

Differential Cohomology and Virasoro Central Extensions

Yu Leon Liu

Harvard University

April 3rd 2022

Based on [[arXiv:2112.10837](https://arxiv.org/abs/2112.10837)]

Joint with Arun Debray and Christoph Weis



Table of Contents

- 1 Motivation
- 2 Virasoro groups and central extensions
- 3 Differential cohomology
- 4 Main theorem

Table of Contents

- 1 Motivation
- 2 Virasoro groups and central extensions
- 3 Differential cohomology
- 4 Main theorem

Motivation

- The **Virasoro groups** describe space-time symmetries of 2d CFTs. As such, it is important to physics (string theory, condensed matter) and mathematics (geometric representation theory).

Motivation

- The **Virasoro groups** describe space-time symmetries of 2d CFTs. As such, it is important to physics (string theory, condensed matter) and mathematics (geometric representation theory).
- **Virasoro groups** is a \mathbf{R} family of central extension of $\text{Diff}^+(S^1)$, the group of orientation preserving smooth automorphism of S^1 .

Motivation

- The **Virasoro groups** describe space-time symmetries of 2d CFTs. As such, it is important to physics (string theory, condensed matter) and mathematics (geometric representation theory).
- **Virasoro groups** is a \mathbf{R} family of central extension of $\text{Diff}^+(S^1)$, the group of orientation preserving smooth automorphism of S^1 .
- The central extension is describe by the **Bott-Thurston cocycle**.

Motivation

- The **Virasoro groups** describe space-time symmetries of 2d CFTs. As such, it is important to physics (string theory, condensed matter) and mathematics (geometric representation theory).
- **Virasoro groups** is a **R** family of central extension of $\text{Diff}^+(S^1)$, the group of orientation preserving smooth automorphism of S^1 .
- The central extension is describe by the **Bott-Thurston cocycle**.
- The goal of this talk is to give a novel geometric description these central extensions, using differential cohomology, affirmatively answering a conjecture of Freed-Hopkins.

Table of Contents

- 1 Motivation
- 2 Virasoro groups and central extensions
- 3 Differential cohomology
- 4 Main theorem

Bott-Thurston cocycles

Recall that $\text{Diff}^+(S^1)$ is the group of orientation preserving smooth automorphism of S^1 . It is an infinite-dimensional Frechet Lie group.

Bott-Thurston cocycles

Recall that $\text{Diff}^+(S^1)$ is the group of orientation preserving smooth automorphism of S^1 . It is an infinite-dimensional Frechet Lie group.

Definition

The Virasoro group Vir_λ , for $\lambda \in \mathbf{R}$, is a $U(1)$ central extension of $\text{Diff}^+(S^1)$, described by the **Bott-Thurston cocycle** $B_\lambda : \text{Diff}^+(S^1) \times \text{Diff}^+(S^1) \rightarrow U(1)$:

Bott-Thurston cocycles

Recall that $\text{Diff}^+(S^1)$ is the group of orientation preserving smooth automorphism of S^1 . It is an infinite-dimensional Frechet Lie group.

Definition

The Virasoro group Vir_λ , for $\lambda \in \mathbf{R}$, is a $U(1)$ central extension of $\text{Diff}^+(S^1)$, described by the **Bott-Thurston cocycle** $B_\lambda : \text{Diff}^+(S^1) \times \text{Diff}^+(S^1) \rightarrow U(1)$:

$$B_\lambda(\gamma_1, \gamma_2) = \exp \left(-\frac{i\lambda}{48\pi} \int_{S^1} \log(\gamma_1' \circ \gamma_2) d(\log(\gamma_2))' \right) \quad (1)$$

for $\gamma_1, \gamma_2 \in \text{Diff}^+(S^1)$, viewed as morphisms $S^1 \rightarrow S^1$.

