold, for example $X = \mathbb{S}^1$.
- A contractible space X is contractible at any point x₀. Since X is path-connected, a path from x to x' defines a homotopy c_x ≃ c_{x'}.
- 4. Any two maps $f, g: X \to Y$ are homotopic if Y is contractible.

Definition 9: Retracts and Deformation Retracts

- A **retract** of X onto a subspace $A \subset X$ is a map $r: X \to A$ such that $r|_A = \mathrm{id}_A$. Equivalently, this is a map $r: X \to X$ such that $r^2 = r$ and r(X) = A.
- A deformation retract of X onto A is the additional datum of a homotopy h: id_X ≃ i ∘ r.

In other words, a deformation retract is a homotopy $h: X \times I \to X$ such that h(x,0) = x and $h(x,1) \in A$ for all $x \in X$ and h(a,1) = a for all $a \in A$. Not all retracts can form deformation retracts. For instance, the retract X onto a point $\{x_0\}$ can be a deformation retract iff X is contractible.

This notion is called **weak** deformation retract. A **strong** deformation retract has the condition h(a,t) = a for all $t \in I$, $a \in A$. i.e. Our notion of a (weak) deformation retract deforms X into A while allowing to deform A to do so, while a strong deformation retract deforms X into A while keeping A fixed at all times

Proposition 5: Deformation Retracts and Homotopies

A deformation retract of X onto A induces a homotopy equivalence $X \simeq A$.

Recall 2: Quotient Space

Let X be a topological space and let \sim be an equivalence relation on X. Then, X/\sim is equipped with the quotient topology and called a **quotient space**. If Z is a closed subset in X, then we can also define the quotient space X/Z.

Another form of quotient spaces: Let $f:Z\to Y$ be a continuous map between a closed subset $Z\subset X$ and Y. Then

$$X \cup_f Y = X \cup Y/z \sim f(z).$$

Example 5: Examples of Quotient Spaces

- The quotient of the *n*-dimensional closed disk by its boundary is the *n*-sphere, i.e. $\mathbb{D}^n/\partial \mathbb{D}^n \cong \mathbb{S}^n$.
- The 2-torus: $\mathbb{R}^2/\mathbb{Z}^2$. The projective space:

 $\mathbb{RP}^n = \mathbb{R}^{n+1} \backslash \{0\} / \sim \text{by the relation } x \sim y \text{ iff there exists some } \lambda \in \mathbb{R}^\times \text{ such that } x = \lambda y. \text{ This corresponds to the space of lines through the origin in } \mathbb{R}^{n+1}.$

Definition 10: Mapping Quotients

Let $f: X \to Y$ be a continuous map.

• Its mapping cylinder is defined as the topological space

$$M_f := (X \times I) \cup Y / \sim$$

where the quotient identifies $(x,0) \sim f(x)$ for any $x \in X$.

• Its **cone** is the further quotient:

$$C_f = M_f/X \times \{1\}.$$

• The **cone** of a topological space X is

$$C_X := C_{\mathrm{id}_X} = X \times I/X \times \{1\}.$$

In other words, the mapping cylinder of $f: X \times Y$ is the pushout of the diagram

$$X \times \{0\} \xrightarrow{f} Y$$

$$\downarrow \qquad \qquad \downarrow$$

$$X \times I \longrightarrow M_{I}$$

Example 5.5: Spheres

Consider the *n*-sphere \mathbb{S}^n with the standard embedding $\mathbb{R}^{n+1}\setminus\{0\}$. Then the map

$$r: \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n, \quad x \mapsto \frac{x}{|x|}$$

is a retract. Indeed, if x has norm |x| = 1, then r(x) = x. For a deformation retract one needs to find a homotopy $h: i \circ r \simeq id_X$. This can easily be realised by following straight-line homotopy:

$$h: \mathbb{R}^{n+1} \setminus \{0\} \times I \to \mathbb{R}^{n+1} \setminus \{0\}, \quad (x,t) \mapsto (1-t)\frac{x}{|x|} + tx.$$

Indeed, h(x,0) = r(x) and h(x,1) = x for all x.

