Differential Cohomology and Virasoro Central Extensions

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Motivation

Virasoro groups is a $\mathbf{R}$ family of central extension of $\mathrm{Diff}^+(S^1)$, the group of orientation preserving smooth automorphism of S^1. The central extension is describe by the Bott-Thurston cocyle. The goal of this talk is to give a novel geometric description these central extensions, using differential cohomology. thus affirmativaly answering a conjecture of Freed-Hopkins.

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Bott-Thurston cocycles

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Definition

The Virasoro group $\widetilde{\Gamma}_\lambda$, for $\lambda \in \mathbf{R}$, is a $U(1)$ central extension of $\mathrm{Diff}^+(S^1)$, described by the Bott-Thurston cocycle
$B_\lambda : \mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1) \to U(1)$:

$$B_\lambda(\gamma_1, \gamma_2) = \exp\left(-\frac{i\lambda}{48\pi}\int_{S^1} \log(\gamma_1' \circ \gamma_2)\, \mathrm{d}(\log(\gamma_2))'\right) \tag{1}$$

for $\gamma_1, \gamma_2 \in \mathrm{Diff}^+(S^1)$, viewed as morphisms $S^1 \to S^1$.

Central Extensions

Let's briefly review what is a central extension:

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Definition

Let G be a group and A be an abelian group, a central extension of G by A is a group $\tilde{G}$ with short exact sequence:

$$0 \to A \to \tilde{G} \to G \to 1 \tag{2}$$

such that subgroup $A \subset \tilde{G}$ is in the center, that is, it commutes with every element of $\tilde{G}$.

Central extension as group cohomology: I

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Proposition

Let G be a discrete group, then the isomorphism class of central extensions of G by A is classified by group cohomology class $H^2(G; A) \simeq H^2(BG; A)$, where BG is the classifying space of G.

Given a cocycle class $b \in C^2(G; A)$, viewed as a map $b : G \times G \to A$ satisying some cocycle conditions. Then $\tilde{G} = G \times A$ as a set, with multiplication $(g, a) \cdot (g', a') := (g \cdot g', a + a' + b(g, g'))$.

Central extension as group cohomology: II

In our case, $\mathrm{Diff}^+(S^1)$ is an infinite dimension Frechet Lie group, and we care about its topology. Smooth $U(1)$ central extensions of Lie groups are NOT classified by ordinary cohomology classes.

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We need a cohomology theory that remembers the smooth structure.

The answer is differential cohomology.

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Sheaves on smooth manifolds

Let M be a manifold, then the ordinary cohomology groups $H^*(M; A)$ depends only on the homotopy classes of M. It is the cohomology of the constant sheave $\underline{A}$ on M. On the other hand, the i-th cohomology form on M, $\Omega^i(M)$ is sensitive to the smooth structure of M.

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We can view both constant sheaves and differential forms as sheaves on *Mfld* the site of smooth manifolds.

Even though Ω^i are not homotopy invariant, the chain complex of sheaves $\Omega^* = 0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots$ is a homotopy invariant, in fact,

Theorem (de Rham)

The chain complex Ω^* is the constant sheave $\mathbb{R}$, as a chain complex concentrated in degree 0.

Sheaves $\mathbb{Z}(n)$

With this in mind, we define the (chain complex of) sheave $\mathbb{Z}(n)$ as

$$\mathbb{Z}(n) = \mathbb{Z} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{n-1} \to 0. \tag{4}$$

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There is also a form of integration: let M be a closed oriented d-dimensional manifold, then there is an integration map:

$$\int_M : H^*(M; \mathbb{Z}(n)) \to H^{*-d}(M; \mathbb{Z}(n-d)). \tag{5}$$

There is also a relative version of this.

$\mathbb{Z}(1)$ and line bundles

For example, a degree cocycle in $C^2(M;\mathbb{Z}(1))$ can be describe as follows: fix an open covering $\{U_i\}$ of M, we have 1-form α_i on the open subsets U_i, and 0-form f_{ij} on intersections $U_i \cap U_j$, such that $\alpha_i - \alpha_j = f_{ij}$

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This precisely describe the data of a $U(1)$ principal bundle on M!

