General Topology Math Notes

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Contents

1 Intro to Topology

1.1 Why Topology?

Topology can appear where we least expect it...

- Algebraic Number Theory - Next to Euclidean topology, can define other topologies on $\mathbb{Q}$ (related to how often primes divide a number). Extends to Adeles, Langlands programme, etc
- Arithmetic Progressions in the Integers - An arithmetic progression of length k is a set $\{a, a+d, \ldots, a+(k-1)d\}$ Finding subsets of $\mathbb{N}$ that contain arbitrarily long APs:
  - $2\mathbb{N}$ or $\mathbb{N}$
  - Primes (Green-Tao Theorem, 2007). Green-Tao theorem relies on **Szemeredi's Theorem**: Any dense enough subset of $\mathbb{N}$ contains arbitrarily long APs

    Furstenburg's idea: Get from $A \subseteq \mathbb{N}$ to $(a_i \in \{0,1\}^{\mathbb{N}})$ with $a_i \begin{cases} 1 & i \in A \\ 0 & \text{else} \end{cases}$

    Use topological dynamics to study this: A topological dynamical system is a triple of X cpt, $T : X \to X$ continuous, and a probability measure μ preserved by T (what)

1.2 Topological Spaces and Examples

Definition 1.2.1: Topological Space

A **topological space** is a pair $(X, \mathcal{T})$, where X is a nonempty set, and $\mathcal{T}$ is a collection of subsets of X which satisfies:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
2. if $U_\lambda \in \mathcal{T}$ for each $\lambda \in A$ (where A is some indexing set), then $\bigcup_{\lambda \in A} U_\lambda \in \mathcal{T}$
3. If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

1.2.2 Examples of Topological Spaces

1. $\mathbb{R}^n$ with the Euclidean Topology - induced by the Euclidean Metric
2. For any set X, $\mathcal{T} = \mathcal{P}(X)$ (discrete topology)
3. For any set X, $\mathcal{T} = \{\emptyset, X\}$ (indiscrete topology)
4. $X = \{0,1,2\}$ with $\mathcal{T} = \{\emptyset, X, \{0\}, \{0,1\}, \{0,2\}\}$
5. $X = \mathbb{R}$ and U open (aka, in $\mathcal{T}$) if $R \backslash U$ is finite or $U = \emptyset$

Proof for 5:

1. $\emptyset \in \mathcal{T}$, $\emptyset$ is finite $\implies X \in \mathcal{T}$
2. Intersections of finite sets are finite
3. Unions of finite sets are finite

Definition 1.2.3: Neighbourhood of a point

A **neighbourhood** of a point $x \in X$ is a subset $N \subseteq X$ s.t. $x \in U \subseteq N$ for some open subset $U \subseteq X$

Definition 1.2.4: Metric Space

A **metric space** (X, d) is a nonempty set X together with a function

$$d : X \times X \to \mathbb{R}$$

with the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

The function d is called the metric. Point 3 is called the *triangle inequality*

For any $x \in X$ and any positive real number r the **open ball** in X with centre x and radius r is defined by

$$B(x, r) = \{y \in X | d(x, y) < r\}$$

We declare a subset U of X to be *open in the metric topology given by* d iff for each $a \in U$ there is an $r > 0$ such that $B(a, r) \subseteq U$

If $(X, \mathcal{T})$ is a topological space, and if X admits a metric whose metric topology is precisely $\mathcal{T}$ we say that $(X, \mathcal{T})$ is **metrisable**. Thus the euclidean spaces with their usual topologies are metrisable

1.2.5 Examples of Metric Spaces

1. Any set X with $d(x, y) = \begin{cases} 0, & x = y \\ 1, & \text{else} \end{cases}$
2. $\mathbb{R}^n$ with $d(x, y) = |x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
3. $C([0, 1])$ with $d(f, g) = \max_{t \in [0,1]} |f(t) - g(t)|$
4. $C([0, 1])$ with $d(f, g) = \sqrt{\int_0^n |f(t) - g(t)|^2 dt}$

