Metric Spaces Exam Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Introduction to Metric Spaces

Theorem 1.0.1: Definition of a Metric

Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** iff for all $x,\ y,\ z \in X$,

- $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

A non-empty set X equipped with a metric d is a **metric space**

Definition 1.0.2: Real Vector Spaces

A **real vector space** V is a set with two operations $(X, +, \cdot)$, where:

- $+$ is addition, and $\cdot$ is scalar multiplication
- $(X, +)$ is an abelian group - i.e. for all (vectors) $x, y, z \in X$:
  - **Closure**: $x + y \in X$
  - **Commutativity**: $x + y = y + x$
  - **Associativity**: $x + (y + z) = (x + y) + z$
  - **Identity**: $\exists 0 \in X$ s.t. for all $x \in X$ we have $0 + x = x + 0 = x$
  - **Inverse**: $\forall x \in X$ we have $-x$ s.t. $x + (-x) = (-x) + x = 0$
- Vector space axioms: for all $x, y, z \in X$ and $\mu, \lambda \in \mathbb{R}$ we have:
  - **Closure-ish thing**: $\lambda x \in X$
  - **Distributivity 1**: $\lambda(x + y) = \lambda x + \lambda y$
  - **Distributivity 2**:$(\lambda + \mu)x = \lambda y + \mu x$
  - **Associativity**: $\lambda(\mu x) = (\lambda\mu)x$
  - **Identity**: $1x = x$

Definition 1.0.3: Normed and Inner Product Spaces

Normed Vector Spaces

A **normed vector space** is a real vector space X equipped with a **norm**, i.e. a function that assigns to every vector $x \in X$ a real number $\|x\|$ so that, for all vectors x and y in X and all real scalars a:

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- $\|ax\| = |a|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

Remark: If $(X, \|\cdot\|)$ is a normed vector space then

$$d(x, y) = \|x - y\|$$

defines a metric in X

Remark: This is a generalisation of the "length of a vector"

Inner Product Spaces

Let X be a real vector space. An **inner product** on X is a function that assigns to every pair $(x, y) \in X \times X$ a real number denoted by $\langle x, y\rangle$ and has the following properties:

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $ax + by, z = a\langle x, z\rangle + b\langle y, z\rangle$

A **real inner product space** is a real vector space equipped with an inner product. If $\langle \cdot, \cdot\rangle$ is an inner product on X, then

- $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm in X
- $d(x, y) = \|x - y\|$ defines a metric in X

Remark: This is a generalisation of the dot product

Definition 1.1.4: n-dimensional Euclidean space

Let $X = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$
For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \text{ (inner product)}$$

Properties of n-inner product: For all vectors $x, y, z \in \mathbb{R}^n$ and all real scalars a, b,

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

For $x = (x_1, x_2, \ldots, .x_n) \in \mathbb{R}^n$ define

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \sqrt{x_1^2 + x_2^2 + \cdots + c_n^2} \text{ (norm)}$$

Properties of n-norm: For $x, y \in \mathbb{R}^n$, $a \in \mathbb{R}$,

- $\|x\|_2 \geq 0$ and $\|x\|_2 = 0 \iff x = 0$
- $\|ax\|_2 = |a|\|x\|_2$
- $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$ (triangle inequality)

Example 1.1.5: Examples of Metric Spaces

Unless stated otherwise let $X = \mathbb{R}^n$. The case $X = \mathbb{R}^2$ is listed in red

Name	Norm and Metric
Standard	$\lvert x\rvert = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$
Taxicab	$\lVert x\rVert_1 = \lvert x_1\rvert + \lvert x_2\rvert + \cdots \lvert x_n\rvert$ $d_1(x, y) = \lvert x_1 - y_1\rvert + \lvert x_2 - y_2\rvert + \cdots + \lvert x_n - y_n\rvert$
Euclidean	$\lVert x\rVert_2 = \sqrt{\lvert x_1\rvert^2 + \lvert x_2\rvert^2 + \cdots \lvert x_n\rvert^2}$ $d_2(x, y) = \sqrt{\lvert x_1 - y_1\rvert^2 + \lvert x_2 - y_2\rvert^2 + \cdots + \lvert x_n - y_n\rvert^2}$
p-metric	$\lVert x\rVert_p = \left(\sum_{k=1}^{n} \lvert x_k\rvert^p\right)^{1/p}$ $d_p(x, y) = \left(\sum_{k=1}^{n} \lvert x_k - y_k\rvert^p\right)^{1/p}$
Chebyshev	$\lVert x\rVert_\infty = \max\{\lvert x_1\rvert, \lvert x_2\rvert, \ldots, \lvert x_n\rvert\}$ $d(x, y) = \max\{\lvert x_1 - y_1\rvert, \lvert x_2 - y_2\rvert, \ldots, \lvert x_n - y_n\rvert\}$
Discrete	Not induced by a metric $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$
Post Office	Not induced by a metric $d(x, y) = \begin{cases} \lVert x\rVert_2 + \lVert y\rVert_2 & x = y \\ 1 & x \neq y \end{cases}$

