Metric Spaces Exam Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Introduction to Metric Spaces

Definition 1: Definition of a Metric

Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** iff for all $x,\ y,\ z \in X$,

- $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

A non-empty set X equipped with a metric d is a **metric space**

Definition A: Real Vector Spaces

A **real vector space** V is a set with two operations $(X, +, \cdot)$, where:

- $+$ is addition, and $\cdot$ is scalar multiplication
- $(X, +)$ is an abelian group - i.e. for all (vectors) $x, y, z \in X$:
  - **Closure**: $x + y \in X$
  - **Commutativity**: $x + y = y + x$
  - **Associativity**: $x + (y + z) = (x + y) + z$
  - **Identity**: $\exists 0 \in X$ s.t. for all $x \in X$ we have $0 + x = x + 0 = x$
  - **Inverse**: $\forall x \in X$ we have $-x$ s.t. $x + (-x) = (-x) + x = 0$
- Vector space axioms: for all $x, y, z \in X$ and $\mu, \lambda \in \mathbb{R}$ we have:
  - **Closure-ish thing**: $\lambda x \in X$
  - **Distributivity 1**: $\lambda(x + y) = \lambda x + \lambda y$
  - **Distributivity 2**:$(\lambda + \mu)x = \lambda y + \mu x$
  - **Associativity**: $\lambda(\mu x) = (\lambda\mu)x$
  - **Identity**: $1x = x$

Definition B: Normed and Inner Product Spaces

Def 5 (Normed Vector Spaces)

A **normed vector space** is a real vector space X equipped with a **norm**, i.e. a function that assigns to every vector $x \in X$ a real number $\|x\|$ so that, for all vectors x and y in X and all real scalars a:

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- $\|ax\| = |a|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

Remark: If $(X, \|\cdot\|)$ is a normed vector space then

$$d(x, y) = \|x - y\|$$

defines a metric in X

Def 6 (Inner Product Spaces)

Let X be a real vector space. An **inner product** on X is a function that assigns to every pair $(x, y) \in X \times X$ a real number denoted by $\langle x, y\rangle$ and has the following properties:

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $ax + by, z = a\langle x, z\rangle + b\langle y, z\rangle$

A **real inner product space** is a real vector space equipped with an inner product. If $\langle\cdot,\cdot\rangle$ is an inner product on X, then

- $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm in X
- $d(x, y) = \|x - y\|$ defines a metric in X

Definition C: n-dimensional Euclidean space

Let $X = \mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1, x_2, \dots, x_n \in \mathbb{R}\}$
For $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$, define

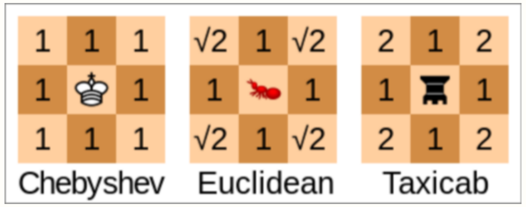
$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \text{ (inner product)}$$

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \sqrt{x_1^2 + x_2^2 + \cdots + c_n^2} \text{(norm)}$$

Example D: Examples of Metric Spaces

Unless stated otherwise let $X = \mathbb{R}^n$. The case $X = \mathbb{R}^2$ is listed in red

Name	Norm and Metric
Standard	$X = \mathbb{R}$ and $\lvert x\rvert$ = Absolute Value $d(x, y) = \lvert x - y\rvert$
Taxicab	$\lVert x\rVert_1 = \lvert x_1\rvert + \lvert x_2\rvert + \cdots \lvert x_n\rvert$ $d_1(x, y) = \lvert x_1 - y_1\rvert + \lvert x_2 - y_2\rvert + \cdots + \lvert x_n - y_n\rvert$
Euclidean	$\lVert x\rVert_2 = \sqrt{\lvert x_1\rvert^2 + \lvert x_2\rvert^2 + \cdots \lvert x_n\rvert^2}$ $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$
p-metric	$\lVert x\rVert_p = \left(\sum_{k=1}^{n} \lvert x_k\rvert^p\right)^{1/p}$ $d_p(x, y) = \left(\sum_{k=1}^{n} \lvert x_k - y_k\rvert^p\right)^{1/p}$
Chebyshev	$\lVert x\rVert_\infty = \max\{\lvert x_1\rvert, \lvert x_2\rvert, \dots, \lvert x_n\rvert\}$ $d(x, y) = \max\{\lvert x_1 - y_1\rvert, \lvert x_2 - y_2\rvert, \dots, \lvert x_n - y_n\rvert\}$
Discrete	$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$
Post Office	$d(x, y) = \begin{cases} \lVert x\rVert_2 + \lVert y\rVert_2 & x \neq y \\ 0 & x = y \end{cases}$



The complex plane

Let $X = \mathbb{C}$, $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$

$$d(z, w) = |z - w|$$

If $z = a + ib, w = c + id,\ a, b, c, d \in \mathbb{R}$, then

$$z - w = (a - c) + i(b - d)$$

therefore,

$$d(z, w) = \sqrt{(a - c)^2 + (b - d)^2}$$

Example E: Sequence Spaces

The space ℓ^1

ℓ^1 is the set of real sequences $(x_n)_{n\in\mathbb{N}}$ where $\sum_{n=1}^{\infty}|x_n|$ converges.
For $x = (x_1, \dots, x_n, \dots) \in \ell^1$, $y = (y_1, \dots, y_n, \dots) \in \ell^1$ we define

