Abstract

We proof a version of abelian duality in topological field theories. The theories are topological sigma models, generalizations of untwisted Dijkgraaf-Witten theory to π-finite spaces and spectra. The dual theories are the sigma models associated to a π-finite spectrum and (shifts of) its Pontryagin (Brown-Comenetz) dual. Our main theorem shows that the two theories are equivalent up to an invertible field theory. Physically, it is a duality between finite abelian gauge theories and gerbes. In low dimensions, the duality recovers finite Fourier transform and character theory for finite abelian groups.

Abelian duality is a major theme in both mathematics and physics. In mathematics, examples of abelian duality includes Pontryagin duality for locally compact topological abelian groups, Cartier duality for abelian group scheme, and abelian Langlands duality. In Physics, abelian duality appears as electro-magnetic duality in 4 dimensions, which generalizes to S-duality, and T-duality in 2 dimensions, which is strongly related to mirror symmetry.

In this thesis, we will discuss a finite, topological version on abelian duality. Mathematically, it is a homotopic version of duality between topological generalizations of finite abelian groups, namely π-finite spectra. Physically, it is a duality between finite abelian gauge and higher gerbe theories. As our groups are finite, the path integral, which normally sums over an infinite dimensional space, is finite and mathematically well-defined. In addition, because our group is discrete, the theory is topological, i.e., it doesn't depend on the geometry of the spacetime, only its topology. As a result, our theories are topological , and can be mathematically constructed under the formalism of topological field theories ([2]). The abelian duality is stated and proven as equivalence of field theories.

Recall that a (unextended) d topological field theory Z is a symmetric monoidal functor

$$Z : Bord_d \to Vect_{\mathbb{C}}. \tag{0.1}$$

$Bord_d$ is the category whose objects are $d-1$-dimensional closed manifolds and morphisms are d-dimensional bordisms. Physically, a TFT is a field theory that doesn't depend of the metric, thus the geometry of the spacetime manifolds. It assigns to a closed d-dimensional manifold M a number, which is the partition function $Z(M)$ evaluated at M. To a closed $d-1$ dimensional manifold N, it assigns a vector space, the space of states associated to the space-slice N.

A famous class of topological field theories are Dijkgraaf-Witten theories [5]: let G be a finite group. Then for every topological space N, we can define $Bun_G(N)$ to be the groupoid of principal G-bundles on N. Recall that a groupoid is a category whose morphisms are all invertible. Objects of $Bun_G(N)$ are principal G bundles, and morphisms between two principal G bundle are isomorphisms. Note that when N is compact, $Bun_G(N)$ is a finite groupoid, that is, it has finite many isomorphism class of objects, and each object has finite automorphisms.

In [5], they defined a d-dimensional TFT Z_{BG} associated to every group G. For a $d-1$ dimensional manifold N, the d-dimensional theory Z_{BG} assigns to N the vector space

$$Z_{BG}(N) \coloneqq \mathbb{C}[Bun_G(N)] \tag{0.2}$$

of locally constant complex-valued functions on $Bun_G(N)$. For a closed d dimensional manifold M, Z_{BG} assigns to M the number

$$Z_{BG}(M) \coloneqq \sum_x \frac{1}{|Aut_{Bun_G(M)}(x)|}, \tag{0.3}$$

where x sums over connected components of $Bun_G(M)$. Note that this is counting the number of isomorphism class of principal G bundles on M, weighted by autmorphisms.

In this paper, we will consider a generalization of Dijkgraaf-Witten theory. Let X be a π-finite space, that is, a topological space (homotopy type) X with finite many connected components, and for every basepoint $x \in X$, the homotopy groups $\pi_i(X, x)$ are nontrivial only in finite many degrees, and each is a finite group. For such X, we define a d-dimensional TFT

$$Z_X : Bord_d \to Vect_{\mathbb{C}} \tag{0.4}$$

In the case that $X = BG$, then Z_{BG} recovers the d-dimensional (untwisted) Dijkgraaf-Witten theory.

There is a special class of π-finite spaces, namely, those that comes from π-finite spectra (see 2.1 for detail). A spectrum $\mathcal{X}$ is finite if $\pi_i(\mathcal{X})$ are nontrivial in finitely many degrees, and each is a finite abelian group. For each π-finite spectra $\mathcal{X}$, we defined the d-dimensional sigma model to $\mathcal{X}$ simply by its 0-th loop space:

$$Z_{\mathcal{X}} \coloneqq Z_{\Omega^\infty \mathcal{X}}. \tag{0.5}$$

When A is an abelian group, then we can define its Pontryagin dual (character dual) $\hat{A}$ as $Hom(A, \mathbb{C}^\times)$. When A is finite, then taking the Pontryagin dual is a duality: $\hat{\hat{A}} \cong A$ (theorem 2.5.12). This can be generalized to spectra. For a spectrum $\mathcal{X}$, there is a canonical dual spectrum associated to it, called the Pontryagin (Brown-Comensatz) dual, $\hat{\mathcal{X}}$, defined in subsection 2.2. When $\mathcal{X} = \Sigma^n HA$, then $\hat{\mathcal{X}} = \Sigma^{-n} H\hat{A}$, where $\hat{A}$ is the Pontryagin dual group of A. Restricted to π-finite spectra, Pontryagin dual is again a duality: $\hat{\hat{\mathcal{X}}} \cong \mathcal{X}$ 2.6.11.

The main theorem states that in d dimensions, the sigma model associated to $\mathcal{X}$ and $\Sigma^{d-1}\hat{\mathcal{X}}$ are equivalent up to an invertible field theory:

Theorem 0.0.1. *Let $\mathcal{X}$ be a π-finite spectra and $\hat{\mathcal{X}}$ its dual. Then there is an equivalence of (suitably oriented) TFTs between d dimensional sigma model associated to $\mathcal{X}$ and $\Sigma^{d-1}\hat{\mathcal{X}}$:*

$$\mathbb{D} : Z_{\mathcal{X}} \cong Z_{\Sigma^{d-1}\hat{\mathcal{X}}} \otimes E_{|\mathcal{X}|}, \tag{0.6}$$

Where is $E_{|\mathcal{X}|}$ is the d dimensional Euler TQFT, which is an invertible TQFT.

Contents We will now describe the outline of the thesis. After the introduction, in section 1 we define d dimensional untwisted sigma models for any π-finite space X. First we recall the formalism of topological field theory. Then we will recall some basic facts about π-finite spaces and define a functions functor. Lastly, we will define the sigma model.

In section 2 we review the two dualities theorem needed for the main theorem. Since the theorems are for spectra, we first review some basics of spectra, then prove some facts for π-finite spectra. Then we discuss generalized orientation and Poincaré duality for spectra. Lastly, we will review Pontryagin duality for finite groups and generalize it to π-finite spectra (which is called Brown-Comentaz duality).

In section 3 we review some basics about Euler characteristics, then define the Euler invertible TQFT.

In section 4 we state and proof the main theorem 0.0.1.

Acknowledgement I want to thank....

Contents

1 Finite sigma model TFTs **5**
1.1 Topological Field Theories 5
1.2 Background on π-finite spaces 7
1.3 Functions on π-finite spaces 10
1.4 Sigma models 15

2 Spectra and duality theorems **18**
2.1 Spectra 18
2.2 Spectra and Cohomology theories 23
2.3 π-finite spectra 26
2.4 Poincaré duality 28
2.5 Pontryagin duality for finite abelian groups 34
2.6 Pontryagin duality for spectra 38

3 Euler characteristic and TQFT **44**
3.1 Euler Characteristics 44
3.2 Euler TQFT 49

4 Abelian duality **53**
4.1 Finite Sigma model for π-finite spectra 53
4.2 $\mathcal{R}$-oriented bordism category 56
4.3 Main theorem 57
4.4 Proof of Lemma 4.3.9 61
4.5 Proof of lemma 4.3.10 67

References **73**

1 Finite sigma model TFTs

In this section we define the d-dimensional finite sigma model TFT Z_X associated to a π-finite space X. In 1.1 we recall the basics of TFTs. In 1.2 we give some background on π-finite spaces. In 1.3 we discuss the $\mathcal{O}$ functor from spans of π-finite groupoids to $Vect_{\mathbb{C}}$. In 1.4 we use $\mathcal{O}$ to define the finite sigma model Z_X for a π-finit space X.

1.1 Topological Field Theories

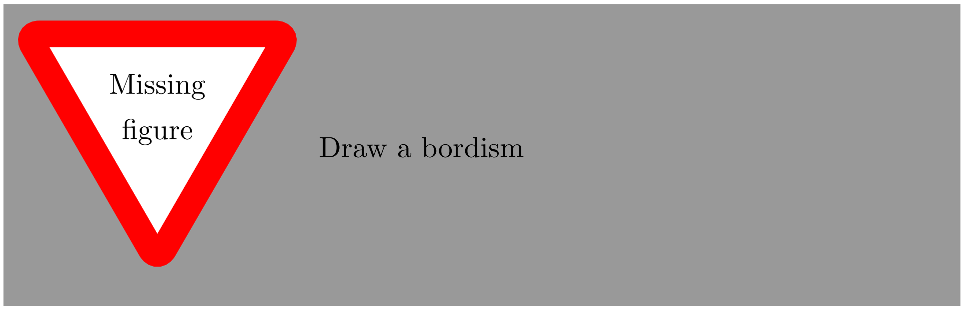
Let us first motivate topological field theories. In a d dimensional quantum theory, we have a partition function Z evaluated at a d-dim spacetime manifold M:

$$Z(M) = \int \mathcal{D}\phi e^{iS[\phi]}, \tag{1.1}$$

where ϕ represents all fields of the theory, $S = \int_M \mathcal{L}(\phi)$ is the action, $\mathcal{L}$ is the Lagrangian. Note that the integral integrates over all field possible field configurations over M. In a quantum theory, the field configurations are often infinitely dimensional, making the measure $\mathcal{D}\phi$ and the integral difficult to mathematically define.

In a topological theory, there is no dependence on the metric. The fields are often discrete and finite, and we can actually evaluate the path integral. Thus for any d dimensional closed manifold M we expect a number $Z(M)$. More generally, given a manifold M with boundary $\partial M \simeq N \sqcup N'$, then we can view N and N' as the incoming and outgoing space-slice, and M an evolution through time. In this case, we should have a complex vector space of initial states $Z(N)$. For a given state $v \in Z(N)$, we can let it evolve and get a state $v' \in Z(N')$. Thus we see that $Z(M)$ gives a map $Z(N) \to Z(N')$.

We can formulate a topological field theory as one that gives a vector spaces of states on closed $d-1$ manifold, and a map of states $Z(M) : Z(N) \to Z(N')$ for a d dimensional manifold M with $\partial M \xrightarrow{\sim} N \sqcup N'$. Note that a d dimensional manifold M with $\partial M \xrightarrow{\sim} N \sqcup N'$ is called a bordism from N to N'. Two bordism M, M' from N to N' are isomorphic if there exists a diffeomorphism from m to m' that restricts to identity on the boundaries.



We can formulate the notions above mathematically. First, note that spaceslices and bordisms between them naturally form a category:

Definition 1.1.1. The d-dimensional unoriented $Bord_d$ be the category whose objects are $d-1$ dimensional unoriented closed manifold, and morphisms between them are isomorphism classes of unoriented bordisms. It is a symmetric monoidal category under disjoint union.

Remark 1.1.2. This is called the unoriented bordism category because the manifolds and bordisms are unoriented. Often than not, the field theory will require additional tangential structure, which as an orientation (for integration), or a $Spin$ structure for the fermionic fields. There is a general notion of tangential structure Θ, and for each Θ, we can define a corresponding bordism category $Bord_d^\Theta$, symmetric monoidal under disjoint union. We won't need tangential structure and $Bord_d^\Theta$ in general, so this won't be defined. See [11] for details. However, for a ring spectrum $\mathcal{R}$ (see 2.1), we will need to the $\mathcal{R}$-oriented bordism category $Bord_d^{\mathcal{R}}$(see 4.2). Currently we only need the unoriented bordism category $Bord_d$ as the finite sigma model Z_X is defined on unoriented manifolds.

Definition 1.1.3. Note that the empty set $\varnothing$ is a closed n dimensional manifold for any $n \geq 0$. When viewed as a n dimensional manifold, we denote it as $\varnothing_n$. $\varnothing_{d-1}$ is the unit object for the symmetric monoidal structure on $Bord_d$ (and its tangential variants). Note that set of endormorphism of the unit object, usually denoted as $\Omega Bord_d$, is $Mor(\varnothing_{d-1}, \varnothing_{d-1})$ the set of isomorphism classes of closed d dimensional manifolds. It is a monoid under disjoint union. Similarly, for any tangential structure Θ, $\Omega Bord_d^\Theta$ is the monoid of isomorphism classes of closed d dimensional manifold with Θ-tangential structure.

Remark 1.1.4. We have defined the unextended bordism category. One can extend this theory down to points. See [11] for such a construction.

We want to formulate a TFT as a symmetric monoidal functor out of the bordism category. We need our target category:

Definition 1.1.5. The category of $\mathbb{C}$-linear vector spaces, $Vect_{\mathbb{C}}$ is the category whose objects are finite dimensional $\mathbb{C}$-linear vector spaces and morphisms are $\mathbb{C}$-linear transformations. It is symmetric monoidal under tensor products $\otimes$.

Now we can define a topological field theory, following [2]:

Definition 1.1.6. A d dimensional (unoriented) topological field theory is a symmetric monoidal functor $Z : Bord_d \to Vect_{\mathbb{C}}$.

Of course, for every tangential structure Θ, one can define a d-dimensional Θ TFT as a symmetric monoidal functor $Z : Bord_d^{\Theta} \to Vect_{\mathbb{C}}$.

Remark 1.1.7. Let Z be a d dimensional TFT. As it s symmetric monoidal, $Z(\varnothing_{d-1}) = \mathbb{C}$. Thus for a closed d dimensional manifold M, viewed as an morphism $\varnothing_{d-1} \to \varnothing_{d-1}$, $Z(M) : \mathbb{C} \to \mathbb{C}$ is given by multiplication by a scalar. Thus we see that a TFT gives numbers on closed d-dimsional manifolds. Similarly, for a bordism $M : N \to N'$, we get a map of states $Z(N) \to Z(N')$. This is exactly what we want from a topological field theory.

Remark 1.1.8. We have the Picard subgroupoid of lines $Lines \subset Vect_{\mathbb{C}}$. The objects of $Lines$ are 1-dimensional $\mathbb{C}$ vector spaces. Morphisms are invertible linear transformation between lines. A d dimensional TQFT

$$Z : Bord_d \to Vect_{\mathbb{C}} \tag{1.2}$$

is invertible if it factors through $Lines$. It means that for every $d-1$ closed manifold N, $Z(N)$ is 1-dimensional, and for every bordism $M : N \to N'$, $Z(M) : Z(N) \to Z(N')$ is an isomorphism. The Euler TQFT E_λ defined in subsection 3.2 is invertible.

1.2 Background on π-finite spaces

In this subsection we define π-finite spaces and proof some basic properties.

Definition 1.2.1. A topological space X is called a π-finite space if $\pi_0(X)$ is finite, and for every $x \in X$, $\pi_i(X, x)$ is nontrivial for only finitely many i, and each one is a finite group.

we denote S the category of spaces, and S^{fin} the full subcategory of π-finite spaces.

Remark 1.2.2. Note that we only care about topological spaces up to weak homotopy equivalence. If $f : X \to Y$ is a weak equivalence (induce isomorphism on all homotopy groups), then it is an isomorphism in S. Thus S can be defined as the category of all topological space Top loocalized at weak equivalences. However, the resulting 1-category of homotopy classes of spaces $Ho(S)$ does not behave well as it doesn't keep track of the higher coherences. Rigorously, S is the model category of CW complexes (and homotopy equivalence), or equivalently, the model category of simplicial sets. Alternatively, one can use the machinery of ∞-categories ([10], [9]). The limits and colimits are always the homotopy limits and colimits.

Example 1.2.3. Let X be a finite set, then as a discrete topological space it is π-finite.

Example 1.2.4. Let G be a (discrete) group. Then we can define the classifying space BG to be a space with $\pi_0(BG) = *$, and for $x \in BG$, we have $\pi_1(BG, x) = G$ and $\pi_i(BG, x) = 0$ for $i \geq 1$. When G is finite, this is a π-finite space. Maps into BG classifies principal G bundles.

Example 1.2.5. Similarly, when A is an abelian group, $n \geq 1$, we have the Eilenberg MacLane space $K(A, n)$ a space with $\pi_0(K(A, n)) = *$, and for $x \in K(A, n)$, we have $\pi_n K(A, n) = A$ and $\pi_i K(A, n) = 0$ for $i \neq n$. When A is finite, $K(A, n)$ is a π-finite space. Maps into $K(A, n)$ classifies cohomology classes with coefficients in A, physically, they classify abelian n-gerbes.

There is a more common notion of finiteness of topological space, that is, homotopic to a finite CW complex. Note that the intersection of these two notion of finiteness are only finite sets. Finite CW complex are spaces that can be build from spheres in finite steps, by contrast, π-finite spaces are spaces that can be build from Eilenberg-MacLane spaces in finite steps.

Next we need to define mapping spaces:

Definition 1.2.6. Let X, Y be a topological space, then we can define the mapping space $Maps(X, Y)$ as follows: as a set it is the set of continuous maps from X to Y. We give it the compact open topology. For nice enough spaces X, Y, $Maps(X, Y)$ behave nicely. Note that we only care about the homotopy type of X, Y, and $Map(X, Y)$.

Remark 1.2.7. Alternatively, we can work with the simplicial set model. In there the mapping space (simplicial set) behave better.

We will need the following lemma:

Lemma 1.2.8. *Let M be a compact manifold with boundary, X a π-finite space, then $Map(M, X)$ is again a π-finite space.*

Proof. Given two spaces X, Y, then

$$Maps(M, X \sqcup Y) \simeq Maps(M, X) \sqcup Maps(M, Y). \tag{1.3}$$

As π-finite spaces are finite disjoint unions of connected π-finite spaces, it is enough to prove the lemma in the case that X is a connected π-finite space.

We will do this by induction. First we look at the case where the homotopy groups of X are concentrated in a single degree i ($i > 0$ as $\pi_0(X) = *$). Thus $X = BG$ for G a finite group or $K(A, n)$ for A a finite abelian group. In the case that $X = BG$, we see that

$$Maps(M, BG) = Bun_G(M) \tag{1.4}$$

the groupoid of principal G-bundles on M. As M is a compact manifold, there are finitely many different isomorphism classes of principal G bundles on M, thus

$$\pi_0(Bun_G(M)) = \pi_0(Maps(M, BG)) \tag{1.5}$$

is finite. For a given principal G-bunle P on M, it also has finite automorphism. It has no higher homotopy groups, thus it is a π-finite space.

In the other case, $X = K(A, n)$ for A a finite abelian group. Then

$$\pi_i(Maps(M, K(A, n))) = H^{n-i}(M, A) \tag{1.6}$$

is the $(n - i)$-th cohomology group of M with A coefficients (this will play a major theme in the main theorem). As M is compact and A finite, they are also finite. One way to see this is by considering a fintie CW complex K homotopic equivalent to M. Then the finite CW cochain complex

$$C^*(K, \mathbb{Z}) \otimes A \tag{1.7}$$

computes $H^*(M, A)$ as its homology. At each degree i , the cochain complex

$$C^i(K, \mathbb{Z}) \otimes A \tag{1.8}$$

is finite, thus so is $H^i(M, A)$. This shows that

$$Maps(M, K(A, n)) \tag{1.9}$$

is a π-finite space.

Now for the inductive step. We assume that for the lemma is true for all connected π-finite space with homotopy group concentrated in degrees less than n (case $n = 2$ is $X = BG$ proven above). Now we will prove the lemma for connected π-finite space X with homtopy group concentrated in degrees less or equal to n. Take $x \in X$ as a basepoint and consider (X, x) as a based space. Let Consider the following fiber sequence

$$\tau_{\geq n}X \to X \to \tau_{\leq n}X. \tag{1.10}$$

Note that the homotopy group of $\tau_{\geq n}X$ is concentrated in degree n, thus

$$\tau_{\geq n}X = K(\pi_n(X), n). \tag{1.11}$$

Note that Maps in to fiber sequence gives a fiber sequence

$$Maps(M, \tau_{\geq n}X) \to Maps(M, X) \to Maps(M, \tau_{\leq n}X). \tag{1.12}$$

This gives a long exact sequence of connected π-finite groups. As both

$$Maps(M, \tau_{\geq n}X) \simeq Maps(M, K(\pi_n(X), n)) \tag{1.13}$$

and

$$Maps(M, \tau_{\leq n}X) \tag{1.14}$$

are π-finite. By long exact sequence of homotopy groups, we see that $Maps(M, X)$ is also π-finite. $\square$

1.3 Functions on π-finite spaces

Let $Vect_{\mathbb{C}}$ be the symmetric monoidal category of $\mathbb{C}$-vector spaces and tensor products. The goal of this subsection is to define the symmetric monoidal functor : $\mathcal{O} : M_1(S^{fin}) \to Vect_{\mathbb{C}}$. First we need to define $M_1(S^{fin})$. Recall that S^{fin} is the category of π-finite spaces.

Definition 1.3.1. The category of spans (correspondences) of π-finite groupoids $M_1(S^{fin})$ the category with objects π-finite spaces X, and a morphism from X to X' is a correspondence

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X & & X'. \end{tikzcd} \tag{1.15}$$

Given two morphisms

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X & & X', \end{tikzcd} \qquad \begin{tikzcd} & Y' \arrow[dl] \arrow[dr] & \\ X' & & X'', \end{tikzcd} \tag{1.16}$$

the composition is the pullback:

$$\begin{tikzcd} & Y \times_{X'} Y' \arrow[dl] \arrow[dr] & \\ X & & X''. \end{tikzcd} \tag{1.17}$$

Note that $M_1(S^{fin})$ is a symmetric monoidal category under Cartesian products.

