Abstract

We investigate abelian duality in topological field theories. The theories are finite homotopy TFTs, generalizations of finite gauge theories to π-finite spaces and spectra. In d dimension, We proof that the theories associated to a $K(A,n)$ and $K(\hat{A}, d-1-n)$ are equivalent up to an invertible field theory, where A is a finite abelian group and $\hat{A}$ its Pontryagin dual group. This is a version of abelian duality for p-form gauge theories, where the gauge groups are finite abelian. In low dimensions, the duality recovers discrete Fourier transform and character theory for finite abelian groups. In addition, using Brown-Comenetz duality, we extend our results to π-finite spectra.

Abelian duality is a major theme in both mathematics and physics. In mathematics, examples of abelian duality includes Pontryagin duality for topological abelian groups and Cartier duality in algebraic geometry. In Physics, abelian duality appears as electromagnetic duality in 4 dimension and T-duality in 2 dimension.

In this thesis, we study a finite, topological version of abelian duality. It is a discrete analogue of the abelian duality for p-form gauge theories [7]. As our gauge groups are finite, the path integral, which normally sums over an infinite dimensional space, is finite and mathematically well-defined. In addition, because our groups are discrete, the theories are topological, i.e., they don't depend on the geometry of the spacetime, only its topology. As a result, our theories can be mathematically constructed under the formalism of topological field theories (TFTs). The duality is stated and proven as equivalence of TFTs.

Let $d \geq 1$ be the dimension of our theory. A (unextended) d-dimensional topological field theory (TFT) Z is a symmetric monoidal functor

$$Z : Bord_d \to Vect_{\mathbb{C}}. \tag{0.0.1}$$

where $Bord_d$ is the category whose objects are closed $(d-1)$-dimensional manifolds and morphisms are d-dimensional bordisms. Physically, a TFT is a field theory that doesn't depend of the metrics, thus the geometry, of the spacetime manifolds. It assigns to a closed d-dimensional manifold M a complex number $Z(M) \in \mathbb{C}$, which is the partition function Z evaluated at M. To a closed $(d-1)$-dimensional manifold N, it assigns the vector space $Z(N) \in Vect_{\mathbb{C}}$ of states associated to the space-slice N.

An important class of topological field theories are finite gauge theories [6, 11]. Let G be a finite group, viewed as our gauge group. for every topological space N, we have $Bun_G(N)$ the groupoid of principal G-bundles on

N. Recall that a groupoid is a category whose morphisms are all invertible. Objects of $Bun_G(N)$ are principal G-bundles, and morphisms are equivalences of principal G-bundles. When N is a compact manifold, $Bun_G(N)$ is a finite groupoid, that is, it has finitely many isomorphism classes of objects, and each object has finite automorphisms.

For every finite group G, the finite gauge theory with gauge group G is a d-dimensional TFT Z_{BG} (see §2.5). For a closed $(d-1)$-dimensional manifold N, the d-dimensional theory Z_{BG} assigns to N the vector space

$$Z_{BG}(N) := \mathbb{C}[Bun_G(N)] \tag{0.0.2}$$

of locally constant complex-valued functions on $Bun_G(N)$. For a closed d-dimensional manifold M, Z_{BG} assigns to M the number

$$Z_{BG}(M) := \sum_{[x]} \frac{1}{|Aut_{Bun_G(M)}([x])|}, \tag{0.0.3}$$

where $[x]$ sums over the isomorphism classes of $Bun_G(M)$. The partition function $Z_{BG}(M)$ counts the number of isomorphism classes of principal G-bundles on M, each one weighted by the size of its automorphism group.

In this paper, we consider a generalization of these finite gauge theories, called finite homotopy TFTs [5, 9, 13, 24]. Let X be a π-finite space, that is, a topological space X with finitely many connected components, and for every point $x \in X$, the homotopy groups $\pi_i(X,x)$ are nontrivial in only finitely many degrees, and each is a finite group. For such X and dimension d, the finite homotopy TFT is a d-dimensional topological field theory (§2.4)

$$Z_X : Bord_d \to Vect_{\mathbb{C}}. \tag{0.0.4}$$

In the case that $X = BG$, then Z_{BG} is the d-dimensional finite gauge theory with gauge group G.

Abelian duality is a duality between p-form gauge theories (§2.2), where the gauge group is typically $U(1)$. In our topological case, the gauge group is finite abelian. Let A be a finite abelian group, and n a natural number. The n-th Eilenberg-MacLane space $K(A,n)$ (§2.2.5) is a π-finite space. The d-dimensional finite homotopy TFT $Z_{K(A,n)}$ counts n-principal A-bundles. It is the discrete analogue of p-form gauge theories.

Under abelian duality, the gauge group swaps to its Pontryagin dual. When A is an abelian group, the Pontryagin dual (character dual) group $\hat{A}$ is defined as $Hom(A, \mathbb{C}^\times)$. In [12], Freed and Teleman proved that the 3-dimensional finite homotopy TFTs $Z_{K(A,1)}$ and $Z_{K(\hat{A},1)}$ are isomorphic. In this thesis, we extend this equivalence to general n and arbitrary dimension d:

Theorem (Abelian duality)**.** *Let A be a finite abelian group and $\hat{A}$ its Pontryagin dual. Let $d \geq 1$ be the dimension of our theory and choose $n < d$. Let $Z_{K(A,n)}$, $Z_{K(\hat{A},d-1-n)}$ be the d-dimensional finite homotopy TFTs associated to $K(A, n)$ and $K(\hat{A}, d-1-n)$. There is an equivalence of oriented topological field theories:*

$$Z_{K(A,n)} \cong Z_{K(\hat{A},d-1-n)} \otimes E_{|K(A,n)|}, \tag{0.0.5}$$

where is $E_{|K(A,n)|}$ is the d dimensional Euler TFT (§4.2), which is an invertible TFT.

Our result extends to π-finite spectra. A spectrum $\mathcal{X}$ is π-finite if its (stable) homotopy groups $\pi_i(\mathcal{X})$ are nontrivial in finitely many degrees (§3.1) and each is a finite abelian group. For a π-finite spectrum $\mathcal{X}$, its underlying space $\Omega^\infty \mathcal{X}$ is a π-finite space. We define the d-dimensional finite homotopy TFT associated to $\mathcal{X}$ as the finite homotopy TFT of its underlying space:

$$Z_{\mathcal{X}} \coloneqq Z_{\Omega^\infty \mathcal{X}} : Bord_d \to Vect_{\mathbb{C}}. \tag{0.0.6}$$

Pontryagin duality can also be extended to π-finite spectra. In [4], Brown and Comenetz defined a dual spectrum $\hat{\mathcal{X}}$ for any spectrum $\mathcal{X}$. Let A be an abelian group and HA be its Eilenberg-MacLane spectrum, then

$$\widehat{HA} = H\hat{A}, \tag{0.0.7}$$

where $\hat{A}$ is the Pontryagin dual group of A. When $\mathcal{X}$ is π-finite, then $\hat{\mathcal{X}}$ is also π-finite.

Our main theorem is an extension of the theorem above to π-finite spectra:

Theorem (Abelian duality)**.** *Let $\mathcal{X}$ be a π-finite spectrum and $\hat{\mathcal{X}}$ its Brown-Comenetz dual. Let $Z_{\mathcal{X}}, Z_{\Sigma^{d-1}\hat{\mathcal{X}}}$ be the corresponding d-dimensional finite homotopy TFTs. There is an equivalence of (suitably oriented) topological field theory:*

$$\mathbb{D} : Z_{\mathcal{X}} \cong Z_{\Sigma^{d-1}\hat{\mathcal{X}}} \otimes E_{|\mathcal{X}|}, \tag{0.0.8}$$

where is $E_{|\mathcal{X}|}$ is the d dimensional Euler TFT (see 4.2), which is an invertible TFT.

Contents We now describe the content of the thesis:

In §1 we review electromagnetic duality in Maxwell's theory and abelian duality of p-form gauge theories. This section is entirely motivational and is not needed for the rest of the thesis.

In §2 we define d-dimensional finite homotopy TFT for any π-finite space X.

In §3 we review two duality theorems in stable homotopy theory: Poincaré duality and Brown-Comenetz duality.

In §4 we review the Euler characteristics and define the Euler TFT.

In §5 we state and proof the main theorem.

In the appendix §A we review the homotopy theory needed for the thesis.

Notations All manifolds are smooth, compact, and possibly with boundaries. A manifold is closed if it has no boundary (and compact). d-dimensional manifolds are also referred to as d-manifolds. Same for bordisms. We will suppress all ∞-category notations. All limits and colimits are homotopy limits and colimits.

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Contents

1 Physical motivations

In this section we review electromagnetic duality in Maxwell's theory and its generalization, abelian duality for p-form gauge fields. This provides the physical motivation for our abelian duality theorem. Note that this section is entirely motivational and can be safely skipped.

1.1 Electromagnetic duality

Let $M = \mathbb{R}^{3,1}$ be the $(3+1)$-dimensional Minkowski spacetime. The Maxwell's equations (in vacuum) are:

$$\vec{\partial} \cdot \vec{\mathbf{E}} = 0 \qquad \vec{\partial} \cdot \vec{\mathbf{B}} = 0 \tag{1.1.1}$$

$$\vec{\partial} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad \vec{\partial} \times \vec{\mathbf{B}} = \frac{\partial \vec{\mathbf{E}}}{\partial t} \tag{1.1.2}$$

where $\vec{\mathbf{E}}$ is the electric field and $\vec{\mathbf{B}}$ is the magnetic field. The Maxwell's equations 1.1.1 are invariant under *electromagnetic duality*:

$$(\vec{\mathbf{E}}, \vec{\mathbf{B}}) \mapsto (\vec{\mathbf{B}}, -\vec{\mathbf{E}}). \tag{1.1.3}$$

In the relativistic notation, we can package the electric and magnetic field into the electromagnetic field strength F, which is a 2-form on M:

$$F^{0i} = -F^{i0} = -E^i \qquad F^{ij} = -\epsilon_{ijk} B^k. \tag{1.1.4}$$

The Maxwell's equations 1.1.1 have compact form in terms of F:

$$\partial_\nu F^{\mu\nu} = 0 \qquad \partial_\mu * F^{\mu\nu} = 0, \tag{1.1.5}$$

where

$$* F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}. \tag{1.1.6}$$

is the Hodge star operator $*$ applied to F. The electromagnetic duality 1.1.3 becomes

$$F^{\mu\nu} \mapsto *F^{\mu\nu} \qquad *F^{\mu\nu} \mapsto -F^{\mu\nu}. \tag{1.1.7}$$

Now we explore what happens in the quantum theory, which is a pure abelian gauge theory. The dynamical field is the $U(1)$ gauge connection (electromagnetic potential) A_μ. The field-strength F is simply dA, the exterior derivative of A. The action of the theory is

$$S = -\frac{1}{2g^2} \int d^4x \ F^{\mu\nu} F_{\mu\nu} = -\frac{1}{2g^2} \int d^4x \ dA \wedge *dA, \tag{1.1.8}$$

where $\wedge$ is the wedge product of differential forms. This the electric picture, as the gauge field can be couple to electrically charged particles.

We move on to the magnetic picture. It is once again an $U(1)$ gauge theory, with gauge field $\widetilde{A}$ and the field strength $\widetilde{F} = d\widetilde{A}$. the action is

$$\widetilde{S} = -\frac{1}{2\widetilde{g}^2} \int d^4x \; d\widetilde{A} \wedge *d\widetilde{A}. \tag{1.1.9}$$

To prove electromagnetic duality, we need to show that these theories are equivalent for some given $(g, \widetilde{g})$, with $\widetilde{F} = *F$. We will sketch a proof of this, and its extension for general p-form gauge theories in the §1.2.

1.2 Abelian duality

In this section we review the formalism of p-form ($U(1)$) gauge theories and abelian duality between them. We present a path integral argument to show that the dual theories are equivalent. This serves as the physical motivation for the thesis, which is a topological formulation of this duality for finite discrete abelian groups. Note that the arguments given here are imprecise and incomplete. See [7] for a more precise treatment.

In §1.1 we reviewed electromagnetism and pure $U(1)$ Yang-Mills on the Minkowski spacetime. The dynamical field is the electromagnetic potential A, which on the flat spacetime is simply a 1-form. Its field strength $F = dA$ is a 2-form.

We want a p-form analogue: a p-form gauge field is a p-form A, and its field strength $F = dA$ is $(p+1)$-form. The gauge symmetries are given by $(p-1)$-forms. Given a gauge symmetry α, the gauge transformation is given by

$$A \mapsto A + d\alpha, \tag{1.2.1}$$

The field strength F is unchanged under a generalized gauge transformation as $d^2\alpha = 0$. The action of our p-form gauge theory is

$$S[A] = \frac{1}{g^2} \int F \wedge *F = \frac{1}{g^2} \int dA \wedge *dA. \tag{1.2.2}$$

The partition function Z is

$$Z = \int \mathcal{D}A \; e^{iS[A]}. \tag{1.2.3}$$

When $p = 1$ we recovered $U(1)$ gauge theory.

Remark 1.2.4. The (higher) gauge group of these p-form gauge theories is $U(1)$. However, on the Minkowski spacetime, we can't see the topological difference between $U(1)$ and $\mathbb{R}$. For example, all $U(1)$ bundles on M are trivial. On a more general spacetime manifold, the higher $U(1)$ gauge fields are defined via differential cohomology, a generalization of differential forms that also remembers the topology of the (higher) gauge bundles. A good reference is [8, §2].

Now we want to extend electromagnetic duality to abelian duality between p-form gauge theories in arbitrary dimension. Fix a dimension d and $p < d$. The abelian dual of the p-form gauge theory should be a q-form gauge theory, with field strength $\widetilde{F} = d\widetilde{A}$ and satisfies the equation

$$\widetilde{F} = *F. \tag{1.2.5}$$

From degree considerations, we see that the degree of $\widetilde{F}$ is $d - p - 1$, thus

$$q = d - p - 2. \tag{1.2.6}$$

For examples, for pure abelian Yang-Mills in 4 dimension, we have $d = 4$ and $p = 1$. Therefore $q = 1$. We see that we should have a duality between $U(1)$ gauge fields.

Now we give a heuristic path integral argument for abelian duality. Given a p-form gauge theory, there is an $(d - p - 2)$-form gauge theory that gives the same physics.

Recall that the action for our p-form theory is

$$S[A] = \frac{1}{g^2} \int F \wedge *F = \frac{1}{g^2} \int dA \wedge *dA. \tag{1.2.7}$$

A is a p-form. Now consider a different action, with fields A and B:

$$S'[A, B] = g^2 \int B \wedge *B + 2i \int B \wedge dA. \tag{1.2.8}$$

B is a $(d-p-1)$-form. As A and B are both dynamical fields, the partition function is

$$Z' = \int \mathcal{D}A \; \mathcal{D}B \; e^{iS'[A,B]}. \tag{1.2.9}$$

Now we can complete the square with respect to B, and we get

$$S'[A,B] = g^2(\int B \wedge *B + \frac{2i}{g^2} B \wedge dA) \tag{1.2.10}$$

$$= g^2(\int (B + \frac{i}{g^2} * dA) \wedge *(B + \frac{i}{g^2} * dA) + \frac{1}{g^4}(*dA \wedge dA)) \tag{1.2.11}$$

$$= g^2 \int (B + \frac{i}{g^2} * dA) \wedge *(B + \frac{i}{g^2} * dA) + \frac{1}{g^2} \int *dA \wedge dA \tag{1.2.12}$$

The the only B-dependent term in the action is the quadratic term

$$g^2 \int (B + \frac{i}{g^2} * dA) \wedge *(B + \frac{i}{g^2} * dA). \tag{1.2.13}$$

For a fix A, as

$$B + \frac{i}{g^2} * dA \tag{1.2.14}$$

is simply a translation in the B space, the integral

$$\int \mathcal{D}B \ e^{ig^2 \int (B + \frac{i}{g^2} * dA) \wedge *(B + \frac{i}{g^2} * dA)} \tag{1.2.15}$$

$$= \int \mathcal{D}B \ e^{ig^2 B \wedge *B} \tag{1.2.16}$$

simply gives a constant, which we will ignore. Thus we see that the partition function

$$Z' = \int \mathcal{D}A \ \mathcal{D}B \ e^{iS'[A,B]} \tag{1.2.17}$$

$$= \int \mathcal{D}A \ \ e^{\frac{i}{g^2} \int *dA \wedge dA} \ \left(\int \mathcal{D}B \ e^{ig^2 \int (B + \frac{i}{g^2} * dA) \wedge *(B + \frac{i}{g^2} * dA)} \right) \tag{1.2.18}$$

$$= \int \mathcal{D}A \ e^{i \frac{i}{g^2} \int *dA \wedge dA} \tag{1.2.19}$$

$$= \int \mathcal{D}A \ e^{iS[A]}. \tag{1.2.20}$$

recovers the original theory Z.

On the other hand, we integrate out A in Equation 1.2.8 and get the constraint

$$dB = 0. \tag{1.2.21}$$

As we are in Minkowski spacetime, this tells us that $B = d\widetilde{A}$ for some $q = d - p - 2$ form $\widetilde{A}$[1]. Thus we see that the theory Z' is equivalent to the q-form gauge theory, with action

$$g^2 \int dA \wedge *dA. \tag{1.2.22}$$

Remark 1.2.23. The coupling constants of the dual theory is $\widetilde{g} \propto \frac{1}{g}$. If the original theory has a large coupling constant $g \gg 1$ (thus nonperturbative), then the abelian dual theory has a small coupling constant. This is the beginning of S-duality (strong-weak duality).

Example 1.2.24. Let $d = 2, p = q = 0$, then the theories are 2d sigma models to circles of radius R and $1/R$. This is the beginning of T-duality. See [16, §11.2].

Example 1.2.25. Let $d - 3, p = 1, q = 0$, we get the well-known duality between sigma models and gauge theories in 3 dimension.

Example 1.2.26. Let $d = 4, p = q = 1$, we get electromagnetic duality for the quantum Maxwell theories.

In the thesis, we will formulate a version of abelian duality where the (higher) gauge groups are discrete and finite, formulated as equivalences of topological field theory.

[1]In general the argument is more delicate, see [16] chapter 11.2 page 251 see an argument where d=2, p=q=0.

2 Finite homotopy TFTs

In this section we define the d-dimensional finite sigma model TFT Z_X associated to a π-finite space X. In §2.1 we recall the definition of topological field theories. In §2.2 we give some background on π-finite spaces. In §2.3 we discuss the $\mathcal{O}$ functor from spans of π-finite spaces to $Vect_{\mathbb{C}}$. In §2.4 we use $\mathcal{O}$ to define the finite sigma model Z_X for a π-finite space X. In §2.5 we compute some examples of finite homotopy TFTs.

2.1 Topological Field Theories

Let us first motivate topological field theories. In a d-dimensional quantum theory, we have a partition function Z. Given a d-dimensional manifold M, which we viewed as the spacetime of the theory, we have:

$$Z(M) = \int \mathcal{D}\phi \; e^{iS[\phi]}, \tag{2.1.1}$$

where ϕ represents all fields of the theory, $S = \int_M \mathcal{L}(\phi)$ is the action, and $\mathcal{L}$ is the Lagrangian. Note that the integral integrates over all field possible field configurations over M. In a quantum theory, the field configurations are often infinitely dimensional, making the measure $\mathcal{D}\phi$ and the integral difficult to mathematically define.

In a topological theory, there is no dependence on the metric. The fields are often discrete and finite, and we can actually evaluate the path integral as a finite sum. Thus for any d dimensional closed manifold M we expect a number $Z(M) \in \mathbb{C}$. More generally, given a manifold M with boundary $\partial M \simeq N \sqcup N'$, then we can view N and N' as the incoming and outgoing space-slice, and M an evolution through time. In this case, we should have a complex vector space of initial states $Z(N)$. For a given state $v \in Z(N)$, we can let it evolve and get a state $v' \in Z(N')$. Thus we see that $Z(M)$ gives a map $Z(N) \to Z(N')$.

A d-manifold M with $\partial M \xrightarrow{\sim} N \sqcup N'$ is called a bordism from N to N'. Two bordism M, M' from N to N' are isomorphic if there exists a diffeomorphism from M to M' that restricts to identity on the boundaries. In addition, given bordisms $M : N \to N'$ and $M' : N' \to N''$, we can glue them together at N' [2]and form a bordism $M \sqcup_{N'} M' : N \to N''$. That is, we can compose bordisms.

[2]To rigorously glue two bordisms together, one needs a collar neighborhood at N'. There is a contractible choice of collar neighborhoods, but such choice is needed.

Definition 2.1.2. The d-dimensional (unoriented) bordism category $Bord_d$ is the category whose objects are closed $(d-1)$-manifold, and morphisms between them are isomorphism classes of unoriented bordisms. Composition is given by gluing bordisms along the common boundary [3]. It is a symmetric monoidal category under disjoint union.

Remark 2.1.3. This is called the unoriented bordism category because the manifolds and bordisms are unoriented. Often than not, the field theory require additional tangential structure, such as an orientation (for integration), or a *Spin* structure (for the fermionic fields). There is a general notion of tangential structures Θ. For each Θ, we can define a corresponding bordism category $Bord_d^\Theta$, symmetric monoidal under disjoint union. We will not need the general notion of tangential structure and $Bord_d^\Theta$. For detail, see [19, §2.4]. However, for a $\mathbb{E}_1$-ring spectrum $\mathcal{R}$, we will define the $\mathcal{R}$-oriented bordism category $Bord_d^{\mathcal{R}}$ (see §5.2). Currently we only need the unoriented bordism category $Bord_d$ as the finite homotopy TFTs Z_X are defined on unoriented manifolds.

Remark 2.1.4. The empty set $\varnothing$ is a closed n-dimensional manifold for any $n \geq 0$. When viewed as a n-dimensional manifold, we denote it as $\varnothing_n$. $\varnothing_{d-1}$ is the unit object for the symmetric monoidal structure on $Bord_d$ (and its tangential variants). The set of endormorphism of the unit object

$$\Omega Bord_d \coloneqq Mor_{Bord_d}(\varnothing_{d-1}, \varnothing_{d-1}) \tag{2.1.5}$$

is the set of isomorphism classes of closed d-manifolds. It is a monoid under disjoint union. Similarly, for any tangential structure Θ, $\Omega Bord_d^\Theta$ is the monoid of isomorphism classes of closed d-manifold with Θ-tangential structure.

Remark 2.1.6. We have defined the unextended bordism categories. See [19, §1.2] for the definitions of the extended bordism categories.

We want to formulate a TFT as a symmetric monoidal functor out of the bordism category. We need our target category:

Definition 2.1.7. The category of $\mathbb{C}$-linear vector spaces, $Vect_{\mathbb{C}}$, is the category whose objects are finite dimensional $\mathbb{C}$-linear vector spaces and morphisms are $\mathbb{C}$-linear transformations. It is symmetric monoidal under the tensor products $\otimes$.

[3]The complication with collar neighborhoods make defining $Bord_d$ difficult. These difficulty will not arise in our constructions. For more detail, see [23, §3.1].

Following [2], we define a topological field theory:

Definition 2.1.8. A d-dimensional (unoriented) topological field theory is a symmetric monoidal functor $Z : Bord_d \to Vect_{\mathbb{C}}$.

Of course, for every tangential structure Θ, one can define a d-dimensional Θ-oriented TFT as a symmetric monoidal functor $Z : Bord_d^{\Theta} \to Vect_{\mathbb{C}}$.

Remark 2.1.9. Let Z be a d-dimensional TFT. As it is symmetric monoidal, $Z(\varnothing_{d-1}) = \mathbb{C}$. For a closed d-manifold M, viewed as an morphism $\varnothing_{d-1} \to \varnothing_{d-1}$, we have $Z(M) : \mathbb{C} \to \mathbb{C}$ given by multiplication by a scalar. Therefore a TFT assigns numbers to closed d-manifolds. More generally, for a bordism $M : N \to N'$, we get a map of states $Z(N) \to Z(N')$. This is exactly what we want from a topological field theory.

Remark 2.1.10. Let *Lines* be the Picard groupoid of $Vect_{\mathbb{C}}$. The objects of *Lines* are 1-dimensional vector spaces, and morphisms are invertible linear transformation between them. A d-dimensional TFT

$$Z : Bord_d \to Vect_{\mathbb{C}} \tag{2.1.11}$$

is called invertible if it factors through *Lines*. If Z is invertible, then for every closed $(d-1)$-manifold N, $Z(N)$ is 1-dimensional; for every bordism $M : N \to N'$,

$$Z(M) : Z(N) \to Z(N') \tag{2.1.12}$$

is an isomorphism. An example of invertible field theory is the the Euler TFT E_λ defined in §4.2.