1.2.6 Topologies on Metric spaces

We want to define a topology on (X, d). For this, we want open balls to be open in the topology

Definition 1.2.7: Base

For a set X, a basis $\mathcal{B}$ is a collection of subsets such that

1. $\bigcup_{B \in \mathcal{B}} B = X$
2. $B_1 \cap B_2 \in \mathcal{B}$ for all $B_1, B_2 \in \mathcal{B}$

The **topology generated by** $\mathcal{B}$ is

$$\mathcal{T} := \{\bigcup_{i\in I} B_i, I \text{ index set}, B_i \in \mathcal{B}\}$$

Note: This is a topology because

$$(\cup_{i\in I} B_i) \cap (\cup_{j\in J} B_j) = \bigcup_{i\in I, j\in J} \underbrace{B_i \cap B_j}_{\in\mathcal{B}} \in \sqcup$$

Definition 1.2.8: Metric Topology

$$\text{Let } \mathcal{B} = \{\bigcap_{i=1}^{n} Br_1(x_1), n \in \mathbb{N},\ r_i > 0,\ x_i \in X,\ \forall i\}$$

The **metric topology** is the topology generated by this basis

Observation A set U is open in the metric topology $\iff \forall x \in U, \exists r > 0$ s.t. $Br(x) \subseteq U$

- $\Longleftarrow$: For each $x \in U$, let r_x s.t. $B_{r_x}(x) \subseteq U$. Then $U = \bigcup_{x\subseteq U} B_{r_x}(x)$ is open
- $\Longrightarrow$: Let $x \in U$ be given. Knwo that $x \in B_{r_1}(x_1) \cup \cdots \cup B_{r_n}(x_n)$ for some n, r_1, x_1. For each i, there is $\delta_i > 0$ s.t. $B_{\delta_i}(x) \leq B_{r_1}(x_1)$.

huh?

Theorem 1.2.9: random ms prop

If X carries metrics $d, \tilde{d}$ such that $ad(x,y) \leq \tilde{d}(x,y) \leq Ad(x,y)$ for some $a, A > 0$, then the induced topologies agree

Definition 1.2.10: Subspace Topology

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be any subset. Then the **subspace topology** on A consists of all sets of the form $U \cap A$ where $U \in \mathcal{T}$

Example: $(-1,1) \subseteq \mathbb{R}$ with euclidean topology. The subspace topology is

$$\{(-1,1) \cap U,\ U \subseteq \mathbb{R} \text{ open}\}$$

$(-1,1)$ is closed in the subspace topology

Theorem 1.2.11: Topology Lemmas

1.3 If $(X, \mathcal{T})$ is a topological space and $U_1, \ldots, U_n$ are open sets, then the intersection $\bigcap_{i=1}^{n} u_i$ is also open

1.6 In order to show that a set $U \subseteq X$ is open, it is enough to show that for every $x \in U$ there is an open set V with $x \in V \subseteq U$

1.6 A subset U of $\mathbb{R}^n$ is *open for the usual topology* iff for each $a \in U$ there exists an $r > 0$ s.t.

$$|x - a| < r \implies x \in U$$

The collection of open sets thus defined is called the **usual topology on** $\mathbb{R}^n$. Note that open balls are open sets under this definition

Definition 1.2.12: Topology Small Definitions

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1.3 Closed sets, Closure, Interior, and Boundary

Definition 1.3.1: Closed Subsets

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is **closed** iff its complement $X \backslash A = A^C := \{x \in X \; x \notin A\}$ is open in X

Note: A set being "closed" has no connection with "not being open"

1.3.2 Examples of open and closed sets

- A set that is neither open nor closed: $[0, 1) \subseteq \mathbb{R}$ under Euclidean topology
- A set that is both closed and open: $\emptyset$ or X