Definition 11: Star-Shaped Spaces

A subset $S \subset \mathbb{R}^n$ is called **star-shaped** at a point $x_0 \in S$, if for any $x \in S$ the line segment from x_0 to x is contained in S, i.e.

$$\{(1-t)x_0 + tx \mid t \in [0,1]\} \subset S$$

If S is star-shaped at every point, then it is called **convex**.

Example 5.6: Star-Shaped Spaces are Contractible

Let S be star-shaped at x_0 and $i:\{x_0\}\leftrightarrow S:r$ be the inclusion and constant maps. Define the straight line homotopy

$$h: S \times I \to S, \quad (x,t) \mapsto (1-t)x_0 + tx$$

which is well-defined by the star-shaped condition. Moreover, $h(x,0)=x_0=r(x)$ and h(x,1)=x for all x. Hence, star-shaped, and in particular convex spaces, are contractible.

Example 5.7: Möbius band

The Möbius band M can be defined as

$$M = I^2 / \sim$$

where the equivalence relation \sim identifies the two vertical edges of I^2 by flipping one, i.e. $(0,b)\sim (1,1-b)$ for $b\in I$. Its core $C\subset M$ is the line $\{[a,1/2]\mid a\in I\}$. Thus, the core is homeomorphic to \mathbb{S}^1 .

The Möbius band deformation retracts onto its core. Indeed, consider the retract $r:M\to C$ defined by r([a,b]):=[(a,1/2)] and the homotopy

$$h: M \times I \to M, \quad ([(a,b)],t) \mapsto \left[\left(a,(1-t)\frac{1}{2}+\right)\right].$$

In particular, $M \simeq \mathbb{S}^1$.

Proposition 6: Retracts of the Mapping Cylinder

Via Definition 1.1.10, the mapping cylinder is formed by the cylinder of X by gluing Y onto the bottom with the map f. The mapping cylinder M_f strongly deformation retracts onto Y.

Proof. Consider the retract:

$$r: M_f \to Y$$

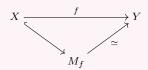
defined by r([x,s]) := [(x,0)] = [f(x)] on the class of $(x,s) \in X \times I$ and r([y]) = y for $y \in Y$. This is well-defined and by definition a retract on Y. Define the homotopy

$$h: M_f \times I \to M_f$$

by h([[x,s)],t) := [(x,st)] for $(x,s) \in X \times I$ and $t \in I$, and by h([y],t) := y for $y \in Y$. In particular, $h(-,0)i \circ r$ and $h(-,1) = \mathrm{id}_{Mt}$. This forms a strong deformation retract. \square

Remark 6: Continuous Maps are Homotopic

Any continuous $f: X \to Y$ can be replaced up to homotopy equivalence by the closed inclusion $X \hookrightarrow M_f$, $x \mapsto [(x,1)]$. More precisely, it factorises through an inclusion and a homotopy equivalence as the following diagram commutes:



Definition 12: Relative Homotopy

Let X, Y be topological spaces and $A \subset X$ be a subset in X. A homotopy $h: X \times I \to y$ is called **relative to** A if h(a, t) is independent of t for all $a \in A$. In particular, this defines homotopies

between maps $f, g: X \to Y$ such that $f|_A = g|_A$.

This definition generalises ordinary homotopies, as an ordinary homotopy is the same as a homotopy relative to \emptyset .

Example 6: Relative Homotopies and Retracts

A strong deformation retract of X onto A is a deformation retract such that the homotopy $h: i \circ r \simeq \operatorname{id}_X$ is relative to A.