- **Norm**: $\|x\|_1 = \sum_{n=1}^{\infty}|x_n|$
- **Metric**: $d_1(x, y) = \|x - y\|_1 = \sum_{n=1}^{\infty}|x_n - y_n|$

The space ℓ^2

ℓ^2 is the set of real seqs $(x_n)_{n\in N}$ where $\sum_{n=1}^{\infty}|x_n|^2$ converges
For $x = (x_1, \dots, x_n, \dots) \in \ell^2$, $y = (y_1, \dots, y_n, \dots) \in \ell^2$ we define

- **Inner product**: $\langle x, y\rangle = \sum_{n=1}^{\infty} x_ny_n$
- **Norm**: $\|x\|_2 = \left(\sum_{n=1}^{\infty}|x_n|^2\right)^{1/2}$
- **Metric**: $d_2(x, y) = \|x - y\|_2 = \left(\sum_{n=1}^{\infty}|x_n - y_n|^2\right)^{1/2}$

Thm: ℓ^2 is a real vector space

The space ℓ^∞

ℓ^∞ is the set of all bounded sequences of real numbers For $x = (x_1, \dots, x_n, \dots)$, $y = (y_1, \dots, y_n, \dots) \in \ell^\infty$

- **Norm**: $\|x\|_\infty = \sup\{|x_1|, \dots, |x_n|, \dots\}$
- **Metric**: $\|x - y\|_\infty = \sup\{|x_1 - y_1|, \dots, |x_n - y_n|, \dots\}$

The space $C([a, b])$

$X = C([a, b])$ is the set of all continuous functions $f : [a, b] \to \mathbb{R}$

- **Norm**: $\|f\|_\infty = \max\{|f(x)| : a \leq x \leq b\}$
- **Metric**: $d_\infty(f, g) = \|f - g\| = \max\{|f(x) - g(x)| : a \leq x \leq b\}$

The L^1 metric

X is the set of all continuous functions $f : [a, b] \to \mathbb{R}$

- **Norm**: $\|f\|_1 = \int_a^b|f(x)|dx$
- **Metric**: $d_2(f, g) = \|f - g\|_1 = \int_a^b|f(x) - g(x)|dx$

The L^2 metric

X is the set of all continuous functions $f : [a, b] \to \mathbb{R}$

- **Inner Product**: $\langle f, g\rangle = \int_a^b f(x)g(x)dx$
- **Norm**: $\|f\|_2 = \langle f, f\rangle^{1/2} = \left(\int_a^b|f(x)|^2dx\right)^{1/2}$
- **Metric**: $d_1(f, g) = \left(\int_a^b|f(x) - g(x)|^2dx\right)^{1/2}$

Definition F: Metric Subspaces

Ex 7: Let (X, d) be a metric space and Y a non-empty subset of X. Define

- $d_Y : Y \times Y \to \mathbb{R}$
- $d_y(y, y') = d(y, y')$

Then d_Y is a metric on Y. d_Y is called the **induced** or **inherited** metric, and (Y, d_Y) is said to be a metric subspace of the metric space (X, d)

Theorem G: a lack of equality or fair treatment in t...

Good old fashioned Triangle Inequality

If it ain't broke...

$|x+y| \leq |x| + |y| \quad |x-y| \geq \big||x|-|y|\big| \quad |x-y| \leq |x-z| + |z-y|$

Cauchy-Schwarz Inequality

For all x and y of an inner product space:

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

Minkowski's Inequality

Let $p \geq 1$, and real numbers x_i, y_i, $(i = 1, \ldots, n)$. Then

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}$$

Ex 56 (Young's Inequality)

Let $1 \leq p, q \leq \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$, and $a, b \leq 0$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Thm 169 (Hölder Inequality)

Let $1 \leq p, q \leq \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$ and $x, y \in \mathbb{R}^n$. Then

$$\sum_{i=1}^n |x_i y_i| \leq \|x\|_p \|y\|_q$$

Definition 166: Equivalent Norms

Two norms on the same real vector space are said to be equivalent iff their corresponding metrics are equivalent

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Thm 167: If $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on the same real vector space X and there exist positive constants C and C' s.t., for all $x \in X$,

$$D\|x\|_1 \leq \|x\|_2 \leq C'\|x\|_1$$

then they are equivalent

Equivalence Theorems of p-metrics

171: Any of the following norms are equivalent:

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p},\ x \in \mathbb{R}^n,\ 1 \leq p < \infty$$
$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\},\ x \in \mathbb{R}^n$$

172: Let $1 \leq p \leq q < \infty$. For all $x \in \mathbb{R}^n$:

$$\|x\|_q \leq \|x\|_p$$

As a consequence,

$$\|x\|_\infty \leq \|x\|_q \leq \|x\|_p \leq \|x\|_1$$

173: All norms in $\mathbb{R}^n$ are equivalent

Definition 8: Open Ball

Let (X, d) be a metric space, c be a point in X, and $r > 0$. The **open ball** with center c and radius r is defined by

$$B(c, r) = \{x \in X : d(c, x) < r\}$$

2 Convergence

Definition 15: Convergent Sequence

Let (X, d) be a metric space, $(x_n)_{n=1}^\infty$ be a sequence in X, and $x \in X$. We say that $(x_n)_{n=1}^\infty$ converges to x iff for every $\epsilon > 0$, there exists an index N s.t. for all $n \geq N$ we have $d(x_n, x) < \epsilon$.

