Abelian Duality in Topological Field Theory

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Abstract

We prove a topological version of abelian duality where the gauge groups are finite abelian. The theories are finite homotopy TFTs, topological analogues of the p-form $U(1)$ gauge theories where the gauge group is finite abelian. Using Brown-Comenetz duality, we extend our results to π-finite spectra.

Contents

Introduction

Abelian duality is a generalization of electromagnetic duality [22, 23] to higher gauge groups in general dimensions. The theories are p-form $U(1)$ gauge theories, free field theories whose dynamical fields are p-forms. In d

dimensions, abelian duality [4, 10, 16] is an equivalence of QFT between p-form and $(d-p-2)$-form abelian gauge theories.

In this paper we study the case where the gauge groups are finite abelian. These theories are topological and can be formulated mathematically as topological field theories (TFT) [3, 20].

Fix a dimension $d \geq 0$, an important class of topological field theories are the d dimensional finite gauge theories [14, 9] Z_{BG}, defined for any finite group G. Given a closed d dimensional manifold M, partition function $Z_{BG}(M)$ counts the equivalence classes of principal G bundles on M, weighted by automorphisms.

For an abelian Lie group A, there are higher analogues of principal A-bundle, called p-principal A-bundle [6, 7]. These bundles usually comes with geometric structures such as a connection, and they can also have higher automorphisms. When A is finite abelian, there are no geometric structures, and they can be understood via ordinary cohomology: the equivalence class of p-principal A-bundle on M is the cohomology group $H^p(M;A)$, the equivalence class of automorphisms of any principal p-bundle is $H^{p-1}(M;A)$, and automorphisms of automorphisms is $H^{p-2}(M;A)$...

In §2, we define the d dimensional, p-form gauge theories $Z_{K(A,p)}$. Similar to the finite gauge theories, the partition function $Z_{K(A,p)}(M)$ counts equivalence classes of p-principal A-bundle on M, weighted by not only the automorphisms of these higher bundles, but also the automorphisms of the automorphisms and so on. They are the topological analogue of the standard p-form gauge theories in QFT.

In QFT, abelian duality is a duality between p-form $U(1)$ gauge theories. In our topological case, the dual groups are Pontryagin dual groups. Let A be a finite abelian group, the Pontryagin dual (character dual) group $\hat{A}$ is defined as $Hom(A, \mathbb{C}^\times)$. In [15], Freed and Teleman proved that the 3 dimensional finite gauge theories $Z_{K(A,1)}$ and $Z_{K(\hat{A},1)}$ are equivalent. We extend this equivalence to general p and arbitrary dimension d:

Theorem (Abelian duality)**.** *Let A be a finite abelian group and $\hat{A}$ its Pontryagin dual. Let $d \geq 1$ be the dimension of our theories and $p \in \mathbb{Z}$. Let $Z_{K(A,p)}$, $Z_{K(\hat{A},d-1-p)}$ be the d dimensional finite homotopy TFTs associated to $K(A,n)$ and $K(\hat{A}, d-1-p)$. There is an equivalence of oriented topological field theories:*

$$Z_{K(A,p)} \simeq Z_{K(\hat{A},d-1-p)} \otimes E_{|A|^{(-1)^p}}, \tag{0.1}$$

where is $E_{|A|^{(-1)^p}}$ is the d dimensional Euler invertible TFT (§3) associated to $|A|^{(-1)^p} \in \mathbb{C}^\times$.

We extend our result to the context of π-finite spectra. A spectrum $\mathcal{X}$ is π-finite (§1) if the stable homotopy groups $\pi_*(\mathcal{X})$ are nontrivial in finitely many degree, and each one is a finite abelian group. The canonical example of a π-finite spectrum is the suspended Eilenberg-MacLane spectrum $\Sigma^p HA$, where A is a finite abelian group. Given a π-finite spectrum $\mathcal{X}$, we define a d dimensional finite homotopy TFT $Z_{\mathcal{X}}$ [8, 11, 13, 21], which counts "$\mathcal{X}$" bundles. Note that these TFTs can be defined for general π-finite spaces, however, we will work in the setting of π-finite spectra. See Remark 2.21 for how they are related.

Pontryagin duality can also be extended to π-finite spectra. In [5], Brown and Comenetz defined a dual spectrum $\hat{\mathcal{X}}$ for any spectrum $\mathcal{X}$. When $\mathcal{X}$ is π-finite, then $\hat{\mathcal{X}}$ is also π-finite. We review this in §5. It is a generalization of Pontryagin duality: let A be an abelian group and HA be its Eilenberg-MacLane spectrum, then

$$\widehat{HA} = H\hat{A}, \tag{0.2}$$

where $\hat{A}$ is the Pontryagin dual group of A.

Our main theorem is an extension of the theorem above to π-finite spectra. See Theorem 6.1 for the rigorous statement.

Theorem (Abelian duality)**.** *Let $\mathcal{X}$ be a π-finite spectrum and $\hat{\mathcal{X}}$ its Brown-Comenetz dual. Let $Z_{\mathcal{X}}, Z_{\Sigma^{d-1}\hat{\mathcal{X}}}$ be the corresponding d dimensional finite homotopy TFTs. There is an equivalence of (suitably oriented) topological field theory:*

$$\mathbb{D} : Z_{\mathcal{X}} \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}} \otimes E_{|\mathcal{X}|}, \tag{0.3}$$

where is $E_{|\mathcal{X}|}$ is the d dimensional Euler invertible TFT (§3).

Remark 0.4. Orientations are needed for abelian duality. When generalizing to π-finite spectra, we need a notion of orientation ([19, 2]) with respect to a ring spectrum. We review this in §4.

Outline In §1 we review the basics of π-finite spectra. In §2 we define the finite homotopy TFTs associated to π-finite spectra. In §3 we define the Euler invertible TFT. In §4 we review orientation for generalized cohomology theories and Poincaré duality. In §5 we review Pontryagin and Brown-Comenetz duality. In §6 we state and prove the main theorem 6.1.

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1 π-finite spectra

In this section, we define π-finite spectra and define their sizes. We assume some familiarity with spectra, see [1, 17]. First we recall some basic facts about stable homotopy theory.

Let S, Sp denote the category of spaces and spectra. $\Sigma^\infty_+ : S \to Sp$ is the infinite suspension functor. Let $\tau_{\geq i}$ ($\tau_{\leq i}$) be the truncation functor that takes a spectrum to its i-th connective cover (truncation). Let A be an abelian group, its Eilenberg-MacLane spectrum is denoted as HA.

Let M be a (unpointed) space, $\mathcal{X}$ a spectrum. We denote the homology and cohomology groups of M with $\mathcal{X}$ coefficients as $\mathcal{X}_*(M)$, $\mathcal{X}^*(M)$. Similarly, let N be a pointed space, then $\widetilde{\mathcal{X}}_*(N)$, $\widetilde{\mathcal{X}}^*(N)$ are the reduced homology and cohomology groups of N with $\mathcal{X}$ coefficients.

Given a cofiber sequence of (unpointed) spaces

$$N \to M \to M/N. \tag{1.1}$$

M/N is canonically pointed. The reduced (co)homology groups of M/N is called the relative (co)homology groups of (M, N), denoted as $\mathcal{X}_*(M, N)$ ($\mathcal{X}^*(M, N)$).

We denote the mapping spectrum $\underline{Maps}(\Sigma^\infty_+ M, \mathcal{X})$ as $\mathcal{X}(M)$, its homotopy groups are cohomology groups:

$$\pi_i(\mathcal{X}(M)) = \mathcal{X}^{-i}(M). \tag{1.2}$$

A spectrum $\mathcal{Y}$ is an extension of $\mathcal{X}, \mathcal{Z}$ if there is a fiber sequence

$$\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}. \tag{1.3}$$

Given a full subcategory C, the extension closure $\overline{C}$ is the smallest full subcategory that contains C and is closed under extensions and suspension. For any property P, if all objects of C satisfies P, and P is closed under extensions and suspension, then inductively, any object in the extension closure $\overline{C}$ satisfies P.

Definition 1.4. A spectrum $\mathcal{X}$ is π-finite if

1. $\pi_i\mathcal{X}$ are non-trivial in only finitely many degrees.

2. $\pi_i \mathcal{X}$ are finite abelian groups for all $i \in \mathbb{Z}$.

We denote the category of π-finite spectra by Sp^{fin}. It is closed under suspensions and extensions. Canonical examples of π-finite spectra are $\Sigma^n HA$ where A is a finite abelian group.

Given a π-finite spectrum $\mathcal{X}$ with homotopy groups concentrated in degrees $\leq n$. Consider the fiber sequence

$$\tau_{\geq n}\mathcal{X} \to \mathcal{X} \to \tau_{\leq n-1}\mathcal{X}. \tag{1.5}$$

As $\tau_{\geq n}\mathcal{X}$ only have nontrivial homotopy groups in degree n, it is a suspended finite Eilenberg-MacLane spectrum $\tau_{\geq n}\mathcal{X} \simeq \Sigma^n H\pi_n(\mathcal{X})$. Do this inductively on n, we see that $\mathcal{X}$ lies in the extension closure of full-subcategory of finite Eilenberg-MacLane spectra. Therefore, Sp^{fin} is the extension closure of finite Eilenberg-MacLane spectra. In fact, as finite abelian group are extensions of $\mathbb{F}_p$, Sp^{fin} is the extension closure of finite Eilenberg-MacLane spectra of the form $H\mathbb{F}_p$.