Definition 13: Homotopic Path

Let $\alpha, \beta: I \to X$ be paths in X such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. A relative homotopy from α to β is a homotopy $h: I \times I \to x$ relative to $\partial I = \{0, 1\}$, i.e.

$$h(-,0) = \alpha, \quad h(-,1) = \beta$$
 (8)

and

$$h(0,t) = \alpha(0) = \beta(0), \quad h(1,t) = \alpha(1) = \beta(1), \quad \forall t \in I.$$
 (9)

In particular, at any point $t \in I$ a relative homotopy h defines a path $h_t := h(-,t) : I \to X$ with start $\alpha(0) = \beta(0)$ and end $\alpha(1) = \beta(1)$. If one omits the relative condition, the start and end points of h_t would be allowed to vary.

Remark 7: Ordinary Homotopies and Paths

Observe that ordinary homotopies are not well suited for paths: Any path $\alpha:I\to X$ is homotopic (relative \emptyset) to a constant. Indeed, the homotopy

$$h: I \times I \to X, \quad (s,t) \mapsto \alpha(st)$$

defines a homotopy from the constant path $c_{\alpha(0)}$ on $\alpha(0)$ to α , i.e. $c_{\alpha(0)} \simeq \alpha$. Hence, (ordinary) homotopy classes of paths in X are in one-to-one correspondence with path-connected components of X

Proposition 7: Homotopic Properties of Paths

Path concatenation is **unital**, **associative**, and **invertible** up to homotopy in the following sense: Let α , β , $\gamma: I \to x$ be paths such that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. Then there exists homotopies relative to $\{0,1\}$:

- 1. Left Unitality: $c_{\alpha(0)} * \alpha \simeq \alpha$
- 2. Right Unitality: $\alpha \simeq c_{\alpha(0)} * \alpha$
- 3. Associativity: $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$
- 4. Right Inverse: $\alpha * \overline{\alpha} \simeq c_{\alpha(0)}$
- 5. Left Inverse: $\overline{\alpha} * \alpha \simeq c_{\alpha(1)}$

where c_x for some $x \in X$ denotes the constant path on x and $\overline{\alpha}$ is the reversed path.

Lemma 3:

Let $\alpha: I \to X$ be a path and $\lambda: I \to I$ a boundary preserving map, i.e. $\lambda(0) = 0$ and $\lambda(1) = 1$. Then,

$$\alpha \circ \lambda \simeq \alpha$$
, rel. ∂I .

Definition 14: Fundamental Group

Let X be a topological space and $x_0 \in X$ some fixed point. The **fundamental group** of X at x_0 i the group of homotopy classes of paths in X that start and end on x_0 . i.e. $\alpha: I \to X$ such that $\alpha(0) = \alpha(1) = x_0$, i.e.

$$\pi_1(X, x) = {\alpha : I \to X \mid \alpha(0) = \alpha(1)}/\sim.$$

Theorem 1: The Fundamental Group is Well Defined

The fundamental group $\pi_1(X, x_0)$ is a well-defined group with:

- Multiplication: $[\alpha] \cdot [\beta] := [\alpha * \beta]$
- Unit: $1 = [c_{x_0}]$
- Inverse: $[\alpha]^{-1} = [\overline{\alpha}]$

Lemma 4: Relative Concated Homotopic Paths

Let $\alpha \simeq \alpha': I \to X$ and $\beta \simeq \beta': I \to X$ be two pairs of relative homotopic paths such that $\alpha(1) = \alpha'(1) = \beta(0) = \beta'(0)$. Then the concatenations are relative homotopic, i.e.

$$\alpha * \beta \simeq \alpha' * \beta'$$
, rel. $\{0, 1\}$.

Proposition 8: Fundamental Group is Point Independent

Let $\gamma:I\to X$ be a path from $\gamma(0)=x$ to $\gamma(1)=x'.$ Then it induces a group isomorphism:

$$(\gamma)_{\#}: \pi(X,x) \to \pi(X,x'), \quad [\alpha] \mapsto [\overline{\gamma} * \alpha * \gamma].$$

We abuse notation to say that for a path-connected space X, $\pi_1(X)$ is the fundamental group omitting the choice of base point.

