Question 4

Suppose that (x_n) is a Cauchy sequence such that x_n is an integer for all $n \in \mathbb{N}$. Prove that (x_n) is eventually constant, that is there exists $a \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $x_n = a$ for all $n \geq N$.

From Lecture notes Def 1.3, a sequence (x_n) of numbers $x_n \in \mathbb{R}$ is said to be Cauchy if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \epsilon \quad \forall n, m \geq N$$

In this sequence, since every $x_n \in \mathbb{Z}$, the difference $|x_n - x_m|$ must also be an integer for any n and m.
Since a Cauchy sequence works for every $\epsilon > 0$, then for values of ϵ that are smaller than 1, e.g. $\epsilon = 0.5$ there must be a point N where the difference $|x_n - x_m|$ must be an integer smaller than 1 for every single n and m past that point, and the only integer where this is possible is 0
From this, the only way this can happen is if past some N, we have that

$$x_n = x_m, \quad \forall n, m$$

or in other words, after some point N there exists some $a \in \mathbb{Z}$ where $x_n = a$ for all $n \geq N$

Question 8

Using the definition of $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$ find them for $x_n = 2 + (-1)^n$

From Lecture notes Def. 1.5, $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ are defined as

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right), \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

- For the sequence $x_n = 2 + (-1)^n$, as you increase n to infinity the values of x_k for $k \geq n$ will oscillate between 3 and 1, but never exceeding 3. Therefore the supremum for every n is 3. Therefore,

$$\limsup_{n\to\infty} x_n = 3$$

- The same can be said for the Infimum, as n goes to infinity the values of x_k for $k \geq n$ oscillate between 1 and 3, but never smaller than 1. Therefore the infimum for every n is 1. Therefore

$$\liminf_{n \to \infty} x_n = 1$$