To define $\mathcal{O}$, we first need to define it on objects:

Definition 1.3.2. Let X be a space, $\mathcal{O}(X)$, the vector space of functions on X, is defined be the vector space of locally constant functions complex valued functions $\mathbb{C}[\pi_0(X)]$ on X.

Note that when X is a π-finite space, then $\mathcal{O}(X)$ is finite dimensional. Now we need to look at the functoriality of $\mathcal{O}$. Given a map $f : X \to Y$, we can naturally pullback functions on Y to functions on X:

Definition 1.3.3. Given $p : X \to Y$ a map of π-finite spacaces, we have the pullback map

$$p^* : \mathcal{O}(Y) \to \mathcal{O}(X) \tag{1.18}$$

defined as follows: given $f : X \to \mathbb{C}$ and $x \in X$,

$$p^*(f)(x) \coloneqq f(p(x)). \tag{1.19}$$

Note that this defines a functor

$$(-)^* : Span(S)^{op} \to Vect_{\mathbb{C}}. \tag{1.20}$$

In addition to pullbacks, we also need to define pushforwards:

Definition 1.3.4. Given $p : X \to Y$, we define pushforward map

$$p_* : \mathcal{O}(X) \to \mathcal{O}(Y) \tag{1.21}$$

defined as follows: let $g : X \to \mathbb{C}$ and $y \in Y$,

$$p_*(g)(y) := \sum_{x \to y} \frac{|\pi_1(Y,y)|}{|\pi_1(X,x)|} \frac{|\pi_2(X,x)|}{|\pi_2(Y,y)|} \cdots g(x), \tag{1.22}$$

where $\sum_{x\to y}$ means summing x over all $\pi_0(X)$ that maps to the the connected component of y in Y.

The pushforward sums over the fibers in a way that keep track the automorphisms. The infinite product is well-defined as π-finite spaces have finitely many nontrivial homotopy groups, and each is finite.

Example 1.3.5. Let $X = BG$ the classifying space of a finite group G. As $\pi_0(BG) = *$, we see that $\mathcal{O}(BG) = \mathbb{C}$, with $1 \in \mathbb{C}$ be the constant function 1. We have the map $p : BG \to *$ and $p_*(1) = \frac{1}{|G|}$.

Proposition 1.3.6. *Pushforward defines a functor*

$$(-)_* : S^{fin} \to Vect_{\mathbb{C}}. \tag{1.23}$$

Proof. Given X, Y, Z π-finite spaces, and maps $p : X \to Y$, and $q : Y \to Z$. We have to show that

$$q_* \circ p_* = (q \circ p)_*. \tag{1.24}$$

For $g : X \to \mathbb{C}$ and $z \in Z$,

$$(q_* \circ p_*(g))(z) = \sum_{y\to z} \frac{|\pi_1(Z,z)|}{|\pi_1(Y,y)|} \frac{|\pi_2(Y,y)|}{|\pi_2(Z,z)|} \cdots (p_*(g))(y) \tag{1.25}$$

$$= \sum_{y\to z} \frac{|\pi_1(Z,z)|}{|\pi_1(Y,y)|} \frac{|\pi_2(Y,y)|}{|\pi_2(Z,z)|} \cdots \tag{1.26}$$

$$\sum_{x\to y} \frac{|\pi_1(Y,y)|}{|\pi_1(X,x)|} \frac{|\pi_2(X,x)|}{|\pi_2(Y,y)|} \cdots g(x) \tag{1.27}$$

$$= \sum_{x\to z} \frac{|\pi_1(Z,z)|}{|\pi_1(X,x)|} \frac{|\pi_2(X,x)|}{|\pi_2(Z,z)|} \cdots g(x) \tag{1.28}$$

$$= ((q \circ p)_*(g))(z) \tag{1.29}$$

□

We need have this compatibility lemma between pullback and pushforwards:

Lemma 1.3.7 (Base-change)**.** *Given $X, Y, Z \in S^{fin}$ and $p : X \to Y$, $q : Z \to Y$, we denote the (homotopy) pullback $W = X \times_Y Z$:*

$$\begin{tikzcd} W \arrow[r, "q'"] \arrow[d, "p'"] & X \arrow[d, "p"] \\ Z \arrow[r, "q"] & Y \end{tikzcd} \tag{1.30}$$

Then we have an equality for maps

$$q^* \circ p_* = p'_* \circ q'^* : \mathcal{O}(X) \to \mathcal{O}(Z) \tag{1.31}$$

Proof. Functions are determined by their values at points. Let $z \in Z$, Note the evaluation of a function at f is z is the same as pull back the function from the map $* \xrightarrow{z} Z$, then evaluating at the point:

$$f(z) = z^*(f)(*). \tag{1.32}$$

As two small pullback diagrams forms a larger pullback diagram, it is suffice to check the lemma with the case when $Z = *$, and $q : Z \to Y$ is given by a point $y \in Y$. In this case, $W = fib_y(f)$ is the fiber of the map (of based spaces). Note that objects in W are $(x, \gamma : f(x) \rightsquigarrow y)$, a point $x \in X$ and a path $f(x) \rightsquigarrow y$ in Y. Let $f \in \mathcal{O}(X)$,

$$\begin{align} (q^* \circ p_*(f))(*) &= p_*(f)(y) \tag{1.33} \\ &= \sum_{x \to y} \frac{|\pi_1(Y, y)|}{|\pi_1(X, x)|} \frac{|\pi_2(X, x)|}{|\pi_2(Y, y)|} \cdots f(x). \tag{1.34} \end{align}$$

On the other hand,

$$(p'_* \circ q'^*(f))(*) = \sum_{x'} \frac{1}{|\pi_1(fib_y(f), x')|} |\pi_2(fib_y(f), x')| \cdots q'^*(f)(x') \tag{1.35}$$

For each $x' = (x, \gamma)$,

$$q'^*(f)(x') = f(x). \tag{1.36}$$

Let's consider the parts of the summation that comes from a specific connected components of X. That is, assume the case when X is connected. In that case, we see that we need to show

$$\sum_{x \to y} \frac{|\pi_1(Y, y)|}{|\pi_1(X, x)|} \frac{|\pi_2(X, x)|}{|\pi_2(Y, y)|} \cdots = \sum_{x'} \frac{1}{|\pi_1(fib_y(f), x')|} |\pi_2(fib_y(f), x')| \cdots \tag{1.37}$$

This follows from the long exact sequence of homotopy group associated to the fiber sequence $fib(f) \to X \to Y$:

$$\cdots \to \pi_*(fib(f)) \to \pi_*(X) \to \pi_*(Y) \to \cdots \tag{1.38}$$

□

With this in hand, we can finally define $\mathcal{O}$ functor: Given a span $Z : X \to Y$

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & X', \end{tikzcd} \tag{1.39}$$

we define $\mathcal{O}(Y) : \mathcal{O}(X) \to \mathcal{O}(X')$ to be $q_* \circ p^*$. This gives us a well defined functor:

Proposition 1.3.8. *This defines a functor $\mathcal{O} : Span(S^{fin}) \to Vect_{\mathbb{C}}$.*

Proof. We have to check that composition agrees. Given two spans

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & & & Y' \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & X', & X' & & X'', \end{tikzcd} \tag{1.40}$$

we have the larger diagram

$$\begin{tikzcd} & & Y \times_{X'} Y' \arrow[dl, "s'"'] \arrow[dr, "q'"] & & \\ & Y \arrow[dl, "p"'] \arrow[dr, "q"] & & Y' \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & X' & & X''. \end{tikzcd} \tag{1.41}$$

The composition of the two span is

$$\begin{tikzcd} & Y \times_{X'} Y' \arrow[dl, "p \circ s'"'] \arrow[dr, "t \circ q'"] & \\ X & & X''. \end{tikzcd} \tag{1.42}$$

The composition is

$$\begin{aligned}
\mathcal{O}(Y') \circ \mathcal{O}(Y) &= t_* \circ s^* \circ q_* \circ p^* && (1.43)\\
&= t_* \circ q'_* \circ s'^* \circ p^* && (1.44)\\
&= (t \circ q')_* \circ (p \circ s')^* && (1.45)\\
&= \mathcal{O}(Y \times_{X'} Y'). && (1.46)
\end{aligned}$$

□

Remark 1.3.9. Note that we had to use exactly use the fact that pullbacks, pushforwards are functors, as well as the base change lemma. The fact that we have a pushforward and the base change lemma holds is a specific case of a more general phenonemon, called ambidexterity, see [13] for detail discussion.

1.4 Sigma models

Let X be a π-finite space, $d \geq 0$ the dimension of our theory. In this subsection, we define an unoriented field theory

$$Z_X : Bord_d \to Vect_{\mathbb{C}}. \tag{1.47}$$

First we need to define the field functor

$$\mathcal{F}_X : Bord_d \to M_1(S^{fin}). \tag{1.48}$$

This is defined as follows: Let $N \in Bord_d$ be a closed $d-1$ manifold. Then

$$\mathcal{F}_X(N) := Maps(N, X). \tag{1.49}$$

Note $Maps(N, X)$ is a π-finite space by proposition 1.2.8. Similarly, for a bordism $M : N \to N'$, we have

$$\mathcal{F}_X(M) := Maps(M, X), \tag{1.50}$$

viewed as a span:

$$\begin{tikzcd}
& Maps(M, X) \arrow[dl] \arrow[dr] & \\
Maps(N, X) & & Maps(N', X).
\end{tikzcd} \tag{1.51}$$

given by restriction. This defines a functor:

Proposition 1.4.1. *$\mathcal{F}$ defines a symmetric monoidal functor: $Bord_d \to Span(S^{fin})$.*

Proof. For $\mathcal{F}_X$ to be a functor, we need to show composition holds. Given two bordisms

$$M : N \to N', \quad M' : N' \to N''. \tag{1.52}$$

We have the composition bordism

$$M \sqcup_{N'} M' : N \to N''. \tag{1.53}$$

Note that

$$\begin{tikzcd} N' \arrow[r] \arrow[d] & M \arrow[d] \\ M' \arrow[r] & M \sqcup_{N'} M' \end{tikzcd} \tag{1.54}$$

is a homotopy pushout diagram as N' is nicely included in M and M'. The $Maps(-, X)$ functor takes homotopy pushout to homotopy pullbacks, thus we have a homotopy pullback diagram:

$$\begin{tikzcd} Maps(M \sqcup_{N'} M', X) \arrow[r] \arrow[d] & Maps(M', X) \arrow[d] \\ Maps(M, X) \arrow[r] & Maps(N', X). \end{tikzcd} \tag{1.55}$$

Recall that in the span category S^{fin}, composition of spans are given by pullbacks. Thus

$$\begin{aligned} \mathcal{F}_X(M \sqcup_{N'} M') &= Maps(M \sqcup_{N'} M', X) && (1.56)\\ &\simeq Maps(M, X) \times_{Maps(N', X)} Maps(M', X) && (1.57)\\ &= \mathcal{F}_X(M') \circ \mathcal{F}_X(M). && (1.58) \end{aligned}$$

In addition, this functor is symmetric monoidal as $Maps(N \sqcup N', X) \simeq Maps(N, X) \times Maps(N', X)$. □

Now we can compose

$$\mathcal{F}_X : Bord_d \to M_1(S^{fin}) \tag{1.59}$$

with the symmetric monoidal functor

$$\mathcal{O} : M_1(S^{fin}) \to Vect_{\mathbb{C}} \tag{1.60}$$

to get a symmetric monoidal functor from $Bord_d$ to $Vect_{\mathbb{C}}$, i.e. a TQFT:

Definition 1.4.2. Given X a π-finite space, then we define the d dim (untwisted) sigma model

$$Z_X := \mathcal{O} \circ \mathcal{F}_X : Bord_d \to Vect_{\mathbb{C}}. \tag{1.61}$$

Remark 1.4.3. In physical terms, $\mathcal{F}(M)$ is the space of fields on M. Note that pushforward corresponds to the path integral. Note that the exponentiated action

$$e^{i \int_M \mathcal{L}} \tag{1.62}$$

is trivial in this case, as we are just summing over fibers.

Remark 1.4.4. This is called the d-dimensional untwisted TFT associated to π-finite space X. To get a twisted theory, we need to have a nontrivial exponential action. To do this we need a "character" $X \to \mathbb{B}^d\mathbb{C}^\times$. Note that to integrate this we will need an orientation. Thus this defines an oriented theory. As this is not needed for the main theorem, it will not be discussed in detail.

Example 1.4.5. Let $X = BG$, the d dim theory is the d dim untwisted Dijkgraaf-Witten theory.

2 Spectra and duality theorems

In this section we introduce the basics of spectra, then prove the two duality theorems needed for the main theorem: Pontryagin (Brown-Comantasz) duality and Poincaré duality. In 2.1 we recall basics of spectra. In 2.2 we relate spectra to (co)homology theories. In 2.3 we define π-finite spectra and establish some basic properties. In 2.4 we develope orientation theory for ring spectrum and proved Poincaré duality. In 2.5 we develop Pontryagin duality for finite abelian groups. In 2.6 we generalize Pontryagin duality to spectra. Note that Pontryagin duality for spectra is also called Brown-Comenatz duality.

2.1 Spectra

In this subsection, we recall some facts about spectra. Most of them are formal (see [7] for a nice introduction). We follow [6] for a large part of this subsection.

Let S be the category of spaces, S_* the category of pointed spaces. Recall that S_* has a symmetric monoidal product $\wedge$ and an inner hom object $Maps$. Wedgeing with $- \wedge S^1$ is the suspension functor $\Sigma -$. Dually, $Maps(S^1, -)$ is the loop functor $\Omega -$.For every $X, Y \in S_*$, We have an equivalence

$$Maps(\Sigma X, Y) \simeq Maps(X, \Omega Y), \tag{2.1}$$

realizing Σ as the left adjoint of Ω.

Definition 2.1.1. A prespectrum $\mathcal{X}$ is a sequence $X_0, X_1, ...$ of pointed topological spaces with canonical map

$$s_n : \Sigma X_n \to X_{n+1}. \tag{2.2}$$

A map of prespectrum $f : \mathcal{X} \to \mathcal{Y}$ is a series of maps $f_n : X_n \to Y_n$ that commutes with s_n.

The homotopy group $\pi_n \mathcal{X}$, $n \in \mathbb{Z}$, is defined as

$$\pi_n \mathcal{X} \coloneqq \varinjlim_{k} \pi_{n+k} X_{n+k} \tag{2.3}$$

where the limit maps is given by

$$\pi_{n+k} X_{n+k} \xrightarrow{\Sigma} \pi_{n+k+1} \Sigma X_{n+k} \xrightarrow{s_{n+k}} \pi_{n+k+1} X_{n+k+1}. \tag{2.4}$$

Note that even when n is negative, X_{n+k} is eventually defined for k large enough.

Given a map $f : \mathcal{X} \to \mathcal{Y}$ of spectrums, we get induced maps on homotopy groups:

$$\pi_n(f) : \pi_n\mathcal{X} \to \pi_n\mathcal{Y}. \tag{2.5}$$

f is a weak homotopy equivalence if it induces isomorphism on homotopy groups.

The category of spectra Sp is the category of prespectrum localizes at weak homotopy equivalence. Effectively, it means that we will only consider things up to weak homotopy equivalence (similar to our treatment of the category of spaces). From now on, we will think about a prespectrum $\mathcal{X}$ as a spectrum (its equivalence class), and refer to them as such.

Remark 2.1.2. The category of spectra Sp is a $(\infty, 1)$-category, that is, it has higher homotopy coherence coming from the localization at weak equivalence. The formalism of $(\infty, 1)$ categories are harder to describe than ordinary categories. The two standard approach are based on model categories and quasi-categories [9]. Thus everything below should be stated in those context, however, for the purpose of this paper, much of the complication won't play a major role. For detail, see [9].

A large family of spectrum comes from pointed spaces:

Example 2.1.3. The suspension spectrum $\Sigma^\infty X$ of a pointed space X is the spectrum associated to the prespectrum

$$(\Sigma^\infty X)_n = \Sigma^n X \tag{2.6}$$

with the canonical maps $\Sigma(\Sigma^{n-1}X) \simeq \Sigma^n X$.

Note that we have a functor $\Sigma^\infty : S_* \to Sp$. Another class of spectrum comes from abelian group.

Example 2.1.4. Let A be an abelian group, then we define the Eilenberg-MacLane spectrum HA as follows:

$$HA_n = K(A, n) \tag{2.7}$$

with canonical map $\Sigma K(A, n) \to K(A, n+1)$ the right adjoint of the isomorphism

$$K(A, n) \simeq \Omega K(A, n+1). \tag{2.8}$$

Definition 2.1.5. An Ω spectrum is spectrum such that the corresponding map

$$X_n \to \Omega X_{n+1} \tag{2.9}$$

associated to $s_n : \Sigma X_n \to X_{n+1}$ is an equivalence.

In fact, every spectrum is weak equivalent to a Ω-spectrum. Given an Ω spectrum $\mathcal{X} = \{X_0, X_1, ..\}$, we can define its 0-th space, denoted as $\Omega^\infty \mathcal{X}$, to be X_0.

This gives a functor $\Omega^\infty : Sp \to S_*$. In fact, it is the right adjoint of Σ^∞:

$$Maps_{Sp}(\Sigma^\infty X, \mathcal{Y}) \simeq Maps_{S_*}(X, \Omega^\infty \mathcal{Y}). \tag{2.10}$$

here $Maps_{Sp}$ and $Maps_{S_*}$ are the categorical homs, that is, they are the space of maps.

We have a suspection functor $\Sigma : Sp \to Sp$ defined as follows: for a spectrum $\mathcal{X}$,

$$(\Sigma \mathcal{X})_n := \Sigma X_n \tag{2.11}$$

with canonical connecting maps. Note that Σ^∞ commutes with $\Sigma : S_* \to S_*$ and $\Sigma : Sp \to Sp$. That is, we have a commutative diagram:

$$\begin{tikzcd} S_* \arrow[r, "\Sigma"] \arrow[d, "\Sigma^\infty"] & S_* \arrow[d, "\Sigma^\infty"] \\ Sp \arrow[r, "\Sigma"] & Sp \end{tikzcd} \tag{2.12}$$

In addition, this is in fact an invertible functor, thus it has an inverse denoted as Σ^{-1} or Ω. By abstract nonsense, we also have this commutative diagram:

$$\begin{tikzcd} Sp \arrow[r, "\Omega"] \arrow[d, "\Omega^\infty"] & Sp \arrow[d, "\Omega^\infty"] \\ S_* \arrow[r, "\Omega"] & S_* \end{tikzcd} \tag{2.13}$$

With this, we can define the sphere spectrum ant its shifts:

$$\mathcal{S}^n := \Sigma^n \Sigma^\infty S^0 \tag{2.14}$$

$n \in \mathbb{Z}$ S^0 the 0-th sphere as a pointed space. When $n = 0$, $\mathcal{S} := \mathcal{S}^0 = \Sigma^\infty S^0$ is called the sphere spectrum.

Similar the category of pointed space S_*, the category of spectra Sp has a symmetric monoidal product product $\otimes : Sp \times Sp \to Sp$, which we called the tensor product (traditionally it's called the smash product). The unit object of the symmetric monoidal proudct is the sphere spectrum $\mathcal{S}$. It also has an internal hom object

$$Maps : Sp^{op} \times Sp \to Sp. \tag{2.15}$$

As any internal hom, it has the universal property that, for a spectra $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, we have

$$Maps(\mathcal{X} \otimes \mathcal{Y}, \mathcal{Z}) \simeq Maps(\mathcal{X}, Maps(\mathcal{Y}, \mathcal{Z})). \tag{2.16}$$

Note that we will denote the categorical hom as $Maps_{Sp}$, and the internal hom as $Maps$. We have

$$Maps_{Sp}(-,-) = \Omega^{\infty} \circ Maps(-,-) \tag{2.17}$$

Note that $\Sigma^{\infty} : (S_*, \wedge) \to (Sp, \otimes)$ is a symmetric monoidal functor.

Now we move on to the notion of fiber sequence:

Definition 2.1.6. A sequence of map $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ is called a fiber sequence if the induced map of homotopy groups

$$\cdots \to \pi_n \mathcal{X} \to \pi_n \mathcal{Y} \to \pi_n \mathcal{Z} \to \cdots \tag{2.18}$$

is a long exact sequence of abelian groups.

Note that if $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ is a fiber sequence, there is a caononical map $\mathcal{Z} \to \Sigma\mathcal{X}$, and $\mathcal{Y} \to \mathcal{Z} \to \Sigma\mathcal{X}$ is also a fiber sequence.

Given $f : \mathcal{X} \to \mathcal{Y}$, there is an essentially unique $fib(f)$ with a canonical map $fib(f) \to \mathcal{X}$ that makes $fib(f) \to \mathcal{X} \to \mathcal{Y}$ a fiber sequence.

Most operations in Sp preserves fiber sequence: let $\mathcal{X}$ be a spectrum, then the functors

$$\Sigma,\ \Omega,\ Maps(\mathcal{X}, -),\ Maps(-, \mathcal{X}),\ \mathcal{X} \otimes - \tag{2.19}$$

all sends fiber sequence to fiber sequence (these are called exact functors).