Example 7: Examples of Fundamental Groups

- Euclidean: $\pi_1(\mathbb{R}^n) \cong 1$.
- Circle: $\pi_1(\mathbb{S}_1) \cong \mathbb{Z}$.
- n-Spheres: $\pi_1(\mathbb{S}^n) \cong 1$ for $n \geq 2$.
- Torus: $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- Projective Spaces: $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ for $n \geq 2$.

Definition 15: Pointed Space

- A **pointed space** is a pair (X, x) consisting of a topological space X and a point $x \in X$.
- A map of pointed spaces $f:(X,x)\to (Y,y)$ is a continu-

ous map $f: X \to Y$ such that f(x) = y.

• The space of pointed maps from (X, x) to (Y, y) is denoted by

$$\operatorname{Map}_*((X, x), (Y, y)) \subset \operatorname{Map}(X, Y).$$

Proposition 9: Point and Path Space Isormophism

We have a group isomorphism:

$$\pi_1(X, x) \cong \pi_0(\Omega X).$$

Similarly, one can iteratively define the n-fold loop space

$$\Omega^n X := \Omega \Omega^{n-1} X = \Omega \cdots \Omega X$$

There is a homeomorphism

$$\Omega^n X \cong \operatorname{Map}_*((\mathbb{S}^{\ltimes}, 1), (X, x))$$

Definition 16: n-th Homotopy Group

The *n*-th homotopy group $\pi_n(X,x)$ is defined by:

$$\pi_n(X,x) := \pi_0(\Omega^n X) \cong \pi_0(\mathrm{Map}_*(\mathbb{S}^n,(X,x))).$$

Definition 17: Simply Connected Space

A path-connected space X is called **simply connected** if its fundamental group is trivial, i.e.

$$\pi_1(X) = 1$$

Some examples are \mathbb{R}^n , \mathbb{S}^n for n>1, and some non-examples are \mathbb{S}^1 , $\mathbb{S}^1\times\mathbb{S}^1$ and \mathbb{RP}^2 .

Theorem 2: Fundamental Group Isomorphism

Let $f:X \to Y$ be a homotopy equivalence and $x \in X$ an arbitrary base point. Then, the map:

$$f_*: \pi_1(X, x) \to \pi_1(Y, f(x)), \quad [\alpha] \mapsto [f \circ \alpha]$$

In particular, for homotopy equivalent spaces $X\simeq Y$ which are path-connected, we get $\pi_1(X)\cong \pi_1(Y)$.

A map of pointed spaces $f:(X,x)\to (Y,y)$ is a **homotopy** equivalence of pointed spaces or homotopy equivalence relative $\{x\}$ if there exists a map of pointed spaces $g:(Y,y)\to (X,x)$ along with relative homotopies

$$h: f \circ g \simeq \mathrm{id}_Y \quad \text{rel. } \{y\}$$

and

$$k: g \circ f \simeq \mathrm{id}_X$$
 rel. $\{x\}$

Example 9: Strong Deformation Retracts Homotopies

A strong deformation retract of X onto a subspace A gives a homotopy equivalence of pointed spaces $(x, a) \rightarrow (A, a)$ for any

choice of $a \in A$. In particular, a contractible space $X \simeq *$ determines a homotopy equivalence of pointed spaces $(X, x) \to *$ for any choice of base point x.

Lemma 5: Pointed Space Isomorphism

Let $f:(X,x)\to (Y,y)$ be a homotopy equivalence of pointed spaces. Then the map

$$f_*: \pi_1(X, x) \to \pi_1(Y, y)$$

is a group isomorphism.

