Stanford CS 229, Public Course, Problem Set 3

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December 27, 2016

1

a)

By the Hoeffding inequality, we know that

$P(|\varepsilon(\hat{h}_i) - \hat{\varepsilon}_{S_{cv}}(\hat{h}_i)| > \gamma) \leq 2\exp(-2\gamma^2\beta m)$

Let A_i denote the event that $|\varepsilon(\hat{h}_i) - \hat{\varepsilon}_{S_{cv}}(\hat{h}_i)| > \gamma$. Then

$$\begin{aligned}
P(\exists \hat{h}_i \in \{\hat{h}_1 ... \hat{h}_k\}.|\varepsilon(\hat{h}_i) - \hat{\varepsilon}_{S_{cv}}(\hat{h}_i)| > \gamma) &= P(A_1 \cup ... \cup A_k) \\
&\leq \sum_k P(A_i) \\
&\leq \sum_k 2\exp(-2\gamma^2\beta m) \\
&= 2k\exp(-2\gamma^2\beta m)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P(\neg\exists \hat{h}_i \in \{\hat{h}_1 ... \hat{h}_k\}.|\varepsilon(\hat{h}_i) - \hat{\varepsilon}_{S_{cv}}(\hat{h}_i)| > \gamma) \\
&= P(\forall \hat{h}_i \in \{\hat{h}_1 ... \hat{h}_k\}.|\varepsilon(\hat{h}_i) - \hat{\varepsilon}_{S_{cv}}(\hat{h}_i)| \leq \gamma) \\
&\geq 1 - 2k\exp(-2\gamma^2\beta m)
\end{aligned}$$

Let $\dfrac{\delta}{2} = 2k\exp(-2\gamma^2\beta m)$. Then

$$\begin{aligned}
\frac{\delta}{4k} &= \exp(-2\gamma^2\beta m) \\
\frac{4k}{\delta} &= \exp(2\gamma^2\beta m) \\
\log\frac{4k}{\delta} &= 2\gamma^2\beta m \\
\frac{1}{2\beta m}\log\frac{4k}{\delta} &= \gamma^2 \\
\gamma &= \sqrt{\frac{1}{2\beta m}\log\frac{4k}{\delta}}
\end{aligned}$$

Therefore,

$P\left(\forall \hat{h}_i \in \{\hat{h}_1 ... \hat{h}_k\}.|\varepsilon(\hat{h}_i) - \hat{\varepsilon}_{S_{cv}}(\hat{h}_i)| \leq \sqrt{\dfrac{1}{2\beta m}\log\dfrac{4k}{\delta}}\right) \geq 1 - \dfrac{\delta}{2}$

b)

From part (a), we have that

$P(|\varepsilon(\hat{h}_i) - \hat{\varepsilon}_{S_{cv}}(\hat{h}_i)| \leq \gamma) \geq 1 - \frac{\delta}{2}$, where $\gamma = \sqrt{\frac{1}{2\beta m} \log \frac{4k}{\delta}}$

Because $\hat{h} \in \{\hat{h}_1 ... \hat{h}_k\}$

$P(|\varepsilon(\hat{h}) - \hat{\varepsilon}_{S_{cv}}(\hat{h})| \leq \gamma) \geq 1 - \frac{\delta}{2}$

Let $h^* = \arg \min_{\hat{h}_i \in \{\hat{h}_1 ... \hat{h}_k\}} \varepsilon(\hat{h}_i)$

Then with probability at least $1 - \frac{\delta}{2}$

$$
\begin{aligned}
\varepsilon(\hat{h}) &\leq \hat{\varepsilon}_{S_{cv}}(\hat{h}) + \gamma \\
&\leq \hat{\varepsilon}_{S_{cv}}(h^*) + \gamma && \text{By the definition of } \hat{h} \text{ it has the lowest } \hat{\varepsilon}_{S_{cv}} \text{ of any } \hat{h}_i \in \{\hat{h}_1 ... \hat{h}_k\} \\
&\leq \varepsilon(h^*) + 2\gamma && \text{By the uniform convergence result proved in part (a)} \\
&= \min_{i=1,\ldots,k} \varepsilon(\hat{h}_i) + 2\gamma && \text{By the definition of } h^* \\
&= \min_{i=1,\ldots,k} \varepsilon(\hat{h}_i) + 2\sqrt{\frac{1}{2\beta m} \log \frac{4k}{\delta}} \\
&= \min_{i=1,\ldots,k} \varepsilon(\hat{h}_i) + \sqrt{\frac{2}{\beta m} \log \frac{4k}{\delta}}
\end{aligned}
$$