Recall that in S_*, there is a notion of cofiber and fiber sequences (and they are not the same). The functor

$$\Sigma^{\infty} : S_* \to Sp \tag{2.20}$$

sends cofiber sequences of pointed spaces to fiber sequences of spectra.

Similarly, we have

$$\Omega^{\infty} : Sp \to S_* \tag{2.21}$$

sends fiber sequences of spectra to fiber sequences of pointed spaces.

We will also need to talk about the Postnikov truncation. We first start with the notion of connective, coconnective spectra:

Definition 2.1.7. A spectrum $\mathcal{X}$ is called n-connective if for every $n < 0$, we have $\pi_n(\mathcal{X}) = 0$. Similarly, a spectrum is called n-coconnective if for every $n > 0$¡ we have $\pi_n(\mathcal{X}) = 0$. When n is 0, they are just called connective and coconnective.

The quintessential example of a connective spectrum is $\Sigma^\infty X$ for X a pointed space.

For a spectrum $\mathcal{X}$ and $n \in \mathbb{Z}$, there is an universal n-connective spectra $\tau_{\geq n}\mathcal{X}$ with a map $\tau_{\geq n}\mathcal{X} \to \mathcal{X}$ that induces isomorphism

$$\pi_i(\tau_{\geq n}\mathcal{X}) \simeq \pi_i\mathcal{X} \tag{2.22}$$

for all $i >\geq n$. Similarly, there is also an universal $n-1$-coconnective spetrum $\tau_{<n}\mathcal{X}$ with a map $\mathcal{X} \to \tau_{<n}\mathcal{X}$ that induces isomorphism

$$\pi_i(\mathcal{X}) \to \pi_i(\tau_{<n}\mathcal{X}) \tag{2.23}$$

for $i < n$. Note that for any n, the canonical maps

$$\tau_{\geq n}\mathcal{X} \to \mathcal{X} \to \tau_{<n}\mathcal{X} \tag{2.24}$$

is a fiber sequence.

Lastly, we need the notion of a ring spectrum. Recall the Sp is a symmetric monoidal category with tensor product $\otimes$. For symmetric monoidal category $(C, \otimes, 1_C)$, there is a notion of associative algebra of $(C, \otimes)$. Heuristically, it is an object $c \in C$ with an associative product $c \otimes c \to c$ and an unital object $1_C \to c$. In addition, if c is an algebra object, then we can define (left) module objects of c. Heuristically, a c-module is an object $d \in C$ and action maps $c \otimes d \to d$ that is unital and compatible with the multiplication of c.

Definition 2.1.8. An $\mathbb{E}_1$-ring spectrum $\mathcal{R}$ is an associative algebra in Sp. We also refer to them as just ring spectra. For a ring spectrum $\mathcal{R}$, a $\mathcal{R}$ module is a $\mathcal{R}$ module object in Sp. The category of $\mathcal{R}$ modules forms a category $Mod_{\mathcal{R}}$.

Remark 2.1.9. As Sp is a higher category, thus there are higher coherence data for ring spectrum and module spectrum that needs to be given. They are essential part of the data. For precise definition see [9].

Example 2.1.10. The quiteseential example of a ring spectrum is the sphere spectrum $\mathcal{S}$. Being the unit of Sp, $\mathcal{S}$ has in fact a fully symmetric monoidal product. In addition, every spectrum is canonically a $\mathcal{S}$ module, thus $Mod_{\mathcal{S}} = Sp$.

Example 2.1.11. Let R be an associative ring, then the Eilenberg-MacLane spectrum HR is a ring spectrum. If N is a R module, then HN is a HR module. Thus $\mathbb{Z}$ is a (commutative) ring spectrum, and any Eilenberg-MacLane spectrum is a module for $\mathbb{Z}$. Note that $Mod_{\mathbb{Z}}$ is the derived category of chain complexes of Abelian groups $D(Ab)$.

2.2 Spectra and Cohomology theories

A huge motivation for spectra was to define a suitable category of (co)homology theories. In this subsection we review this relationship. This subsection largely follows [7]. We first recall the definition of a homology/cohomology theory. We first start with homology theory:

Let CW be the ordinary 1-category of finite pointed CW complexes (not localized at weak equivalence), CW_* be the category of finite pointed CW complexes. A reduced homology theory is a sequence of functors $\widetilde{E}_n : CW \to Ab$ that satisfies the Eilenberg-Steenrod Axioms:

Definition 2.2.1. An reduced (extraordinary) homology theory $\widetilde{E}_*$ is a sequence of functors $\widetilde{E}_n : CW_* \to Ab$, $n \in \mathbb{Z}$, such that

1. A homotopy equivalence of pointed finite CW complexes $f : X \xrightarrow{\sim} Y$ induces an isomorphism
$$WE_n(f) : \widetilde{E}_n(X) \xrightarrow{\sim} \widetilde{E}_n(Y) \tag{2.25}$$
for every n.

2. For any two finite CW complexes X and Y, the canonical map exhibits isomorphism
$$\widetilde{E}_n(X \vee Y) \simeq \widetilde{E}_n(X) \oplus \widetilde{E}_n(Y) \tag{2.26}$$
for all n.

3. For any finite CW complex X, we have canonical isomorphism
$$\widetilde{E}_{n+1}(\Sigma X) \simeq \widetilde{E}_n(X). \tag{2.27}$$

4. Let $X \to Y \to Z$ be a cofiber sequence of pointed finite CW complexes. Then we the sequence
$$\widetilde{E}_n(X) \to \widetilde{E}_n(Y) \to \widetilde{E}_n(Z) \tag{2.28}$$
is exact at $\widetilde{E}_n(Y)$.

Note that if $X \to Y \to Z$ is a cofiber sequence, then so is $Y \to Z \to \Sigma X$. As $\widetilde{E}_{n+1}(\Sigma X) \simeq \widetilde{E}_n(X)$, we see that we have a long exact sequence of homology groups
$$\cdots \to \widetilde{E}_n(X) \to \widetilde{E}_n(Y) \to \widetilde{E}_n(Z) \to \widetilde{E}_{n-1}(X) \to \cdots \tag{2.29}$$
A homology theory $\widetilde{E}_*$ is called ordinary if $\widetilde{E}_n(S^0) = 0$ for $n \neq 0$.

From a reduced homology theory we can define a nonreduced homology theory:

Definition 2.2.2. Let $\widetilde{E}_*$ be a reduced homology theory. Let X be a (unpointed) CW complex. Then the nonreduced homology groups $E_i(X)$ is defined as

$$E_i(X) \coloneqq \widetilde{E}_i(X_+), \tag{2.30}$$

where X_+ is the X with an added basepoint. Functorially, E_n is the composition

$$CW \xrightarrow{-_+} CW_* \xrightarrow{\widetilde{E}_n} Ab. \tag{2.31}$$

Similarly, we can define a cohomology theory:

Definition 2.2.3. An reduced (extraordinary) cohomology theory $\widetilde{E}^*$ is a sequence of functors $\widetilde{E}^n : CW^{op} \to Ab$ such that

1. A homotopy equivalence of pointed finite CW complexes $f : X \xrightarrow{\sim} Y$ induces an isomorphism

$$\widetilde{E}^n(f) : \widetilde{E}^n(Y) \xrightarrow{\sim} \widetilde{E}^n(X) \tag{2.32}$$

for every n.

2. For any two finite CW complexes X and Y, the canonical map exhibits isomorphism

$$\widetilde{E}^n(X \vee Y) \simeq \widetilde{E}^n(X) \oplus \widetilde{E}^n(Y) \tag{2.33}$$

for all n.

3. For any finite CW complex X, we have canonical isomorphism

$$\widetilde{E}^{n-1}(\Sigma X) \simeq \widetilde{E}^n(X). \tag{2.34}$$

4. Let $X \to Y \to Z$ be a cofiber sequence of pointed finite CW complexes. Then we the sequence

$$\widetilde{E}^n(Z) \to \widetilde{E}^n(Y) \to \widetilde{E}^n(X) \tag{2.35}$$

is exact at $\widetilde{E}^n(Y)$.

Simiarly, given a cofiber sequence $X \to Y \to Z$, we have a long exact sequence of cohomology groups

$$\cdots \to \widetilde{E}_n(X) \to \widetilde{E}_n(Y) \to \widetilde{E}_n(Z) \to \widetilde{E}_{n-1}(X) \to \cdots \tag{2.36}$$

A cohomology theory $\widetilde{E}^*$ is called ordinary if $\widetilde{E}^n(S^0) = 0$ for $n \neq 0$.

From a reduced cohomology theory we can define a nonreduced cohomology theory:

Definition 2.2.4. Let $\widetilde{E}^*$ be a reduced cohomology theory. Let X be a (unpointed) CW complex. Then the nonreduced cohomology groups $E^i(X)$ is defined as

$$E^i(X) \coloneqq \widetilde{E}^i(X_+), \tag{2.37}$$

where X_+ is the X with an added basepoint. Functorially, E^n is the composition

$$CW^{op} \xrightarrow{-_+} CW_*^{op} \xrightarrow{\widetilde{E}^n} Ab. \tag{2.38}$$

One of the original motivation for spectra is that they give homology and cohomology theories:

Construction 2.2.5. Let $\mathcal{X}$ be a spectrum, X a pointed finite CW space. Then we define the n-th reduced homology of X with coefficients $\mathcal{X}$ as

$$\widetilde{\mathcal{X}}_n(X) \coloneqq \pi_n(\Sigma^\infty X \otimes \mathcal{X}). \tag{2.39}$$

Functorially, consider the composition

$$\widetilde{\mathcal{X}}_n : CW_* \xrightarrow{\Sigma^\infty} Sp \xrightarrow{\otimes \mathcal{X}} Sp \xrightarrow{\pi_n} Ab. \tag{2.40}$$

Similarly, we define the n-th reduced cohomology group of X with coefficients in $\mathcal{X}$ as

$$\widetilde{\mathcal{X}}^n(N) \coloneqq \pi_{-n}(Maps(\Sigma^\infty X, \mathcal{X})). \tag{2.41}$$

Functorially, consider the composition

$$\widetilde{\mathcal{X}}^n : CW_*^{op} \xrightarrow{\Sigma^\infty} Sp^{op} \xrightarrow{Maps(-,\mathcal{X})} Sp \xrightarrow{\pi_{-n}} Ab. \tag{2.42}$$

One can check that $\widetilde{\mathcal{X}}_n, \widetilde{\mathcal{X}}^n$ satisfies the Eilenberg-Steenrod axioms, thus they define homology and cohomology theory, respectively. Thus we can also define nonreduced homology and cohomology:

Definition 2.2.6. Thus if X is an unpointed CW complex, the nonreduced homology $\mathcal{X}_i(X)$ of X with coefficients in $\mathcal{X}$ is

$$\mathcal{X}_i(X) \coloneqq \pi_i(\Sigma^\infty N \wedge \mathcal{X}). \tag{2.43}$$

And the nonreduced cohomology $\mathcal{X}^i(X)$ of X with coefficients in $\mathcal{X}$ is

$$\mathcal{X}^n(N) \coloneqq \pi_{-n}(Maps(\Sigma_+^\infty X, \mathcal{X})). \tag{2.44}$$

Conversely, given a homology/cohomology, theory, we can find a spectrum representing it. This is the statements of Brown's representability theorems [3]:

Theorem 2.2.7 (Brown's Representability)**.** *For any (reduced) homology theory $\widetilde{E}_n$, there is an essentially unique spectrum representing it. That is, there is a spectrum $\mathcal{X}$ such that $\widetilde{\mathcal{X}}_n \simeq \widetilde{E}_n$ compatible with all data.*

Note

$$\pi_n(\mathcal{X}) = \widetilde{\mathcal{X}}_n(S^0) \simeq \widetilde{E}_n(S^0). \tag{2.45}$$

Thus if $\widetilde{E}_*$ is an ordinary homology theory with $\widetilde{E}_0(S^0) = A$, then it is the homology theory associated to the Eilenberg-MacLane spectrum HA. This is why it's called ordinary.

We have a similar theorem for cohomology theories

Theorem 2.2.8. *For any (reduced) cohomology theory $\widetilde{E}^n$, there is an essentially unique spectrum representing it. That is, there is a spectrum $\mathcal{X}$ such that $\widetilde{\mathcal{X}}^n \simeq E^n$ compatible with all data.*

Note that

$$\pi_n(\mathcal{X}) = \widetilde{\mathcal{X}}^{-n}(S^0) \simeq \widetilde{E}^{-n}(S^0). \tag{2.46}$$

Thus if $\widetilde{E}^*$ is an ordinary cohomology theory with $\widetilde{E}^0(S^0) = A$, then it is the cohomology theory associated to HA. We will not prove Brown's Representability theorems, see [3], [9] for details.

2.3 π-finite spectra

In this subsection we define π-finite spectra and prove some basic facts about them:

Definition 2.3.1. Let $\mathcal{X}$ be a spectrum. It is π-finite if the (stable) homotopy groups $\pi_*\mathcal{X}$ are nonzero only in finitely many degrees, and each $\pi_i\mathcal{X}$ is a finite abelian group.

We will denote the full subcategory of π-finite spectra as Sp^{fin}.

Remark 2.3.2. There is a notion of finite spectrum as dualizable objects in Sp. Note that that is a different notion. For example, suspension spectrum of finite CW complexes are dualizable, but they are not always π-finite. For example, the sphere spectrum $\mathcal{S}^0$ is not π-finite.

Definition 2.3.3. Let $C \subset Sp$ be a full subcategory of Sp. Then the extension closure of C, denoted as C^c, is the smallest full subcategory such that

1. C^c is closed under suspension (and desuspension).

2. if $\mathcal{X}, \mathcal{X}'' \in C^c$, and we have a fiber sequence $\mathcal{X} \to \mathcal{X}' \to \mathcal{X}''$ of spectra, then $\mathcal{X}' \in C^c$.

The objects in C^c are are finite extensions of spectrum in C.

Proposition 2.3.4. $(Sp^{fin})^c = Sp^{fin}$.

Proof. Sp^{fin} is clearly closed under suspension. Given $\mathcal{X} \to \mathcal{X}' \to \mathcal{X}''$ a fiber sequence of spectra. If $\mathcal{X}$, $\mathcal{X}''$ are π-finite, we need to show that $\mathcal{X}$ is also π-finite. This is due to the long exact sequence of homotopy groups:

$$\cdots \to \pi_* \mathcal{X} \to \pi_* \mathcal{X}' \to \pi_* \mathcal{X}'' \to \pi_{*-1} \mathcal{X} \to \cdots . \tag{2.47}$$

□

As π-finite spaces are finite extensions of finite Eilenberg MacLane spaces, π-finite spectra are finite extensions of finite Eilenberg-MacLane spectrum:

Proposition 2.3.5. *Let C be the full subcategory of finite Eilenberg-MacLane spectrum (spectrum of the form HA for some A finite abelian group), then C^c is the full subcategory of π-finite spaces.*

Proof. Clearly Sp^{fin} it contains C. Thus it is suffice to show that all π-finite spectra are finite extensions of finite Eilenberg-MacLane spectrum. Let $\mathcal{X}$ be a π-finite spectrum.

We will do induction on the range where the homotopy groups of $\mathcal{X}$ are nontrivial. On the base case, where the homotopy groups of $\mathcal{X}$ is concentrated in a single degree, then it is precisely a suspension of HA, where A is an finite abelian groups. These are in C^c as C^c is closed under suspension.

We shift $\mathcal{X}$ so that its lowest nontrival homotopy group is concentrated in degree 0. Assume we have proven the case for all $\mathcal{X}$ where $\mathcal{X}$'s homotopy groups are concentrated in degree 0 to i. Now for a π-spectrum $\mathcal{X}$ with highest nontrivial homotopy group in degree $i+1$: consider the fiber sequence

$$\tau_{\geq i+1} \mathcal{X} \to \mathcal{X} \to \tau_{\leq i} \mathcal{X}. \tag{2.48}$$

Note that the homotopy groups of $\tau_{\geq i+1} \mathcal{X}$ is concentrated in a single degree, namely $i+1$. Thus it is a suspension of HA, and it is in C^c. By induction, $\tau_{\leq i} \mathcal{X}$ is a π-fintie spectrum whose nontrivial homotopy group is concentrated in degree 0 to i, thus it is also in C^c. But the extension property, so is $\mathcal{X}$. □

Lastly, we need the following lemma, analogous to lemma 1.2.8:

Lemma 2.3.6. *Let N be a compact manifold with boundary, $\mathcal{X}$ a π-finte spectrum. Then we have the mapping spectrum*

$$\mathcal{X}(M) \coloneqq Maps(\Sigma^{\infty}_{+} M, \mathcal{X}). \tag{2.49}$$

Then $\mathcal{X}(M)$ is a π-finite space

Proof. Let C be the full subcategory whose objects are π-finite spaces such that for all N compact manifold with boundary, $\mathcal{X}(M)$ is π-finite. Note that $C^c = C$, that is, it is closed under suspension and extensions. Suspension is easy as

$$(\Sigma\mathcal{X})(M) = \Sigma(\mathcal{X}(M)). \tag{2.50}$$

For extensions, given a fiber sequence

$$\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}, \tag{2.51}$$

then we have a corresponding fiber sequence

$$\mathcal{X}(M) \to \mathcal{Y}(M) \to \mathcal{Z}(M). \tag{2.52}$$

Now if $\mathcal{X}(M)$ and $\mathcal{Z}(M)$ are in C, then so is $\mathcal{Y}(M)$ by looking at the long exact sequence of homotopy groups.

Thus by proposition 2.3.5, it is suffice to show that finite Eilenberg MacLane spectrums HA are in C, where A is a finite abelian group. Note that the homotopy groups

$$\pi_{-i}\ HA(M) = H^i(M, A) \tag{2.53}$$

are the ordinary cohomology groups of M with A coefficients. These are concentrated in degree 0 to $dimM$ and are finite as M is compact. One way to see this is M is homotopy equivalent to a finite CW complex, thus the CW cochain complex that computes $H^*(M, A)$ is finite. Another way is to use Cech cohomology. □

2.4 Poincaré duality

In this subsection, we define the relative cap product, introduce the notion of R-orientation. and state the relative Poincaré duality theorem 2.4.17.

Let S, S_*, Sp be the category of spaces, pointed spaces, and spectra. The basics of spectra and (co)homology theories are developed in 2.1 and 2.2. Recall that Spectra defines generalized (reduced) homology and cohomology theories:

Definition 2.4.1. Let $\mathcal{X}$ be a spectra, N a pointed CW complex. Then we define the i-th reduced homology on N with coefficients in $\mathcal{X}$:

$$\widetilde{\mathcal{X}}_i(N) \coloneqq \pi_i(\Sigma^\infty N \wedge \mathcal{X}). \tag{2.54}$$

Similarly, we define the i-th reduced cohomology group of N with coefficients in $\mathcal{X}$:

$$\widetilde{\mathcal{X}}^i(N) \coloneqq \pi_{-i}(Maps(\Sigma^\infty N, \mathcal{X})). \tag{2.55}$$

Now for unpointed spaces, thus nonreduced (co)homology:

Definition 2.4.2. Let $\mathcal{X}$ be a spectra, N a (unpointed) CW complex. Then we define the i-th homology on N with coefficients in $\mathcal{X}$:

$$\mathcal{X}_i(N) \coloneqq \pi_i(\Sigma^\infty_+ N \wedge \mathcal{X}). \tag{2.56}$$

Similarly, we define the i-th cohomology group of N with coefficients in $\mathcal{X}$:

$$\mathcal{X}^i(N) \coloneqq \pi_{-i}(Maps(\Sigma^\infty_+ N, \mathcal{X})). \tag{2.57}$$

Recall that cofiber sequence of based spaces gives long exact sequence of (co)homology groups: Let $N \to N' \to N''$ be a confiber sequence in based spaces. Then we have a long exact sequence of homology groups:

$$\cdots \to \widetilde{\mathcal{X}}_*(N) \to \widetilde{\mathcal{X}}_*(N') \to \widetilde{\mathcal{X}}_*(N'') \to \widetilde{\mathcal{X}}_{*-1}(N) \to \cdots . \tag{2.58}$$

We also have long exact sequence of cohomology groups:

$$\cdots \to \widetilde{\mathcal{X}}^*(N'') \to \widetilde{\mathcal{X}}^*(N') \to \widetilde{\mathcal{X}}^*(N) \to \widetilde{\mathcal{X}}^{*+1}(N) \to \cdots . \tag{2.59}$$

Example 2.4.3. Let $i : N \hookrightarrow M$ be a "nice" inclusion (like inclusion of a boundary component of a manifold). Then we have a cofiber sequence of base spaces

$$N_+ \to M_+ \to M/N. \tag{2.60}$$

The quotient M/N is canonicall a based space with the base point being the image of N under $p : M \to M/N$. We define the relative homology, cohomology group for the pair (M, N) as:

$$\mathcal{X}_*(M, N) \coloneqq \widetilde{\mathcal{X}}_*(M/N),\ \mathcal{X}^*(M, N) \coloneqq \widetilde{\mathcal{X}}^*(M/N). \tag{2.61}$$

By equation 2.58 We have long exact sequence of homology groups:

$$\cdots \to \mathcal{X}_*(N) \to \mathcal{X}_*(M) \to \mathcal{X}_*(M, N) \to \mathcal{X}_{*-1}(N) \to \cdots , \tag{2.62}$$

and cohomology groups (equation 2.59):

$$\cdots \to \mathcal{X}^*(N) \to \mathcal{X}^*(M) \to \mathcal{X}^*(M, N) \to \mathcal{X}^{*+1}(N) \to \cdots . \tag{2.63}$$

Now we will construct the cap product.