Given a suitably finite graded abelian group, we have a notion of "homotopic size":

Definition 1.6. Let $A_\bullet = \bigoplus A_i$ be a $\mathbb{Z}$-graded abelian group. $A_\bullet$ is finite if all but finitely many A_i are trivial and each A_i is a finite abelian group. The size of $A_\bullet$ is

$$|A_\bullet| := \prod_i |A_i|^{(-1)^i}, \tag{1.7}$$

where $|A_i|$ is the cardinality of A_i.

We have a neat algebraic fact:

Lemma 1.8. *Given an exact sequence of finite graded abelian group $H^0_\bullet \to H^1_\bullet \to H^2_\bullet$, that is, a long exact sequence*

$$\cdots \to H^0_* \to H^1_* \to H^2_* \to H^0_{*-1} \to \cdots, \tag{1.9}$$

then

$$|H^0_\bullet||H^2_\bullet| = |H^1_\bullet|. \tag{1.10}$$

Alternatively, viewed the pieces of a long exact sequence as a finite graded abelian group $H_\bullet$, then

$$|H_\bullet| = 1 \tag{1.11}$$

Let $\mathcal{X}$ be a π-finite spectrum, then $\pi_\bullet(\mathcal{X})$ is a finite graded abelian group. We can define its size:

Definition 1.12. The size of $\mathcal{X}$, denoted as $|\mathcal{X}|$, is the size of its homotopy groups

$$|\pi_\bullet(\mathcal{X})| = \prod_i |\pi_i(X)|^{(-1)^i} = \cdots \frac{|\pi_0\mathcal{X}|}{|\pi_{-1}\mathcal{X}|} \frac{|\pi_2\mathcal{X}|}{|\pi_1\mathcal{X}|} \cdots . \tag{1.13}$$

Proposition 1.14. *Given a fiber sequence $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ of π-finite spectra, we have that $|\mathcal{X}|\ |\mathcal{Z}| = |\mathcal{Y}|$.*

Proof. This follows from applying Lemma 1.8 to the long exact sequence of homotopy groups associated to the fiber sequence $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$. □

Example 1.15. Let $\mathcal{X}$ be a π-finite spectrum. We have a fiber sequence

$$\tau_{\geq i}\mathcal{X} \to \mathcal{X} \to \tau_{\leq i-1}\mathcal{X} \tag{1.16}$$

of π-finite spaces. By Proposition 1.14 we have

$$|\tau_{\geq i}\mathcal{X}|\ |\tau_{\leq i-1}\mathcal{X}| = |\mathcal{X}|. \tag{1.17}$$

Proposition 1.18. *Let M be a d dimensional compact manifold (possibly with boundary) and $\mathcal{X}$ a π-finite spectrum, then the mapping spectrum $\mathcal{X}(M)$ is a π-finite spectrum of size*

$$|\mathcal{X}(M)| = |\mathcal{X}|^{\chi(M)}. \tag{1.19}$$

Proof. Fix M, we first proof that $\mathcal{X}(M)$ is π-finite in the case that $\mathcal{X} = HA$. As M is compact, $\pi_{-i}HA(M) \simeq H^i(M; A)$ are finite in each degree. It is also trivial when outside of degrees $d \leq i \leq 0$. Therefore $HA(M)$ is a π-finite spectrum.

Consider $\mathcal{X}(M)$ being π-finite as a property on $\mathcal{X}$. This property is clearly closed under suspension. It is also closed under extension: given fiber sequence $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$, we have the following fiber sequence

$$\mathcal{X}(M) \to \mathcal{Y}(M) \to \mathcal{Z}(M). \tag{1.20}$$

If $\mathcal{X}(M), \mathcal{Z}(M)$ are π-finite, then so is $\mathcal{Y}(M)$. Since Sp^{fin} is the extension closure of finite Eilenberg-MacLane spectra, we see that $\mathcal{X}(M)$ is π-finite for all $\mathcal{X} \in Sp^{fin}$.

Now we prove that $|\mathcal{X}(M)| = |\mathcal{X}|^{\chi(M)}$. First consider the case that $\mathcal{X} = H\mathbb{F}_p$, then

$$\pi_i(H\mathbb{F}_p(M)) = H^{-i}(M; H\mathbb{F}_p) \tag{1.21}$$

are finite dimensional $\mathbb{F}_p$ vector spaces. Let d be its dimension, then

$$|\pi_i(H\mathbb{F}_p(M))| = p^d. \tag{1.22}$$

As

$$\sum_i (-1)^i dim_{\mathbb{F}_p} H^i(M; H\mathbb{F}_p) = \chi(M) \tag{1.23}$$

It follows that

$$|H\mathbb{F}_p(M)| = |H\mathbb{F}_p|^{\chi(M)}. \tag{1.24}$$

Consider equation $|\mathcal{X}(M)| = |\mathcal{X}|^{\chi(M)}$ as a property on spectrum $\mathcal{X}$. As both sides of the equation are multiplicative with respect to extensions, this property is closed under suspension and extension. Since Sp^{fin} is the extension closure of spectra of the form $H\mathbb{F}_p$, it holds for all $\mathcal{X} \in Sp^{fin}$. $\square$

2 Finite Homotopy TFT

In this section we construct the d dimensional (unoriented) TFT $Z_{\mathcal{X}}$ associated to a π-finite spectrum $\mathcal{X}$. Recall that a d dimensional (unoriented) topological field theory (TFT) [3, 20, 18] is a symmetric monoidal functor

$$Z : Bord_d \to Vect_{\mathbb{C}}, \tag{2.1}$$

where

1. $Bord_d$ is the d dimensional bordism category. Its objects are closed $d-$1 dimensional manifolds, and morphisms are bordisms. It is symmetric monoidal under disjoint union.
2. $Vect_{\mathbb{C}}$ is the category of finite dimensional $\mathbb{C}$-linear vector spaces, symmetric monoidal under tensor products.

Construction 2.2. Fix a dimension $d \geq 0$, consider the following assignment: for any closed $d-1$ dimensional manifold N, we assign the vector space $\mathbb{C}[\mathcal{X}^0(N)]$; for any bordism $M : N \to N'$, the map

$$Z_{\mathcal{X}}(M) : \mathbb{C}[\mathcal{X}^0(N)] \to \mathbb{C}[\mathcal{X}^0(N')] \tag{2.3}$$

is defined as follows: for $a \in \mathcal{X}^0(N)$, considered as a basis element in $\mathbb{C}[\mathcal{X}^0(N)]$, $Z_{\mathcal{X}}(M)$ takes

$$a \mapsto \frac{|\mathcal{X}^1(N')|}{|\mathcal{X}^1(M)|}\frac{|\mathcal{X}^2(M)|}{|\mathcal{X}^2(N')|}\frac{|\mathcal{X}^3(N')|}{|\mathcal{X}^3(M)|}\cdots\sum_{b\to a} q^*b \tag{2.4}$$

$$= \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|}\sum_{b\to a} q^*b \tag{2.5}$$

$$= \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|}\sum_{a'}\sum_{b\to a, b\to a'} a', \tag{2.6}$$

where $\sum_{b\to a}$ means sum over all $b \in \mathcal{X}^0(M)$ such that b maps to a.

This defines a topological field theory:

Proposition 2.7. *The assignment above defines a symmetric monoidal functor $Z_{\mathcal{X}} : Bord_d \to Vect_{\mathbb{C}}$, that is, a topological field theory.*

Proof. We first show that it is a functor, that is, the assignment $Z_{\mathcal{X}}$ composes:

Given two bordism $M : N \to N'$, $M' : N' \to N''$, the composite is $M \sqcup_{N'} M' : N \to N''$. We need to show that

$$Z_{\mathcal{X}}(M') \circ Z_{\mathcal{X}(M)} = Z_{\mathcal{X}}(M \sqcup_{N'} M') \tag{2.8}$$

Given $a \in \mathcal{X}^0(N)$, considered as a basis vector on $\mathbb{C}[\mathcal{X}^0(N)]$, then

$$Z_{\mathcal{X}}(M') \circ Z_{\mathcal{X}}(M)\ a = Z_{\mathcal{X}}(M') \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{a'} \sum_{b\to a, b\to a'} a' \tag{2.9}$$

$$= \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \frac{|\tau_{\geq 1}\mathcal{X}(M')|}{|\tau_{\geq 1}\mathcal{X}(N'')|} \tag{2.10}$$

$$\sum_{a'} \sum_{b\to a, b\to a'} \sum_{a''} \sum_{b'\to a', b'\to a''} a'' \tag{2.11}$$

On the hand,

$$Z_{\mathcal{X}}(M \sqcup_{N'} M')a = \frac{|\tau_{\geq 1}\mathcal{X}(M \sqcup_{N'} M')|}{|\tau_{\geq 1}\mathcal{X}(N'')|} \sum_{a''} \sum_{b\to a, b\to a'} a'' \tag{2.12}$$

Fix $a'' \in \mathcal{X}(N'')$, it is suffice to show that

$$\frac{|\tau_{\geq 1}\mathcal{X}(M)||\tau_{\geq 1}\mathcal{X}(M')|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{b,b',f^*b=g^*b'} 1 = |\tau_{\geq 1}\mathcal{X}(M \sqcup_{N'} M')| \sum_{c\to a, c\to a''} 1 \tag{2.13}$$

The sum $\sum_{b,b',f^*b=g^*b'}$ sums over pairs $b \in \mathcal{X}^0(M)$, $b' \in \mathcal{X}^0(M')$ that pulls back to the same element in $\mathcal{X}^0(N')$. This is a (homotopy) pushout diagram:

$$\begin{tikzcd} N' \arrow[r, "f'"] \arrow[d, "g'"] & M' \arrow[d, "g"] \\ M \arrow[r, "f"] & M \sqcup_{N'} N' \end{tikzcd} \tag{2.14}$$

Consider the following truncated Mayer-Vietoris sequence:

$$\begin{aligned} &\cdots \to \mathcal{X}^{-1}(M \sqcup_{N'} M') \to \mathcal{X}^{-1}(M) \oplus \mathcal{X}^{-1}(M') \to \mathcal{X}^{-1}(N) \\ &\to \mathcal{X}^{0}(M \sqcup_{N'} M') \to ker(\mathcal{X}^{0}(M) \oplus \mathcal{X}^{0}(M') \to \mathcal{X}^{0}(N)) \to 0 \end{aligned} \tag{2.15}$$

The elements of $ker(\mathcal{X}^0(M) \oplus \mathcal{X}^0(M') \to \mathcal{X}^0(N))$ are exactly pairs $b, b', f^*b = g^*b'$. From the exactness at $ker(\mathcal{X}^0(M) \oplus \mathcal{X}^0(M') \to \mathcal{X}^0(N))$, given a, a'', there is a c that pulls back to a, a'' iff there is $b, b', f^*b = g^*b'$ that pulls back to a, a'' respectively. Therefore for each a, a'', the right hand side of 2.13 vanishes iff the left hand side does.

