Differential Cohomology and Virasoro Central Extensions

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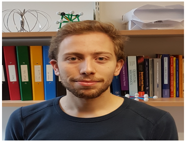
April 3rd 2022

Based on [arXiv:2112.10837]
Joint with Arun Debray and Christoph Weis



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Motivation

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- The central extension is describe by the Bott-Thurston cocyle.
- The goal of this talk is to give a novel geometric description these central extensions, using differential cohomology, affirmativaly answering a conjecture of Freed-Hopkins.

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Bott-Thurston cocycles

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The Virasoro group Vir_λ, for $\lambda \in \mathbf{R}$, is a $U(1)$ central extension of $\mathrm{Diff}^+(S^1)$, described by the Bott-Thurston cocycle $B_\lambda : \mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1) \to U(1)$:

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$$B_\lambda(\gamma_1, \gamma_2) = \exp\left(-\frac{i\lambda}{48\pi} \int_{S^1} \log(\gamma_1' \circ \gamma_2)\, \mathrm{d}(\log(\gamma_2))'\right) \tag{1}$$

for $\gamma_1, \gamma_2 \in \mathrm{Diff}^+(S^1)$, viewed as morphisms $S^1 \to S^1$.

Central Extensions

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Definition

Let G be a group and A be an abelian group, a central extension of G by A is a group $\tilde{G}$ with short exact sequence:

$$0 \to A \to \tilde{G} \to G \to 1 \tag{2}$$

such that subgroup $A \subset \tilde{G}$ is in the center, that is, it commutes with every element of $\tilde{G}$.

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Proposition

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Given a cocycle class $b \in C^2(G; A)$, viewed as a map $b : G \times G \to A$ satisying some cocycle conditions. Then $\tilde{G} = G \times A$ as a set, with multiplication $(g, a) \cdot (g', a') := (g \cdot g', a + a' + b(g, g'))$.

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The map $\tilde{G} \to G$ is $(g, a) \mapsto g$.

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The answer is differential cohomology.

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Sheaves on smooth manifolds

Let M be a manifold, then the ordinary cohomology groups $H^*(M; A)$ depends only on the homotopy classes of M. It is the cohomology of the constant sheave $\underline{A}$ on M. On the other hand, the i-th differential form on M, $\Omega^i(M)$ is sensitive to the smooth structure of M.

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We can view both constant sheaves and differential forms as sheaves on *Mfld*, the site of smooth manifolds.

Even though Ω^i are not homotopy invariant, the chain complex of sheaves $\Omega^* = 0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \ldots$ is a homotopy invariant, in fact,

Theorem (de Rham)

The chain complex Ω^* is the constant sheave $\underline{\mathbb{R}}$, as a chain complex concentrated in degree 0.

Sheaves $\mathbb{Z}(n)$

With this in mind, we define the (chain complex of) sheave $\mathbb{Z}(n)$ as

$$\mathbb{Z}(n) = \underline{\mathbb{Z}} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{n-1} \to 0. \tag{4}$$

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These sheaves $\mathbb{Z}(n)$ are both sensitive to topology (from $\mathbb{Z}$) and the smooth structure (from Ω^i).

There is also a form of integration. let M be a closed oriented d-dimensional manifold, then there is an integration map:

$$\int_M : H^*(M; \mathbb{Z}(n)) \to H^{*-d}(*; \mathbb{Z}(n-d)). \tag{5}$$

There is also a relative version of this.

Example: $\mathbb{Z}(1)$ and line bundles

A cocycle in $C^2(M;\mathbb{Z}(1))$ can be describe as follows: fix an open covering $\{U_i\}$ of M, we have 0-form ($\mathbb{R}$-valued functions) a on the open subsets $U_{ij} = U_i \cap U_j$, and $\mathbb{Z}$-valued functions f_{ijk} on intersections U_{ijk}, such that $a_{ij} - a_{jk} + a_{ik} = f_{ijk}$.

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This precisely describe the data of a $U(1)$ principal bundle on M!

Proposition

$H^2(M;\mathbb{Z}(1))$ is the group of isomorphism classes of $U(1)$ principal bundles on M.

Furthermore, $H^2(M;\mathbb{Z}(2))$ is the group of isomorpism classes of $U(1)$ principal bundles with connections on M (Hint: for the cocycle here we need also 1-form α_i on U_i, with $\alpha_i - \alpha_j = da_{ij}$).

$\mathbb{Z}(1)$ and central extensions

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Theorem

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$B_\bullet G$ is the classifying space of G, viewed as a sheave on $Mfld$.

Differential characteristic classes

Let G be a Lie group, and $B_\bullet G$ the classifying space. Then $H^*(B_\bullet G; \mathbb{Z})$ are the characteristic classes of G. Similiarly, $H^*(B_\bullet G; \mathbb{Z}(n))$ are differential characetristic classes of G.