Corollary 1: Let $r: X \to A$ be a strong deformation retract of X onto $A \subset X$. Then

$$\pi(X, a) \cong \pi(A, a)$$

for any $a \in A$. In particular, contractible spaces are simply connected

Lemma 6: Identity Homomorphic Isormorphism

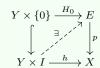
Let $f: X \to X$ be a continuous map which is homotopic to id_X . Then, the map

$$f_*: \pi_1(X, x_0) \to \pi_1(X, f(x_0))$$

is a group isomorphism for any choice of base point $x_0 \in X$.

Definition 18: Homotopy Lifting Property

A continuous map $p: E \to X$ satisfies the **homotopy lifting property** (HLP) with respect to a topological space Y if for any commuting diagram:



There exists a map $H:Y\times I\to E$ such that both triangles commute, i.e. $H|_{Y\times\{0\}}=H_0$ and $p\circ H=h.$

In other wowrds, the map $p: E \to X$ has the HLP if for any homotopy $h: Y \times I \to X$ of maps $h(-,0) := f_0$ and $h(-,1) := f_1$ of maps $Y \to X$ and a choice of a lift H_0 of f_0 , then the homotopy h lifts to a homotopy $H: Y \times I \to E$.

In particular, if $f_0 \simeq f_1 : Y \to X$ and H_0 is a lift of f_0 , we find $H_0 \simeq H_1$ where H_1 lifts f_1 .

Example 10: Example of the HLP

Trivially the identity map $\mathrm{id}_X:X\to X$ has the HLP with respect to any space Y.

Definition 19: Covering Space

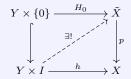
A **covering space** of X is a topological space \overline{X} along with a continuous map $p: X \to x$ such that for any point $x \in X$ there exists an open neighbourhood $U \subset X$ whose preimage $p^{-1}(U) = \bigcup_{j \in J} V_j$ and the opens $V_j \subset \overline{X}$ map homeomorphically to U under p. In other words, a covering space of X looks locally like a product of X with a discrete space.

Example 11: Example of a Covering Space

- The projection map p: X × Z → X is a covering map if Z is a discrete topological space. If Z is not discrete, then this is not a covering map in general.
- 2. The identity map $\mathrm{id}_X:X\to X$ is always trivially a covering map.
- 3. While the projection of $p: X \times I \to X$ from the cylinder is not a covering map, its restriction to the boundary $\partial(X \times I) = X \times \{0,1\} =: \overline{X}$ gives a trivial (2-fold) covering of X.
- 4. Recall that the Möbius band M deformation retracts onto its core \mathbb{S}^1 . Restricting to the boundary $\partial M = \mathbb{S}^1$, one obtains a (non-trivial) covering map $\mathbb{S}^1 \to \mathbb{S}^1$. This map coincides with $z \mapsto z^2$ if we identify S^1 as the unit circle in \mathbb{C} .

Theorem 3: Unique HLPs from Covering Maps

Let $p: \tilde{X} \to X$ be a covering map and Y any topological space. Then p satisfies the HLP uniquely:



i.e. the lift H not only exists, but it is also unique.

Corollary 2:

- 1. Let $\gamma: I \to X$ be a path and fix a point $\tilde{x_0} \in \tilde{X}$ such that $p(\tilde{x_0}) = \gamma(0)$. Then, there exists a unique path $\tilde{\gamma}: I \to \tilde{X}$ which starts at $\tilde{x_0}$ and lifts γ i.e. $p \circ qa\tilde{m}ma = \gamma$
- 2. Let $h: I \times I \to X$ be a (relative) homotopy of paths $h(-,0) =: \gamma_0$ and $h(-,1) =: \gamma_1$, and fix a point $\tilde{x_0}$ such that $p(\tilde{x_0}) = h(0,t) = \gamma_0(0) = \gamma_1(0)$. Suppose $\tilde{\gamma_0}: I \to X$ is a lift of γ starting at $\tilde{\gamma_0}(0) = \tilde{x_0}$. Then, there exists a unique homotopy of paths $\tilde{h}: I \times I \to \tilde{X}$ which lifts h and $\tilde{h}(-,0) = \tilde{\gamma_0}$