The complex plane

Let $X = \mathbb{C}$, $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$

$$d(z, w) = |z - w|$$

If $z = a + ib, w = c + id$, $a, b, c, d \in \mathbb{R}$, then

$$z - w = (a - c) + i(b - d)$$

therefore,

$$d(z, w) = \sqrt{(a - c)^2 + (b - d)^2}$$

Example 1.1.6: Sequence Spaces

The space ℓ^1

ℓ^1 is the set of real sequences $(x_n)_{n\in\mathbb{N}}$ where $\sum_{n=1}^{\infty}|x_n|$ converges.
For $x = (x_1, \ldots, x_n, \ldots) \in \ell^1$, $y = (y_1, \ldots, y_n, \ldots) \in \ell^1$ we define

- **Norm**: $\|x\|_1 = \sum_{n=1}^{\infty}|x_n|$
- **Metric**: $d_1(x, y) = \|x - y\|_1 = \sum_{n=1}^{\infty}|x_n - y_n|$

The space ℓ^2

ℓ^2 is the set of real seqs $(x_n)_{n\in N}$ where $\sum_{n=1}^{\infty}|x_n|^2$ converges
For $x = (x_1, \ldots, x_n, \ldots) \in \ell^2$, $y = (y_1, \ldots, y_n, \ldots) \in \ell^2$ we define

- **Inner product**: $\langle x, y\rangle = \sum_{n=1}^{\infty} x_ny_n$
- **Norm**: $\|x\|_2 = \left(\sum_{n=1}^{\infty}|x_n|^2\right)^{1/2}$
- **Metric**: $d_2(x, y) = \|x - y\|_2 = \left(\sum_{n=1}^{\infty}|x_n - y_n|^2\right)^{1/2}$

Thm: ℓ^2 is a real vector space

The space ℓ^∞

ℓ^∞ is the set of all bounded sequences of real numbers For $x = (x_1, \ldots, x_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots) \in \ell^\infty$

- **Norm**: $\|x\|_\infty = \sup\{|x_1|, \ldots, |x_n|, \ldots\}$
- **Metric**: $\|x - y\|_\infty = \sup\{|x_1 - y_1|, \ldots, |x_n - y_n|, \ldots\}$

The space $C([a, b])$

$X = C([a, b])$ is the set of all continuous functions $f : [a, b] \to \mathbb{R}$

- **Norm**: $\|f\|_\infty = \max\{|f(x)| : a \leq x \leq b\}$
- **Metric**: $d_\infty(f, g) = \|f - g\| = \max\{|f(x) - g(x)| : a \leq x \leq b\}$

The L^1 metric

X is the set of all continuous functions $f : [a, b] \to \mathbb{R}$

- **Norm**: $\|f\|_1 = \int_a^b |f(x)|dx$
- **Metric**: $d_2(f, g) = \|f - g\|_1 = \int_a^b |f(x) - g(x)|dx$

The L^2 metric

X is the set of all continuous functions $f : [a, b] \to \mathbb{R}$

- **Inner Product**: $\langle f, g\rangle = \int_a^b f(x)g(x)dx$
- **Norm**: $\|f\|_2 = \langle f, f\rangle^{1/2} = \left(\int_a^b |f(x)|^2dx\right)^{1/2}$
- **Metric**: $d_1(f, g) = \left(\int_a^b |f(x) - g(x)|^2dx\right)^{1/2}$

Definition 1.1.7: Metric Subspaces

Let (X, d) be a metric space and Y a non-empty subset of X. Define

- $d_Y : Y \times Y \to \mathbb{R}$
- $d_y(y, y') = d(y, y')$

Then d_Y is a metric on Y. d_Y is called the **induced** or **inherited** metric, and (Y, d_Y) is said to be a metric subspace of the metric space (X, d)

Definition 1.1.8: Open Ball

Let (X, d) be a metric space, c be a point in X, and $r > 0$. The **open ball** with center c and radius r is defined by

$$B(c, r) = \{x \in X : d(c, x) < r\}$$

2 Convergence

2.1 Convergent Sequences in Metric Spaces

On the real line, $x_n \to x$ iff for every positive ϵ, there exists an index N such that for all indices n where $n \geq N$, we have $|x_n - x| < \epsilon$.

Definition 2.1.1: Convergent Sequence

Let (X, d) be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in X, and $x \in X$. We say that $(x_n)_{n=1}^{\infty}$ converges to x iff for every positive ϵ, there exists an index N s.t. for all indices n with $n \geq N$a we have $d(x_n, x) < \epsilon$.
Observe that:

- $d(x_n, x) < \epsilon$ is equivalent to $x_n \in B(x, \epsilon)$.
- $x_n \to x$ in (X, d) iff $d(x_n, x) \to 0$ on the real line

Theorem 2.1.2: Uniqueness of metric limit

- Let (X, d) be a metric space, and $x, x' \in X$, $x \neq x'$. Then there exists a positive radius r s.t. $B(x, r) \cap B(x', r) = \emptyset$
- A sequence in a metric space can have at most one limit

Example 2.1.3: convergence in $(\mathbb{R}^N, d_2)$

A sequence

$$x_1 = (x_{11}, \ldots, x_{1j}, \ldots x_{1N})$$
$$x_2 = (x_{21}, \ldots, x_{2j}, \ldots x_{2N})$$
$$\vdots$$
$$x_n = (x_{n1}, \ldots, x_{nj}, \ldots x_{nN})$$
$$\vdots$$
$$\downarrow$$
$$x = (x_1, \ldots, x_j, \ldots, x_N)$$

in $\mathbb{R}^N, d_2$ converges to $x = (x_1, \ldots, x_j, \ldots, x_N)$ iff for each j,

$$x_{nj} \xrightarrow[j \to +\infty]{} x_j$$

Definition 2.1.4: Bounded Sequence

A sequence in a metric space is said to be **bounded** iff there exists an open ball that contains all of its terms