Observe that:

- $d(x_n, x) < \epsilon$ is equivalent to $x_n \in B(x, \epsilon)$.
- $x_n \to x$ in (X, d) iff $d(x_n, x) \to 0$ on the real line

Theorem 16: Uniqueness of metric limit

- Let (X, d) be a metric space, and $x, x' \in X$, $x \neq x'$. Then there exists a positive radius r s.t. $B(x, r) \cap B(x', r) = \emptyset$
- A sequence in a metric space can have at most one limit

Definition 19: Bounded Sequence

A sequence in a metric space is said to be **bounded** iff there exists an open ball that contains all of its terms

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Note: this is the same definition as "sequence is bounded if there is upper and lower bound", as open ball implies the same thing

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Thm 20: Every convergent sequence is bounded

Definition 21: Cauchy Sequence

A sequence $(x_n)_{n=1}^\infty$ in a metric space (X, d) is said to be a **Cauchy sequence** iff for every positive ϵ, there exists an index N, s.t. for all indices n, m with $n, m \geq N$,

$$d(x_n, x_m) < \epsilon$$

Thm 22: If a sequence in a metric space converges, then it is a Cauchy sequence. **Note**: the converse is not true

Definition 24: Complete Metric Spaces

A metric space is said to be **complete** if and only if every Cauchy Sequence is convergent

Example 25: Examples of Complete Metric Spaces

- $\mathbb{R}$ with the standard metric is complete
- $\mathbb{Q}$ with the standard metric is not complete
- $(0, 1)$ with the standard metric is not complete
- $[0, 1]$ with the standard metric is complete
- $\mathbb{R}^n$, ℓ^p, $C([a, b])$ is complete (proof later)

Definition 26: Open Sets and Closed Sets

Let (X, d) be a metric space.

- A subset G of X is said to be **open** iff for every point x in G there exists a positive radius r such that $B(x, r) \subseteq G$.
- A subset F of X is said to be **closed** iff F^c is open

Definition 31: Discrete Spaces and Clopens

A metric space is called **discrete** iff all its subsets are open (equiv. all subsets are closed)

Example: $[0, 1] \cap (2, 3)$

Def 33: A set that is both open and closed is called **clopen**

Theorem 34: Properties of open and closed sets

Let (X, d) be a metric space

1. The union of **any family** of open sets is an open set
2. The intersection of **finitely many** open sets is an open set
3. The intersection of **any family** of closed sets is an closed set
4. The union of **finitely many** closed sets is an closed set

Remark 35: Infinite open sets

The intersection of infinitely many open sets isn't always an open set e.g., let $G_n = (-\frac{1}{n}, \frac{1}{n}), n = 1, 2, \ldots$ on $\mathbb{R}$ with the standard metric. Each G_n is open but

$$\bigcap_{n=1}^\infty G_n = \{0\}$$

Theorem 18: Relatively open sets

Let (X, d) be a metric space and A a nonempty subset of X equipped with the induced metric d_A. Let $G \subseteq A$. Then G is open in (A, d_A) iff there exists a subset O of X, open in (X, d), s.t. $G = A \cap O$

The open sets of (A, d_A) are referred to as **relatively open**

Theorem 36

Let (X, d) be a metric space, $(x_n)_{n=1}^\infty$ be a sequence in X and x be a point in X.

$x_n \to x$ iff every open set that contains x contains eventually all terms of the sequence

Definition H: Neighbourhoods of points

An **open neighbourhood** of a point x is any open set that has x.

$x_n \to x$ iff every open neighbourhood of x contains eventually all terms of the sequence.

A **neighbourhood** of a point x is a set that contains an open neighbourhood of x.

$x_n \to x$ iff every neighbourhood of x contains eventually all terms of the sequence.

Remark 38: Infinite closed sets

The union of infinitely many closed sets is not always a closed set. For example, let $F_n = [\frac{1}{n}, 1], n = 1, 2, \ldots$, on the real line with the standard metric. Each F_n is closed but

$$\bigcup_{n=1}^\infty F_n = (0, 1]$$

is not closed.

Theorem 41

A subset F of a metric space is closed iff the limit of every convergent sequence of elements of F belongs to F

- In any metric space (X,d), singletons $F=\{x\}$ are closed.
- In any metric space, any finite set is closed because

$$\{x_1,\dots,x_n\}=\{x_1\}\cup\cdots\cup\{x_n\}$$

Definition 43: Closure

Let (X,d) be a metric space and $A\subseteq X$. The **closure** of A, deonted by $\overline{A}$, is the smallest closed subset of X that contains A

There exists at least one closed subset of X that contains A, namely X itself. The smallest closed subset of X that contains A is

$$\bigcap_{\substack{A\subseteq F\subseteq X\\ F\text{closed}}} F$$

Theorem 44: Properties of Closure

Let (X,d) be a metric space and $A,\ B\subseteq X$.

1. $\overline{\emptyset}=\emptyset$ and $\overline{X}=X$
2. $A\subseteq\overline{A}$ and $\overline{A}$ is closed
3. A is closed iff $A=\overline{A}$
4. $\overline{\overline{A}}=\overline{A}$
5. If $A\subseteq B$, then $\overline{A}\subseteq\overline{B}$
6. $\overline{A\cup B}=\overline{A}\cup\overline{B}$

Definition 49: Dense Subset of a Metric Space

Let (X,d) be a metric space. A subset $D\subseteq X$ is **dense** iff $\overline{D}=X$

Random Fact: In $\mathbb{R}^n$ with the Euclidean metric d_2, $\mathbb{Q}^n$ is dense.

Theorem 50: Adherent Points

Let (X,d) be a metric space, $A\subseteq X, x\in X$. The following are equiv.