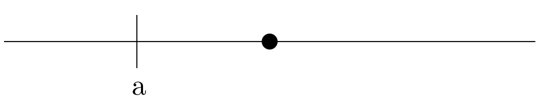
c)

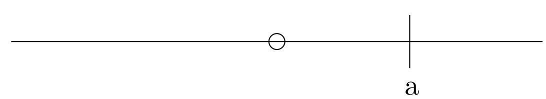
2

a)

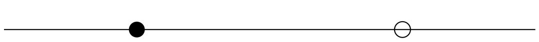
$h(x) = 1\{a < x\}, \quad a \in \mathbb{R}$

$h(x)$ can shatter a set of 1 point:





There is no set of 2 points that $h(x)$ can shatter because in the labeling situation shown below, no choice of a will successfully label both the points:

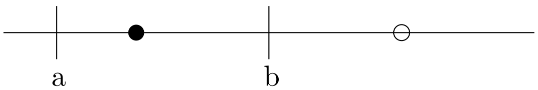


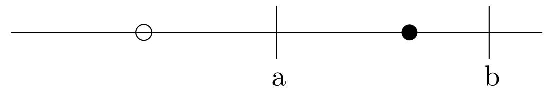
Therefore $VC(h(x)) = 1$

b)

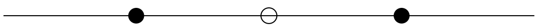
$h(x) = 1\{a < x < b\}, \quad a, b \in \mathbb{R}$

$h(x)$ can shatter a set of 2 points:





There is no set of 3 points that $h(x)$ can shatter because in the labeling situation shown below, no choice of a and b will successfully label all the points:



Therefore, $VC(h(x)) = 2$

c)

$h(x) = 1\{a \sin x > 0\}, \quad a \in \mathbb{R}$

$h(x)$ can shatter a set of 1 point by manipulating the sign of a. This will ensure that $h(x)$ can evaluate to a 1 or a 0 no matter where the point lies.

There is no set of 2 points that $h(x)$ can shatter. $h(x)$ divides the input space into π-wide sections which alternate evaluating to 1 or 0. E.g. with $a = 1$, $x \in (0, \pi)$ will evaluate to 1, $x \in [\pi, 2\pi]$ will evaluate to 0, $x \in (2\pi, 3\pi)$ will evaluate to 1, etc. Because of this, with any set of two points there is a labeling which $h(x)$ cannot achieve. There are two cases to consider: the two points both lie within sections that evaluate the same (i.e. both 1 or both 0) or they lie within sections that evaluate differently. In the first case, if the points are labeled differently $h(x)$ will not be able to achieve the labeling because no matter how you manipulate a both points will evaluate to 1 or both points will evaluate to 0. Similarly in the second case, if the points are labeled the same $h(x)$ will not be able to achieve the labeling because no matter how you manipulate a one point will evaluate to 1 and the other point will evaluate to 0.

Therefore $VC(h(x)) = 1$

d)

$h(x) = 1\{\sin(x + 1) > 0\}, \quad a \in \mathbb{R}$

$h(x)$ can shatter a set of 1 point:

$$\begin{aligned} x = \{0\} \quad & \text{label } \{1\}, \quad a = \frac{\pi}{2} \\ & \text{label } \{0\}, \quad a = 0 \end{aligned}$$

$h(x)$ can shatter a set of 2 points:

$$\begin{aligned} x = \{0, \frac{\pi}{2}\} \quad & \text{label } \{0, 0\}, \quad a = -\frac{\pi}{2} \\ & \text{label } \{0, 1\}, \quad a = 0 \\ & \text{label } \{1, 0\}, \quad a = \frac{\pi}{2} \\ & \text{label } \{1, 1\}, \quad a = \frac{\pi}{4} \end{aligned}$$

There is no set of 3 points which $h(x)$ can shatter because the points must span a distance of π to realize the labeling $\{0, 1, 0\}$, but in that case they cannot realize the labeling $\{1, 1, 1\}$.