---

Note: this is the same definition as "sequence is bounded if there is upper and lower bound", as open ball implies the same thing

Theorem 2.1.5

Every convergence is bounded

2.2 Cauchy Sequences

Convergence: For every ϵ, there is an N such that for $n \geq N$, $d(x_n, x) < \epsilon$

$$x_1 \quad x_2 \quad \cdots \quad x_N \quad \cdots x_n \quad \cdots \quad \to x$$

Replace x by any x_m with $m \geq N$

$$x_1 \quad x_2 \quad \cdots \quad x_N \quad \cdots x_n \quad \cdots \quad x_m \quad \cdots$$

'$d(x_n, x) < \epsilon$' becomes '$\forall m \geq N$, $d(x_n, x_m) < \epsilon$'

Definition 2.2.1: Cauchy Sequence

A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) is said to be a **Cauchy sequence** iff for every positive ϵ, there exists an index N, s.t. for all indices n, m with $n, m \geq N$,

$$d(x_n, x_m) < \epsilon$$

Theorem 2.2.2

If a sequence in a metric space converges, then it is a Cauchy sequence

Definition 2.2.3: Complete Metric Spaces

A metric space is said to be **complete** if and only if every Cauchy Sequence is convergent

Examples:

- $\mathbb{R}$ with the standard metric is complete
- $\mathbb{Q}$ with the standard metric is not complete
- $(0, 1)$ with the standard metric is not complete
- $[0, 1]$ with the standard metric is complete
- $\mathbb{R}^n$, ℓ^p, $C([a, b])$ is complete (proof later)

2.3 Open sets and closed sets

Definition 2.3.1: Open Sets and Closed Sets

Let (X, d) be a metric space.

- A subset G of X is said to be **open** iff for every point x in G there exists a positive radius r such that $B(x, r) \subseteq G$.
- A subset F of X is said to be **closed** iff F^c is open

Definition 2.3.2: Discrete Metric Space

A metric space is called **discrete** iff all its subsets are open (equiv. all subsets are closed)

Example: $[0, 1] \cap (2, 3)$

Theorem 2.3.3: Properties of open sets

Let (X, d) be a metric space

1. The union of any family of open sets is an open set
2. The intersection of finitely many open sets is an open set

Proof for 1: Let $(G_i)_{i \in I}$ be a family of open sets and define $G = \bigcup_{i \in I} G_i$. If $x \in G$, then $x \in G_i$ for some i. Since G_i is open, there exists a positive r such that $B(x, r) \subseteq G_i$. Then $B(x, r) \subseteq G$
Proof for 2: Let $G_1, \ldots, G_n$ be open sets. Define $G = G_1 \cap \cdots \cap G_n$. If $x \in G$, then $x \in G_i$ for all i. Since each G_i is open, there exists a positive r_i such that $B(x, r_i) \subseteq G_i$. Let $r = \min\{r_1, \ldots, r_n\}$. For each i,

$$B(x, r) \subseteq B(x, r_i) \subseteq G_i$$

Therefore, $B(x, r) \subseteq G_1 \cap \cdots \cap G_n = G$

Theorem 2.3.4: Infinite open sets

The intersection of infinitely many open sets is not always an open set
For example, let $G_n = (-\frac{1}{n}, \frac{1}{n}), n = 1, 2, \ldots$ on the real line with the standard metric.
Each G_n is open but

$$\bigcap_{n=1}^{\infty} G_n = \{0\}$$

Theorem 2.3.5: Relatively open sets

Let (X, d) be a metric space and A be a non-empty subset of X equipped with the induced metric d_A. Let $G \subseteq A$. G is open in (A, d_A) iff there exists a subset O of X, open in (X, d), such that $G = A \cap O$
The open sets of (A, d_A) are sometimes referred to as **relatively open**

Theorem 2.3.6

Let (X, d) be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in X and x be a point in X.
$x_n \to x$ iff every open set that contains x contains eventually all terms of the sequence

Proof: Assume $x_n \to x$. Let G be any open set with $x \in G$. There is a positive r such that $B(x, r) \subseteq G$. There is an N such that for all n with $n \geq N$ we have $x_n \in B(x, r)$, hence, $x_n \in G$.
Conversely, assume that every open set containing x contains eventually all terms of the sequence. Every open ball centered at x is an open set, therefore it contains eventually all terms of the sequence. It follows that $x_n \to x$.

Definition 2.3.7: Neighbourhoods of points

An **open neighbourhood** of a point x is any open set that contains x. $x_n \to x$ iff every open neighbourhood of x contains eventually all terms of the sequence.

A **neighbourhood** of a point x is a set that contains an open neighbourhood of x. $x_n \to x$ iff every neighbourhood of x contains eventually all terms of the sequence.

Theorem 2.3.8: Properties of Closed sets

Let (X, d) be a metric space.

1. The intersection of any family of closed sets is a closed set
2. The union of finitely many closed sets is a closed set.