In addition, for each a, a'', if the two sides doesn't vanish, I claim that the sum is independent of a, a''. For left hand side, this is because the set of $c \in \mathcal{X}^0(M \sqcup_{N'} M')$ that maps to a, a'' is a torsor for the kernel of the map $\mathcal{X}^0(M \sqcup_{N'} M') \to \mathcal{X}^0(N) \oplus \mathcal{X}^0(N'')$. The same is true for the left hand side.

Combining this two facts, it is suffice to show the sum over all a, a'' of the two sides are equal, that is, it is suffice to show that

$$\frac{|\tau_{\geq 1}\mathcal{X}(M)||\tau_{\geq 1}\mathcal{X}(M')|}{|\tau_{\geq 1}\mathcal{X}(N')|} |ker(\mathcal{X}^0(M) \oplus \mathcal{X}^0(M') \to \mathcal{X}^0(N))| \tag{2.16}$$

$$= |\tau_{\geq 1}\mathcal{X}(M \sqcup_{N'} M')||\mathcal{X}^0(M \sqcup_{N'} M')| \tag{2.17}$$

which follows directly from applying Lemma 1.8 to the exact sequence 2.15.

Now we show that it is symmetric monoidal. On object, it follows from

$$\chi^0(N \sqcup N') \simeq \chi^0(N) \times \chi^0(N') \tag{2.18}$$

and

$$\mathbb{C}[X \times Y] \simeq \mathbb{C}[X] \otimes \mathbb{C}[Y] \tag{2.19}$$

On morphisms, as the maps are natural, it follows that they are compatible with the isomorphism on objects. □

Example 2.20. Let $\mathcal{X} = \Sigma^p HA$ where A is a finite abelian group. Recall that $H^p(M; A)$ classifies p-principal A-bundles on A, and $H^{p-i}(M; A)$ classifies level i automorphism of the bundles.

Given a closed d manifold M, the partition function $Z_{\Sigma^p HA}(M)$ counts the number of p-principal A-bundles on M, weighted with multiplicities. For a closed $d-1$ manifold N, $Z_{\Sigma^p HA}(N)$, the space of states on N, is $\mathbb{C}[H^n(M; A)]$. This is the $Z_{K(A,p)}$ theory that we described in the introduction, the p-form gauge theories where the gauge group is finite.

Remark 2.21. Let $\Omega^\infty : Sp \to S$ be the underlying space functor. The theory $Z_{\mathcal{X}}$ depends only on the space $\Omega^\infty \mathcal{X}$, which is a π-finite space, that is, it has finitely many connected components, and each component has finite homotopy groups nontrivial in only finite degrees.

In fact, finite homotopy TFTs Z'_X [8, 11, 13, 21] can be defined for any π-finite space X. For a π-finite spectrum $\mathcal{X}$, the TFT we defined $Z_{\mathcal{X}}$ agrees with the more general version:

$$Z_{\mathcal{X}} \simeq Z'_{\Omega^\infty \mathcal{X}}. \tag{2.22}$$

3 Euler TFT

In this section, we define the d dimensional Euler invertible TFT associated to a nonzero complex number $\lambda \in \mathbb{C}^\times$.

Recall that a TFT Z is invertible if

1. for every closed $d-1$ manifold N, $Z(N)$ is a one dimensional vector space (a line).
2. for every bordism $M : N \to N'$, $Z(M) : Z(N) \to Z(N')$ is an isomorphism of lines.

Let M be a d dimensional compact manifold (possibly with boundary), its Euler characteristic $\chi(M)$ is defined as the alternating sum

$$\sum^{i} (-1)^i dim_F H^i(M; F) \tag{3.1}$$

for any field F. This is well-defined as the cohomology groups are finite dimensional F vector spaces and nonzero only in degrees from 0 to d.

Definition 3.2. Let $\lambda \in \mathbb{C}^\times$ be a nonzero complex number, the d dimensional Euler TFT E_λ is defined as follows: for any closed $d-1$ manifold N,

$$E_\lambda(N) := \mathbb{C}. \tag{3.3}$$

For a bordism $M : N \to N'$,

$$E_\lambda(M) : \mathbb{C} \to \mathbb{C} \tag{3.4}$$

is given by multiplication by $\lambda^{\chi(M)-\chi(N)} \in \mathbb{C}^\times$.

It is a topological field theory by the following lemma:

Lemma 3.5. *Given closed $d-1$ manifolds N, N', N'' and bordisms $M : N \to N'$ and $M' : N' \to N''$, then*

$$\chi(M \sqcup_{N'} M') - \chi(N) = \chi(M) - \chi(N) + \chi(M') - \chi(N'). \tag{3.6}$$

Remark 3.7. The Euler TFTs E_λ are invertible field theories.

Example 3.8. Let $\lambda \neq 1, -1$. In even dimensions $d = 2n$, the d dimensional sphere S^d has Euler characteristic $\chi(S^d) = 2$. Therefore $E_\lambda(S^d) = \lambda^2 \neq 1$. We see that in even dimensions, the Euler TFT E_λ is nontrivial.

In odd dimensions, by Poincaré duality (with $\mathbb{F}_2$ coefficients), the Euler characteristic of a closed d manifold is 0, so $Z(M) = 1$ for every closed d manifold M. In fact, we have a stronger statement:

Proposition 3.9. *Let d be odd. For any $\lambda \in \mathbb{C}^\times$, the d dimensional Euler TFT is trivial, that is, $E_\lambda \simeq Z_{triv}$.*

4 Orientation and Poincaré duality for spectra

As discussed in the introduction, the main theorem 6.1 needs orientation and Poincaré duality in an essential way. In this section we review the general theory of orientation [2, 19]. The new ingredient here is a $\mathbb{E}_1$-ring spectrum [17], which is a homotopic generalization of associative ring.

Let M be a d manifold, $\mathcal{R}$ a $\mathbb{E}_1$-ring spectrum, note $\pi_*\mathcal{R}$ inherits a graded ring structure. An $\mathcal{R}$-orientation on M is a homology class

$$[M] \in \mathcal{R}_d(M, \partial M) \tag{4.1}$$

satisfying the following condition: for every interior point $x \in M^o$ a point in the interior, the image of $[M]$ under

$$\mathcal{R}_d(M, \partial M) \to \mathcal{R}_d(M, M - x) \simeq \widetilde{\mathcal{X}}_d(S^d) \simeq \pi_0(\mathcal{R}) \tag{4.2}$$

is an multiplicative unit in the ring $\pi_*(\mathcal{R})$. An $\mathcal{R}$-orientation on a d-dimensional manifold gives a $\mathcal{R}$-orientation on the boundary:

Proposition 4.3. *Let M be a d manifold. A $\mathcal{R}$-orientation on M, $[M] \in \mathcal{R}_d(M, \partial M)$, gives a class $\partial[M] \in \mathcal{R}_{d-1}(N)$ via the natural boundary map*

$$\partial : \mathcal{R}_*(M, \partial M) \to \mathcal{R}_{*-1}(\partial M). \tag{4.4}$$

The class $\partial[M] \in \mathcal{R}_{*-1}(\partial M)$ is a $\mathcal{R}$-orientation on the boundary ∂M.

For any $\mathbb{E}_1$-ring spectrum $\mathcal{R}$, we can define the d dimensional $\mathcal{R}$-oriented bordism category $Bord_d^{\mathcal{R}}$: the object closed $\mathcal{R}$-oriented $d-1$ manifold $(N,[N])$, and a morphism

$$(M,[M]) : (N,[N]) \to (N',[N']) \tag{4.5}$$

a $\mathcal{R}$-oriented bordism M whose orientation class $[M]$ restricts to $[N]$ and $-[N']$ on the boundaries. The minus sign for $[N']$ is necessary for $\mathcal{R}$-oriented bordisms to compose. It is symmetric monoidal under disjoint union. See [18] for details.

Definition 4.6. A $\mathcal{R}$-oriented topological field theory is a symmetric monoidal functor

$$Z : Bord_d^{\mathcal{R}} \to Vect_{\mathbb{C}} \tag{4.7}$$

The forgetful functor is a symmetric monoidal functor $Bord_d^{\mathcal{R}} \to Bord_d$, thus any (unoriented) TFT pullback to a $\mathcal{R}$ oriented TFT.

