Mathematics 3 **Honours Analysis** **2023/24**

Workshop 3 – Series of real numbers and some questions on functions

1. True or false? (Give reasons.)
(i) the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence (a_n) is convergent;
(ii) the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $a_n \to 0$ as $n \to \infty$;
(iii) the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} |a_n|$ converges.

2. $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if p satisfies ...?

3. State the **ratio test** and explain how it is related to geometric series.

4. Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that $\sum_{n=1}^{\infty} |a_n|^p$ converges for all $p \geq 1$.

5. Suppose that $\sum_{n=1}^{\infty} a_n$ converges conditionally. Prove that $\sum_{n=1}^{\infty} n^p a_n$ diverges for all $p > 1$.

6. Let $f(x) = x^2$ when x is rational and $f(x) = 0$ when x is irrational. Discuss the continuity and differentiability of f.

7. Decide which of the following statements are true and which are false. Prove the true one and find counterexamples for the false ones.

   (i) If f is continuous on $[a, b]$ and $J = f([a, b])$ then J is a closed bounded interval.

   (ii) If f, g is continuous on $[a, b]$ and $f(a) < g(a)$, $f(b) > g(b)$ then $\exists c \in (a, b)$ such that $f(c) = g(c)$.

   (iii) Suppose that f, g are defined and finite valued on an open interval I containing a point a. Assume also that f is continuous at a and that $f(a) \neq 0$. Then g is continuous at a if and only if fg is continuous at a.

8. State carefully the mean value theorem for a function $f : [0, 1] \to \mathbb{R}$. Why is it called the "mean value" theorem?

9. Let $f : (0,1) \to \mathbb{R}$ be a function and let $a \in (0,1)$. Match each statement in Group A with a statement from Group B which means the same thing.

Group A:

(i) $\forall \epsilon > 0, \exists \delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(ii) $\forall \epsilon > 0, \forall \delta > 0$, $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(iii) $\exists \epsilon > 0$ such that $\forall \delta > 0$, $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(iv) $\exists \epsilon > 0$ and $\exists \delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(v) $\forall \delta > 0, \exists \epsilon > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(vi) $\exists \delta > 0$ such that $\forall \epsilon > 0$, $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

Group B:

(a) f is continuous at a.

(b) f is bounded on $(0,1)$.

(c) f is constant on $(0,1)$

(d) There is some neighbourhood of a on which f is bounded.

(e) There is some neighbourhood of a on which f is constant.

Assessment task to be handed in on Thursday 12/10/2023 at noon): Questions 4 and 9.