General Topology Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Topological Spaces and Examples

Definition 1.1: Topological Space

A **topological space** is a pair $(X, \mathcal{T})$, where X is a nonempty set, and $\mathcal{T}$ is a collection of subsets of X which satisfies:

a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

b) if $U_\lambda \in \mathcal{T}$ for each $\lambda \in \Lambda$ (where Λ is some indexing set), then $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$

c) if $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

The collection $\mathcal{T}$ is called the **topology** of the topological space, and the members of $\mathcal{T}$ are called the **open sets** of the topology

Example 1.7: Euclidean Spaces

Let $\mathbb{R}^n$ enote the n-dimensional Euclidean vector space with elements $x = (x_1, x_2, \dots, x_n)$ and $x_i \in \mathbb{R}$, and let

$$|x| = \sqrt{\sum_{i=1}^{n} x_i^2} \geq 0$$

be the length of x. ($\mathbb{R}^1 = \mathbb{R}$ is the real line). A subset U of $\mathbb{R}^n$ is **open (for the usual topology)** iff for each $a \in U$ there exists an $r > 0$ such that

$$|x - a| < r \implies x \in U.$$

The collection of open sets thus defined is called the **usual topology** on $\mathbb{R}^n$. Note that open balls $B(y, \rho) = \{x \in \mathbb{R}^n : |x - y| < \rho\}$ are open sets under this definition.

Example 1.8: Metric Spaces

A **metric space** (X, d) is a nonempty set X together with a function $d : X \times X \to \mathbb{R}$ with the following properties:

a) $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$

b) $d(x, y) = d(y, x)$

c) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

The function d is called the **metric**.

Let (X, d) be a metric space, x be a point in X, and $r > 0$. The **open ball** with center x and radius r is defined by

$$B(x, r) = \{y, , \in X : d(x, y) < r\}.$$

A subset U of X is **open (in the metric topology given by d)** iff for each $a \in U$ there is an $r > 0$ such that $B(a, r) \subseteq U$. Just like euclidean spaces, open balls are open in this sense.

Example 1.0.1: Other Topologies and Metrics

If $(X, \mathcal{T})$ is a topological space, and if X admits a metric whose metric topology is precisely $\mathcal{T}$, then we say that $(X, \mathcal{T})$ is **metrisable**

- Euclidean spaces with their usual topologies are metrisable.

1.9) The **Discrete Topology** is the topology of all subsets of a set X. We can define the **discrete metric** of X to be

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}.$$

1.10) The **Trivial** or **Indiscrete Topology** is the topology $\mathcal{T} := \{\emptyset, X\}$ for a set X. This is a non-metrisable topology when X has more than one member.

1.14) Let $X = \{a, b, c\}$, where a, b, c are distinct. Then

$$\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$$

is a topology on X

1.15) Give $\mathbb{R}$ the topolgoy whose open subsets $U \subseteq \mathbb{R}$ are precisesly the subsets with finite complement $\mathbb{R} \backslash U$, or $U = \emptyset$. Then $\mathbb{R}$ with this topology is not metrisable. This is an example of a **Zariski Topology**

Proposition 1.11: Topology Equality

Let d, d' be metrics on the same set X, and let $\mathcal{T}$, $\mathcal{T}'$ be the corresponding metric topologies. If for real numbers $A, B > 0$ we have

$$d(x, y) \leq Ad'(x, y),\ d'(x, y) \leq Bd(x, y) \text{ for all } x,\ y \in X,$$

then $\mathcal{T} = \mathcal{T}'$.

Example 1.12: Example of Topology Equality

- The **Euclidean metric** on $\mathbb{R}^n$ is defined as:

$$d\big((x_1, x_2, \dots, x_n),\ (y_1, y_2, \dots, y_n)\big) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

- The **Box metric** on $\mathbb{R}^n$ is defined as:

$$d(x, y) \leq \sqrt{n} d'(x, y),\ d'(x, y) \leq d(x, y)$$

By 1, these have the same topology.

Definition 1.16: Subspace Topology

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be any subset. Then teh **subspace topology** on A consists of all sets of the form $U \cap A$ where $U \in \mathcal{T}$.

Definition 1.17: Closed Set

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is **closed** iff its complement $X \backslash A := \{x \in X \mid x \notin A\}$ is open in X. Note that a set being *closed* does not mean it isn't *open*. Sets that are both *closed* and *open* are called **clopen**.

