Counterexamples to HKR in Characteristic p

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We will continue from last time, looking at how would HKR work in characteristic $p \neq 0$.

Contents

1 Last time

Let k be a field, and A is a k-algebra. We introduce the Hochschild complex: $HH(A/k) \coloneqq A \otimes_{A \otimes A} A$ (note all tensor products now are derived), and the i-th homology of $HH(A/k)$ is called the i-th homology.

We can also generalize this to X a smooth scheme, and we have the weak HKR isomorphism theorem:

Theorem 1.1. *If X/k is a smooth scheme, then $H^{-i}(HH(X/k)) \cong \Omega^i_{X/k}$.*

Question: What does this tell us about $HH_i(X/k)$?

In a general setting: if $\mathcal{F}$ be a chain complex of quasi-coherent sheaves on X. What is the relationship between $H^s(X; H^t(\mathcal{F}))$ relate to $H^?(X, \mathcal{F}) \coloneqq H^?(R\Gamma(X; \mathcal{F}))$.

A: They are related by a spectral sequence.

We might guess that: $HH(X/k) = \sum_{s-t=n} H^t(X; \Omega^s_X)$.

This is called the Hodge decomposition. In characterstic 0, we always gets this result, which is the strong HKR that we discuss last time:

Theorem 1.2. *$HH(X/k) \cong \bigoplus_{i\geq 0} \Omega^i_X[i]$ if X is smooth, if not, we replace Ω with the cotangent complex: $HH_n(X/k) \equiv \bigoplus_{s-t=n} H^2(X; \wedge^t L_{X/k})$.*

This is related to the Hodge decomposition of Kahler manifolds. But in there there is a S^1 action, periodic cyclic homology, which we are not discussing today.

2 characteristic p

Q: when char $k = p > 0$ a perfect field, does the Hodge decomposition of HH still exist?

A: Yes, if X^d smoioth projective over k, and $d \leq p$. No, in general. The no situation is what we will discuss today.

Theorem 2.1 (Antieau-Bhatt-Mathew,'19)**.** *There exists a smooth projective $2p-dim$ scheme X/k such that Hodge decomposition for HH does not hold for X.*

The proof comes in two steps: 1. Find a classifying stack counterexample. 2. Approximate by smooth scheme.

Let's do step one:

We are going to look at classfiying stack BG, where G is finite. In our case, $G = \mu_p$ the group scheme of roots of unity. $\mu(R) = \{x \in R | x^p = 1\}$.

Rough sketch of 2: V a f.d. G representation, consider $\mathbb{P}(V)$. By a Bertini-type theorem: there exists $X \subset \mathbb{P}(V)$ smooth complete intersection, such that G act on X freely:

We have $X/G \cong [X/G] \hookrightarrow [\mathbb{P}(V)/G] \to BG$. We have X/G a smooth projective scheme, the inclusion is a smooth locally complete intersection, and the last map a projective bundle. Because it is a projective bundle, the pullback on cohomology is injective. Because Hodge decomposition fails for BG, and the Hodge decomposition fails for $[\mathbb{P}(V)/G]$. Because of Lefschetz principle, X/G would have the same lower cohomology as $[\mathbb{P}(V)/G]$, thus we see that Hodge decomposition fails for X/G.

Remark (Sam)*.* The agreement of the cohomology in low degrees is based on Lefschetz principle. The idea of doing this kind of manuveur from BG

is from Serre. The idea is that BG for G a finite group has no rational cohomology, only torsion. Thus pulling back from BG will give you a lot of torsion cohomology classes.

Now we are going to discuss step 1:

Remark (Sam). This is from talking with Bhargav. We want to compare $HH_*(B\mu_p)$ and $H^*(B\mu_p, \wedge^* L_{B\mu_p})$. We want to show that RHS is not the direct sum of things on the left hand side. We are going to show that the LHS essentially vanishes.