Orientation gives Poincaré duality via the cap product construction:

Construction 4.8. Let $\mathcal{R}$ be a $\mathbb{E}_1$ ringed spectrum, and $\mathcal{X}$ a left $\mathcal{R}$ module. Let

$$f : N \to N' \wedge N'' \tag{4.9}$$

be a map of pointed spaces. The cap product [1] is a map

$$- \frown - : \widetilde{\mathcal{R}}_n(N) \otimes \widetilde{\mathcal{X}}^n(N') \to \widetilde{\mathcal{X}}_{m-n}(N''). \tag{4.10}$$

In our setting, let M be a $\mathcal{R}$-oriented d-manifold with boundary $\partial M = N \sqcup N'$. We have the orientation class $[M] \in \mathcal{R}_d(M, \partial M)$. Consider the map

$$M/\partial M \to M/N \wedge M/N' \tag{4.11}$$

of pointed spaces. From Equation 4.10 we get a map

$$[M] \frown - : \mathcal{X}^*(M,N) \to \mathcal{X}_{d-*}(M,N'). \tag{4.12}$$

We denote this map by $\int_{[M,N]}$. In the case that $N = \varnothing$, then we will denote this as $\int_{[M]}$. Poincaré duality says taking cap product give functorial isomorphism:

Theorem 4.13. *[?] Let $\mathcal{R}$ be a $\mathbb{E}_1$-ring spectrum and $\mathcal{X}$ a left $\mathcal{R}$-module spectrum. Let M be a $\mathcal{R}$-oriented d-manifold with boundary $\partial M = N \sqcup N'$.* give references

We denote the orientation class as $[M]$. It restricts to orientations $[N], [N']$ on the boundaries. Cap product gives isomorphisms of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^*(M, N') \arrow[r] \arrow[d, "\int_{[M,N']}"] & \hat{\mathcal{X}}^*(M) \arrow[r] \arrow[d, "\int_{[M]}"] & \hat{\mathcal{X}}^*(N') \arrow[r] \arrow[d, "\int_{[N']}"] & \cdots \arrow[d] \\
\cdots \arrow[r] & \hat{\mathcal{X}}_{d-*}(M, N) \arrow[r] & \hat{\mathcal{X}}_{d-*}(M, \partial M) \arrow[r] & \hat{\mathcal{X}}_{d-1-*}(N') \arrow[r] & \cdots
\end{tikzcd} \tag{4.14}$$

Example 4.15. Let $\mathcal{R} = H\mathbb{F}_2$. There is only one unit in $\pi_0(H\mathbb{F}_2) = \mathbb{F}_2$, the unique local choice glue together to give the $H\mathbb{F}_2$-orientation. In another word, any manifold is uniquely $H\mathbb{F}_2$-oriented. The $H\mathbb{F}_2$ bordism category is simply the unoriented bordism category $Bord_d$. π-finite $H\mathbb{F}_2$-module spectrum can be represented as bounded chain complexes of finite dimensional $\mathbb{F}_2$-vector spaces.

Example 4.16. Let $\mathcal{R} = H\mathbb{Z}$, then $H\mathbb{Z}$-orientation is the standard notion of orientation, and the $H\mathbb{Z}$-oriented bordism category is the oriented bordism category. π-finite $H\mathbb{Z}$-module spectrum can be represented as bounded chain complexes of finitely abelian groups.

Example 4.17. Let $\mathcal{R} = \mathcal{S}$ the sphere spectrum, then a $\mathcal{S}$-orientation is a trivialization of the Thom spectrum of the normal bundle. π-finite $\mathcal{S}$ modules are simply π-finite spectra.

5 Brown-Comenetz Duality

In this section we review Pontryagin duality for finite abelian groups and Brown-Comenetz duality [5] for π-finite spectra. We first start with Pontryagin duality: Let A be a abelian group, $\mathbb{C}^\times$ the group of nonzero complex number. The Pontryagin dual group $\hat{A}$ is the group of characters $Hom(A, \mathbb{C}^\times)$. Let Ab be the category of abelian group, taking Pontryagin dual defines a contravariant functor:

$$\hat{D} \coloneqq Hom(-, \mathbb{C}^\times) : Ab \to (Ab)^{op}. \tag{5.1}$$

The canonical pairing

$$\mu : A \otimes \hat{A} \to \mathbb{C}^\times. \tag{5.2}$$

gives rise to a map $\alpha_\mu : \mathbb{C}[A] \otimes \mathbb{C}[\hat{A}] \to \mathbb{C}$. On basis vectors it sends $a \otimes \alpha \to \mu(a, \alpha) = \alpha(a)$, where we identify an element $a \in A$ with the standard basis vector in $\mathbb{C}[A]$.

Let Ab^{fin} be the full subcategory of fintie abelian group. We now assume that A is a finite abelian group. In this case, the pairing is a nondegenerate pairing, that is, the induce map $(\mathbb{C}[A])^* \to \mathbb{C}[\hat{A}]$ is an isomorphism. Taking dimension, this implies that the dual groups have equal cardinality: $|A| = |\hat{A}|$. We see that Pontryagin dual restricts to a functor:

$$\hat{D} : Ab^{fin} \to ((Ab)^{fin})^{op}. \tag{5.3}$$

In addition, we have the corollary:

Corollary 5.4. *The map*

$$\begin{aligned} \mathbb{C}[A] &\to \mathbb{C}[\hat{A}] \\ a &\mapsto \sum_{\alpha} \mu(a, \alpha)\ \alpha \end{aligned} \tag{5.5}$$

is an isomorphism.

The nondegeneracy of the pairing also implies that taking Pontryagin dual $\hat{D}$ is an involution on Ab^{fin}:

Theorem 5.6. *Restricted to Ab^{fin}, $\hat{D}^2 \simeq id$.*

As a corollary, we get that Pontryagin duality is in fact a duality on finite abelian groups:

Corollary 5.7. *$\hat{D} : Ab^{fin} \to (Ab^{fin})^{op}$ is an equivalence of categories.*

Remark 5.8. Normally, the Pontryagin dual of A is defined to be $Hom(A, \mathbb{Q}/\mathbb{Z})$. For a finite abelian group A, the natural map

$$Hom(A, \mathbb{Q}/\mathbb{Z}) \to Hom(A, \mathbb{C}^\times) \tag{5.9}$$

is an isomorphism and the two notions coincide. We choose $\mathbb{C}^\times$ over $\mathbb{Q}/\mathbb{Z}$ as our TFTs are complex-valued.

Now we will generalize Pontryagin duality to π-finite spectra. There is a spectrum $I\mathbb{C}^\times$ that plays the role of $\mathbb{C}^\times$:

Theorem 5.10. *[5] There exists a spectrum $I\mathbb{C}^\times$ with the following property: for any spectra $\mathcal{X}$, there is a functorial equivalence*

$$\pi_{-*}(\underline{Maps}(\mathcal{X}, I\mathbb{C}^\times)) \simeq \widehat{\pi_*(\mathcal{X})}. \tag{5.11}$$

More precisely, we view both sides as families of functors $Sp \to Ab$, and we claim that there is a natural isomorphism between these two families of functors, compatible with the connecting homomorphisms.

Definition 5.12. Let $\mathcal{X}$ be a spectrum. The Brown-Comenetz dual spectrum $\hat{\mathcal{X}}$ is defined to be the mapping spectrum $\underline{Maps}(\mathcal{X}, I\mathbb{C}^\times)$.

This defines a contravariant functor

$$\hat{\mathcal{D}} \coloneqq \underline{Maps}(-, I\mathbb{C}^\times) : Sp \to Sp^{op}. \tag{5.13}$$

Example 5.14. Let $\mathcal{X}$ be the sphere spectrum $\mathcal{S}$. Then

$$\hat{\mathcal{S}} = \underline{Maps}(\mathcal{S}, I\mathbb{C}^\times) \simeq I\mathbb{C}^\times. \tag{5.15}$$

Therefore $I\mathbb{C}^\times$ is the Brown-Comenetz dual of the sphere spectrum $\mathcal{S}$. This is similar to the fact that $\mathbb{C}^\times$ is the Pontryagin dual group of $\mathbb{Z}$.

Remark 5.16. The common approach to Brown-Comenetz uses $I\mathbb{Q}/\mathbb{Z}$ rather than $I\mathbb{C}^\times$. As with the abelian group case, they give the same answers on π-finite spectra. We use $I\mathbb{C}^\times$ over $I\mathbb{Q}/\mathbb{Z}$ because the target of our TFTs are complex-valued and $I\mathbb{C}^\times$ is the natural target for invertible TFTs (see [12]).