Central Extensions

Let's briefly review what is a central extension:

Let's briefly review what is a central extension:

Definition

Let G be a group and A be an abelian group, a central extension of G by A is a group \tilde{G} with short exact sequence:

$$0 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \quad (2)$$

such that subgroup $A \subset \tilde{G}$ is in the center, that is, it commutes with every element of \tilde{G} .

Central extension as group cohomology: I

As many other things, central extensions can be classified by cohomology groups:

Central extension as group cohomology: I

As many other things, central extensions can be classified by cohomology groups:

Proposition

Let G be a discrete group, then the isomorphism class of central extensions of G by A is classified by group cohomology class $H^2(G; A) \simeq H^2(BG; A)$, where BG is the classifying space of G .

Central extension as group cohomology: I

As many other things, central extensions can be classified by cohomology groups:

Proposition

Let G be a discrete group, then the isomorphism class of central extensions of G by A is classified by group cohomology class $H^2(G; A) \simeq H^2(BG; A)$, where BG is the classifying space of G .

Given a cocycle class $b \in C^2(G; A)$, viewed as a map $b : G \times G \rightarrow A$ satisfying some cocycle conditions. Then $\tilde{G} = G \times A$ as a set, with multiplication $(g, a) \cdot (g', a') := (g \cdot g', a + a' + b(g, g'))$.

Central extension as group cohomology: I

As many other things, central extensions can be classified by cohomology groups:

Proposition

Let G be a discrete group, then the isomorphism class of central extensions of G by A is classified by group cohomology class $H^2(G; A) \simeq H^2(BG; A)$, where BG is the classifying space of G .

Given a cocycle class $b \in C^2(G; A)$, viewed as a map $b : G \times G \rightarrow A$ satisfying some cocycle conditions. Then $\tilde{G} = G \times A$ as a set, with multiplication $(g, a) \cdot (g', a') := (g \cdot g', a + a' + b(g, g'))$. The map $\tilde{G} \rightarrow G$ is $(g, a) \mapsto g$.

Central extension as group cohomology: II

Problem: Smooth $U(1)$ central extensions of nondiscrete Lie groups are **NOT** classified by ordinary cohomology classes.

Central extension as group cohomology: II

Problem: Smooth $U(1)$ central extensions of nondiscrete Lie groups are **NOT** classified by ordinary cohomology classes.

We need a cohomology theory that remembers the smooth structures.

Central extension as group cohomology: II

Problem: Smooth $U(1)$ central extensions of nondiscrete Lie groups are **NOT** classified by ordinary cohomology classes.

We need a cohomology theory that remembers the smooth structures.

The answer is **differential cohomology**.

Table of Contents

- 1 Motivation
- 2 Virasoro groups and central extensions
- 3 Differential cohomology
- 4 Main theorem

Sheaves on smooth manifolds

Let M be a manifold, then the ordinary cohomology groups $H^*(M; A)$ depends only on the homotopy classes of M . It is the cohomology of the constant sheave \underline{A} on M . On the other hand, the i -th cohomology form on M , $\Omega^i(M)$ is sensitive to the smooth structure of M .

Sheaves on smooth manifolds

Let M be a manifold, then the ordinary cohomology groups $H^*(M; A)$ depends only on the homotopy classes of M . It is the cohomology of the constant sheave \underline{A} on M . On the other hand, the i -th cohomology form on M , $\Omega^i(M)$ is sensitive to the smooth structure of M .

We can view both constant sheaves and differential forms as sheaves on $Mfld$, the site of smooth manifolds.

Sheaves on smooth manifolds

Let M be a manifold, then the ordinary cohomology groups $H^*(M; A)$ depends only on the homotopy classes of M . It is the cohomology of the constant sheave \underline{A} on M . On the other hand, the i -th cohomology form on M , $\Omega^i(M)$ is sensitive to the smooth structure of M .

We can view both constant sheaves and differential forms as sheaves on $Mfld$, the site of smooth manifolds.