2.2 π-finite spaces

In this section we define the π-finite spaces and proof some closure properties. The basics of the homotopy theory of topological spaces is reviewed in appendix A.1. Let S be the (∞-)category of spaces, and

$$Maps(-,-) : S^{op} \times S \to S \tag{2.2.1}$$

be the mapping space functor.

Definition 2.2.2. A topological space X is π-finite space if $\pi_0(X)$ is finite, and for every $x \in X$, $\pi_i(X, x)$ is nontrivial for only finitely many i, and each one is a finite group.

The full subcategory of π-finite spaces is denoted as S^{fin}.

Example 2.2.3. Let X be a finite set, then as a discrete topological space it is π-finite.

Example 2.2.4. Let G be a (discrete) group. The classifying space BG is a topological space with the following property:

1. $\pi_0(BG) = *$.

2. For $x \in BG$, we have $\pi_1(BG, x) \simeq G$.

3. $\pi_i(BG, x) = 0$ for $i \geq 1$.

Maps into BG classifies principal G bundles. When G is finite, this is a π-finite space.

Example 2.2.5. Let A is an abelian group, $n \geq 1$, The n-th Eilenberg MacLane space $K(A, n)$ is a topological space with the following property:

1. $\pi_0(K(A, n)) = *$.

2. for $x \in K(A, n)$, we have $\pi_n(K(A, n), x) \simeq A$.

3. $\pi_i(K(A, n), x) = 0$ for $i \neq n$

When A is finite, $K(A, n)$ is a π-finite space. Maps into $K(A, n)$ classifies n-th cohomology classes with A coefficients:

$$\pi_0(Maps(-, K(A, n))) \simeq H^n(-, A). \tag{2.2.6}$$

Geometrically, $K(A, n)$ classify n-principal A bundles.

Now we discuss some closure properties of S^{fin}.

Proposition 2.2.7. *Let $p : X \to Z$, $q : Y \to Z$ be maps π-finite spaces, If*

$$\begin{array}{ccc} W & \xrightarrow{q'} & X \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{q} & Z \end{array} \tag{2.2.8}$$

is a (homotopy) pullback diagram in S, then W is also a π-finite space.

Proof. It is easy to see that W has finite connected components. Take a point $w \in W$, then we have a commutative diagram of based spaces:

$$\begin{array}{ccc} (W, w) & \xrightarrow{q'} & (X, q'(w)) \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ (Y, p'(w)) & \xrightarrow{q} & (Z, p \circ q'(w)). \end{array} \tag{2.2.9}$$

This is a pullback diagram in S_*, the category of pointed spaces. Note that the homotopy groups of the based spaces $\pi_*(W,w)$ is the same as the homotopy groups of W at w. The finiteness of $\pi_*(W,w)$ follows from the Mayer-Vietoris long exact sequence

$$\cdots \to \pi_*(W,w) \to \pi_*(X,q'(w)) \oplus \pi_*(Y,p'(w)) \to \pi_*(Z,p\circ q'(w)) \to \cdots \tag{2.2.10}$$

and the fact that X,Y,Z are π-finite spaces. $\square$

We will need the following lemma:

Lemma 2.2.11. *Let M be a (compact) manifold, X a π-finite space, then the mapping space $Maps(M,X)$ is again a π-finite space.*

Proof. As

$$Maps(M \sqcup M', X) \simeq Maps(M,X) \times Maps(M',X). \tag{2.2.12}$$

and

$$\pi_i(X \times Y) = \pi_i(X) \times \pi_i(Y), \tag{2.2.13}$$

we see that it is suffice to proof the case where M is connected, which we assume from now on. Given two spaces X,Y, then

$$Maps(M, X \sqcup Y) \simeq Maps(M,X) \sqcup Maps(M,Y). \tag{2.2.14}$$

As π-finite spaces are finite disjoint unions of connected π-finite spaces, it is enough to prove the case where X is a connected π-finite space.

We will do this by induction. First we look at the case where the homotopy groups of X are concentrated in a single degree i ($i > 0$ as $\pi_0(X) = *$). Therefore either $X = BG$ for G a finite group or $X = K(A,n)$ for A a finite abelian group. In the case that $X = BG$, we see that

$$Maps(M, BG) = |Bun_G(M)| \tag{2.2.15}$$

is the classifying space of the groupoid of principal G-bundles on M (see §2.5.3). As M is a compact manifold, there are finitely many different isomorphism classes of principal G bundles on M, then $\pi_0(Maps(M,BG))$ is a finite set. As each principal G-bundle P on M has finite automorphism group, we see that $\pi_1(Maps(M,BG),x)$ is finite for any $x \in Maps(M,BG)$. In addition, it has no higher homotopy groups, therefore it is a π-finite space.

In the other case, let $X = K(A,n)$ for A a finite abelian group. Then

$$\pi_i(Maps(M, K(A,n))) = H^{n-i}(M,A) \tag{2.2.16}$$

is the $(n-i)$-th cohomology group of M with A coefficients (this will play a major role in the main theorem). As M is compact and A finite, they are also finite. One way to see this is by considering a finite CW complex K homotopic equivalent to M. Then the finite CW cochain complex

$$C^*(K, \mathbb{Z}) \otimes A \tag{2.2.17}$$

has cohomology $H^*(M, A)$. At each degree i , the cochain complex

$$C^i(K, \mathbb{Z}) \otimes A \tag{2.2.18}$$

is a finite abelian group. Therefore the subquotient $H^i(M, A)$ is also finite. This shows that

$$Maps(M, K(A, n)) \tag{2.2.19}$$

is a π-finite space.

Now we use induction. We assume the lemma is true for all connected π-finite space with homotopy group concentrated in degrees less than n (case $n = 2$ is $X = BG$ proven above). We will prove the lemma for connected π-finite space X with homotopy group concentrated in degrees less or equal to n. Take $x \in X$ and consider (X, x) as a based space. Consider the following fiber sequence

$$\tau_{\geq n} X \to X \to \tau_{\leq n} X. \tag{2.2.20}$$

As the homotopy group of $\tau_{\geq n} X$ is concentrated in degree n, we have

$$\tau_{\geq n} X = K(\pi_n(X), n). \tag{2.2.21}$$

As $Maps(M, -) : S \to S$ takes fiber sequences to fiber sequences, we have a fiber sequence

$$Maps(M, \tau_{\geq n} X) \to Maps(M, X) \to Maps(M, \tau_{\leq n} X). \tag{2.2.22}$$

Note that

$$Maps(M, \tau_{\geq n} X) \simeq Maps(M, K(\pi_n(X), n)) \tag{2.2.23}$$

and

$$Maps(M, \tau_{\leq n} X) \tag{2.2.24}$$

are both π-finite. By the long exact sequence of homotopy groups, we see that $Maps(M, X)$ is also π-finite. □

2.3 Functions on π-finite spaces

Let $Vect_{\mathbb{C}}$ be the symmetric monoidal category of finite dimensional $\mathbb{C}$-vector spaces, S^{fin} the category of π-finite spaces. The goal of this section is to define the symmetric monoidal functor

$$\mathcal{O} : Span(S^{fin}) \to Vect_{\mathbb{C}}. \tag{2.3.1}$$

First we need to define the span category $Span(S^{fin})$.

Definition 2.3.2. The category of spans (correspondences) of π-finite spaces $Span(S^{fin})$ the category with objects π-finite spaces X, and a morphism from X to X' is a span

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X & & X'. \end{tikzcd} \tag{2.3.3}$$

Given two morphisms

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & & & Y' \arrow[dl] \arrow[dr] & \\ X & & X', & X' & & X'', \end{tikzcd} \tag{2.3.4}$$

the composition is the (homotopy) pullback [4]:

$$\begin{tikzcd} & Y \times_{X'} Y' \arrow[dl] \arrow[dr] & \\ X & & X''. \end{tikzcd} \tag{2.3.5}$$

$Span(S^{fin})$ is a symmetric monoidal category under Cartesian products: The product of objects $X, Y \in Span(S^{fin})$ is $X \times Y$. The product of two span

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & & & W \arrow[dl] \arrow[dr] & \\ X & & X', & Z & & Z' \end{tikzcd} \tag{2.3.6}$$

[4]Since S is a ∞-category, so is $Span(S^{fin})$. There are also addition coherence issues since the (homotopy) pullback is only well-defined up to a contractible choice. See [15] for more detail. These complications will not arise in our discussion.

is

$$\begin{tikzcd} & Y \times W \arrow[dl] \arrow[dr] & \\ X \times Z & & X' \times Z'. \end{tikzcd} \tag{2.3.7}$$

Now we define $\mathcal{O}$ on objects:

Definition 2.3.8. Let X be a space, then $\mathcal{O}(X)$ is the vector space of locally constant complex valued functions $\mathbb{C}[\pi_0(X)]$ on X.

Example 2.3.9. Let $X = BG$ be the classifying space of a group G. As $\pi_0(BG) = *$, we see that evaluation at the only connected component gives an equivalence

$$\mathcal{O}(BG) \xrightarrow{\sim} \mathbb{C}, \tag{2.3.10}$$

with $1 \in \mathbb{C}$ corresponds to the constant function $\underline{1}$ on BG.

When X is a π-finite space, then $\mathcal{O}(X)$ is finite dimensional.

Definition 2.3.11. Given $p : X \to Y$ a map of π-finite spaces, We define the pullback map

$$p^* : \mathcal{O}(Y) \to \mathcal{O}(X) \tag{2.3.12}$$

as follows: given $f : Y \to \mathbb{C}$ and $x \in X$,

$$p^*(f)(x) \coloneqq f(p(x)). \tag{2.3.13}$$

This defines a functor

$$(-)^* : (S^{fin})^{op} \to Vect_{\mathbb{C}}. \tag{2.3.14}$$

We also need to define pushforwards.

Definition 2.3.15. Let X and Y be π-finite spaces. Given $p : X \to Y$, we define pushforward map

$$p_* : \mathcal{O}(X) \to \mathcal{O}(Y) \tag{2.3.16}$$

as follows: let $g : X \to \mathbb{C}$ and $y \in Y$,

$$p_*(g)(y) \coloneqq \sum_{[x]\to[y]} \frac{|\pi_1(Y,y)|}{|\pi_1(X,x)|} \frac{|\pi_2(X,x)|}{|\pi_2(Y,y)|} \cdots g(x), \tag{2.3.17}$$

where $\sum_{[x]\to[y]}$ means summing over all $[x] \in \pi_0(X)$ that maps to the the connected component $[y] \in \pi_0(Y)$ of y.

For a fix connected component $[x]$, we pick a point x in the connected component. As $|\pi_i(X, x)|$ only depends on the connected component $[x]$, $|\pi_i(X, x)|$ is well-defined in Equation 2.3.17. $g(x)$ is also well-defined as g is a locally constant function.

The pushforward map sums over the fibers in a way that keeps track the sizes of the higher homotopy groups. The infinite product is actually finite as π-finite spaces have finitely many nontrivial homotopy groups, and each homotopy group is finite.

Example 2.3.18. Continuing the BG example, we have the projection map $p : BG \to *$. Recall that $\mathcal{O}(*) = \mathbb{C}[*] = \mathbb{C}\underline{1}$, where $\underline{1}$ is the constant function on $*$ with value at 1. Similarly, $\mathcal{O}(BG) = \mathbb{C}\underline{1}_{BG}$. We have $p^*(\underline{1}) = \underline{1}_{BG}$ and $p_*(\underline{1}_{BG}) = \frac{1}{|G|}\underline{1}$.

Proposition 2.3.19. *The pushforward maps define a functor*

$$(-)_* : S^{fin} \to Vect_{\mathbb{C}}. \tag{2.3.20}$$

Proof. Let maps $p : X \to Y,\ q : Y \to Z$ be maps between π-finite spaces. We have to show that

$$q_* \circ p_* = (q \circ p)_*. \tag{2.3.21}$$

For $g : X \to \mathbb{C}$ and $z \in Z$,

$$(q_* \circ p_*(g))(z) = \sum_{[y]\to[z]} \frac{|\pi_1(Z, z)|}{|\pi_1(Y, y)|} \frac{|\pi_2(Y, y)|}{|\pi_2(Z, z)|} \cdots (p_*(g))(y) \tag{2.3.22}$$

$$= \sum_{[y]\to[z]} \frac{|\pi_1(Z, z)|}{|\pi_1(Y, y)|} \frac{|\pi_2(Y, y)|}{|\pi_2(Z, z)|} \cdots \tag{2.3.23}$$

$$\sum_{[x]\to[y]} \frac{|\pi_1(Y, y)|}{|\pi_1(X, x)|} \frac{|\pi_2(X, x)|}{|\pi_2(Y, y)|} \cdots g(x) \tag{2.3.24}$$

$$= \sum_{[x]\to[z]} \frac{|\pi_1(Z, z)|}{|\pi_1(X, x)|} \frac{|\pi_2(X, x)|}{|\pi_2(Z, z)|} \cdots g(x) \tag{2.3.25}$$

$$= ((q \circ p)_*(g))(z) \tag{2.3.26}$$

□

We have this compatibility lemma between pullback and pushforward maps:

Lemma 2.3.27 (Base-change)**.** *Let*

$$\begin{array}{ccc} W & \xrightarrow{q'} & X \\ \Big\downarrow{\scriptstyle p'} & & \Big\downarrow{\scriptstyle p} \\ Z & \xrightarrow{q} & Y \end{array} \tag{2.3.28}$$

be a (homotopy) pullback diagram of π*-finite spaces. Then we have an equality of maps*

$$q^* \circ p_* = p'_* \circ q'^* : \mathcal{O}(X) \to \mathcal{O}(Z) \tag{2.3.29}$$

Proof. Functions are determined by their values at points. Let $z \in Z$, consider as a map $* \xrightarrow{z} Z$, then

$$f(z) = z^*(f)(*). \tag{2.3.30}$$

As two small pullback diagrams forms a larger pullback diagram, it is suffice to check the lemma with the case where $Z = *$, and $q : Z \to Y$ is given by a point $y \in Y$. In this case, $W = fib_y(p)$ is the fiber of the map $p : X \to Y$ at $y \in Y$. The points in W are pairs

$$(x, \gamma : f(x) \rightsquigarrow y), \tag{2.3.31}$$

where x is a point in X and γ is a path $f(x) \rightsquigarrow y$ in Y. Let $f \in \mathcal{O}(X)$,

$$\begin{aligned} (q^* \circ p_*(f))(*) &= p_*(f)(y) && (2.3.32)\\ &= \sum_{[x]\to[y]} \frac{|\pi_1(Y,y)|}{|\pi_1(X,x)|}\frac{|\pi_2(X,x)|}{|\pi_2(Y,y)|}\cdots f(x). && (2.3.33) \end{aligned}$$

On the other hand,

$$(p'_* \circ q'^*(f))(*) = \sum_{[x']} \frac{1}{|\pi_1(fib_y(p),x')|}|\pi_2(fib_y(p),x')|\cdots q'^*(f)(x'). \tag{2.3.34}$$

For each $x' = (x, \gamma)$,

$$q'^*(f)(x') = f(x). \tag{2.3.35}$$

We can look at one components of X at a time. That is, we assume X is connected. Take a point $x \in X$ and $y = p(x)$, We need to show that

$$\frac{|\pi_1(Y,y)|}{|\pi_1(X,x)|}\frac{|\pi_2(X,x)|}{|\pi_2(Y,y)|}\cdots = \sum_{[x']\to[x]} \frac{1}{|\pi_1(fib_y(p),x')|}|\pi_2(fib_y(p),x')|\cdots \tag{2.3.36}$$

This follows from the long exact sequence of homotopy group associated to the fiber sequence $fib_y(p) \to X \xrightarrow{p} Y$:

$$\cdots \to \pi_*(fib_y(f)) \to \pi_*(X) \to \pi_*(Y) \to \cdots \tag{2.3.37}$$

□

We can finally define $\mathcal{O}$ functor: given a span

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & X', \end{tikzcd} \tag{2.3.38}$$

we define $\mathcal{O}(Y) : \mathcal{O}(X) \to \mathcal{O}(X')$ to be

$$q_* \circ p^* : \mathcal{O}(X) \xrightarrow{p^*} \mathcal{O}(Y) \xrightarrow{q_*} \mathcal{O}(X'). \tag{2.3.39}$$

We need to show that this is well-defined:

Proposition 2.3.40. *This defines a functor $\mathcal{O} : Span(S^{fin}) \to Vect_{\mathbb{C}}$.*

Proof. We have to check that composition agrees. Given two spans

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & & & Y' \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & X', & X' & & X'', \end{tikzcd} \tag{2.3.41}$$

we have the larger diagram

$$\begin{tikzcd} & & Y \times_{X'} Y' \arrow[dl, "s'"'] \arrow[dr, "q'"] & & \\ & Y \arrow[dl, "p"'] \arrow[dr, "q"] & & Y' \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & X' & & X''. \end{tikzcd} \tag{2.3.42}$$

The composition of the two span is

$$\begin{tikzcd} & Y \times_{X'} Y' \arrow[dl, "p \circ s'"'] \arrow[dr, "t \circ q'"] & \\ X & & X''. \end{tikzcd} \tag{2.3.43}$$

Lastly, we have

$$\begin{aligned}\mathcal{O}(Y') \circ \mathcal{O}(Y) &= t_* \circ s^* \circ q_* \circ p^* && (2.3.44)\\ &= t_* \circ q'_* \circ s'^* \circ p^* && (2.3.45)\\ &= (t \circ q')_* \circ (p \circ s')^* && (2.3.46)\\ &= \mathcal{O}(Y \times_{X'} Y'). && (2.3.47)\end{aligned}$$

□

Remark 2.3.48. In the last equation, We exactly used the functoriality of pullback, pushfoward, and base change lemma. The fact that we have a pushforward map and the base change lemma holds is a specific case of a more general phenomenon, called ambidexterity ([21]).

2.4 Finite homotopy TFTs

Let X be a π-finite space. In this section, we define the d-dimensional finite homotopy TFT Z_X associated to X, which is a d-dimensional (unoriented) topological field theory

$$Z_X : Bord_d \to Vect_{\mathbb{C}}. \tag{2.4.1}$$

First we need to define the field functor $\mathcal{F}_X$.

Definition 2.4.2. We define

$$\mathcal{F}_X : Bord_d \to Span(S^{fin}) \tag{2.4.3}$$

as follows: let $N \in Bord_d$ be a closed $(d-1)$-manifold. Then

$$\mathcal{F}_X(N) \coloneqq Maps(N, X). \tag{2.4.4}$$

$Maps(N, X)$ is a π-finite space by Proposition 2.2.11. Similarly, for a bordism $M : N \to N'$, we define

$$\mathcal{F}_X(M) \coloneqq Maps(M, X) \tag{2.4.5}$$

as a span:

$$\begin{tikzcd} & Maps(M,X) \arrow[dl] \arrow[dr] & \\ Maps(N,X) & & Maps(N',X). \end{tikzcd} \tag{2.4.6}$$

Proposition 2.4.7. *$\mathcal{F}_X$ defines a symmetric monoidal functor: $Bord_d \to Span(S^{fin})$.*

Proof. We need to show composition holds. Given two bordisms

$$M : N \to N', \quad M' : N' \to N''. \tag{2.4.8}$$

We have the composition bordism

$$M \sqcup_{N'} M' : N \to N''. \tag{2.4.9}$$

Note that

$$\begin{tikzcd} N' \arrow[r] \arrow[d] & M \arrow[d] \\ M' \arrow[r] & M \sqcup_{N'} M' \end{tikzcd} \tag{2.4.10}$$

is a pushout diagram in S. As $Maps(-, X) : S^{op} \to S$ takes pushouts to pullbacks, we have a pullback diagram:

$$\begin{tikzcd} Maps(M \sqcup_{N'} M', X) \arrow[r] \arrow[d] & Maps(M', X) \arrow[d] \\ Maps(M, X) \arrow[r] & Maps(N', X). \end{tikzcd} \tag{2.4.11}$$

In the span category $Span(S^{fin})$, composition of spans are given by pullbacks. Therefore

$$\begin{aligned} \mathcal{F}_X(M \sqcup_{N'} M') &= Maps(M \sqcup_{N'} M', X) && (2.4.12) \\ &\simeq Maps(M, X) \times_{Maps(N', X)} Maps(M', X) && (2.4.13) \\ &= \mathcal{F}_X(M') \circ \mathcal{F}_X(M). && (2.4.14) \end{aligned}$$

In addition, this functor is symmetric monoidal as $Maps(N \sqcup N', X) \simeq Maps(N, X) \times Maps(N', X)$. □

Now we can compose

$$\mathcal{F}_X : Bord_d \to Span(S^{fin}) \tag{2.4.15}$$

with the symmetric monoidal functor

$$\mathcal{O} : Span(S^{fin}) \to Vect_{\mathbb{C}} \tag{2.4.16}$$

to get a symmetric monoidal functor from $Bord_d$ to $Vect_{\mathbb{C}}$, i.e. a TFT:

Definition 2.4.17. Given X a π-finite space, the d-dimensional finite homotopy TFT associated to X is

$$Z_X \coloneqq \mathcal{O} \circ \mathcal{F}_X : Bord_d \to Vect_{\mathbb{C}}. \tag{2.4.18}$$

Remark 2.4.19. In physical terms, $\mathcal{F}_X(M)$ is the space of fields of this theory Z_X on M. The exponentiated action

$$e^{i\int_M \mathcal{L}} \tag{2.4.20}$$

is trivial, as we are just summing over the number of fields (weighted by their automorphisms). See [5] Quinn's lecture 4 for details.

Remark 2.4.21. Given a π-finite space X and a "character" $\chi : X \to B^d\mathbb{C}^\times$, we can define a d-dimensional "twisted theory" as follows: let M be a closed d-manifold and $ev_M : M \times Maps(M, X) \to X$ be the evaluation map. Using this, we can pullback the character

$$ev_M^* \ \chi : X \times Maps(M, X) \to B^d\mathbb{C}^\times. \tag{2.4.22}$$

Given an orientation $[M]$ and M, then we can integrate ev_M^* over $[M]$:

$$\int_{[M]} ev_M^* \ \chi : Maps(M, X) \to \mathbb{C}^\times. \tag{2.4.23}$$

As $Maps(M, X)$ are the fields in this theory, we have an exponentiated action:

$$e^{i\int_M \mathcal{L}(\mathcal{F})} \coloneqq (\int_{[M]} ev_M^* \ \chi)(\mathcal{F}) \in \mathbb{C}^\times, \tag{2.4.24}$$

where $\mathcal{F} \in Maps(M, X)$ is a field. This defines a classical invertible field theory. For the quantum theory, we integrate over all fields $\mathcal{F} \in Maps(M, X)$. Therefore partition function is

$$Z(M) = \int \mathcal{D}\mathcal{F} \ (\int_{[M]} ev_M^* \ \chi)(\mathcal{F}). \tag{2.4.25}$$

This theory is an oriented TFT as it needs the orientation $[M]$ on M. It is a generalization of Dijkgraaf-Witten theory [6] to π-finite spaces. See [5] Quinn's lecture 5 for the full construction. Our finite homotopy TFTs are simply the cases where χ is the constant map.

2.5 Examples of finite homotopy TFTs

In this section we calculate some examples of finite homotopy TFTs.

Example 2.5.1. Let T be a finite set and M a compact manifold with boundary. The mapping space $Maps(N,T)$ is homotopic equivalent to the discrete set $T^{\pi_0(N)}$. If N is a closed $(d-1)$-manifold, then

$$Z_T(N) = \mathbb{C}[T^{\pi_0(N)}] \tag{2.5.2}$$

is a vector space of dimension $|T|^{\pi_0(N)}$. This is a trivial sigma model where the fields are maps into a discrete set.

Example 2.5.3. Let G be a finite group, and BG its classifying space. Let M be a manifold, then $Maps(M,BG)$ is the classifying space of the groupoid $Bun_G(M)$. Recall that $Bun_G(M)$ the groupoid of principal G bundles on M. Its object are principal G-bundles on M, and morphisms are equivalence of principal G-bundles. This is a groupoid, that is, its morphisms are all invertible. Given a groupoid, one can take its classifying space, which is a 1-truncated space. In this case, the classifying space of $Bun_G(M)$ is

$$|Bun_G(M)| \simeq Maps(M,BG), \tag{2.5.4}$$

where $\simeq$ means homotopy equivalence. We see that $\pi_0(Maps(M,BG))$ is the set of isomorphism classes of principal G bundles, and for a given principal G-bundle P, viewed as an object in $P \in Maps(M,BG)$, we have

$$\pi_1(Maps(M,BG),P) = Aut(P), \tag{2.5.5}$$

the group of automorphism of P.