Proof for 1: Let $(F_i)_{i\in I}$ be a family of closed sets. Then each F_i^c is open, therefore, $\bigcup_{i\in I} F_i^c$ is open, therefore $\left(\bigcup_{i\in I} F_i^c\right)$ is closed. By De Morgan's rule, $\left(\bigcup_{i\in I} F_i^c\right)^c = \bigcap_{i\in I} F_i$. Therefore, $\bigcap_{i\in I} F_i$ is closed.
Proof for 2: Let $F_1, \ldots, F_n$ be closed sets. Then $F_1^c, .., F_n^c$ are open, therefore $F_1^c \cap \cdots \cap F_n^c$ is open, therefore $(F_1^c \cap \cdots \cap F_n^c)^c$ is closed. By de Morgan's rule, $(F_1^c \cap \cdots \cap F_n^c)^c = F \cup \cdots \cup F_n$. Therefore, $F \cup \cdots \cup F_n$ is closed

Theorem 2.3.9: Infinite closed sets

The union of infinitely many closed sets is not always a closed set.
For example, let $F_n = [\frac{1}{n}, 1], n = 1, 2, \ldots,$ on the real line with the standard metric. Each F_n is closed but

$$\bigcup_{n=1}^{\infty} F_n = (0, 1]$$

is not closed.

Watch lecture recording 30/01 for examples

Theorem 2.3.10

A subset F of a metric space is closed iff the limit of every convergent sequence of elements of F belongs to F

Proof $\implies$: Assume F is closed, and let $(x_n)_{n=1}^{\infty}$ be a convergent sequence of elements of F. Let x be its limit. We wish to show that $x \in F$. We argue by contradiction. Suppose $x \notin F$. Then $x \in F^c$, and since F^c is open, there exists a positive r such that $B(x, r) \subseteq F^c$. Then $B(x, r)$ contains no terms of the sequence - contradiction
Proof $\impliedby$: assume that the limit of every convergent sequence of elements of F belongs to F. We wish to show that F is closed.
We show that F^c is open. Let $x \in F^c$. We need to show that there exists a postive r such that $B(x, r) \subseteq F^c$. If not, then for every r there exists a point in $B(x, r)$ that belongs to F.
Using this with $r = \frac{1}{n}, n = 1, 2, 3, \ldots,$ we find points x_n with $x_n \in B(x, 1/n)$ and $x_n \in F$. Then $x_n \to x$ but $x \notin F$a. Contradiction
Watch lecture recording 30/01 for examples

- In any metric space (X, d), singletons $F = \{x\}$ are closed.
- In any metric space, any finite set is closed because

$$\{x_1, \ldots, x_n\} = \{x_1\} \cup \cdots \cup \{x_n\}$$

2.4 Closure

Definition 2.4.1: Closure

Let (X, d) be a metric space and $A \subseteq X$. The **closure** of A, deonted by $\overline{A}$, is the smallest closed subset of X that contains A
There exists at least one closed subset of X that contains A, namely X itself. The smallest closed subset of X that contains A is

$$\bigcap_{A\subseteq F\subseteq X,\ F\text{closed}} F$$

Theorem 2.4.2: Properties of Closure

Let (X, d) be a metric space and $A,\ B \subseteq X$.

1. $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$
2. $A \subseteq \overline{A}$ and $\overline{A}$ is closed
3. A is closed iff $A = \overline{A}$
4. $\overline{\overline{A}} = \overline{A}$
5. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$
6. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Lecture 30/01 45m for proofs
Example: $X = \mathbb{R}$, $d(x, y) = |x - y|$, $A = (0, 1)$. We claim that $\overline{A} = [0, 1]$
$A \subseteq [0, 1]$ and $[0, 1]$ is a closed set. The smallest such set is $\overline{A}$. Therefore $\overline{A} \subseteq [0, 1]$.
Next we show that $[0, 1] \subseteq \overline{A}$. clearly, $(0, 1) = A \subseteq \overline{A}$
$(1/2, 1/3, \ldots, 1/n \ldots) \to 0$, each term belongs to $\overline{A}$, and $\overline{A}$ is closed, therefore $0 \in \overline{A}$. Similarly, $1 \in \overline{A}$
Watch lecture recording 02/02 10m for more in-depth examples of closure things

- On the real line with the standard metric, $\overline{(a, b)} = [a, b]$
- In $\mathbb{R}^n$ with the Euclidean metric d_2, the closure of the open ball $B(c, r)$ is the closed ball $\{x \in \mathbb{R}^n : d_2(x, c) \leq r\}$
- On the complex plane with its standard metric, the closure of an open disc is the corresponding closed disc
- Let X be a non-empty set with the discrete metric, $c \in X$ and $r = 1$. Then $B(c, 1) = \{c\}$, therefore $\overline{B(c, 1)} - \overline{\{c\}} = \{c\}$, while

$$\{x \in X : d(x, c) \leq 1\} = X$$

  The closure of an open ball is not always equal to the corresponding closed ball
- $X = \mathbb{R}$, $d(x, y) = |x - y|$. $\overline{\mathbb{Q}} = \mathbb{R}$

Definition 2.4.3: Dense Subset of a Metric Space

Let (X, d) be a metric space. A subset D of X is said to be **dense** iff $\overline{D} = X$

Random fact: In $\mathbb{R}^n$ with the Euclidean metric d_2, $\mathbb{Q}^n$ is dense.