Therefore $VC(h(x)) = 2$

3

a)

$$
\begin{aligned}
J(\theta) &= \frac{1}{2}||X\bar{\theta} + X_i\theta_i - \vec{y}||_2^2 + \lambda||\bar{\theta}||_1 + \lambda s_i\theta_i \\
&= \frac{1}{2}(X\bar{\theta} + X_i\theta_i - \vec{y})^T(X\bar{\theta} + X_i\theta_i - \vec{y}) + \lambda||\bar{\theta}||_1 + \lambda s_i\theta_i \\
&= \frac{1}{2}(\bar{\theta}^T X^T + X_i^T\theta_i - \vec{y}^T)(X\bar{\theta} + X_i\theta_i - \vec{y}) + \lambda||\bar{\theta}||_1 + \lambda s_i\theta_i \\
&= \frac{1}{2}(\bar{\theta}^T X^T X\bar{\theta} + \bar{\theta}^T X^T X_i\theta_i - \bar{\theta}^T X^T\vec{y} + X_i^T\theta_i X\bar{\theta} + X_i^T\theta_i X_i\theta_i - X_i^T\theta_i\vec{y} - \vec{y}^T X\bar{\theta} - \vec{y}X_i\theta_i + \vec{y}^T\vec{y}) + \lambda||\bar{\theta}||_1 + \lambda s_i\theta_i
\end{aligned}
$$

Find $\frac{\partial}{\partial\theta_i}J(\theta)$ when $s_i = 1$:

$$
\frac{\partial}{\partial\theta_i}J(\theta) = \frac{1}{2}(\bar{\theta}^T X^T X_i + X_i^T X\bar{\theta} + 2X_i^T X_i\theta_i - 2X_i^T\vec{y}) + \lambda
$$

Set equal to 0:

$$
\begin{aligned}
0 &= \frac{1}{2}(\bar{\theta}^T X^T X_i + X_i^T X\bar{\theta} + 2X_i^T X_i\theta_i - 2X_i^T\vec{y}) + \lambda \\
-X_i^T X_i\theta_i &= \frac{1}{2}\bar{\theta}^T X^T X_i + \frac{1}{2}X_i^T X\bar{\theta} - X_i^T\vec{y} + \lambda \\
\theta_i &= \frac{-1}{X_i^T X_i}(\frac{1}{2}\bar{\theta}^T X^T X_i + \frac{1}{2}X_i^T X\bar{\theta} - X_i^T\vec{y} + \lambda)
\end{aligned}
$$

When $s_i = 1$, $\theta_i = \max\left(0, \quad \frac{-1}{X_i^T X_i}(\frac{1}{2}\bar{\theta}^T X^T X_i + \frac{1}{2}X_i^T X\bar{\theta} - X_i^T\vec{y} + \lambda)\right)$

Following the same process for $s_i = -1$, we find

when $s_i = -1$, $\theta_i = \min\left(0, \quad \frac{-1}{X_i^T X_i}(\frac{1}{2}\bar{\theta}^T X^T X_i + \frac{1}{2}X_i^T X\bar{\theta} - X_i^T\vec{y} - \lambda)\right)$

b)

See q3/l1ls.m

c)

Following are the thetas arrived at by the coordinate descent algorithm from part (b), vertical dots indicate areas of contiguous 0s.

$$\lambda = .001, \quad [\text{index}, \theta] = \begin{bmatrix} 1 & 0.67077 \\ 2 & 0.81233 \\ 3 & -0.82544 \\ 4 & -0.81632 \\ 5 & -0.93817 \\ 6 & -0.00437 \\ & \vdots \\ 12 & -0.00632 \\ 13 & -0.00213 \\ 14 & 0.00000 \\ 15 & -0.03042 \\ & \vdots \\ 33 & 0.00349 \\ 34 & -0.01144 \\ & \vdots \\ 39 & 0.00203 \\ & \vdots \\ 48 & 0.01879 \\ & \vdots \\ 51 & 0.00512 \\ & \vdots \\ 71 & 0.00002 \\ 72 & 0.00000 \\ 73 & 0.00033 \\ 74 & -0.02040 \\ & \vdots \\ 77 & 0.00023 \\ & \vdots \\ 81 & 0.00059 \\ 82 & 0.01227 \\ 83 & -0.01478 \\ & \vdots \\ 86 & -0.00727 \\ & \vdots \\ 91 & 0.00026 \\ & \vdots \\ 94 & -0.00046 \\ & \vdots \end{bmatrix} \qquad \lambda = .01, \quad [\text{index}, \theta] = \begin{bmatrix} 1 & 0.66890 \\ 2 & 0.81693 \\ 3 & -0.81803 \\ 4 & -0.81682 \\ 5 & -0.94517 \\ 6 & -0.00377 \\ & \vdots \\ 12 & -0.00393 \\ 13 & -0.00015 \\ 14 & 0.00000 \\ 15 & -0.03132 \\ & \vdots \\ 34 & -0.00887 \\ & \vdots \\ 39 & 0.00176 \\ & \vdots \\ 48 & 0.01794 \\ & \vdots \\ 71 & 0.00093 \\ & \vdots \\ 74 & -0.02258 \\ & \vdots \\ 81 & 0.00188 \\ 82 & 0.01066 \\ 83 & -0.01606 \\ & \vdots \\ 86 & -0.00284 \\ 87 & 0.00000 \\ & \vdots \\ 95 & 0.00194 \\ & \vdots \end{bmatrix}$$