1. $x\in\overline{A}$
2. For every positive r, $B(x,r)\cap A\neq\emptyset$
3. There exists a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n\in A$ for all n, such that $a_n\to x$

A point x with any of these properties is called an **adherent point** of A. So, $\overline{A}$ is the set of all adherent points of A.

Definition 52: Limit points of sets

Let (X,d) be a metric space, $A\subseteq X$ and $x\in X$. We say that x is a **limit point** or an **accumulation point** of A iff every open ball centered at x contains an element of A distinct from x, i.e.

$$\forall r>0\quad (B(x,r)\backslash\{x\})\cap A\neq\emptyset$$

The set of all limit points of A is called the **derived set** of A and is denoted by A' or $\tilde{A}$.

Thm 78: Let (X,d_X) and (Y,d_Y) be metric spaces, x_0 be a limit point of X, $y_0\in Y$ and $f:X\to Y$ be a function.

We say that $\lim_{x\to x_0}f(x)=y_0$ iff for all $\epsilon>0$, there exists $\delta>0$ such that for all $x\in B_X(x_0,\delta)\backslash\{x_0\}$ we have

$$f(x)\in B_Y(y_0,\epsilon)$$

Definition 54: Continuity at a point

Let (X,d_X), (Y,d_Y) be metric spaces and $f:X\to Y$ be a function. We say that f is **continuous at a point** x_0 in X iff...

- for every $\epsilon>0$, there exists a $\delta>0$, such that, for all $x\in X$ with $d_X(x,x_0)<\delta$ we have

$$d_Y(f(x),f(x_0))<\epsilon$$

- for every $\epsilon>0$, there exists a $\delta>0$, such that, for all $x\in B_X(x_0,\delta)$ we have

$$f(x)\in B_Y(f(x_0),\epsilon)$$

- **Thm 57**: for every open nbhd G of $f(x_0)$, there exists an open nbhd O of x_0 such that, for all $x\in O$, we have $f(x)\in G$

Def 55 (Continuity of a Function)

Let $(X,d_X),(Y,d_Y)$ be metric spaces. A function $f:X\to Y$ is said to be **continuous** iff it is continuous at every point in X

Theorem 58: Continuity and Convergence

Let (X,d_X), (Y,d_Y) be metric spaces, x_0 be a point in X, and $f:X\to Y$ be a function. The following are equivalent:

1. f is continuous at x_0
2. For every sequence $(x_n)_{n=1}^{\infty}$ in X, if $x_n\xrightarrow[n\to+\infty]{}$ in (X,d_X), then $f(x_n)\xrightarrow[n\to+\infty]{}f(x_0)$ in (Y,d_Y)

Theorem 59: Continuity and Open Sets

Let (X,d_X), (Y,d_Y) be metric spaces. A function $f:X\to Y$ is continuous iff the inverse image $f^{-1}(G)$ of any open subset G of Y is an open subset of X

Definition 60: Topological Space

A **topological space** is a set X together with a family $\mathcal{T}$ of subsets of X that has the following properties:

- $\emptyset, X\in\mathcal{T}$
- Any union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$
- Any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$

$\mathcal{T}$ is called a **topology** and the elements of $\mathcal{T}$ are called **open sets**

Definition 61: Continuity of Topological Spaces

- Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be two topological spaces. A function $f:X\to Y$ is said to be **continuous** iff for every G in $\mathcal{T}_Y$ the pre-image $f^{-1}(G)$ is an element of $\mathcal{T}_X$.
- f is said to be a **homeomorphism** iff it is a continuous bijection and its inverse is continuous.
- If such a homeomorphism exists then $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ are said to be **homeomorphic**

Theorem 66: $d:X\times X\to\mathbb{R}$ is continuous

Let (X,d) be a metric space. $f:X\times X\to\mathbb{R}$ is continuous, where

- $\mathbb{R}$ is equipped with the standard metric.
- $X\times X$ is equipped with the product metric

Definition 67: Bounded Linear Operators

A linear operator $T:X\to Y$ is said to be **bounded** iff there exists a positive constant C such that, for all $x\in X$,

$$\|T(x)\|_Y\leq C\|x\|_X$$

Thm 68: Let $T:X\to Y$ be a linear operator. The following are equivalent:

1. T is continuous
2. T is continu. at 0
3. T is bounded

Definition 70: Lipschitz Functions

Let (X,d_X), (Y,d_Y) be metric spaces. A function $f:X\to Y$ is said to be a **Lipschitz** function iff there exists a constant L such that for all $x,x'\in X$,

$$d_Y(f(x),f(x'))\leq Ld_X(x,x')$$

If $L<1$, f is said to be a **contraction**

If $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function and x is any point in $\mathbb{R}$, then for any $x\in\mathbb{R}$ we have

$$|f(x)-f(x')|\leq L|x-x'|$$

For $x\geq x'$ this can be expanded to

$$f(x')-L(x-x')\leq f(x)\leq f(x')+L(x-x')$$

Lipschitz Theorem Bank

71: Every Lipschitz function is continuous

175: Let (X,d_X) and (Y,d_Y) be two metric spaces, and $f:X\to Y$ be a Lipschitz function. Then there exists a smallest Lipschitz constant of f

176: Let I be a non-degenerate open interval on the real line and let $f:I\to\mathbb{R}$ be a differentiable function. Then f is Lipschitz iff f' is bounded. When that is the case,

$$|f|_{\text{Lip}}=\sup\{|f'(x)|:x\in I\}$$

Definition 72: Fixed Points

A **fixed point** of a function $f:S\to S$ where S is a non-empty set, is any element x of S such that $f(x)=x$