By equation 5.11 and the fact that Pontryagin dual of a finite abelian group is also finite, we see that taking Brown-Comenetz dual restricts to a functor

$$\hat{\mathcal{D}} \coloneqq \underline{Maps}(-, I\mathbb{C}^\times) : Sp^{fin} \to (Sp^{fin})^{op}. \tag{5.17}$$

There is a natural transformation $id \to \hat{\mathcal{D}}^2$, given by

$$\begin{aligned} &\mathcal{X} \to \hat{\hat{\mathcal{X}}} = \underline{Maps}(\underline{Maps}(\mathcal{X}, I\mathbb{C}^\times), I\mathbb{C}^\times) \\ &x \mapsto (\alpha \mapsto \alpha(a)). \end{aligned} \tag{5.18}$$

Restricting to π-finite spectra, this natural transformation is an isomorphism:

Theorem 5.19. *For π-finite spectrum $\mathcal{X}$, the natural map 5.18 is an isomorphism. Therefore, we have an equivalence of functors*

$$\hat{\mathcal{D}}^2 \simeq id : Sp^{fin} \to Sp^{fin}. \tag{5.20}$$

As a corollary, we get that Brown-Comenetz duality is in fact a duality on finite abelian groups:

Corollary 5.21. *$\hat{D} : Sp^{fin} \to (Sp^{fin})^{op}$ is an equivalence of categories.*

We also have the following corollary:

Corollary 5.22. *Let $\mathcal{X}$ be a π-finite spectra and $N \to M \to M/N$ a cofiber sequence of finite CW complexes. The Pontryagin dual of the long exact sequence of cohomology group with $\mathcal{X}$ coefficients:*

$$\cdots \to \mathcal{X}^*(M,N) \to \mathcal{X}^*(M) \to \mathcal{X}^*(N) \to \cdots \tag{5.23}$$

is canonically isomorphic to the long exact sequence of homology group with $\hat{\mathcal{X}}$ coefficients:

$$\cdots \leftarrow \hat{\mathcal{X}}_*(M,N) \leftarrow \hat{\mathcal{X}}_*(M) \leftarrow \hat{\mathcal{X}}_*(N) \leftarrow \cdots . \tag{5.24}$$

6 Abelian Duality

6.1 Statement of Main Theorem and Proof

In this section we state and prove the main theorem 6.1. Fix $d \geq 1$ the dimension of our theory. Let $\mathcal{R}$ be a $\mathbb{E}_1$-ring spectrum and $\mathcal{X}$ a π-finite left $\mathcal{R}$-module spectrum. The Brown-Comenetz dual $\hat{\mathcal{X}}$ is a π-finite right $\mathcal{R}$-module. In §2, we defined the d-dimensional finite homotopy TFTs $Z_{\mathcal{X}}$ and $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}$ associated to $\mathcal{X}$ and $\Sigma^{d-1}\hat{\mathcal{X}}$. In addition, if λ is a nonzero complex number, we have the d-dimensional Euler invertible TFT E_λ (§3).

In §4 we defined the bordism category $Bord_d^{\mathcal{R}}$ of $\mathcal{R}$-oriented manifolds and bordisms. Any unoriented TFT can be viewed as a $\mathcal{R}$-oriented TFT by precomposing with the forgetful map $Bord_d^{\mathcal{R}} \to Bord_d$. We can view $Z_{\mathcal{X}}, Z_{\Sigma^{d-1}\hat{\mathcal{X}}}, E_{|\mathcal{X}|}$ as $\mathcal{R}$-oriented theories.

Theorem 6.1 (Abelian duality)**.** *There is an equivalence of $\mathcal{R}$-oriented TFTs*

$$\mathbb{D} : Z_{\mathcal{X}} \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}} \otimes E_{|\mathcal{X}|}. \tag{6.2}$$

Remark 6.3. In general, $Z_{\mathcal{X}}$ and $Z_{\Sigma^{d-1}\hat{\mathcal{X}}} \otimes E_{|\mathcal{X}|}$ are not equivalent as unoriented theories, despite both sides can be extended to unoriented theories. This is because we need to use Poincaré duality in an essential way. For example, they give different partition functions for the $d = 2$ theories on the Klein bottle.

Here's some consequences of the theorem:

Corollary 6.4. *When d is odd, we have an equivalence of $\mathcal{R}$-oriented TFTs*

$$Z_{\mathcal{X}} \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}}. \tag{6.5}$$

Proof. By Proposition 3.9, when d is odd, E_λ is isomorphic to the trivial theory. Therefore $Z_{\mathcal{X}} \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}}$. □

Recall that $H\mathbb{Z}$ orientation is the same as the classical notion of orientation on manifolds. Thus if $\mathcal{X}$ is a π-finite $H\mathbb{Z}$-module, then we have an equivalence of oriented theories. A large example of π-finite $H\mathbb{Z}$-module are $\Sigma^p HA$ where A is a finite abelian group. Recall that we also denote $Z_{\Sigma^p HA}$ as $Z_{K(A,p)}$. Apply Theorem 6.1 to $\mathcal{X} = \Sigma^p HA$, we get:

Corollary 6.6. *Let A be a finite abelian group and $\hat{A}$ the Pontryagin dual. In d dimension, we have an equivalence of d dimensional oriented TFTs:*

$$Z_{K(A,p)} \simeq Z_{K(\hat{A},d-1-p)} \otimes E_{|A|^{(-1)^p}}. \tag{6.7}$$

This is the Theorem **??** in the introduction. The rest of the paper is about proving the main theorem:

Proof of Theorem 6.1. From now on, all manifolds, bordisms are $\mathcal{R}$-oriented. We will suppressed the $\mathcal{R}$-orientation notations.

To give an equivalence, we will need to define an isomorphism of states

$$Z_{\mathcal{X}}(N) \to Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N) \otimes E_{|\mathcal{X}|}(N), \tag{6.8}$$

and check that it is compatible with bordisms. As $E_{|\mathcal{X}|}(N) = \mathbb{C}$, it is suffice to give maps

$$\mathbb{D}(N) : Z_{\mathcal{X}}(N) \to Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N). \tag{6.9}$$

This is done as follows:

Construction 6.10. By Pontryagin duality, there is a pairing

$$ev_N(-,-) : \mathcal{X}^*(N) \times \hat{\mathcal{X}}_*(N) \to \mathbb{C}^\times. \tag{6.11}$$

Note that this exist for any topological space N. As N is a (compact) manifold, the homology and cohomology groups are finite. This pairing is exhibits $\mathcal{X}^*(N)$ and $\hat{\mathcal{X}}_*(N)$ as Pontryagin dual of each other. Compose this with the Poincaré duality isomorphism 4.13:

$$\int_{[N]} : \hat{\mathcal{X}}^{d-1-*}(N) \xrightarrow{\sim} \hat{\mathcal{X}}_*(N), \tag{6.12}$$

we get a pairing

$$\mathcal{X}^*(N) \times \hat{\mathcal{X}}^{d-1-*}(N) \to \mathbb{C}^\times \tag{6.13}$$

$$(a,\alpha) \mapsto ev_N(a, \int_{[N]} \alpha) \tag{6.14}$$

When $* = 0$, we denote this pairing as

$$\langle -,- \rangle_N : \mathcal{X}^0(N) \times \hat{\mathcal{X}}^{d-1}(N) \to \mathbb{C}^\times. \tag{6.15}$$

It exhibits $\mathcal{X}^0(N)$ and $\hat{\mathcal{X}}^{d-1}(N)$ as the Pontryagin dual of each other. Note that this denotes on the orientation class of N, reversing the orientation inverts this pairing.

Recall that

$$Z_{\mathcal{X}}(N) = \mathbb{C}[\mathcal{X}^0(N)] \tag{6.16}$$

and

$$Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N) = \mathbb{C}[\hat{\mathcal{X}}^{d-1}(N)]. \tag{6.17}$$

We will denote elements of $\mathcal{X}^0(N)$ as a, and $\hat{\mathcal{X}}^{d-1}(N)$ as α, and view them as basis vectors for $Z_{\mathcal{X}}(N)$ and $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N)$ respectively. Now we can define the isomorphism on states:

$$\begin{aligned} \mathbb{D}(N) : \mathbb{C}[\mathcal{X}^0(N)] &\to \mathbb{C}[\hat{\mathcal{X}}^{d-1}(N)] \\ a &\mapsto |\tau_{\geq 1}\mathcal{X}(N)| \sum_\alpha \langle a, \alpha \rangle_N \; \alpha. \end{aligned} \tag{6.18}$$

This is an isomorphism of vector spaces by Corollary 5.4.

It remains to show that this intertwines with bordisms. Given $M : N \to N'$ in $Bord_d$, with the inclusion maps $p : N \hookrightarrow M$ and $q : N' \hookrightarrow M$. We have to show that the following diagram commute:

$$\begin{tikzcd} Z_{\mathcal{X}}(N) \arrow[r, "Z_{\mathcal{X}}(M)"] \arrow[d, "\mathbb{D}(N)"] & Z_{\mathcal{X}}(N') \arrow[d, "\mathbb{D}(N')"] \\ Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N) \arrow[r, "Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) * |\mathcal{X}|^{\chi(M)-\chi(N)}"'] & Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N') \end{tikzcd} \tag{6.19}$$

Note that we have canonically identified

$$Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N) \otimes E_{|\mathcal{X}|}(N) \simeq Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(N). \tag{6.20}$$

The factor

$$|\mathcal{X}|^{\chi(M)-\chi(N)} \tag{6.21}$$

in the bottom arrow comes from

$$E_{|\mathcal{X}|}(M) : E_{|\mathcal{X}|}(N) = \mathbb{C} \to \mathbb{C} = E_{|\mathcal{X}|}(N'). \tag{6.22}$$

We will proof that diagram 6.19 commutes in two lemmas:

Lemma 6.23. $\mathbb{D}(N') \circ Z_{\mathcal{X}}(M)$ *and* $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N)$ *differ by a constant* $\lambda(M)$.

Lemma 6.24. $\lambda(M) = |\mathcal{X}|^{\chi(M)-\chi(N)}$.

The two lemmas are proven in §6.2 and §6.3 respectively. $\square$

6.2 Proof of Lemma 1

We borrow the notation from above. This section is devoted to proving Lemma 6.23:

Lemma 6.25. $\mathbb{D}(N') \circ Z_{\mathcal{X}}(M)$ *and* $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N)$ *differ by a constant* $\lambda(M)$.

Proof. From now on, we will denote elements of

$$\mathcal{X}^0(N),\ \mathcal{X}^0(M),\ \mathcal{X}^0(N') \tag{6.26}$$

as a, b, and a'. Similarly, we will denote elements of

$$\hat{\mathcal{X}}^{d-1}(N),\ \hat{\mathcal{X}}^{d-1}(M),\ \hat{\mathcal{X}}^{d-1}(N') \tag{6.27}$$

as α, β, and α'. We also use the summing convention that $\sum_b$ means summing over all $b \in \mathcal{X}^0(M)$, and $\sum_{b\to a}$ means summing over all $b \in \mathcal{X}^0(M)$ such that $p^*(b) = a$.

