Question 4

Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that $\sum_{n=1}^{\infty} |a_n|^p$ converges for all $p \geq 1$

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, by definition it means $\sum_{n=1}^{\infty} |a_n|$ converges.

So we are actually proving that $\sum_{n=1}^{\infty} (b_n)^p$ converges, where $b_n > 0, \ \forall n \in \mathbb{N}$ and $p \geq 1$

Via, the limit comparison test:

Thm: Limit Comparison test

Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be two real sequences with $a_n \geq 0$ and $b_n \geq 0$ for all n. Assume that $\frac{a_n}{b_n} \to L$ for some $L \in (0, \infty)$. Then, $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges.

Suppose we have our absolutely convergent series $\sum_{n=1}^{\infty} a_n$. We now define two sequences (b_n) where $b_n = |a_n|$, and a sequence (c_n) where $c_n = (b_n)^p$, $p \geq 1$ ($\sum_{n=1}^{\infty} b_n$ is convergent as stated above)

The fraction $\frac{(b_n)^p}{b_n}$ will simplify to $(b_n)^{p-1}$. Since $b_n \in (0, \infty)$, and $p \geq 1$ this must mean that we also have that $(b_n)^{p-1} \in (0, \infty)$.

Therefore via the limit comparison test, since $\sum_{n=1}^{\infty} b_n$ converges, then so must $\sum_{n=1}^{\infty} c_n$.

Since $c_n = (b_n)^p = |a_n|^p$, this is equivalent in saying that $\sum_{n=1}^{\infty} |a_n|^p$ converges.

Question 9

Let $f : (0,1) \to \mathbb{R}$ be a function and let $a \in (0,1)$. Match each statement in Group A with a statement in Group B which means the same thing:

Group A

i) $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$

ii) $\forall \epsilon > 0$, $\forall \delta > 0$, $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$

iii) $\exists \epsilon > 0$ such that $\forall \delta > 0$, $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$

iv) $\exists \epsilon > 0$ and $\exists \delta > 0$ such that $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$

v) $\forall \delta > 0$, $\exists \epsilon > 0$ such that $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$

vi) $\exists \delta > 0$ such that $\forall \epsilon > 0$, $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$

Group B

a) f is continuous at a

b) f is bounded on $(0,1)$

c) f is constant on $(0,1)$

d) There is some neighbourhood of a on which f is bounded.

e) There is some neighbourhood of a on which f is constant.

i - a
ii - c
iii -b
iv - d
v -b
vi - e