Theorem 2.4.4: Closure Equivalence

Let (X, d) be a metric space, $A \subseteq X, x \in X$. The following are equivalent

1. $x \in \overline{A}$
2. For every positive r, $B(x, r) \cap A \neq \emptyset$
3. There exists a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A$ for all n, such that $a_n \to x$

A point x with any of these properties is called an **adherent point** of A. So, $\overline{A}$ is the set of all adherent points of A.

Example: $X = \mathbb{R}$, $d(x, y) = |x - y|$, $A = (0, 1) \cup \{2\}$, $\overline{A} = [0, 1] \cup \{2\}$
2 is an adherent point of A. 0 is an adherent point of A.
Observe: $2 \in A, 0 \notin A$
Proof: $1 \implies 2$
Assume $x \in \overline{A}$. Fix a positive r. We show: $B(x, r) \cap A \neq \emptyset$.
The set $\overline{A} \backslash B(x, r)$ is closed and $\overline{A} \backslash B(x, r) \subsetneq \overline{A}$
Therefore, $A \not\subseteq \overline{A} \backslash B(x, r)$
Therefore there exists an element $a \in A$ s.t. $a \notin \overline{A} \backslash B(x, r)$. But $a \in \overline{A}$. Therefore $a \in B(x, r)$
Proof: $2 \implies 3$
If A intersects every open ball centered at x, then for every n there is a point a_n that belongs to A and to $B(x, 1/n)$. Then $d(a_n, x) < 1/n$, therefore $a_n \to x$
Proof: $3 \implies 1$ Assume that there is a sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \in A$ for all n, and $a_n \to x$. We show that $x \in \overline{A}$.
For each n we have $a_n \in \overline{A}$. Also, $a_n \to x$ and $\overline{A}$ is closed. Therefore $x \in \overline{A}$

Definition 2.4.5: Limit points of sets

Let (X, d) be a metric space, $A \subseteq X$ and $x \in X$. We say that x is a **limit point** or an **accumulation point** of A iff every open ball centered at x contains an element of A distinct from x, i.e.

$$\forall r > 0 \quad (B(x, r) \backslash \{x\}) \cap A \neq \emptyset$$

The set of all limit points of A is called the **derived set** of A and is denoted by A' or $\dot{A}$.

Note w/o proof: x is a limit point of A iff there exists a sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \in A, a_n \neq x$ for all n, and $a_n \to x$
Note w/o proof: Let (X, d) be a metric space and $A \subseteq X$. Then $\overline{A} = A \cup A'$
Example: On the real line with the standard metric, let $A = (0, 1) \cup \{2\}$. Then $\overline{A} = [0, 1] \cup \{2\}$, so $0, 2 \in \overline{A}$ 0 is a limit point of A 2 isn't a limit point of A

2.5 Continuous functions between metric spaces

Definition 2.5.1: Continuity at a point

Let (X, d_X), (Y, d_Y) be metric spaces and $f : X \to Y$ be a function. We say that f is **continuous at a point** x_0 in X iff for for every positive ϵ, there exists a positive δ, s.t., for all $x \in X$ with $d_X(x, x_0) < \delta$ we have $d_Y(f(x), f(x_0)) < \epsilon$

Alternatively, f is **continuous at a point** $x_0 \in X$ iff, for every positive ϵ, there exists a positive δ, such that, for all $x \in B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$

Definition 2.5.2: Continuity of a function

Let $(X, d_X), (Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is said to be **continuous** iff it is continuous at every point in X

Example: Let (X, d) be a metric space and p be a point in X. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, p)$. f is continuous.
Watch lecture recording 02/02 40m for proof

Theorem 2.5.3

Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f : X \to Y$ be a function and x_0 be a point in X. Then f is continuous at x_0 iff for every open neighbourhood G of $f(x_0)$ there exists an open neighbourhood O of x_0 such that, for all $x \in O$, we have $f(x) \in G$

Proof. Assume f is continuous at x_0. Let G be an open set in Y with $f(x_0) \in G$. There exists a positive ϵ such that $B_Y(f(x_0), \epsilon) \subseteq G$. By continuity, there exists a positive δ such that for all $x \in B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$. Let $O = B_X(x_0, \delta)$. For all $x \in O$ we have $f(x) \in G$ □

Conversely, assume that for every open neighbourhood G of $f(x_0)$ there exists an open neighbourhood O of x_0 s.t. for all $x \in O$, we have $f(x) \in G$. We wish to show that f is continuous at x_0
Let ϵ be positive. Apply our hypothesis with $G = B_Y(f(x_0), \epsilon)$ to see that there exists an open set O in X with $x_0 \in O$, s.t. for all $x \in O$ we have $f(x) \in G$.
Since O is open, there exists a positive δ such that $B_X(x_0, \delta) \subseteq O$.
For all x in $B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$

Theorem 2.5.4: Continuity and Convergence

Let (X, d_X), (Y, d_Y) be metric spaces, x_0 be a point in X, and $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous at x_0
2. For every sequence $(x_n)_{n=1}^{\infty}$ in X, if $x_n \xrightarrow[n\to+\infty]{}$ in (X, d_X), then $f(x_n) \xrightarrow[n\to+\infty]{} f(x_0)$ in (Y, d_Y)