Theorem 1.3.3

Let $(X, \mathcal{T})$ be a topological space. Then

1. $\emptyset$ and X are closed.
2. The union of finitely many closed sets is a closed set
3. The intersection of any collection of closed sets is a closed set

$\bigcup_{i \in I} A_i$ is not necessarily closed.

$$A_n = \left[-2 + \frac{1}{n}, 2 - \frac{1}{n}\right] \text{ in } \mathbb{R}, \bigcup_{i=1}^{n} A_i = (-2, 2)$$

1.3.4 Topological proof that there are infinitely many primes (Furstenberg)

Proof. Look at $\mathbb{Z}$ with

$$\mathcal{B} := \{S(a, b),\, a \neq 0,\, b \in \mathbb{Z}\} \quad \text{and} \quad S(a, b) = \{an + b,\, n \in \mathbb{Z}\}$$

Let the open sets be the one generated by this basis. We can show

1. $S(a, b)$ is both open and closed.
2. All open sets are infinite.

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1. $S(a, b) = \mathbb{Z} \backslash \bigcup_{i=1}^{a-1} S(a, b - 1)$
2. Clear

Thus:

$$\underbrace{\mathbb{Z} \backslash \{-1, 1\}}_{\text{not closed}} = \bigcup_{p \text{ primes}} \overbrace{S(p, 0)}^{\text{closed}}$$

If there were only finitely many primes, right hand side would be closed $\square$

1.4 Closure and stuff

Definition 1.4.1: Closure, Interior, Boundary

Let $(X, \mathcal{T})$ be a topological space.

1. The **closure** of a subset $A \subseteq X$ is the smallest closed set such that $A \subseteq \overline{A}$.

$$\overline{A} := \bigcap_{\substack{C \subseteq X \text{closed} \\ A \subseteq C}} C$$

2. The (open) **interior** of a subset $A \subseteq X$ is the biggest open set U contained in A

$$\text{int } A = A^{\circ} := \bigcap_{\substack{U \subseteq X \text{open} \\ U \subseteq A}} C$$

3. The **boundary** or **frontier** of a subset $A \subseteq X$ is

$$\partial A := \overline{A} \backslash A^{\circ}$$

4. A subset A of X is **dense** in X iff $\overline{A} = X$

   E.g.: $\mathbb{Q} \subseteq \mathbb{R}$ with the Euclidean topology

Theorem 1.4.2: Closure and Interior of Complement

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then

1. The closure of the complement is the complement of the interior:

$$\overline{X \backslash A} = X \backslash (A^{\circ})$$

2. the interior of the complement is the complement of the closure:

$$(X \backslash A)^{\circ} = X \backslash \overline{A}$$

Definition 1.4.3: Limits in Topological spaces

A sequence (x_n) converges to $x \in X$ if $\forall U$ open with $x \in U$, $\exists N$ s.t. $x_n \in U$ for all $n \geq N$

Definition 1.4.4: Limit Set

$\overline{A}$ can be thought as the set of limits. Formally define the **Limit Set** as

$$A' = \{x \in X : \forall U \subseteq X, x \in U, \text{ open } \exists a \in A, \text{ s.t. } a \in U\}$$

But, limits in general are much worse behaved in topological spaces, e.g. no unique limit point
Example: a topological space X and a sequence (x_n) which does not have a unique limit (i.e. $\exists x \neq y$ s.t. $x_n \to x$ and $x_n \to y$ in the sense defined): Nontrivial X with the indiscrete topology $\{\emptyset, X\}$

1.5 Hausdorff Spaces

Problem: Non-unique limits are nasty :(

Definition 1.5.1: Hausdorff Space

A topological space $(X, \mathcal{T})$ is **Hausdorff** if for each $x, y \in X$ with $x \neq y$ there exist *disjoint* open sets U and V s.t. $x \in U$ and $y \in V$
This space has *unique limits*!