Let N be a closed $(d-1)$-manifold, then the field theory Z_{BG} evaluated at N is

$$Z_{BG}(N) = \mathbb{C}[\pi_0(Maps(M,BG))] \tag{2.5.6}$$

is the vector space of locally constant functions on the groupoid $Bun_G(M)$. For a closed d-dimensional manifold M, Z_{BG} assigns to M the number

$$Z_{BG}(M) := \sum_{[x]} \frac{1}{|Aut_{Bun_G(M)}([x])|}, \tag{2.5.7}$$

where $[x]$ sums over the isomorphism classes of $Bun_G(M)$. The partition function $Z_{BG}(M)$ counts the number of isomorphism classes of principal G-bundles on M, each one weighted by the size of its automorphism group.

This is a finite gauge theory that counts principal G-bundles. In $d=2$ dimension, it is closely related to the character theory for finite groups. We will review the abelian version of this in Example 5.4.8.

Example 2.5.8. Let A be a finite abelian group and $K(A,n)$ its n-th Eilenberg-MacLane space. The theory $Z_{K(A,n)}$ counts cohomology classes $H^n(-,A)$ (see Example 5.1.9). As $H^1(-;A)$ classifies principal A-bundles, $H^n(-;A)$ classifies n-principal A-bundles. Therefore $Z_{K(A,n)}$ is the discrete analogue of the n-form gauge theories defined in §1.2.

3 Duality theorems in stable homotopy theory

In this section we review two duality theorems: Brown-Comenetz duality and Poincaré duality. In §3.1 we define π-finite spectra and proof some basic properties. In §3.2 we develop orientation theory for $\mathbb{E}_1$-ring spectrum and prove Poincaré duality. In §3.3 we develop Pontryagin duality for finite abelian groups. In §3.4 we generalize Pontryagin duality to Brown-Comenetz duality [4] for π-finite spectra.

3.1 π-finite spectra

Let Sp be the ($\infty-$)category of spectra.

Definition 3.1.1. Let $\mathcal{X}$ be a spectrum. It is called π-finite if the (stable) homotopy groups $\pi_*\mathcal{X}$ are nonzero only in finitely many degrees, and each $\pi_i\mathcal{X}$ is a finite abelian group.

Remark 3.1.2. If $\mathcal{X}$ is a π-finite spectrum, then the underlying space $\Omega^\infty\mathcal{X}$ is a π-finite space.

We will denote the full subcategory of π-finite spectra as Sp^{fin}.

Example 3.1.3. Let A be a abelian group, and HA its the Eilenberg MacLane spectrum. If A is finite, then HA and its shifts are π-finite spectra.

Remark 3.1.4. Another common notion of finiteness for spectrum is dualizability. Note that π-finiteness is a different notion. For example, the sphere spectrum $\mathcal{S}^0$ is not π-finite.

Now we show some closure properties of Sp^{fin}.

Definition 3.1.5. Let $C \subset Sp$ be a full subcategory of Sp. Then the extension closure of C, denoted as C^c, is the smallest full subcategory such that

1. C^c is closed under suspension (and desuspension).
2. If $\mathcal{X}, \mathcal{X}'' \in C^c$, and we have a fiber sequence $\mathcal{X} \to \mathcal{X}' \to \mathcal{X}''$ of spectra, then $\mathcal{X}' \in C^c$.

The objects in C^c are finite extensions of spectra in C.

Proposition 3.1.6. $(Sp^{fin})^c = Sp^{fin}$.

Proof. Sp^{fin} is clearly closed under suspension. Given $\mathcal{X} \to \mathcal{X}' \to \mathcal{X}''$ a fiber sequence of spectra. If $\mathcal{X}$, $\mathcal{X}''$ are π-finite, we need to show that $\mathcal{X}$ is also π-finite. This is due to the long exact sequence of homotopy groups:

$$\cdots \to \pi_*\mathcal{X} \to \pi_*\mathcal{X}' \to \pi_*\mathcal{X}'' \to \pi_{*-1}\mathcal{X} \to \cdots . \tag{3.1.7}$$

□

π-finite spectra are finite extensions of finite Eilenberg-MacLane spectrum:

Proposition 3.1.8. *Let C be the full subcategory of finite Eilenberg-MacLane spectrum (spectrum of the form HA for some A finite abelian group), then $C^c = Sp^{fin}$.*

Proof. Sp^{fin} contains C. Thus it is suffice to show that all π-finite spectra are finite extensions of finite Eilenberg-MacLane spectrum.

Let $\mathcal{X}$ be a π-finite spectrum. We will do induction on the range where the homotopy groups of $\mathcal{X}$ are nontrivial. On the base case, where the homotopy groups of $\mathcal{X}$ is concentrated in a single degree, then it is precisely a suspension of HA, where A is an finite abelian groups. These are in C^c as C^c is closed under suspension.

We shift $\mathcal{X}$ so that its lowest nontrival homotopy group is concentrated in degree 0. Assume we have proven the case for all $\mathcal{X}$ where $\mathcal{X}$'s homotopy groups are concentrated in degree 0 to i. For a π-finite spectrum $\mathcal{X}$ with highest nontrivial homotopy group in degree $i+1$, consider the fiber sequence

$$\tau_{\geq i+1}\mathcal{X} \to \mathcal{X} \to \tau_{\leq i}\mathcal{X}. \tag{3.1.9}$$

Note that the homotopy groups of $\tau_{\geq i+1}\mathcal{X}$ is concentrated in a single degree, namely $i+1$. Therefore it is a suspension of HA and it is in C^c. On the other hand, as $\tau_{\leq i}\mathcal{X}$ is a π-fintie spectrum whose nontrivial homotopy groups are concentrated in degree 0 to i, it is in C^c by induction. As both $\tau_{\geq i+1}\mathcal{X}$ and $\tau_{\leq i}\mathcal{X}$ are in C^c, so is $\mathcal{X}$. □

Lastly, we need the following lemma:

Lemma 3.1.10. *Let N be a (compact) manifold, $\mathcal{X}$ a π-finte spectrum. The mapping spectrum*

$$\mathcal{X}(M) := Maps(\Sigma^\infty_+ M, \mathcal{X}). \tag{3.1.11}$$

is a π-finite space.

Proof. Let $C \subset Sp$ be the full subcategory of π-finite spectra satisfying the following property: for any compact manifold with boundary M, $\mathcal{X}(M)$ is a π-finite spectrum. We claim that $C^c = C$, that is, it is closed under suspension and extensions. Suspension is easy as

$$(\Sigma\mathcal{X})(M) = \Sigma(\mathcal{X}(M)). \tag{3.1.12}$$

For extensions, given a fiber sequence

$$\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}, \tag{3.1.13}$$

then we have a corresponding fiber sequence

$$\mathcal{X}(M) \to \mathcal{Y}(M) \to \mathcal{Z}(M). \tag{3.1.14}$$

If $\mathcal{X}(M)$ and $\mathcal{Z}(M)$ are in Sp^{fin}, by proposition 3.1.8, so is $\mathcal{Y}(M)$.

By proposition 3.1.8, it is suffice to show that finite Eilenberg-MacLane spectra are in C. Let A be a finite abelian group and HA its Eilenberg-MacLane spectrum. The homotopy groups

$$\pi_{-i}\ HA(M) = H^i(M, A) \tag{3.1.15}$$

are the ordinary cohomology groups of M with A coefficients. These cohomology are concentrated in degree 0 to $dim\ M$. In addition, each cohomology group is finite as M is compact. Therefore $HA(M)$ is a π-finite spectrum. □

3.2 Poincaré duality

In this section, we define the relative cap product, introduce the notion of $\mathcal{R}$-orientation, and state the relative Poincaré duality theorem (Theorem 3.2.55). We will take the classical approach to orientation theory ([22]). See [1] for a more modern approach.

Let S, S_*, Sp be the categories of spaces, pointed spaces, and spectra, $\mathcal{S}$ the sphere spectrum. In appendix A.3, we reviewed the relationship between spectra and generalized (co)homology theories. Recall that spectra define generalized (reduced) homology and cohomology theories:

Definition 3.2.1. Let $\mathcal{X}$ be a spectrum and N a pointed topological space. The i-th reduced homology on N with coefficients in $\mathcal{X}$ is

$$\widetilde{\mathcal{X}}_i(N) \coloneqq \pi_i(\Sigma^\infty N \wedge \mathcal{X}). \tag{3.2.2}$$

The i-th reduced cohomology group of N with coefficients in $\mathcal{X}$ is

$$\widetilde{\mathcal{X}}^i(N) \coloneqq \pi_{-i}(Maps(\Sigma^\infty N, \mathcal{X})). \tag{3.2.3}$$

For unpointed spaces, we get nonreduced (co)homology theories:

Definition 3.2.4. Let $\mathcal{X}$ be a spectrum and N a (unpointed) topological space. The i-th homology on N with coefficients in $\mathcal{X}$ is

$$\mathcal{X}_i(N) \coloneqq \pi_i(\Sigma_+^\infty N \wedge \mathcal{X}). \tag{3.2.5}$$

The i-th cohomology group of N with coefficients in $\mathcal{X}$ is

$$\mathcal{X}^i(N) \coloneqq \pi_{-i}(Maps(\Sigma_+^\infty N, \mathcal{X})). \tag{3.2.6}$$

Let $N \to N' \to N''$ be a cofiber sequence in pointed spaces. Then we have a long exact sequence of homology groups:

$$\cdots \to \widetilde{\mathcal{X}}_*(N) \to \widetilde{\mathcal{X}}_*(N') \to \widetilde{\mathcal{X}}_*(N'') \to \widetilde{\mathcal{X}}_{*-1}(N) \to \cdots . \tag{3.2.7}$$

We also have long exact sequence of cohomology groups:

$$\cdots \to \widetilde{\mathcal{X}}^*(N'') \to \widetilde{\mathcal{X}}^*(N') \to \widetilde{\mathcal{X}}^*(N) \to \widetilde{\mathcal{X}}^{*+1}(N) \to \cdots . \tag{3.2.8}$$

Now we can define relative (co)homology groups:

Definition 3.2.9. Let $N \to N'$ be a map of (unpointed) topological spaces. Let N'' be their cofiber. It is canonically a pointed space. Then the relative homology of pair (N', N) with coefficients in spectrum $\mathcal{X}$ is

$$\mathcal{X}_*(N', N) \coloneqq \widetilde{\mathcal{X}}_i(N''). \tag{3.2.10}$$

Similarly, the relative cohomology of pair (N', N) with coefficients in spectrum $\mathcal{X}$ is

$$\mathcal{X}^*(N', N) \coloneqq \widetilde{\mathcal{X}}^i(N''). \tag{3.2.11}$$

As $N_+ \to N'_+ \to N''$ is a cofiber sequence in pointed spaces, by Equation 3.2.7, we have a long exact sequence of homology groups:

$$\cdots \to \mathcal{X}_*(N) \to \mathcal{X}_*(M) \to \mathcal{X}_*(M, N) \to \mathcal{X}_{*-1}(N) \to \cdots . \tag{3.2.12}$$

Similarly, by Equation 3.2.8, we have a long exact sequence of cohomology groups:

$$\cdots \to \mathcal{X}^*(N) \to \mathcal{X}^*(M) \to \mathcal{X}^*(M, N) \to \mathcal{X}^{*+1}(N) \to \cdots . \tag{3.2.13}$$

Remark 3.2.14. Let $i : N \hookrightarrow M$ be a “nice” inclusion (such as an inclusion of a boundary component of a manifold), then the cofiber of i is homotopy equivalent of M/N.

Now we will construct the cap product.

Construction 3.2.15. Let $\mathcal{R}$ be a $\mathbb{E}_1$-ring spectrum (see §A.2 for definition) and $\mathcal{X}$ a left $\mathcal{R}$-module spectrum. We have the action map

$$act : \mathcal{R} \wedge \mathcal{X} \to \mathcal{X}. \tag{3.2.16}$$

Let N, N', N'' be pointed spaces and

$$f : N \to N' \wedge N'' \tag{3.2.17}$$

be a map of pointed spaces. Given

$$\sigma : \mathcal{S} \to \Sigma^{-m}\mathcal{R} \wedge \Sigma^{\infty} N \tag{3.2.18}$$

a map that represents the homology class

$$[\sigma] \in \pi_0(\Sigma^{-m}\mathcal{R} \wedge \Sigma^{\infty} N) = \widetilde{\mathcal{R}}_m(N), \tag{3.2.19}$$

and

$$\alpha : \Sigma^{\infty} N' \to \Sigma^n \mathcal{X} \tag{3.2.20}$$

representing the cohomology class

$$[\alpha] \in \widetilde{\mathcal{X}}^n(N'). \tag{3.2.21}$$

Consider the following composition

$$\begin{aligned} \sigma \frown \alpha : \mathcal{S} &\xrightarrow{\sigma} \Sigma^{-m}\mathcal{R} \wedge \Sigma^{\infty} N && (3.2.22)\\ &\xrightarrow{id \wedge f} \Sigma^{-m}\mathcal{R} \wedge \Sigma^{\infty} N' \wedge \Sigma^{\infty} N'' && (3.2.23)\\ &\xrightarrow{id \wedge \alpha \wedge id} \Sigma^{-m}\mathcal{R} \wedge \Sigma^{n}\mathcal{X} \wedge \Sigma^{\infty} N'' && (3.2.24)\\ &\xrightarrow{act \wedge id} \Sigma^{-m+n}\mathcal{X} \wedge \Sigma^{\infty} N''. && (3.2.25) \end{aligned}$$

This represents a class in

$$[\sigma \frown \alpha] \in \pi_0(\Sigma^{-m+n}\mathcal{X} \wedge \Sigma^{\infty} N'') = \widetilde{\mathcal{X}}_{m-n}(N''). \tag{3.2.26}$$

This cohomology class does not depends on the representatives σ and α. Therefore we have a well-defined map:

$$- \frown - : \widetilde{\mathcal{R}}_n(N) \otimes \widetilde{\mathcal{X}}^n(N') \to \widetilde{\mathcal{X}}_{m-n}(N''). \tag{3.2.27}$$

This is called the cap product.

Now we move on to the notion of $\mathcal{R}$-orientation, when $\mathcal{R}$ is a $\mathbb{E}_1$-ring spectrum. First we need the following lemma:

Lemma 3.2.28. *Let M be a d-manifold and $x \in M^o = M - \partial M$ an interior point in M. We denote $M - x$ the complement of x in M. For any spectrum $\mathcal{X}$, $\mathcal{X}_*(M, M-x) \simeq \widetilde{\mathcal{X}}_*(S^d) \simeq \pi_{*-d}(\mathcal{X})$.*

Proof. We have to compute the cofiber of $M - x \hookrightarrow M$. As $M - x \subset M$ is not a "nice inclusion", we have to homotopic it to be one. This is local in x. Since x is in the interior of M, we can replace (M, x) with a local coordinate $(B^d,\ 0)$, where B^d is the d-dimensional ball and $0 \in B$ is the origin. We have

$$cofib\ (M - x \hookleftarrow M) \simeq cofib\ (B^d - x \hookleftarrow B^d) \tag{3.2.29}$$

$$\simeq B^d/(\partial B^d) \tag{3.2.30}$$

$$\simeq S^{d-1}. \tag{3.2.31}$$

Therefore

$$\mathcal{X}_*(M, M-x) \simeq \mathcal{X}_*(S^d) \tag{3.2.32}$$

$$\simeq \pi_{*-d}(\mathcal{X}). \tag{3.2.33}$$

□

Now we can define $\mathcal{R}$-orientation on manifolds (possibly with boundaries) [22, §5]:

Definition 3.2.34. Let M be a d-manifold, $\mathcal{R}$ a $\mathbb{E}_1$-ring spectrum. Note that $\pi_*\mathcal{R}$ inherits a graded ring structure.. An $\mathcal{R}$-orientation on M is a homology class

$$[M] \in \mathcal{R}_d(M, \partial M) \tag{3.2.35}$$

satisfying the following condition: for every interior point $x \in M^o$ a point in the interior, the image of $[M]$ under

$$\mathcal{R}_d(M, \partial M) \to \mathcal{R}_d(M, M-x) \simeq \pi_0(\mathcal{R}) \tag{3.2.36}$$

is an multiplicative unit in the ring $\pi_*(\mathcal{R})$. The isomorphism is by Lemma 3.2.28.

Example 3.2.37. Let $\mathcal{R}$ be $H\mathbb{Z}/2\mathbb{Z}$. Every manifold is $H\mathbb{Z}/2\mathbb{Z}$-oriented. Let $\mathcal{R}$ be $H\mathbb{Z}$, then $H\mathbb{Z}$-orientation is the usual notion of orientation for manifolds. Let $\mathcal{R}$ be $\mathcal{S}$ the sphere spectrum, then a $\mathcal{S}$-orientation on N is a trivialization of the Thom spectra of the normal bundle of N.

Remark 3.2.38. If N is a closed d-manifold. Then an $\mathcal{R}$-orientation lives in $\mathcal{R}_d(N)$.

Remark 3.2.39. Given a ring homomorphism of ring spectrum $f : \mathcal{R} \to \mathcal{R}'$, then an $\mathcal{R}$-orientation gives a $\mathcal{R}'$-orientation via the pushforward map:

$$f_* : \mathcal{R}'(-) \to \mathcal{R}'(-). \tag{3.2.40}$$

An $\mathcal{R}$-orientation on a d-dimensional manifold gives a $\mathcal{R}$-orientation on the boundary:

Proposition 3.2.41. *Let M be a d manifold. A $\mathcal{R}$-orientation on M, $[M] \in \mathcal{R}_d(M, \partial M)$, gives a class $\partial[M] \in \mathcal{R}_{d-1}(N)$ via the natural boundary map*

$$\partial : \mathcal{R}_*(M, \partial M) \to \mathcal{R}_{*-1}(\partial M). \tag{3.2.42}$$

The class $\partial[M] \in \mathcal{R}_{*-1}(\partial M)$ is a $\mathcal{R}$-orientation on the boundary ∂M.

Proof. A proof of this is given in [20, §21.3]. □

With the notion of $\mathcal{R}$-orientation and cap product, we can define the Poincaré isomorphism map:

Let $\mathcal{R}$ be a $\mathbb{E}_1$-ring spectrum and $\mathcal{X}$ a left $\mathcal{R}$-module spectrum. Let N be a $\mathcal{R}$-oriented d-manifold. We denote the orientation class as $[N] \in \mathcal{R}_{d-1}(N)$. Let f be the diagonal map:

$$N_+ \to N_+ \wedge N_+ \simeq (N \times N)_+. \tag{3.2.43}$$

Then the cap product (see Equation 3.2.27) with $[N]$ gives maps

$$\int_{[N]} : \mathcal{X}^*(N) \to \mathcal{X}_{d-*}(N). \tag{3.2.44}$$

Here's the Poincaré duality theorem for manifold without boundary:

Theorem 3.2.45 (Poincaré duality)**.** *For every $*$, the map*

$$\int_{[N]} : \mathcal{X}^*(N) \to \mathcal{X}_{d-*}(N) \tag{3.2.46}$$

is an isomrphism.

Proof. See [22, §V.2]. □

We need a more general form Poincaré duality for manifolds with boundaries. Let M be a $\mathcal{R}$-oriented d-manifold with boundary $\partial M = N \sqcup N'$. We have the orientation class $[M] \in \mathcal{R}_d(M, \partial M)$. Consider the map

$$M/\partial M \to M/N \wedge M/N' \tag{3.2.47}$$

of pointed spaces. From Equation 3.2.27 we get a map

$$[M] \frown - : \mathcal{X}^*(M, N) \to \mathcal{X}_{d-*}(M, N'). \tag{3.2.48}$$

We denote this map by $\int_{[M,N]}$.

Theorem 3.2.49 (Poincaré duality)**.** *The maps*

$$\int_{[M,N]} : \mathcal{X}^*(M, N) \to \mathcal{X}_{d-*}(M, N') \tag{3.2.50}$$

are isomorphisms.

Proof. The general case for ordinary homology theory is given in [**Lef**]. $\square$

There are two special examples. Let $N = \partial M$ and $N' = \varnothing$. We have the following corollary:

Corollary 3.2.51. *The maps*

$$\int_{[M,\partial M]} : \mathcal{X}^*(M, \partial M) \to \mathcal{X}_{d-*}(M) \tag{3.2.52}$$

are isomorphisms.

Similarly, let $N = \varnothing$, and $N = \partial M$. We have:

Corollary 3.2.53. *The maps*

$$\int_{[M]} : \mathcal{X}^*(M) \to \mathcal{X}_{d-*}(M, \partial M) \tag{3.2.54}$$

are isomorphisms.

Lastly, we need to know the functoriality of the Poincaré duality isomorphisms:

Theorem 3.2.55. *Let $\mathcal{R}$ be a $\mathbb{E}_1$-ring spectrum and $\mathcal{X}$ a left $\mathcal{R}$-module spectrum. Let M be a $\mathcal{R}$-oriented d-manifold with boundary $\partial M = N \sqcup N'$. We denote the orientation class as $[M]$. It gives orientations $[N], [N']$ on the boundaries. Poincaré duality isomorphism maps give an equivalence of long exact sequences:*

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^*(M, N') \arrow[r] \arrow[d, "\int_{[M,N']}"] & \hat{\mathcal{X}}^*(M) \arrow[r] \arrow[d, "\int_{[M]}"] & \hat{\mathcal{X}}^*(N') \arrow[r] \arrow[d, "\int_{[N']}"] & \cdots \arrow[d] \\ \cdots \arrow[r] & \hat{\mathcal{X}}_{d-*}(M, N) \arrow[r] & \hat{\mathcal{X}}_{d-*}(M, \partial M) \arrow[r] & \hat{\mathcal{X}}_{d-1-*}(N') \arrow[r] & \cdots \end{tikzcd} \tag{3.2.56}$$

Proof. For orindary cohomology theories this is proven in [20, §21.4]. □

3.3 Pontryagin duality

In this section we review Pontryagin duality for finite abelian groups. We will write the group multiplication additively. We denote the category of abelian group as Ab, and the full subcategory of finite abelian group as Ab^{fin}. Note that given $A, B \in Ab$, the set of homomorphism from A to B, $Hom(A, B)$ has a group structure by point-wise multiplication (Ab has internal hom). Let $\mathbb{C}^\times$ denote the multiplicative group of nonzero complex number. This is an injective object in Ab.

Definition 3.3.1. Let A be an abelian group. The Pontryagin dual group $\hat{A}$ is defined to be $Hom(A, \mathbb{C}^\times)$.

Note that taking Pontryagin dual gives an exact contravariant functor

$$D := Hom(-, \mathbb{C}^\times) : Ab \to Ab^{op}. \tag{3.3.2}$$

It is exact as $\mathbb{C}^\times$ is an injective object.

Remark 3.3.3. Normally, the Pontryagin dual of A is defined as $Hom(A, \mathbb{Q}/\mathbb{Z})$. For a finite abelian group A, the natural map

$$Hom(A, \mathbb{Q}/\mathbb{Z}) \to Hom(A, \mathbb{C}^\times) \tag{3.3.4}$$

is an isomorphism. Thus the two notions coincide. We choose $\mathbb{C}^\times$ over $\mathbb{Q}/\mathbb{Z}$ as our TFTs are complex-valued.

Example 3.3.5. Let $A = \mathbb{Z}$, then $\hat{A} = Hom(A, \mathbb{C}^\times) = \mathbb{C}^\times$.

Example 3.3.6. Let $A = \mathbb{Z}/n\mathbb{Z}$, then $\hat{A} = \mu_n \subset \mathbb{C}^\times$ is the subgroup of n-th root of unity.

Lemma 3.3.7. *Let $A, B \in Ab$, then $\widehat{A \times B} \simeq \hat{A} \times \hat{B}$.*

Proof. Note that product and coproduct coincide in Ab. Thus

$$\begin{aligned} \widehat{A \times B} &= Hom(A \times B, \mathbb{C}^\times) && (3.3.8) \\ &\simeq Hom(A, \mathbb{C}^\times) \times Hom(B, \mathbb{C}^\times) && (3.3.9) \\ &= \hat{A} \times \hat{B}. && (3.3.10) \end{aligned}$$

□

Note that there is a canonical bilinear pairing

$$(-,-)_A : A \times \hat{A} \to \mathbb{C}^\times. \tag{3.3.11}$$

Bilinear means that $(aa', \alpha)_A = (a, \alpha)_A (a', \alpha)_A$. It might looks strange for bilinearity because we write the group operation multiplicatively rather than additively, which is more common when we view abelian group as $\mathbb{Z}$ modules. It is equivalent to a group homomorphism $A \otimes \hat{A} \to \mathbb{C}^\times$.

This pairing gives a universal characterization for the Pontryagin dual:

Definition 3.3.12. Let $A, B \in Ab$ be two abelian groups. A pairing is a bilinear map $\mu : A \times B \to \mathbb{C}^\times$. This induce a map $\phi_\mu : B \to \hat{A}$, given by $\phi_\mu(b)(a) := \mu(a, b)$. The pairing μ says to exhibit B as the Pontryagin dual of A if ϕ_μ is an isomorphism.