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We will need the following key fact:

Theorem (Bott, Freed-Hopkins)

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They are the differential first Pontryagin classes.

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We want to get a $\mathbb{R}$ family of central extension of $\mathrm{Diff}^+(S^1)$ by $U(1)$, therefore we want a $\mathbb{R}$ family in $H^3(B_\bullet \mathrm{Diff}^+(S^1), \mathbb{Z}(1))$. We get this by pullback and integration:

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- The quotient $S^1/\mathrm{Diff}^+(S^1)$ has a map to $B_\bullet\mathrm{Diff}^+(S^1) = */\mathrm{Diff}^+(S^1)$. Since the action of $\mathrm{Diff}^+(S^1)$ on S^1 is orientation preserving, this is a oriented S^1 fiber bundle.

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- The tangent bundle of S^1 gives a map $TS^1 : S^1 \to B_\bullet \mathrm{GL}_1^+(\mathbb{R})$. Since the action of $\mathrm{Diff}^+(S^1)$ on S^1 is smooth, the tangent bundle is $\mathrm{Diff}^+(S^1)$-equivariant. Equivalently, the tangent bundle factors through the quotient as a map $TS^1 : S^1/\mathrm{Diff}^+(S^1) \to B_\bullet \mathrm{GL}_1^+(\mathbb{R})$.

Key Idea II

To summarize, we have a span of maps:

$$\begin{tikzcd} S^1/\mathrm{Diff}^+(S^1) \arrow[r, "TS^1"] \arrow[d] & B_\bullet \mathrm{GL}_1^+(\mathbb{R}) \\ B_\bullet \mathrm{Diff}^+(S^1). \end{tikzcd} \tag{7}$$

Note the vertical map is a S^1 fibration, something we can integrate against. Therefore we get a map:

$$\begin{tikzcd} H^4(S^1/\mathrm{Diff}^+(S^1); \mathbb{Z}(2)) \arrow[d, "\int_{S^1}"] & H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \arrow[l] \\ H^3(B_\bullet \mathrm{Diff}^+(S^1); \mathbb{Z}(1)). \end{tikzcd} \tag{8}$$

Main theorem

Finally, we can state the conjecture of Freed and Hopkins that we proved:

Theorem (Y.L., Arun Debray, Christoph Weis)

The image of map $\mathbb{R} \simeq H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \to H^3(B_\bullet \mathrm{Diff}^+(S^1); \mathbb{Z}(1))$ *are the Virasoro central extensions* Vir_λ*.*

Furthermore, we explicitly recovers the Bott-Thurston cocylces when calculating the map on cocycles.

Proof sketch I

We construct explicit cocycles and compute the map on the level of cocycles.

- We find 1-form cocycles for $H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2))$, using the canonical simplicial resolution of $B_\bullet \mathrm{GL}_1^+(\mathbb{R})$.

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- We find 1-form cocycles for $H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2))$, using the canonical simplicial resolution of $B_\bullet \mathrm{GL}_1^+(\mathbb{R})$.
- We pullback to 1-form cocycles on $S^1/\mathrm{Diff}^+(S^1)$, using the simplicial realization of $S^1 = Fr_+(S^1)/\mathrm{GL}_1^+(\mathbb{R})$, where $Fr_+(S^1)$ is the oriented frame bundle.

Proof sketch II

Here's the key point of the proof:

- Now we move the cocycles across the double complex associated to the bisimplicial object $S^1/\mathrm{Diff}^+(S^1) = \mathrm{GL}_1^+(\mathbb{R})\backslash Fr_+(S^1)/\mathrm{GL}_1^+(\mathbb{R})$, to get cocycles on the simplicial resolution for $S^1/\mathrm{Diff}^+(S^1)$.

$$
\begin{tikzcd}
 & & \Omega^1(F\times\mathbb{R}^{\times 2}) \\
 & & -\log(\gamma_1'\circ\gamma_2)\,\mathrm{d}\log(\gamma_2) \\
 & \Omega^1(\Gamma\times F\times\mathbb{R}) & \Omega^1(F\times\mathbb{R}) \arrow[l] \arrow[uu] \\
 & {\color{blue}x\log\gamma'-} & \log(v)\ \mathrm{d}\log(\gamma') \arrow[l, mapsto, blue] \arrow[uu, mapsto] \\
 & {\color{red}x\log\gamma'}=0 & \\
\Omega^1(\Gamma^{\times 2}\times F) & \Omega^1(\Gamma\times F) \arrow[l] \arrow[uuu] & \\
x_2\mathrm{d}x_1 & x\ \mathrm{d}\log(v) \arrow[l, mapsto] \arrow[uu, mapsto, red] &
\end{tikzcd}
$$

Proof sketch III

- Lastly, we integrate over S^1 and immediately see that we recover the Bott-Thurston cocycles! Q.E.D.

Last slide

Thank you for listening!