Construction 2.4.4. Let $\mathcal{R}$ be a $\mathbb{E}_1$ ring spectrum (see 2.1 for definition) and $\mathcal{X}$ a $\mathcal{R}$-module spectrum. We have the action map

$$act : \mathcal{R} \wedge \mathcal{X} \to \mathcal{X}. \tag{2.64}$$

Let N, N', N'' be based spaces and

$$f : N \to N' \wedge N'' \tag{2.65}$$

be a map of based spaces. Recall that $\mathcal{S}$ is the sphere spectrum. Let

$$\sigma : \mathcal{S} \to \Sigma^{-m}\mathcal{R} \wedge \Sigma^{\infty} N \tag{2.66}$$

be a map representing the homology class

$$[\sigma] \in \pi_0(\Sigma^{-n}\mathcal{R} \wedge \Sigma^{\infty} N) = \widetilde{\mathcal{R}}_n(N). \tag{2.67}$$

Similarly, let

$$\alpha : N' \to \Sigma^n \mathcal{X} \tag{2.68}$$

represent the cohomology class

$$[\alpha] \in \widetilde{\mathcal{X}}^n(N'). \tag{2.69}$$

Then consider the following composition

$$\begin{aligned}
\sigma \frown \alpha : S &\xrightarrow{\sigma} \Sigma^{-m}\mathcal{R} \wedge \Sigma^{\infty} N && (2.70)\\
&\xrightarrow{id \wedge f} \Sigma^{-m}\mathcal{R} \wedge \Sigma^{\infty} N' \wedge \Sigma^{\infty} N'' && (2.71)\\
&\xrightarrow{id \wedge \alpha \wedge if} \Sigma^{-m+n}\mathcal{R} \wedge \mathcal{X} \wedge \Sigma^{\infty} N'' && (2.72)\\
&\xrightarrow{act \wedge id} \Sigma^{-m+n}\mathcal{X} \wedge \Sigma^{\infty} N''. && (2.73)
\end{aligned}$$

This represents a class in

$$[\sigma \frown \alpha] \in \pi_0(\Sigma^{-m+n}\mathcal{X} \wedge \Sigma^{\infty} N'') = \widetilde{\mathcal{X}}_{m-n}(N''). \tag{2.74}$$

This cohomology class does not depends on the representatives σ and α. Thus we have a well-defined map:

$$- \frown - : \widetilde{\mathcal{R}}_n(N) \otimes \widetilde{\mathcal{X}}^n(N') \to \widetilde{\mathcal{X}}_{m-n}(N''). \tag{2.75}$$

This is called the cap product.

Now we introduce the notion of $\mathcal{R}$-orientation, when $\mathcal{R}$ is a ring spectrum. First we need the following lemma:

Lemma 2.4.5. *Let M be a d dimensional manifold with boundary. Then let $x \in M^o = M - \partial M$ be an interior point in M. We denote $M - x$ is the complement of M. Then for any spectrum $\mathcal{X}$, $\mathcal{X}_*(M, M - x) \simeq \widetilde{\mathcal{X}}_*(S^d) \simeq \pi_{*-d}\mathcal{X}$.*

Proof. We have to compute the cofiber $M/M - x$. Note as $M - x \subset M$ is not a "nice inclusion", we have to homotopic it to be one. As this is local in x, and x is in the interior of M. We can replace M with a local corodinate B^d the d-dimensional ball, and x be the origin. Then

$$\begin{aligned} M/(M - x) &\simeq B^d/(B^d - x) && (2.76)\\ &\simeq B^d/(\partial B^d) && (2.77)\\ &\simeq S^{d-1}. && (2.78)\end{aligned}$$

Thus

$$\begin{aligned} \mathcal{X}_*(M, M - x) &\simeq \mathcal{X}_*(S^d) && (2.79)\\ &\simeq \pi_{*-d}(\mathcal{X}). && (2.80)\end{aligned}$$

□

Now we can define $\mathcal{R}$-orientation on manifolds with boundary:

Definition 2.4.6. Let M be a d dimensional manifold with boundary, $\mathcal{R}$ a ring spectrum. Recall that $\pi_*(\mathcal{R})$ gets a graded ring structure. Then an $\mathcal{R}$ orientation on M is a homology class

$$[M] \in \mathcal{R}_d(M, \partial M) \tag{2.81}$$

satisfying the following condition: for every $x \in M^o$ a point in the interior, the image of $[M]$ under

$$\mathcal{R}_d(M, \partial M) \to \mathcal{R}_d(M, M - x) \simeq \pi_0(\mathcal{R}) \tag{2.82}$$

is an multiplicative unit in the ring $\pi_*(\mathcal{R})$.

Remark 2.4.7. See [14] for details about $\mathcal{R}$. A modern approach to $\mathcal{R}$-orientation is described in [1].

Example 2.4.8. Let $\mathcal{R}$ be $H\mathbb{Z}/2\mathbb{Z}$, then every manifold is $H\mathbb{Z}/2\mathbb{Z}$-oriented. Let $\mathcal{R}$ be $H\mathbb{Z}$, then $H\mathbb{Z}$ orientation is the usual notion of orientation for manifolds. Let $\mathcal{R}$ be $\mathcal{S}$ the sphere spectrum, then a $\mathcal{S}$ orientation on N is a trivialization of the Thom spectra of the normal bundle of N. See [1] for details.

Remark 2.4.9. If N is a d dimensional manifold without boundary. Then an $\mathcal{R}$ orientation lives in $\mathcal{R}_d(N)$.

Remark 2.4.10. Given a ring homomorphism of ring spectrum $f : \mathcal{R} \to \mathcal{R}'$, then an $\mathcal{R}'$ orientation gives a $\mathcal{R}'$ orientation via pushforward:

$$f_* : \mathcal{R}'(-) \to \mathcal{R}'(-). \tag{2.83}$$

It turns out an $\mathcal{R}$ orientation on a d-dimensional manifold gives a $\mathcal{R}$ orientation on the boundary:

Proposition 2.4.11. *Let M be a d dimensional manifold with boundary. Then a $\mathcal{R}$ orientation on M, $[M] \in \mathcal{R}_d(M, \partial M)$ gives a class $\partial[M]$ in $\mathcal{R}_{d-1}(N)$ via the natural boundary map*

$$\partial : \mathcal{R}_*(M, \partial M) \to \mathcal{R}_*(\partial M). \tag{2.84}$$

This class $\partial[M]$ is a $\mathcal{R}$ orientation on the boundary ∂M.

Proof. A proof of this is given in [12]. □

With the notion of $\mathcal{R}$ orientation and cap product, we can now define Poincaré isomorphism map:

Let $\mathcal{R}$ be a ring spectrum, $\mathcal{X}$ a module spectrum for $\mathcal{R}$. Let N be a $\mathcal{R}$-oriented d-dimensional manifold. We denote the orientation class as $[N] \in \mathcal{R}_{d-1}(N)$. Let f be the diagonal map:

$$N_+ \to N_+ \wedge N_+ \simeq (N \times N)_+. \tag{2.85}$$

Then the cap product (see equation 2.75) with $[N]$ gives a map

$$\int_{[N]} : \mathcal{X}^*(N) \to \mathcal{X}_{d-*}(N). \tag{2.86}$$

Here's the Poincaré duality for manifold without boundary:

Theorem 2.4.12 (Poincaré duality)**.** *For every $*$, the map*

$$\int_{[N]} : \mathcal{X}^*(N) \to \mathcal{X}_{d-*}(N) \tag{2.87}$$

is an isomrphism.

Proof. This is (for ordinary cohomology) given in [12], the proof applies the same way. ☐

Remark 2.4.13. We write the cap product as $\int_{[N]}$ because it is a form of intergration. In de Rham cohomology, this is explicitly given by intergration, once given an orientation.

We a more general form Poincaré duality for manifolds with boundaries. Let M be a d-dimensional $\mathcal{R}$-oriented manifold with boundary $\partial M = N \sqcup N'$. We have the orientation class $[M] \in \mathcal{R}_d(M, \partial M)$. Consider the map

$$M/\partial M \to M/N \wedge M/N' \tag{2.88}$$

of pointed spaces. From 2.75 we get a map

$$[M] \frown - : \mathcal{X}^*(M, N) \to \mathcal{X}_{d-*}(M, N') \tag{2.89}$$

We denote this map by $\int_{[M,N]}$. Poincaré duality for manifold with boundary states $\int_{[M,N]}$ is an isomorphism.

Theorem 2.4.14 (Poincaré duality)**.** *The map*

$$\int_{[M,N]} : \mathcal{X}^*(M, N) \to \mathcal{X}_{d-*}(M, N') \tag{2.90}$$

are isomorphism.

Proof. The case that $N = \partial M$ and $N = \varnothing$ is given in [12]. The general case for ordinary homology theory is given in [**Lef**]. ☐

There are two special examples: let $N = \partial M$ and $N' = \varnothing$. Then we have the following corollary:

Corollary 2.4.15. *The map*

$$\int_{[M,\partial M]} : \mathcal{X}^*(M, \partial M) \to \mathcal{X}_{d-*}(M). \tag{2.91}$$

is an isomorphism.

Similarly, let $N = \varnothing$, and $N = \partial M$. Then Poincaré duality gives:

Corollary 2.4.16.

$$\int_{[M]} : \mathcal{X}^*(M) \to \mathcal{X}_{d-*}(M, \partial M). \tag{2.92}$$

Lastly, for our main theorem, we will need to know the functoriality of Poincaré duality:

Theorem 2.4.17. *Let $\mathcal{R}$ be a ring spectrum and $\mathcal{X}$ be a module spectrum. Let M be a $\mathcal{R}$-oriented manifold with boundary $\partial M = N \sqcup N'$. We denote the orientation class as $[M]$. Note that it also gives orientation on the boundaries $[N], -[N']$. Poincaré duality isomorphism maps give an equivalence of long exact sequences:*

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^{d-1}(M, N') \arrow[r] \arrow[d, "\int_{[M,N']}"] & \hat{\mathcal{X}}^{d-1}(M) \arrow[r, "\hat{p}^*"] \arrow[d, "\int_{[M]}"] & \hat{\mathcal{X}}^{d-1}(N') \arrow[r] \arrow[d, "\int_{[N']}"] & \cdots \arrow[d] \\ \cdots \arrow[r] & \hat{\mathcal{X}}_1(M, N) \arrow[r, "\hat{\mu}_*"] & \hat{\mathcal{X}}_1(M, \partial M) \arrow[r, "\hat{\nu}_*"] & \hat{\mathcal{X}}_0(N') \arrow[r] & \cdots \end{tikzcd} \tag{2.93}$$

Proof. A version of this is proven in [12]. □

2.5 Pontryagin duality for finite abelian groups

In this subsection we review Pontryagin duality for finite abelian groups:

We denote the category of Abelian group as Ab, and the full subcategory of finite abelian group as Ab^{fin}. Note that given $A, B \in Ab$, the set of homomorphism from A to B $Hom(A, B)$ has a group structure by pointwise multiplication (Ab has internal hom). Let $\mathbb{C}^\times$ denote the group of units in $\mathbb{C}$. This is an injective obejct in Ab.

Definition 2.5.1. Let A be an abelian group. The Pontryagin dual group $\hat{A}$ is defined to be $Map(A, \mathbb{C}^\times)$.

Note that taking Pontryagin dual gives an exact contravariant functor $D := Hom(-, \mathbb{C}^\times) : Ab \to Ab^{op}$. It is exact as $\mathbb{C}^\times$ is an injective object.

Remark 2.5.2. Another candidate for $\hat{A}$ is $Hom(A, \mathbb{Q}/\mathbb{Z})$. They are the same for finite abelian group, which is the case that we care about.

Example 2.5.3. Let $A = \mathbb{Z}$, then $\hat{A} = Hom(A, \mathbb{C}^\times) = \mathbb{C}^\times$.

Example 2.5.4. Let $A = \mathbb{Z}/n\mathbb{Z}$, then $\hat{A} = \mu_n \subset \mathbb{C}^\times$ the subgroup of n-th root of unity.

Lemma 2.5.5. *Let $A, B \in Ab$, then $\widehat{A \times B} \simeq \hat{A} \times \hat{B}$.*

Proof. Note that product and coproduct are the same in Ab. Thus

$$\widehat{A \times B} = Hom(A \times B, \mathbb{C}^{\times}) \tag{2.94}$$
$$\simeq Hom(A, \mathbb{C}^{\times}) \times Hom(B, \mathbb{C}^{\times}) \tag{2.95}$$
$$= \hat{A} \times \hat{B}. \tag{2.96}$$

□

Note that there is a canonical bilinear pairing

$$(-,-)_A : A \times \hat{A} \to \mathbb{C}^{\times}. \tag{2.97}$$

Bilinear means that $(aa', \alpha)_A = (a, \alpha)_A (a', \alpha)_A$. It might looks strange for bilinearity because we describe the group operation multiplicatively rather than additively, which is more common when we view abelian group as $\mathbb{Z}$ modules. It is equivalent to a group homomorphism $A \otimes \hat{A} \to \mathbb{C}^{\times}$.

This pairing gives a universal characterization of the Pontryagin dual:

Definition 2.5.6. Let $A, B \in Ab$ be two abelian groups. A pairing is a bilinear map $\mu : A \times B \to \mathbb{C}^{\times}$. This induce a map $\phi_\mu : B \to \hat{A}$, given by $\phi_\mu(b)(a) := \mu(a, b)$. The pairing μ says to exhibit B as the Pontryagin dual of A if ϕ_μ is an isomorphism.

Example 2.5.7. By above, $B = \hat{A}$, $\mu = (-,-)_A$ exhibits $\hat{A}$ as the Pontryagin dual of A. This is the universal example as $\phi_\mu : \hat{A} \to \hat{A}$ is identity.

Given A, B and $\mu : A \times B \to \mathbb{C}^{\times}$ a bilinear pairing. Let $\mathbb{C}[A]$ be the free vector space generated by (the set) A. Similarly for $\mathbb{C}[B]$. The pairing μ give rise to a bilinear pairing $\alpha_\mu : \mathbb{C}[A] \otimes \mathbb{C}[B] \to \mathbb{C}$. On the basis it sends $a \otimes b \to \mu(a, b)$.

Now we restrict to finite abelian groups.

Proposition 2.5.8. *If A is a finite abelian group. Then $\hat{A}$ is also an finite abelian group. In addition, $|A| = |\hat{A}|$.*

Proof. By the classification of finite abelian group, we know that A is a product of $\mathbb{Z}/n\mathbb{Z}$. Note that for a single $\mathbb{Z}/n\mathbb{Z}$, its dual is μ_n, which is of the same size. For a product of $\mathbb{Z}/n\mathbb{Z}$, proposition 2.5.5 implies the result. □

Thus taking Pontryagin dual restricts to a functor from Ab^{fin} to $(Ab^{fin})^{op}$. We denote this functor by

$$D : Ab^{fin} \to (Ab^{fin})^{op}. \tag{2.98}$$

Remark 2.5.9. In fact, for A finite, then $\hat{A}$ is noncanonically isomorphic to A. But it is best to view them as different groups. This is similar to the case of duals of a finite dimensional vector space.

Recall from above that A, B and $\mu : A \times B \to \mathbb{C}^\times$ a bilinear pairing gives a bilinear map $\alpha_\mu : \mathbb{C}[A] \otimes \mathbb{C}[B] \to \mathbb{C}$. Now assume A and B finite. Recall that a pairing $\alpha : V \otimes V' \to \mathbb{C}^\times$ of finite dimensional vector spaces is called nondegenerate if the natural map $V' \to V^* = Hom_{\mathbb{C}}(V^*, \mathbb{C})$ is an isomorphism.

Proposition 2.5.10. *μ exhibits B as the Pontryagin dual of A iff*

$$\alpha_\mu : \mathbb{C}[A] \otimes \mathbb{C}[B] \to \mathbb{C} \tag{2.99}$$

is a nondegenerate pairing of finite dimensional vector spaces.

Proof. If the pairing By Proposition 2.5.8, $dim(\mathbb{C}[A]) = |A| = |B| = dim(\mathbb{C}[B])$. Thus it is suffice to that the map $\mathbb{C}[B] \to \mathbb{C}[A]^*$ is surjective. Recall that we view $a_i \in A$ and $b_j \in B$ as basis elements for $\mathbb{C}[A]$ and $\mathbb{C}[B]$. Let a^i be the dual basis for $\mathbb{C}[A]^*$. It is suffice to see that image of the map

$$\mathbb{C}[B] \to \mathbb{C}[A]^* \tag{2.100}$$

includes basis vectors a^i. Equivalently, exists vectos $v_i = v^j b_j \in \mathbb{C}[B]$ so that

$$\alpha_\mu(a_{i'}, v_i) = \delta_{i,i'}. \tag{2.101}$$

We first do this for $a_i = e$ the identity element. We take

$$v_e = \frac{1}{|B|} \sum_j b_j. \tag{2.102}$$

Then

$$\alpha_\mu(v_e, a_i) = \frac{1}{|B|} \sum_j \mu(a_i, b_j). \tag{2.103}$$

For $a_i \neq e$,

$$\sum_j \mu(b_j, a_i) = 0 \tag{2.104}$$

as we sum over all character of a nonidentity element. For $a_i = e$, then

$$\alpha_\mu(v_e, e) = \frac{1}{|B|}\sum_j \mu(e, b_j) \tag{2.105}$$

$$= \frac{1}{|B|}\sum_j 1 \tag{2.106}$$

$$= 1. \tag{2.107}$$

Thus v_e maps to a^e the dual basis vector of e. Now for a general a_i, then we let

$$v_i = \frac{1}{|B|}\sum_j \mu(a_i, b_j) b_j. \tag{2.108}$$

Same calculation shows that v_i maps to a^i.

For the converse, note that if the pairing is nondegenerate, then

$$\mathbb{C}[B] \simeq \mathbb{C}[A]^* \simeq \mathbb{C}[\hat{A}]. \tag{2.109}$$

The composition $\mathbb{C}[B] \xrightarrow{\sim} \mathbb{C}\hat{A}$ is induce by the map $\phi_\mu : B \to \hat{A}$. As the map of vector spaces is an isomorphism, so is the map of groups. □

Corollary 2.5.11. *μ exhibits B as the Pontryagin dual of A iff the map*

$$\begin{aligned} \mathbb{C}[A] &\to \mathbb{C}[B] \\ a &\mapsto \sum_b \mu(a, b)\ b \end{aligned} \tag{2.110}$$

is an isomorphism.

Lastly, note that we have a natural transformation $id \to D^2$ of functors $Ab \to Ab$. This natural transformation is given as such: given A, the map is

$$\begin{aligned} A &\to \hat{\hat{A}} \\ a &\mapsto (\alpha \mapsto \alpha(a)). \end{aligned} \tag{2.111}$$

The Pontryagin dual functor on finite abelian groups is a duality:

Theorem 2.5.12. *Restricted to Ab^{fin}, this natural transformation is an isomorphism. Thus $D^2 \simeq id$.*

Proof. I claim the pairing $(-,-)_A : \hat{A} \times A \to \mathbb{C}^\times$ exhibits A as the Pontryagin dual of $\hat{A}$. By Proposition 2.5.10, we see that this is equivalent to $\alpha_\mu : \mathbb{C}[A] \otimes \mathbb{C}[\hat{A}] \to \mathbb{C}$ being a nondegenerate pairing. Note that for finite dimensional vector spaces, this condition is symmetric in its variables. Thus A is the Pontryagin dual of $\hat{A}$. □

2.6 Pontryagin duality for spectra

In this subsection we review Pontryagin duality for spectra. Let Sp be the category of spectra. It also has a internal mapping objects $Maps$. To define Pontryagin dual, we first have to find a replace for $\mathbb{C}^\times$. Such replacement is $I\mathbb{C}^\times$:

Proposition 2.6.1. *There exists a spectrum $I\mathbb{C}^\times$ such that for any spectra $\mathcal{X}$, there is a functorial equivalence*

$$\pi_{-*}(Maps(\mathcal{X}, I\mathbb{C}^\times)) \simeq \widehat{\pi_*(\mathcal{X})}. \tag{2.112}$$

Proof. Let $\mathcal{S}$ be the sphere spectrum. Consider the homology theory represented by $\mathcal{S}$ (stable homotopy theory)

$$\widetilde{\mathcal{S}}_* : CW_* \xrightarrow{\Sigma^\infty} Sp \xrightarrow{\pi_*} Ab \tag{2.113}$$

As $\mathbb{C}^\times$ is an injective object in the category of abelian groups, Pontryagin duality functor $D : Ab \to Ab^{op}$ is an exact functor (it preserves long exact sequences of abelian groups). Thus the composition

$$\hat{\mathcal{S}}_* : CW_*^{op} \xrightarrow{\widetilde{\mathcal{S}}_*} Ab^{op} \xrightarrow{D} Ab \tag{2.114}$$

is a cohomology theory. By Brown representability 2.2.8, there exists a spectrum $\hat{\mathcal{S}}$ representing this cohomology theory

$$N \to \widehat{\mathcal{X}_i(N)} = \pi_*\widehat{(\Sigma^\infty N)}, \tag{2.115}$$

N is any finite pointed CW complex. Let $I\mathbb{C}^\times := \hat{\mathcal{S}}$. We see that for any finite (not π-finite) suspension spectrum $\Sigma^\infty N$, we have

$$\pi_{-*}(Maps(\Sigma^\infty N, I\mathbb{C}^\times)) = \widetilde{I\mathbb{C}^\times}^*(N) \tag{2.116}$$

$$\simeq \pi_*\widehat{(\Sigma^\infty N)}. \tag{2.117}$$

Thus we see that $I\mathbb{C}^\times$ satisfies equation 2.112 for finite suspension spectra. Now a general spectra $\mathcal{X}$ can be written as a colimit

$$\mathcal{X} \simeq \varinjlim \mathcal{X}_i \tag{2.118}$$

over finite suspension spectra $\mathcal{X}_i$ (in fact, we only need the sphere spectrums $\mathcal{S}^n$ for $n \geq 0$). Thus

$$Maps(\mathcal{X}, I\mathbb{C}^\times) \simeq Maps(\varinjlim \mathcal{X}_i, I\mathbb{C}^\times) \tag{2.119}$$

$$\simeq \varprojlim Maps(\mathcal{X}_i, I\mathbb{C}^\times) \tag{2.120}$$

$$\tag{2.121}$$

I don't really know what to do from here. Because filtered limits doesn't commute with taking homotopy groups (I think). I think we need to rig the system somehow so that it could commute with that, then we are done!