Solving equations can sometimes be reduced to finding fixed points

Theorem 75: Banach's Fixed Point Theorem

Let (X,d) be a complete metric space and let $f:X\to X$ be a contraction. Then f has a unique fixed point

Definition 76: Equivalent Metrics

Two metrics on the same non-empty set X are said to be **equivalent** iff they have the same open sets

Thm 77: Let d_1 and d_2 be metrics on the same non-empty set X. If there exist positive constants C and C' such that for all x,y in X,

$$Cd_1(x,y)\leq d_2(x,y)\leq C'd_1(x,y)$$

then d_1 and d_2 are equivalent

3 Completeness

Theorem I: Completeness of the Classical Spaces

Some examples of complete metric spaces:

79: $(\mathbb{R}^n, d_2)$	**80**: ℓ^2	**81**: ℓ^p	**82**: $C([a,b])$	**83**: ℓ^∞

Exercise 31

- Let (X, d_X) and (Y, d_Y) be two metric spaces and assume that (Y, d_Y) is complete.
- Let $C(X,Y)$ be the set of all continuous and bounded functions from X to Y. For $f, g \in C(X,Y)$ define

$$D(f,g) = \sup\{d_Y(f(x), g(x)) : x \in X\}$$

- Then D is a metric and the metric space $(C(X,Y), D)$ is complete

Definition 83: The product space X^n

Let (X,d) be a metric space and $n \in \mathbb{N}$. Define $D : X^n \to \mathbb{R}$ by

$$D(x_1, x_2) = d(x_{11}, x_{21}) + d(x_{12}, x_{22}) + \cdots + d(x_{1n}, x_{2n})$$

Lemma Bank

Ex.33: D is a metric and a sequence converges in (X^n, D) iff it converges componentwise

Ex.34: If (X,d) is complete then (X^n, D) is complete

Definition 84: The product space $X^{\mathbb{N}}$

Let B^A, where A, B are sets, be the set of all functions from A to B

Def 85: Let (X,d) be a metric space. Define a metric $D : X^{\mathbb{N}} \times X^{\mathbb{N}} \to \mathbb{R}$ by

$$D(x_1, x_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d(x_{1n}, x_{2n})}{1 + d(x_{1n}, x_{2n})}$$

- $x_1 = (x_{11}, \ldots, x_{1n}, \ldots)$, $x_2 = (x_{21}, \ldots, x_{2n}, \ldots)$
- $(X^{\mathbb{N}}, D)$ is called a **product space**

Theorem J: Product space Convergence & Completeness

Thm 86 (Convergence)

Let (X,d) be a metric space, let $(x_k)_{k=1}^\infty$ be a sequence in $X^{\mathbb{N}}$ and let $x \in X^{\mathbb{N}}$. Write $x_k = (x_{k1}, \ldots, x_{kn}, \ldots)$ and $x = (l_1, \ldots, l_n, \ldots)$.

Then, $x_k \xrightarrow[k\to+\infty]{(X^{\mathbb{N}}, D)} x$ if and only if, for all n, $x_{kn} \xrightarrow[k\to+\infty]{(X^{\mathbb{N}}} l_n$

Thm 87 (Completeness)

Let (X,d) be a complete metric space. Then the product space $(X^{\mathbb{N}}, D)$ is complete.

Theorem K: Completeness of $\mathbb{R}$

- **Thm (Least Upper Bound Principle)**: Every non-empty bounded above subset of $\mathbb{R}$ has a least upper bound
- **Thm 88 (Monotone Convergence)**: Every bounded monotone sequence of real numbers has a limit
- **Thm/Ex. 36 (ϵ-convergence)**: Let A be a non-empty boudned subset of $\mathbb{R}$ and let ϵ be positive. If the distance between any two elements of A is $< \epsilon$, then

$$\sup(A) - \inf(A) \leq \epsilon$$

- **Thm 89**: Every Cauchy sequence of real numbers is convergent

Definition L: Limit Superior and Inferior

Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathbb{R}$. Then $(x_n)_{n=1}^\infty$ is bounded. Define:

$$I_n = \inf\{x_n, x_{n+1}, \ldots\} \quad S_n = \sup\{x_n, x_{n+1}, \ldots\}$$

Thm: $(S_n)_{n=1}^\infty$ and $(I_n)_{n=1}^\infty$ are monotone and bounded

$$I_1 \leq I_n \leq S_n \leq S_1, \quad n = 1, 2, \ldots$$

Therefore $I_n \to I$ and $S_n \to S$ for some reals I and S. Since $S_n - I_n \to 0$ we have $S = I$. We also have $x_n \to S = I$

Def 90: Limsup and Liminf

- The limit of the sequence $(I_n)_{n=1}^\infty$ is called the **limit inferior** of $(x_n)_{n=1}^\infty$ and is denoted by $\liminf x_n$

$$\liminf x_n = \lim_{n\to+\infty} I_n = \lim_{n\to+\infty} \inf\{x_n, x_{n+1}, \ldots\}$$

- The limit of the sequence $(S_n)_{n=1}^\infty$ is called the **limit superior** of $(x_n)_{n=1}^\infty$ and is denoted by $\limsup x_n$

$$\limsup x_n = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \sup\{x_n, x_{n+1}, \ldots\}$$

- $\liminf x_n$ is the smallest subsequential limit of $(x_n)_{n=1}^\infty$
- $\limsup x_n$ is the largest subsequential limit of $(x_n)_{n=1}^\infty$
- $(x_n)_{n=1}^\infty$ converges iff $\liminf x_n = \limsup x_n$