Even though Ω^i are not homotopy invariant, the chain complex of sheaves $\Omega^* = 0 \rightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots$ is a homotopy invariant, in fact,

Theorem (de Rham)

The chain complex Ω^ is the constant sheave $\underline{\mathbb{R}}$, as a chain complex concentrated in degree 0.*

With this in mind, we define the (chain complex of) sheave $\mathbb{Z}(n)$ as

$$\mathbb{Z}(n) = \underline{\mathbb{Z}} \rightarrow \Omega^0 \rightarrow \Omega^1 \rightarrow \cdots \rightarrow \Omega^{n-1} \rightarrow 0. \quad (4)$$

With this in mind, we define the (chain complex of) sheave $\mathbb{Z}(n)$ as

$$\mathbb{Z}(n) = \underline{\mathbb{Z}} \rightarrow \Omega^0 \rightarrow \Omega^1 \rightarrow \cdots \rightarrow \Omega^{n-1} \rightarrow 0. \quad (4)$$

These sheaves $\mathbb{Z}(n)$ are both sensitive to topology (from $\underline{\mathbb{Z}}$) and the smooth structure (from Ω^i).

With this in mind, we define the (chain complex of) sheave $\mathbb{Z}(n)$ as

$$\mathbb{Z}(n) = \underline{\mathbb{Z}} \rightarrow \Omega^0 \rightarrow \Omega^1 \rightarrow \cdots \rightarrow \Omega^{n-1} \rightarrow 0. \quad (4)$$

These sheaves $\mathbb{Z}(n)$ are both sensitive to topology (from $\underline{\mathbb{Z}}$) and the smooth structure (from Ω^i).

There is also a form of integration. let M be a closed oriented d -dimensional manifold, then there is an integration map:

$$\int_M : H^*(M; \mathbb{Z}(n)) \rightarrow H^{*-d}(*; \mathbb{Z}(n-d)). \quad (5)$$

There is also a relative version of this.

Example: $\mathbb{Z}(1)$ and line bundles

A cocycle in $C^2(M; \mathbb{Z}(1))$ can be describe as follows: fix an open covering $\{U_i\}$ of M , we have 0-form (\mathbb{R} -valued functions) a on the open subsets $U_{ij} = U_i \cap U_j$, and \mathbb{Z} -valued functions f_{ijk} on intersections U_{ijk} , such that $a_{ij} - a_{jk} + a_{ik} = f_{ijk}$.

Example: $\mathbb{Z}(1)$ and line bundles

A cocycle in $C^2(M; \mathbb{Z}(1))$ can be describe as follows: fix an open covering $\{U_i\}$ of M , we have 0-form (\mathbb{R} -valued functions) a on the open subsets $U_{ij} = U_i \cap U_j$, and \mathbb{Z} -valued functions f_{ijk} on intersections U_{ijk} , such that $a_{ij} - a_{jk} + a_{ik} = f_{ijk}$.

This precisely describe the data of a $U(1)$ principal bundle on M !

Proposition

$H^2(M; \mathbb{Z}(1))$ is the group of isomorphism classes of $U(1)$ principal bundles on M .

Furthermore, $H^2(M; \mathbb{Z}(2))$ is the group of isomorphism classes of $U(1)$ principal bundles with connections on M (Hint: for the cocycle here we need also 1-form α_i on U_i , with $\alpha_i - \alpha_j = da_{ij}$).

$\mathbb{Z}(1)$ and central extensions

While $H^2(-; \mathbb{Z}(1))$ classifies $U(1)$ principal bundle, $H^3(-; \mathbb{Z}(1))$ classifies $U(1)$ central extensions:

$\mathbb{Z}(1)$ and central extensions

While $H^2(-; \mathbb{Z}(1))$ classifies $U(1)$ principal bundle, $H^3(-; \mathbb{Z}(1))$ classifies $U(1)$ central extensions:

Theorem

Let G be a smooth (possibly infinite dimensional) Lie group, $B_\bullet G$ its classifying space. Then $H^3(BG; \mathbb{Z}(1))$ classifies smooth central extensions of G by $U(1)$.

