Question 1

Show that the space curve

$$x(s) = \begin{pmatrix} \frac{\sqrt{3}}{2}\sin(s) \\ \cos(s) \\ \frac{1}{2}\sin(s) \end{pmatrix}$$

is arc-length parameterised. Compute its curvature, torsion and Frenet-Serret frame

[3 Marks]

Arc-length parameterisation

A curve is arc-length parameterised if $x(s)$ is a unit-speed parameterisation, i.e. if $v(s) = 1$ everywhere for $v(s) = \sqrt{x'(s) \cdot x'(s)}$

Differentiating $x(s)$,

$$x'(s) = \begin{pmatrix} \frac{\sqrt{3}}{2}\cos(s) \\ -\sin(s) \\ \frac{1}{2}\cos(s) \end{pmatrix}$$

Therefore,

$$\begin{aligned} v(s) &= \sqrt{\frac{3}{4}\cos^2(s) + \sin^2(s) + \frac{1}{4}\cos^2(s)} \\ &= \sqrt{\cos^2(s) + \sin^2(s)} \\ &= \sqrt{1} \\ v(s) &= 1 \end{aligned}$$

Therefore, since $v(s) = 1$ everywhere, it means that $x(s)$ is arc-length parameterised.

Curvature

The curvature is defined as $\kappa(s) = |T'(s)|$. Since $x(s)$ is unit-speed, $T(s) = x'(s)$ therefore $T'(s) = x''(s)$. Differentiating $x'(s)$ we get

$$x''(s) = \begin{pmatrix} -\frac{\sqrt{3}}{2}\sin(s) \\ -\cos(s) \\ -\frac{1}{2}\sin(s) \end{pmatrix}$$

Therefore,

$$\begin{aligned} \kappa(s) &= \sqrt{\frac{3}{4}\sin^2(s) + \cos^2(s) + \frac{1}{4}\sin^2(x)} \\ &= \sqrt{\cos^2(s) + \sin^2(s)} \\ \kappa(s) &= 1 \end{aligned}$$

Frenet-Serret frame

From our previous calculations, $T(s) = \{\frac{\sqrt{3}}{2}\cos(s), -\sin(s), \frac{1}{2}\cos(s)\}$, $T'(s) = \{-\frac{\sqrt{3}}{2}\sin(s), -\cos(s), -\frac{1}{2}\sin(s)\}$, κ=1.
Therefore, the principal normal is:

$$N(s) = \frac{T'(s)}{1} = \begin{pmatrix} -\frac{\sqrt{3}}{2}\sin(s) \\ -\cos(s) \\ -\frac{1}{2}\sin(s) \end{pmatrix}$$

Following that, the binormal is:

$$B(s) = \begin{pmatrix} \frac{\sqrt{3}}{2}\cos(s) \\ -\sin(s) \\ \frac{1}{2}\cos(s) \end{pmatrix} \times \begin{pmatrix} -\frac{\sqrt{3}}{2}\sin(s) \\ -\cos(s) \\ -\frac{1}{2}\sin(s) \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{2}\sin^2(s) + \frac{1}{2}\cos^2(s) \\ -\frac{\sqrt{3}}{4}\cos(s)\sin(s) + \frac{\sqrt{3}}{4}\cos(s)\sin(s) \\ -\frac{\sqrt{3}}{2}\cos^2(s) - \frac{\sqrt{3}}{2}\sin^2(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{\sqrt{3}}{2} \end{pmatrix}$$

Therefore, the Frenet-Serret Frame $\{T(s), N(s), B(s)\}$ is:

$$\left\{ \begin{pmatrix} \frac{\sqrt{3}}{2}\cos(s) \\ -\sin(s) \\ \frac{1}{2}\cos(s) \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2}\sin(s) \\ -\cos(s) \\ -\frac{1}{2}\sin(s) \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \right\}$$

Torsion

The torsion is defined as $\tau = -B' \cdot N$
Since B is made up of constants, $B' = 0$, Therefore $\tau = 0$

Question 2

A *sphere curve* is a curve $x(s)$ in Euclidean space which lies on the surface of a sphere,

$$|x(s) - p|^2 = R^2$$

with radius R centred at p. Prove that the curvature of a sphere curve $\kappa \geq R^{-1}$. Next, prove that a sphere curve with constant curvature must be part of a circle.
[Hint: differentiate (1) a few times and use the structure equations]
[4 Marks]

$$|x(s) - p|^2 = \left(\sqrt{(x(s)-p)\cdot(x(s)-p)}\right)^2 = (x(s)-p)\cdot(x(s)-p) = R^2$$

via dot product rules ($\frac{d}{dx}(a \cdot b) = \frac{da}{dx} \cdot b + a \cdot \frac{db}{dx}$), differentiating w.r.t. s gives

$$\begin{aligned}
\frac{d}{dx}(x(s)-p)\cdot(x(s)-p) + \frac{d}{dx}(x(s)-p)\cdot(x(s)-p) &= 0 \\
x'(s)\cdot(x(s)-p) + x'(s)\cdot(x(s)-p) &= 0 \\
2(x'(s)\cdot(x(s)-p))) &= 0 \\
x'(s)\cdot(x(s)-p) &= 0
\end{aligned}$$