We denote the inclusion maps $p : N \hookrightarrow M$ and $q : N' \hookrightarrow M$. We have pullback maps

$$p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N), \quad q^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N'). \tag{6.28}$$

Similarly we have

$$\hat{p}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N), \quad \hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N'). \tag{6.29}$$

First we will calculate $\mathbb{D}(N') \circ Z_{\mathcal{X}}(M)$. By Construction 2.2, $Z_{\mathcal{X}}(M)$ sends

$$a \mapsto \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{b\to a} q^* b \tag{6.30}$$

$$= \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{a'} \sum_{b\to a, b\to a'} a', \tag{6.31}$$

Recall that $\mathbb{D}(N')$ takes

$$a' \mapsto |\tau_{\geq 1}\mathcal{X}(N')| \sum_{\alpha'} \langle a', \alpha' \rangle_N \ \alpha'. \tag{6.32}$$

Thus the composition $\mathbb{D}(N') \circ Z_{\mathcal{X}}(M)$ sends

$$a \mapsto \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N')|} \sum_{b \to a} (|\tau_{\geq 1}\mathcal{X}(N')| \sum_{\alpha'} \langle q^* b, \alpha' \rangle_{N'} \ \alpha') \tag{6.33}$$

$$= |\tau_{\geq 1}\mathcal{X}(M)| \sum_{b \to a} \sum_{\alpha'} \langle q^* b, \alpha' \rangle_{N'} \ \alpha'. \tag{6.34}$$

Now for $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N)$. $\mathbb{D}(N)$ sends:

$$a \mapsto |\tau_{\geq 1}\mathcal{X}(N)| \sum_{\alpha} \langle a, \alpha \rangle_N \ \alpha. \tag{6.35}$$

$Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M)$ takes

$$\alpha \mapsto \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|} \sum_{\beta, \beta \to \alpha} \hat{q}^* \beta. \tag{6.36}$$

Thus the composition $Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N)$ is

$$a \mapsto |\tau_{\geq 1}\mathcal{X}(N)| \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|} \sum_{\alpha'} \sum_{\beta \to \alpha'} \langle a, \hat{p}^* \beta \rangle_N \ \alpha'. \tag{6.37}$$

We are reduce to showing the following lemma:

Lemma 6.38. *For every a and α', $\sum_{b \to a} \langle q^* b, \alpha' \rangle_{N'}$ and $\sum_{\beta \to \alpha'} \langle a, \hat{p}^* \beta \rangle_N$ differ a nonzero constant multiplicative C that doesn't depend on a or α'.*

Proof. Note that if a has no preimage $b \mapsto a$. Then

$$\sum_{b \to a} \langle q^* b, \alpha' \rangle_{N'} = 0. \tag{6.39}$$

In this case, Lemma 6.75 (stated and proven below) precise says that

$$\sum_{\beta \to \alpha'} \langle a, \hat{p}^* \beta \rangle_N = 0. \tag{6.40}$$

Similarly, if α' has no preimage $\beta \mapsto \alpha'$, then both sides are also zero. Thus we are reduced to the case that a lies in the image of

$$p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N) \tag{6.41}$$

and α' lies in the image of

$$\hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N). \tag{6.42}$$

There are

$$|kp| \coloneqq |ker(p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N))| \tag{6.43}$$

many preimage of a. Similarly, there are

$$|kq| \coloneqq |ker(\hat{q}^* : \Sigma^{d-1}\hat{\mathcal{X}}^0(M) \to \Sigma^{d-1}\hat{\mathcal{X}}^0(N'))| \tag{6.44}$$

many preimages of α'.

On one side, we have

$$\sum_{b \to a} \langle q^*(b), \alpha' \rangle_{N'} \tag{6.45}$$

$$= |kq|^{-1} \sum_{b \to a} \sum_{\beta \to \alpha'} \langle q^*(b), q^*\beta \rangle_{N'} \tag{6.46}$$

$$= |kq|^{-1} \sum_{b \to a} \sum_{\beta \to \alpha'} \langle p^*(b), p^*\beta \rangle_{N}. \tag{6.47}$$

The last equation is by Lemma 6.60. On the other side, we have

$$\sum_{\beta \to \alpha'} \langle a, p^*\beta \rangle_{N} \tag{6.48}$$

$$= |kp|^{-1} \sum_{b \to a} \sum_{\beta \to \alpha'} \langle p^*(b), p^*\beta \rangle_{N}. \tag{6.49}$$

We see that they differ by a constant $C = |kp|/|kq|$. □

Therefore

$$\mathbb{D}(N') \circ Z_{\mathcal{X}}(M) = \lambda(M)\ Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N), \tag{6.50}$$

with

$$\lambda(M) = \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N)|} \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|} \frac{|kp|}{|kq|}. \tag{6.51}$$

□

Now we need to proof the two Lemmas 6.75, 6.60 used above. We need the following lemma:

Lemma 6.52. *The natural maps*

$$f : \hat{\mathcal{X}}_1(M, \partial M) \to \hat{\mathcal{X}}_0(N) \to \hat{\mathcal{X}}_0(M) \tag{6.53}$$

and

$$g : \hat{\mathcal{X}}_1(M, \partial M) \to \hat{\mathcal{X}}_0(N') \to \hat{\mathcal{X}}_0(M) \tag{6.54}$$

are inverses to each other. That is, $f + g = 0$.

Proof. Consider the triple $\partial M \to M \to (M, \partial M)$, where $(M, \partial M)$ represents the cofiber. We have a long exact sequence

$$\cdots \to \hat{\mathcal{X}}_1(M, \partial M) \to \hat{\mathcal{X}}_0(\partial M) \to \hat{\mathcal{X}}_0(M) \to \cdots . \tag{6.55}$$

In particular, this means that the composition

$$h : \hat{\mathcal{X}}_1(M, \partial M) \to \hat{\mathcal{X}}_0(\partial M) \to \hat{\mathcal{X}}_0(M) \tag{6.56}$$

is the zero homomorphism $h = 0$. As $\partial M = N \sqcup N'$, we see that $h = f + g$. □

We first show lemma 6.60. Because it might have independence interest, we recall the notations: we have $M : N \to N'$ a bordism between N and N', with the inclusion maps $p : N \hookrightarrow M$ and $q : N' \hookrightarrow M$. We have pullback maps

$$p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N), \quad q^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N'). \tag{6.57}$$

Similarly we have

$$\hat{p}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N), \quad \hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N'). \tag{6.58}$$

We will denote elements of $\mathcal{X}^0(M)$ as b and $\hat{\mathcal{X}}^{d-1}(M)$ as β. Given b and β, we have the two pairings (Equation 6.15):

$$\langle p^* b, \hat{p}^* \beta \rangle_N, \quad \langle q^* b, \hat{q}^* \beta \rangle_{N'}. \tag{6.59}$$

Here's the lemma that we need to show:

Lemma 6.60. $\langle p^* b, \hat{p}^* \beta \rangle_N = \langle q^* b, \hat{q}^* \beta \rangle_{N'}$.

Proof. Recall that the orientation class $[M]$ restricts to $[N]$ on N and $-[N']$ on N'. We will first consider $\langle p^*b, \hat{p}^*\beta\rangle_N$. By Poincaré duality (Theorem 4.13), there is an isomorphism of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^{d-1}(M) \arrow[r, "\hat{p}^*"] \arrow[d, "\int_{[M]}"] & \hat{\mathcal{X}}^{d-1}(N) \arrow[r] \arrow[d, "\int_{[N]}"] & \hat{\mathcal{X}}^{d}(M,N) \arrow[r] \arrow[d] & \cdots \arrow[d] \\
\cdots \arrow[r] & \hat{\mathcal{X}}_1(M,\partial M) \arrow[r, "\hat{\mu}_*"] & \hat{\mathcal{X}}_0(N) \arrow[r] & \hat{\mathcal{X}}_0(M,N') \arrow[r] & \cdots
\end{tikzcd} \tag{6.61}$$

By definition of $\langle -,-\rangle_N$, we have

$$\langle p^*b, \hat{p}^*\beta\rangle_N = ev_N(p^*b, \int_{[N]} \hat{p}^*\beta) \tag{6.62}$$

$$= ev_N(p^*b, \hat{\mu}_* \int_{[M]} \beta) \tag{6.63}$$

Now consider the long exact sequence:

$$... \to \mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N) \to \mathcal{X}^1(M,N) \to ... \tag{6.64}$$

By Brown-Comenetz duality (Corollary 5.6), taking Pontryagin dual termwise is isormorphic to the following long exact sequence

$$... \leftarrow \hat{\mathcal{X}}_0(M) \xleftarrow{\hat{p}_*} \hat{\mathcal{X}}_0(N) \leftarrow \hat{\mathcal{X}}_1(M,N) \leftarrow ... \tag{6.65}$$

The dual long exact sequences are connected by the "projection formula": given $b \in \mathcal{X}^0(M)$ and $\gamma \in \hat{\mathcal{X}}_0(N)$, then

$$ev_N(p^*b, \gamma) = ev_M(b, \hat{p}_*\gamma). \tag{6.66}$$

Put it together with Equation 6.63:

$$\langle p^*b, \hat{p}^*\beta\rangle_N = ev_N(p^*b, \hat{\mu}_* \int_{[M]} \beta) \tag{6.67}$$

$$= ev_M(b, \hat{p}_* \circ \hat{\mu}_* \int_{[M]} \beta) \tag{6.68}$$

The same argument shows that

$$\langle q^*b, \hat{q}^*\beta\rangle_{N'} = ev_M(b, -\hat{q}_* \circ \hat{\nu}_* \int_{[M]} \beta) \tag{6.69}$$

The minus sign comes from the fact that $[M]$ restricts to $-[N']$.

