Differential Cohomology and Virasoro Central Extensions

Yu Leon Liu

Harvard University

April 3rd 2022

Based on [arXiv:2112.10837]
Joint work with Arun Debray and Christopher Weis

Table of Contents

1. Motivation
2. Virasoro groups and central extensions
3. Differential cohomology
4. Main theorem

Table of Contents

1. Motivation
2. Virasoro groups and central extensions
3. Differential cohomology
4. Main theorem

Motivation

- The Virasoro groups describe space-time symmetries of 2d CFTs. As such, it is important to physics (string theory, condensed matter) and mathematics (geometric representation theory).

Motivation

- The Virasoro groups describe space-time symmetries of 2d CFTs. As such, it is important to physics (string theory, condensed matter) and mathematics (geometric representation theory).
- Virasoro groups is a **R** family of central extension of $\mathrm{Diff}^+(S^1)$, the group of orientation preserving smooth automorphism of S^1.

Motivation

- The Virasoro groups describe space-time symmetries of 2d CFTs. As such, it is important to physics (string theory, condensed matter) and mathematics (geometric representation theory).
- Virasoro groups is a **R** family of central extension of $\mathrm{Diff}^+(S^1)$, the group of orientation preserving smooth automorphism of S^1.
- The central extension is describe by the Bott-Thurston cocyle.

Motivation

- The Virasoro groups describe space-time symmetries of 2d CFTs. As such, it is important to physics (string theory, condensed matter) and mathematics (geometric representation theory).
- Virasoro groups is a **R** family of central extension of $\mathrm{Diff}^{+}(S^1)$, the group of orientation preserving smooth automorphism of S^1.
- The central extension is describe by the Bott-Thurston cocyle.
- The goal of this talk is to give a novel geometric description these central extensions, using differential cohomology, affirmativaly answering a conjecture of Freed-Hopkins.

Table of Contents

1. Motivation
2. Virasoro groups and central extensions
3. Differential cohomology
4. Main theorem

Bott-Thurston cocycles

Recall that $\mathrm{Diff}^+(S^1)$ is the group of orientation preserving smooth automorphism of S^1. It is an infinite-dimensional Frechet Lie group.

Bott-Thurston cocycles

Recall that $\mathrm{Diff}^+(S^1)$ is the group of orientation preserving smooth automorphism of S^1. It is an infinite-dimensional Frechet Lie group.

Definition

The Virasoro group Vir_λ, for $\lambda \in \mathbf{R}$, is a $U(1)$ central extension of $\mathrm{Diff}^+(S^1)$, described by the Bott-Thurston cocycle $B_\lambda : \mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1) \to U(1)$:

Bott-Thurston cocycles

Recall that $\mathrm{Diff}^+(S^1)$ is the group of orientation preserving smooth automorphism of S^1. It is an infinite-dimensional Frechet Lie group.

Definition

The Virasoro group Vir_λ, for $\lambda \in \mathbf{R}$, is a $U(1)$ central extension of $\mathrm{Diff}^+(S^1)$, described by the Bott-Thurston cocycle $B_\lambda : \mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1) \to U(1)$:

$$B_\lambda(\gamma_1, \gamma_2) = \exp\left(-\frac{i\lambda}{48\pi} \int_{S^1} \log(\gamma_1' \circ \gamma_2)\, \mathrm{d}(\log(\gamma_2))'\right) \tag{1}$$

for $\gamma_1, \gamma_2 \in \mathrm{Diff}^+(S^1)$, viewed as morphisms $S^1 \to S^1$.

Central Extensions

Let's briefly review what is a central extension:

Central Extensions

Let's briefly review what is a central extension:

Definition

Let G be a group and A be an abelian group, a central extension of G by A is a group $\tilde{G}$ with short exact sequence:

$$0 \to A \to \tilde{G} \to G \to 1 \tag{2}$$

such that subgroup $A \subset \tilde{G}$ is in the center, that is, it commutes with every element of $\tilde{G}$.