Example 3.3.13. By above, $B = \hat{A}$, $\mu = (-,-)_A$ exhibits $\hat{A}$ as the Pontryagin dual of A, and $\phi_\mu : \hat{A} \to \hat{A}$ is the identity map. This is the universal example.

Given A, B and $\mu : A \times B \to \mathbb{C}^\times$ a bilinear pairing. Let $\mathbb{C}[A]$ be the free vector space generated by (the set) A. Similarly for $\mathbb{C}[B]$. The pairing μ give rise to a bilinear pairing $\alpha_\mu : \mathbb{C}[A] \otimes \mathbb{C}[B] \to \mathbb{C}$. On basis vectors it sends $a \otimes b \to \mu(a, b)$, where we identity an element $a \in A$ with the standard basis in $\mathbb{C}[A]$.

Now we restrict to finite abelian groups.

Proposition 3.3.14. *If A is a finite abelian group. Then $\hat{A}$ is also an finite abelian group. In addition, $|A| = |\hat{A}|$.*

Proof. By the classification of finite abelian group, we know that A is a product of $\mathbb{Z}/n\mathbb{Z}$. Note that for a single $\mathbb{Z}/n\mathbb{Z}$, its dual is μ_n, which is of the same size. For a product of $\mathbb{Z}/n\mathbb{Z}$, proposition 3.3.7 implies the result. □

Taking Pontryagin dual restricts to a functor from Ab^{fin} to $(Ab^{fin})^{op}$. We denote this functor by

$$D : Ab^{fin} \to (Ab^{fin})^{op}. \tag{3.3.15}$$

Remark 3.3.16. In fact, for A finite, $\hat{A}$ is noncanonically isomorphic to A. But it is best to view them as different groups. This is similar to the case of duals of a finite dimensional vector space.

Recall from above that A, B and $\mu : A \times B \to \mathbb{C}^\times$ a bilinear pairing gives a bilinear map $\alpha_\mu : \mathbb{C}[A] \otimes \mathbb{C}[B] \to \mathbb{C}$. Now assume A and B finite. Recall that a pairing $\alpha : V \otimes V' \to \mathbb{C}^\times$ of finite dimensional vector spaces is called nondegenerate if the natural map $V' \to V^* = Hom_{\mathbb{C}}(V^*, \mathbb{C})$ is an isomorphism.

Proposition 3.3.17. *μ exhibits B as the Pontryagin dual of A iff*

$$\alpha_\mu : \mathbb{C}[A] \otimes \mathbb{C}[B] \to \mathbb{C} \tag{3.3.18}$$

is a nondegenerate pairing of finite dimensional vector spaces.

Proof. By Proposition 3.3.14, $dim(\mathbb{C}[A]) = |A| = |B| = dim(\mathbb{C}[B])$. Thus it is suffice to that the map $\mathbb{C}[B] \to \mathbb{C}[A]^*$ is surjective. Recall that we view $a_i \in A$ and $b_j \in B$ as basis elements for $\mathbb{C}[A]$ and $\mathbb{C}[B]$. Let a^i be the dual basis for $\mathbb{C}[A]^*$. It is suffice to see that image of the map

$$\mathbb{C}[B] \to \mathbb{C}[A]^* \tag{3.3.19}$$

includes basis vectors a^i. Equivalently, exists vectors $v_i \in \mathbb{C}[B]$ so that

$$\alpha_\mu(a_{i'}, v_i) = \delta_{i,i'}. \tag{3.3.20}$$

We first do this for $a_i = e$ the identity element. We take

$$v_e = \frac{1}{|B|} \sum_j b_j. \tag{3.3.21}$$

Then

$$\alpha_\mu(a_i, v_e) = \frac{1}{|B|} \sum_j \mu(a_i, b_j). \tag{3.3.22}$$

For $a_i \neq e$, we have

$$\sum_j \mu(a_i, b_j) = 0 \tag{3.3.23}$$

as we sum over values of all the character at a non-identity element a_i. For $a_i = e$, then

$$\alpha_\mu(e, v_e) = \frac{1}{|B|} \sum_j \mu(e, b_j) \tag{3.3.24}$$
$$= \frac{1}{|B|} \sum_j 1 \tag{3.3.25}$$
$$= 1. \tag{3.3.26}$$

Thus v_e maps to a^e to the dual basis vector of e. Now for a general a_i, then we let

$$v_i = \frac{1}{|B|} \sum_j \mu(a_i^{-1}, b_j) \cdot b_j. \tag{3.3.27}$$

Same calculation shows that v_i maps to a^i.

For the converse, note that if the pairing is nondegenerate, then

$$\mathbb{C}[B] \simeq \mathbb{C}[A]^* \simeq \mathbb{C}[\hat{A}]. \tag{3.3.28}$$

The composition $\mathbb{C}[B] \xrightarrow{\sim} \mathbb{C}[\hat{A}]$ is induce by the map $\phi_\mu : B \to \hat{A}$. As the map of vector spaces is an isomorphism, so is the map of groups. $\square$

Corollary 3.3.29. *μ exhibits B as the Pontryagin dual of A iff the map*

$$\begin{aligned} \mathbb{C}[A] &\to \mathbb{C}[B] \\ a &\mapsto \sum_b \mu(a,b)\ b \end{aligned} \tag{3.3.30}$$

is an isomorphism.

Lastly, note that we have a natural transformation $id \to D^2$ of functors $Ab \to Ab$. This natural transformation is given as such: let A be an abelian group, then we have

$$\begin{aligned} A &\to \hat{\hat{A}} \\ a &\mapsto (\alpha \mapsto \alpha(a)). \end{aligned} \tag{3.3.31}$$

Theorem 3.3.32. *Restricted to Ab^{fin}, this natural transformation is an isomorphism. Thus $D^2 \simeq id$.*

Proof. I claim the pairing $(-,-)_A : \hat{A} \times A \to \mathbb{C}^\times$ exhibits A as the Pontryagin dual of $\hat{A}$. By Proposition 3.3.17, we see that this is equivalent to $\alpha_\mu : \mathbb{C}[A] \otimes \mathbb{C}[\hat{A}] \to \mathbb{C}$ being a nondegenerate pairing. Note that for finite dimensional vector spaces, the nondegeneracy condition is symmetric in its variables. Thus A is the Pontryagin dual of $\hat{A}$. $\square$

As a corollary, we get that Pontryagin duality is in fact a duality on finite abelian groups:

Corollary 3.3.33. *$\hat{D} : Ab^{fin} \to (Ab^{fin})^{op}$ is an equivalence of categories.*

3.4 Brown-Comenetz duality

In this subsection we review Brown-Comenetz duality for spectra. This duality is originally found in [4].

Let Sp be the category of spectra and Ab be the category of abelian groups. They have internal homs $Maps$ and Hom respectively. First we need the following proposition:

Proposition 3.4.1. *Let K be an injective abelian group. Then there exists an essentially unique spectrum IK with the following property: for any spectra $\mathcal{X}$, there is a functorial equivalence*

$$\pi_{-*}(Maps(\mathcal{X}, I\mathbb{C}^\times)) \simeq Hom(\pi_*(\mathcal{X}), K). \tag{3.4.2}$$

More precisely, we view both sides as familes of functors $Sp \to Ab$, and there is an natural isomorphism between these two families of functors.

Proof. We want to apply Brown Representability theorem ([17] Theorem 1.4.1.2) applied to the functor

$$F : Sp^{op} \xrightarrow{\pi_0} Ab^{op} \xrightarrow{Hom(-,K)} Ab \xrightarrow{fgt} Set. \tag{3.4.3}$$

This takes a spectrum $\mathcal{X}$ to $F(\mathcal{X}) := Hom(\pi_0(\mathcal{X}), K)$. We need to show that this functor satisfies the following conditions:

1. For every collection of spectra $\mathcal{X}_\beta$, the map $F(\coprod_\beta \mathcal{X}_\beta) \to \prod_\beta F(\mathcal{X}_\beta)$ is a bijection.

2. For every pushout square

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathcal{Z} \arrow[r] & \mathcal{W}, \end{tikzcd} \tag{3.4.4}$$

in Sp, the induced map $F(\mathcal{W}) \to F(\mathcal{Y}) \times_{F(\mathcal{Z})} F(\mathcal{Z})$ is surjective.

We will first proof (1): π_0 takes coproducts in spectra to coproducts (direct sums) in Ab. $Hom(-,K)$ takes colimits to limits, and the forgetful functor preserves limits. Thus F takes coproducts to products.

Now for (2): given a pushout square 3.4.4, as Sp is stable, this is also a pullback diagram. Thus we have a Mayer-Vietoris long exact sequence:

$$\cdots \pi_0\mathcal{X} \to \pi_0\mathcal{Y} \oplus \pi_0\mathcal{Z} \to \pi_0\mathcal{W} \cdots \tag{3.4.5}$$

As this sequence is exactly, this implies that the natural map

$$\pi_0\mathcal{Y} \oplus_{\pi_0\mathcal{X}} \pi_0\mathcal{W} \to \pi_0\mathcal{Z} \tag{3.4.6}$$

is injective.

As K is injective, $Hom(-,K)$ takes an injective map to a surjective map (this is where the injectivity of K is needed). In addition, the forgetful functor preserves surjectivity. We see that $F(\mathcal{W}) \to F(\mathcal{Y}) \times_{F(\mathcal{Z})} F(\mathcal{Z})$ is surjective. □

Example 3.4.7. As $\mathbb{C}^\times$ is an injective abelian group, we get a spectrum $I\mathbb{C}^\times$, with the functorial equivalence

$$\pi_{-*}(Maps(\mathcal{X}, I\mathbb{C}^\times)) \simeq \widehat{\pi_*(\mathcal{X})}. \tag{3.4.8}$$

Recall that $\hat{(-)}$ is taking the Pontryagin dual.

We define the Pontryagin dual group as follows:

Definition 3.4.9. Let $\mathcal{X}$ be a spectrum. The Pontryagin dual spectrum $\hat{\mathcal{X}}$ is defined to be the mapping spectrum $Maps(\mathcal{X}, I\mathbb{C}^\times)$.

Note that this defines a functor

$$\mathbb{D} := Maps(-, I\mathbb{C}^\times) : Sp \to Sp^{op}. \tag{3.4.10}$$

Example 3.4.11. Let $\mathcal{X}$ be the sphere spectrum $\mathcal{S}$. Then

$$\hat{\mathcal{S}} = Maps(\mathcal{S}, I\mathbb{C}^\times) \simeq I\mathbb{C}^\times. \tag{3.4.12}$$

Thus $I\mathbb{C}^\times$ is the Pontryagin dual of the sphere spectrum $\mathcal{S}$. This is similar to the fact that $\mathbb{C}^\times$ is the Pontryagin dual group of $\mathbb{Z}$.

Remark 3.4.13. The common approach to Brown-Comenetz uses $I\mathbb{Q}/\mathbb{Z}$ rather than $I\mathbb{C}^\times$. As with the abelian group case (see remark 3.3.3), they give the same answers on π-finite spectra. Once again, we use $I\mathbb{C}^\times$ over $I\mathbb{Q}/\mathbb{Z}$ because the target of our TFTs are complex-valued.

We can calculate the homotopy group of $I\mathbb{C}^\times$. Let $\mathcal{X}$ be the sphere spectrum $\mathcal{S}$, then by proposition 3.4.1, we have

$$\pi_{-i}(Maps(\hat{\mathcal{S}}, I\mathbb{C}^\times)) = \pi_{-i}I\mathbb{C}^\times \simeq \widehat{\pi_i\mathcal{S}}. \tag{3.4.14}$$

Thus we see that

$$\pi_0 I\mathbb{C}^\times = \widehat{\pi_0\mathcal{S}} = Hom(\mathbb{Z}, \mathbb{C}^\times) = \mathbb{C}^\times. \tag{3.4.15}$$

The negative homotopy groups

$$\pi_{-i}I\mathbb{C}^\times = \widehat{\pi_i\mathcal{S}} \tag{3.4.16}$$

are finite abelian groups (non-canonically isomorphic the i-th homotopy group of spheres), and the positive homotopy groups are trivial. We see that $I\mathbb{C}^\times$ is a co-connective spectra, and there is a canonical map

$$H\mathbb{C}^\times \simeq \tau_{\geq 0} I\mathbb{C}^\times \to I\mathbb{C}^\times, \tag{3.4.17}$$

where $H\mathbb{C}^\times$ is the Eilenberg-MacLane spectrum corresponding to $\mathbb{C}^\times$.

We also have the canonical pairing

$$ev_{\mathcal{X}} : \mathcal{X} \wedge \hat{\mathcal{X}} \to I\mathbb{C}^\times \tag{3.4.18}$$

We use this to give a universal characterization of $\hat{\mathcal{X}}$:

Construction 3.4.19. Let $\mathcal{X}$, $\mathcal{Y}$ be two spectra with a pairing

$$\mu : \mathcal{X} \wedge \mathcal{Y} \to I\mathbb{C}^\times. \tag{3.4.20}$$

Let $\mathcal{Z}$ be another spectra. we denote

$$\mathcal{X}^*(\mathcal{Z}) \coloneqq \pi_{-*}Maps(\mathcal{Z}, \mathcal{X}). \tag{3.4.21}$$

Similarly, we denote

$$\mathcal{X}_*(\mathcal{Z}) \coloneqq \pi_*(\mathcal{Z} \wedge \mathcal{X}). \tag{3.4.22}$$

Now consider the composition:

$$\begin{aligned} \mathcal{Z} \wedge \mathcal{X} \wedge Maps(\mathcal{Z}, \mathcal{Y}) &\xrightarrow{\sim} \mathcal{X} \wedge (\mathcal{Z} \wedge Maps(\mathcal{Z}, \mathcal{Y})) && (3.4.23) \\ &\xrightarrow{ev_{\mathcal{Z}}} \mathcal{X} \wedge \mathcal{Y} && (3.4.24) \\ &\xrightarrow{\mu} I\mathbb{C}^\times && (3.4.25) \end{aligned}$$

In general, there is a map

$$\pi_*(\mathcal{Z}_1) \otimes \pi_{-*}(\mathcal{Z}_2) \to \pi_0(\mathcal{Z}_1 \wedge \mathcal{Z}_2). \tag{3.4.26}$$

Apply to our case, we get

$$\begin{aligned}
\mathcal{X}_*(\mathcal{Z}) \otimes \mathcal{Y}^*(\mathcal{Z}) &= \pi_*(\mathcal{Z} \wedge \mathcal{X}) \otimes \pi_{-*} Maps(\mathcal{Z}, \mathcal{Y}) && (3.4.27)\\
&\to \pi_0(\mathcal{Z} \wedge \mathcal{X} \wedge Maps(\mathcal{Z}, \mathcal{Y})) && (3.4.28)\\
&\to \pi_0(I\mathbb{C}^\times) && (3.4.29)\\
&= \mathbb{C}^\times. && (3.4.30)
\end{aligned}$$

Thus there is a natural transformation $\phi_\mu(-) : \mathcal{Y}^*(-) \to \widehat{\mathcal{X}_*(-)}$. Where $\mathcal{Y}^*(-)$, $\widehat{\mathcal{X}_*(-)}$ are viewed as functors $Sp \to Ab^{op}$. Note that $\widehat{\mathcal{X}_*(-)}$ is the composition $\mathcal{X}_*(-) : Sp \to Ab$, and $D : Ab \to Ab^{op}$ the Pontryagin dual group functor.

Definition 3.4.31. The pairing $\mu : \mathcal{X} \wedge \mathcal{Y} \to I\mathbb{C}^\times$ exhibits $\mathcal{Y}$ as the Pontryagin dual of $\mathcal{X}$ if the natural pairing $\mathcal{X}_*(\mathcal{Z}) \otimes \mathcal{Y}^*(\mathcal{Z}) \to \mathbb{C}^\times$ exhibits $\mathcal{Y}^*(\mathcal{Z})$ as the Pontryagin dual group of $\mathcal{X}_*(\mathcal{Z})$ for every $\mathcal{Z}$. Alternatively, the map $\phi_\mu(-) : \mathcal{Y}^*(-) \to \widehat{\mathcal{X}_*(-)}$ is an isomorphism for every $\mathcal{Z} \in Sp$.

This gives an universal characterization of the Pontryagin dual spectrum:

Proposition 3.4.32. *Let $\mathcal{Y} = \hat{\mathcal{X}}$, and $\mu = ev_{\mathcal{X}} : \mathcal{X} \wedge \hat{\mathcal{X}} \to I\mathbb{C}^\times$. Then μ exhibits $\hat{\mathcal{X}}$ as the Pontryagin dual of $\mathcal{X}$.*

Proof. Let $\mathcal{Z} \in Sp$ be a spectra. We have

$$\begin{aligned}
Maps(\mathcal{Z}, \hat{\mathcal{X}}) &= Maps(\mathcal{Z}, Maps(\mathcal{X}, I\mathbb{C}^\times)) && (3.4.33)\\
&\xrightarrow{\sim} Maps(\mathcal{Z} \wedge \mathcal{X}, I\mathbb{C}^\times). && (3.4.34)
\end{aligned}$$

Thus we have

$$\begin{aligned}
\hat{\mathcal{X}}^*(\mathcal{Z}) &\xrightarrow{\sim} \pi_{-*} Maps(\mathcal{Z}, \hat{\mathcal{X}})\\
&\xrightarrow{\sim} \pi_{-*} Maps(\mathcal{Z} \wedge \mathcal{X}, I\mathbb{C}^\times)\\
&\xrightarrow{\sim} \pi_* \widehat{\mathcal{Z} \wedge \mathcal{X}}\\
&= \widehat{\mathcal{X}_*(\mathcal{Z})}
\end{aligned} \tag{3.4.35}$$

We used the proposition 3.4.1 in the last arrow. □

Remark 3.4.36. Note that this proof essentially only uses the corresponding property of $I\mathbb{C}^\times$ (proposition 3.4.1). This is a theme in duality theorems.

Corollary 3.4.37. $\pi_i(\hat{\mathcal{X}}) \simeq \widehat{\pi_{-i}\mathcal{X}}$.

Proof. Let $\mathcal{Z}$ be the sphere spectrum $\mathcal{S}$, then formula 3.4.35 above gives the equivalence. $\square$

Let N be a CW complex and $\mathcal{X}$ a spectrum. Recall that we have the nonreduced homology $\mathcal{X}_*$ and cohomology $\mathcal{X}^*$ associated to X (3.2):

$$\mathcal{X}_*(N) := \mathcal{X}_*(\Sigma^\infty_+ N), \quad \mathcal{X}^*(N) := \mathcal{X}^*(\Sigma^\infty_+ N), \tag{3.4.38}$$

where $\Sigma^\infty_+ N$ is the suspension spectrum associated to N.

We have the following corollary:

Corollary 3.4.39. *When a pairing $\mu : \mathcal{X} \wedge \mathcal{Y} \to I\mathbb{C}^\times$ exhibits $\mathcal{Y}$ as the Pontryagin dual of $\mathcal{X}$, we have an equivalence of cohomology theory*

$$\hat{\mathcal{Y}}^*(-) \to \widehat{\mathcal{X}_*(-)}. \tag{3.4.40}$$

Thus given a cofiber sequence $N \to M \to (M, N)$, we have the long exact sequence of homology groups:

$$\cdots \to \mathcal{X}_*(N) \to \mathcal{X}_*(M) \to \mathcal{X}_*(M, N) \to \cdots, \tag{3.4.41}$$

then its Pontryagin dual long exact sequence (apply $\hat{D}$ termwise) give the long exact sequence of cohomology groups:

$$\cdots \leftarrow \hat{\mathcal{Y}}^*(N) \leftarrow \hat{\mathcal{Y}}^*(M) \leftarrow \hat{\mathcal{Y}}^*(M, N) \leftarrow \cdots. \tag{3.4.42}$$

Example 3.4.43. Let A be an abelian group, HA the Eilenberg MacLane spectrum. Then the Pontryagin dual spectrum $\widehat{HA}$ has homotopy groups concentrated in degree 0, and $\pi_0(\widehat{HA}) \simeq \hat{A}$. Thus we see that $\widehat{HA} \simeq H\hat{A}$. For generally, the Pontryagin dual of $\Sigma^n HA$ is $\Sigma^{-n} H\hat{A}$.

The Pontryagin dual operation $\mathbb{D} := Maps(-, I\mathbb{C}^\times)$ is functorial, it defines an exact contravariant functor $\mathbb{D} : Sp \to Sp^{op}$. Recall that we have the Pontryagin dual map $\hat{D} : Ab \to Ab$, and the embedding of Eilenberg-MacLane spectrums $H : Ab \to Sp$. By the example 3.4.43 above, we see that we have a commutative diagram of functors:

$$\begin{tikzcd} Ab \arrow[r, "D"] \arrow[d, "H"] & Ab^{op} \arrow[d, "H^{op}"] \\ Sp \arrow[r, "\mathbb{D}"] & Sp^{op} \end{tikzcd} \tag{3.4.44}$$

Now we turn to π-finite spectra. Let $Sp^{fin} \subset Sp$ be the full subcategory of π-finite spectra and $Ab^{fin} \subset Ab$ the full subcategory of finite abelian groups. Notice that Eilenberg-MacLane functor restricts to a functor $H : Ab^{fin} \to Sp^{fin}$. Note by corollary 3.4.37, Pontryagin duality functor restricts to a functor $\mathbb{D} : Sp^{fin} \to (Sp^{fin})^{op}$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} Ab^{fin} & \xrightarrow{D} & (Ab^{fin})^{op} \\ \Big\downarrow{\scriptstyle H} & & \Big\downarrow{\scriptstyle H^{op}} \\ Sp^{fin} & \xrightarrow{\mathbb{D}} & (Sp^{fin})^{op} \end{array} \tag{3.4.45}$$

Recall that D is a duality for finite abelian groups: $D^2 \simeq id$ (theorem 3.3.32). We will show the same for π-finite spectra. There is a natural transformation $id \to \mathbb{D}^2$ between the identity functor and the double dual functor on Sp, given by

$$\begin{aligned} &\mathcal{X} \to Maps(Maps(\mathcal{X}, I\mathbb{C}^\times), I\mathbb{C}^\times) = \hat{\hat{\mathcal{X}}} \\ &x \mapsto (\alpha \mapsto \alpha(a)). \end{aligned} \tag{3.4.46}$$

Restricts to π-finite spectra, we have the following:

Theorem 3.4.47. *For π-finite spectrum $\mathcal{X}$, the natural map 3.4.46 is an isomorphism. Therefore, restricted to Sp^{fin}, we have $\hat{\mathbb{D}}^2 \simeq id$.*

Proof. Recall π-finite spectra is generated finite Eilenberg-MacLane spectra HA under extensions (Proposition 3.3.14). By Theorem 3.3.32, we know that in our theorem is true when $\mathcal{X} = HA$. So it is suffice to show that

1. if $\mathcal{X} \simeq \mathbb{D}^2(\mathcal{X}) = \hat{\hat{\mathcal{X}}}$, then so are $\Sigma^n \mathcal{X}$.
2. if our theorem holds for $\mathcal{X}$ and $\mathcal{X}''$, and we have a fiber sequence $\mathcal{X} \to \mathcal{X}' \to \mathcal{X}''$, then it is also true for $\mathcal{X}'$.