Theorem 4: Fundamental Group of the Circle

The fundamental group of the circle is $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. It is generated by the class of

$$\alpha: I \to \mathbb{S}^1, \quad t \mapsto e^{2\pi i t}.$$

Theorem 5: Brouwer's Fixed Point Theorem

Any continuous map $f:\mathbb{D}^2\to\mathbb{D}^2$ has a fixed point, i.e. there exists $x\in\mathbb{D}^2$ such that f(x)=x.

Theorem 6: Fundamental Theorem of Algebra

Every non-constant complex polynomial $p \in \mathbb{C}[z]$ has at least one root, i.e. $p(z_0) = 0$ for some z_0 .

Theorem 7: Fundamental Group of \mathbb{S}^n

The fundamental group of \mathbb{S}^n is trivial for $n\geq 2,$ i.ke. $\pi_1(\mathbb{S}^2)\cong 1$ for $n\geq 2$

Lemma 7: Closed Paths Homotopic to Loops

Let (X,x_0) be a topological space with an open cover $\{U_j\}_{j\in J}$ such that U_j are path-connected neighbourhoods of x_0 and $U_j\cap U_{j'}$ is path-connected for any $j,j'\in J$. Then, any closed path γ based at x_0 is homotopic to a concatenation $\gamma_1*\gamma_2*\cdots*\gamma_n$ of loops at x_0 each of them contained in a single U_j

Corollary 3: Homemorphisms between \mathbb{R}^2 and \mathbb{R}^n

There is no homeomorphism between \mathbb{R}^2 and \mathbb{R}^n for $n \neq 2$

Recall 4: Defining the Real Projective Space

1. The space \mathbb{RP}^2 is the quotient space:

$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$$

where $x \sim y$ if there exists $\lambda \in \mathbb{R}$ such that $x = \lambda y$. In other words, the real projective *n*-space represents the lines in \mathbb{R}^{n+1} through the origin.

- 2. Picking representatives that lie in the unit *n*-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1} \setminus \{0\}$, we obtain $\mathbb{RP}^n \cong \mathbb{S}^n / \sim$ where $x \sim -x$ for all $x \in \mathbb{S}^n$, i.e. identifying antipodal points on the *n*-sphere.
- 3. Further restricting to the upper half $\mathbb{D}^n \subset \mathbb{S}^n$ we obtain:

$$\mathbb{RP}^n \cong \mathbb{D}^n / \sim$$

where $x \sim -x$ for any boundary points $x \in \partial \mathbb{D}^n \cong \mathbb{S}^{n-1}$

For example, \mathbb{RP}^0 is a one point space, $\mathbb{RP}^1 \cong \mathbb{S}^1$, while \mathbb{RP}^n are different than spheres for larger n.

Definition 20: Lift of a Path

- A lift of a path $\alpha:I\to\mathbb{RP}^n$ is a pth $\tilde{\alpha}:I\to\mathbb{S}^n$ such that $p\circ\tilde{\alpha}=\alpha$
- If α is a closed path, then $[\tilde{\alpha}(0)] = [\tilde{\alpha}(1)]$ which implies

 $\tilde{\alpha}(0) = \pm \tilde{\alpha}(1)$. The **sign** of α is defined by

$$\operatorname{sgn}(\alpha) \mapsto \begin{cases} +1 & \tilde{\alpha}(0) = \tilde{\alpha}(1) \\ -1 & \tilde{\alpha}(0) = -\tilde{\alpha}(1) \end{cases}$$

Theorem 8: Group Homomorphism of the Sign

The sign induces a surjective group homomorphism

$$\operatorname{sgn}: \pi_1(\mathbb{RP}^n) \to \mathbb{Z}_2, \quad [\alpha] \mapsto \operatorname{sgn}(\alpha)$$

which is an isomorphism for $n \geq 2$.