Proposition

$H^2(M;\mathbb{Z}(1))$ is the group of isomorphism classes of $U(1)$ principal bundles on M.

Furthermore, $H^2(M;\mathbb{Z}(2))$ is the group of isomorpism classes of $U(1)$ principal bundles with connections on M.

$\mathbb{Z}(1)$ and central extensions

While $H^2(-;\mathbb{Z}(1))$ classifies $U(1)$ principal bundle, $H^3(-;\mathbb{Z}(1))$ classifies $U(1)$ central extensions:

Theorem

Let G be a smooth (possibly infinite dimensional) Lie group, BG its classifying space. Then $H^3(BG;\mathbb{Z}(1))$ classifies smooth central extensions of G by $U(1)$.

Differential characteristic classes

Let G be a Lie group, and $B_\bullet G$ the classifying space. Then $H^*(B_\bullet G;\mathbb{Z})$ are the characteristic classes of G. Similiarly, $H^*(B_\bullet G;\mathbb{Z}(n))$ are differential characeteristic classes of G.

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We will need the following key fact:

Theorem (Bott)

$H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}(2)) \simeq \mathbb{R}$.

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Key Idea I

We want to get a $\mathbb{R}$ family of central extension of $\mathrm{Diff}^+(S^1)$ by $U(1)$, therefore we want a $\mathbb{R}$ family in $H^3(B_\bullet \mathrm{Diff}^+(S^1), \mathbb{Z}(1))$. We get this by pullback and integration:

Key Idea I

We want to get a $\mathbb{R}$ family of central extension of $\mathrm{Diff}^+(S^1)$ by $U(1)$, therefore we want a $\mathbb{R}$ family in $H^3(B_\bullet\mathrm{Diff}^+(S^1), \mathbb{Z}(1))$. We get this by pullback and integration:

Consider the canonical $\mathrm{Diff}^+(S^1)$ action on S^1, note that

- The quotient $S^1/\mathrm{Diff}^+(S^1)$ has a map to $B_\bullet\mathrm{Diff}^+(S^1) = */\mathrm{Diff}^+(S^1)$. Since the action of $\mathrm{Diff}^+(S^1)$ on S^1 is orientation preserving, this is a oriented S^1 fiber bundle.
- The tangent bundle of S^1 gives a map $TS^1 : S^1 \to B_\bullet\mathrm{GL}_1^+(\mathbb{R})$. Since the action of $\mathrm{Diff}^+(S^1)$ on S^1 is smooth, the tangent bundle is $\mathrm{Diff}^+(S^1)$-equivariant. Equivalently, the tangent bundle factors through the quotient as a map $TS^1 : S^1/\mathrm{Diff}^+(S^1) \to B\mathrm{GL}_1^+(\mathbb{R})$.

Key Idea II

To summarize, we have a span of maps:

$$\begin{tikzcd} S^1/\mathrm{Diff}^+(S^1) \arrow[r, "TS^1"] \arrow[d] & B_\bullet \mathrm{GL}_1^+(\mathbb{R}) \\ B_\bullet \Gamma. & \end{tikzcd} \tag{7}$$

Note the vertical map is a S^1 fibration, something we can integrate against. Therefore we get a map:

$$\begin{tikzcd} H^4(S^1/\mathrm{Diff}^+(S^1); \mathbb{Z}(2)) \arrow[d, "\int_{S^1}"] & H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \arrow[l] \\ H^3(B_\bullet \Gamma; \mathbb{Z}(1)). & \end{tikzcd} \tag{8}$$

Main theorem

Finally, we can state the conjecture of Freed and Hopkins that we proved:

Theorem (Y.L., Arun Debray, Christoph Weis)

The image of map $\mathbb{R} \simeq H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \to H^3(B_\bullet \mathrm{Diff}^+(S^1); \mathbb{Z}(1))$ *are the Virasoro central extensions* $\widetilde{\Gamma}_\lambda$.

Furthermore, we explicitly recovers the Bott-Thurston cocylces when considering the map on cocycles.

Last slide

Thank you for listening!