HA's Brown representability 1.4.1.2 does the job

. ☐

We define the Pontryagin dual group as follows:

Definition 2.6.2. Let $\mathcal{X}$ be a spectrum. The Pontryagin dual spectrum $\hat{\mathcal{X}}$ is defined to be the mapping spectrum $Maps(\mathcal{X}, I\mathbb{C}^\times)$.

Note that this defines a functor

$$\mathbb{D} := Maps(-, I\mathbb{C}^\times) : Sp \to Sp^{op}. \tag{2.122}$$

Example 2.6.3. Let $\mathcal{X}$ be the sphere spectrum $\mathcal{S}$. Then

$$\hat{\mathcal{S}} = Maps(\mathcal{S}, I\mathbb{C}^\times) \simeq I\mathbb{C}^\times. \tag{2.123}$$

Thus $I\mathbb{C}^\times$ is the Pontryagin dual of the sphere spectrum $\mathcal{S}$. This is similar to the fact that $\mathbb{C}^\times$ is the Pontryagin dual group of $\mathbb{Z}$.

We can calculate the homotopy group of $I\mathbb{C}^\times$: let $\mathcal{X}$ be the sphere spectrum $\hat{\mathcal{S}}$, then by proposition 2.6.1,

$$\pi_{-i}(Maps(\hat{\mathcal{S}}, I\mathbb{C}^\times)) = \pi_{-i}I\mathbb{C}^\times \simeq \widehat{\pi_i\mathcal{S}}. \tag{2.124}$$

Thus we see that

$$\pi_0 I\mathbb{C}^\times = \widehat{\pi_0\mathcal{S}} = Hom(\mathbb{Z}, \mathbb{C}^\times) = \mathbb{C}^\times. \tag{2.125}$$

The negative homotopy groups

$$\pi_{-i}I\mathbb{C}^\times = \widehat{\pi_i\mathcal{S}} \tag{2.126}$$

are finite abelian groups (noncanonically isomorphic the i-th homotopy group of spheres), and the positive homotopy groups are trivial. Thus $I\mathbb{C}^\times$ is a coconnective spectra, and there is a canonical map

$$H\mathbb{C}^\times \simeq \tau_{\geq 0}I\mathbb{C}^\times \to I\mathbb{C}^\times, \tag{2.127}$$

where $H\mathbb{C}^\times$ is the Eilenberg-MacLane spectrum corresponding to $\mathbb{C}^\times$.

We also have the canonical pairing

$$ev_{\mathcal{X}} : \mathcal{X} \wedge \hat{\mathcal{X}} \to I\mathbb{C}^\times \tag{2.128}$$

We use this to give a universal characterization of $\hat{\mathcal{X}}$:

Construction 2.6.4. Let $\mathcal{X}$, $\mathcal{Y}$ be two spectra with a pairing

$$\mu : \mathcal{X} \wedge \mathcal{Y} \to I\mathbb{C}^{\times}. \tag{2.129}$$

Let $\mathcal{Z}$ be another spectra. Inspired by spectra as homology and cohomology theories 2.2, we denote

$$\mathcal{X}^*(\mathcal{Z}) := \pi_{-*}Maps(\mathcal{Z}, \mathcal{X}). \tag{2.130}$$

Similarly, we denote

$$\mathcal{X}_*(\mathcal{Z}) := \pi_*(\mathcal{Z} \wedge \mathcal{X}). \tag{2.131}$$

Now consider the composition:

$$\begin{aligned} \mathcal{Z} \wedge \mathcal{X} \wedge Maps(\mathcal{Z}, \mathcal{Y}) &\xrightarrow{\sim} \mathcal{X} \wedge (\mathcal{Z} \wedge Maps(\mathcal{Z}, \mathcal{Y})) && (2.132)\\ &\xrightarrow{ev_{\mathcal{Z}}} \mathcal{X} \wedge \mathcal{Y} && (2.133)\\ &\xrightarrow{\mu} I\mathbb{C}^{\times} && (2.134) \end{aligned}$$

In general, there is a map

$$\pi_*(\mathcal{Z}_1) \otimes \pi_{-*}(\mathcal{Z}_2) \to \pi_0(\mathcal{Z}_1 \wedge \mathcal{Z}_2). \tag{2.135}$$

Apply to our case, we get

$$\begin{aligned} \mathcal{X}_*(\mathcal{Z}) \otimes \mathcal{Y}^*(\mathcal{Z}) &= \pi_*(\mathcal{Z} \wedge \mathcal{X}) \otimes \pi_{-*}(Maps(\mathcal{Z}, \mathcal{Y})) && (2.136)\\ &\to \pi_0(\mathcal{Z} \wedge \mathcal{X} \wedge Maps(\mathcal{Z}, \mathcal{Y})) && (2.137)\\ &\to \pi_0(I\mathbb{C}^{\times}) && (2.138)\\ &= \mathbb{C}^{\times}. && (2.139) \end{aligned}$$

Thus there is a natural transformation $\phi_\mu(-) : \mathcal{Y}^*(-) \to \widehat{\mathcal{X}_*(-)}$. Where $\mathcal{Y}^*(-)$, $\widehat{\mathcal{X}_*(-)}$ are viewed as functors $Sp \to Ab^{op}$. Note that $\widehat{\mathcal{X}_*(-)}$ is the composition $\mathcal{X}_*(-) : Sp \to Ab$, and $D : Ab \to Ab^{op}$ the Pontryagin dual group functor.

Definition 2.6.5. The pairing $\mu : \mathcal{X} \wedge \mathcal{Y} \to I\mathbb{C}^{\times}$ exhibits $\mathcal{Y}$ as the Pontryagin dual of $\mathcal{X}$ if the natural pairing $\mathcal{X}_*(\mathcal{Z}) \otimes \mathcal{Y}^*(\mathcal{Z}) \to \mathbb{C}^{\times}$ exhibits $\mathcal{Y}^*(\mathcal{Z})$ as the Pontryagin dual group of $\mathcal{X}_*(\mathcal{Z})$ for every $\mathcal{Z}$. Alternatively, the map $\phi_\mu(-) : \mathcal{Y}^*(-) \to \widehat{\mathcal{X}_*(-)}$ is an isomorphism for every $\mathcal{Z} \in Sp$.

This gives an universal characterization of the Pontryagin dual spectrum:

Proposition 2.6.6. *Let $\mathcal{Y} = \hat{\mathcal{X}}$, and $\mu = ev_{\mathcal{X}} : \mathcal{X} \wedge \hat{\mathcal{X}} \to I\mathbb{C}^{\times}$. Then μ exhibits $\hat{\mathcal{X}}$ as the Pontryagin dual of $\mathcal{X}$.*

Proof. Let $\mathcal{Z} \in Sp$ be a spectra. We have

$$Maps(\mathcal{Z}, \hat{\mathcal{X}}) = Maps(\mathcal{Z}, Maps(\mathcal{X}, I\mathbb{C}^\times)) \tag{2.140}$$
$$\xrightarrow{\sim} Maps(\mathcal{Z} \wedge \mathcal{X}, I\mathbb{C}^\times). \tag{2.141}$$

Thus we have

$$\hat{\mathcal{X}}^*(\mathcal{Z}) \xrightarrow{\sim} \pi_{-*}Maps(\mathcal{Z}, \hat{\mathcal{X}}) \tag{2.142}$$
$$\xrightarrow{\sim} \pi_{-*}Maps(\mathcal{Z} \wedge \mathcal{X}, I\mathbb{C}^\times) \tag{2.143}$$
$$\xrightarrow{\sim} \pi_*\widehat{\mathcal{Z} \wedge \mathcal{X}} \tag{2.144}$$
$$= \widehat{\mathcal{X}_*(\mathcal{Z})} \tag{2.145}$$

Where we used the proposition 2.6.1 in the last arrow. □

Remark 2.6.7. Note that this proof essentially only uses the corresponding property of $I\mathbb{C}^\times$. This is a theme in duality theorems.

Corollary 2.6.8. $\pi_i(\hat{\mathcal{X}}) \simeq \widehat{\pi_{-i}\mathcal{X}}$.

Proof. Let $\mathcal{Z}$ be the sphere spectrum $\mathcal{S}$, then the proposition 2.145 above gives the equivalence. □

Let N be a CW complex and $\mathcal{X}$ a spectrum. Recall that we have the nonreduced homology $\mathcal{X}_*$ and cohomology $\mathcal{X}^*$ associated to X (2.2). Note that

$$\mathcal{X}_*(N) \coloneqq \mathcal{X}_*(\Sigma^\infty_+ N), \quad \mathcal{X}^*(N) \coloneqq \mathcal{X}^*(\Sigma^\infty_+ N), \tag{2.146}$$

where $\Sigma^\infty_+ N$ is the suspension spectrum associated to N.

We have the following corollary:

Corollary 2.6.9. *When a pairing $\mu : \mathcal{X} \wedge \mathcal{Y} \to I\mathbb{C}^\times$ exhibits $\mathcal{Y}$ as the Pontryagin dual of $\mathcal{X}$, there is $\hat{\mathcal{Y}}^*(-) \to \widehat{\mathcal{X}_*(-)}$ gives an equivalence of cohomology theory. Thus given a cofiber sequence $N \to M \to (M, N)$, we have the long exact sequence of homology groups:*

$$\cdots \to \mathcal{X}_*(N) \to \mathcal{X}_*(M) \to \mathcal{X}_*(M, N) \to \cdots, \tag{2.147}$$

then its Pontryagin dual long exact sequence (apply $\hat{D}$ termwise) give the long exact sequence of cohomology groups:

$$\cdots \leftarrow \hat{\mathcal{X}}^*(N) \leftarrow \hat{\mathcal{X}}^*(M) \leftarrow \hat{\mathcal{X}}^*(M, N) \leftarrow \cdots \tag{2.148}$$

Example 2.6.10. Let A be an abelian group, HA the Eilenberg MacLane spectrum. Then the Pontryagin dual spectrum $\widehat{HA}$ has homotopy groups concentrated in degree 0, and $\pi_0(\widehat{HA}) \simeq \hat{A}$. Thus we see that $\widehat{HA} \simeq H\hat{A}$. For generally, the Pontryagin dual of $\Sigma^n HA$ is $\Sigma^{-n} H\hat{A}$.

Note that the Pontryagin dual operation $\mathbb{D} := Maps(-, I\mathbb{C}^{\times})$ is functorial, it defines an exact contravariant functor :$Sp \to Sp^{op}$. Recall that we have the Pontryagin dual map $\hat{D} : Ab \to Ab$, and the embedding of Eilenberg-MacLane spectrums $H : Ab \to Sp$. By the example 2.6.10 above, we see that we have a commutative diagram of functors:

$$\begin{tikzcd} Ab \arrow[r, "D"] \arrow[d, "H"] & Ab^{op} \arrow[d, "H^{op}"] \\ Sp \arrow[r, "\mathbb{D}"] & Sp^{op} \end{tikzcd} \tag{2.149}$$

Now we turn to π-finite spectra. Let $Sp^{fin} \subset Sp$ be the full subcategory of π-finite spectra and $Ab^{fin} \subset Ab$ the full subcategory of finite abelian groups. Notice that Eilenberg-MacLane functor restricts to a functor $H : Ab^{fin} \to Sp^{fin}$. Note by corollary 2.6.8, Pontryagin duality functor restricts to a functor $\mathbb{D} : Sp^{fin} \to (Sp^{fin})^{op}$. Thus we have the following commutative diagram:

$$\begin{tikzcd} Ab^{fin} \arrow[r, "D"] \arrow[d, "H"] & (Ab^{fin})^{op} \arrow[d, "H^{op}"] \\ Sp^{fin} \arrow[r, "\mathbb{D}"] & (Sp^{fin})^{op} \end{tikzcd} \tag{2.150}$$

Recall that D is a duality for finite abelian groups: $D^2 \simeq id$. We will show the same for π-finite spectra. Note that there is a natural transformation $id \to \mathbb{D}^2$ between the identity functor and the double dual functor on Sp, given by

$$\begin{aligned} \mathcal{X} &\to Maps(Maps(\mathcal{X}, I\mathbb{C}^{\times}), I\mathbb{C}^{\times}) \\ x &\mapsto (\alpha \mapsto \alpha(a)). \end{aligned} \tag{2.151}$$

Restricts to π-finite spectra, we have the following:

Theorem 2.6.11. *For π-finite spectrum $\mathcal{X}$, the natural transformation 2.151 is an isomorphism.*

Proof. Recall π-finite spectra is generated finite Eilenberg-MacLane spectra HA under extensions (Proposition 2.3.5). By Theorem 2.5.12, we know that in our theorem is true when $\mathcal{X} = HA$. So it is suffice to show that

1. if $\mathcal{X} \simeq \mathbb{D}^2(\mathcal{X}) = \hat{\hat{\mathcal{X}}}$, then so are $\Sigma^n \mathcal{X}$.

2. if our theorem holds for $\mathcal{X}$ and $\mathcal{X}''$, and we have a fiber sequence $\mathcal{X} \to \mathcal{X}' \to \mathcal{X}''$, then it is also true for $\mathcal{X}'$.

For (1), simply note that $\widehat{\Sigma^n \mathcal{X}} \simeq \Sigma^{-n} \hat{\mathcal{X}}$.

For (2), Note that there is map of long exact sequences of homotopy groups:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \pi_* \mathcal{X}' \arrow[r] \arrow[d, "\alpha_{\mathcal{X}'}"] & \pi_* \mathcal{X} \arrow[r] \arrow[d, "\alpha_{\mathcal{X}}"] & \pi_* \mathcal{X}'' \arrow[r] \arrow[d, "\alpha_{\mathcal{X}''}"] & \cdots \arrow[d] \\
\cdots \arrow[r] & \pi_* \hat{\hat{\mathcal{X}}}' \arrow[r] & \pi_* \hat{\hat{\mathcal{X}}} \arrow[r] & \pi_* \hat{\hat{\mathcal{X}}}'' \arrow[r] & \cdots
\end{tikzcd} \tag{2.152}$$

If $\alpha_{\mathcal{X}'}$ and $\alpha_{\mathcal{X}''}$ are isomorphisms, then the 2-out-of-3 lemma implies that so is $\alpha_{\mathcal{X}}$. Recall a map of spectra is an equivalence if all the induced maps on homotopy groups are isomorphisms. □

Remark 2.6.12. An alternative definition for the Pontryagin dual for a spectra is $Maps(\mathcal{X}, I\mathbb{Q}/\mathbb{Z})$, where $I\mathbb{Q}/\mathbb{Z}$ is the Brown-Comenetz spectra (very similar to our $I\mathbb{C}^\times$). They give the same dual spectrum for π-finite spectrum. This duality is also called Brown-Comenetz duality [4].

Lastly, we get the following corollary, which is what we need in our main theorem:

Corollary 2.6.13. *Let $\mathcal{X}$ be a π-finite spectra, $N \to M \to (M, N)$ a cofiber sequence, then the Pontryagin dual of the long exact sequence of cohomology group with $\mathcal{X}$ coefficients:*

$$\cdots \to \mathcal{X}^*(M, N) \to \mathcal{X}^*(M) \to \mathcal{X}^*(N) \to \cdots \tag{2.153}$$

is canonically isomorphic to the long exact sequence of homology group with $\hat{\mathcal{X}}$ coefficients:

$$\cdots \leftarrow \hat{\mathcal{X}}_*(M, N) \leftarrow \hat{\mathcal{X}}_*(M) \leftarrow \hat{\mathcal{X}}_*(N) \leftarrow \cdots \tag{2.154}$$

Proof. As $\mathcal{X}$ is a π-finite spectra, we have the Pontryagin dual of $\hat{\mathcal{X}}$ is identified with $\mathcal{X}$. Now apply Corollary 2.6.9 above the Pontryagin dual pair $(\hat{\mathcal{X}}, \mathcal{X})$. □

3 Euler characteristic and TQFT

In this section we discuss some properties of Euler characteristics, then define the Euler TQFT, which is an invertible field theory (in any dimension). This is needed to proof abelian duality as an equivalence of field theories.

In 3.1 we define the Euler characteristic of a manifold, and show some basic properties. In 3.2 we introduce the notion of invertible field theory and define the Euler TQFT. Lastly, we show that the Euler TQFT is trivial in odd dimensions.

3.1 Euler Characteristics

In this section, we collect some facts about the Euler characteristic of compact manifolds and size (homotopy cardinality) of π-finite spectra. We first start with Euler characteristic of a finite graded abelian group:

Definition 3.1.1. Let k be a field and $H^\bullet = \bigoplus H^i$ be a $\mathbb{Z}$-graded k-vector space. $H^\bullet$ is called finite if all but finitely many $H^i = 0$ and each H^i is finite dimensional. The Euler character $\chi(H)$ of a finite graded vector space $H^\bullet$ is

$$\chi(H) = \sum_i (-1)^i dim_k H_i. \tag{3.1}$$

Here's a similiar notion for graded finite abelian groups:

Definition 3.1.2. Let $A^\bullet = \bigoplus A^i$ be a $\mathbb{Z}$-graded abelian groups. $A^\bullet$ is called finite if all but finitely many $A^i = 0$ and each A^i is finite. The size of A, is

$$|A| = \prod_i |A^i|^{(-1)^i}, \tag{3.2}$$

where $|A^i|$ is the cardinality of A^i.

In this section, all $H^\bullet$ and $A^\bullet$ will satisfy the finiteness assumption above, and we will implicitly assume this condition throughout the section.

Remark 3.1.3. If $H = \bigoplus H^i$ with each H^i finite dimensional $\mathbb{F}_q$ vector space, where $\mathbb{F}_q$ is the finite field of cardinality q, then

$$|H| = q^{\chi(H)}. \tag{3.3}$$

A large class of example of graded k-vector spaces comes from chain complexes:

Definition 3.1.4. Let

$$C^* : \cdots \to C^i \to C^{i+1} \to \cdots \tag{3.4}$$

be a cochain complex of k-vector spaces. It is called finite if it is finite when considered as a graded vector space. Its Euler characteristic $\chi(C^*)$ is the Euler characteristic defined above for finite graded vector spaces.

Given C^* be a finite cochain complex of k-vector space, then its cohomology H^* is a finite graded vector space, thus we can assign to it Euler character $\chi(H^*)$. The next proposition shows they are the same:

Lemma 3.1.5. *Let C^* be a finite cochain complex of k vector spaces, H^* its cohomology. Then $\chi(C^*) = \chi(H^*)$.*

Proof. Let $d^i : C^i \to C^{i+1}$ denote the i-th differential. We have a (non-canonical) decomposition

$$C^i \simeq im(d^i) \oplus H^i \oplus im(d^{i-1}). \tag{3.5}$$

Thus

$$\begin{aligned}
\chi(C^*) &= \sum_i (-1)^i dim(C^i) && (3.6)\\
&= \sum_i (-1)^i (dim(im(d^i)) + dimH^i + dim(im(d^{i-1}) && (3.7)\\
&= \sum_i (-1)^i (dimH^i) && (3.8)\\
&= \chi(H^*). && (3.9)
\end{aligned}$$

□

Remark 3.1.6. A similar argument works for a finite chain complex of abelian groups. Note that in that case there will not be a splitting like Equation 3.5 but short exact sequences. However, that is suffice.

The notion of Euler characteristic also behave well with long exact sequences:

Lemma 3.1.7. *Given a long exact sequence of k-vector spaces $H_0^\bullet \to H_1^\bullet \to H_2^\bullet$, that is, a long exact sequence*

$$\cdots \to H_0^* \to H_1^* \to H_2^* \to H_0^{*+1} \to \cdots \tag{3.10}$$

Then we have

$$\chi(H) + \chi(H'') = \chi(H). \tag{3.11}$$

A similar result also holds for long exact sequences of finite graded abelian groups and their sizes.

Proof. We will work with finite k-vector spaces. The finite abelian groups proof is the same. Consider the entire long exact sequence as a chain complex K^*. As it is exact, $H^*(K) = 0$. By the lemma above 3.1.5, we see that $\chi(K) = 0$. However,

$$\chi(K) = \chi(H_0) + \chi(H_2) - \chi(H_1). \tag{3.12}$$

Thus

$$\chi(H_0) + \chi(H_2) = \chi(H_1). \tag{3.13}$$

□

Now we can define the Euler characteristic of a compact manifold (possibly with boundaries):

Definition 3.1.8. Let M be a compact manifold (possibly with boundaries) and k a field. Then we have finite graded k vector spaces $H^*(M, k)$. The Euler characteristic $\chi(M)$ is defined to be $\chi(H^*(M, k))$. As

$$H^*(M, k) = Hom_k(H_*(M, k), k), \tag{3.14}$$

we see that $\chi(M)$ is also $\chi(H_*(M, k))$.

