Com S 330 Discrete Computational Structures

Fall Semester 2012

Final Exam

Wednesday, December 12, 2012

Time: 2 hours

Name: ______________________________

ID: ____________________

The exam is **closed book. No notes or calculators are allowed.** Please go over all the questions in the exam before you start working on it. Attempt the questions that seem easier first. The exam has a total of 120 points but you will only be scored over 100 points. This means that you can get a total of 20 **extra credit** points. If you see yourself getting stuck on one question, continue on and come back to it later if you have time. *Always explain your answers!* Good luck and have a great winter break!

Note: For the computational problems, leaving answers in the form of $P(n, r)$ or $C(n, r)$ will be acceptable. *Show all intermediate steps.*

1	2	3	4	5	6	7	8	Total
20	16	18	12	16	12	14	12	120

1. Counting Functions and Relations [**20 Points**]

   Let S and T be sets where $|S| = 5$, $|T| = 6$. *Show your work.*

   (a) [**6 Pts**] How many different functions are possible from S to T? How many of these functions are one-to-one?

   (b) [**6 Pts**] How many different functions are possible from T to S? How many of these functions are onto?

   (c) [**4 Pts**] How many different binary relations are possible on S?

   (d) [**4 Pts**] How many equivalence relations on set T are possible where each equivalence class is of size 2?

2. Counting Arguments [**16 Points**]

Prove that $P(n,k)C(n-k,k) = C(n,2k)P(2k,k)$, where $2k \leq n$.

(a) [**8 Pts**] using an algebraic argument based on the formulae for $C(n,k)$ and $P(n,k)$.

(b) [**8 Pts**] using a counting or combinatorial argument.

Hint: Describe a problem that can be counted in two ways, and the lhs and rhs of the equation gives the two different ways.

3. Combinatorial Techniques I [**18 Points**]

   For the following problems, you have a sack containing 40 coins, with 10 of each of four different denominations (pennies, nickels, dimes and quarters).

   (a) [**6 Pts**] How many coins do you have to pick to make sure you have (i) *at least* 5 coins of the same kind? (ii) *at least* 5 dimes?

   (b) [**4 Pts**] How many ways can we pick a *set* of 10 coins from the sack?

   (c) [**8 Pts**] How many of these sets have (i) at least 3 dimes and at least 3 nickels? (ii) at least 3 dimes but at most 2 nickels?

4. Combinatorial Techniques II [**12 Points**]

For the next few problems, pick a *sequence* of 8 coins from the same sack of 40 coins, containing equal numbers of pennies, nickels, dimes and quarters.

(a) [**4 Pts**] How many different sequences have *exactly* 3 quarters?

(b) [**4 Pts**] How many different sequences have *exactly* 3 quarters **and** *exactly* 3 nickels?

(c) [**4 Pts**] How many different sequences have *exactly* 3 quarters **or** *exactly* 3 nickels?

5. Combinatorial Techniques II [**16 Points**]

(a) [**8 Pts**] How many ways can we place 10 different books into 5 boxes with 2 books in each (i) if the boxes are distinct? (ii) if the boxes are identical?

(b) [**8 Pts**] Now place 5 identical books into 3 boxes with no restrictions on the number that go into each box. How many ways can you do this (i) if the boxes are identical? (ii) if the boxes are distinct?

6. Graph Properties [**12 Points**]

(a) [**6 Pts**] If G is a graph with n vertices and n edges, is G connected? If *yes*, give a short justification. If *no*, give a counterexample.

(b) [**6 Pts**] Let G be a simple, undirected graph with 20 edges. Five vertices have degree 4, two vertices have degree 3, four vertices have degree 2, and the rest have degree 1. How many vertices does G have with degree 1? Justify your answer. Drawing a graph will not be sufficient!

7. Recursive Definitions and Induction I [**14 Points**]

   Consider this recursiive definition for S, a set of ordered pairs of natural numbers.

   Base Case: $(0, 0) \in S$.

   Recursive Step: if $(a, b) \in S$, then $(a + 1, b + 2) \in S$ and $(a + 2, b + 1) \in S$.

   (a) [**8 Pts**] Prove that if $(a, b) \in S$ then $a + b$ is divisible by 3.

      i. [**3 Pts**] Prove the basis step.

      ii. [**6 Pts**] Prove the recursive step. State what you assume clearly.

   (b) [**5 Pts**] Show that the converse of the statement above is not true, *i.e.*, if $a, b \in \mathcal{N}$, and $a + b$ is divisible by 3, it does not follow that $(a, b) \in S$. Modify the recursive definition of S to make the converse true.

8. Recursive Definitions and Induction II [**12 Points**]

Consider this recursive definition of a *complete ternary tree (CTT)* of height h.

- A *complete ternary tree* of height 0 is a single vertex.
- A *complete ternary tree* of height $h+1$ is a tree whose root node has three children, and each child is the root of a subtree which is a *complete ternary tree* of height h.

(a) [**3 Pts**] Give a recursive definition of $leaves(T)$, the number of leaves in tree T, where T is a CTT, as defined above. *In other words, define* $leaves(T)$ *in terms of* $leaves(T_1)$, $leaves(T_2)$ *and* $leaves(T_3)$, *where* T_1, T_2 *and* T_3 *are the subtrees of* T.

If T is the singleton vertex, $leaves(T) = 1$. *This is the base case of your definition.*

(b) [**9 Pts**] Now prove, by induction on h, that a CTT of height h has 3^h leaves. *Use the recursive definitions above. Justify each step.*

i. [**3 Pts**] First, prove the basis step, where $h = 0$.

ii. [**6 Pts**] Then, prove the inductive step.