Central extension as group cohomology: I

As many other things, central extensions can be classified by cohomology groups:

Central extension as group cohomology: I

As many other things, central extensions can be classified by cohomology groups:

Proposition

Let G be a discrete group, then the isomorphism class of central extensions of G by A is classified by group cohomology class $H^2(G; A) \simeq H^2(BG; A)$, where BG is the classifying space of G.

Central extension as group cohomology: I

As many other things, central extensions can be classified by cohomology groups:

Proposition

Let G be a discrete group, then the isomorphism class of central extensions of G by A is classified by group cohomology class $H^2(G;A) \simeq H^2(BG;A)$, where BG is the classifying space of G.

Given a cocycle class $b \in C^2(G;A)$, viewed as a map $b : G \times G \to A$ satisying some cocycle conditions. Then $\tilde{G} = G \times A$ as a set, with multiplication $(g,a) \cdot (g',a') := (g \cdot g', a + a' + b(g,g'))$.

Central extension as group cohomology: I

As many other things, central extensions can be classified by cohomology groups:

Proposition

Let G be a discrete group, then the isomorphism class of central extensions of G by A is classified by group cohomology class $H^2(G; A) \simeq H^2(BG; A)$, where BG is the classifying space of G.

Given a cocycle class $b \in C^2(G; A)$, viewed as a map $b : G \times G \to A$ satisying some cocycle conditions. Then $\tilde{G} = G \times A$ as a set, with multiplication $(g, a) \cdot (g', a') := (g \cdot g', a + a' + b(g, g'))$.
The map $\tilde{G} \to G$ is $(g, a) \mapsto g$.

Central extension as group cohomology: II

Problem: Smooth $U(1)$ central extensions of nondiscret Lie groups are NOT classified by ordinary cohomology classes.

Central extension as group cohomology: II

Problem: Smooth $U(1)$ central extensions of nondiscret Lie groups are NOT classified by ordinary cohomology classes.

We need a cohomology theory that remembers the smooth structures.

Central extension as group cohomology: II

Problem: Smooth $U(1)$ central extensions of nondiscret Lie groups are NOT classified by ordinary cohomology classes.

We need a cohomology theory that remembers the smooth structures.

The answer is differential cohomology.

Table of Contents

1. Motivation
2. Virasoro groups and central extensions
3. Differential cohomology
4. Main theorem

Sheaves on smooth manifolds

Let M be a manifold, then the ordinary cohomology groups $H^*(M; A)$ depends only on the homotopy classes of M. It is the cohomology of the constant sheave $\underline{A}$ on M. On the other hand, the i-th cohomology form on M, $\Omega^i(M)$ is sensitive to the smooth structure of M.

Sheaves on smooth manifolds

Let M be a manifold, then the ordinary cohomology groups $H^*(M; A)$ depends only on the homotopy classes of M. It is the cohomology of the constant sheave $\underline{A}$ on M. On the other hand, the i-th cohomology form on M, $\Omega^i(M)$ is sensitive to the smooth structure of M.

We can view both constant sheaves and differential forms as sheaves on $Mfld$, the site of smooth manifolds.

Sheaves on smooth manifolds

Let M be a manifold, then the ordinary cohomology groups $H^*(M; A)$ depends only on the homotopy classes of M. It is the cohomology of the constant sheave $\underline{A}$ on M. On the other hand, the i-th cohomology form on M, $\Omega^i(M)$ is sensitive to the smooth structure of M.

We can view both constant sheaves and differential forms as sheaves on *Mfld*, the site of smooth manifolds.

Even though Ω^i are not homotopy invariant, the chain complex of sheaves $\Omega^* = 0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots$ is a homotopy invariant, in fact,

Theorem (de Rham)

The chain complex Ω^* is the constant sheave $\underline{\mathbb{R}}$, as a chain complex concentrated in degree 0.