For (1), this follows from the observation $\widehat{\Sigma^n \mathcal{X}} \simeq \Sigma^{-n} \hat{\mathcal{X}}$. For (2), there is map of long exact sequences of homotopy groups:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_* \mathcal{X}' & \longrightarrow & \pi_* \mathcal{X} & \longrightarrow & \pi_* \mathcal{X}'' & \longrightarrow & \cdots \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_{\mathcal{X}'}} & & \Big\downarrow{\scriptstyle \alpha_{\mathcal{X}}} & & \Big\downarrow{\scriptstyle \alpha_{\mathcal{X}''}} & & \Big\downarrow \\ \cdots & \longrightarrow & \pi_* \hat{\hat{\mathcal{X}}}' & \longrightarrow & \pi_* \hat{\hat{\mathcal{X}}} & \longrightarrow & \pi_* \hat{\hat{\mathcal{X}}}'' & \longrightarrow & \cdots \end{array} \tag{3.4.48}$$

If $\alpha_{\mathcal{X}'}$ and $\alpha_{\mathcal{X}''}$ are isomorphisms, then the 2-out-of-3 lemma implies that so is $\alpha_{\mathcal{X}}$. Recall a map of spectra is an equivalence if all the induced maps on homotopy groups are isomorphisms. ☐

Theorem 3.4.47 shows that Brown-Comenetz dual is a duality on π-finite spectra:

Corollary 3.4.49. *$\hat{\mathcal{D}} : Sp^{fin} \to (Sp^{fin})^{op}$ is an equivalence of categories.*

Lastly, we get the following corollary, which we need in our main theorem:

Corollary 3.4.50. *Let $\mathcal{X}$ be a π-finite spectra and $N \to M \to M/N$ a cofiber sequence of finite CW complexes. The Pontryagin dual of the long exact sequence of cohomology group with $\mathcal{X}$ coefficients:*

$$\cdots \to \mathcal{X}^*(M, N) \to \mathcal{X}^*(M) \to \mathcal{X}^*(N) \to \cdots \tag{3.4.51}$$

is canonically isomorphic to the long exact sequence of homology group with $\hat{\mathcal{X}}$ coefficients:

$$\cdots \leftarrow \hat{\mathcal{X}}_*(M, N) \leftarrow \hat{\mathcal{X}}_*(M) \leftarrow \hat{\mathcal{X}}_*(N) \leftarrow \cdots \tag{3.4.52}$$

Proof. As $\hat{\mathcal{X}}$ is a π-finite spectra, by theorem 3.4.47 the Pontryagin dual of $\hat{\mathcal{X}}$ can be identified with $\mathcal{X}$. Apply Corollary 3.4.39 above the Pontryagin dual pair $(\hat{\mathcal{X}}, \mathcal{X})$, we have the Pontryagin dual of the long exact sequence

$$\cdots \leftarrow \hat{\mathcal{X}}_*(M, N) \leftarrow \hat{\mathcal{X}}_*(M) \leftarrow \hat{\mathcal{X}}_*(N) \leftarrow \cdots \tag{3.4.53}$$

is

$$\cdots \to \mathcal{X}^*(M, N) \to \mathcal{X}^*(M) \to \mathcal{X}^*(N) \to \cdots \tag{3.4.54}$$

As $M, N, M/N$ are finite CW complexes, all homology and cohomology groups above are finite. As Pontryagin duality is a duality on finite abelian groups (theorem 3.3.32), we see that the Pontryagin dual of long exact sequence 3.4.54 is canonically isomorphic to long exact sequence 3.4.53. $\square$

4 Euler characteristic and TFT

In §4.1 we define the Euler characteristic of a manifold, and show some basic properties. In §4.2 we define the Euler TFT (in any dimension). Lastly, we show that the Euler TFT is trivial in odd dimensions.

4.1 Euler characteristic

In this section, we collect some facts about the Euler characteristics of compact manifolds and the sizes (homotopy cardinality) of π-finite spectra. We first start with the Euler characteristic of a finite graded vector space:

Definition 4.1.1. Let k be a field and $H^\bullet = \bigoplus H^i$ be a $\mathbb{Z}$-graded k-vector space. $H^\bullet$ is called finite if all but finitely many $H^i = 0$ and each H^i is finite dimensional. The Euler character $\chi(H^\bullet)$ of a finite graded vector space $H^\bullet$ is

$$\chi(H^\bullet) \coloneqq \sum_i (-1)^i dim_k H_i. \tag{4.1.2}$$

Here's a similiar notion for finite graded abelian groups:

Definition 4.1.3. Let $A^\bullet = \bigoplus A^i$ be a $\mathbb{Z}$-graded abelian groups. $A^\bullet$ is called finite if all but finitely many $A^i = 0$ and each A^i is finite. The size of $A^\bullet$ is

$$|A^\bullet| \coloneqq \prod_i |A^i|^{(-1)^i}, \tag{4.1.4}$$

where $|A^i|$ is the cardinality of A^i.

In this section, all $H^\bullet$ and $A^\bullet$ will satisfy the finiteness assumption above, and we will implicitly assume this condition throughout the section.

Remark 4.1.5. If $H^\bullet = \bigoplus H^i$ is a finite graded $\mathbb{F}_q$ vector space, where $\mathbb{F}_q$ is the finite field of cardinality q, then $H^\bullet$ is a finite graded abelian group with size

$$|H| = q^{\chi(H)}. \tag{4.1.6}$$

A large class of example of graded k-vector spaces comes from chain complexes:

Definition 4.1.7. Let

$$C^* : \cdots \to C^i \to C^{i+1} \to \cdots \tag{4.1.8}$$

be a cochain complex of k-vector spaces. It is called finite if it is finite when considered as a graded vector space. Its Euler characteristic $\chi(C^*)$ is the Euler characteristic defined above for finite graded vector spaces.

Given C^* be a finite cochain complex of k-vector space, then its cohomology H^* is a finite graded vector space, thus we can assign to it Euler character $\chi(H^*)$. The next proposition shows they are the same:

Lemma 4.1.9. *Let C^* be a finite cochain complex of k vector spaces, H^* its cohomology. Then $\chi(C^*) = \chi(H^*)$.*

Proof. Let $d^i : C^i \to C^{i+1}$ denote the i-th differential. We have a (non-canonical) decomposition

$$C^i \simeq im(d^i) \oplus H^i \oplus im(d^{i-1}). \tag{4.1.10}$$

Thus

$$\begin{aligned}
\chi(C^*) &= \sum_i (-1)^i dim(C^i) && (4.1.11)\\
&= \sum_i (-1)^i (dim(im(d^i)) + dim H^i + dim(im(d^{i-1}) && (4.1.12)\\
&= \sum_i (-1)^i (dim H^i) && (4.1.13)\\
&= \chi(H^*). && (4.1.14)
\end{aligned}$$

□

Remark 4.1.15. A similar argument works for a finite chain complex of abelian groups. In that case there will not be a splitting like Equation 4.1.10. However, there are still short exact sequences.

The notion of Euler characteristic also behave well with long exact sequences:

Lemma 4.1.16. *Given a long exact sequence of k-vector spaces $H_0^\bullet \to H_1^\bullet \to H_2^\bullet$, that is, a long exact sequence*

$$\cdots \to H_0^* \to H_1^* \to H_2^* \to H_0^{*+1} \to \cdots \tag{4.1.17}$$

Then we have

$$\chi(H) + \chi(H'') = \chi(H). \tag{4.1.18}$$

A similar result also holds for long exact sequences of finite graded abelian groups and their sizes.

Proof. We will work with finite k-vector spaces. The finite abelian groups case follows from the same argument. Consider the entire long exact sequence 4.1.17 as a chain complex K^*. It is exact $H^*(K) = 0$. By the lemma 4.1.9, we see that $\chi(K) = 0$. However,

$$\chi(K) = \chi(H_0) + \chi(H_2) - \chi(H_1). \tag{4.1.19}$$

Thus

$$\chi(H_0) + \chi(H_2) = \chi(H_1). \tag{4.1.20}$$

□

Now we can define the Euler characteristic of a compact manifold with boundaries:

Definition 4.1.21. Let M be a compact manifold with boundaries and k a field. Then we have finite graded k vector spaces $H^*(M, k)$. The Euler characteristic $\chi(M)$ is defined to be $\chi(H^*(M, k))$. As

$$H^*(M, k) = Hom_k(H_*(M, k), k), \tag{4.1.22}$$

we see that $\chi(H_*(M, k)) = \chi(M)$, that is, homology groups also computes the Euler characteristic of M.

Note that there is no mention of k in the notation of $\chi(M)$, this is due to the following lemma:

Lemma 4.1.23. *$\chi(H^*(M, k))$ is independent of k.*

Proof. As M is a compact smooth manifold, there is a finte CW complex X homotopy equivalent to M . The CW cochain complex $C^*(X, \mathbb{Z})$ is a bounded cochain complex of finite dimensionsal free $\mathbb{Z}$ modules with cohomology $H^*(M, \mathbb{Z})$. Furthermore,

$$C^*(M, k) \coloneqq C^*(M, \mathbb{Z}) \otimes k \tag{4.1.24}$$

is a finite k-cochain complex that computes $H^*(M, k)$. By lemma 4.1.9, we see that

$$\chi(H^*(M, k)) = \chi(C^*(M, k)) \tag{4.1.25}$$

$$= \sum_i (-1)^i \; dim_k(C^*(M, \mathbb{Z} \otimes k)) \tag{4.1.26}$$

$$= \sum_i (-1)^i \; rank(C^*(M, \mathbb{Z})). \tag{4.1.27}$$

Note the last equation is true because each C^i are free $\mathbb{Z}$ modules. $rank$ is the usual notion of rank of an finitely generated abelian group. As

$$\sum_i (-1)^i \ rank(C^*(M,\mathbb{Z})) \tag{4.1.28}$$

is independent of k, so is

$$\chi(H^*(M,k)). \tag{4.1.29}$$

□

The Euler characteristic also behave with composition of bordisms:

Lemma 4.1.30. *Given closed $(d-1)$-dimensional manifolds N, N', N'', and bordisms $M : N \to N'$ and $M' : N' \to N''$, then the composition $M \sqcup_{N'} M'$ has*

$$\chi(M \sqcup_{N'} M') = \chi(M) + \chi(M') - \chi(N'). \tag{4.1.31}$$

Proof. We have a Mayer Vietoris sequence

$$\cdots \to H^*(M \sqcup_{N'} M') \to H^*(M) \oplus H^*(M') \to H^*(N') \to \cdots \tag{4.1.32}$$

By 4.1.16, we get

$$\chi(M \sqcup_{N'} M') + \chi(N') = \chi(M) + \chi(M'). \tag{4.1.33}$$

□

Similiar to the Euler characteristic of M, we can define the size (also called homotopy cardinality) of a π-finite spectrum:

Definition 4.1.34. If $\mathcal{X}$ is a π-finite spectrum, then $\pi^\bullet(\mathcal{X}) = \pi_i(\mathcal{X})$ forms a finite graded abelian group. The size of $\mathcal{X}$, denoted as $|\mathcal{X}|$, is defined to be $|\pi^\bullet(\mathcal{X})|$.

The notion of size of π-finite spectrum behave well with fiber sequences:

Proposition 4.1.35. *Given a fiber sequence $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$, we have that $|\mathcal{X}| \ |\mathcal{Z}| = |\mathcal{Y}|$.*

Proof. This is due to proposition 4.1.16 apply to long exact sequence

$$\cdots \to \pi_*(\mathcal{X}) \to \pi_*(\mathcal{Y}) \to \pi_*(\mathcal{Z}) \to \cdots \tag{4.1.36}$$

□

Example 4.1.37. Let $\mathcal{X}$ be a π-finite spectrum. We have a fiber sequence

$$\tau_{\geq i}\mathcal{X} \to \mathcal{X} \to \tau_{\leq i-1}\mathcal{X} \tag{4.1.38}$$

of π-finite spaces. By proposition 4.1.35 we have

$$|\tau_{\geq i}\mathcal{X}|\ |\tau_{\leq i-1}\mathcal{X}| = |\mathcal{X}|. \tag{4.1.39}$$

Example 4.1.40. Let M be a compact manifold with boundary, and $\mathcal{X}$ a π-finite spectrum. Then the mapping spectrum

$$\mathcal{X}(M) \coloneqq Maps(\Sigma^\infty_+ M, \mathcal{X}), \tag{4.1.41}$$

is a π-finite spectrum (3.3.7) with homotopy groups

$$\pi_i(\mathcal{X}(M)) = \mathcal{X}^{-i}(M), \tag{4.1.42}$$

where $\mathcal{X}^i(M)$ is the i-th generalized cohomology group of M with coefficients $\mathcal{X}$. Thus

$$|\mathcal{X}(M)| = \cdots \frac{|\mathcal{X}^0(M)|}{|\mathcal{X}^{-1}(M)|}\frac{|\mathcal{X}^2(M)|}{|\mathcal{X}^1(M)|}\cdots. \tag{4.1.43}$$

The size of $|\mathcal{X}(M)|$ relates to the size of $|\mathcal{X}|$ and the Euler characteristic of M as follows:

Proposition 4.1.44. *$|\mathcal{X}(M)| = |\mathcal{X}|^{\chi(M)}$.*

Proof. By proposition 3.3.14 and the fact that every finite abelian group is a finite extension of $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$'s, we see that S^{fin} is generated by $H\mathbb{F}_p$ by finite extensions and (de)-suspensions.

First we look at the case $\mathcal{X} = H\mathbb{F}_p$. As noted in remark 4.1.5,

$$\begin{aligned}
|\mathcal{X}(M)| &= \prod_i |H^i(X, \mathbb{F}_p)|^{(-1)^i} && (4.1.45)\\
&= \prod_i p^{(-1)^i dim H^i(M,\mathbb{F}_p)} && (4.1.46)\\
&= p^{\chi(M)} && (4.1.47)\\
&= |\mathcal{X}|^{\chi(M)}. && (4.1.48)
\end{aligned}$$

Next, if the hypothesis holds for $\mathcal{X}$, it also holds for $\Sigma\mathcal{X}$:

$$\begin{aligned}
|\Sigma\mathcal{X}(M)| &= |\mathcal{X}(M)|^{-1} && (4.1.49)\\
&= |\mathcal{X}|^{-\chi(M)} && (4.1.50)\\
&= (|\mathcal{X}|^{-1})^{\chi(M)} && (4.1.51)\\
&= |\Sigma\mathcal{X}|^{\chi(M)}. && (4.1.52)
\end{aligned}$$

Same thing holds for desuspension. Lastly, for extension, if we have a fiber sequence

$$\mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \tag{4.1.53}$$

and $\mathcal{X}, \mathcal{Z}$ satisfies the hypothesis, then we also have a fiber sequence of π-finite spectra

$$\mathcal{X}(M) \to \mathcal{Y}(M) \to \mathcal{Z}(M). \tag{4.1.54}$$

Thus

$$\begin{aligned} |\mathcal{Y}(M)| &= |\mathcal{X}(M)|\ |\mathcal{Z}(M)| && (4.1.55)\\ &= |\mathcal{X}|^{\chi(M)}\ |\mathcal{Z}|^{\chi(M)} && (4.1.56)\\ &= (|\mathcal{X}|\ |\mathcal{Z}|)^{\chi(M)} && (4.1.57)\\ &= |\mathcal{Y}|^{\chi(M)}, && (4.1.58) \end{aligned}$$

where the first and last equality is due to proposition 4.1.35. As the hypothesis holds for $H\mathbb{F}_p, 0$, and remains true under suspensions and extensions, it holds for any π-finite spectrum. □

4.2 Euler TFT

In this subsection we define the Euler TFT, and show that it is trivial in odd dimensions.

Definition 4.2.1. Let $\lambda \in \mathbb{C}^\times$ be a nonzero complex number, then we define an d dimensional unoriented TFT E_λ as follows: for any $d-1$ dimensional manifold N,

$$E_\lambda(N) := \mathbb{C}. \tag{4.2.2}$$

For a bordism $M : N \to N'$,

$$E_\lambda(M) : \mathbb{C} \to \mathbb{C} \tag{4.2.3}$$

is given by multiplication by scalar $\lambda^{\chi(M)-\chi(N)}$.

We have to check that the composition behaves, which boils down this following lemma:

Lemma 4.2.4. *Given $d-1$ dimensional manifolds N, N', N'', and bordisms $M : N \to N'$ and $M' : N' \to N''$, then*

$$\chi(M \sqcup_{N'} M') - \chi(N) = \chi(M) - \chi(N) + \chi(M') - \chi(N'). \tag{4.2.5}$$

Proof. By lemma 4.1.30 we have that

$$\chi(M \sqcup_{N'} M') = \chi(M) + \chi(M') - \chi(N'). \tag{4.2.6}$$

Thus

$$\chi(M \sqcup_{N'} M') - \chi(N) = \chi(M) + \chi(M') - \chi(N') - \chi(N) \tag{4.2.7}$$
$$= \chi(M) - \chi(N) + \chi(M') - \chi(N'). \tag{4.2.8}$$

□

Example 4.2.9. Let $\lambda \neq 1, -1$. In even dimensions $d = 2n$, the Euler TFT E_λ is nontrivial (not isomorphic to the trivial theory). This can be checked on the partition function (the value of E_λ on closed d-dim manifolds): let $M = S^d$, then $\chi(M) = 2$. Thus we have $E_\lambda(M) \neq 1$ and the theory is not trivial. In fact, as S^d is oriented, E_λ is nontrivial as an oriented TFT.

However, in odd dimensions, the Euler characteristic of a closed d dimensional manifold is 0, this is due to Poincare duality (with $\mathbb{F}_2$ coefficients, as all manifolds are $\mathbb{F}_2$ oriented). In fact,we have a stronger statement:

Proposition 4.2.10. *For any $\lambda \in \mathbb{C}^\times$, $E_\lambda \simeq Z_{triv}$ as unoriented theories.*

Proof. To show that $E_\lambda \simeq Z_{triv}$, we have to give an natural isomorphism $\alpha : E_\lambda \xrightarrow{\sim} Z_{triv}$ between the two functors. First let's see what kind of data is needed for such α and what conditions it needs to satisfy:

For every closed (d-1)-manifold N, we have

$$\alpha(N) : Z_{triv}(N) = \mathbb{C} \xrightarrow{\sim} \mathbb{C} = E_\lambda(N), \tag{4.2.11}$$

which sends $1 \in \mathbb{C} = Z_{triv}(N)$ to a nonzero elements

$$\alpha_N := \alpha(N)(1). \tag{4.2.12}$$

α need to satisfy the following compatibility condition: given a bordism $M : N \to N'$, we have a commutative diagram

$$\begin{tikzcd} Z_{triv}(N) \arrow[r, "Z_{triv}(M)"] \arrow[d, "\alpha(N)"] & Z_{triv}(N') \arrow[d, "\alpha(N')"] \\ E_\lambda(N) \arrow[r, "E_\lambda(M)"] & E_\lambda(N'). \end{tikzcd} \tag{4.2.13}$$

Tracking where

$$1 \in \mathbb{C} = Z_{triv}(N) \tag{4.2.14}$$

goes, we see that we need to show that

$$\alpha_{N'} = \lambda^{\chi(M)-\chi(N)}\alpha_N. \tag{4.2.15}$$

I claim that for

$$\alpha_N = \lambda^{\frac{1}{2}\chi(N')}, \tag{4.2.16}$$

equation 4.2.15 is satisfied. Thus we need to show that

$$\chi(M) = \frac{1}{2}(\chi(N) + \chi(N')) = \frac{1}{2}\chi(\partial M). \tag{4.2.17}$$

Let $k = \mathbb{F}_2$, as every manifold is k-oriented, we have Poincare duality (3.2):

$$H^*(M,k) \simeq H_{d-*}(M, \partial M, k). \tag{4.2.18}$$

As d is odd, we see that

$$\begin{aligned}
\chi(M) &= \chi(H^*(M,k)) && (4.2.19)\\
&= \chi(H_{d-*}(M,\partial M,k)) && (4.2.20)\\
&= -\chi(H_*(M,\partial M,k) && (4.2.21)\\
&= -\chi(M,\partial M), && (4.2.22)
\end{aligned}$$

Finally, consider the long exact sequence associated to the cofiber sequence $\partial M \to M \to (M, \partial M)$:

$$\cdots \to H^*(M, \partial M, k) \to H^*(M) \to H^*(N) \to \cdots .$$

By 4.1.30, we see that

$$\begin{aligned}
\chi(M) &= \chi(M,\partial M) + \chi(\partial M) && (4.2.23)\\
&= -\chi(M) + \chi(\partial M). && (4.2.24)
\end{aligned}$$

Thus

$$\chi(M) = \frac{1}{2}\chi(\partial M). \tag{4.2.25}$$

□

5 Abelian duality

Fix dimension $d \geq 1$ of our theories. Given any π-finite space X,there is a d-dimensional unoriented finite homotopy TFT $Z_X : Bord_d \to Vect_{\mathbb{C}}$. For a π-finite spectrum $\mathcal{X}$, its 0-th space $\Omega^\infty \mathcal{X}$ is a π-finite space. We define the d-dimensional unoriented finite homotopy TFT associated to $\mathcal{X}$ as

$$Z_{\mathcal{X}} \coloneqq Z_{\Omega^\infty \mathcal{X}} : Bord_d \to Vect_{\mathbb{C}}. \tag{5.0.1}$$

In 3.4, we define the Brown-Comenetz dual spectrum $\hat{\mathcal{X}}$. It is also a π-finite spectra. Now assume that we have a ring spectrum $\mathcal{R}$ and $\mathcal{X}$ is a right $\mathcal{R}$ module spectrum. Note that $\hat{\mathcal{X}}$ gets a left $\mathcal{R}$ module structure. In 5.2 we define the $\mathcal{R}$-oriented bordism category $Bord_d^{\mathcal{R}}$. It has a forgetful map

$$Bord_d^{\mathcal{R}} \to Bord_d. \tag{5.0.2}$$

We can use this map to pullback unoriented theories to $\mathcal{R}$-oriented theories. The theories of interests are $\mathcal{R}$-oriented theories

$$Z_{\mathcal{X}}, Z_{\Sigma^{d-1}\hat{\mathcal{X}}} : Bord_d^{\mathcal{R}} \to Bord_d \to Vect_{\mathbb{C}} \tag{5.0.3}$$

associated to $\mathcal{X}$ and $\Sigma^{d-1}\hat{\mathcal{X}}$.

Recall that if λ is a nonzero complex number, then E_λ is the d-dim Euler TFT defined in 4.2, and we view it as a $\mathcal{R}$-oriented theory. Here's the main theorem of the thesis:

Theorem 5.0.4 (Abelian duality)**.** *There is an equivalence of $\mathcal{R}$-oriented TFTs:*

$$\mathbb{D} : Z_{\mathcal{X}} \cong Z_{\Sigma^{d-1}\hat{\mathcal{X}}} \otimes E_{|\mathcal{X}|}. \tag{5.0.5}$$

This section is devoted to stating and proving main theorem 5.0.4. In §5.1 we define the finite homotopy TFT $Z_{\mathcal{X}}$ for any π-finite spectrum $\mathcal{X}$ and do some basic computations. In §5.2 we define the $\mathcal{R}$-oriented bordism category $Bord_d^{\mathcal{R}}$ for any ring spectrum $\mathcal{R}$. In §5.3, we proof the main theorem, borrowing two lemmas 5.3.24 and 5.3.25. In 5.4, we give examples of the main theorem in low dimensions. In §5.5 we proof lemma 5.3.24. In 5.6, we proof lemma 5.3.25.

5.1 Finite homotopy TFTs for π-finite spectra

In this subsection we define a d-dimensional unoriented TFT $Z_{\mathcal{X}}$ for a π-finite spectrum $\mathcal{X}$ and do some basic calculations.

Recall that we have the underlying space functor $\Omega^\infty : Sp \to S_*$. If $\mathcal{X}$ is a π-finite spectrum, then is a π-finite space. In 2.4 we define the d-dimensional finite homotopy TFT Z_X associated to any π-fintie space X.

Definition 5.1.1. The d-dimensional finite homotopy TFT associated to π-finite spectrum $\mathcal{X}$ is

$$Z_{\mathcal{X}} \coloneqq Z_{\Omega^\infty \mathcal{X}} : Bord_d \to Vect_{\mathbb{C}} \tag{5.1.2}$$

Note that $Z_{\mathcal{X}}$ doesn't not see the non-connective part of $\mathcal{X}$, as

$$\Omega^\infty \mathcal{X} \simeq \Omega^\infty(\tau_{\geq 0}\mathcal{X}). \tag{5.1.3}$$

Recall that we can think about $\mathcal{X}$ as a generalized cohomology theory, with

$$\mathcal{X}^n(N) \coloneqq \pi_{-n}(Maps(\Sigma^\infty_+ N, \mathcal{X})). \tag{5.1.4}$$

Note that this is the nonreduced cohomology. We also use $\mathcal{X}(N)$ to denote the mapping spectrum $Maps(\Sigma^\infty_+ N, \mathcal{X})$.

Let N be a $d-1$ dimensional closed manifold, then

$$\begin{aligned} Z_{\mathcal{X}}(N) &= Z_{\Omega^\infty \mathcal{X}}(N) && (5.1.5)\\ &= \mathbb{C}[\pi_0(Maps(N, \Omega^\infty \mathcal{X}))] && (5.1.6)\\ &= \mathbb{C}[\pi_0(Maps(\Sigma^\infty_+ N, \mathcal{X}))] && (5.1.7)\\ &= \mathbb{C}[\mathcal{X}^0(N)] && (5.1.8) \end{aligned}$$

The states of the finite homotopy TFT associated to $\mathcal{X}$ are related to cohomologies with $\mathcal{X}$ coefficients.

Example 5.1.9. Let A be a finite abelian group, and HA the Eilenberg-MacLane spectrum. For $n \leq 0$, we have $\Omega^\infty \Sigma^n HA = *$ and the theory associated to $\Sigma^n HA$ is trivial. When $n \geq 0$, then

$$\Omega^\infty \Sigma^n HA = K(A, n) \tag{5.1.10}$$

is the n-th Eilenberg-MacLane space. For N a closed $(d-1)$-manifold, by equation 5.1.5, we see that

$$Z_{\Sigma^n HA}(N) = \mathbb{C}[\Sigma^n HA^0(N)] = H^n(N, A). \tag{5.1.11}$$

This is a theory that counts n-principal A bundles. Thus $Z_{\Sigma^n HA}$ is the topological analogue of n-form gauge theories, discussed in subsection 1.2.