If (X, d) is a metric space then it is automatically Hausdorff, so any metrisable space is Hausdorff. The trivial topology on a set with more than one element is not Hausdorff. Not every Hausdorff space is metrisable
Non-Hausdorff spaces are a lot more annoying to work with - for example you can have multiple limits in non-Hausdorff spaces

Theorem 1.5.2: Open sets on $\mathbb{R}$ with Euclidean Topology

- A set U is open iff there are open intervals I_j s.t.

$$U = \bigcup_{j=1}^{\infty} I_j$$

- A set A is closed iff there are F_j (union of two closed intervals) s.t.

$$A = \bigcap_{j=1}^{\infty} F_j$$

Definition 1.5.3: Convergence of Hausdorff Spaces

A sequence (x_n) of members of a topological space X converges to $x \in X$ if for every open set U containing x, there exists an N such that $n \geq N \implies x_n \in U$

Theorem 1.5.4: Haussdorf Convergence Uniqueness

Suppose $(X, \mathcal{T})$ is Hausdorff. Then a sequence (x_n) can converge to at most one limit.

Being Hausdorff is what's called a *topological property*, which means whether it is true in a particular case depends only on the open sets of the space in question.
In contrast, the property of *completeness* of a metric space is not a topological property as there exist sets upon which one can put two distinct metrics, one complete and one not, yet for which the metric topologies coincide

Definition 1.5.5: Cauchy Sequences

Let (X, d) be a metric space

1. A **Cauchy Sequence** is a sequence (x_n) with each $x_n \in X$ with the property that for each $\epsilon > 0$, there exists an N s.t. $m, n \geq N \implies d(x_m, x_n) < \epsilon$
2. (X, d) is **complete** if every Cauchy Sequence converges

Caveat: In general, this does not have to converge to an $x \in X$
Example: $\mathbb{Q}$ with the Euclidean metric.

Definition 1.5.6: Complete Space

A metric space is called **complete** if all Cauchy sequences converge to a point in the space

Definition 1.5.7: Closure in Metric Spaces

Let (X, d) be a complete metric space and $A \subseteq X$. A point x is in the **closure** of $A \iff \exists x_i \to x$ with $x \in A$

2 Continuity

2.1 Continuity

Definition 2.1.1: Continuity

Let $(X, \mathcal{T}), (Y, \tilde{\mathcal{T}})$ be topological spaces and $f : X \to Y$. f is **continuous** if for all $U \in \tilde{\mathcal{T}}$, $f^{-1}(U) \in \mathcal{T}$

Equivalently:

- $U \subseteq Y$ open $\implies f^{-1}(U)$ open
- $A \subseteq Y$ closed $\implies f^{-1}(A)$ closed

2.1.2 Why take f^{-1}

Properties: For U, V sets in Y,

- $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(U)$
- $f^{-1}(U^C) = f^{-1}(U)^C$
- $f^{-1}(U \cup U) = f^{-1}(U) \cup f^{-1}(U)$

Example: $\mathbb{R}$ with Euclidean Topology

Proof. "Proof" that $[-1, 1]$ is open:
Take $[-1, 1]$ with the subspace topology $\mathcal{T} := \{[-1, 1] \cap U, U \subseteq \mathbb{R} \text{ open}\}$
Embedding $i : [-1, 1] \to \mathbb{R}$, $x \mapsto x$ is continuous
$[-1, 1]$ open in subspace topology
i cont $\implies i([-1, 1])$ is open this is actually wrong! U open $\not\Longrightarrow f(U)$ open
But $i([-1, 1]) = [-1, 1] \subseteq \mathbb{R}$ ∎

Definition 2.1.3: Formal Definition of Continuity

Let $(X, d), (Y, d)$ be metric spaces with the metric topology. $f : X \to Y$ is continuous as above iff $\forall x \in X$, $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$$