Sheaves $\mathbb{Z}(n)$

With this in mind, we define the (chain complex of) sheave $\mathbb{Z}(n)$ as

$$\mathbb{Z}(n) = \underline{\mathbb{Z}} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{n-1} \to 0. \tag{4}$$

Sheaves $\mathbb{Z}(n)$

With this in mind, we define the (chain complex of) sheave $\mathbb{Z}(n)$ as

$$\mathbb{Z}(n) = \underline{\mathbb{Z}} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{n-1} \to 0. \tag{4}$$

These sheaves $\mathbb{Z}(n)$ are both sensitive to topology (from $\mathbb{Z}$) and the smooth structure (from Ω^i).

Sheaves $\mathbb{Z}(n)$

With this in mind, we define the (chain complex of) sheave $\mathbb{Z}(n)$ as

$$\mathbb{Z}(n) = \underline{\mathbb{Z}} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{n-1} \to 0. \tag{4}$$

These sheaves $\mathbb{Z}(n)$ are both sensitive to topology (from $\mathbb{Z}$) and the smooth structure (from Ω^i).

There is also a form of integration. let M be a closed oriented d-dimensional manifold, then there is an integration map:

$$\int_M : H^*(M; \mathbb{Z}(n)) \to H^{*-d}(*; \mathbb{Z}(n-d)). \tag{5}$$

There is also a relative version of this.

Example: $\mathbb{Z}(1)$ and line bundles

A cocycle in $C^2(M;\mathbb{Z}(1))$ can be describe as follows: fix an open covering $\{U_i\}$ of M, we have 0-form ($\mathbb{R}$-valued functions) a on the open subsets $U_ij = U_i \cap U_j$, and $\mathbb{Z}$-valued functions f_{ijk} on intersections U_{ijk}, such that $a_{ij} - a_{jk} + a_{ik} = f_{ijk}$.

Example: $\mathbb{Z}(1)$ and line bundles

A cocycle in $C^2(M;\mathbb{Z}(1))$ can be describe as follows: fix an open covering $\{U_i\}$ of M, we have 0-form ($\mathbb{R}$-valued functions) a on the open subsets $U_ij = U_i \cap U_j$, and $\mathbb{Z}$-valued functions f_{ijk} on intersections U_{ijk}, such that $a_{ij} - a_{jk} + a_{ik} = f_{ijk}$.

This precisely describe the data of a $U(1)$ principal bundle on M!

Proposition

$H^2(M;\mathbb{Z}(1))$ is the group of isomorphism classes of $U(1)$ principal bundles on M.

Furthermore, $H^2(M;\mathbb{Z}(2))$ is the group of isomorpism classes of $U(1)$ principal bundles with connections on M (Hint: for the cocycle here we need also 1-form α_i on U_i, with $\alpha_i - \alpha_j = da_{ij}$).

$\mathbb{Z}(1)$ and central extensions

While $H^2(-;\mathbb{Z}(1))$ classifies $U(1)$ principal bundle, $H^3(-;\mathbb{Z}(1))$ classifies $U(1)$ central extensions:

$\mathbb{Z}(1)$ and central extensions

While $H^2(-;\mathbb{Z}(1))$ classifies $U(1)$ principal bundle, $H^3(-;\mathbb{Z}(1))$ classifies $U(1)$ central extensions:

Theorem

Let G be a smooth (possibly infinite dimensional) Lie group, $B_\bullet G$ its classifying space. Then $H^3(BG;\mathbb{Z}(1))$ classifies smooth central extensions of G by $U(1)$.

$\mathbb{Z}(1)$ and central extensions

While $H^2(-;\mathbb{Z}(1))$ classifies $U(1)$ principal bundle, $H^3(-;\mathbb{Z}(1))$ classifies $U(1)$ central extensions:

Theorem

Let G be a smooth (possibly infinite dimensional) Lie group, $B_\bullet G$ its classifying space. Then $H^3(BG;\mathbb{Z}(1))$ classifies smooth central extensions of G by $U(1)$.