Given a bordism $M : N \to N'$. We have

$$Z_{\mathcal{X}}(M) : Z_{\mathcal{X}}(N) \to Z_{\mathcal{X}}(N'). \tag{5.1.12}$$

In our basis, we have a map

$$Z_{\mathcal{X}}(M) : \mathbb{C}[\mathcal{X}^0(N)] \to \mathbb{C}[\mathcal{X}^0(N')]. \tag{5.1.13}$$

In fact, there is a simple formula to calculate this maps:

Proposition 5.1.14. *Let a, b, a' denote the elements of $\mathbb{C}[\mathcal{X}^0(N)]$, $\mathbb{C}[\mathcal{X}^0(M)]$, $\mathbb{C}[\mathcal{X}^0(N')]$, we also view them as basis vectors for the correspond vector spaces. Under*

$$Z_{\mathcal{X}}(M) : \mathbb{C}[\mathcal{X}^0(N)] \to \mathbb{C}[\mathcal{X}^0(N')], \tag{5.1.15}$$

we have

$$a \mapsto \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{b \to a} q^* b \tag{5.1.16}$$

$$= \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{a'} \sum_{b \to a, b \to a'} a', \tag{5.1.17}$$

where $\sum_a$ means sum over all $a \in \mathbb{C}[\mathcal{X}^0(N)]$, $\sum_{b \to a}$ means sum over all $b \in \mathbb{C}[\mathcal{X}^0(M)]$ such that $p^*(b) = a$.

Proof. Consider the span of π-finite spaces:

$$\begin{tikzcd} & Maps(M, \Omega^\infty \mathcal{X}) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Maps(N, \Omega^\infty \mathcal{X}) & & Maps(N', \Omega^\infty \mathcal{X}). \end{tikzcd} \tag{5.1.18}$$

Recall that

$$Z_{\mathcal{X}}(M) : \mathbb{C}[\mathcal{X}^0(N)] \to \mathbb{C}[\mathcal{X}^0(N')] \tag{5.1.19}$$

is defined to be the composition $q_* \circ p^*$. First we compute p^*:

$$p^* : \mathbb{C}[\mathcal{X}^0(N)] \to \mathbb{C}[\mathcal{X}^0(M)] \tag{5.1.20}$$

$$a \mapsto \sum_{b \to a} b. \tag{5.1.21}$$

For q_*, we need to understand the homotopy groups of $Maps(M, \Omega^\infty \mathcal{X})$ and $Maps(N', \Omega^\infty \mathcal{X})$. We have

$$\pi_0(Maps(M, \Omega^\infty \mathcal{X})) = \mathcal{X}^0(M), \tag{5.1.22}$$

for any $a \in Maps(M, \Omega^\infty \mathcal{X}$, we have

$$\pi_n(Maps(M, \Omega^\infty \mathcal{X}), a) \simeq \pi_n(Maps(\Sigma^\infty_+ M, \mathcal{X})) = \pi_n(\mathcal{X}(M)). \tag{5.1.23}$$

This is because $Maps(M, \Omega^\infty \mathcal{X}) = \Omega^\infty \mathcal{X}(M)$ is an infinite loop space, and all the connected components of $Maps(M, \Omega^\infty \mathcal{X})$ are isomorphic to each other. Same argument is also true for $Maps(N', \Omega^\infty \mathcal{X})$. Thus

$$\begin{aligned} q_* : \mathbb{C}[\mathcal{X}^0(M)] &\to \mathbb{C}[\mathcal{X}^0(N')] \\ b \mapsto \frac{|\pi_1(Maps(N', \Omega^\infty \mathcal{X}))|}{|\pi_1(Maps(M, \Omega^\infty \mathcal{X}))|} &\frac{|\pi_2(Maps(M, \Omega^\infty \mathcal{X}))|}{|\pi_2(Maps(N', \Omega^\infty \mathcal{X})|} \cdots q^* b \\ &= \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} q^* b. \end{aligned} \tag{5.1.24}$$

Composing p^* and q_*, we get:

$$Z_{\mathcal{X}}(M) : a \mapsto \sum_{b \to a} b \tag{5.1.25}$$

$$\mapsto \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{b \to a} q^* b \tag{5.1.26}$$

$$= \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{a'} \sum_{b \to a, b \to a'} a'. \tag{5.1.27}$$

□

5.2 $\mathcal{R}$-oriented bordism category

Let $\mathcal{R}$ be a ring spectrum (see 3.1 for definitions), in this subsection, we define the d-dimensional $\mathcal{R}$-oriented bordism category $Bord_d^{\mathcal{R}}$ and $\mathcal{R}$-oriented TFTs. Recall the notion of $\mathcal{R}$-orientation for a manifold M (see 3.4):

Definition 5.2.1. Let M be a d-manifold, $\mathcal{R}$ a ringed spectrum. Then an $\mathcal{R}$-orientation on M is a homology class $[M] \in \mathcal{R}_d(M, \partial M)$ such that for every $x \in M^o$ a point in the interior, the image of $[M]$ under $\mathcal{R}_d(M, \partial M) \to \mathcal{R}_d(M, M - x) \simeq \pi_0(\mathcal{R})$ is an multiplicative unit in the ring $\pi_*(\mathcal{R})$.

From now on, we will say orientation for $\mathcal{R}$-orientation unless explicitly said otherwise.

Note that if N is a closed $d-1$-manifold, then an orientation $[N]$ lives in $\mathcal{R}_{d-1}(N)$. If $[N] \in \mathcal{R}_{d-1}(N)$ is an orientation, then so is $-[N] \in \mathcal{R}_{d-1}(N)$.

Let N and N' be two closed $d-1$ dimensional manifold, and $M : N \to N'$ is a bordism. An $\mathcal{R}$ orientation $[M]$ on M, by proposition 3.2.41, gives an orientation on $\partial M \simeq N \sqcup N'$ via the boundary map

$$\mathcal{R}_d(M, \partial M) \to \mathcal{R}_{d-1}(\partial M). \tag{5.2.2}$$

Thus an orientation $[M]$ on M gives an orientation on both N and N'.

Definition 5.2.3. Let N and N' be closed oriented $d-1$-manifolds with orientation $[N]$ and $[N']$. Then an oriented bordism is a bordism $M : N \to N'$ with an orientation $[M]$ that restricts to $[N]$ on N and $-[N']$ on N'. An isormophism between oriented bordisms is an isomorphism of the underlying unoriented bordisms such that is compatible with the orientation classes on the bordisms.

The reason why there is a minus sign on $-[N']$ is so that oriented bordisms can compose:

Proposition 5.2.4. *Given two oriented bordisms $M : N \to N'$ and $M' : N' \to N''$, then the composition $M \sqcup_{N'} M'$ has a canonical orientation and is an oriented bordism from N to N''.*

Proof. We just have to show that we can glue the two orientation class $[M]$ and $[M']$ to an orientation class $[M \sqcup_{N'} M']$ on $M \sqcup_{N'} M'$. As M and M' are glued at N', we just have to do that locally. Locally, note that M and M' looks like $N' \times I$, where I is the interval. Note that $N' \times I$ has a canonical orientation that restricts to $-[N]$ on $N \times 0$ and $[N]$ on $N \times 1$. Two of these oriented cylinders $N' \times I$ can compose iff the orientation are reversed on the boundary they glue on. This is exactly our situation. □

As oriented bordisms compose, we can define the oriented bordism category:

Definition 5.2.5. The d-dimensional $\mathcal{R}$-oriented cateogry $Bord_d^{\mathcal{R}}$ is the category with objects closed $\mathcal{R}$-oriented $d-1$-manifolds, and morphisms are isomorphism classes of oriented bordisms. It is symmetric monoidal under disjoint union.

Example 5.2.6. Let $\mathcal{R} = H\mathbb{Z}$, then a $H\mathbb{Z}$-orientation is the same as the usual notion of orientation for manifolds. Thus we recovered the oriented bordism category $Bord_d^{or}$.

Example 5.2.7. Let $\mathcal{R} = H\mathbb{Z}/2\mathbb{Z}$, then every manifold is $H\mathbb{Z}/2\mathbb{Z}$-oriented. Thus $Bord_d^{H\mathbb{Z}/2\mathbb{Z}} = Bord_d$.

Example 5.2.8. Let $\mathcal{R} = \mathcal{S}$, then $\mathcal{S}$-orientation is a trivialization of the thom spectra of the (stable) normal bundle. Since $\mathcal{S}$ is the intial ring spectrum, a $\mathcal{S}$-orientation implies $\mathcal{R}$-orientation for any ring spectrum $\mathcal{R}$.

Remark 5.2.9. Framed manifolds are $\mathcal{S}$ oriented. Thus they are $\mathcal{R}$ oriented for any ring spectrum $\mathcal{R}$.

Lastly, we can define $\mathcal{R}$-oriented TFTs:

Definition 5.2.10. A $\mathcal{R}$-oriented topological field theory Z is a symmetric monoidal functor

$$Z : Bord_d^{\mathcal{R}} \to Vect_{\mathbb{C}} \tag{5.2.11}$$

Remark 5.2.12. Note that there is a symmetric monoidal map $Bord_d^{\mathcal{R}} \to Bord_d$ by forgetting the orientation structure. Thus any unoriented TFT gives a $\mathcal{R}$-oriented TFT.

5.3 Main theorem

Let $d \geq 1$ be the dimension of our theory. Let $\mathcal{R}$ be a ring spectrum and $\mathcal{X}$ a π-finite right $\mathcal{R}$ module spectrum. We have the Pontryagin dual spectrum $\hat{\mathcal{X}}$. Note that $\hat{\mathcal{X}}$ is naturally a left $\mathcal{R}$ module. We have unoriented TFTs $Z_{\mathcal{X}}$ and $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}$ associated to $\mathcal{X}$ and $\Sigma^{d-1}\hat{\mathcal{X}}$ (see 5.1). In addition, if λ is a nonzero complex number, we have E_λ is the d-dim Euler TFT defined in 4.2.

in 5.2 we define the bordism category $Bord_d^{\mathcal{R}}$ of $\mathcal{R}$-oriented manifolds and bordisms. Any unoriented TFT can be viewed as a $\mathcal{R}$-oriented TFT by precomposing with the forgetful map $Bord_d^{\mathcal{R}} \to Bord_d$. Now we view $Z_{\mathcal{X}}, Z_{\Sigma^{d-1}\hat{\mathcal{X}}}, E_{|\mathcal{X}|}$ as $\mathcal{R}$-oriented theories. Then there is an equivalence of theories:

Theorem 5.3.1 (Abelian duality)**.** *There is an equivalence of $\mathcal{R}$-oriented TFTs*

$$\mathbb{D} : Z_{\mathcal{X}} \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}} \otimes E_{|\mathcal{X}|}. \tag{5.3.2}$$

Remark 5.3.3. Note that in general, they are not equivalent as unoriented theories, despite both sides can be extended to unoriented theories. This is because we need to use Poincaré duality in an essential way. For example, they give different partition functions for the $d = 2$ theories on the Klein bottle.

Remark 5.3.4. For reader familiar with the duality between p and $(d-p-2)$-form gauge theories, it might seen strange that we suspend that $\hat{A}$ side $d - p - 1$ times. This means all the dualities are off by one dimension. For example, in 2 dimension, T-duality swaps sigma models to S^1, which are 0-form gauge theories. In 4 dimension, electromagnetic duality swaps $U(1)$ gauge theories, which are 1-form gauge theories. However, as will be discussed in 5.4, in this topological version, we have duality of a sigma models in 1 dimension, and duality of a gauge theories in 3 dimension.

The seemingly off-by-one-dimension is explain by the fact that, the Pontryagin dual of $U(1)$ **is not** $U(1)$, but rather $\mathbb{Z}$. As $U(1) = B\mathbb{Z}$ we see that this off-by-one-dimension exactly cancels the off-by-one-dimension above.

Here's some consequences of the theorem:

Corollary 5.3.5. *When d is odd, we have eqiovalence of $\mathcal{R}$-oriented TFTs*

$$Z_{\mathcal{X}} \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}}. \tag{5.3.6}$$

Proof. By proposition 4.2.10, when d is odd, E_λ is isomorphic to the trivial theory, thus $Z_{\mathcal{X}} \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}}$. □

Example 5.3.7. As any manifold is $H\mathbb{Z}/2\mathbb{Z}$ oriented. If $\mathcal{X}$ is a π-finite $H\mathbb{Z}/2\mathbb{Z}$ module, then the main theorem 5.3.1 gives an equivalence of unoriented theories.

Example 5.3.8. $H\mathbb{Z}$ orientation is the same as the classicial notion of orientation on manifolds. Thus if $\mathcal{X}$ is a π-finite $H\mathbb{Z}$ module (e.g. $K(A,n)$), then we have an equivalence of oriented theories.

We will look at examples of theorem 5.3.1 in 5.4. The rest of the section is devoted to the proof of theorem 5.3.1:

Proof of Theorem 5.3.1. For the rest of the subsection, all manifolds, bordisms are $\mathcal{R}$-oriented. We will suppressed the $\mathcal{R}$-orientation notations.

To give an equivalence, we will need to define an isomorphism of states

$$Z_{\mathcal{X}}(N) \to Z_{\Sigma^{d-1}\hat{\mathcal{X}}} \otimes E_{|\mathcal{X}|}(N), \tag{5.3.9}$$

and check that it is compatible with bordisms. As $E_{|\mathcal{X}|}(N) = \mathbb{C}$, it is suffice to give maps

$$\mathbb{D}(N) : Z_{\mathcal{X}}(N) \to Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N). \tag{5.3.10}$$

This is done as follows:

Construction 5.3.11. By Pontryagin duality, there is a pairing

$$ev_N(-,-) : \mathcal{X}^*(N) \times \hat{\mathcal{X}}_*(N) \to \mathbb{C}^\times. \tag{5.3.12}$$

Note that this exist for any topological space N. As N is a (compact) manifold, the homology and cohomology groups are finite. This pairing is exhibits $\mathcal{X}^*(N)$ and $\hat{\mathcal{X}}_*(N)$ as Pontryagin dual of each other. Compose this with the Poincaré duality isomorphism 3.2.45(this requires a orientation $[N]$ on N):

$$\int_{[N]} : \hat{\mathcal{X}}^{d-1-*}(N) \xrightarrow{\sim} \hat{\mathcal{X}}_*(N), \tag{5.3.13}$$

we get a pairing

$$\mathcal{X}^*(N) \times \hat{\mathcal{X}}^{d-1-*}(N) \to \mathbb{C}^\times \tag{5.3.14}$$

$$(a, \alpha) \mapsto ev_N(a, \int_{[N]} \alpha) \tag{5.3.15}$$

When $* = 0$, we denote this pairing as

$$\langle -, - \rangle_N : \mathcal{X}^0(N) \times \hat{\mathcal{X}}^{d-1}(N) \to \mathbb{C}^\times. \tag{5.3.16}$$

It exhibits $\mathcal{X}^0(N)$ and $\hat{\mathcal{X}}^{d-1}(N)$ as the Pontryagin dual of each other.

Note that this denotes on the orientation class of N, reversing the orientation inverts this pairing. Recall that

$$Z_{\mathcal{X}}(N) = \mathbb{C}[\mathcal{X}^0(N)] \tag{5.3.17}$$

and

$$Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N) = \mathbb{C}[\hat{\mathcal{X}}^{d-1}(N)]. \tag{5.3.18}$$

We will denote elements of $\mathcal{X}^0(N)$ as a, and $\hat{\mathcal{X}}^{d-1}(N)$ as α, and view them as basis vectors for $Z_{\mathcal{X}}(N)$ and $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N)$ respectively. Now we can define the isomorphism on states:

$$\begin{aligned} \mathbb{D}(N) : \mathbb{C}[\mathcal{X}^0(N)] &\to \mathbb{C}[\hat{\mathcal{X}}^{d-1}(N)] \\ a &\mapsto |\tau_{\geq 1}\mathcal{X}(N)| \sum_\alpha \langle a, \alpha \rangle_N \ \alpha. \end{aligned} \tag{5.3.19}$$

This is an isomorphism of vector spaces as the pairing $\langle -, - \rangle_N$ is nondegenerate 3.3.29.

It remains to show that this intertwines with morphisms. Given $M : N \to N'$ in $Bord_d$, with the inclusion maps $p : N \hookrightarrow M$ and $q : N' \hookrightarrow M$. We have to show that the following diagram commute:

$$\begin{tikzcd} Z_{\mathcal{X}}(N) \arrow[r, "Z_{\mathcal{X}}(M)"] \arrow[d, "\mathbb{D}(N)"] & Z_{\mathcal{X}}(N') \arrow[d, "\mathbb{D}(N')"] \\ Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N) \arrow[r, "Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) * |\mathcal{X}|^{\chi(M)-\chi(N)}"'] & Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N') \end{tikzcd} \tag{5.3.20}$$

Note as we canonically identified

$$Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N) \otimes E_{|\mathcal{X}|}(N) \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N). \tag{5.3.21}$$

The factor

$$|\mathcal{X}|^{\chi(M)-\chi(N)} \tag{5.3.22}$$

in the bottom arrow comes from

$$E_{|\mathcal{X}|}(M) : E_{|\mathcal{X}|}(N) = \mathbb{C} \to \mathbb{C} = E_{|\mathcal{X}|}(N'). \tag{5.3.23}$$

We will do this in two lemmas:

Lemma 5.3.24. *$\mathbb{D}(N') \circ Z_{\mathcal{X}}(M)$ and $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N)$ differ by a constant $\lambda(M)$.*

Lemma 5.3.25. *$\lambda(M) = |\mathcal{X}|^{\chi(M)-\chi(N)}$.*

The two lemmas are proven in sections 5.5 and 5.6 respectively. □

5.4 Examples in low dimensions

In this subsection we work out the main theorem 5.0.4 in $d = 1, 2, 3$ dimensions for shifts of Eilenberg-MacLane spectra. We will see how it recovers finite Fourier theory and character theory (for finite abelian group) in low dimensions.

Let A be a finite abelian group, and $\hat{A}$ its dual group. The theories in consideration are the finite homotopy TFTs associated to shifts of HA and $H\hat{A}$.

Recall that in d dimension, the dual theories are $Z_{\Sigma^n HA}$ and $Z_{\Sigma^{d-1-n}HA}$. Note that $\Omega^\infty \Sigma^n HA = *$ for $n < 0$, thus $Z_{\Sigma^n HA} = Z_*$ is the trivial theory. On the other hand, if $n \geq d$, then the dual theory $Z_{\Sigma^{d-1-n}HA} = Z_*$ becomes the trivial theory. Using 5.0.4, we see that

Corollary 5.4.1. *When $n \geq d$, then $Z_{\Sigma^n HA} \cong E_{|A|^{-1^n}}$ as oriented TFTs.*

Thus when $n < 0$ or $n \geq d$, $Z_{\Sigma^n HA}$ is an invertible field theory. The most interesting cases are when $0 \leq n < d$, which we now investigate. We will consider the cases $(d = 1, n = 0)$, $(d = 2, n = 0, 1)$, and $(d = 3, n = 1)$.

Example 5.4.2. Let $d = 1$ and $n = 0$, then the spectra of interest are HA and $H\hat{A}$. The corresponding spaces are A and $\hat{A}$, thus these are sigma models to discrete sets A and $\hat{A}$, discussed in 2.5.1. As $*$ is a 0 dimensional manifold, we have

$$Z_{HA}(*) = \mathbb{C}[A] \qquad\qquad Z_{H\hat{A}}(*) = \mathbb{C}[\hat{A}]. \tag{5.4.3}$$

As d is odd, the Euler TQFT is trivial. Thus by corollary 5.3.5 we have

$$Z_{HA} \simeq Z_{H\hat{A}} \tag{5.4.4}$$

as oriented TFTs. As 1 dimensional oriented TFT is determined by the vector space on the positively oriented point $+$, this is equivalent to the isomorphism of states:

$$\mathbb{C}[A] \xrightarrow{\sim} \mathbb{C}[\hat{A}]. \tag{5.4.5}$$

This is given in Construction 5.3.11. In our case, the construction is simple, it is

$$\begin{aligned} \mathbb{C}[A] &\to \mathbb{C}[\hat{A}] \\ a &\mapsto \sum_{\alpha} \alpha(a)\ \alpha \end{aligned} \tag{5.4.6}$$

where $\sum_\alpha$ runs through $\alpha \in \hat{A}$. This is a form of discrete Fourier transform.

Remark 5.4.7. Recall that both theories Z_{HA} and $Z_{H\hat{A}}$ can be extended to unoriented theories. An $1d$ unoriented theory Z is equivalent to a symmetric nondegenerate pairing on $Z(*)$. One can check that the natural pairing on $Z_{HA}(*) = \mathbb{C}[A]$ and $Z_{H\hat{A}} = \mathbb{C}[\hat{A}]$ are not equivalent. That is, this equivalence doesn't extend to an equivalence of unoriented theory (in fact, the isomorphism of states 5.4.6 depends on the orientation of the point).

Example 5.4.8. Let $d = 2$ and $n = 1$. The dual spectra are ΣHA and $H\hat{A}$. Their underlying spaces are BA and $\hat{A}$. Thus we have an equivalence between finite gauge theory and sigma model. By swapping A and $\hat{A}$, we also covers the $d = 2$, $n = 0$ case.

The main theorem 5.0.4 gives an equivalence of oriented TFTs:

$$Z_{BA} \simeq Z_{\hat{A}} \otimes E_{|A|^{-1}}. \tag{5.4.9}$$

Let's take a look at their values at S^1:

$$Z_{BA} = \mathbb{C}[H^1(S^1, HA)] \simeq \mathbb{C}[A], \tag{5.4.10}$$

where the isomorphism uses the orientation of S^1 by taking the monodromy around that loop. On the other side, we have

$$Z_{\hat{A}} = \mathbb{C}[\hat{A}]. \tag{5.4.11}$$

The equivalence of TFTs gives an isomorphism:

$$\mathbb{C}[A] \xrightarrow{\sim} \mathbb{C}[\hat{A}] \tag{5.4.12}$$

$$a \mapsto \frac{1}{|A|} \sum_{\alpha} \alpha(a)\ \alpha \tag{5.4.13}$$

in fact, it differs from isomorphism of states on the 1d theories (see equation 5.4.6 by a factor of $|A|$. The pair of pants bordism gives commutative multiplication on both sides. On the A side, we see that the multiplication is

$$\mathbb{C}[A] \otimes \mathbb{C}[A] \to \mathbb{C}[A] \tag{5.4.14}$$

$$a \otimes b \mapsto \frac{1}{|A|}ab, \tag{5.4.15}$$

where again we identify $a \in A$ with the basis vector in $\mathbb{C}[A]$. This is the multiplication associated to the group algebra $\mathbb{C}[A]$. On the $\hat{A}$ side, the multiplication is

$$\mathbb{C}[\hat{A}] \otimes \mathbb{C}[\hat{A}] \to \mathbb{C}[A] \tag{5.4.16}$$

$$\alpha \otimes \beta \mapsto \delta_{ab}. \tag{5.4.17}$$

This is the point-wise multiplication of functions on the set $\hat{A}$. Under the isomorphism 5.4.12, the two multiplication differ by $|A|$, which comes from the Euler TFT $E_{|BA|}$.

For a finite abelian group A, the isomorphism between the group algebra $\mathbb{C}[A]$ and $\hat{A}$ is often the starting point of the character theory. If we extend these finite homotopy TFTs, then the extended version of our main theorem 5.0.4 implies that $Rep(A) \simeq Vect_{\hat{A}}$, where $Rep(A)$ is the category of representations of A, and $Vect_{\hat{A}}$ is the category of vector spaces on $\hat{A}$. Under this equivalence, the irreducible representation labelled by $\alpha \in \hat{A}$ is send to the vector bundle that is rank 1 over $\alpha \in \hat{A}$ and rank 0 over all other points. See [**???**] for a detail discussion.

Example 5.4.18. Let $d = 3, n = 1$, then the dual spectrums are ΣHA and $\Sigma H\hat{A}$. The corresponding spaces are BA and $B\hat{A}$, and we have a duality of gauge theories. Once again, as Euler TFT is trivial in odd dimensions, we have an equivalence of oriented theory

$$Z_{BA} \simeq Z_{B\hat{A}} \tag{5.4.19}$$

by Corollary 5.3.5.

This is the starting point of [12], where they showed that certain 2d lattice gauge theories, generalization of the Ising model, are boundary field theories for the $3d$ finite gauge theory Z_{BA}. In addition, they showed that Kramers-Wannier duality, which is a duality between these lattice gauge theories, can be understood as the isomorphism of boundary theories coming from the 3d equivalence $Z_{BA} \simeq Z_{B\hat{A}}$. See [12] for details.