$$\lambda = .1, \quad [\text{index}, \theta] = \begin{bmatrix} 1 & 0.65289 \\ 2 & 0.81030 \\ 3 & -0.80756 \\ 4 & -0.80795 \\ 5 & -0.95211 \\ & \vdots \\ 12 & -0.00537 \\ 13 & -0.00216 \\ 14 & 0.00000 \\ 15 & -0.02394 \\ & \vdots \\ 18 & 0.00292 \\ & \vdots \\ 34 & -0.00294 \\ & \vdots \\ 48 & 0.00484 \\ & \vdots \\ 74 & -0.01429 \\ & \vdots \\ 83 & -0.00298 \\ & \vdots \\ 86 & -0.00261 \\ & \vdots \end{bmatrix} \qquad \lambda = 1, \quad \theta = \begin{bmatrix} 0.49876 \\ 0.65611 \\ -0.79058 \\ -0.65564 \\ -0.89192 \\ \vdots \end{bmatrix}$$

$$\lambda = 10, \quad \theta = \begin{bmatrix} 0.00000 \\ 0.00000 \\ -0.41080 \\ 0.00000 \\ -0.07083 \\ \vdots \end{bmatrix}$$

For comparison,

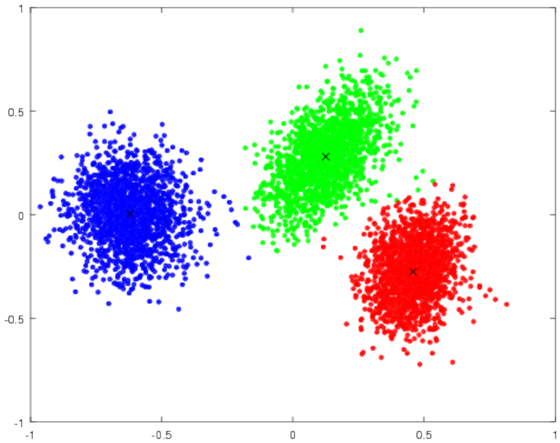
$$\theta_{\text{true}} = \begin{bmatrix} 0.68372018 \\ 0.84110202 \\ -0.83028605 \\ -0.85031124 \\ -0.93904984 \\ \vdots \end{bmatrix}$$

As λ is increased, the θ output by the algorithm becomes more sparse. This algorithm could be used for feature selection because most of the values in θ are set to zero, meaning those features are effectively ignored. By tuning λ you can control approximately how many features are kept by the algorithm.