$B_\bullet G$ is the classifying space of G, viewed as a sheave on $Mfld$.

Differential characteristic classes

Let G be a Lie group, and $B_\bullet G$ the classifying space. Then $H^*(B_\bullet G;\mathbb{Z})$ are the characteristic classes of G. Similiarly, $H^*(B_\bullet G;\mathbb{Z}(n))$ are differential characetristic classes of G.

Differential characteristic classes

Let G be a Lie group, and $B_\bullet G$ the classifying space. Then $H^*(B_\bullet G; \mathbb{Z})$ are the characteristic classes of G. Similiarly, $H^*(B_\bullet G; \mathbb{Z}(n))$ are differential characetristic classes of G.

We will need the following key fact:

Theorem (Bott, Freed-Hopkins)

$H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \simeq \mathbb{R}$.

Differential characteristic classes

Let G be a Lie group, and $B_\bullet G$ the classifying space. Then $H^*(B_\bullet G; \mathbb{Z})$ are the characteristic classes of G. Similiarly, $H^*(B_\bullet G; \mathbb{Z}(n))$ are differential characetristic classes of G.

We will need the following key fact:

Theorem (Bott, Freed-Hopkins)

$H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \simeq \mathbb{R}$.

They are the differential first Pontryagin classes.

Table of Contents

1 Motivation

2 Virasoro groups and central extensions

3 Differential cohomology

4 Main theorem

Key Idea I

We want to get a $\mathbb{R}$ family of central extension of $\mathrm{Diff}^+(S^1)$ by $U(1)$, therefore we want a $\mathbb{R}$ family in $H^3(B_\bullet \mathrm{Diff}^+(S^1), \mathbb{Z}(1))$. We get this by pullback and integration:

Key Idea I

We want to get a $\mathbb{R}$ family of central extension of $\mathrm{Diff}^+(S^1)$ by $U(1)$, therefore we want a $\mathbb{R}$ family in $H^3(B_\bullet \mathrm{Diff}^+(S^1), \mathbb{Z}(1))$. We get this by pullback and integration:

Consider the canonical $\mathrm{Diff}^+(S^1)$ action on S^1, note that

Key Idea I

We want to get a $\mathbb{R}$ family of central extension of $\mathrm{Diff}^+(S^1)$ by $U(1)$, therefore we want a $\mathbb{R}$ family in $H^3(B_\bullet \mathrm{Diff}^+(S^1), \mathbb{Z}(1))$. We get this by pullback and integration:

Consider the canonical $\mathrm{Diff}^+(S^1)$ action on S^1, note that

- The quotient $S^1/\mathrm{Diff}^+(S^1)$ has a map to $B_\bullet \mathrm{Diff}^+(S^1) = */\mathrm{Diff}^+(S^1)$. Since the action of $\mathrm{Diff}^+(S^1)$ on S^1 is orientation preserving, this is a oriented S^1 fiber bundle.

Key Idea I

We want to get a $\mathbb{R}$ family of central extension of $\mathrm{Diff}^+(S^1)$ by $U(1)$, therefore we want a $\mathbb{R}$ family in $H^3(B_\bullet \mathrm{Diff}^+(S^1), \mathbb{Z}(1))$. We get this by pullback and integration:

Consider the canonical $\mathrm{Diff}^+(S^1)$ action on S^1, note that

- The quotient $S^1/\mathrm{Diff}^+(S^1)$ has a map to $B_\bullet \mathrm{Diff}^+(S^1) = */\mathrm{Diff}^+(S^1)$. Since the action of $\mathrm{Diff}^+(S^1)$ on S^1 is orientation preserving, this is a oriented S^1 fiber bundle.
- The tangent bundle of S^1 gives a map $TS^1 : S^1 \to B_\bullet \mathrm{GL}_1^+(\mathbb{R})$. Since the action of $\mathrm{Diff}^+(S^1)$ on S^1 is smooth, the tangent bundle is $\mathrm{Diff}^+(S^1)$-equivariant. Equivalently, the tangent bundle factors through the quotient as a map $TS^1 : S^1/\mathrm{Diff}^+(S^1) \to B_\bullet \mathrm{GL}_1^+(\mathbb{R})$.