Proof. $1 \implies 2$: Assume f is continuous at x_0 and let $x_n \to x_0 \in X$
Let ϵ be positive. There exists a positive δ such that, for all $x \in B_X(x_0, \delta)$, $f(x) \in B_Y(f(x_0), \epsilon)$. Eventually all x_n belong to $B_X(x_0, \delta)$. Therefore eventually all $f(x_n)$ belong to $B_Y(f(x_0), \epsilon)$ $2 \implies 1$: Contrapositive - not $1 \implies$ not 2
Assume that f is not continuous at x_0. Then

$$\text{not } (\forall\epsilon, \exists\delta, \forall x \in B_X(x_0, \delta) \quad f(x) \in B_Y(f(x_0), \epsilon))$$

i.e.

$$\exists\epsilon, \forall\delta, \exists x \in B_X(x_0, \delta) \quad f(x) \notin B_Y(f(x_0), \epsilon)$$

Apply this with $\delta = 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots$ to see that there exists $x_1, x_2, \dots, x_n$, such that

$$x_n \in B_X(x_0, 1/n) \text{ and } f(x_n) \notin B_Y(f(x_0), \epsilon)$$

Then $x_n \to x_0$ in X and $f(x_n) \not\to f(x_0)$ in Y, so, not 2 □

Theorem 2.5.5: Continuity and Open Sets

Let (X, d_X), (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous iff the inverse image $f^{-1}(G)$ of any open subset G of Y is an open subset of X

Proof. Assume f is continuous and let G be an open subset of Y. Let $x_0 \in f^{-1}(G)$. Then $f(x_0) \in G$, therefore there exists a positive ϵ such that $B_Y(f(x_0), \epsilon) \subseteq G$. Since f is continuous at x_0, there exists a positive δ such that, for all $x \in B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$, therefore $f(x) \in G$, therefore $x \in f^{-1}(G)$. This shows that $B_X(x_0, \delta) \subseteq f^{-1}(G)$.
Conversely, assume that the inverse image of every open subset of Y is an open subset of X.
Fix a point $x_0 \in x$. We show that f is continuous at x_0.
Let ϵ be positive. The open ball $B_Y(f(x_0), \epsilon)$ is an open subset of Y, therefore $f^{-1}(B_Y(f(x_0), \epsilon))$ is an open subset of X that contains x_0.
Therefore, there exists a positive δ such that

$$B_X(x_0, \delta) \subseteq f^{-1}(B_Y(f(x_0), \epsilon))$$

For any $x \in B_X(x_0, \delta)$ we have $x \in f^{-1}(B_Y(f(x_0), \epsilon))$, therefore $f(x) \in B(f(x_0), \epsilon)$ □

Exercise: Let $(X, d_X), (Y, d_Y), Z_{d_Z}$ be three metric spaces. Let $f : X \to Y$ and $g : Y \to Z$ be two continuous functions. Then $g \circ f : X \to Z$ is continuous

3 Topology!!!

3.1 Homeomorphisms and Topological Properties

Definition 3.1.1: Topological Space

A **topological space** is a set X together with a family $\mathcal{T}$ of subsets of X that has the following properties:

- $\emptyset, X \in \mathcal{T}$
- Any union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$
- Any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$

$\mathcal{T}$ is called a **topology** and the elements of $\mathcal{T}$ are called **open sets**

Definition 3.1.2: Continuity of Topological Spaces

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. A function $f : X \to Y$ is said to be **continuous** iff for every G in $\mathcal{T}_Y$ the pre-image $f^{-1}(G)$ is an element of $\mathcal{T}_X$.
f is said to be a **homeomorphism** iff it is a continuous bijection and its inverse is continuous.
If such a homeomorphism exists then $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are said to be **homeomorphic**

If $(X, \mathcal{T}_X)$ and (Y, T_Y) are homeomorphic, and one of them is compact or connected or separable etc etc, then so is the other
Properties that are preserved by homeomorphisms are called topological properties

3.2 Just kidding back to metric spaces

Example: Let (X, d_X) be a discrete metric space and (Y, d_Y) be any metric space. Show that every function $f : X \to Y$ is continuous.
Indeed, the inverse image $f^{-1}(G)$ of any open subset G of Y is an open subset of X (all subsets of X are open)
Example: Let $X = \mathbb{R}$ equipped with the standard metric d, and $Y = \mathbb{R}$ equipped with the discrete metric ρ. Show the function $f : X \to Y$, $f(x) = x$ is not continuous.

