Geometry 2023 Workshop Sheet 1

Dual Spaces and Implicit Function Theorem

The purpose of this workshop is to familiarise you with dual vector spaces and the Implicit Function Theorem. It will likely be useful to look at the statement of the Implicit Function Theorem in the lecture notes at some point.

1. Suppose we have a vector space V, in which we have chosen a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, giving rise to a dual basis $\alpha^1, \dots, \alpha^n$ for V^*. As discussed in the lecture notes, this means that we can write any vector $\mathbf{v} \in V$ as

$$\mathbf{v} = [\mathbf{e}_1 \ \dots \ \mathbf{e}_n] \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix},$$

and vice versa any (dual) vector $\alpha \in V^*$ as

$$\alpha = [b_1 \ \dots \ b_n] \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{bmatrix}.$$

Here $\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$ and $[b_1 \ \dots \ b_n]$ are matrices whose entries are scalars (real numbers), and $[\mathbf{e}_1 \ \dots \ \mathbf{e}_n]$ and $\begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{bmatrix}$ are matrices of vectors (with entries in respectively V and V^*).

We now want to think about what happens if we change the basis for V. Suppose have another basis $\widetilde{\mathbf{e}}_1, \dots, \widetilde{\mathbf{e}}_n$ for V, with corresponding dual basis $\widetilde{\alpha}^1, \dots, \widetilde{\alpha}^n$ for V^*. With respect to these bases, we have

$$\mathbf{v} = [\widetilde{\mathbf{e}}_1 \ \dots \ \widetilde{\mathbf{e}}_n] \begin{bmatrix} \widetilde{v}^1 \\ \vdots \\ \widetilde{v}^n \end{bmatrix},$$

and

$$\alpha = \left[\widetilde{b}_1 \ \dots \ \widetilde{b}_n\right] \begin{bmatrix} \widetilde{\alpha}^1 \\ \vdots \\ \widetilde{\alpha}^n \end{bmatrix}.$$

This gives rise to a change-of-basis matrix A (an $n \times n$-matrix with real coefficients) such that

$$\begin{bmatrix} \widetilde{v}^1 \\ \vdots \\ \widetilde{v}^n \end{bmatrix} = A \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}.$$

In this set-up, how are

$$[\widetilde{\mathbf{e}}_1 \ \dots \ \widetilde{\mathbf{e}}_n] \qquad \text{and} \qquad [\mathbf{e}_1 \ \dots \ \mathbf{e}_n]\,,$$

$$\begin{bmatrix} \widetilde{\alpha}^1 \\ \vdots \\ \widetilde{\alpha}^n \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{bmatrix}$$

and

$$\left[\widetilde{b}_1 \ \dots \ \widetilde{b}_n\right] \qquad \text{and} \qquad [b_1 \ \dots \ b_n]$$

related to each other?

2. Continuing on from the above, suppose now that V is further equipped with a Euclidean structure $\langle .,. \rangle$, and that the bases $\mathbf{e}_1, \dots \mathbf{e}_n$ and $\widetilde{\mathbf{e}}_1, \dots, \widetilde{\mathbf{e}}_n$ are both orthonormal. Show that in this case the identification of V and V^* induced by identifying $\mathbf{e}_1, \dots, \mathbf{e}_n$ and $\alpha^1, \dots, \alpha^n$ is independent of the choice of the basis, and in fact coincides with the identification of V and V^* given by using $\langle .,. \rangle$.

3. Suppose we are interested in solutions to the equations

$$\begin{aligned} x^2 - y^2 - u^3 + v^2 + 4 &= 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 &= 0 \end{aligned}$$

Can we express u and v as functions of x and y for all solutions near $(x, y, u, v) = (2, -1, 2, 1)$?