$\mathbb{Z}(1)$ and central extensions

While $H^2(-; \mathbb{Z}(1))$ classifies $U(1)$ principal bundle, $H^3(-; \mathbb{Z}(1))$ classifies $U(1)$ central extensions:

Theorem

Let G be a smooth (possibly infinite dimensional) Lie group, $B_\bullet G$ its classifying space. Then $H^3(BG; \mathbb{Z}(1))$ classifies smooth central extensions of G by $U(1)$.

$B_\bullet G$ is the classifying space of G , viewed as a sheave on $Mfld$.

Differential characteristic classes

Let G be a Lie group, and $B_\bullet G$ the classifying space. Then $H^*(B_\bullet G; \mathbb{Z})$ are the characteristic classes of G . Similarly, $H^*(B_\bullet G; \mathbb{Z}(n))$ are differential characteristic classes of G .

Differential characteristic classes

Let G be a Lie group, and $B_\bullet G$ the classifying space. Then $H^*(B_\bullet G; \mathbb{Z})$ are the characteristic classes of G . Similarly, $H^*(B_\bullet G; \mathbb{Z}(n))$ are differential characteristic classes of G .

We will need the following key fact:

Theorem (Bott, Freed-Hopkins)

$$H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \simeq \mathbb{R}.$$

Differential characteristic classes

Let G be a Lie group, and $B_\bullet G$ the classifying space. Then $H^*(B_\bullet G; \mathbb{Z})$ are the characteristic classes of G . Similarly, $H^*(B_\bullet G; \mathbb{Z}(n))$ are differential characteristic classes of G .

We will need the following key fact:

Theorem (Bott, Freed-Hopkins)

$$H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \simeq \mathbb{R}.$$

They are the differential first Pontryagin classes.

Table of Contents

- 1 Motivation
- 2 Virasoro groups and central extensions
- 3 Differential cohomology
- 4 Main theorem

Key Idea I

We want to get a \mathbb{R} family of central extension of $\text{Diff}^+(S^1)$ by $U(1)$, therefore we want a \mathbb{R} family in $H^3(B_\bullet \text{Diff}^+(S^1), \mathbb{Z}(1))$. We get this by pullback and integration:

Key Idea I

We want to get a \mathbb{R} family of central extension of $\mathrm{Diff}^+(S^1)$ by $U(1)$, therefore we want a \mathbb{R} family in $H^3(B_\bullet \mathrm{Diff}^+(S^1), \mathbb{Z}(1))$. We get this by pullback and integration:

Consider the canonical $\mathrm{Diff}^+(S^1)$ action on S^1 , note that

Key Idea I

We want to get a \mathbb{R} family of central extension of $\text{Diff}^+(S^1)$ by $U(1)$, therefore we want a \mathbb{R} family in $H^3(B_\bullet \text{Diff}^+(S^1), \mathbb{Z}(1))$. We get this by pullback and integration:

Consider the canonical $\text{Diff}^+(S^1)$ action on S^1 , note that

- The quotient $S^1/\text{Diff}^+(S^1)$ has a map to $B_\bullet \text{Diff}^+(S^1) = */\text{Diff}^+(S^1)$. Since the action of $\text{Diff}^+(S^1)$ on S^1 is orientation preserving, this is a oriented S^1 fiber bundle.

Key Idea I

We want to get a \mathbb{R} family of central extension of $\text{Diff}^+(S^1)$ by $U(1)$, therefore we want a \mathbb{R} family in $H^3(B_\bullet \text{Diff}^+(S^1), \mathbb{Z}(1))$. We get this by pullback and integration:

Consider the canonical $\text{Diff}^+(S^1)$ action on S^1 , note that

- The quotient $S^1/\text{Diff}^+(S^1)$ has a map to $B_\bullet \text{Diff}^+(S^1) = */\text{Diff}^+(S^1)$. Since the action of $\text{Diff}^+(S^1)$ on S^1 is orientation preserving, this is a oriented S^1 fiber bundle.
- The tangent bundle of S^1 gives a map $TS^1 : S^1 \rightarrow B_\bullet \text{GL}_1^+(\mathbb{R})$. Since the action of $\text{Diff}^+(S^1)$ on S^1 is smooth, the tangent bundle is $\text{Diff}^+(S^1)$ -equivariant. Equivalently, the tangent bundle factors through the quotient as a map $TS^1 : S^1/\text{Diff}^+(S^1) \rightarrow B_\bullet \text{GL}_1^+(\mathbb{R})$.