Definition 21: Right Lifting Property

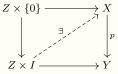
A map $p: X \to Y$ satisfies the **right lifting property** (RLP) with respect to a map $i: A \to B$ if any commutative square



has a solution to the lifting problem making both triangles commute. Explicitly, if $f: B \to Y$ and $g: A \to X$ such that $f \circ i = p \circ g$, then there exists a map $l: B \to X$ satisfying $l \circ i = g$ and $p \circ l = f$. Dually, the map $i: A \to B$ is said to satisfy the **left lefting property** (LLP) with respect to $p: X \to Y$.

Example 13: Homotopy Lifting Property WRT Spaces

1. A map $p:X\to Y$ satisfies the homotopy lifting property with respect to a space Z iff it has the RLP ith respect to the inclusion map $i:Z\times\{0\}\hookrightarrow Z\times I$, i.e. solves the following lifting problem:



In other words, given a homotopy $h: Z \times I \to Y$ and a lift $\tilde{f}: Z \to X$ of h(-,0) =: f, there is a homotopy lift $\tilde{h}: Z \times I \to X$ with $\tilde{h}(-,0) = \tilde{f}$.

 Dually, a map i: A → b satisfies the homotopy extension property (HEP) with respect to a space Z iff it has the LLP with respect to the map

$$p: Z^I \to Z, \quad \gamma \mapsto \gamma(0)$$

where $Z^I := \text{Map}(I, Z)$ is the space of paths in Z. In other words, one can solve the following lifting problem:



Note that a map $A \to Z^I$ is the same datum as a homotopy $h: A \times I \to Z$. Given an extension $\tilde{f}: B \to Z$ of h(-,0) along i, the existence of a map $B \to Z^I$ which makes both triangles commute provides an extension of the homotopy h

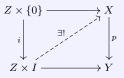
to a homotopy $\tilde{h}: B \times I \to Z$ along i.

Definition 22: Fibration

- 1. A map $p:X\to Y$ is a **fibration** if it satisfies the HLP with respect to all spaces Z. In other words, it has the RLP with respect to the set of maps $\{i:Z\times\{0\}\hookrightarrow Z\times I\}_Z$ where Z runs over all topological spaces
- 2. Dually, a map $i: A \to B$ is a **cofibration** if it satisfies the HEP with respect to all spaces Z.

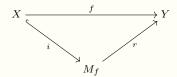
Theorem 9: Covering Maps are Fibrations

A covering map $p: \tilde{X} \to X$ is a fibration. Additionally, the homotopy lifts are unique:



Example 14: Examples of Fibrations

- 1. By Theorem 9, fibrations include all covering maps
- 2. The projection map $p: X \times F \to X$ is always a fibration. However this map is a covering map iff F is a discrete space. Hence, this includes examples of fibrations that are not coming from covering maps.
- 3. An important example of a cofibration is the inclusion $i: X \to M_f$ where M_f is the mapping cylinder of $f: X \times Y$. We have seen that any continuous map $f: X \times Y$ factors through the mapping cylinder:



In particular, every map factors through a cofibration and a homotopy equivalence.

Example 15: Covering Spaces

- 1. The projection map $p: X \times D \to X$ where D is a discrete space. Note that $X \times D$ cannot be path-connected unless D is a one-point set.
- 2. The covering map $\mathbb{R} \to \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$ which we can use to compute the fundamental group of \mathbb{S}^1 .
- 3. The degree-n map $F_n: \mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto z^n$ provides an n-fold covering of \mathbb{S}^1 by itself.
- 4. The product of two covering maps $p_i: \tilde{X}_i \to X_i$ is also a covering map $p_1 \times p_2: \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$
- 5. The product of F_n and F_m in the third example also provides a self covering of the torus:

$$T^2 = \mathbb{S}^1 \times \mathbb{S}^1 \to T^2, \quad (z, w) \mapsto (z^n, w^m)$$