Note that the map

$$\hat{p}_* \circ \hat{\mu}_* : \hat{\mathcal{X}}_1(M, \partial M) \to \hat{\mathcal{X}}_0(M) \tag{6.70}$$

is precisely the map f in Lemma 6.52. Similarly, $\hat{q}_* \circ \hat{\nu}_* = g$. By Lemma 6.52, we see that

$$\hat{p}_* \circ \hat{\mu}_* \int_{[M]} \beta = -\hat{q}_* \circ \hat{\nu}_* \int_{[M]} \beta, \tag{6.71}$$

therefore

$$\langle p^* b, \hat{p}^* \beta \rangle_N = \langle q^* b, \hat{q}^* \beta \rangle_{N'}. \tag{6.72}$$

□

Remark 6.73. Heuristically, since the orientation class $[M]$ for M is a homotopy from $p_*[N]$ to $q_*[N] \in \mathcal{R}_{d-1}(M)$, therefore

$$\langle p^* b, \hat{p}^* \beta \rangle_N \approx \langle b, \beta \rangle_{p_*[N]} \approx \langle b, \beta \rangle_{q_*[N']} \approx \langle q^* b, \hat{q}^* \beta \rangle_{N'}. \tag{6.74}$$

Now we proof the following lemma:

Lemma 6.75. *Let $a \in \mathcal{X}^0(N)$ and $\alpha' \in \hat{\mathcal{X}}^{d-1}(N')$. If a is not in the image of $p^* : \mathcal{X}^0(M) \to \mathcal{X}^0(N)$, then*

$$\sum_{\beta \to \alpha'} \langle a, \hat{p}^* \beta \rangle_N = 0, \tag{6.76}$$

where β sums over $\hat{\mathcal{X}}^{d-1}(M)$.

Proof. If α' has no preimage in

$$\hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N'), \tag{6.77}$$

then the sum is trivially 0. If α' has a preimage, say β'_α. Then all other preimages of α are of the form $\beta'_\alpha + \beta_0$, where $\beta_0 \in ker(\hat{q}^*)$. Thus

$$\sum_{\beta \to \alpha'} \langle a, \hat{p}^* \beta \rangle_N = \sum_{\beta_0 \in ker(\hat{q}^*)} \langle a, \hat{p}^* (\beta'_\alpha + \beta_0) \rangle_N \tag{6.78}$$

$$= (\langle a, \hat{p}^* \beta'_\alpha \rangle_N) \sum_{\beta_0 \in ker(\hat{q}^*)} \langle a, \hat{p}^* \beta_0 \rangle_N. \tag{6.79}$$

Therefore it is suffice to show that

$$\sum_{\beta_0 \in ker(\hat{q}^*)} \langle a, \hat{p}^* \beta_0 \rangle_N = 0, \tag{6.80}$$

i.e. the case where $\alpha' = 0$.

Poincaré duality (Theorem 4.13) gives an isomorphism of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^{d-1}(M,N') \arrow[r] \arrow[d] & \hat{\mathcal{X}}^{d-1}(M) \arrow[r, "\hat{q}^*"] \arrow[d, "\int_{[M]}"] & \hat{\mathcal{X}}^{d-1}(N') \arrow[r] \arrow[d, "\int_{[N']}"] & \cdots \arrow[d] \\
\cdots \arrow[r] & \hat{\mathcal{X}}_1(M,N) \arrow[r, "\hat{\mu}_*"] & \hat{\mathcal{X}}_1(M,\partial M) \arrow[r, "\hat{\nu}_*"] & \hat{\mathcal{X}}_0(N') \arrow[r] & \cdots
\end{tikzcd} \tag{6.81}$$

Note that $\hat{\mu}_*$ represents a different map from the proof of Lemma 6.60.

Under Poincaré duality, $ker(\hat{q}^*)$ corresponds to $ker(\hat{\nu}_*) = im(\hat{\mu}_*)$. Given $\beta_0 \in ker(\hat{q}^*)$ with

$$\int_{[M]} \beta_0 = \hat{\mu}_*\gamma, \quad \gamma \in \hat{\mathcal{X}}_1(M,N), \tag{6.82}$$

By definition of $\langle -,- \rangle_N$, we have:

$$\langle a, \hat{p}^*\beta_0 \rangle_N = ev_N(a, \hat{\lambda}_* \int_{[M]} \beta) \tag{6.83}$$

$$= ev_N(a, (\hat{\lambda}_* \circ \hat{\mu}_*)\gamma). \tag{6.84}$$

$\hat{\lambda}_*$ is the canonical map $\mathcal{X}_0(M) \to \mathcal{X}_0(N)$. The composition

$$\hat{\lambda}_* \circ \hat{\mu}_* : \hat{\mathcal{X}}_1(M,N) \to \hat{\mathcal{X}}_0(N) \tag{6.85}$$

is the Pontryagin dual of the

$$\partial^* : \mathcal{X}^0(N) \to \mathcal{X}^1(M,N). \tag{6.86}$$

Therefore by Equation 6.84

$$\langle a, \hat{p}^*\beta_0 \rangle_N = ev_N(a, (\hat{\lambda}_* \circ \hat{\mu}_*)\gamma) \tag{6.87}$$

$$= ev_{(M,N)}(\partial^* a, \gamma). \tag{6.88}$$

Thus

$$|ker(\hat{\mu}_*)| \sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^*\beta_0 \rangle_N = \sum_{\gamma} ev_{(M,N)}(\partial^* a, \gamma), \tag{6.89}$$

where γ sums over $\hat{\mathcal{X}}_1(M,N)$. Now consider the the long exact sequence:

$$\cdots \to \mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N) \xrightarrow{\partial^*} \mathcal{X}^1(M,N) \to \cdots \tag{6.90}$$

By hypothesis, a is not in the image of $p^*: \mathcal{X}^0(M) \to \mathcal{X}^0(N)$, therefore

$$\partial^* a \in \mathcal{X}^1(M,N) \tag{6.91}$$

is not the identity element. Thus

$$ev_{(M,N)}(\partial^* a, -): \hat{\mathcal{X}}_1(M,N) \to \mathbb{C}^\times \tag{6.92}$$

is a nontrivial character on $\hat{\mathcal{X}}_1(M,N)$. As the sum over all elements of the group paired with a nontrivial character is 0, we see that

$$\sum_\gamma ev_{(M,N)}(\partial^* a, \gamma) = 0. \tag{6.93}$$

By Equation 6.89, we get

$$\sum_{\beta_0 \in ker(\hat{p}^*)} \langle a, \hat{p}^* \beta_0 \rangle_N = 0. \tag{6.94}$$

□

6.3 Proof of Lemma 2

We will borrow notation from last section §6.2. Recall from last section we have

$$\mathbb{D}(N') \circ Z_{\mathcal{X}}(M) = \lambda(M)\ Z_{\Sigma^{d-1}\hat{\mathcal{X}}}(M) \circ \mathbb{D}(N) \tag{6.95}$$

with

$$\lambda(M) = \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N)|} \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|} \frac{|kp|}{|kq|}. \tag{6.96}$$

To finish the proof of the main theorem, we need the following lemma (see previous sections §6 for notations):

Lemma 6.97. $\lambda(M) = |\mathcal{X}|^{\chi(M)-\chi(M)}$.

Proof. Recall that

$$\lambda(M) = \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N)|} \frac{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|}{|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|} \frac{|kp|}{|kq|}. \tag{6.98}$$

First we will move everything in $\hat{\mathcal{X}}$ to $\mathcal{X}$.

The first term is

$$|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')|. \tag{6.99}$$

By Poincaré duality (Theorem 4.13) we have

$$\hat{\mathcal{X}}^*(N') \simeq \hat{\mathcal{X}}_{d-1-*}(N'). \tag{6.100}$$

Therefore

$$|\hat{\mathcal{X}}^i(N')| = |\hat{\mathcal{X}}_{d-1-i}(N')| = |\mathcal{X}^{d-1-i}(N')|. \tag{6.101}$$

The cardinality of Pontryagin dual groups are equal. Thus

$$|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(N')| = \frac{|\hat{\mathcal{X}}^{d-3}(N')|}{|\hat{\mathcal{X}}^{d-2}(N')|}\frac{|\hat{\mathcal{X}}^{d-5}(N')|}{|\hat{\mathcal{X}}^{d-4}(N')|}\cdots \tag{6.102}$$

$$= \frac{|\hat{\mathcal{X}}_2(N')|}{|\hat{\mathcal{X}}_1(N')|}\frac{|\hat{\mathcal{X}}_4(N')|}{|\hat{\mathcal{X}}_3(N')|}\cdots \tag{6.103}$$

$$= \frac{|\mathcal{X}^2(N')|}{|\mathcal{X}^1(N')|}\frac{|\mathcal{X}^4(N')|}{|\mathcal{X}^3(N')|}\cdots \tag{6.104}$$

$$= |\tau_{\leq -1}\mathcal{X}(N')|. \tag{6.105}$$

Next we will work on

$$|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|^{-1}. \tag{6.106}$$

Similar to above, we have

$$|\hat{\mathcal{X}}^i(M)| = |\hat{\mathcal{X}}_{d-i}(M,\partial M)| = |\mathcal{X}^{d-i}(M,\partial M)|. \tag{6.107}$$