Proof. $\implies$ **Direction**
Recall: U open in metric topology if $\forall x \in U$, $\exists r > 0$ s.t. $B_r(x) \subseteq U$, where $B_r(x) = \{y \in X : d(x, y) < r\}$
$\implies$ Let $x \in X$ be given, $\epsilon > 0$. Let $y = f(x) \in Y$, $U = B_\epsilon(y) = \{y' \in Y : \tilde{d}(y, y') < \epsilon\}$.
f cont $\implies f^{-1}(U)$ is open. $x \in f^{-1}(U) \implies \exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(U)$
$\implies \forall x' \in X$ s.t. $d(x, x') < \delta$, $x' \in B_\delta(x) \subseteq f^{-1}(U)$.
$\implies f(x') \in B_\delta(f(x)) \implies \tilde{d}(f(x), f(x')) < \epsilon$
$\impliedby$ **Direction**
Let U be open in Y. WTS: $f^{-1}(U)$ is open.
So it is enough to show for all $x \in f^{-1}(U)$, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(U)$.
Let x be given, $y := f(x) \in U$. U open $\implies \exists \epsilon > 0$ s.t. $B_\epsilon(y) \subseteq U$.
By assumption $\exists \delta > 0$ s.t.

$$d(x', x) < \delta \implies \tilde{d}(f(x'), f(x)) < \epsilon$$

But, $\{y' : d(y', f(x)) < \epsilon\} \subseteq U$ by choice of ϵ.
$\implies B_\delta(x) \subseteq f^{-1}(U)$ □

2.2 Homeomorphisms

Definition 2.2.1: Homeomorphism

Let $(X, \mathcal{T}), (Y, \tilde{\mathcal{T}})$ be topological spaces. A function $f : X \to Y$ is a **homeomorphism** (or **bi-continuous**) if f is bijective, f is continuous, and $f^{-1} Y \to X$ is continuous

A "Great goal of Topology": Understand topological spaces up to homeomorphisms.
Say that a property of a topological space is a **topological invariant** if it is preserved by homeomorphism. Example: Being Hausdorff

Example 2.2.2: Examples of Homeomorphisms

- $(X, \mathcal{T})$ topological space, $\mathrm{id} : X \to X$, $x \mapsto x$
- $X = \mathbb{R}^n$, $A : \mathbb{R}^n \to \mathbb{R}^n$ Linear + Invertible
- Example which is **not** a homeomorphism:

$$f : \underbrace{\mathbb{R}}_{\text{metric topology}} \to \underbrace{\mathbb{R}}_{\text{indiscrete topology } \{\emptyset, \mathbb{R}\}}, \; x \mapsto x$$

  Problem: f^{-1} is not continuous

Definition 2.2.3: Another continuity definition

Let $(X,d),(Y,\tilde{d})$ be metric spaces with the metric topology. $f: X \to Y$ is continuous iff

$$\forall x \in X,\, \forall \epsilon > 0,\, \exists \delta > 0 \ \text{ s.t. } \forall y \in X,\, d(x,y) < \delta \implies \tilde{d}(f(x), f(y)) < \epsilon$$

Observe: $\forall y \in X$ is equivalent to

$$B_\delta(x) \subseteq f^{-1}(\tilde{B}_\epsilon(f(x)))$$

Why? Let A, B be things which can be true for $y \in X$. i.e.

$$A \implies B \text{ is equivalent to } \{y : A \text{ true} \subseteq \{y : B \text{ true}\}\}$$

Then: $B_\delta(x) = \{y, \underbrace{d(x,y) < \delta}_{A}\}$, $f^{-1}(\tilde{B}_\epsilon(f(x))) = \{y \in X : \underbrace{\tilde{d}(f(x), f(y)) < \epsilon}_{B}\}$

WTS: U open $\iff \forall x \in U, \exists r > 0$ s.t. $B_\delta(x) \subseteq U$

$\implies$ "Let x, ϵ be given, WTS that $\exists \delta$ s.t. $B_\delta(x) \subseteq f^{-1}(\tilde{B}_\epsilon f(x))$