Theorem 1.19: Properties of open and closed sets

Let $(X, \mathcal{T})$ be a topological space.

1. $\emptyset$ and X are closed.
2. The union of **finitely many** closed sets is an closed set.
3. The intersection of **any collection** of closed sets is a closed set.
4. The union of **any collection** of open sets is an open set.
5. The intersection of **finitely many** open sets is an open set

Definition 1.20: Properties of Topological Spaces

1. The **closure** of a set $A \subseteq X$ is
$$\overline{A} := \bigcap_{C \subseteq X \text{ closed};\ A \subseteq C} C.$$
2. The **interior** of a set $A \subseteq X$ is
$$\operatorname{int} A = A^\circ := \bigcap_{C \subseteq X \text{ open};\ A \subseteq C} C.$$
3. The **boundary** (or **frontier**) of a subset $A \subseteq X$ is
$$\partial A := \overline{A} \backslash A^\circ.$$
4. A subset A of X is **dense** in X iff $\overline{A} = X$.

$\overline{A}$ is closed, and contains A and is the smallest set with this property. So A is closed iff $\overline{A} = A$.
A° is open, and is contained in A, and is the largest set with this proprety. So A is open iff $A^\circ = A$.

Proposition 1.22: Relating Topological Properties

The closure of the complement is the complement of the interior:

$$\overline{X \backslash A} = X \backslash (A^\circ).$$

The interior of the complement is the complement of the closure:

$$(X \backslash A)^\circ = X \backslash \overline{A}.$$

Definition 1.23: Limit Points

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be a subset. A **limit point** of A is a point $x \in X$ s.t. for every open subset $U \subseteq X$ with $x \in U$ there exists an element $a \in A \cup U$ with $a \neq x$.
Let A' be the set of limit points of A.
Note that this has nothing to do with limits of sequences.

Lemma 1.24: Limit Points and Open Balls

An element $x \in X$ in a metric space (X, d) is a limit point of a subset $A \subseteq X$ iff for every $\epsilon > 0$ there exists $a \in A$ with $0 < d(x, a) < \epsilon$, or iff there exists a sequence $a_1, a_2, a_3, \cdots$ of elements $a_i \in A$, with $a_i \neq x$ for all i, such that $d(x_i, a_i) \to 0$ as $i \to \infty$. This interpretation does not extend to general topological spaces.

Example 1.0.2: Examples of limit points

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Proposition 1.26: Union of Limit points

Let $(X, \mathcal{T})$ be a topological space, and suppose $A \subseteq X$. Then

$$\overline{A} = A \cup A'$$

Corollary 1.27

A subset $A \subseteq X$ is closed iff it contains all its limit points.

Theorem 1.30: Open and Closed sets in $\mathbb{R}$

Consider $\mathbb{R}$ with the usual topology.

1. A nonempty set U is open iff it can be written as a countable union of disjoint nonempty open intervals I_j:

$$U = \bigcup_{j=1}^{\infty} I_j.$$

2. A set F is closed iff it can be written as a countable intersection

$$F = \bigcap_{j=1}^{\infty} F_j$$

where each F_j is a finite union of closed intervals.

Definition 1.32: Hausdorff Spaces

A topological space $(X, \mathcal{T})$ is **Hausdorff** if for each $x, y \in X$ with $x \neq y$ there exist **disjoint** open sets U and V such that $x \in U$ and $y \in V$.

Any metrisable space is Hausdorff, The trivial topology n a set with more than one element is not Hausdorff.

Definition 1.33: Convergence of a Topological space

A sequence (x_n) of members of a topological space X converges to $x \in X$ if for every open set U containing x, there exists an N such that $n \geq N \implies x_n \in U$

Proposition 1.34: Convergence of Hausdorff Spaces

Suppose $(X, \mathcal{T})$ is Hausdorff. Then a sequence (x_n) can converge to at most one limit.

Definition 1.36: Cauchy and Completeness

Let (X, d) be a metric space.

1. A **Cauchy sequence** is a sequence (x_n) with each $x_n \in X$ with the property that for each $\epsilon > 0$, there exists an N such that $m, n \in N \implies d(x_m, x_n) < \epsilon$
2. (X, d) is **complete** if every Cauchy sequence converges.

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