Galois Theory Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Galois Groups

Definition 1.1.1: Conjugate Numbers

Two complex numbers z and z' are **conjugate over** $\mathbb{R}$ if for all polynomials p with coefficients in $\mathbb{R}$,

$$p(z) = 0 \iff p(z') = 0$$

Lemma 1.1.2: Characterising Conjugates

$z, z' \in \mathbb{C}$ are conjugate over $\mathbb{R}$ iff either $z = z'$ or $\overline{z} = z'$

Definition 1.1.9: Conjugacy in $\mathbb{Q}$

$z, z' \in \mathbb{C}$ are **conjugate over** $\mathbb{Q}$ if $\forall p(t) \in \mathbb{Q}[t]$

$$p(z) = 0 \iff p(z') = 0$$

Definition 1.1.9: Conjugacy for sets

$(z_1, \dots, z_n), z_i, z_i' \in \mathbb{C}$ is conjugate over $\mathbb{Q}$ to $(z_1', \dots, z_n')$ if $\forall p(t_1, \dots, t_n) \in \mathbb{Q}[t_1, \dots, t_n]$

Additionally, if $(z_1, \dots, z_n)$ conjugate to $(z_1', \dots, z_n')$, then z_i is conjugate to z_i' for all i

Definition 1.2.1: Galois Group

Let f be a polynomia lwith coefficients in $\mathbb{Q}$. Write $\alpha_1, \dots, \alpha_k$ for its distinct roots in $\mathbb{C}$. The **Galois group** of f is

$$\text{Gal}(g) = \{\sigma \in S_n \mid (\alpha_1, \dots, \alpha_n) \text{ conjugate to } (\alpha_{S(1)}, \dots, \alpha_{\sigma(n)})\}$$

Note: distinct roots mean that we ignore any repetition of roots.

Definition 1.3.0: Solvability (Simple Definition)

A complex number is **radical** if it can be obtained from the rationals using only the usual arithmetic operations and kth roots. A polynomial over $\mathbb{Q}$ is **solvable (or soluble) by radicals** if all of its complex roots are radical.

Theorem 1.3.5: Galois

Let f be a polynomial over $\mathbb{Q}$. Then

$$f \text{ is solvable by radicals} \iff \text{Gal}(f) \text{ is a solvable group.}$$

2 Groups, Rings, and Fields

Definition 2.1.1: Group Action

Let G be a group and X a set. An **action** of G on X is a function $G \times X \to X$, written as $(g, x) \mapsto gx$ such that

$$(gh)x = g(hx)$$

for all $g, h \in G$ and $x \in X$ and

$$1x = x$$

for all $x \in X$, where 1 is the identity of G

Definition 2.1.7: Faithful Actions

An action of a group G on a set X is **faithful** if for $g, h \in G$,

$$gx = hx \text{ for all } x \in X \implies g = h$$

Faithfulness means that if two elements of the group *do* the same, they *are* the same.

Lemma 2.1.8: Faithful Properties

For an action of a group G on a set X, the following are equivalent:

1. The action is faithful
2. For $g \in G$, if $gx = x$ for all $x \in X$ then $g = 1$
3. The homomorphism $\Sigma : G \to \text{Sym}(X)$ is injective
4. $\ker \Sigma$ is trivial.

Lemma 2.1.11: Isomorphisms of Faithful Groups

Let G be a group acting faithfully on a set X. then G is isomorphic to the subgroup

$$\operatorname{im} \Sigma = \{\overline{g} \mid g \in G\}$$

of $\text{Sym}(X)$, where $\Sigma : G \to \text{Sym}(X)$ and $\overline{g}$ are defined as above.