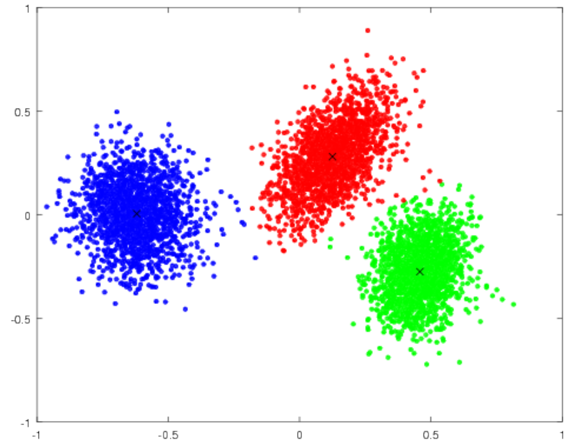
4

See q4/k_means.m for code

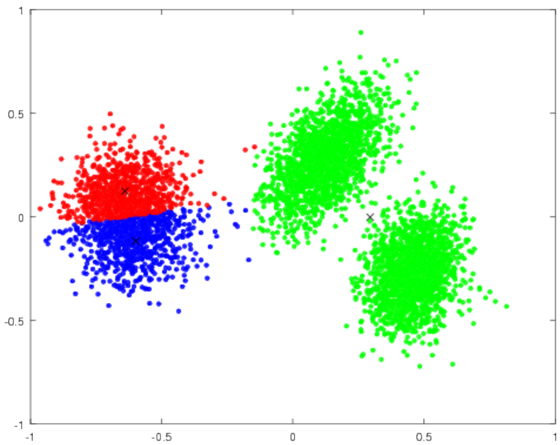
k = 3



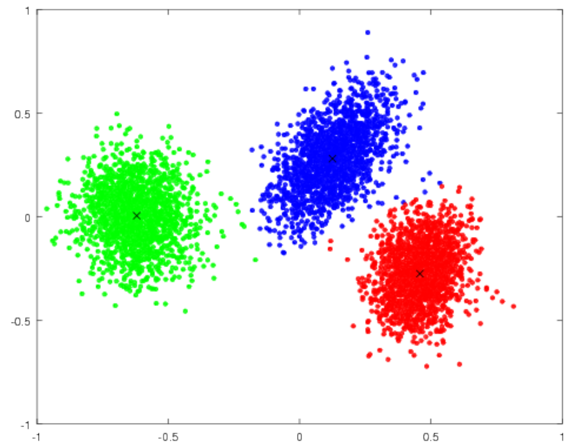
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1



1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

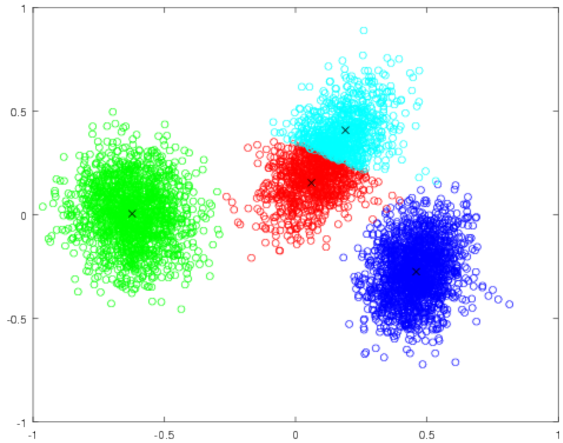


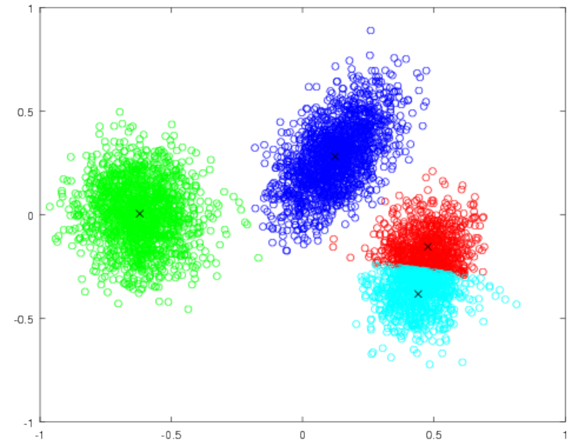
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

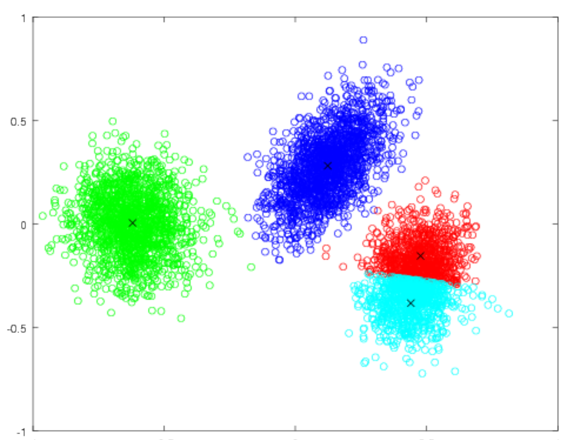


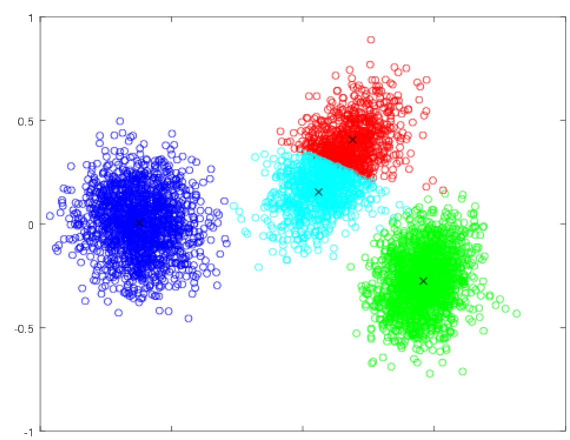
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

k = 4









5

a)