Proof. The set $\{0\}$ is open in Y, but the set $f^{-1}(\{0\}) = \{0\}$ is not open in X
Actually, for any point $x_0 \in X$, we have $x_0 + \frac{1}{n} \to x_0 \in X$, but

$$f\left(x_0 + \frac{1}{n}\right) = x_0 + \frac{1}{n} \not\to x_0 = f(x_0) \text{ in } Y$$

Therefore, f is not continuous at x_0 □

Watch lecture recording 06/02 for examples of continuous functions

Theorem 3.2.1: $d : X \times X \to \mathbb{R}$ is continuous

Let (X, d) be a metric space. The function $f : X \times X \to \mathbb{R}$ is continuous.
$\mathbb{R}$ is equipped with the standard metric. $X \times X$ is equipped with the product metric

Proof. Fix $(x, x') \in X \times X$. We'll show that d is continuous at (x, x').
Let $(x_n, x'_n) \to (x, x')$ in $(X \times X, D)$. We'll show that

$$d(x_n, x'_n) \to d(x, x') \text{ in } \mathbb{R}$$

By exercise 25, $x_n \to x$ and $x'_n \to x'$ in (X, d). By exercise 26,

$$|d(x_n, x'_n) - d(x, x')| \leq d(x_n, x) + d(x'_n, x') \to 0 + 0 = 0$$

□

Let $X = Y = \mathbb{R}^n$, both equipped with the Euclidean metric d_2.
Let A be an $n \times n$ matrix, and define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = Ax$. Then T is continuous.

Proof. Fix $x_0 \in \mathbb{R}^n$. For all $x \in \mathbb{R}^n$ we have

$$\begin{aligned} d_2(T(x), T(x_0)) &= \|T(x) - T(x_0)\|_2 = \|T(x - x_0)\|_2 \\ &= \|A(x - x_0)\|_2 \leq C\|x - x_0\|_2 = Cd_2(x, x_0) \end{aligned}$$

Where C is a positive constant (independent of x, x_0).
Let $\epsilon > 0$. Define $\delta = \frac{\epsilon}{C}$. for all x with $d_2(x, x_0) < \delta$ we have

$$d_2(T(x), T(x_0)) \leq Cd_2(x, x_0) < C\delta = \epsilon$$

□

We need: For every $n \times n$ matrix A there exists a constant C such that, for all vectors $x \in \mathbb{R}^n$

$$\|Ax\|_2 \leq C\|x\|_2$$

Proof. The i-th component of Ax is $(Ax)_i = \sum_{j=1}^{n} a_{ij}x_j$. By Cauchy-Schwarz,

$$|(Ax)_i|^2 \leq \left(\sum_{j=1}^{n}|a_{ij}|^2\right)\left(\sum_{j=1}^{n}|x_j|^2\right) = \left(\sum_{j=1}^{n}|a_{ij}|^2\right)\|x\|_2^2$$

Summing over i we have

$$\|Ax\|_2^2 = \sum_{i=1}^{n}|(Ax)_i|^2 \leq \underbrace{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)}_{=C^2}\|x\|_2^2$$

□

3.2.2 Continuity of linear operators between normed vector spaces

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed vector spaces. Recall that $d_X : X \times X \to \mathbb{R}$, $d(x, x') = \|x - x'\|_X$, and $d_Y : Y \times Y \to \mathbb{R}$, $d_Y(y, y') = \|y - y'\|_Y$ are metrics

Definition 3.2.3: Bounded Linear Operators

A linear operator $T : X \to Y$ is said to be **bounded** iff there exists a positive constant C such that, for all $x \in X$,

$$\|T(x)\|_Y \leq C\|x\|_X$$

Theorem 3.2.4: Linear Operator Equivalence

Let $T : X \to Y$ be a linear operator. The following are equivalent:

1. T is continuous
2. T is continuous at 0
3. T is bounded

Proof. $1 \implies 2$: Trivial
$2 \implies 3$: Assume that T is continuous at 0. We wish to show:

$$\exists C \forall x \|T(x)\|_Y \leq C\|x\|_X$$

If not, then

$$\forall C, \exists x \|T(x)\|_Y > C\|x\|_X$$

Observe that the x is $\neq 0$. Apply with $C = 1, 2, \ldots$, to see that there exists a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that, for all n,

$$\|T(x_n)\|_Y > n\|x_n\|_X$$

Define $x'_n = \frac{1}{n}\frac{x_n}{\|x_n\|_X}$. Then $d_X(x'_n, 0) = \|x'_n\|_X = \frac{1}{n} \to 0$, therefore, $x'_n \to 0 \in X$, but $T(x'_n) \not\to 0 \in Y$ because $T(x'_n)$ is bigger than 1
$3 \implies 1$: Assume T is bounded. Fix $x_0 \in X$. Let $\epsilon > 0$. Define $\delta = \frac{\epsilon}{C}$. For all x with $d_X(x, x_0) < \delta$ we have

$$\begin{aligned}
d_Y(T(x), T(x_0)) &= \|T(x) - T(x_0)\|_Y \\
&= \|T(x - x_0)\|_Y \\
&\leq C\|x - x_0\|_X \\
&= Cd_X(x, x_0) \\
&< C\delta \\
&= \epsilon
\end{aligned}$$

□

Watch lecture recording 09/02 for proofs on examples:

- Let $(X, \|\cdot\|)$ be a normed vector space and define $f : \mathbb{R} \times X \to X$ by $f(\lambda, x) = \lambda x$. Define $g : X \times X \to X$ by $g(x, y) = x + y$. f and g are continuous

3.3 Fixed Points and Lipschitz

Definition 3.3.1: Lipschitz Functions

Let (X, d_X), (Y, d_Y) be metric spaces. A function $f : X \to Y$ is said to be a **Lipschitz** function iff there exists a constant L such that for all $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq Ld_X(x, x')$$

If $L < 1$, f is said to be a **contraction**

Note: Magnus uses non-standard terminology here:

- When the equation is satisifed and $L < 1$, Magnus calls f a **strict contraction**
- He uses **contraction** for a functino f that satisfies the weaker condition: for all $x, x' \in X$ with $x \neq x'$

$$d_Y(f(x), f(x')) < d_X(x, x')$$

Theorem 3.3.2: Lipschitz Continuity

Every Lipschitz function is continuous

Definition 3.3.3: Fixed Points

A **fixed point** of a function $f : S \to S$ where S is a non-empty set, is any element x of S such that $f(x) = x$
Solving equations can sometimes be reduced to finding fixed points

Watch lecture recording 06/03 for more in-depth examples

- Newton's Method for solving $f(x) = 0$
- Picard's Method for solving the Initial Value Problem

Theorem 3.3.4: Metric Space Unique Fixed Points

Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction. Then f has a unique fixed point

Proof. Let $x_1 \in X$ and define $x_{n+1} = f(x_n)$, $n = 1, 2, \ldots$
$(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Observe first that, for all n,

$$d(x_{n+1}, d + n) = d(f(x_n, f(x_{n-1})) \leq Ld(x_n, x_{n-1}))$$

Therefore, for all n,

$$d(x_{n+1}, x_n) \leq Ld(x_n, x_{n-1}) \leq L^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq L^{n-1} d(x_2, x_1)$$

This goes on for like 10 more lines, watch 09/06 42 min □

3.4 Equivalent Metrics

Definition 3.4.1: Equivalent Metrics

Two metrics on the same non-empty set X are said to be **equivalent** iff they have teh same open sets

Exercise: Let X be a non-empty set and d_1, d_2 be two metrics on X. Prove that d_1 and d_2 are equivalent iff the identity function

$$i : (X, d_1) \to (X, d_2)$$

is a homeomorphism (i.e. i is continuous and its inverse $i^{-1} = i : (X, d_2) \to (X, d_1)$ is continuous)
Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.
Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.
Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.
Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.
Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.
Suspendisse vel felis. Ut lorem lorem, interdum eu, tincidunt sit amet, laoreet vitae, arcu. Aenean faucibus pede eu ante. Praesent enim elit, rutrum at, molestie non, nonummy vel, nisl. Ut lectus eros, malesuada sit amet, fermentum eu, sodales cursus, magna. Donec eu purus. Quisque vehicula, urna sed ultricies auctor, pede lorem egestas dui, et convallis elit erat sed nulla. Donec luctus. Curabitur et nunc. Aliquam dolor odio, commodo pretium, ultricies non, pharetra in, velit. Integer arcu est, nonummy in, fermentum faucibus, egestas vel, odio.
Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.
Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Donec odio elit, dictum in, hendrerit sit amet, egestas sed, leo. Praesent feugiat sapien aliquet odio. Integer vitae justo. Aliquam vestibulum fringilla lorem. Sed neque lectus, consectetuer at, consectetuer sed, eleifend ac, lectus. Nulla facilisi. Pellentesque eget lectus. Proin eu metus. Sed porttitor. In hac habitasse platea dictumst. Suspendisse eu lectus. Ut mi mi, lacinia sit amet, placerat et, mollis vitae, dui. Sed ante tellus, tristique ut, iaculis eu, malesuada ac, dui. Mauris nibh leo, facilisis non, adipiscing quis, ultrices a, dui.
Morbi luctus, wisi viverra faucibus pretium, nibh est placerat odio, nec commodo wisi enim eget quam. Quisque libero justo, consectetuer a, feugiat vitae, porttitor eu, libero. Suspendisse sed mauris vitae elit sollicitudin malesuada. Maecenas ultricies eros sit amet ante. Ut venenatis velit. Maecenas sed mi eget dui varius euismod. Phasellus aliquet volutpat odio. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Pellentesque sit amet pede ac sem eleifend consectetuer. Nullam elementum, urna vel imperdiet sodales, elit ipsum pharetra ligula, ac pretium ante justo a nulla. Curabitur tristique arcu eu metus. Vestibulum lectus. Proin mauris. Proin eu nunc eu urna hendrerit faucibus. Aliquam auctor, pede consequat laoreet varius, eros tellus scelerisque quam, pellentesque hendrerit ipsum dolor sed augue. Nulla nec lacus.
Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia ve-

lit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi.

Sed feugiat. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Ut pellentesque augue sed urna. Vestibulum diam eros, fringilla et, consectetuer eu, nonummy id, sapien. Nullam at lectus. In sagittis ultrices mauris. Curabitur malesuada erat sit amet massa. Fusce blandit. Aliquam erat volutpat. Aliquam euismod. Aenean vel lectus. Nunc imperdiet justo nec dolor.

Etiam euismod. Fusce facilisis lacinia dui. Suspendisse potenti. In mi erat, cursus id, nonummy sed, ullamcorper eget, sapien. Praesent pretium, magna in eleifend egestas, pede pede pretium lorem, quis consectetuer tortor sapien facilisis magna. Mauris quis magna varius nulla scelerisque imperdiet. Aliquam non quam. Aliquam porttitor quam a lacus. Praesent vel arcu ut tortor cursus volutpat. In vitae pede quis diam bibendum placerat. Fusce elementum convallis neque. Sed dolor orci, scelerisque ac, dapibus nec, ultricies ut, mi. Duis nec dui quis leo sagittis commodo.