STAT 212: Principles of Statistics II

Lecture Notes: Chapter 4

Nonparametric Methods
('Distribution-Free' Tests)

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Distribution-Free Procedures

Throughout 211 and 212 we have made assumptions such as *"the data are a random sample from the Normal distribution"* and *"the errors are Normally distributed."*

The validity of the tests and confidence intervals we've discussed relies to some extent on these assumptions. *Validity definitely relies on the assumptions when the sample size is not very big.*

The motivation for *distribution-free, or nonparametric* procedures is that they retain their validity under very general conditions. In other words, *we don't have to make restrictive assumptions for these procedures to be valid.*

Sign test

We go back to a 211 problem: testing a hypothesis about the "center" of a population.

Two common measures of center are the mean and median.

The *t-test* was used to test hypotheses about the population *mean*.

The *sign test* will be used to test hypotheses about the population *median*.

When the population is symmetric, the mean and median are, of course, the same.

If the mean and median are different, as with skewed distributions, the t-test and sign test are testing different hypotheses.

Let X be a continuous random variable with median $\tilde{\mu}$. This means that

$$P(X \geq \tilde{\mu}) = P(X \leq \tilde{\mu}) = \frac{1}{2}.$$

Suppose that $X_1, \ldots, X_n$ is a random sample from the distribution possessed by X.

Goal is to test:

$$H_0 : \tilde{\mu} = c \quad \text{vs.} \quad H_a : \tilde{\mu} > c.$$

The motivation for the sign test is as follows:

- *When H_0 is true, around 1/2 of the X_is will be larger than c.*

- *When H_a is true, the fraction of X_is larger than c will tend to be more than 1/2.*

Our test statistic will be:

$$Y = \text{number of } X_i\text{'s larger than } c.$$

We will reject H_0 when Y is 'too big'. How big is 'too big'?

Under H_0, Y has a binomial distribution with number of trials equal to n and success probability 1/2.

As long as $n \geq 10$, it is reasonable to use the Normal approximation to the binomial to carry out the test.

When H_0 is true, the distribution of

$$\frac{Y - n/2}{\sqrt{n}/2}$$

is approximately standard Normal. So, H_0 is rejected at level α if:

$$\frac{Y - n/2}{\sqrt{n}/2} \geq z_\alpha.$$

Of course, we can also test:

$$H_0 : \tilde{\mu} = c \quad \text{vs.} \quad H_a : \tilde{\mu} < c.$$

and

$$H_0 : \tilde{\mu} = c \quad \text{vs.} \quad H_a : \tilde{\mu} \neq c.$$

We use the same test statistic as before but different rejection regions. These are, respectively,

$$\frac{Y - n/2}{\sqrt{n}/2} \leq -z_\alpha$$

and

$$\frac{|Y - n/2|}{\sqrt{n}/2} \geq z_{\alpha/2}.$$

Example 20: *Distribution of pH values*

Observations: pH values of synovial fluid taken from the knees of arthritis sufferers

7.02	7.35	7.34	7.17	7.28	7.77	7.09
7.22	7.45	6.95	7.40	7.10	7.32	7.14

True median pH for nonarthritic individuals: 7.39

Does the median pH for arthritis sufferers appear to differ from that for nonarthritic individuals?

$$H_0 : \tilde{\mu} = 7.39 \quad \text{vs.} \quad H_a : \tilde{\mu} \neq 7.39$$

There are three data values larger than 7.39, so $Y = 3$. The test statistic is

$$\frac{Y - n/2}{\sqrt{n}/2} = \frac{3 - 7}{\sqrt{14}/2} = -2.13809.$$

The P-value is:

$$P = 2P(Z \geq 2.13809) = 0.0325.$$

H_0 would be rejected for any $\alpha \geq 0.0325$. So, there is significant evidence that the median of arthritis sufferers differs from 7.39.

The sample median is 7.25, suggesting that the median pH for arthritis sufferers is less than that of nonarthritic people.

Pros and cons of the sign test

Pros

- The only assumption needed is that the data are a random sample.
- Don't need to assume anything about the distribution.
- When the population is "heavy-tailed," the sign test is usually more powerful than the t-test.

Cons

- Suppose that the population *is* Normally distributed. Then, of course, a t-test will be more powerful than the sign test.