4 Compactness

Definition 96: Open Covers and Subcovers

An **open cover** of a set S in a metric space is a family $(G_i)_{i\in I}$ of open sets such that $S \subset \bigcup_{i \in I} G_i$. A **subcover** of an open cover $(G_i)_{i\in I}$ is a sub-family $(G_i)_{i\in I'}$ where $I' \subset I$, such that $S \subseteq \bigcup_{i\in I'} G_i$

Definition M: Compacting Compactness

Def 91 (Compactness)

Let $X = \mathbb{R}$ and d be the standard metric. A subset K of $\mathbb{R}$ is said to be **compact** iff every sequence of elements of K has a subsequence that converges to an element of K

Def 102 (Sequential Compactness)

1. K is **sequentially compact** iff every sequence in K has a subsequence that converges to an element of K

   For the case $K = X$ it's just the definition (1) defined above
2. K is **compact** iff every open cover of K has a finite subcover

Def 111 (Uniform Continuity)

3. Let $(X, d_X), (Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is said to be **uniformly continuous** iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that, for all x, x' in X with $d_X(x, x') < \delta$ we have

$$d_Y(f(x), f(x')) < \epsilon$$

Def 117 (Totally bounded Spaces)

117: A metric space (X,d) is said to be **totally bounded** iff for every positive δ, X can be covered by a finite number of open balls of radius δ.

118: If (X,d) is totally bounded then it is bounded, but the converse is not necessarily true

Example N: Examples of compactness

Compact sets	Not Compact sets
$[a,b]$ is compact	$(0,1)$ is not compact
$\emptyset$ is compact	$\mathbb{R}$ is not compact
$\mathbb{R} \cup \{-\infty, +\infty\}$ is compact!	

Theorem 116: Lebesgue's Lemma

Let (X,d) be a sequentially compact metric space and $X = \bigcup_{i\in I} G_i$ be an open cover of X. There exists a $\delta > 0$ such that for any two points $x, y \in X$ with $d(x,y) < \delta$ there exists an i such that $x, y \in G_i$. Any such δ is called a **Lebesgue number** of the open cover

Ex.44: Let (X,d) be a sequentially compact m.s. and $X = \bigcup_{i\in I} G_i$ be an open cover of X. Then there exists a $\delta > 0$ s.t. any nonempty subset of X of diameter $< \delta$ can be covered by a single G_i

Theorem O: big theorem bank of obvious shit

Regular Compactness

- For a set K in $\mathbb{R}$ with the standard metric:
  - **93**: K is compact $\iff$ K is closed and bounded
  - **100**: K is compact $\iff$ every open cover of K has a finite cover
- For a set K in $\mathbb{R}^n$ with the Euclidean metric:

Ex.38: K is compact $\iff$ K is closed and bounded

- For a set K in $\mathbb{R}$
  - **101**: Every open cover of K has a finite subcover $\implies$ K is closed and bounded $\implies$ K is compact

99: Every open cover of the interval $[a,b]$, where $a, b \in \mathbb{R}$, $a \leq b$ has a finite subcover

Continuous Functions

- Let $K \subseteq \mathbb{R}$ be compact, and $f : K \to \mathbb{R}$ continuous:
  - **94**: f is bounded
  - **95**: f has a maximum and minimum (EVT)
- Let (X,d) be a metric space, K be a sequentially compact subset of X and $f : K \to \mathbb{R}$ be a continous function:
  - **110**: f has a maximum and a minimum. In particular, f is bounded. (EVT ..again)

Sequential compactness stuff

Let (X,d) be a metric space, and $K \subseteq X$:

- Let $K \neq \emptyset$, and let d_K be the induced metric on K.

Ex.39: K (seq.) compact $\iff$ the M.S. (K, d_K) is (seq.) compact

105: K sequentially compact $\implies$ K is closed and bounded

107: (X,d) and K are both sequentially compact $\iff$ K is closed

108: (X,d) is sequentially compact $\implies$ (X,d) is complete

115: K is compact $\iff$ K is sequentially compact

x42: (X,d) is compact $\implies$ (X,d) is sequentially compact

x43: (X,d) is compact, and let A be an infinite subset of X $\implies$ A has at least one limit point

Thm 114 (Uniform Continuity)

Let (X, d_X) be a sequentially compact metric space, (Y, d_Y) be a metric space and $f : X \to Y$ be a continuous function. Then f is uniformly continuous

Totally Compact Spaces

Let (X,d) be a metric space:

120: (X,d) is sequentially compact $\implies$ (X,d) is totally bounded

122: (X,d) is compact $\iff$ (X,d) complete and totally bounded

121: Every sequentially compact metric space is compact.