Idea for LHS: $Qcoh(BG) = RepG$. If $G = \mu_p$, what are the representations? We have the trivial rep, and the canonical one ($\mu_p \in \mathbb{G}_m$). We will call this $k(1)$, then if we denote n tensor of them to be $k(n)$, then we see that $k(p) = triv$.

Claim: $\mathrm{Rep}(\mu_p) = \mathbb{Z}/p$-graded vector spaces. $\mathrm{Rep}(\mu_p) = \mathcal{O}(\mu_p)$-comod $= \mathcal{O}(\mu_p)^*$-mod, and $\mathcal{O}(\mu_p) \cong \Pi_{i=0}^{p-1} k$.

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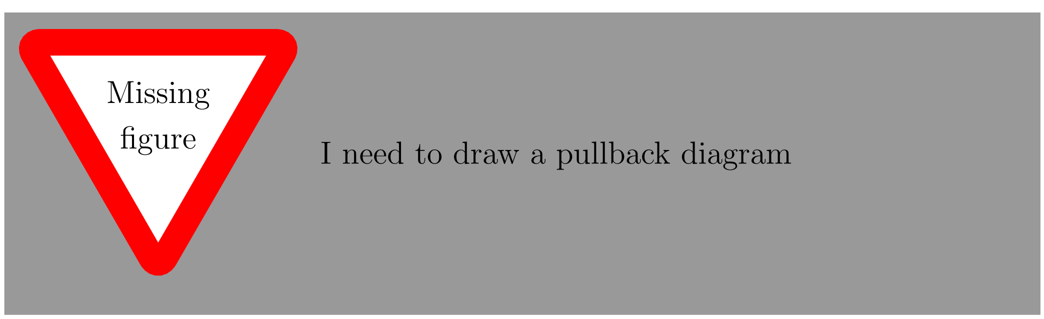
We see that HH depends only on the category of quasicoherent sheaves. $HH(B\mu_p) = \Pi_{i=0}^{p-1} k$.

Now we want to show that $H^*(B\mu_p, \wedge^* L_{B\mu_p})$ is huge.

We will do this calculation for arbitrary G.

Given $* \to eG$, then we define $coLin(G) := e^* L_{G/k}$, which has a G action, which is the coadjoint representation. If G is smooth, then $coLie(G) = \mathfrak{g}^*$. $G = \mu_p$ is not smooth in characterstic p. We have a lemma:

Lemma 2.1. *$R\Gamma(BG; \wedge^i L_{BG}) \cong R\Gamma(G; coLie(G)[-i]) = Sym^i(coLie(G))^G[-1]$*

Proof. 

I need to draw out the BG pullback diagram

$coLie(G) = e^*L_{G/k} = e^*p^*L_{*/BG} = L_{*/BG}$, due to left exactness of L, we see that $* \to BG \to *$ gives us a fiber sequence: $\pi^*L_{BG/k} \to L_{*/*} = 0 \to L_{*/BG} \Rightarrow L_{*/BG} \cong \pi^*L_{BG/k}[1]$.

We know that π^*, when interpreted as from $Rep(G) \to Vect_k$, it is the forgetful map.

Thus we see that $Sym^i(coLie(G) \cong Sym^i(L_{BG/k}[1]) \cong \wedge^i L_{BG/k}[i]$. The last thing is the thing that we are trying to calculate. For $G = \mu_p$, we want to calculate the invariance of this representation.

For $G = \mu_p = speck[t]/(t^p - 1)$, then we have a two term resolution $k[t] \to t \mapsto t^p - 1k[t]$. Pass to L: $k[t]dt \to k[t]dt$ sends $dt \mapsto d(t^p - 1) = pt^{p-1}dt = 0$.

We see that $e^*L_{\mu_p/k} \cong k \oplus k[1]$.

Now we see that $H^*(BG; \wedge^* BG) = \wedge_k^*(d) \otimes Sym_k(c)$.

Note that in our context, taking the invariance is exactly taking the degree 0 case. In our case, this representation is trivial because G is abelian thus the co-adjoint representation is concentrated in the 0 case.

$\blacksquare$

Todo list