Therefore

$$|\tau_{\geq 1}\Sigma^{d-1}\hat{\mathcal{X}}(M)|^{-1} = \frac{|\hat{\mathcal{X}}^{d-2}(M')|}{|\hat{\mathcal{X}}^{d-3}(M')|}\frac{|\hat{\mathcal{X}}^{d-4}(M')|}{|\hat{\mathcal{X}}^{d-5}(M')|}\cdots \tag{6.108}$$

$$= \frac{|\hat{\mathcal{X}}_2(M,\partial M)|}{|\hat{\mathcal{X}}_3(M,\partial M)|}\frac{|\hat{\mathcal{X}}_4(M,\partial M)|}{|\hat{\mathcal{X}}_5(M,\partial M)|}\cdots \tag{6.109}$$

$$= \frac{|\mathcal{X}^2(M,\partial M)|}{|\mathcal{X}^3(M,\partial M)|}\frac{|\mathcal{X}^4(M,\partial M)|}{|\mathcal{X}^5(M,\partial M)|}\cdots \tag{6.110}$$

$$= |\tau_{\leq -1}\mathcal{X}(N')|. \tag{6.111}$$

Lastly, we have

$$|kq| \coloneqq |ker(\hat{q}^* : \hat{\mathcal{X}}^{d-1}(M) \to \hat{\mathcal{X}}^{d-1}(N'))|. \tag{6.112}$$

By Poincare duality (Theorem 4.13): we have that an isomorphism of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \hat{\mathcal{X}}^*(M,N') \arrow[r] \arrow[d] & \hat{\mathcal{X}}^*(M) \arrow[r, "q^*"] \arrow[d] & \hat{\mathcal{X}}^*(N') \arrow[r] \arrow[d] & \cdots \arrow[d] \\
\cdots \arrow[r] & \hat{\mathcal{X}}_{d-*}(M,N) \arrow[r] & \hat{\mathcal{X}}_{d-*}(M,\partial M) \arrow[r] & \hat{\mathcal{X}}_{d-1-*}(N') \arrow[r] & \cdots
\end{tikzcd} \tag{6.113}$$

Thus

$$|kq| = |ker(\hat{\mathcal{X}}_1(M,\partial M) \to \hat{\mathcal{X}}_0(N'))| \tag{6.114}$$

$$= |im(\hat{\mathcal{X}}_1(M,N) \to \hat{\mathcal{X}}_1(M,\partial M)|. \tag{6.115}$$

By Brown-Comenetz duality (Corollary 5.6), the long exact sequence

$$\cdots \to \hat{\mathcal{X}}_{d-*}(M,N) \to \hat{\mathcal{X}}_{d-*}(M,\partial M) \to \hat{\mathcal{X}}_{d-1-*}(N') \to \cdots \tag{6.116}$$

is the Pontryagin dual of

$$\cdots \leftarrow \mathcal{X}^{d-*}(M,N) \leftarrow \mathcal{X}^{d-*}(M,\partial M) \leftarrow \mathcal{X}^{d-1-*}(N') \leftarrow \cdots \tag{6.117}$$

Thus

$$|kq| = |im(\hat{\mathcal{X}}_1(M,N) \to \hat{\mathcal{X}}_1(M,\partial M)| \tag{6.118}$$

$$= |im(\mathcal{X}^1(M,\partial M) \to \mathcal{X}^1(M,N)| \tag{6.119}$$

$$= |ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|. \tag{6.120}$$

To recap, we have

$$\lambda(M) = \frac{|\tau_{\geq 1}\mathcal{X}(M)|}{|\tau_{\geq 1}\mathcal{X}(N)|} \frac{|ker(\mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|} |\tau_{\leq -1}\mathcal{X}(N')||\tau_{\leq -1}\mathcal{X}(N')|. \tag{6.121}$$

Now we will factor out $|\mathcal{X}|^{\chi(M)-\chi(M)}$ from $\lambda(M)$. For any π-finite space $\mathcal{Y}$, we have a fiber sequence

$$\tau_{\geq i}\mathcal{Y} \to \mathcal{Y} \to \tau_{\leq i-1}\mathcal{Y} \tag{6.122}$$

of π-finite spaces. By Example 1.15 we have

$$|\tau_{\geq i}\mathcal{Y}|\ |\tau_{\leq i-1}\mathcal{Y}| = |\mathcal{Y}|. \tag{6.123}$$

In our case,

$$|\tau_{\geq 1}\mathcal{X}(M)| = \frac{|\mathcal{X}(M)|}{|\tau_{\leq 0}\mathcal{X}(M)|}. \tag{6.124}$$

Similarly,

$$|\tau_{\geq 1}\mathcal{X}(N)|^{-1} = \frac{|\tau_{\leq 0}\mathcal{X}(N)|}{|\mathcal{X}(N)|}. \tag{6.125}$$

By Proposition 1.18 we have

$$|\mathcal{X}(M)| = |\mathcal{X}|^{\chi(M)} \tag{6.126}$$

and

$$|\mathcal{X}(N)| = |\mathcal{X}|^{\chi(N)}. \tag{6.127}$$

Putting it all together, we see that

$$\lambda(M) = \lambda'(M)\ |\mathcal{X}|^{\chi(M)-\chi(M)}, \tag{6.128}$$

where

$$\begin{aligned}\lambda'(M) = \frac{|\tau_{\leq 0}\mathcal{X}(N)|}{|\tau_{\leq 0}\mathcal{X}(M)|}|\tau_{\leq -1}\mathcal{X}(N')|\ |\tau_{\leq -2}\mathcal{X}(M,\partial M)| \\ \frac{|ker(\mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|}\end{aligned} \tag{6.129}$$

It remains to show that $\lambda'(M) = 1$.

As

$$\partial M = N \sqcup N', \tag{6.130}$$

we have

$$|\mathcal{X}^*(\partial M)| = |\mathcal{X}^*(N)|\ |\mathcal{X}^*(N')|. \tag{6.131}$$

Therefore

$$|\tau_{\leq 0}\mathcal{X}(N)|\ |\tau_{\leq -1}\mathcal{X}(N')| = |\mathcal{X}^0(N)|\ |\tau_{\leq -1}\mathcal{X}(\partial M)|. \tag{6.132}$$

Now consider the exact sequences

$$0 \to ker\ p^* \to \mathcal{X}^0(M) \xrightarrow{p^*} \mathcal{X}^0(N) \to coker\ p^* \to 0, \tag{6.133}$$

We see that the terms

$$|ker\ p^*|\ |\tau_{\leq 0}\mathcal{X}(M)|^{-1}\ |\tau_{\leq 0}\mathcal{X}(N)| = |coker\ p^*|. \tag{6.134}$$

Lastly, we rewrite

$$|coker\ p^*| = |ker\ \mathcal{X}^1(M,N) \to \mathcal{X}^1(M)|. \tag{6.135}$$

Thus

$$\begin{aligned} \lambda'(M) = \frac{|\tau_{\leq -1}\mathcal{X}(\partial M)|}{|\tau_{\leq -1}\mathcal{X}(M)|}\ |\tau_{\leq -2}\mathcal{X}(M,\partial M)| \\ \frac{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(M))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|} \end{aligned} \tag{6.136}$$

I claim that

$$\frac{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(M))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|} = |ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M))|^{-1}. \tag{6.137}$$

First notice that the canonical map

$$\mathcal{X}^1(M,N) \to \mathcal{X}^1(N) = 0, \tag{6.138}$$

therefore

$$|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))| = |ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(\partial M))|. \tag{6.139}$$

Note that

$$\mathcal{X}^1(M,N) \to \mathcal{X}^1(\partial M) \tag{6.140}$$

is the composition of the two terms

$$(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M)) \circ (\mathcal{X}^1(M,N) \to \mathcal{X}^1(M)) \tag{6.141}$$

on the RHS. Therefore we are trying to show this:

$$|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(\partial M)) = |ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(M))| \tag{6.142}$$

$$|ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M))|. \tag{6.143}$$

We have the following algebraic fact: given

$$f : A \to B,\ g : B \to C \tag{6.144}$$

then

$$|ker(g \circ f)| = |kerf|\ |kerg| \tag{6.145}$$

iff

$$ker(g) \subset im(f). \tag{6.146}$$

In our case, if an element $a \in \mathcal{X}^1(M)$ maps to 0 in $\mathcal{X}^1(\partial M)$, then it maps to 0 in $\mathcal{X}^1(N)$. Since

$$\mathcal{X}^1(M,N) \to \mathcal{X}^1(M) \to \mathcal{X}^1(N)$$

is a part of a long exact sequence, it is exact at $\mathcal{X}^1(M)$. That means that there exists $b \in \mathcal{X}^1(M,N)$ which maps to a. Thus we satisfy the algebraic condition, and we have

$$\frac{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(M))|}{|ker(\mathcal{X}^1(M,N) \to \mathcal{X}^1(N'))|} = |ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M))|^{-1}. \tag{6.147}$$

So

$$\lambda'(M) = \frac{|\tau_{\leq -1}\mathcal{X}(\partial M)|}{|\tau_{\leq -1}\mathcal{X}(M)|} \frac{|\tau_{\leq -2}\mathcal{X}(M,\partial M)|}{|ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M))|}. \tag{6.148}$$

Finally, consider the following long exact sequence:

$$\begin{aligned} 0 \to ker(\mathcal{X}^1(M) \to \mathcal{X}^1(\partial M)) \to \mathcal{X}^1(M) \to \mathcal{X}^1(\partial M) \quad (6.149)\\ \to \mathcal{X}^2(M,\partial M) \to \mathcal{X}^2(M) \to \mathcal{X}^2(\partial M) \to \cdots . \quad (6.150)\end{aligned}$$

By Lemma 1.8 the alternating size of the finite abelian groups in a long exact sequence is 1. The alternating size of the long exact sequence 6.149 above is precisely $\lambda'(M)$, thus

$$\lambda'(M) = 1. \tag{6.151}$$

□

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