5.5 Proof of Lemma 5.3.24

We borrow the notation from subsection 5.3. This section is devoted to proving lemma 5.3.24:

Lemma 5.5.1. *$\mathbb{D}(N')\circ Z_{\mathcal{X}}(M)$ and $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M)\circ\mathbb{D}(N)$ differ by a constant $\lambda(M)$.*

Proof. First we have to calculate them. From now on, we will denote elements of

$$\mathcal{X}^0(N),\ \mathcal{X}^0(M),\ \mathcal{X}^0(N') \tag{5.5.2}$$

as a, b, and a'. Similarly, we will denote elements of

$$\hat{\mathcal{X}}^{d-1}(N),\ \hat{\mathcal{X}}^{d-1}(M),\ \hat{\mathcal{X}}^{d-1}(N') \tag{5.5.3}$$

as α, β, and α'. We also use the summing convention that $\sum_b$ means summing over all $b \in \mathcal{X}^0(M)$, and $\sum_{b\to a}$ means summing over all $b \in \mathcal{X}^0(M)$ such that $p^*(b) = a$.

First we will calculate $\mathbb{D}(N')\circ Z_{\mathcal{X}}(M)$. By proposition 5.1.14, $Z_{\mathcal{X}}(M)$ sends

$$a \mapsto \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|}\sum_{b\to a} q^*b. \tag{5.5.4}$$

Recall that $\mathbb{D}(N')$ takes

$$a' \mapsto |\tau_{\geq 1}\mathcal{X}(N')|\ \sum_{\alpha'}\langle a', \alpha'\rangle_N\ \alpha'. \tag{5.5.5}$$

Thus the composition $\mathbb{D}(N')\circ Z_{\mathcal{X}}(M)$ sends

$$a \mapsto \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|}\sum_{b\to a}(|\tau_{\geq 1}\mathcal{X}(N')|\ \sum_{\alpha'}\langle q^*b, \alpha'\rangle_{N'}\ \alpha') \tag{5.5.6}$$

$$= |\tau_{\geq 1}\mathcal{X}(M)|\sum_{b\to a}\sum_{\alpha'}\langle q^*b, \alpha'\rangle_{N'}\ \alpha'. \tag{5.5.7}$$

Now for $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M)\circ\mathbb{D}(N)$. $\mathbb{D}(N)$ sends:

$$a \mapsto |\tau_{\geq 1}\mathcal{X}(N)|\sum_{\alpha}\langle a, \alpha\rangle_N\ \alpha. \tag{5.5.8}$$

$Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M)$ takes

$$\alpha \mapsto \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|}\sum_{\beta,\beta\to\alpha}\hat{q}^*b. \tag{5.5.9}$$

Thus the composition $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N)$ sends:

$$a \mapsto |\tau_{\geq 1}\mathcal{X}(N)| \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|} \sum_{\alpha'} \sum_{\beta \to \alpha'} \langle a, p^*\beta \rangle_N \; \alpha'. \tag{5.5.10}$$

Thus we are reduce to showing the following lemma:

Lemma 5.5.11. *For every a and α', $\sum_{b \to a} \langle q^* b, \alpha' \rangle_{N'}$ and $\sum_{\beta \to \alpha'} \langle a, p^*\beta \rangle_N$ differ a nonzero constant multiplicative C that doesn't depend on a or α'.*

Proof. Note that if a has no preimage $b \mapsto a$. Then

$$\sum_{b \to a} \langle q^* b, \alpha' \rangle_{N'} = 0. \tag{5.5.12}$$

In this case, lemma 5.5.42 (stated and proven below) precise says that

$$\sum_{\beta \to \alpha'} \langle a, p^*\beta \rangle_N = 0. \tag{5.5.13}$$

Similarly, if α' has no preimage $\beta \mapsto \alpha'$, then both sides are also zero. Thus we are reduced to the case that a lies in the image of

$$p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N) \tag{5.5.14}$$

and α' lies in the image of

$$\hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N). \tag{5.5.15}$$

There are

$$|kp| := |ker(p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N))| \tag{5.5.16}$$

many preimage of a. Similiarly, there are

$$|kq| := |ker(\hat{q}^* : \Sigma^{d-1}\hat{\mathcal{X}}^0(M) \to \Sigma^{d-1}\hat{\mathcal{X}}^0(N'))| \tag{5.5.17}$$

preimages of α'.

On one side, we have

$$\sum_{b \to a} \langle q^*(b), \alpha' \rangle_{N'} \tag{5.5.18}$$

$$= |kq|^{-1} \sum_{b \to a} \sum_{\beta \to \alpha'} \langle q^*(b), q^*\beta \rangle_{N'} \tag{5.5.19}$$

$$= |kq|^{-1} \sum_{b \to a} \sum_{\beta \to \alpha'} \langle p^*(b), p^*\beta \rangle_N. \tag{5.5.20}$$

Note that $\langle q^*(b), q^*\beta\rangle_{N'} = \langle p^*(b), p^*\beta\rangle_N$ is due to lemma 5.5.28 (stated and proven below)

On the other side, we have

$$\sum_{\beta\to\alpha'} \langle a, p^*\beta\rangle_N \tag{5.5.21}$$

$$= |kp|^{-1} \sum_{b\to a} \sum_{\beta\to\alpha'} \langle p^*(b), p^*\beta\rangle_N. \tag{5.5.22}$$

Thus we see that LHS and RHS differ by a constant $C = |kp|/|kq|$. □

Thus we see that the two sides differ by a constant $\lambda(M)$:

$$\mathbb{D}(N') \circ Z_{\mathcal{X}}(M) = \lambda(M)\ Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N), \tag{5.5.23}$$

with

$$\lambda(M) = \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N)|} \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|} |kp|/|kq|. \tag{5.5.24}$$

□

Now we need to proof the two lemmas 5.5.42, 5.5.28 above. We first show lemma 5.5.28. Because it might have independence interest, we recall the notations:

Given $M : N \to N'$ a bordism between N and N', with the inclusion maps $p : N \hookrightarrow M$ and $q : N' \hookrightarrow M$. We have pullback maps

$$p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N), \quad q^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N'). \tag{5.5.25}$$

Similarly we have

$$\hat{p}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N), \quad \hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N'). \tag{5.5.26}$$

We will denote elements of $\mathcal{X}^0(M)$ as b and $\hat{\mathcal{X}}^{d-1}(M)$ as β. Given b and β, we have two pairings:

$$\langle p^*b, \hat{p}^*\beta\rangle_N, \quad \langle q^*b, \hat{q}^*\beta\rangle_{N'}. \tag{5.5.27}$$

Here's the lemma that we need to show:

Lemma 5.5.28. $\langle p^*b, \hat{p}^*\beta\rangle_N = \langle q^*b, \hat{q}^*\beta\rangle_{N'}$.

Proof. We will show this equality by equating both side to something that depends only on M, b, and β. By Poincaré duality (3.2.55), there is an isomorphism of long exact sequences:

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^{d-1}(M) \arrow[r, "\hat{p}^*"] \arrow[d, "\int_{[M]}"] & \hat{\mathcal{X}}^{d-1}(N) \arrow[r] \arrow[d, "\int_{[N]}"] & \hat{\mathcal{X}}^{d}(M,N) \arrow[r] \arrow[d] & \cdots \arrow[d] \\ \cdots \arrow[r] & \hat{\mathcal{X}}_1(M,\partial M) \arrow[r, "\hat{a}_*"] & \hat{\mathcal{X}}_0(N) \arrow[r] & \hat{\mathcal{X}}_0(M,N') \arrow[r] & \cdots \end{tikzcd} \tag{5.5.29}$$

The isomorphism $\int_{[M]} : \hat{\mathcal{X}}^{d-1}(M) \cong \hat{\mathcal{X}}_{-1}(M,\partial M)$ depends only on the orientation class of M.

By definition of $\langle -,- \rangle_N$, we have

$$\langle p^*b, \hat{p}^*\beta\rangle_N = ev_N(p^*b, \int_{[N]} \hat{p}^*\beta) \tag{5.5.30}$$

$$= ev_N(p^*b, \hat{a}_* \int_{[M]} \beta) \tag{5.5.31}$$

Now consider the long exact sequence:

$$... \to \mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N) \to \mathcal{X}^1(M,N) \to ... \tag{5.5.32}$$

By Pontryagin duality (3.4.50), taking Pontryagin dual term-wise gives long exact sequence:

$$... \leftarrow \hat{\mathcal{X}}_0(M) \xleftarrow{\hat{p}_*} \hat{\mathcal{X}}_0(N) \leftarrow \hat{\mathcal{X}}_{-1}(M,N) \leftarrow ... \tag{5.5.33}$$

The dual long exact sequences are connected by the "projection formula": given $b \in \mathcal{X}^0(M)$ and $\gamma \in \hat{\mathcal{X}}_0(N)$, then

$$ev_N(p^*b, \gamma) = ev_M(b, \hat{p}_*\gamma). \tag{5.5.34}$$

Put it together with equation 5.5.31:

$$\langle p^*b, \hat{p}^*\beta\rangle_N = ev_N(p^*b, \hat{a}_* \int_{[M]} \beta) \tag{5.5.35}$$

$$= ev_M(b, \hat{p}_* \circ \hat{a}_* \int_{[M]} \beta) \tag{5.5.36}$$

Note that

$$\partial_* := \hat{p}_* \circ \hat{a}_* : \hat{\mathcal{X}}_{-1}(M,\partial M) \to \hat{\mathcal{X}}_0(M) \tag{5.5.37}$$

is the boundary map associated to the triple $\partial M \to M \to (M, \partial M)$, thus it is independent of N. The same argument work for $\langle q^* b, \hat{q}^* \beta\rangle_{N'}$. Thus

$$\langle p^* b, \hat{p}^* \beta\rangle_N = ev_M(b, \partial_* \int_{[M]} \beta) \tag{5.5.38}$$

$$= \langle q^* b, \hat{q}^* \beta\rangle_{N'}. \tag{5.5.39}$$

□

Remark 5.5.40. Heuristically, $\langle p^* b, \hat{p}^* \beta\rangle_N = \langle q^* b, \hat{q}^* \beta\rangle_{N'}$ because the orientation class $[M]$ for M is exactly a homotopy from $p_*[N] \in H_{d-1}(M)$ to $q_*[N] \in H_{d-1}(M)$. Thus we have

$$\langle p^* b, \hat{p}^* \beta\rangle_N \approx \langle b, \beta\rangle_{p_*[N]} \approx \langle b, \beta\rangle_{q_*[N']} \approx \langle q^* b, \hat{q}^* \beta\rangle_{N'}. \tag{5.5.41}$$

The actual proof is just a way to make this heuristic rigorous.

We also used this following lemma:

Lemma 5.5.42. *Let $a \in \mathcal{X}^0(N)$ and $\alpha' \in \hat{\mathcal{X}}^{d-1}(N')$. If a is not in the image of $p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N)$, then*

$$\sum_{\beta \to \alpha'} \langle a, \hat{p}^* \beta\rangle_N = 0, \tag{5.5.43}$$

where β sums over $\hat{\mathcal{X}}^{d-1}(M)$.

Proof. If α' has no preimage in

$$\hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N'), \tag{5.5.44}$$

then the sum is trivially 0. If α' has a preimage, say β'_α. Then all other preimages of α are of the form $\beta'_\alpha + \beta_0$, where $\beta_0 \in ker(\hat{p}^*)$. Thus

$$\sum_{\beta \to \alpha'} \langle a, \hat{p}^* \beta\rangle_N = \sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^*(\beta'_\alpha + \beta_0)\rangle_N \tag{5.5.45}$$

$$= (\langle a, \hat{p}^* \beta'_\alpha\rangle_N) \sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^* \beta_0\rangle_N. \tag{5.5.46}$$

Thus it is suffice to show that

$$\sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^* \beta_0\rangle_N = 0, \tag{5.5.47}$$

i.e. the case where $\alpha' = 0$.

Poincaré duality 3.2.55 gives an isomorphism of long exact sequences:

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^{d-1}(M,N') \arrow[r] \arrow[d] & \hat{\mathcal{X}}^{d-1}(M) \arrow[r, "\hat{p}^*"] \arrow[d, "\int_{[M]}"] & \hat{\mathcal{X}}^{d-1}(N') \arrow[r] \arrow[d, "\int_{[N']}"] & \cdots \arrow[d] \\ \cdots \arrow[r] & \hat{\mathcal{X}}_1(M,N) \arrow[r, "\hat{\mu}_*"] & \hat{\mathcal{X}}_1(M,\partial M) \arrow[r, "\hat{\nu}_*"] & \hat{\mathcal{X}}_0(N') \arrow[r] & \cdots \end{tikzcd} \tag{5.5.48}$$

Under Poincaré duality, $ker(\hat{p}^*)$ corresponds to $ker(\hat{a}_*) = im(\hat{\mu}_*)$. Given $\beta_0 \in ker(\hat{p}^*)$ with

$$\int_{[M]} \beta = \hat{\mu}_* \gamma, \quad \gamma \in \hat{\mathcal{X}}_1(M,N), \tag{5.5.49}$$

By definition of $\langle -,- \rangle_N$, we have:

$$\langle a, \hat{p}^* \beta_0 \rangle_N = ev_N(a, \hat{\nu}_* \int_{[M]} \beta) \tag{5.5.50}$$

$$= ev_N(a, \hat{\nu}_* \circ \hat{\mu}_* \gamma). \tag{5.5.51}$$

The composition

$$\hat{\nu}_* \circ \hat{\mu}_* : \hat{\mathcal{X}}_1(M,N) \to \hat{\mathcal{X}}_0(N) \tag{5.5.52}$$

is the Pontryagin dual of the

$$\partial^* : \mathcal{X}^0(N) \to \mathcal{X}^1(M,N). \tag{5.5.53}$$

Thus by above equation (5.5.51) we have

$$\langle a, \hat{p}^* \beta_0 \rangle_N = ev_N(a, \hat{\nu}_* \hat{\mu}_* \gamma) \tag{5.5.54}$$

$$= ev_{(M,N)}(\partial^* a, \gamma). \tag{5.5.55}$$

Thus

$$|ker(\hat{\mu}_*)| \sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^* \beta_0 \rangle_N = \sum_{\gamma} ev_{(M,N)}(\partial^* a, \gamma), \tag{5.5.56}$$

where γ sums over $\hat{\mathcal{X}}_1(M,N)$. Now consider the the long exact sequence:

$$... \to \mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N) \xrightarrow{\partial^*} \mathcal{X}^1(M,N) \to ... \tag{5.5.57}$$

As a is not in the image of $p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N)$, $\partial^* a \in \mathcal{X}^1(M,N)$ is not the identity element. Thus

$$ev_{(M,N)}(\partial^* a, -) : \hat{\mathcal{X}}_1(M,N) \to \mathbb{C}^\times \tag{5.5.58}$$

is a nontrivial character on $\hat{\mathcal{X}}_1(M,N)$. As the sum over all elements of the group paired with a nontrivial character is 0, we see that

$$\sum_{\gamma} ev_{(M,N)}(\partial^* a, \gamma) = 0. \tag{5.5.59}$$

By equation (5.5.56), we see that

$$\sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^* \beta_0 \rangle_N = 0. \tag{5.5.60}$$

□

5.6 Proof of lemma 5.3.25

Recall from last section we have

$$\mathbb{D}(N') \circ Z_{\mathcal{X}}(M) = \lambda(M)\ Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N) \tag{5.6.1}$$

with

$$\lambda(M) = \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N)|} \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|} |kp|/|kq|. \tag{5.6.2}$$

To finish the proof of the main theorem, we need the following lemma (see previous section 5.4 for notations):

Lemma 5.6.3. *Let M be a bordism from N to N'. Then*

$$\lambda(M) = |\mathcal{X}|^{\chi(M) - \chi(M)}. \tag{5.6.4}$$

Proof. Recall that

$$\lambda(M) = \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N)|} \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|} |kp|/|kq|. \tag{5.6.5}$$

First we will move everything in $\hat{\mathcal{X}}$ to $\mathcal{X}$ by Poincaré duality 3.2.55.

The first term is

$$|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|. \tag{5.6.6}$$

Note that Poincaré duality (3.2.45) shows that

$$\hat{\mathcal{X}}^*(N') \cong \hat{\mathcal{X}}_{d-1-*}(N'), \tag{5.6.7}$$

thus

$$|\hat{\mathcal{X}}^i(N')| = |\hat{\mathcal{X}}_{d-1-i}(N')| = |\mathcal{X}^{d-1-i}(N'). \tag{5.6.8}$$

Note that the cardinalities of Pontryagin dual groups are equal by proposition 3.3.14. Thus

$$\begin{aligned} |\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')| &= \frac{|\hat{\mathcal{X}}^{d-3}(N')|}{|\hat{\mathcal{X}}^{d-2}(N')|}\frac{|\hat{\mathcal{X}}^{d-5}(N')|}{|\hat{\mathcal{X}}^{d-4}(N')|}\cdots && (5.6.9)\\ &= \frac{|\hat{\mathcal{X}}_2(N')|}{|\hat{\mathcal{X}}_1(N')|}\frac{|\hat{\mathcal{X}}_4(N')|}{|\hat{\mathcal{X}}_3(N')|}\cdots && (5.6.10)\\ &= \frac{|\mathcal{X}^2(N')|}{|\mathcal{X}^1(N')|}\frac{|\mathcal{X}^4(N')|}{|\mathcal{X}^3(N')|}\cdots && (5.6.11)\\ &= |\tau_{\leq -1}\mathcal{X}(N')|. && (5.6.12) \end{aligned}$$

Next we will work on

$$|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|^{-1}. \tag{5.6.13}$$

Similar to above, we have

$$|\hat{\mathcal{X}}^i(M)| = |\hat{\mathcal{X}}_{d-i}(M,\partial M)| = |\mathcal{X}^{d-i}(M,\partial M)| \tag{5.6.14}$$

Thus

$$\begin{aligned} |\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|^{-1} &= \frac{|\hat{\mathcal{X}}^{d-2}(M')|}{|\hat{\mathcal{X}}^{d-3}(M')|}\frac{|\hat{\mathcal{X}}^{d-4}(M')|}{|\hat{\mathcal{X}}^{d-5}(M')|}\cdots && (5.6.15)\\ &= \frac{|\hat{\mathcal{X}}_2(M,\partial M)|}{|\hat{\mathcal{X}}_3(M,\partial M)|}\frac{|\hat{\mathcal{X}}_4(M,\partial M)|}{|\hat{\mathcal{X}}_5(M,\partial M)|}\cdots && (5.6.16)\\ &= \frac{|\mathcal{X}^2(M,\partial M)|}{|\mathcal{X}^3(M,\partial M)|}\frac{|\mathcal{X}^4(M,\partial M)|}{|\mathcal{X}^5(M,\partial M)|}\cdots && (5.6.17)\\ &= |\tau_{\leq -2}\mathcal{X}(M,\partial M)| && (5.6.18) \end{aligned}$$

Lastly, we have

$$|kq| := |ker(\hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N'))|. \tag{5.6.19}$$

By Poincare duality (3.2.55): we have that an isomorphism of long exact sequences:

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^*(M,N') \arrow[r] \arrow[d] & \hat{\mathcal{X}}^*(M) \arrow[r, "q^*"] \arrow[d] & \hat{\mathcal{X}}^*(N') \arrow[r] \arrow[d] & \cdots \arrow[d] \\ \cdots \arrow[r] & \hat{\mathcal{X}}_{d-*}(M,N) \arrow[r] & \hat{\mathcal{X}}_{d-*}(M,\partial M) \arrow[r] & \hat{\mathcal{X}}_{d-1-*}(N') \arrow[r] & \cdots \end{tikzcd} \tag{5.6.20}$$

Thus

$$|kq| = |ker\ (q' : \hat{\mathcal{X}}_1(M, \partial M) \to \hat{\mathcal{X}}_0(N'))| \tag{5.6.21}$$
$$= |im(\hat{\mathcal{X}}_1(M, N) \to \hat{\mathcal{X}}_1(M, \partial M)|. \tag{5.6.22}$$

Now, under Pontraygin duality 3.4.50, we know that the long exact sequence

$$\cdots \to \hat{\mathcal{X}}_{d-*}(M, N) \to \hat{\mathcal{X}}_{d-*}(M, \partial M) \to \hat{\mathcal{X}}_{d-1-*}(N') \to \cdots \tag{5.6.23}$$

is the Pontryagin dual of

$$\cdots \leftarrow \mathcal{X}^{d-*}(M, N) \leftarrow \mathcal{X}^{d-*}(M, \partial M) \leftarrow \mathcal{X}^{d-1-*}(N') \leftarrow \cdots \tag{5.6.24}$$

Thus

$$|kq| = |im(\hat{\mathcal{X}}_1(M, N) \to \hat{\mathcal{X}}_1(M, \partial M)| \tag{5.6.25}$$
$$= |im(\mathcal{X}^1(M, \partial M) \to \mathcal{X}^1(M, N)| \tag{5.6.26}$$
$$= |ker(\mathcal{X}^1(M, N) \to \mathcal{X}^1(N'))|. \tag{5.6.27}$$

Now we will factor out $|\mathcal{X}|^{\chi(M)-\chi(M)}$ from $\lambda(M)$. Recall that given any π-finite space $\mathcal{Y}$. We have a fiber sequence

$$\tau_{\geq i}\mathcal{Y} \to \mathcal{Y} \to \tau_{\leq i-1}\mathcal{Y} \tag{5.6.28}$$

of π-finite spaces. By Example 4.1.37 we have

$$|\tau_{\geq i}\mathcal{Y}|\ |\tau_{\leq i-1}\mathcal{Y}| = |\tau_{\leq i-1}\mathcal{Y}|. \tag{5.6.29}$$

In our case we have:

$$|\tau_{\geq 1}\mathcal{X}(M)| = \frac{|\mathcal{X}(M)|}{|\tau_{\leq 0}\mathcal{X}(M)|}. \tag{5.6.30}$$

Similarly,

$$|\tau_{\geq 1}\mathcal{X}(N)|^{-1} = \frac{|\tau_{\leq 0}\mathcal{X}(N)|}{|\mathcal{X}(N)|}. \tag{5.6.31}$$

Recall that by 4.1.44 we have

$$|\mathcal{X}(M)| = |\mathcal{X}|^{\chi(M)} \tag{5.6.32}$$

and

$$|\mathcal{X}(N)| = |\mathcal{X}|^{\chi(N)}. \tag{5.6.33}$$

Putting it all together, we see that

$$\lambda(M) = \lambda'(M)\ |\mathcal{X}|^{\chi(M)-\chi(M)}, \tag{5.6.34}$$

where

$$\begin{aligned}\lambda'(M) = \frac{|\tau_{\leq 0}\mathcal{X}(N)|}{|\tau_{\leq 0}\mathcal{X}(M)|}|\tau_{\leq -1}\mathcal{X}(N')|\ |\tau_{\leq -2}\mathcal{X}(M,\partial M)| \\ \frac{|ker(\mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|}\end{aligned} \tag{5.6.35}$$

It remains to show that $\lambda'(M) = 1$.