Definition 123: Countable and Uncountable Sets

A set S is said to be:

- **Infinitely countable** iff there is a bijection $f : \mathbb{N} \to S$
- **Countable** if it is finite or infinitely countable
- **Uncountable** iff it isn't countable

Example 124

- $\{1, 2, 3\}$ and $\mathbb{R}$ are countable sets
- $\mathbb{Q}$ is infinitely countable
- $\mathbb{R}$ is uncountable

Theorem or rather Ex 45: Dense Subset equivalence

Let (X, d) be a metric space, $D \subseteq X$. The following are equivalent:

1. D is dense
2. For every $x \in X$ and $\epsilon > 0$ there exists $y \in D$ s.t. $d(x, y) < \epsilon$
3. For every $x \in X$ there is a sequence $(y_n)_{n=1}^{\infty}$ of elements of D s.t. $y_n \to x$
4. For every element $x \in X$ and every open nbhd G of x, $G \cap D \neq \emptyset$
5. D intersects every non-empty open set

Definition 125: Separable spaces

A metric space is **separable** iff it has a countable dense subset

Examples

- $\mathbb{R}$ with the standard metric is a separable metric because $\mathbb{Q}$ is dense and countable
- $\mathbb{R}^n$ with the Euclidean metric is a separable metric space because $\mathbb{Q}^n$ is dense and countable
- $\mathbb{C}$ with its standard metric is a separable metric space because $\{z \in \mathbb{C} : \mathrm{Re}(z), \mathrm{Im}(z) \in \mathbb{Q}\}$
- ℓ^2 is separable, and ℓ^p is separable for $1 \leq p < \infty$

Theorem P: Polynomials

Thm 130 (Weierstrass Approximation Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\epsilon > 0$. There exists a polynomial p with *real* coefficients s.t. for all $x \in [a, b]$

$$|f(x) - p(x)| < \epsilon$$

Thm 131 (literally same thing but with $\mathbb{Q}$)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\epsilon > 0$. There exists a polynomial p with *rational* coefficients s.t. for all $x \in [a, b]$

$$|f(x) - p(x)| < \epsilon$$

More Theorems

- **Ex 47**: The set of all polynomials (of one variable and any degree) with rational coefficients is countable
- **Thm 132**: $C([a, b])$ is separable

Theorem 133: Separability of subspaces

Let (X, d) be a separable metric space, $A \subseteq X$, $A \neq \emptyset$, and d_A be the induced metric on A. Then the metric space (A, d_A) is separable

---

Thm 135: Every compact metric space is separable (compact $\implies$ separable)

Theorem 136: Open Ball countability

Let (X, d) be a separable metric space and let D be a countable dense subset of X. Let

$$\mathcal{B} = \{B(c, r) : c \in D, r \in \mathbb{Q}^+\}$$

be the set of all open balls with centers in D and rational radii. Then $\mathcal{B}$ is countable and every open set in X can be written as a union of elements of $\mathcal{B}$

Definition Q: Open Bases and Second Countability

Def 137 (Open Bases)

Let $(X, \mathcal{T})$ be a topological space. An **open base** (or **base**) for the topology $\mathcal{T}$, is a family $\mathcal{B}$ of open sets such that every open set in $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$

Def 139 (Second Countability)

A topological space $(X, \mathcal{T})$ satisfies the **second Axiom of Countability**, or is **second countable** iff it has a countable open base

Other theorems

- **Thm 140**: In a separable metric space, every family of pairwise disjoint non-empty open sets is countable
- **Thm 141**: On the real line with the standard metric, every open set can be written as a countable union of disjoint open intervals

Theorem 142: Continuous Extensions

Let $(X, d_X), (Y, d_Y)$ be metric spaces, D be a dense subset of X, $f, g : X \to Y$ continuous functions s.t. $f(x) = g(x)$ for all $x \in D$. Then $f = g$

Thm 143: Let $(X, d_X), (Y, d_Y)$ be metric spaces, $D \subseteq X$ be dense, $f : D \to Y$ be uniformly continuous, and assume that (Y, d_Y) is complete. Then f has a unique continuous extension $F : X \to Y$

Theorem R: Properties of Complete Metric Spaces

- **144**: Let (X, d) be a metric space, F be a nonempty subset of X and d_F be the induced metric on F. If the metric space (F, d_F) is complete then F is a closed subset of X
- **145**: Let (X, d) be a complete metric space, F be a nonempty subset of X, and d_F be the induced metric on F. If F is a closed subset of X, then the metric space (F, d_F) is complete
- **146**: Let (X, d) be a complete metric space, $A \subseteq X$, $A \neq \emptyset$. Then
  1. The metric space $(\overline{A}, d_{\overline{A}})$ is complete
  2. If $A \subseteq B \subseteq X$ and (B, d_B) is complete, then $\overline{A} \subseteq B$

Definition 147: Isometries

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called a **isometry** iff for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

The metric spaces (X, d_X) and (Y, d_Y) are said to be **isometric** iff there exists an isometry f from X onto Y

Isometry Theorems

- **Thm 148**: Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be an isometry. Then f is an injection. If, moreover, f is a surjection (hence f bij.) then $f^{-1} : Y \to X$ is also an isometry
- **Fun Fact**: if two metric spaces are isometric and one of them is complete/compact/connected/... then so is the other

Theorem 150: Isometry completion

Let (X, d) be a bounded metric space and let $C(X, \mathbb{R})$ be the set of all bounded continuous functions $f : X \to \mathbb{R}$ equipped with the metric

$$d_\infty(f_1, f_2) = \sup\{|f_1(t) - f_2(t)| : t \in X\}$$

For each $x \in X$, define $F_X : X \to \mathbb{R}$ be $F_X(x') = d(x, x')$. Then

1. $F_X \in C(X, \mathbb{R})$
2. The map $X \to C(X, \mathbb{R}), x \mapsto F_X$ is an isometry
3. $X^* = \{F_X : x \in X\}$, equipped with the induced metric, is a subspae of $C(X\mathbb{R})$ isometric to X
4. The closure $\overline{X^*}$ of X^* in $C(X, \mathbb{R})$, equipped with the induced metric, is a complete metric space
5. X^* is dense in $\overline{X^*}$

