> Question 5

A real number is algebraic if it satisfies some polynomial equation with integer coefficients. Why is the set of algebraic numbers countable?

Proof

- The set of algebraic numbers is the union of the set of every degree polynomial $\cup A_n$ where n is a natural number corresponding to the degree.
- For A_1, or the set of polynomials with degree 1, i.e. any equation that satisfies $n = N$, clearly the set is just the natural numbers, implying countable
- For A_2 or the set of polynomials with degree 2, i.e. any equation that satisfies $n + ax = N$, there are in total $\mathbb{N}^2$ different polynomials with at most 2 distinct roots, which is also countable.
- It follows that in general for A_n, or the set of polynomials with degree n, there are in total $\mathbb{N}^n$ elements of the set which is countable.
- Since the set of algebraic numbers is $\cup A_n, n \in \mathbb{N}$ and a countable union of countable sets is also countable, this implies that the set of algebraic numbers is also countable.

> Question 6a

Let (a_n) be a sequence of real numbers and $a \in \mathbb{R}$. Suppose $a_n \to a$. Show that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \to a$$

Proof

Let $|a_n| \leq M$, $\forall n$. /given $\epsilon > 0$ find N such that $\forall n \geq N$,

$$|a_n - a| < \epsilon$$

From the triangle inequality,

$$\left|\frac{a_1+a_2+\cdots+a_n}{n}-a\right| \leq \frac{1}{n}\sum_{k=1}^{n}|a_k-a|$$

Splitting the sum up, we get

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}|a_k-a| &= \frac{1}{n}\sum_{k=1}^{N-1}|a_k-a| + \frac{1}{n}\sum_{k=N}^{n}|a_k-a| \\ &\leq \frac{2(N-1)M}{n} + \frac{(n-N+1)\epsilon}{n}\end{aligned}$$

for some fixed number M and N. when we take $n \to \infty$, the first term converges to 0 and the second is less than ϵ. Therefore for large n we have

$$\left|\frac{a_1+a_2+\cdots+a_n}{n}-a\right| < 2\epsilon$$

Question 6b

Find a sequence such that (a_n) does not converge but

$$\frac{a_1+a_2+\cdots+a_n}{n}$$

does

The sequence

$$a_n = \begin{cases} 1 & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$$

doesn't converge as it is an oscillating function, but the equation

$$\frac{0+1+0+1+\cdots}{n} \quad \text{or} \quad \frac{\overbrace{1+1+\cdots+1}^{n/2}}{n}$$

converges to $\frac{1}{2}$