- 6. Similarly, there is a covering $\mathbb{R}^2 \to T^2$.
- 7. The 2-fold covering $\mathbb{S}^n \to \mathbb{RP}^n$ which was used to compute the fundamental group of \mathbb{RP}^n

Theorem 10: Homomorphism of Covering Maps

Let $p: \tilde{X} \to X$ be a covering map. The induced group homomorphism

$$p_*: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$$

is injective for any $\tilde{x}_0 \in p^{-1}(x_0)$. Its image consists of (classes of) loops in X based at x_0 that lift to loops in \tilde{X} based at \tilde{x}_0

Remark 5: Notation for Covers

Fix a covering map $p: \tilde{X} \to X$ and $x_0 \in X$ a fixed point. We will write

$$G := \pi_1(X, x_0)$$

for the fundamental group of X at x_0 and

$$H := p_*\pi_1(\tilde{X}, \tilde{x}_0) \subset G$$

for the subgroup determined by the covering map. The subgroup H depends on the choice of fiber point $\tilde{x}_0 \in p^{-1}(x_0)$ and we shall see that it subgroups for different fiber points are conjugate to each other. Finally, the fiber over x_0 will be denoted by

$$F_{x_0} := p^{-1}(x_0)$$

Lemma 8: Transitive Actions

If \tilde{X} is path-connected, then the G-action on F_{x_0} is transitive, i.e. for any $\tilde{x}, \tilde{x}' \in F_{x_0}$, there exists a $\alpha \in G$ such that $\tilde{x}.\alpha = \tilde{x}'$.

Theorem 11: Path-Connected Correspondence

If \tilde{X} is apth-connected, then there is a one-to-one correspondence between right cosets and fiber poitns, i.e. a bijection

$$G_{\tilde{x}} \backslash G \to F_{x_0}, \quad G_{\tilde{x}} \cdot g \mapsto \tilde{x}.g$$

Thus the index of $G_{\bar{x}}$ in G coincides with the cardinality of the fiber F_{x_0} :

$$[G:G_{\tilde{x}}] = |F_{x_0}| \tag{20}$$

Corollary 4: If \tilde{X} is simply-connected, then there is a bijection

$$G \to F_{x_0}$$

Equation (20) becomes

$$|G| = |F_{x_0}| \tag{21}$$

Definition 23: Deck Transformation

A deck transformation of a covering map $p: \tilde{X} \to X$ is a self-homeomorphism $D: \tilde{X} \stackrel{\cong}{\longrightarrow} \tilde{X}$ such that $p \circ D = p$.

Deck transformations form a group Deck(p). Indeed, for any two deck transformations D, D' their composite is also a deck transformation since $p \circ D \circ D' = p \circ D' = p$. Moreover, if D is a deck transformation then so is its inverse D^{-1} as $p \circ D^{-1} = p \circ D \circ D' = p$.

For example, deck transformations of the covering map $\mathbb{R} \to \mathbb{S}^1$ are precisely translations by integers:

$$D_n: \mathbb{R} \to \mathbb{R}, \quad t \mapsto t + n$$

In particular, the group of deck transformations is \mathbb{Z} . More generally, the group of deck transformations $\operatorname{Deck}(p)$ of a universal covering p is isomorphic to the fundamental group G. From now on $p: \tilde{X} \to x$ will be a covering with \tilde{X} path-connected and X

path-connected and locally path-connected.

Example 16: Topologist's Sine Curve

Recall that the topologist's sine curve

$$X = \{0\} \times [-1, 1] \cup \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid 0 < x \le \frac{1}{2\pi} \right\} \subset \mathbb{R}^2$$

is an example of a connected, but not path-connected space. Let Z be the quotient of X by identifying the points $(0,0) \sim (\frac{1}{2\pi},0)$. Z is a path-connected space but not locally path-connected.

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