Example: f cont + bijective but not a homeomorphism:

$$f : \underbrace{X}_{\text{discrete topology - every set is open}} \to \overbrace{X}^{\text{indiscrete topology: only } \emptyset \text{ and } X \text{ are open}} \text{ identity}$$

Lemma 2.2.4: Homeomorphism-condition

For a set X with topologies $\mathcal{T}$, $\tilde{\mathcal{T}}$. The identity map $(X, \mathcal{T}) \to (X, \tilde{\mathcal{T}})$, $x \mapsto x$ is

- continuous $\iff \tilde{\mathcal{T}} \subseteq T$
- a homeo $\iff \tilde{\mathcal{T}} = \mathcal{T}$

Theorem 2.2.5: Mapping prop

- Let $f: X \to Y$, $g: Y \to Z$ continuous. The map $f \circ g$ is continuous

$$\text{As } (f \circ g)^{-1}(U) = f^{-1}(g^{-1}(U))$$

- If $f: X \to Y$ is constant, then f is continuous
- In particular, f, g homeo $\implies f \circ g$ is a homeo

3 The Clark Barwick Era

Theorem 3.0.1: Clark Barwick Quotes List

"Shadows are harshest when there is only one lamp" - 04/10/24

3.1 More top

3.1.1 Something weird

$$
\begin{aligned}
&[0, 2\pi) \to S^1 = \{z \in \mathbb{Z} : \|z\| = 1\} \\
&[0, 2\pi) \to [0, 1) \quad \text{is open, and is also creepy}
\end{aligned}
$$

Not a homeomorphism
Claim: A continuous bijection in which the **image** of every open set is open is a homeomorphism

Definition 3.1.2: Subspace Topology

For X a topological space, and $T \subseteq X$, $\mathcal{U} \subseteq T$ is open iff $\exists V \subseteq X$ open and $\mathcal{U} = V \cap T$

Definition 3.1.3: Impromptu Set Theory - Products

$\mathcal{F}$ is a family of sets. We can talk about a product

$$\prod_{x \in \mathcal{F}} X = \{(a_x)_{x \in \mathcal{F}} : a_X \in X\}$$

Example:

$$\mathbb{R}^\infty = \prod_{i=1}^{\infty} \mathbb{R} = \{(x_1, x_2, \dots) : x_i \in \mathbb{R}\}$$

$$\prod_{x \in \mathcal{F}} = \{\phi : \mathcal{F} \implies \bigcup_{x \in \mathcal{F}} X : \phi(X) \in X\}$$

Note: the $\mathcal{F}$ notation is pretty creepy - Clark

3.1.4 Topologising the above thing

$$\prod_{i \in I} X_i \to X_j$$

3.2 Week 4 Lecture 1

Definition 3.2.1: Quotient Topology

Define X with $\sim$ a relation on X. We have a function

$$
\begin{aligned}
g : X &\to X/\sim \\
x &\mapsto [x]
\end{aligned}
$$

$\mathcal{U} \subseteq X/\sim$ is open iff $g^{-1}(\mathcal{U}) \subseteq X$ is open

3.3 Week 4 Lecture 2

Definition 3.3.1: Coarser and Finer

- Coarse: There are more open sets

Definition 3.3.2: Product Topology

$$
\begin{tikzcd}
 & X \times Y \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
X & & Y
\end{tikzcd}
$$

The **Product Topology** is the coarsest possible topology such that pr_1 and pr_2 are both continuous

Definition 3.3.3: Quotient Topology

For X a topological space and $\sim$ a relation, define a function $q : X \to X/\sim$ where the quotient top is the finest topology such that q is continuous

Definition 3.3.4: Coarse and Fine Topologies

For X a topological space and Y a set:

- $f : X \to Y$ means there exists a unique finest topology s.t. f is continuous
- $g : Y \to X$ means that there exists a unique coarsest topology s.t. g is continuous

Lemma 3.3.5: Hausdorff Coarseness

τ_1 is coarser than τ_2 and τ_1 is Hausdorff $\implies$ τ_2 is Hausdorff