For a unit-speed curve, $x'(s) = T(s)$, therefore

$$T(s)\cdot(x(s)-p) = 0$$

Differentiating again w.r.t. s gives:

$$\begin{aligned}
\frac{d}{dx}(T(s))\cdot(x(s)-p) + \frac{d}{dx}(x(s)-p)\cdot T(s) &= 0 \\
T'(s)\cdot(x(s)-p) + x'(s)\cdot T(s) &= 0 \\
T'(s)\cdot(x(s)-p) + T(s)\cdot T(s) &= 0 \\
\text{(via unit speed, } T(s)\cdot T(s) = 1) \quad T'(s)\cdot(x(s)-p) + 1 &= 0 \\
\text{(substitute FS equation)} \quad (\kappa N(s))\cdot(x(s)-p) &= -1 \\
(\kappa \text{ is a scalar/constant}) \quad \kappa(N(s))\cdot(x(s)-p) &= -1
\end{aligned}$$

From this, we can rearrange to isolate κ

$$\implies \kappa = -\frac{1}{N(s)\cdot(x(s)-p)}$$

Since κ is always positive,

$$\kappa = \left| -\frac{1}{N(s) \cdot (x(s) - p)} \right| = \frac{1}{|N(s) \cdot (x(s) - p|}$$

From the definition of the dot product,

$$N(s) \cdot (x(s) - p) = |N(s)||x(s) - p| \cos \theta$$

From the original equation we have that $|x(s) - p| = R$, and also $|N(s)| = 1$ since it is a unit vector, therefore

$$N(s) \cdot (c(s) - p) = R \cos \theta$$

Furthermore, since we are taking the absolute value,

$$|N(s) \cdot (c(s) - p)| = |R \cos \theta| = R|\cos \theta| \quad \text{(cannot have negative radius)}$$

Therefore, finally we can substitute back into the sum to get

$$\kappa = \frac{1}{R|\cos \theta|} \geq \frac{1}{R}$$

as required.

$$\begin{aligned} \kappa(N(s)) \cdot (x(s) - p) &= -1 \\ N(s) \cdot (x(s) - p) &= -\frac{1}{\kappa} \\ \frac{d}{dx}(N(s)) \cdot (x(s) - p) + \frac{d}{dx}(x(s) - p) \cdot (N(s)) = 0 \\ (N'(s)) \cdot (x(s) - p) + (x'(s)) \cdot (N(s)) = 0 \\ (-\kappa T(s) + \tau B(s)) \cdot (x(s) - p) + T(s) \cdot N(s) = 0 \\ \color{red}{T(s) \text{ and } N(s) \text{ are perpendicular so } T(s) \cdot N(s) = 0} \\ (-\kappa T(s)) \cdot (x(s) - p) + (\tau B(s)) \cdot (x(s) - p) = 0 \end{aligned}$$

somehow τ goes to 0

Therefore, by the fundamental theorem of curves, since $\tau = 0$ and κ is constant, it means the curve is a circle.

Question 3

Let $c : [0, 2\pi] \to \mathbb{R}^3$ be the helix $c(t) = (\cos(t), \sin(t), t)$ and consider the 1-form on $\mathbb{R}^3$

$$\alpha = 2x^1x^2dx^1 + (x^1)^2dx^2 + x^3dx^3$$

Find the tangent vector $c'(t)$ at each point along the curve. Hence evaluate

the line integral of the 1-form α along the curve c
[3 Marks]

First, finding the tangent vector $c'(t)$, we get

$$c(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix} \quad \implies \quad c'(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix}$$

Changing the notation of α, let $x^1 = x$, $x^2 = y$, $x^3 = z$.

$$\begin{aligned} \alpha &= 2x^1x^2dx^1 + (x^1)^2dx^2 + x^3dx^3 \\ \alpha &= 2xy \cdot dx + x^2 \cdot dy + zdz \end{aligned}$$

Substituting values of c we get:

$$\begin{aligned} \alpha &= 2xydx + x^2dy + zdz \\ a(c'(t)) &= 2 \cdot \cos(t) \cdot \sin(t) \cdot (-\sin(t)) + \cos^2(t) \cdot \cos(t) + t \cdot 1 \\ &= -2\sin^2(t)\cos(t) + \cos^3(t) + t \end{aligned}$$

Therefore,

$$\int_c \alpha = \int_a^b \alpha(c'(t))\,dt = \int_0^{2\pi} -2\sin^2(t)\cos(t) + \cos^3(t) + t\,dt$$

Solving the first part:

$$-2\int_0^{2\pi} \sin^2(t)\cos(t)\,dx, \qquad \text{let } u = \sin(t),\ du = \cos(t)$$

$$\implies -2\int_0^0 u^2 = 0$$

Solving the second part

$$\int_0^{2\pi} \cos^3(t)\,dt = \int_0^{2\pi} \cos(t)(1 - \sin^2(t))\,dt, \qquad \text{let } u = \sin(t),\ du = (\cos(t))$$

$$\implies \int_0^0 (1 - u^2)\,du = 0$$

Solving the third part

$$\int_0^{2\pi} t\,dt = 2\pi^2$$

Therefore, the line integral is evaluated as

$$\int_c \alpha = 2\pi^2$$