To summarize, we have a span of maps:

$$
\begin{tikzcd}
S^1/\mathrm{Diff}^+(S^1) \arrow[r, "TS^1"] \arrow[d] & B_\bullet \mathrm{GL}_1^+(\mathbb{R}) \\
B_\bullet \mathrm{Diff}^+(S^1). &
\end{tikzcd}
\tag{7}
$$

Note the vertical map is a S^1 fibration, something we can integrate against. Therefore we get a map:

$$
\begin{tikzcd}
H^4(S^1/\mathrm{Diff}^+(S^1); \mathbb{Z}(2)) \arrow[d, "\int_{S^1}"] & H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \arrow[l] \\
H^3(B_\bullet \mathrm{Diff}^+(S^1); \mathbb{Z}(1)). &
\end{tikzcd}
\tag{8}
$$

Main theorem

Finally, we can state the conjecture of Freed and Hopkins that we proved:

Theorem (Y.L., Arun Debray, Christoph Weis)

The image of map $\mathbb{R} \simeq H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \to H^3(B_\bullet \mathrm{Diff}^+(S^1); \mathbb{Z}(1))$ *are the Virasoro central extensions* Vir_λ.

Furthermore, we explicitly recovers the Bott-Thurston cocylces when calculating the map on cocycles.

Proof sketch I

We construct explicit cocycles and compute the map on the level of cocycles.

- We find 1-form cocycles for $H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2))$, using the canonical simplicial resolution of $B_\bullet \mathrm{GL}_1^+(\mathbb{R})$.

Proof sketch I

We construct explicit cocycles and compute the map on the level of cocycles.

- We find 1-form cocycles for $H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2))$, using the canonical simplicial resolution of $B_\bullet \mathrm{GL}_1^+(\mathbb{R})$.
- We pullback to 1-form cocycles on $S^1/\mathrm{Diff}^+(S^1)$, using the simplicial realization of $S^1 = Fr_+(S^1)/\mathrm{GL}_1^+(\mathbb{R})$, where $Fr_+(S^1)$ is the oriented frame bundle.

Proof sketch II

Here's the key point of the proof:

- Now we move the cocycles across the double complex associated to the bisimplicial object $S^1/\mathrm{Diff}^+(S^1) = \mathrm{GL}_1^+(\mathbb{R})\backslash Fr_+(S^1)/\mathrm{GL}_1^+(\mathbb{R})$, to get cocycles on the simplicial resolution for $S^1/\mathrm{Diff}^+(S^1)$.

$$\begin{tikzcd}
 & & \Omega^1(F\times\mathbb{R}^{\times 2}) \\
 & & -\log(\gamma_1'\circ\gamma_2)\,\mathrm{d}\log(\gamma_2) \\
 & \Omega^1(\Gamma\times F\times\mathbb{R}) \arrow[uu, phantom] & \Omega^1(F\times\mathbb{R}) \arrow[l] \arrow[uu] \\
 & x\log\gamma' - x\log\gamma' = 0 & \log(v)\ \mathrm{d}\log(\gamma') \arrow[l, mapsto] \arrow[uu, mapsto] \\
\Omega^1(\Gamma^{\times 2}\times F) & \Omega^1(\Gamma\times F) \arrow[l] \arrow[uu] & \\
x_2\mathrm{d}x_1 & x\ \mathrm{d}\log(v) \arrow[l, mapsto] \arrow[uu, mapsto] &
\end{tikzcd}$$

Proof sketch III

- Lastly, we integrate over S^1 and immediately see that we recover the Bott-Thurston cocycles! Q.E.D.

Last slide

Thank you for listening!