Key Idea II

To summarize, we have a span of maps:

$$\begin{array}{ccc} S^1/\mathrm{Diff}^+(S^1) & \xrightarrow{TS^1} & B_\bullet \mathrm{GL}_1^+(\mathbb{R}) \\ \downarrow & & \\ B_\bullet \mathrm{Diff}^+(S^1). & & \end{array} \quad (7)$$

Note the vertical map is a S^1 fibration, something we can integrate against. Therefore we get a map:

$$\begin{array}{ccc} H^4(S^1/\mathrm{Diff}^+(S^1); \mathbb{Z}(2)) & \longleftarrow & H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \\ \downarrow \int_{S^1} & & \\ H^3(B_\bullet \mathrm{Diff}^+(S^1); \mathbb{Z}(1)). & & \end{array} \quad (8)$$

Main theorem

Finally, we can state the conjecture of Freed and Hopkins that we proved:

Theorem (Y.L., Arun Debray, Christoph Weis)

The image of map $\mathbb{R} \simeq H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \rightarrow H^3(B_\bullet \mathrm{Diff}^+(S^1); \mathbb{Z}(1))$ are the Virasoro central extensions Vir_λ .

Furthermore, we explicitly recovers the Bott-Thurston cocycles when calculating the map on cocycles.

Proof sketch I

We construct explicit cocycles and compute the map on the level of cocycles.

- We find 1-form cocycles for $H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2))$, using the canonical simplicial resolution of $B_\bullet \mathrm{GL}_1^+(\mathbb{R})$.

Proof sketch I

We construct explicit cocycles and compute the map on the level of cocycles.

- We find 1-form cocycles for $H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2))$, using the canonical simplicial resolution of $B_\bullet \mathrm{GL}_1^+(\mathbb{R})$.
- We pullback to 1-form cocycles on $S^1/\mathrm{Diff}^+(S^1)$, using the simplicial realization of $S^1 = Fr_+(S^1)/\mathrm{GL}_1^+(\mathbb{R})$, where $Fr_+(S^1)$ is the oriented frame bundle.

Proof sketch II

Here's the key point of the proof:

- Now we move the cocycles across the double complex associated to the bisimplicial object $S^1/\text{Diff}^+(S^1) = \text{GL}_1^+(\mathbb{R}) \backslash Fr_+(S^1) / \text{GL}_1^+(\mathbb{R})$, to get cocycles on the simplicial resolution for $S^1/\text{Diff}^+(S^1)$.

$$\begin{array}{ccccc}
 & & \Omega^1(F \times \mathbb{R}^{\times 2}) & & \\
 & & \uparrow -\log(\gamma'_1 \circ \gamma_2) d\log(\gamma_2) & & \\
 & \Omega^1(\Gamma \times F \times \mathbb{R}) & \longleftarrow & \Omega^1(F \times \mathbb{R}) & \\
 & \uparrow \begin{array}{l} \text{blue } x \log \gamma' - \\ \text{red } x \log \gamma' = 0 \end{array} & & \uparrow \log(v) \quad d\log(\gamma') & \\
 & \Omega^1(\Gamma^{\times 2} \times F) & \longleftarrow & \Omega^1(\Gamma \times F) & \\
 & \downarrow x_2 dx_1 & \longleftarrow & \downarrow x \quad d\log(v) &
 \end{array}$$

Proof sketch III

- Lastly, we integrate over S^1 and immediately see that we recover the Bott-Thurston cocycles! Q.E.D.

Thank you for listening!