Prove that $\ell(\theta^{(t+1)}) \geq \ell(\theta^{(t)})$

Because we chose $Q_i(z^{(i)}) := p(z^{(i)}|x^{(i)};\theta^{(t)})$ in the E-step,

$$\ell(\theta^{(t)}) = \sum_i \sum_{z^{(i)}} Q_i^{(t)}(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \theta^{(t)})}{Q_i^{(t)}(z^{(i)})}$$

The parameters $\theta^{(t+1)}$ are obtained by the gradient ascent update step

$$\theta^{(t+1)} = \theta^{(t)} + \alpha \nabla_\theta \sum_i \sum_{z^{(i)}} Q_i^{(t)}(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \theta^{(t)})}{Q_i^{(t)}(z^{(i)})}$$

Because we assume α is small enough to not decrease the objective function, $\theta^{(t+1)}$ will result in a greater than or equal value of the objective function. Using similar logic as in the lecture notes:

$$\begin{aligned}
\ell(\theta^{(t+1)}) &\geq \sum_i \sum_{z^{(i)}} Q_i^{(t)}(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \theta^{(t+1)})}{Q_i^{(t)}(z^{(i)})} \quad \text{(by Jensen's inequality)} \\
&\geq \sum_i \sum_{z^{(i)}} Q_i^{(t)}(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \theta^{(t)})}{Q_i^{(t)}(z^{(i)})} \quad \text{(because of our gradient ascent step as noted above)} \\
&= \ell(\theta^{(t)}) \quad \text{(because } Q_i^{(t)} \text{ was chosen to make Jensen's inequality hold with equality at } \theta^{(t)} \text{ as noted above)}
\end{aligned}$$

b)

To show that the update steps are the same we need to show that
$\nabla_\theta \sum_i \log \sum_{z^{(i)}} p(x^{(i)}, z^{(i)}; \theta) = \nabla_\theta \sum_i \sum_{z^{(i)}} Q_i(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \theta)}{Q_i(z^{(i)})}$

We start with taking the gradient of the first term:

$$\begin{aligned}
&\nabla_\theta \sum_i \log \sum_{z^{(i)}} p(x^{(i)}, z^{(i)}; \theta) \\
&= \sum_i \frac{1}{\sum_{z^{(i)}} p(x^{(i)}, z^{(i)}; \theta)} \frac{\partial}{\partial \theta} \sum_{z^{(i)}} p(x^{(i)}, z^{(i)}; \theta) \\
&= \sum_i \frac{1}{p(x^{(i)}; \theta)} \frac{\partial}{\partial \theta} p(x^{(i)}; \theta)
\end{aligned}$$

And then the gradient of the second term:

$$\nabla_\theta \sum_i \sum_{z^{(i)}} Q_i(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \theta)}{Q_i(z^{(i)})}$$

$$= \sum_i \sum_{z^{(i)}} Q_i(z^{(i)}) \left(\frac{Q_i(z^{(i)})}{p(x^{(i)}, z^{(i)}; \theta)} \right) \frac{\partial}{\partial \theta} \frac{p(x^{(i)}, z^{(i)}; \theta)}{Q_i(z^{(i)})}$$

But we have defined $Q_i(z^{(i)}) = p(z^{(i)}|x^{(i)}; \theta) = \dfrac{p(x^{(i)}, z^{(i)}; \theta)}{p(x^{(i)}; \theta)}$, therefore

$$= \sum_i \sum_{z^{(i)}} \frac{p(x^{(i)}, z^{(i)}; \theta)}{p(x^{(i)}; \theta)} \frac{p(x^{(i)}, z^{(i)}; \theta)}{p(x^{(i)}; \theta)} \frac{1}{p(x^{(i)}, z^{(i)}; \theta)} \frac{\partial}{\partial \theta} p(x^{(i)}, z^{(i)}; \theta) \frac{p(x^{(i)}; \theta)}{p(x^{(i)}, z^{(i)}; \theta)}$$

$$= \sum_i \sum_{z^{(i)}} \frac{p(x^{(i)}, z^{(i)}; \theta)}{p(x^{(i)}; \theta) p(x^{(i)}; \theta)} \frac{\partial}{\partial \theta} p(x^{(i)}; \theta)$$

$$= \sum_i \frac{1}{p(x^{(i)})} \frac{\partial}{\partial \theta} p(x^{(i)}; \theta)$$

Thus the gradient of both terms is the same and the update functions yield the exact same update.