Definition 152: Completion of a Metric Space

Let (X, d) be a metric space. A **completion** of (X, d_X) is any metric space (Y, d_Y) with the following properties

1. (Y, d_Y) is complete
2. (Y, d_Y) has a subspace X^* isometric to (X, d_X)
3. X^* is dense in Y

It can be shown that any two completions of X are isometric to each other, i.e. a completion is unique up to isometries

Definition S: Construction of Completion via Cauchy

Let (X, d) be a metric space and let $\mathcal{C}$ be the set of all Cauchy sequences of elements of X

We define an equivalence relation $\sim$ in $\mathcal{C}$ as follows: Let $x = (x_n)_{n \in \mathbb{N}}$, $y = (y_n)_{n \in \mathbb{N}} \in \mathcal{C}$. We say that $x \sim y$ iff $d(x_n, y_n) \to 0$

Distinct equivalence classes are disjoint and partition $\mathcal{C}$

The set of all equivalence classes is called the **quotient space**, denoted $\mathcal{C}/\sim$

Define a metric D on $\mathcal{C}/\sim$ as follows:

Let $\alpha, \beta \in \mathcal{C}/\sim$. Then

$$\alpha = [(x_1, \dots, x_n, \dots)] \text{ and } \beta = [(y_1, \dots, y_n, \dots)]$$

for some $(x_1, \dots, x_n, \dots), (y_1, \dots, y_n, \dots) \in \mathcal{C}$. Define

$$D(\alpha, \beta) = \lim_{n \to +\infty} d(x_n, y_n)$$

$(\mathcal{C}/\sim, D)$ is complete. Additionally, the following is an isometry:

$$X \to \mathcal{C}/\sim \qquad x \mapsto ([x, x, \dots, x, \dots])$$

Let X^* be its range. The metric space (X^*, D_{X^*}) is isometric to (X, d), $(\overline{X^*}, D_{\overline{X^*}})$ is a complete metric space, and X^* is dense in $\overline{X^*}$

Definition 153: Connected and Disconnected Spaces

A metric space (X, d) is said to be **disconnected** iff there exists non-empty disjoint open sets G_1 and G_2 such that

$$X = G_1 \cup G_2$$

Otherwise the metric space is called **connected**

A non-empty subset A of a metric space (X, d) is said to be **disconnected** iff the metric space (A, d_A), where d_A is the induced metric, is disconnected

Theorem T: Connected Theorems

A subset O of A is open in (A, d_A) iff $O = A \cup G$ for some G that is open in X. Therefore, A is disconnected iff there exist open subsets G_1, G_2 of X s.t.

- $A = (A \cap G_1) \cup (A \cap G_2)$, which is equivalent to $A \subseteq G_1 \cup G_2$
- $A \cap G_1 \neq \emptyset$, $A \cap G_2 \neq \emptyset$
- $(A \cap G_1) \cap (A \cap G_2) = \emptyset$, which is equivalent to $A \cap G_1 \cap G_2 = \emptyset$

Connected Theorems

- **Thm 154**: $\mathbb{R}$ with the standard metric is connected
- **Ex.53**: On the real line with the standard metric, all intervals are connected sets
- **Thm 155**: A non-empty subset of the real line is connected iff it is an interval
- **Thm 157**: A metric space (X, d) is connected iff the only subsets of X with empty boundary are $\emptyset$ and X
- **Thm 158**: Let (X, d_X) be a connected metric space, (Y, d_Y) be a metric space and $f : X \to Y$ be a continuous surjection. Then (Y, d_Y) is connected as well
- **Thm 160**: A metric space (X, d) is connected iff the only clopen subsets are $\emptyset, X$

Theorem 159: Intermediate Value Theorem

Let (X, d) be a connected metric space and $f : X \to \mathbb{R}$ be a continuous function. If $x_1, x_2 \in X$ with $f(x_1 \neq f(x_2))$ and y is a real number between $f(x_1)$ and $f(x_2)$, then there exists an $x \in X$ such that $f(x) = y$

Definition U: Connected Components

Let (X, d) be a metric space. We define an equivalence relation $\sim$ in X as follows: $x \sim x'$ iff there exists a connected subset C of X that contains both x and x'

Ex.55: If $(C_i)_{i \in I}$ is a family of connected subsets of X with nonempty intersection, then $\bigcup_{i \in I} C_i$ is connected

Theorem 161: Big equivalence classes

The equivalence class of any point in X is the largest connected subset of X that contains that point (what point?)

Definition 162: Path Connected Metric Spaces

Let (X, d) be a metric space and $x_0, x_1 \in X$.

- A **path** in X from x_0 to x_1 is a continuous function $\gamma : [0, 1] \to X$ s.t. $\gamma(0) = x_0$, $\gamma(1) = x_1$
- (X, d) is **path-connected** iff for any two points x_0, x_1 in X there is a path in X from x_0 to x_1
- A non-empty subset A of X is **path-connected** iff the metric space (A, d_A), where d_A is the induced metric, is path connected

Thm 163 (Path Connected Theorem)

- Every path-connected metric space is connected
- Not every connected metric space is necessarily path-connected

5 Applications

Theorem 5.0.1: Picard's Theorem

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous and boudned function, and t_0, x_0 be real numbers. Assume that there exists a positive constant L s.t. for all real t, x_1, x_2 we have:

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$$

Then, there exists a positive δ and a unique differentiable function $x : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ s.t. for all $t \in [t_0 - \delta, t_0 + \delta]$,

$$x'(t) = f(t, x(t)) \quad \text{and} \quad x(t_0) = x_0$$