Definition 2.1.1: Fixed Set

Let G be a group acting on a set X. Let $S \subseteq G$. The **fixed set** of S is

$$\text{Fix}(S) = \{x \in X \mid sx = x \text{ for all } s \in S\}$$

Lemma 2.1.15: Normal Fixed Sets

Let G be a group acting on a set X, let $S \subseteq G$, and let $g \in G$. Then $\text{Fix}(gSg^{-1}) = g\,\text{Fix}(S)$.
Here, $gSg^{-1} = \{gsg^{-1} \mid s \in S\}$ and $g\,\text{Fix}(S) = \{gx \mid x \in \text{Fix}(S)\}$

Definition 2.2.1: Ring Homomorphism

Given rings R and S, a **homomorphism** from R to S is a function $\phi : R \to S$ satisfying the following equations for all $r, r' \in R$:

- $\phi(r + r') = \phi(r) + \phi(r')$
- $\phi(rr') = \phi(r)\phi(r')$
- $\phi(0) = 0$, $\phi(1) = 1$
- $\phi(-r) = -\phi(r)$

A **subring** of a ring R is a subset $S \subseteq R$ that contains 0 and 1 and is closed under addition, multiplication, and negatives. Whenever S is a subring of R, the inclusion $\iota : S \to R$ (defined by $\iota(s) = s$) is a homomorphism.

Lemma 2.2.3: Intersection of Subrings

Let R be a ring and let $\mathcal{S}$ be any set (perhaps infinite) of subrings of R. Then their intersection $\bigcap_{S \in \mathcal{S}} S$ is also a subring of R.

Recall 2.0.1: Ideals and Quotient Rings

Let R be a ring. $I \subseteq R$ is an **ideal**, $I \trianglelefteq R$, if the following hold:

1. $I \neq \emptyset$
2. I is closed under subtraction
3. for all $i \in I$ and $r \in R$ we have $ri, ir \in I$

Every ring homomorphism $\phi : R \to S$ has an image $\operatorname{im} \phi$, which is a subring of S, and a kernel $\ker \phi$, which is an ideal of R.

Given an ideal $I \trianglelefteq R$, we obtain the quotient ring R/I and the canonical homomorphism $\pi_I : R \to R/I$ which is surjective and hs kernel I.
Universal Prop: Given any ring S and any homomorphism $\phi : R \to S$ satisfying $\ker \phi \supseteq I$, there is exactly one homomorphism $\overline{\phi} : R/I \to S$ such that this diagram commununtes.

$$\begin{tikzcd} R \arrow[d, "\pi_I"'] \arrow[dr, "\phi"] & \\ R/I \arrow[r, "\overline{\phi}"'] & S \end{tikzcd}$$

Recall 2.0.2: Integral Domain

An **integral domain** is a ring R such that $0_R \neq 1_R$ and for $r, r' \in R$,

$$rr' = 0 \implies r = 0 \text{ or } r' = 0$$

Recall 2.0.3: Generated Ideal

Let Y be a subset of a ring R. The **ideal** $\langle Y \rangle$ **generated by** Y is defined as the intersection of all the ideals of R containing Y.

- Ideals of the form $\langle r \rangle$ are called **principal ideals**. A **principle ideal domain** is an integral domain where every ideal is principal.
- Let r and s be elements of a ring R. We say that r **divides** s, and write $r \mid s$ if there exists $a \in R$ such that $s = ar$. This condition is equivalent to $s \in \langle r \rangle$, and to $\langle s \rangle \supseteq \langle r \rangle$.
- An element $u \in R$ is a **unit** if it has a multiplicative inverse, or equivalently, if $\langle u \rangle = R$. The units form a group $R^{\times}$ under multiplication.
- Elements r and s of a ring are **coprime** if for $a \in R$,
$$a \mid r \text{ and } a \mid s \implies a \text{ is a unit}$$

Lemma 2.2.11: Characterisation of Generated Ideals

Let R be a ring and let $Y = \{r_1, \ldots, r_n\}$ be a finite subset. Then
$$\langle Y \rangle = \{a_1 r_1 + \cdots + a_n r_n : a_1, \ldots, a_n \in R\}$$

Proposition 2.2.16: Coprime and PIDs

Let R be a principal ideal domain and $r, s \in R$. Then
$$r \text{ and } s \text{ are coprime} \iff ar + bs = 1 \text{ for some } a, b \in R$$

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