Abelian Duality

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Σ a closed Riemann Surface with metric g. We have fields valued in S^1, $\phi : \Sigma \to S^1$. The Lagrangian $\mathcal{L}(\phi) := R^2/(4\pi) \int_\Sigma d\phi \wedge d * \phi$. Where R is the radius of the circle S^1, which I think about as a connection.

Plan of todays talk: First we will take out the gauge invariance, then we introduce new fields in a bigger theory, integrate out the new field to recover the old theory, then integrate out the old theory to get a new theory, which is the dual.

1 setup

Let ϕ be our field, we are going to couple it to electromagnetism: $A \in mathcal A \cong \Omega^1_\Sigma(\mathbb{R})$, then we have the covariant derivative $D_A : \phi \mapsto d\phi + A$.

I introduce another field $\sigma : \Sigma \mapsto S^1$. So we have introduce $U(1)$ connection together with another field. Notes that $d\sigma$ is well defined globally.

The new Langrangian is :

$$\mathcal{L}(A, \phi, \sigma) = R^2/(4\phi) \int_\Sigma D_A\phi \wedge *D_A\phi - i/(2\pi) \int_{Sigma} \sigma \wedge F_A$$

, the last part is equal to $i/(2\pi) \int_\Sigma d\sigma \wedge A$.

The Partition function is

$$Z = 1/vol(\mathcal{G}) \int D\phi DAD\sigma \exp \mathcal{L}(A, \phi, \sigma)$$

We are going to show that this is equivalent to the old theory. Notice that σ looks like a Lagrange multiplier term, introduce to be integrate out to give a delta function of the connection at the trivial connection.

Choose $\sigma_n : \Sigma \to S^1$ s.t. $d\sigma_n \in [d\sigma] \subset H^1(\sigma;\mathbb{R})$, and $\sigma_n(P) = 0 \ mod \ 2\pi\mathbb{Z}$. We see that $\sigma = \sigma_n + \sigma_{\mathbb{R}}$, where $\sigma_{\mathbb{R}} \in \Omega^0_\Sigma(\mathbb{R})$. Note that $1/(2\pi)\int_C d\sigma = 1/(2\pi)\int_C d\sigma_n \in \mathbb{Z}$.

Choose (γ_j) basis for $H^1(\Sigma;\mathbb{Z})$, then we have that $1/(2\pi)d\sigma_n = \sum_j m_j\gamma_j$, then if we integrate out σ, we get

$$\int D\sigma \exp -i/2\pi \int (\sigma wedge F_A) = \int D\sigma \exp -i/2\pi \int (\sigma_n + \sigma_{\mathbb{R}} wedge F_A) = \sum_{[d\sigma_k]\in H^1(\Sigma;2\pi\mathbb{Z}} \exp -i/2\pi$$

$\int D\sigma_{\mathbb{R}} \exp -i/2\pi \int \sigma_{\mathbb{R}} \wedge F_A(1)$Note that the first piece become

$$\prod_j \sum_{m_j\in\mathbb{Z}} \exp^{j\int \gamma_j \wedge A}$$

Where $\int \gamma_j \wedge A$ is the holonomy of A around C_j, where C_j is the corresponding curve to the harmonic 1-form. Notes that $\prod_j \sum_{m_j} \exp -im_jh_j = \prod_j \delta(hol_{C_j}(A) = 0$

The second part, we are just going to integrate $\sigma_{\mathbb{R}}$ out and we get $\delta F_A = 0$, aka the curvature is zero.

So we see that the connection can have no curvature, nor holonomy. Thus it is a trivial (flat) connection.

Gauge fix and integrate out the gauge symmetry we see that $A = 0$, and we recover the old theory.

2 Integrate out ϕ and A

Gauge fix: we can gauge fix such that $\phi = 0$. And the $1\{vol\mathcal{G}$ goes away.

So we are left with

$$Z = \int DAD\sigma \exp -R^2/4\pi \int_\Sigma (A \wedge Awedge * A - i/2\pi \int_\Sigma A \wedge d\sigma$$

Now we want to complete the square:

$$A' = A = i/R^2{*}d\sigma = \int DA' D\sigma \exp(-R^2/2\pi \int |A|^2) \exp(-1/4\pi R^2 \int |d\sigma|^2))$$

We see that the out theory went inverted the radius. This is the duality that Leon discussed last time.

Another question we can ask is where did the operators go?

$d\phi(x) \mapsto D_A\phi(x) \mapsto ?$, where did it go? We need to integrate out ϕ and A.

The vacuum expectation value of $D_A\phi(x)$ is

$$\int DAD\phi D\sigma D_A\phi(X) exp(-S)$$

, when we integrate out $D\phi$ and $D\sigma$, we see that $D_A\phi(x)$ becomes $d\phi(x)$. Now let's see where $D_A\phi(x)$ lands on the other side?

It is not very hard to show that it becomes $-i/R^2 * d\sigma$. Basically set $\phi = 0$ and $A' = A + i/R^2 * d\sigma = 0$ when we integrate it out.

Another operator $exp(i\phi)(p)$