Note that there is no mention of k in the notation of $\chi(M)$, this is due to the following lemma:

Lemma 3.1.9. $\chi(H^*(M, k))$ *is independent of* k.

Proof. As M is a compact smooth manifold, there is a fintie CW complex X homotopy equivalent to M (see apendix of [8]). The CW cochain complex $C^*(X, \mathbb{Z})$ is a bounded cochain complex of finite dimensionsal free $\mathbb{Z}$ modules. Its cohomology of $H^*(M, \mathbb{Z})$. Furthermore, $C^*(M, k) := C^*(M, \mathbb{Z}) \otimes k$ is a finite k-cochain complex that computes $H^*(M, k)$. By lemma 3.1.5, we see that

$$\chi(H^*(M, k)) = \chi(C^*(M, k)) \tag{3.15}$$

$$= \sum_i (-1)^i \; dim_k(C^*(M, \mathbb{Z} \otimes k)) \tag{3.16}$$

$$= \sum_i (-1)^i \; rank(C^*(M, \mathbb{Z})). \tag{3.17}$$

Note the last equation is true because each C^i are free $\mathbb{Z}$ modules. *rank* is the usual notion of rank of an finitely generated abelian group. As

$$\sum_i (-1)^i \ rank(C^*(M, \mathbb{Z})) \tag{3.18}$$

is independent of k, so is

$$\chi(H^*(M, k)). \tag{3.19}$$

□

We also need how the Euler characteristic behave with composition of bordisms:

Lemma 3.1.10. *Given $d-1$ dimensional manifolds N, N', N'', and bordisms $M : N \to N'$ and $M' : N' \to N''$, then the composition $M \sqcup_{N'} M'$ has*

$$\chi(M \sqcup_{N'} M') = \chi(M) + \chi(M') - \chi(N'). \tag{3.20}$$

Proof. Thus we have a Mayer Vietoris sequence

$$\cdots \to H^*(M \sqcup_{N'} M') \to H^*(M) \oplus H^*(M') \to H^*(N') \to \cdots \tag{3.21}$$

By 3.1.7, we see that we get that

$$\chi(M \sqcup_{N'} M') + \chi(N') = \chi(M) + \chi(M'). \tag{3.22}$$

□

Similiar to the Euler characteristic of M, we can also previously defined the size (also called homotopy cardinality) of a π-finite spectrum (see 2.3 for definition of π-finite spectrum):

Definition 3.1.11. If $\mathcal{X}$ is a π-finite spectrum, $\pi^\bullet(\mathcal{X}) = \pi_i(\mathcal{X})$ forms a finite graded abelian group. Recall that the size of $\mathcal{X}$, denoted as $|\mathcal{X}|$, is defined to be $|\pi^\bullet(\mathcal{X})|$.

Note that size of π-finite spectrum behave well with fibers (analogues to show Euler characteristic of compact manifold behave well with cofibers):

Proposition 3.1.12. *Given a fiber sequence $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$, we have that $|\mathcal{X}| \ |\mathcal{Z}| = |\mathcal{Y}|$.*

Proof. This is due to proposition 3.1.7 apply to long exact sequence

$$\cdots \to \pi_*(\mathcal{X}) \to \pi_*(\mathcal{Y}) \to \pi_*(\mathcal{Z}) \to \cdots \tag{3.23}$$

□

Example 3.1.13. Let $\mathcal{X}$ be a π-finite spectrum. We have a fiber sequence

$$\tau_{\geq i}\mathcal{X} \to \mathcal{X} \to \tau_{\leq i-1}\mathcal{X} \tag{3.24}$$

of π-finite spaces. Thus by proposition 3.1.12 we have that

$$|\tau_{\geq i}\mathcal{X}|\ |\tau_{\leq i-1}\mathcal{X}| = |\mathcal{X}|. \tag{3.25}$$

Example 3.1.14. Let M be a compact manifold with boundary, and $\mathcal{X}$ a π-finite spectrum. Then the mapping spectrum

$$Maps(\Sigma^\infty_+ M, \mathcal{X}), \tag{3.26}$$

also denoted as

$$\mathcal{X}(M) \tag{3.27}$$

is a π-finite spectrum (2.3.6) with homotopy groups

$$\pi_i(\mathcal{X}(M)) = \mathcal{X}^{-i}(M), \tag{3.28}$$

where $\mathcal{X}^i(M)$ is the i-th generalized cohomology group of M with coefficients $\mathcal{X}$. Thus

$$|\mathcal{X}(M)| = \cdots \frac{|\mathcal{X}^0(M)|}{|\mathcal{X}^{-1}(M)|}\frac{|\mathcal{X}^2(M)|}{|\mathcal{X}^1(M)|}\cdots \tag{3.29}$$

The size of $|\mathcal{X}(M)|$ relates to the size of $|\mathcal{X}|$ and the Euler characteristic of M as follows:

Proposition 3.1.15. $|\mathcal{X}(M)| = |\mathcal{X}|^{\chi(M)}$.

Proof. As every π-finite spectrum is generated by HA by extensions and (de)-suspensions, where $A = \mathbb{Z}/p\mathbb{Z}$, which we viewed as $\mathbb{F}_p$ (use proposition 2.3.5 and the fact that every finite abelian group is extensions of $\mathbb{Z}/p\mathbb{Z}$).

First we look at the case $\mathcal{X} = H\mathbb{F}_p$. As noted in remark 3.1.3,

$$|\mathcal{X}(M)| = \prod_i |H^i(X, \mathbb{F}_p)|^{(-1)^i} \tag{3.30}$$

$$= \prod_i p^{(-1)^i dim H^i(M,\mathbb{F}_p)} \tag{3.31}$$

$$= p^{\chi(M)} \tag{3.32}$$

$$= |\mathcal{X}|^{\chi(M)}. \tag{3.33}$$

Next, note that if the hypothesis holds for $\mathcal{X}$, it also holds for $\Sigma\mathcal{X}$:

$$|\Sigma\mathcal{X}(M)| = |\mathcal{X}(M)|^{-1} \tag{3.34}$$
$$= |\mathcal{X}|^{-\chi(M)} \tag{3.35}$$
$$= (|\mathcal{X}|^{-1})^{\chi(M)} \tag{3.36}$$
$$= |\Sigma\mathcal{X}|^{\chi(M)}. \tag{3.37}$$

Same thing holds for desuspension. Lastly, for extension, if we have a fiber sequence

$$\mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \tag{3.38}$$

and $\mathcal{X}, \mathcal{Z}$ satisfies the hypothesis. Then we also have a fiber sequence of π-finite spectra

$$\mathcal{X}(M) \to \mathcal{Y}(M) \to \mathcal{Z}(M). \tag{3.39}$$

Thus

$$|\mathcal{Y}(M)| = |\mathcal{X}(M)|\ |\mathcal{Z}(M)| \tag{3.40}$$
$$= |\mathcal{X}|^{\chi(M)}\ |\mathcal{Z}|^{\chi(M)} \tag{3.41}$$
$$= (|\mathcal{X}|\ |\mathcal{Z}|)^{\chi(M)} \tag{3.42}$$
$$= |\mathcal{Y}|^{\chi(M)}, \tag{3.43}$$

where the first and last equality is due to 3.1.12. As the hypothesis holds for $K(\mathbb{F}_p, 0)$, and is true under suspensions and extensions, it holds for any π-finite spectrum. □

3.2 Euler TQFT

In this subsection we will define the Euler TQFT, and show that it is trivial in odd dimensions and for framed manifolds.

Definition 3.2.1. Let $\lambda \in \mathbb{C}^\times$ be a nonzero complex number, then we define an d dimensional unoriented TQFT E_λ as follows: for any $d-1$ dimensional manifold N,

$$E_\lambda(N) := \mathbb{C}. \tag{3.44}$$

For a bordism $M : N \to N'$,

$$E_\lambda(M) : \mathbb{C} \to \mathbb{C} \tag{3.45}$$

is given by multiplication by scalar $\lambda^{\chi(M)-\chi(N)}$.

We have to check that the composition behaves, which boils down this following lemma:

Lemma 3.2.2. *Given $d-1$ dimensional manifolds N, N', N'', and bordisms $M : N \to N'$ and $M' : N' \to N''$, then*

$$\chi(M \sqcup_{N'} M') - \chi(N) = \chi(M) - \chi(N) + \chi(M') - \chi(N'). \tag{3.46}$$

Proof. By lemma 3.1.10 we have that

$$\chi(M \sqcup_{N'} M') = \chi(M) + \chi(M') - \chi(N'). \tag{3.47}$$

Thus

$$\chi(M \sqcup_{N'} M') - \chi(N) = \chi(M) + \chi(M') - \chi(N') - \chi(N) \tag{3.48}$$
$$= \chi(M) - \chi(N) + \chi(M') - \chi(N'). \tag{3.49}$$

□

Example 3.2.3. Let $\lambda \neq 1$. In even dimensions $d = 2n$, the Euler TQFT E_λ is nontrivial (not isomorphic to the trivial theory). This can be checked on the partition function (the value of E_λ on closed d-dim manifolds): let $M = S^d$, then $\chi(M) = 2$. Thus we have $E_\lambda(M) \neq 1$ and the theory is not trivial.

However, in odd dimensions, the Euler characteristic of a closed d dimensional manifold is 0, this is due to Poincare duality (with $\mathbb{F}_2$ coefficients, as all manifolds are $\mathbb{F}_2$ oriented). In fact,we have a stronger statement:

Proposition 3.2.4. *For any $\lambda \in \mathbb{C}^\times$, $E_\lambda \cong Z_{triv}$ as unoriented theories.*

Proof. To show that $E_\lambda \cong Z_{triv}$, we have to give an isomorphic natural transformation α between the two functors. First let's see what kind of data is needed for such α and what conditions it needs to satisfy:

For every N a d-1 closed manifold, we have

$$\alpha(N) : Z_{triv}(N) = \mathbb{C} \to \mathbb{C} = E_\lambda(N), \tag{3.50}$$

which sends $1 \in \mathbb{C}$ to a nonzero elements

$$\alpha_N \coloneqq \alpha(N)(1). \tag{3.51}$$

The compatibility condition on α is the following: given a bordism $M : N \to N'$, we need to have a commutative diagram

$$\begin{tikzcd} Z_{triv}(N) \arrow[r, "Z_{triv}(M)"] \arrow[d, "\alpha(N)"] & Z_{triv}(N') \arrow[d, "\alpha(N')"] \\ E_\lambda(N) \arrow[r, "E_\lambda(M)"] & E_\lambda(N'). \end{tikzcd} \tag{3.52}$$

Tracking where does

$$1 \in \mathbb{C} = Z_{triv}(N) \tag{3.53}$$

goes, we see that we need to show that

$$\alpha_{N'} = \lambda^{\chi(M)-\chi(N)} \alpha_N. \tag{3.54}$$

I claim that for

$$\alpha_N = \lambda^{\frac{1}{2}\chi(N')}, \tag{3.55}$$

equation 3.54 is satisfied. Thus we are reduce to show that

$$\chi(M) = \frac{1}{2}(\chi(N) + \chi(N')) = \frac{1}{2}\chi(\partial M). \tag{3.56}$$

We who this via using Poincaré duality:

Let $k = \mathbb{F}_2$, as every manifold is k-oriented, we can use Poincare duality (2.4):

$$H^*(M, k) \simeq H_{d-*}(M, \partial M, k). \tag{3.57}$$

As d is odd, we see that

$$\chi(M) = \chi(H^*(M, k)) \tag{3.58}$$
$$= \chi(H_{d-*}(M, \partial M, k)) \tag{3.59}$$
$$= -\chi(H_*(M, \partial M, k) \tag{3.60}$$
$$= -\chi(M, \partial M), \tag{3.61}$$

as homology groups also computes Euler characteristic. Finally, consider the the long exact sequence associated to the cofiber sequence $\partial M \to M \to (M, \partial M)$:

$$\cdots \to H^*(M, \partial M, k) \to H^*(M) \to H^*(N) \to \cdots .$$

By 3.1.10, we see that

$$\chi(M) = \chi(M, \partial M) + \chi(\partial M) \tag{3.62}$$
$$= -\chi(M) + \chi(\partial M). \tag{3.63}$$

Thus

$$\chi(M) = \frac{1}{2}\chi(\partial M). \tag{3.64}$$

□

The Euler TQFT is also trivial on framed manifolds and bordisms. This is due to the following lemma:

Lemma 3.2.5. *Let M be a framed manifold with boundary, then $\chi(M) = 0$.*

Proof. If M is a manifold without boundary, then a framing gives nonvanishing vector fields, thus by Poincaré -Hopf theorem M has 0 Euler characteristics. If M has boundary, then we can glue M and M along the boundaries to get a closed manifold $M' = M \sqcup_{\partial M} M$. Note that M' is framed, thus $\chi(M') = 0$. Now ∂M is a closed manifold of one dimension lower, thus $\chi(\partial M) = 0$. By lemma 3.1.10 we have

$$\chi(M \sqcup_{\partial M} M) = \chi(M) + \chi(M) - \chi(\partial M). \tag{3.65}$$

Thus

$$\chi(M) = 0. \tag{3.66}$$

□

In fact much less is needed than a framing for $\chi(M) = 0$. Here's a corollary:

Corollary 3.2.6. *E_λ, restricted to framed manifolds, is the trivial TFT.*

4 Abelian duality

Let $d \geq 1$ be the dimensional of our theories. Recall that given any π-finite space X, we can define a d-dimensional unoriented (untwisted) sigma model $Z_X : Bord_d \to Vect_{\mathbb{C}}$. For a π-finite spectrum $\mathcal{X}$, its 0-th space $\Omega^\infty \mathcal{X}$ is a π-finite space. We define the d-dimensional unoriented sigma model associated to $\mathcal{X}$ as

$$Z_{\mathcal{X}} \coloneqq Z_{\Omega^\infty \mathcal{X}} : Bord_d \to Vect_{\mathbb{C}}. \tag{4.1}$$

Recall that we have the Pontryagin dual spectrum $\hat{\mathcal{X}}$. It is also a π-finite spectra. Now assume that we have a ring spectrum $\mathcal{R}$ and $\mathcal{X}$ is a $\mathcal{R}$ module spectrum. In 4.2 we define the $\mathcal{R}$-oriented bordism category $Bord_d^{\mathcal{R}}$. It has a forgetful map

$$Bord_d^{\mathcal{R}} \to Bord_d. \tag{4.2}$$

We can use this map to pullback unoriented theories to $\mathcal{R}$-oriented theories. The theories of interests are $\mathcal{R}$-oriented theories

$$Z_{\mathcal{X}}, Z_{\Sigma^{d-1}\hat{\mathcal{X}}} : Bord_d^{\mathcal{R}} \to Bord_d \to Vect_{\mathbb{C}} \tag{4.3}$$

associated to $\mathcal{X}$ and $\Sigma^{d-1}\hat{\mathcal{X}}$.

Recall that if λ is a nonzero complex number, then E_λ is the d-dim Euler TQFT defined in 3.2, and we view it as a $\mathcal{R}$-oriented theory. Here's the main theorem of the thesis:

Theorem 4.0.1 (Abelian duality)**.** *There is an equivalence of $\mathcal{R}$-oriented TQFTs:*

$$\mathbb{D} : Z_{\mathcal{X}} \cong Z_{\Sigma^{d-1}\hat{\mathcal{X}}} \otimes E_{|\mathcal{X}|}. \tag{4.4}$$

This section is devoted to stating and proving main theorem 4.0.1. In 4.1 we define the finite sigma model $Z_{\mathcal{X}}$ for any π-finite spectrum $\mathcal{X}$ and do some basic computations. In 4.2 we define the $\mathcal{R}$-oriented bordism category $Bord_d^{\mathcal{R}}$ for any ring spectrum $\mathcal{R}$. In 4.3, we proof the main theorem, borrowing two lemmas 4.3.9 and 4.3.10. In 4.4 we proof lemma 4.3.9. In 4.5, we proof lemma 4.3.10.

4.1 Finite Sigma model for π-finite spectra

In this subsection we define a d-dimensional unoriented TFT $Z_{\mathcal{X}}$, the finite sigma model to $\mathcal{X}$, for a π-finite spectrum $\mathcal{X}$ and do some basic calculations.

Recall that we have the 0-th mapping space functor $\Omega^\infty : Sp \to S_*$. If $\mathcal{X}$ is a π-finite spectrum, then $\Omega^\infty\mathcal{X}$ is a π-finite space. Thus we have a d-dimensional unoriented TFT associated to $Z_{\Omega^\infty\mathcal{X}}$. We define

$$Z_{\mathcal{X}} \coloneqq Z_{\Omega^\infty\mathcal{X}} : Bord_d \to Vect_{\mathbb{C}}. \tag{4.5}$$

Note that $Z_{\mathcal{X}}$ doesn't not see the nonconnective part of $\mathcal{X}$, as

$$\Omega^\infty\mathcal{X} \simeq \Omega^\infty(\tau_{\geq 0}\mathcal{X}). \tag{4.6}$$

Recall that we can think about $\mathcal{X}$ as an (extraordinary) cohomology theory, with

$$\mathcal{X}^n(N) \coloneqq \pi_{-n}(Maps(\Sigma^\infty_+ N, \mathcal{X})). \tag{4.7}$$

Note that $\mathcal{X}^n(N)$ denotes the nonreduced cohomology. We also use $\mathcal{X}(N)$ to denote the mapping spectrum $Maps(\Sigma^\infty_+ N, \mathcal{X})$.

Let N be a $d-1$ dimensional closed manifold, then

$$\begin{aligned} Z_{\mathcal{X}}(N) &= Z_{\Omega^\infty\mathcal{X}}(N) && (4.8)\\ &= \mathbb{C}[\pi_0(Maps(N, \Omega^\infty\mathcal{X}))] && (4.9)\\ &= \mathbb{C}[\pi_0(Maps(\Sigma^\infty_+ N, \mathcal{X}))] && (4.10)\\ &= \mathbb{C}[\mathcal{X}^0(N)] && (4.11)\end{aligned}$$

Thus we see that the states of this sigma model relates to cohomology groups.

Given a bordism $M : N \to N'$. We have

$$Z_{\mathcal{X}}(M) : Z_{\mathcal{X}}(N) \to Z_{\mathcal{X}}(N'). \tag{4.12}$$

In our basis, we have

$$Z_{\mathcal{X}}(M) : \mathbb{C}[\mathcal{X}^0(N)] \to \mathbb{C}[\mathcal{X}^0(N')]. \tag{4.13}$$

In fact, there is a simple formula to calculate this maps:

Proposition 4.1.1. *Let a, b, a' denote the elements of $\mathbb{C}[\mathcal{X}^0(N)]$, $\mathbb{C}[\mathcal{X}^0(M)]$, $\mathbb{C}[\mathcal{X}^0(N')]$, we also view them as basis of correspond vector space. Under*

$$Z_{\mathcal{X}}(M) : \mathbb{C}[\mathcal{X}^0(N)] \to \mathbb{C}[\mathcal{X}^0(N')], \tag{4.14}$$

we have

$$\begin{aligned} a &\mapsto \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|}\sum_{b\to a} q^* b && (4.15)\\ &= \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|}\sum_{a'}\sum_{b\to a, b\to a'} a', && (4.16)\end{aligned}$$

where $\sum_a$ means sum over all $a \in \mathbb{C}[\mathcal{X}^0(N)]$, $\sum_{b\to a}$ means sum over all $b \in \mathbb{C}[\mathcal{X}^0(M)]$ such that $p^*(b) = a$.

Proof. Consider the span of π-finite spaces:

$$\begin{tikzcd} & Maps(M,X) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Maps(N,X) & & Maps(N',X). \end{tikzcd} \tag{4.17}$$

Recall that

$$Z_{\mathcal{X}}(M) : \mathbb{C}[\mathcal{X}^0(N)] \to \mathbb{C}[\mathcal{X}^0(N')]. \tag{4.18}$$

is defined to be the composition $q_* \circ p^*$. First we compute p^*:

$$p^* : \mathbb{C}[\mathcal{X}^0(N)] \to \mathbb{C}[\mathcal{X}^0(M)] \tag{4.19}$$

$$a \mapsto \sum_{b \to a} b. \tag{4.20}$$

For q_*, we need to understand the homotopy groups of $Maps(M, \Omega^\infty \mathcal{X})$. Note that

$$\pi_0(Maps(M, \Omega^\infty \mathcal{X})) = \mathcal{X}^0(M), \tag{4.21}$$

for any $a \in \mathcal{X}^0(M)$, we ahve

$$\pi_n(Maps(M, \Omega^\infty \mathcal{X}), a) \simeq \pi_n(Maps(\Sigma_+^\infty M, \mathcal{X})) = \pi_n(\mathcal{X}(M)). \tag{4.22}$$

Recall that $\mathcal{X}(M) = Maps(\Sigma_+^\infty M, \mathcal{X})$ is the mapping spectrum. This is because all connected components of $Maps(M, \Omega^\infty \mathcal{X})$ are isomorphic to each other, as $Maps(M, \Omega^\infty \mathcal{X})$ is an infinite loop space. Same argument is also true for $Maps(N', \Omega^\infty \mathcal{X})$. Thus

$$\begin{aligned} q_* : \mathbb{C}[\mathcal{X}^0(M)] &\to \mathbb{C}[\mathcal{X}^0(N')] \\ b \mapsto \frac{|\pi_1(Maps(N', \Omega^\infty \mathcal{X}))|}{|\pi_1(Maps(M, \Omega^\infty \mathcal{X}))|} & \frac{|\pi_2(Maps(M, \Omega^\infty \mathcal{X}))|}{|\pi_2(Maps(N', \Omega^\infty \mathcal{X})|} \cdots q^* b \\ &= \frac{|\tau_{\leq 1}\mathcal{X}(M)|}{|\tau_{\leq 1}\mathcal{X}(N')|} q^* b. \end{aligned} \tag{4.23}$$

$\tau_{\geq n}$ is the Postnikov truncation defined in 2.1. Compose p^* and q_*, we get that:

$$a \mapsto \sum_{b \to a} b \tag{4.24}$$

$$\mapsto \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{b \to a} q^* b \tag{4.25}$$

$$= \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{a'} \sum_{b \to a, b \to a'} a'. \tag{4.26}$$

under $Z_{\mathcal{X}}(M)$. $\square$

4.2 $\mathcal{R}$-oriented bordism category

Let $\mathcal{R}$ be a ring spectrum (see 2.1 for definitions), in this subsection, we define the d-dimensional $\mathcal{R}$-oriented bordism category $Bord_d^{\mathcal{R}}$ and $\mathcal{R}$-oriented TQFTs. Recall 2.4 the notion of $\mathcal{R}$-orientation for a manifold M with boundary:

Definition 4.2.1. Let M be a d dimensional manifold with boundary.$\mathcal{R}$ a ringed spectrum. Then an $\mathcal{R}$ orientation on M is a homology class $[M] \in \mathcal{R}_d(M, \partial M)$ such that for every $x \in M^o$ a point in the interior, the image of $[M]$ under $\mathcal{R}_d(M, \partial M) \to \mathcal{R}_d(M, M - x) \simeq \pi_0(\mathcal{R})$ is an multiplicative unit in the ring $\pi_*(\mathcal{R})$.