First, notice that

$$\partial M = N \sqcup N', \tag{5.6.36}$$

thus we have

$$|\mathcal{X}^*(\partial M)| = |\mathcal{X}^*(N)|\ |\mathcal{X}^*(N')|. \tag{5.6.37}$$

So

$$|\tau_{\leq 0}\mathcal{X}(N)|\ |\tau_{\leq -1}\mathcal{X}(N')| = |\mathcal{X}^0(N)|\ |\tau_{\leq -1}\mathcal{X}(\partial M)|. \tag{5.6.38}$$

Now consider the exact sequences

$$0 \to ker\ p^* \to \mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N) \to coker\ p^* \to 0, \tag{5.6.39}$$

We see that the terms

$$|ker\ p^*|\ |\tau_{\leq 0}\mathcal{X}(M)|^{-1}\ |\tau_{\leq 0}\mathcal{X}(N)| = |coker\ p^*|. \tag{5.6.40}$$

Lastly, we rewrite

$$|coker\ p^*| = |ker\ \mathcal{X}^1(M,N) \to \mathcal{X}^1(M)|. \tag{5.6.41}$$

Thus

$$\begin{aligned}\lambda'(M) = \frac{|\tau_{\leq -1}\mathcal{X}(\partial M)|}{|\tau_{\leq -1}\mathcal{X}(M)|}\ |\tau_{\leq -2}\mathcal{X}(M,\partial M)| \\ \frac{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(M))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|}\end{aligned} \tag{5.6.42}$$

I claim that

$$\frac{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(M))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|} = |ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M))|^{-1}. \tag{5.6.43}$$

To see this, first notice that the canonical map

$$\mathcal{X}^1(M,N) \to \mathcal{X}^1(N) = 0, \tag{5.6.44}$$

thus

$$|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))| = |ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(\partial M))|. \tag{5.6.45}$$

Note

$$\mathcal{X}^1(M,N) \to \mathcal{X}^1(\partial M) \tag{5.6.46}$$

is the composition of the two terms

$$(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M)) \circ (\mathcal{X}^1(M,N) \to \mathcal{X}^1(M)). \tag{5.6.47}$$

on the RHS. Now we have the following algebraic fact: given

$$f : A \to B,\ \ g : B \to C \tag{5.6.48}$$

then

$$|ker(g \circ f)| = |ker f|\ |ker g| \tag{5.6.49}$$

iff

$$ker(g) \subset im(f). \tag{5.6.50}$$

In our case, if an element $a \in \mathcal{X}^1(M)$ maps to 0 in $\mathcal{X}^1(\partial M)$, then it maps to 0 in $\mathcal{X}^1(N)$. Since

$$\mathcal{X}^1(M,N) \to \mathcal{X}^1(M) \to \mathcal{X}^1(N)$$

is a part of a long exact sequence, it is exact at $\mathcal{X}^1(M)$. That means that there exists $b \in \mathcal{X}^1(M,N)$ which maps to a. Thus we satisfy the algebraic condition, and we have

$$\frac{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(M))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|} = |ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M))|^{-1}. \tag{5.6.51}$$

So we have

$$\lambda'(M) = \frac{|\tau_{\leq -1}\mathcal{X}(\partial M)|}{|\tau_{\leq -1}\mathcal{X}(M)|} \frac{|\tau_{\leq -2}\mathcal{X}(M,\partial M)|}{|ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M))|} \tag{5.6.52}$$

Finally, consider the following long exact sequence:

$$\begin{aligned} 0 \to ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M)) \to \mathcal{X}^1(M) \to \mathcal{X}^1(\partial M) \\ \to \mathcal{X}^2(M,\partial M) \to \mathcal{X}^2(M) \to \mathcal{X}^2(\partial M) \to \cdots . \end{aligned}$$

Recall that the alternating size of the finite abelian groups in a long exact sequence is 1 (by lemma 4.1.9 and the fact that this sequence is exact). Note the alternating size of the long exact sequence above is precisely $\lambda'(M)$, thus

$$\lambda'(M) = 1. \tag{5.6.53}$$

$\square$

A Background on Homotopy theory

In the appendix we recall the basic facts of the (∞-)categories of spaces S and spectra Sp.

Note that both S and Sp are ∞-categories in the sense of [18]. We use ∞-category theory as it has nice categorical properties, however, in the expense of having higher coherences. The complications will not enter our picture, and we will treat S and Sp that appears as ordinary categories.

In subsection A.1 we review the category of spaces S. In subsection A.2 we review the category of spectra Sp.

A.1 The category of spaces

In homotopy theory, the basic objects are topological spaces. However, we have a weaker notion of equivalence, namely weak homootpy equivalence...

Introduce weak homotopy equivalence. Introduce "localizing at weak equivalence". Introduce cofiber sequence??? Introduce S_* the category of based spaces. Introduce fiber sequence of based spaces. Introduce the mapping space. Note that it takes cofiber sequence in first variable to fiber sequences. In addition, it takes fiber sequence in second variable to fiber sequences.

discuss BG and K(A,n) Say:

we use the ∞-categorical model because it has good properties (Maps, fiber sequence, cofiber sequence).

Remark A.1.1. Note that we only care about topological spaces up to weak homotopy equivalence. If $f : X \to Y$ is a weak equivalence (induce isomorphism on all homotopy groups), then it is an isomorphism in S. Thus S can be defined as the category of all topological space Top loocalized at weak equivalences. However, the resulting 1-category of homotopy classes of spaces $Ho(S)$ does not behave well as it doesn't keep track of the higher coherences. Rigorously, S is the model category of CW complexes (and homotopy equivalence), or equivalently, the model category of simplicial sets. Alternatively, one can use the machinery of ∞-categories ([18], [17]). The limits and colimits are always the homotopy limits and colimits.

[4] S is a ∞-category in the sense of [18]. We use ∞-category theory as it has nice categorical properties, however, in the expanse of having higher coherences. The complications will not enter our picture, and we will treat the ∞-categories that appears as ordinary category. The limits and colimits in ∞-categories are homotopy limits and colimits, and we will name them as such. See [18], [17].

A.2 The category of spectra

In this subsection, we recall some facts about spectra. Most of them are formal (see [14] for a nice introduction). We follow [10] for a large part of this subsection.

Let S be the category of spaces, S_* the category of pointed spaces. Recall that S_* has a symmetric monoidal product $\wedge$ and an inner hom object $Maps$. Wedgeing with $- \wedge S^1$ is the suspension functor $\Sigma -$. Dually, $Maps(S^1, -)$ is the loop functor $\Omega -$.For every $X, Y \in S_*$, We have an equivalence

$$Maps(\Sigma X, Y) \simeq Maps(X, \Omega Y), \tag{A.2.1}$$

realizing Σ as the left adjoint of Ω.

Definition A.2.2. A prespectrum $\mathcal{X}$ is a sequence $X_0, X_1, ...$ of pointed topological spaces with canonical map

$$s_n : \Sigma X_n \to X_{n+1}. \tag{A.2.3}$$

A map of prespectrum $f : \mathcal{X} \to \mathcal{Y}$ is a series of maps $f_n : X_n \to Y_n$ that commutes with s_n.

The homotopy group $\pi_n \mathcal{X}$, $n \in \mathbb{Z}$, is defined as

$$\pi_n \mathcal{X} := \varinjlim_k \pi_{n+k} X_{n+k} \tag{A.2.4}$$

where the limit maps is given by

$$\pi_{n+k} X_{n+k} \xrightarrow{\Sigma} \pi_{n+k+1} \Sigma X_{n+k} \xrightarrow{s_{n+k}} \pi_{n+k+1} X_{n+k+1}. \tag{A.2.5}$$

Note that even when n is negative, X_{n+k} is eventually defined for k large enough.

Given a map $f : \mathcal{X} \to \mathcal{Y}$ of spectrums, we get induced maps on homotopy groups:

$$\pi_n(f) : \pi_n \mathcal{X} \to \pi_n \mathcal{Y}. \tag{A.2.6}$$

f is a weak homotopy equivalence if it induces isomorphism on homotopy groups.

The category of spectra Sp is the category of prespectrum localizes at weak homotopy equivalence. Effectively, it means that we will only consider things up to weak homotopy equivalence (similar to our treatment of the category of spaces). From now on, we will think about a prespectrum $\mathcal{X}$ as a spectrum (its equivalence class), and refer to them as such.

Remark A.2.7. The category of spectra Sp is a $(\infty, 1)$-category, that is, it has higher homotopy coherence coming from the localization at weak equivalence. The formalism of $(\infty, 1)$ categories are harder to describe than ordinary categories. The two standard approach are based on model categories and quasi-categories [17]. Thus everything below should be stated in those context, however, for the purpose of this paper, much of the complication won't play a major role. For detail, see [17].

A large family of spectrum comes from pointed spaces:

Example A.2.8. The suspension spectrum $\Sigma^\infty X$ of a pointed space X is the spectrum associated to the prespectrum

$$(\Sigma^\infty X)_n = \Sigma^n X \tag{A.2.9}$$

with the canonical maps $\Sigma(\Sigma^{n-1}X) \simeq \Sigma^n X$.

Note that we have a functor $\Sigma^\infty : S_* \to Sp$. Another class of spectrum comes from abelian group.

Example A.2.10. Let A be an abelian group, then we define the Eilenberg-MacLane spectrum HA as follows:

$$HA_n = K(A, n) \tag{A.2.11}$$

with canonical map $\Sigma K(A, n) \to K(A, n+1)$ the right adjoint of the isomorphism

$$K(A, n) \simeq \Omega K(A, n+1). \tag{A.2.12}$$

Definition A.2.13. An Ω spectrum is spectrum such that the corresponding map

$$X_n \to \Omega X_{n+1} \tag{A.2.14}$$

associated to $s_n : \Sigma X_n \to X_{n+1}$ is an equivalence.

In fact, every spectrum is weak equivalent to a Ω-spectrum. Given an Ω spectrum $\mathcal{X} = \{X_0, X_1, ..\}$, we can define its 0-th space, denoted as $\Omega^\infty \mathcal{X}$, to be X_0.

This gives a functor $\Omega^\infty : Sp \to S_*$. In fact, it is the right adjoint of Σ^∞:

$$Maps_{Sp}(\Sigma^\infty X, \mathcal{Y}) \simeq Maps_{S_*}(X, \Omega^\infty \mathcal{Y}). \tag{A.2.15}$$

here $Maps_{Sp}$ and $Maps_{S_*}$ are the categorical homs, that is, they are the space of maps.

We have a suspection functor $\Sigma : Sp \to Sp$ defined as follows: for a spectrum $\mathcal{X}$,

$$(\Sigma\mathcal{X})_n \coloneqq \Sigma X_n \tag{A.2.16}$$

with canonical connecting maps. Note that Σ^∞ commutes with $\Sigma : S_* \to S_*$ and $\Sigma : Sp \to Sp$. That is, we have a commutative diagram:

$$\begin{tikzcd} S_* \arrow[r, "\Sigma"] \arrow[d, "\Sigma^\infty"] & S_* \arrow[d, "\Sigma^\infty"] \\ Sp \arrow[r, "\Sigma"] & Sp \end{tikzcd} \tag{A.2.17}$$

In addition, this is in fact an invertible functor, thus it has an inverse denoted as Σ^{-1} or Ω. By abstract nonsense, we also have this commutative diagram:

$$\begin{tikzcd} Sp \arrow[r, "\Omega"] \arrow[d, "\Omega^\infty"] & Sp \arrow[d, "\Omega^\infty"] \\ S_* \arrow[r, "\Omega"] & S_* \end{tikzcd} \tag{A.2.18}$$

With this, we can define the sphere spectrum ant its shifts:

$$\mathcal{S}^n \coloneqq \Sigma^n \Sigma^\infty S^0 \tag{A.2.19}$$

$n \in \mathbb{Z}$ S^0 the 0-th sphere as a pointed space. When $n = 0$, $\mathcal{S} \coloneqq \mathcal{S}^0 = \Sigma^\infty S^0$ is called the sphere spectrum.

Similar the category of pointed space S_*, the category of spectra Sp has a symmetric monoidal product product $\otimes : Sp \times Sp \to Sp$, which we called the tensor product (traditionally it's called the smash product). The unit object of the symmetric monoidal proudct is the sphere spectrum $\mathcal{S}$. It also has an internal hom object

$$Maps : Sp^{op} \times Sp \to Sp. \tag{A.2.20}$$

As any internal hom, it has the universal property that, for a spectra $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, we have

$$Maps(\mathcal{X} \otimes \mathcal{Y}, \mathcal{Z}) \simeq Maps(\mathcal{X}, Maps(\mathcal{Y}, \mathcal{Z})). \tag{A.2.21}$$

Note that we will denote the categorical hom as $Maps_{Sp}$, and the internal hom as $Maps$. We have

$$Maps_{Sp}(-,-) = \Omega^\infty \circ Maps(-,-) \tag{A.2.22}$$

Note that $\Sigma^\infty : (S_*, \wedge) \to (Sp, \otimes)$ is a symmetric monoidal functor.

Now we move on to the notion of fiber sequence:

Definition A.2.23. A sequence of map $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ is called a fiber sequence if the induced map of homotopy groups

$$\cdots \to \pi_n\mathcal{X} \to \pi_n\mathcal{Y} \to \pi_n\mathcal{Z} \to \cdots \tag{A.2.24}$$

is a long exact sequence of abelian groups.

Note that if $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ is a fiber sequence, there is a caononical map $\mathcal{Z} \to \Sigma\mathcal{X}$, and $\mathcal{Y} \to \mathcal{Z} \to \Sigma\mathcal{X}$ is also a fiber sequence.

Given $f : \mathcal{X} \to \mathcal{Y}$, there is an essentially unique $fib(f)$ with a canonical map $fib(f) \to \mathcal{X}$ that makes $fib(f) \to \mathcal{X} \to \mathcal{Y}$ a fiber sequence.

Most operations in Sp preserves fiber sequence: let $\mathcal{X}$ be a spectrum, then the functors

$$\Sigma,\ \Omega,\ Maps(\mathcal{X}, -),\ Maps(-, \mathcal{X}),\ \mathcal{X} \otimes - \tag{A.2.25}$$

all sends fiber sequence to fiber sequence (these are called exact functors).

Recall that in S_*, there is a notion of cofiber and fiber sequences (and they are not the same). The functor

$$\Sigma^\infty : S_* \to Sp \tag{A.2.26}$$

sends cofiber sequences of pointed spaces to fiber sequences of spectra.

Similarly, we have

$$\Omega^\infty : Sp \to S_* \tag{A.2.27}$$

sends fiber sequences of spectra to fiber sequences of pointed spaces.

We will also need to talk about the Postnikov truncation. We first start with the notion of connective, coconnective spectra:

Definition A.2.28. A spectrum $\mathcal{X}$ is called n-connective if for every $n < 0$, we have $\pi_n(\mathcal{X}) = 0$. Similarly, a spectrum is called n-coconnective if for every $n > 0$¡ we have $\pi_n(\mathcal{X}) = 0$. When n is 0, they are just called connective and coconnective.

The quintessential example of a connective spectrum is $\Sigma^\infty X$ for X a pointed space.

For a spectrum $\mathcal{X}$ and $n \in \mathbb{Z}$, there is an universal n-connective spectra $\tau_{\geq n}\mathcal{X}$ with a map $\tau_{\geq n}\mathcal{X} \to \mathcal{X}$ that induces isomorphism

$$\pi_i(\tau_{\geq n}\mathcal{X}) \simeq \pi_i\mathcal{X} \tag{A.2.29}$$

for all $i >\geq n$. Similarly, there is also an universal $n-1$-coconnective spetrum $\tau_{<n}\mathcal{X}$ with a map $\mathcal{X} \to \tau_{<n}\mathcal{X}$ that induces isomorphism

$$\pi_i(\mathcal{X}) \to \pi_i(\tau_{<n}\mathcal{X}) \tag{A.2.30}$$

for $i < n$. Note that for any n, the canonical maps

$$\tau_{\geq n}\mathcal{X} \to \mathcal{X} \to \tau_{<n}\mathcal{X} \tag{A.2.31}$$

is a fiber sequence.

Lastly, we need the notion of a ring spectrum. Recall the Sp is a symmetric monoidal category with tensor product $\otimes$. For symmetric monoidal category $(C, \otimes, 1_C)$, there is a notion of associative algebra of $(C, \otimes)$. Heuristically, it is an object $c \in C$ with an associative product $c \otimes c \to c$ and an unital object $1_C \to c$. In addition, if c is an algebra object, then we can define (left) module objects of c. Heuristically, a c-module is an object $d \in C$ and action maps $c \otimes d \to d$ that is unital and compatible with the multiplication of c.

Definition A.2.32. An $\mathbb{E}_1$-ring spectrum $\mathcal{R}$ is an associative algebra in Sp. We also refer to them as just ring spectra. For a ring spectrum $\mathcal{R}$, a $\mathcal{R}$ module is a $\mathcal{R}$ module object in Sp. The category of $\mathcal{R}$ modules forms a category $Mod_{\mathcal{R}}$.

Remark A.2.33. As Sp is a higher category, thus there are higher coherence data for ring spectrum and module spectrum that needs to be given. They are essential part of the data. For precise definition see [17].

Example A.2.34. The quiteseential example of a ring spectrum is the sphere spectrum $\mathcal{S}$. Being the unit of Sp, $\mathcal{S}$ has in fact a fully symmetric monoidal product. In addition, every spectrum is canonically a $\mathcal{S}$ module, thus $Mod_{\mathcal{S}} = Sp$.

Example A.2.35. Let R be an associative ring, then the Eilenberg-MacLane spectrum HR is a ring spectrum. If N is a R module, then HN is a HR module. Thus $\mathbb{Z}$ is a (commutative) ring spectrum, and any Eilenberg-MacLane spectrum is a module for $\mathbb{Z}$. Note that $Mod_{\mathbb{Z}}$ is the derived category of chain complexes of Abelian groups $D(Ab)$.

A.3 Spectra and Cohomology theories

A huge motivation for spectra was to define a suitable category of (co)homology theories. In this subsection we review this relationship. This subsection largely follows [14]. We first recall the definition of a homology/cohomology theory. We first start with homology theory:

Let CW be the ordinary 1-category of finite pointed CW complexes (not localized at weak equivalence), CW_* be the category of finite pointed CW complexes. A reduced homology theory is a sequence of functors $\widetilde{E}_n : CW \to Ab$ that satisfies the Eilenberg-Steenrod Axioms:

Definition A.3.1. An reduced (extraordinary) homology theory $\widetilde{E}_*$ is a sequence of functors $\widetilde{E}_n : CW_* \to Ab$, $n \in \mathbb{Z}$, such that

1. A homotopy equivalence of pointed finite CW complexes $f : X \xrightarrow{\sim} Y$ induces an isomorphism
$$WE_n(f) : \widetilde{E}_n(X) \xrightarrow{\sim} \widetilde{E}_n(Y) \tag{A.3.2}$$
for every n.

2. For any two finite CW complexes X and Y, the canonical map exhibits isomorphism
$$\widetilde{E}_n(X \vee Y) \simeq \widetilde{E}_n(X) \oplus \widetilde{E}_n(Y) \tag{A.3.3}$$
for all n.

3. For any finite CW complex X, we have canonical isomorphism
$$\widetilde{E}_{n+1}(\Sigma X) \simeq \widetilde{E}_n(X). \tag{A.3.4}$$

4. Let $X \to Y \to Z$ be a cofiber sequence of pointed finite CW complexes. Then we the sequence
$$\widetilde{E}_n(X) \to \widetilde{E}_n(Y) \to \widetilde{E}_n(Z) \tag{A.3.5}$$
is exact at $\widetilde{E}_n(Y)$.

Note that if $X \to Y \to Z$ is a cofiber sequence, then so is $Y \to Z \to \Sigma X$. As $\widetilde{E}_{n+1}(\Sigma X) \simeq \widetilde{E}_n(X)$, we see that we have a long exact sequence of homology groups
$$\cdots \to \widetilde{E}_n(X) \to \widetilde{E}_n(Y) \to \widetilde{E}_n(Z) \to \widetilde{E}_{n-1}(X) \to \cdots \tag{A.3.6}$$
A homology theory $\widetilde{E}_*$ is called ordinary if $\widetilde{E}_n(S^0) = 0$ for $n \neq 0$.

From a reduced homology theory we can define a nonreduced homology theory:

Definition A.3.7. Let $\widetilde{E}_*$ be a reduced homology theory. Let X be a (unpointed) CW complex. Then the nonreduced homology groups $E_i(X)$ is defined as
$$E_i(X) \coloneqq \widetilde{E}_i(X_+), \tag{A.3.8}$$
where X_+ is the X with an added basepoint. Functorially, E_n is the composition
$$CW \xrightarrow{-_+} CW_* \xrightarrow{\widetilde{E}_n} Ab. \tag{A.3.9}$$

Similarly, we can define a cohomology theory:

Definition A.3.10. An reduced (extraordinary) cohomology theory $\widetilde{E}^*$ is a sequence of functors $\widetilde{E}^n : CW^{op} \to Ab$ such that

1. A homotopy equivalence of pointed finite CW complexes $f : X \xrightarrow{\sim} Y$ induces an isomorphism
$$\widetilde{E}^n(f) : \widetilde{E}^n(Y) \xrightarrow{\sim} \widetilde{E}^n(X) \tag{A.3.11}$$
for every n.

2. For any two finite CW complexes X and Y, the canonical map exhibits isomorphism
$$\widetilde{E}^n(X \vee Y) \simeq \widetilde{E}^n(X) \oplus \widetilde{E}^n(Y) \tag{A.3.12}$$
for all n.

3. For any finite CW complex X, we have canonical isomorphism
$$\widetilde{E}^{n-1}(\Sigma X) \simeq \widetilde{E}^n(X). \tag{A.3.13}$$

4. Let $X \to Y \to Z$ be a cofiber sequence of pointed finite CW complexes. Then we the sequence
$$\widetilde{E}^n(Z) \to \widetilde{E}^n(Y) \to \widetilde{E}^n(X) \tag{A.3.14}$$
is exact at $\widetilde{E}^n(Y)$.

Simiarly, given a cofiber sequence $X \to Y \to Z$, we have a long exact sequence of cohomology groups
$$\cdots \to \widetilde{E}_n(X) \to \widetilde{E}_n(Y) \to \widetilde{E}_n(Z) \to \widetilde{E}_{n-1}(X) \to \cdots \tag{A.3.15}$$
A cohomology theory $\widetilde{E}^*$ is called ordinary if $\widetilde{E}^n(S^0) = 0$ for $n \neq 0$.

From a reduced cohomology theory we can define a nonreduced cohomology theory:

Definition A.3.16. Let $\widetilde{E}^*$ be a reduced cohomology theory. Let X be a (unpointed) CW complex. Then the nonreduced cohomology groups $E^i(X)$ is defined as
$$E^i(X) := \widetilde{E}^i(X_+), \tag{A.3.17}$$
where X_+ is the X with an added basepoint. Functorially, E^n is the composition
$$CW^{op} \xrightarrow{-_+} CW_*^{op} \xrightarrow{\widetilde{E}^n} Ab. \tag{A.3.18}$$

One of the original motivation for spectra is that they give homology and cohomology theories:

Construction A.3.19. Let $\mathcal{X}$ be a spectrum, X a pointed finite CW space. Then we define the n-th reduced homology of X with coefficients $\mathcal{X}$ as

$$\widetilde{\mathcal{X}}_n(X) \coloneqq \pi_n(\Sigma^\infty X \otimes \mathcal{X}). \tag{A.3.20}$$

Functorially, consider the composition

$$\widetilde{\mathcal{X}}_n : CW_* \xrightarrow{\Sigma^\infty} Sp \xrightarrow{\otimes \mathcal{X}} Sp \xrightarrow{\pi_n} Ab. \tag{A.3.21}$$

Similarly, we define the n-th reduced cohomology group of X with coefficients in $\mathcal{X}$ as

$$\widetilde{\mathcal{X}}^n(N) \coloneqq \pi_{-n}(Maps(\Sigma^\infty X, \mathcal{X})). \tag{A.3.22}$$

Functorially, consider the composition

$$\widetilde{\mathcal{X}}^n : CW_*^{op} \xrightarrow{\Sigma^\infty} Sp^{op} \xrightarrow{Maps(-,\mathcal{X})} Sp \xrightarrow{\pi_{-n}} Ab. \tag{A.3.23}$$

One can check that $\widetilde{\mathcal{X}}_n, \widetilde{\mathcal{X}}^n$ satisfies the Eilenberg-Steenrod axioms, thus they define homology and cohomology theory, respectively. Thus we can also define nonreduced homology and cohomology:

Definition A.3.24. Thus if X is an unpointed CW complex, the nonreduced homology $\mathcal{X}_i(X)$ of X with coefficients in $\mathcal{X}$ is

$$\mathcal{X}_i(X) \coloneqq \pi_i(\Sigma^\infty N \wedge \mathcal{X}). \tag{A.3.25}$$

And the nonreduced cohomology $\mathcal{X}^i(X)$ of X with coefficients in $\mathcal{X}$ is

$$\mathcal{X}^n(N) \coloneqq \pi_{-n}(Maps(\Sigma^\infty_+ X, \mathcal{X})). \tag{A.3.26}$$

Conversely, given a homology/cohomology, theory, we can find a spectrum representing it. This is the statements of Brown's representability theorems [3]:

Theorem A.3.27 (Brown's Representability)**.** *For any (reduced) homology theory $\widetilde{E}_n$, there is an essentially unique spectrum representing it. That is, there is a spectrum $\mathcal{X}$ such that $\widetilde{\mathcal{X}}_n \simeq \widetilde{E}_n$ compatible with all data.*

Note

$$\pi_n(\mathcal{X}) = \widetilde{\mathcal{X}}_n(S^0) \simeq \widetilde{E}_n(S^0). \tag{A.3.28}$$

Thus if $\widetilde{E}_*$ is an ordinary homology theory with $\widetilde{E}_0(S^0) = A$, then it is the homology theory associated to the Eilenberg-MacLane spectrum HA. This is why it's called ordinary.

We have a similar theorem for cohomology theories

Theorem A.3.29. *For any (reduced) cohomology theory $\widetilde{E}^n$, there is an essentially unique spectrum representing it. That is, there is a spectrum $\mathcal{X}$ such that $\widetilde{\mathcal{X}}^n \simeq E^n$ compatible with all data.*

Note that

$$\pi_n(\mathcal{X}) = \widetilde{\mathcal{X}}^{-n}(S^0) \simeq \widetilde{E}^{-n}(S^0). \tag{A.3.30}$$

Thus if $\widetilde{E}^*$ is an ordinary cohomology theory with $\widetilde{E}^0(S^0) = A$, then it is the cohomology theory associated to HA. We will not prove Brown's Representability theorems, see [3], [17] for details.

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