From now on, we will say orientation for $\mathcal{R}$ orientation unless explicitly said otherwise.

Note that if N is a closed $d-1$ dimensional manifold, then an orientation $[N]$ lives in $\mathcal{R}_{d-1}(N)$. Note that if $[N] \in \mathcal{R}_{d-1}(N)$ is an orientation, then so is $-[N] \in \mathcal{R}_{d-1}(N)$.

Let N and N' be two closed $d-1$ dimensional manifold, and $M : N \to N'$ is a bordism. Given an $\mathcal{R}$ orientation $[M]$ on M, by proposition 2.4.11 this gives an orientation on $\partial M \simeq N \sqcup N'$ via the boundary map

$$\mathcal{R}_d(M, \partial M) \to \mathcal{R}_{d-1}(\partial M). \tag{4.27}$$

Thus an orientation $[M]$ on M gives an orientation on N and N'. With this, we can define oriented bordisms:

Definition 4.2.2. Let N and N' be closed $d-1$ dimensional oriented manifolds with orientation $[N]$ and $[N']$. Then an oriented bordism is a bordism $M : N \to N'$ with an orientation $[M]$ that restricts to $[N]$ on N and $-[N']$ on N'. An isormophism of oriented bordism is an isomorphism of the underlying unoriented bordism such that the orientation class agree.

The reason why there is a minus sign on $-[N']$ is so that oriented bordisms can compose:

Proposition 4.2.3. *Given two oriented bordisms $M : N \to N'$ and $M' : N' \to N''$, then the composition $M \sqcup_{N'} M'$ has a canonical orientation and is an oriented bordism from N to N''.*

Proof. We just have to show that we can glue the two orientation class $[M]$ and $[M']$ to an orientation class $[M \sqcup_{N'} M']$ on $M \sqcup_{N'} M'$. As M and M' are glued on at N', we just have to do that locally. Locally, note that M

and M' looks like $N' \times I$, where I is the interval. Note that $N' \times I$ has a canonical orientation that restricts to $-[N]$ on $N \times 0$ and $[N]$ on $N \times 1$. Note that two of these cylinders $N' \times I$ can compose if the orientation are reversed on the boundary they glue on. This is exactly our situation. $\square$

As oriented bordisms compose, we can define the oriented bordism category:

Definition 4.2.4. The d dimensional $\mathcal{R}$-oriented cateogry $Bord_d^{\mathcal{R}}$ is the category with objects $d-1$ dimensional closed $\mathcal{R}$-oriented manifolds, and bordisms are isomorphism classes of oriented bordisms. It is symmetric monoidal under disjoint union.

Example 4.2.5. When $\mathcal{R} = H\mathbb{Z}$, then $H\mathbb{Z}$-orientation is the same as the usual notion of orientation for manifolds. Thus we recovered the oriented bordism category $Bord_d^{or}$.

Example 4.2.6. When $\mathcal{R} = H\mathbb{Z}/2\mathbb{Z}$, then every manifold is $H\mathbb{Z}/2\mathbb{Z}$-oriented. Thus we recovered the unoriented bordism category $Bord_d$.

Example 4.2.7. When $\mathcal{R} = \mathcal{S}$ is the Sphere spectrum, then $\mathcal{S}$-orientation is a trivialization of the thom spectra of the (stable) normal bundle. Since $\mathcal{S}$ is the intial ring spectrum, a $\mathcal{S}$ orientation implies $\mathcal{R}$ orientation for any ring spectrum $\mathcal{R}$.

Remark 4.2.8. Framed manifolds are $\mathcal{S}$ oriented. Thus they are $\mathcal{R}$ oriented for any ring spectrum $\mathcal{R}$.

Lastly, we can define $\mathcal{R}$-oriented TFTs:

Definition 4.2.9. A $\mathcal{R}$-oriented topological field theory Z is a symmetric monoidal functor

$$Z : Bord_d^{\mathcal{R}} \to Vect_{\mathbb{C}} \tag{4.28}$$

Remark 4.2.10. Note that there is a symmetric monoidal map $Bord_d^{\mathcal{R}} \to Bord_d$ by forgetting the orientation structure. Thus any unoriented TFT gives a $\mathcal{R}$-oriented TFT.

4.3 Main theorem

Let $d \geq 1$ be the dimension of our theory. Let $\mathcal{R}$ be a ring spectrum and $\mathcal{X}$ a π-finite (left) module spectrum. We have the Pontryagin dual spectrum $\hat{\mathcal{X}}$. Note that $\hat{\mathcal{X}}$ is naturally a right $\mathcal{R}$ module. By section 4.2, we have the bordism category $Bord_d^{\mathcal{R}}$ of $\mathcal{R}$-oriented manifolds and bordisms. We have unoriented TFTs $Z_{\mathcal{X}}$ and $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}$ associated to $\mathcal{X}$ and $\Sigma^{d-1}\hat{\mathcal{X}}$ (see 4.1). In addition, if λ is a nonzero complex number, we have E_λ is the d-dim Euler TFT defined in 3.2. Now we view $Z_{\mathcal{X}}, Z_{\Sigma^{d-1}\hat{\mathcal{X}}}, E_{|\mathcal{X}|}$ as $\mathcal{R}$-oriented theories. Then there is an equivalence of theories:

Theorem 4.3.1 (Abelian duality)**.** *There is an equivalence of $\mathcal{R}$-oriented TFTs*

$$\mathbb{D} : Z_{\mathcal{X}} \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}} \otimes E_{|\mathcal{X}|}. \tag{4.29}$$

Remark 4.3.2. Note that in general, they are not equivalent as unoriented theories, despite both sides can be extended to unoriented theories. This is because we need to use Poincaré duality in an essential way. For example, they give different partition functions for the $d = 2$ theories on the Klein bottle.

Here's some consequences of the theorem:

Corollary 4.3.3. *When d is odd, we have eqiovalence of $\mathcal{R}$-oriented TFTs*

$$Z_{\mathcal{X}} \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}}. \tag{4.30}$$

Proof. By proposition 3.2.4, when d is odd, E_λ is isomorphic to the trivial theory, thus $Z_{\mathcal{X}} \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}}$. □

Example 4.3.4. As any manifold is $H\mathbb{Z}/2\mathbb{Z}$ oriented. If $\mathcal{X}$ is a π-finite $H\mathbb{Z}/2\mathbb{Z}$ module, then the main theorem 4.0.1 gives an equivalence of unoriented theories.

Example 4.3.5. $H\mathbb{Z}$ orientation is the same as the classicial notion of orientation on manifolds. Thus if $\mathcal{X}$ is a π-finite $H\mathbb{Z}$ module (e.g. $K(A, n)$), then we have an equivalence of oriented theories. Note that these are theories of (finite abelian) gauge fields and gerbes.

For a general π-finite spectrum $\mathcal{X}$, framed manifolds are always $\mathcal{X}$-orientable. Note that Euler TQFT restricted framed manifolds is trivial (3.2.6), as the Euler chacateristic of a framed manifold is 0. Thus we have the following corollary:

Corollary 4.3.6. *For any π-finite spectrum $\mathcal{X}$. Then we have an equivalence of framed theories*

$$Z_{\mathcal{X}} \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}}. \tag{4.31}$$

Remark 4.3.7. The main theorems 4.0.1 in the case of $d = 1$ and $d = 2$ are related to discrete Fourier transform and character theory for finite abelian groups respectively.

The rest of the section is devoted to the proof of main theorem 4.0.1:

Proof of Theorem 4.0.1. For the rest of the subsection, all manifolds, bordisms are $\mathcal{X}$ oriented. We will supprressed the $\mathcal{X}$ orientation notations.

To give an equivalence, we will need to define an isomorphism of states

$$Z_{\mathcal{X}}(N) \to Z_{\Sigma^{d-1}\hat{\mathcal{X}}} \otimes E_{|\mathcal{X}|}(N), \tag{4.32}$$

and check that it commutes with bordisms. As $E_{|\mathcal{X}|}(N) = \mathbb{C}$, it is suffice to give maps

$$\mathbb{D}(N) : Z_{\mathcal{X}}(N) \to Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N). \tag{4.33}$$

This is done as follows:

Construction 4.3.8. By Pontryagin duality, there is a nondegenerate pairing

$$ev_N(-,-) : \mathcal{X}^*(N) \times \hat{\mathcal{X}}_*(N) \to \mathbb{C}^\times. \tag{4.34}$$

Note that this exist for any topological space N. Compose this with the Poincaré duality isomorphism 2.4.12(this requires a $\mathcal{X}$-orientation $[N]$ on N):

$$\int_{[N]} : \hat{\mathcal{X}}^{d-1-*}(N) \xrightarrow{\sim} \hat{\mathcal{X}}_*(N), \tag{4.35}$$

we get a pairing

$$\mathcal{X}^*(N) \times \hat{\mathcal{X}}^{d-1-*}(N) \to \mathbb{C}^\times \tag{4.36}$$

$$(a, \alpha) \mapsto ev_N(a, \int_{[N]} \alpha) \tag{4.37}$$

When $* = 0$, we denote this pairing as

$$\langle -,- \rangle_N : \mathcal{X}^0(N) \times \hat{\mathcal{X}}^{d-1}(N) \to \mathbb{C}^\times. \tag{4.38}$$

Note that this denotes on the orientation class of N, reversing the orientation inverts this pairing. Recall that

$$Z_{\mathcal{X}}(N) = \mathbb{C}[\mathcal{X}^0(N)] \tag{4.39}$$

and

$$Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N) = \mathbb{C}[\hat{\mathcal{X}}^{d-1}(N)]. \tag{4.40}$$

We will denote elements of $\mathcal{X}^0(N)$ as a, and $\hat{\mathcal{X}}^{d-1}(N)$ as α, and view them as basis vectors for $Z_{\mathcal{X}}(N)$ and $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N)$ respectively. Now we can define the isomorphism on states:

$$\begin{aligned}\mathbb{D}(N) : \mathbb{C}[\mathcal{X}^0(N)] &\to \mathbb{C}[\hat{\mathcal{X}}^{d-1}(N)] \\ a &\mapsto |\tau_{\geq 1}\mathcal{X}(N)| \sum_{\alpha} \langle a, \alpha \rangle_N \ \alpha.\end{aligned} \tag{4.41}$$

This is an isomorphism of vector spaces as the pairing $\langle -, - \rangle_N$ is nondegenerate 2.5.11.

It remains to show that this intertwines with morphisms. Given $M : N \to N'$ in $Bord_d$, with the inclusion maps $p : N \hookrightarrow M$ and $q : N' \hookrightarrow M$. We have to show that the following diagram commute:

$$\begin{tikzcd} Z_{\mathcal{X}}(N) \arrow[r, "Z_{\mathcal{X}}(M)"] \arrow[d, "\mathbb{D}(N)"] & Z_{\mathcal{X}}(N') \arrow[d, "\mathbb{D}(N')"] \\ Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N) \arrow[r, "Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) * |\mathcal{X}|^{\chi(M)-\chi(N)}"'] & Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N') \end{tikzcd} \tag{4.42}$$

Note as we canonically identified

$$Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N) \otimes E_{|\mathcal{X}|}(N) \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N). \tag{4.43}$$

The factor

$$|\mathcal{X}|^{\chi(M)-\chi(N)} \tag{4.44}$$

in the bottom arrow comes from

$$E_{|\mathcal{X}|}(M) : E_{|\mathcal{X}|}(N) = \mathbb{C} \to \mathbb{C} = E_{|\mathcal{X}|}(N'). \tag{4.45}$$

We will do this in two lemmas:

Lemma 4.3.9. *$\mathbb{D}(N') \circ Z_{\mathcal{X}}(M)$ and $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N)$ differ by a constant $\lambda(M)$.*

Lemma 4.3.10. *$\lambda(M) = |\mathcal{X}|^{\chi(M)-\chi(N)}$.*

The two lemmas are proven in 4.4 and 4.5 respectively. $\square$

4.4 Proof of Lemma 4.3.9

We borrow the notation from last subsection 4.1. This section is devoted in proving lemma 4.3.9:

Lemma 4.4.1. *$\mathbb{D}(N') \circ Z_{\mathcal{X}}(M)$ and $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N)$ differ by a constant $\lambda(M)$.*

Proof. First we have to calculate them. From now on, we will denote elements of

$$\mathcal{X}^0(N),\ \mathcal{X}^0(M),\ \mathcal{X}^0(N') \tag{4.46}$$

as a, b, and a'. Similarly, we will denote elements of

$$\hat{\mathcal{X}}^{d-1}(N),\ \hat{\mathcal{X}}^{d-1}(M),\ \hat{\mathcal{X}}^{d-1}(N') \tag{4.47}$$

as α, β, and α'. We also use the summing convention that $\sum_b$ means summing over all $b \in \mathcal{X}^0(M)$, and $\sum_{b\to a}$ means summing over all $b \in \mathcal{X}^0(M)$ such that $p^*(b) = a$.

First we will calculate $\mathbb{D}(N') \circ Z_{\mathcal{X}}(M)$. By proposition 4.1.1, $Z_{\mathcal{X}}(M)$ sends

$$a \mapsto \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{b\to a} q^* b. \tag{4.48}$$

Recall that $\mathbb{D}(N')$ takes

$$a' \mapsto |\tau_{\geq 1}\mathcal{X}(N')| \sum_{\alpha'} \langle a', \alpha' \rangle_N \ \alpha'. \tag{4.49}$$

Thus the composition $\mathbb{D}(N') \circ Z_{\mathcal{X}}(M)$ sends

$$a \mapsto \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{b\to a} (|\tau_{\geq 1}\mathcal{X}(N')| \sum_{\alpha'} \langle q^* b, \alpha' \rangle_{N'} \ \alpha') \tag{4.50}$$

$$= |\tau_{\geq 1}\mathcal{X}(M)| \sum_{b\to a} \sum_{\alpha'} \langle q^* b, \alpha' \rangle_{N'} \ \alpha'. \tag{4.51}$$

Now for $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N)$. $\mathbb{D}(N)$ sends:

$$a \mapsto |\tau_{\geq 1}\mathcal{X}(N)| \sum_{\alpha} \langle a, \alpha \rangle_N \ \alpha. \tag{4.52}$$

$Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M)$ takes

$$\alpha \mapsto \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|} \sum_{\beta, \beta\to\alpha} \hat{q}^* b. \tag{4.53}$$

Thus the composition $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N)$ sends:

$$a \mapsto |\tau_{\geq 1}\mathcal{X}(N)| \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|} \sum_{\alpha'} \sum_{\beta \to \alpha'} \langle a, p^*\beta \rangle_N \ \alpha'. \tag{4.54}$$

Thus we are reduce to showing the following lemma:

Lemma 4.4.2. *For every a and α', $\sum_{b\to a}\langle q^*b, \alpha'\rangle_{N'}$ and $\sum_{\beta\to\alpha'}\langle a, p^*\beta\rangle_N$ differ a nonzero constant multiplicative C that doesn't depend on a or α'.*

Proof. Note that if a has no preimage $b \mapsto a$. Then

$$\sum_{b\to a}\langle q^*b, \alpha'\rangle_{N'} = 0. \tag{4.55}$$

In this case, lemma 4.4.5 (stated and proven below) precise says that

$$\sum_{\beta\to\alpha'}\langle a, p^*\beta\rangle_N = 0. \tag{4.56}$$

Similarly, if α' has no preimage $\beta \mapsto \alpha'$, then both sides are also zero. Thus we are reduced to the case that a lies in the image of

$$p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N) \tag{4.57}$$

and α' lies in the image of

$$\hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N). \tag{4.58}$$

There are

$$|kp| \coloneqq |ker(p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N))| \tag{4.59}$$

many preimage of a. Similiarly, there are

$$|kq| \coloneqq |ker(\hat{q}^* : \Sigma^{d-1}\hat{\mathcal{X}}^0(M) \to \Sigma^{d-1}\hat{\mathcal{X}}^0(N'))| \tag{4.60}$$

preimages of α'.

On one side, we have

$$\sum_{b\to a}\langle q^*(b), \alpha'\rangle_{N'} \tag{4.61}$$

$$= |kq|^{-1}\sum_{b\to a}\sum_{\beta\to\alpha'}\langle q^*(b), q^*\beta\rangle_{N'} \tag{4.62}$$

$$= |kq|^{-1}\sum_{b\to a}\sum_{\beta\to\alpha'}\langle p^*(b), p^*\beta\rangle_{N}. \tag{4.63}$$

Note that $\langle q^*(b), q^*\beta\rangle_{N'} = \langle p^*(b), p^*\beta\rangle_N$ is due to lemma 4.4.3 (stated and proven below)

On the other side, we have

$$\sum_{\beta\to\alpha'} \langle a, p^*\beta\rangle_N \tag{4.64}$$

$$= |kp|^{-1} \sum_{b\to a} \sum_{\beta\to\alpha'} \langle p^*(b), p^*\beta\rangle_N. \tag{4.65}$$

Thus we see that LHS and RHS differ by a constant $C = |kp|/|kq|$. $\square$

Thus we see that the two sides differ by a constant $\lambda(M)$:

$$\mathbb{D}(N') \circ Z_{\mathcal{X}}(M) = \lambda(M)\ Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N), \tag{4.66}$$

with

$$\lambda(M) = \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N)|} \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|} |kp|/|kq|. \tag{4.67}$$

$\square$

Now we need to proof the two lemmas 4.4.5, 4.4.3 above. We first show lemma 4.4.3. Because it might have independence interest, we recall the notations:

Given $M : N \to N'$ a bordism between N and N', with the inclusion maps $p : N \hookrightarrow M$ and $q : N' \hookrightarrow M$. We have pullback maps

$$p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N), \quad q^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N'). \tag{4.68}$$

Similarly we have

$$\hat{p}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N), \quad \hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N'). \tag{4.69}$$

We will denote elements of $\mathcal{X}^0(M)$ as b and $\hat{\mathcal{X}}^{d-1}(M)$ as β. Given b and β, we have two pairings:

$$\langle p^*b, \hat{p}^*\beta\rangle_N, \quad \langle q^*b, \hat{q}^*\beta\rangle_{N'}. \tag{4.70}$$

Here's the lemma that we need to show:

Lemma 4.4.3. $\langle p^*b, \hat{p}^*\beta\rangle_N = \langle q^*b, \hat{q}^*\beta\rangle_{N'}$.

Proof. We will show this equality by equating both side to something that depends only on M, b, and β. By Poincaré duality (2.4.17), there is an isomorphism of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^{d-1}(M) \arrow[r, "\hat{p}^*"] \arrow[d, "\int_{[M]}"] & \hat{\mathcal{X}}^{d-1}(N) \arrow[r] \arrow[d, "\int_{[N]}"] & \hat{\mathcal{X}}^{d}(M,N) \arrow[r] \arrow[d] & \cdots \arrow[d] \\
\cdots \arrow[r] & \hat{\mathcal{X}}_1(M,\partial M) \arrow[r, "\hat{a}_*"] & \hat{\mathcal{X}}_0(N) \arrow[r] & \hat{\mathcal{X}}_0(M,N') \arrow[r] & \cdots
\end{tikzcd} \tag{4.71}$$

The isomorphism $\int_{[M]} : \hat{\mathcal{X}}^{d-1}(M) \cong \hat{\mathcal{X}}_{-1}(M,\partial M)$ depends only on the orientation class of M.

By definition of $\langle -,-\rangle_N$, we have

$$\langle p^*b, \hat{p}^*\beta\rangle_N = ev_N(p^*b, \int_{[N]} \hat{p}^*\beta) \tag{4.72}$$

$$= ev_N(p^*b, \hat{a}_* \int_{[M]} \beta) \tag{4.73}$$

Now consider the long exact sequence:

$$... \to \mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N) \to \mathcal{X}^1(M,N) \to ... \tag{4.74}$$

By Pontryagin duality (2.6.13), taking Pontryagin dual term-wise gives long exact sequence:

$$... \leftarrow \hat{\mathcal{X}}_0(M) \xleftarrow{\hat{p}_*} \hat{\mathcal{X}}_0(N) \leftarrow \hat{\mathcal{X}}_{-1}(M,N) \leftarrow ... \tag{4.75}$$

The dual long exact sequences are connected by the "projection formula": given $b \in \mathcal{X}^0(M)$ and $\gamma \in \hat{\mathcal{X}}_0(N)$, then

$$ev_N(p^*b, \gamma) = ev_M(b, \hat{p}_*\gamma). \tag{4.76}$$

Put it together with equation 4.73:

$$\langle p^*b, \hat{p}^*\beta\rangle_N = ev_N(p^*b, \hat{a}_* \int_{[M]} \beta) \tag{4.77}$$

$$= ev_M(b, \hat{p}_* \circ \hat{a}_* \int_{[M]} \beta) \tag{4.78}$$

Note that

$$\partial_* \coloneqq \hat{p}_* \circ \hat{a}_* : \hat{\mathcal{X}}_{-1}(M,\partial M) \to \hat{\mathcal{X}}_0(M) \tag{4.79}$$

is the boundary map associated to the triple $\partial M \to M \to (M, \partial M)$, thus it is independent of N. The same argument work for $\langle q^* b, \hat{q}^* \beta \rangle_{N'}$. Thus

$$\langle p^* b, \hat{p}^* \beta \rangle_N = ev_M(b, \partial_* \int_{[M]} \beta) \tag{4.80}$$

$$= \langle q^* b, \hat{q}^* \beta \rangle_{N'}. \tag{4.81}$$

□

Remark 4.4.4. Heuristically, $\langle p^* b, \hat{p}^* \beta \rangle_N = \langle q^* b, \hat{q}^* \beta \rangle_{N'}$ because the orientation class $[M]$ for M is exactly a homotopy from $p_*[N] \in H_{d-1}(M)$ to $q_*[N] \in H_{d-1}(M)$. Thus we have

$$\langle p^* b, \hat{p}^* \beta \rangle_N \approx \langle b, \beta \rangle_{p_*[N]} \approx \langle b, \beta \rangle_{q_*[N']} \approx \langle q^* b, \hat{q}^* \beta \rangle_{N'}. \tag{4.82}$$

The actual proof is just a way to make this heuristic rigorous.

We also used this following lemma:

Lemma 4.4.5. *Let $a \in \mathcal{X}^0(N)$ and $\alpha' \in \hat{\mathcal{X}}^{d-1}(N')$. If a is not in the image of $p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N)$, then*

$$\sum_{\beta \to \alpha'} \langle a, \hat{p}^* \beta \rangle_N = 0, \tag{4.83}$$

where β sums over $\hat{\mathcal{X}}^{d-1}(M)$.

Proof. If α' has no preimage in

$$\hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N'), \tag{4.84}$$

then the sum is trivially 0. If α' has a preimage, say β'_α. Then all other preimages of α are of the form $\beta'_\alpha + \beta_0$, where $\beta_0 \in ker(\hat{p}^*)$. Thus

$$\sum_{\beta \to \alpha'} \langle a, \hat{p}^* \beta \rangle_N = \sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^* (\beta'_\alpha + \beta_0) \rangle_N \tag{4.85}$$

$$= (\langle a, \hat{p}^* \beta'_\alpha \rangle_N) \sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^* \beta_0 \rangle_N. \tag{4.86}$$

Thus it is suffice to show that

$$\sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^* \beta_0 \rangle_N = 0, \tag{4.87}$$

i.e. the case where $\alpha' = 0$.

Poincaré duality 2.4.17 gives an isomorphism of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^{d-1}(M,N') \arrow[r] \arrow[d] & \hat{\mathcal{X}}^{d-1}(M) \arrow[r, "\hat{p}^*"] \arrow[d, "\int_{[M]}"] & \hat{\mathcal{X}}^{d-1}(N') \arrow[r] \arrow[d, "\int_{[N']}"] & \cdots \arrow[d] \\
\cdots \arrow[r] & \hat{\mathcal{X}}_1(M,N) \arrow[r, "\hat{\mu}_*"] & \hat{\mathcal{X}}_1(M,\partial M) \arrow[r, "\hat{\nu}_*"] & \hat{\mathcal{X}}_0(N') \arrow[r] & \cdots
\end{tikzcd} \tag{4.88}$$

Under Poincaré duality, $ker(\hat{p}^*)$ corresponds to $ker(\hat{a}_*) = im(\hat{\mu}_*)$. Given $\beta_0 \in ker(\hat{p}^*)$ with

$$\int_{[M]} \beta = \hat{\mu}_* \gamma, \quad \gamma \in \hat{\mathcal{X}}_1(M,N), \tag{4.89}$$

By definition of $\langle -, - \rangle_N$, we have:

$$\langle a, \hat{p}^* \beta_0 \rangle_N = ev_N(a, \hat{\nu}_* \int_{[M]} \beta) \tag{4.90}$$

$$= ev_N(a, \hat{\nu}_* \circ \hat{\mu}_* \gamma). \tag{4.91}$$

The composition

$$\hat{\nu}_* \circ \hat{\mu}_* : \hat{\mathcal{X}}_1(M,N) \to \hat{\mathcal{X}}_0(N) \tag{4.92}$$

is the Pontryagin dual of the

$$\partial^* : \mathcal{X}^0(N) \to \mathcal{X}^1(M,N). \tag{4.93}$$

Thus by above equation (4.91) we have

$$\langle a, \hat{p}^* \beta_0 \rangle_N = ev_N(a, \hat{\nu}_* \hat{\mu}_* \gamma) \tag{4.94}$$

$$= ev_{(M,N)}(\partial^* a, \gamma). \tag{4.95}$$

Thus

$$|ker(\hat{\mu}_*)| \sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^* \beta_0 \rangle_N = \sum_{\gamma} ev_{(M,N)}(\partial^* a, \gamma), \tag{4.96}$$

where γ sums over $\hat{\mathcal{X}}_1(M,N)$. Now consider the the long exact sequence:

$$... \to \mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N) \xrightarrow{\partial^*} \mathcal{X}^1(M,N) \to ... \tag{4.97}$$

As a is not in the image of $p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N)$, $\partial^* a \in \mathcal{X}^1(M,N)$ is not the identity element. Thus

$$ev_{(M,N)}(\partial^* a, -) : \hat{\mathcal{X}}_1(M,N) \to \mathbb{C}^\times \tag{4.98}$$

is a nontrivial character on $\hat{\mathcal{X}}_1(M,N)$. As the sum over all elements of the group paired with a nontrivial character is 0, we see that

$$\sum_{\gamma} ev_{(M,N)}(\partial^* a, \gamma) = 0. \tag{4.99}$$

By equation (4.96), we see that

$$\sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^* \beta_0 \rangle_N = 0. \tag{4.100}$$

□

4.5 Proof of lemma 4.3.10

Recall from last section we have

$$\mathbb{D}(N') \circ Z_{\mathcal{X}}(M) = \lambda(M)\ Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N) \tag{4.101}$$

with

$$\lambda(M) = \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N)|} \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|} |kp|/|kq|. \tag{4.102}$$

To finish the proof of the main theorem, we need the following lemma (see previous section 4.4 for notations):

Lemma 4.5.1. *Let M be a bordism from N to N'. Then*

$$\lambda(M) = |\mathcal{X}|^{\chi(M) - \chi(M)}. \tag{4.103}$$

Proof. Recall that

$$\lambda(M) = \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N)|} \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|} |kp|/|kq|. \tag{4.104}$$

First we will move everything in $\hat{\mathcal{X}}$ to $\mathcal{X}$ by Poincaré duality 2.4.17.

The first term is

$$|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|. \tag{4.105}$$

Note that Poincaré duality (2.4.12) shows that

$$\hat{\mathcal{X}}^*(N') \cong \hat{\mathcal{X}}_{d-1-*}(N'), \tag{4.106}$$

thus

$$|\hat{\mathcal{X}}^i(N')| = |\hat{\mathcal{X}}_{d-1-i}(N')| = |\mathcal{X}^{d-1-i}(N'). \tag{4.107}$$

Note that the cardinalities of Pontryagin dual groups are equal by proposition 2.5.8. Thus

$$|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')| = \frac{|\hat{\mathcal{X}}^{d-3}(N')|}{|\hat{\mathcal{X}}^{d-2}(N')|}\frac{|\hat{\mathcal{X}}^{d-5}(N')|}{|\hat{\mathcal{X}}^{d-4}(N')|}\cdots \tag{4.108}$$

$$= \frac{|\hat{\mathcal{X}}_2(N')|}{|\hat{\mathcal{X}}_1(N')|}\frac{|\hat{\mathcal{X}}_4(N')|}{|\hat{\mathcal{X}}_3(N')|}\cdots \tag{4.109}$$

$$= \frac{|\mathcal{X}^2(N')|}{|\mathcal{X}^1(N')|}\frac{|\mathcal{X}^4(N')|}{|\mathcal{X}^3(N')|}\cdots \tag{4.110}$$

$$= |\tau_{\leq -1}\mathcal{X}(N')|. \tag{4.111}$$

Next we will work on

$$|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|^{-1}. \tag{4.112}$$

Similar to above, we have

$$|\hat{\mathcal{X}}^i(M)| = |\hat{\mathcal{X}}_{d-i}(M, \partial M)| = |\mathcal{X}^{d-i}(M, \partial M)| \tag{4.113}$$

Thus

$$|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|^{-1} = \frac{|\hat{\mathcal{X}}^{d-2}(M')|}{|\hat{\mathcal{X}}^{d-3}(M')|}\frac{|\hat{\mathcal{X}}^{d-4}(M')|}{|\hat{\mathcal{X}}^{d-5}(M')|}\cdots \tag{4.114}$$

$$= \frac{|\hat{\mathcal{X}}_2(M, \partial M)|}{|\hat{\mathcal{X}}_3(M, \partial M)|}\frac{|\hat{\mathcal{X}}_4(M, \partial M)|}{|\hat{\mathcal{X}}_5(M, \partial M)|}\cdots \tag{4.115}$$

$$= \frac{|\mathcal{X}^2(M, \partial M)|}{|\mathcal{X}^3(M, \partial M)|}\frac{|\mathcal{X}^4(M, \partial M)|}{|\mathcal{X}^5(M, \partial M)|}\cdots \tag{4.116}$$

$$= |\tau_{\leq -2}\mathcal{X}(M, \partial M)| \tag{4.117}$$

Lastly, we have

$$|kq| \coloneqq |ker(\hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N'))|. \tag{4.118}$$

By Poincare duality (2.4.17): we have that an isomorphism of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^*(M, N') \arrow[r] \arrow[d] & \hat{\mathcal{X}}^*(M) \arrow[r, "q^*"] \arrow[d] & \hat{\mathcal{X}}^*(N') \arrow[r] \arrow[d] & \cdots \arrow[d] \\
\cdots \arrow[r] & \hat{\mathcal{X}}_{d-*}(M, N) \arrow[r] & \hat{\mathcal{X}}_{d-*}(M, \partial M) \arrow[r] & \hat{\mathcal{X}}_{d-1-*}(N') \arrow[r] & \cdots
\end{tikzcd} \tag{4.119}$$

Thus

$$|kq| = |ker\ (q' : \hat{\mathcal{X}}_1(M, \partial M) \to \hat{\mathcal{X}}_0(N'))| \tag{4.120}$$
$$= |im(\hat{\mathcal{X}}_1(M, N) \to \hat{\mathcal{X}}_1(M, \partial M)|. \tag{4.121}$$

Now, under Pontraygin duality 2.6.13, we know that the long exact sequence

$$\cdots \to \hat{\mathcal{X}}_{d-*}(M, N) \to \hat{\mathcal{X}}_{d-*}(M, \partial M) \to \hat{\mathcal{X}}_{d-1-*}(N') \to \cdots \tag{4.122}$$

is the Pontryagin dual of

$$\cdots \leftarrow \mathcal{X}^{d-*}(M, N) \leftarrow \mathcal{X}^{d-*}(M, \partial M) \leftarrow \mathcal{X}^{d-1-*}(N') \leftarrow \cdots \tag{4.123}$$

Thus

$$|kq| = |im(\hat{\mathcal{X}}_1(M, N) \to \hat{\mathcal{X}}_1(M, \partial M)| \tag{4.124}$$
$$= |im(\mathcal{X}^1(M, \partial M) \to \mathcal{X}^1(M, N)| \tag{4.125}$$
$$= |ker(\mathcal{X}^1(M, N) \to \mathcal{X}^1(N'))|. \tag{4.126}$$

Now we will factor out $|\mathcal{X}|^{\chi(M) - \chi(M)}$ from $\lambda(M)$. Recall that given any π-finite space $\mathcal{Y}$. We have a fiber sequence

$$\tau_{\geq i}\mathcal{Y} \to \mathcal{Y} \to \tau_{\leq i-1}\mathcal{Y} \tag{4.127}$$

of π-finite spaces. By Example 3.1.13 we have

$$|\tau_{\geq i}\mathcal{Y}|\ |\tau_{\leq i-1}\mathcal{Y}| = |\tau_{\leq i-1}\mathcal{Y}|. \tag{4.128}$$

In our case we have:

$$|\tau_{\geq 1}\mathcal{X}(M)| = \frac{|\mathcal{X}(M)|}{|\tau_{\leq 0}\mathcal{X}(M)|}. \tag{4.129}$$

Similarly,

$$|\tau_{\geq 1}\mathcal{X}(N)|^{-1} = \frac{|\tau_{\leq 0}\mathcal{X}(N)|}{|\mathcal{X}(N)|}. \tag{4.130}$$

Recall that by 3.1.15 we have

$$|\mathcal{X}(M)| = |\mathcal{X}|^{\chi(M)} \tag{4.131}$$

and

$$|\mathcal{X}(N)| = |\mathcal{X}|^{\chi(N)}. \tag{4.132}$$

Putting it all together, we see that

$$\lambda(M) = \lambda'(M)\ |\mathcal{X}|^{\chi(M)-\chi(M)}, \tag{4.133}$$

where

$$\begin{gathered}\lambda'(M) = \frac{|\tau_{\leq 0}\mathcal{X}(N)|}{|\tau_{\leq 0}\mathcal{X}(M)|}|\tau_{\leq -1}\mathcal{X}(N')|\ |\tau_{\leq -2}\mathcal{X}(M,\partial M)| \\ \frac{|ker(\mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|}\end{gathered} \tag{4.134}$$

It remains to show that $\lambda'(M) = 1$.

First, notice that

$$\partial M = N \sqcup N', \tag{4.135}$$

thus we have

$$|\mathcal{X}^*(\partial M)| = |\mathcal{X}^*(N)|\ |\mathcal{X}^*(N')|. \tag{4.136}$$

So

$$|\tau_{\leq 0}\mathcal{X}(N)|\ |\tau_{\leq -1}\mathcal{X}(N')| = |\mathcal{X}^0(N)|\ |\tau_{\leq -1}\mathcal{X}(\partial M)|. \tag{4.137}$$

Now consider the exact sequences

$$0 \to ker\ p^* \to \mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N) \to coker\ p^* \to 0, \tag{4.138}$$

We see that the terms

$$|ker\ p^*|\ |\tau_{\leq 0}\mathcal{X}(M)|^{-1}\ |\tau_{\leq 0}\mathcal{X}(N)| = |coker\ p^*|. \tag{4.139}$$

Lastly, we rewrite

$$|coker\ p^*| = |ker\ \mathcal{X}^1(M,N) \to \mathcal{X}^1(M)|. \tag{4.140}$$

Thus

$$\begin{gathered}\lambda'(M) = \frac{|\tau_{\leq -1}\mathcal{X}(\partial M)|}{|\tau_{\leq -1}\mathcal{X}(M)|}\ |\tau_{\leq -2}\mathcal{X}(M,\partial M)| \\ \frac{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(M))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|}\end{gathered} \tag{4.141}$$

I claim that

$$\frac{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(M))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|} = |ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M))|^{-1}. \tag{4.142}$$

To see this, first notice that the canonical map

$$\mathcal{X}^1(M,N) \to \mathcal{X}^1(N) = 0, \tag{4.143}$$

thus

$$|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))| = |ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(\partial M))|. \tag{4.144}$$

Note

$$\mathcal{X}^1(M,N) \to \mathcal{X}^1(\partial M) \tag{4.145}$$

is the composition of the two terms

$$(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M)) \circ (\mathcal{X}^1(M,N) \to \mathcal{X}^1(M)). \tag{4.146}$$

on the RHS. Now we have the following algebraic fact: given

$$f : A \to B, \;\; g : B \to C \tag{4.147}$$

then

$$|ker(g \circ f)| = |ker f| \; |ker g| \tag{4.148}$$

iff

$$ker(g) \subset im(f). \tag{4.149}$$

In our case, if an element $a \in \mathcal{X}^1(M)$ maps to 0 in $\mathcal{X}^1(\partial M)$, then it maps to 0 in $\mathcal{X}^1(N)$. Since

$$\mathcal{X}^1(M,N) \to \mathcal{X}^1(M) \to \mathcal{X}^1(N)$$

is a part of a long exact sequence, it is exact at $\mathcal{X}^1(M)$. That means that there exists $b \in \mathcal{X}^1(M,N)$ which maps to a. Thus we satisfy the algebraic condition, and we have

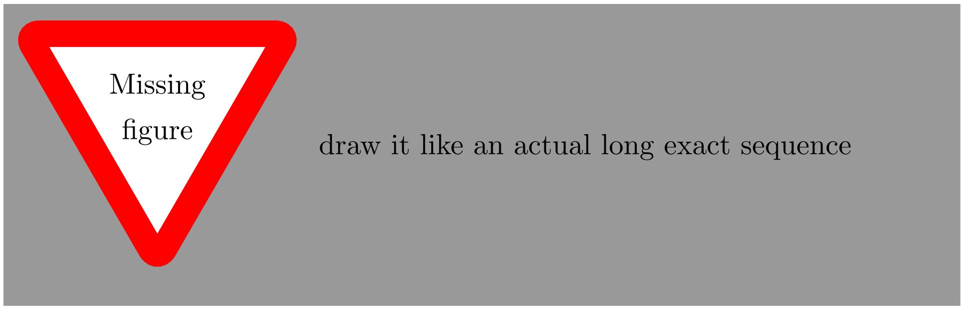
$$\frac{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(M))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|} = |ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M))|^{-1}. \tag{4.150}$$

So we have

$$\lambda'(M) = \frac{|\tau_{\leq -1}\mathcal{X}(\partial M)|}{|\tau_{\leq -1}\mathcal{X}(M)|} \frac{|\tau_{\leq -2}\mathcal{X}(M, \partial M)|}{|ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M))|} \tag{4.151}$$

Finally, consider the following long exact sequence:

$$\begin{aligned} 0 \to ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M)) \to \mathcal{X}^1(M) \to \mathcal{X}^1(\partial M) \\ \to \mathcal{X}^2(M, \partial M) \to \mathcal{X}^2(M) \to \mathcal{X}^2(\partial M) \to \cdots . \end{aligned}$$



Recall that the alternating size of the finite abelian groups in a long exact sequence is 1 (by lemma 3.1.5 and the fact that this sequence is exact). Note the alternating size of the long exact sequence above is precisely $\lambda'(M)$, thus

$$\lambda'(M) = 1. \tag{4.152}$$

$\square$

References

[1] Matthew Ando et al. "An -categorical approach to R -line bundles, R -module Thom spectra, and twisted R -homology". In: *Journal of Topology* 7.3 (Aug. 2013), pp. 869–893. ISSN: 1753-8416. DOI: 10.1112/jtopol/jtt035. URL: http://dx.doi.org/10.1112/jtopol/jtt035.

[2] Michael F. Atiyah. "Topological quantum field theory". en. In: *Publications Mathématiques de l'IHÉS* 68 (1988), pp. 175–186. URL: http://www.numdam.org/item/PMIHES_1988__68__175_0.

[3] Edgar H. Brown. "Cohomology Theories". In: *Annals of Mathematics* 75.3 (1962), pp. 467–484. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970209.

[4] Edgar H. Brown and Michael Comenetz. "Pontrjagin Duality for Generalized Homology and Cohomology Theories". In: *American Journal of Mathematics* 98.1 (1976), pp. 1–27. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2373610.

[5] Robbert Dijkgraaf and Edward Witten. "Topological gauge theories and group cohomology". In: *Comm. Math. Phys.* 129.2 (1990), pp. 393–429. URL: https://projecteuclid.org:443/euclid.cmp/1104180750.

[6] Daniel S. Freed and Michael J. Hopkins. *Reflection positivity and invertible topological phases*. 2019. arXiv: 1604.06527 [hep-th].

[7] Rok Gregoric. "Spectra are your friends". In: (). URL: https://web.ma.utexas.edu/users/gregoric/Spectra%20Are%20Your%20Friends.pdf.

[8] A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: https://books.google.com/books?id=BjKs86kosqgC.

[9] Jacob Lurie. *Higher Algebra*. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[10] Jacob Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press, 2009. ISBN: 9780691140490. URL: http://www.jstor.org/stable/j.ctt7s47v.

[11] Jacob Lurie. "On the classification of topological field theories". In: Current Developments in Mathematics Volume 2008 (Oct. 2009), pp. 129–280. URL: https://projecteuclid.org/euclid.cdm/1254748657.

[12] J.P. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999. ISBN: 9780226511832. URL: https://books.google.com/books?id=g8SG03R1bpgC.

[13] Jacob Lurie Michael Hopkin. "Ambidexterity in $K(n)$-Local Stable Homotopy Theory". In: (). URL: http://people.math.harvard.edu/~lurie/papers/Ambidexterity.pdf.

[14] Y.B. Rudyak. *On Thom Spectra, Orientability, and Cobordism*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2007. ISBN: 9783540777519. URL: https://books.google.com/books?id=M1IOBwAAQBAJ.