Com S 230 Discrete Computational Structures

Spring Semester 2017

Exam 2 Solutions

Friday, March 31, 2017

Time: 50 minutes

Name: ________________________________

The exam is **closed book. No notes or calculators are allowed.** Please go over all the questions in the exam before you start working on it. Attempt the questions that seem easier first. The exam has a total of 120 points but you will only be scored over 100 points. This means that you can get a total of 20 **extra credit** points. If you see yourself getting stuck on one question, continue on and come back to it later if you have time. Try to stay as brief and to the point as possible. Good luck!

1	2	3	4	5	6	Total
18	30	16	18	18	20	100

1. Short Answers [**18 Points**] Answer these questions briefly. No justification is required.

   (a) [**6 Pts**] Give an *inductive definition* for a_i of the sequence $5, 12, 26, 54, 110, \ldots$, where a_0 is the first term.

   $a_0 = 5$
   $a_{i+1} = 2a_i + 2$ for all $i \geq 0$.

   (b) [**6 Pts**] Give an *inductive definition* for S, the set of ordered pairs (a, b) where a and b are integers, and $a = b$.

   Base: $(0, 0) \in S$.

   Ind: if $(a, b) \in S$ *then*

   $$(a+1, b+1) \in S \textit{ and } (a-1, b-1) \in S.$$

   (c) [**6 Pts**] Consider this inductive definition for a set of integers A.

   Base Case: $7 \in A$.

   Inductive Step: if m is in A and n is in A, then mn is in A.

   What are the elements in A? Define A using set-builder notation.

   $A = \{7^i \mid i \in \mathcal{Z}^+\}$

2. Equivalence Relations and Partial Orders [**30 Points**]

   Consider the following relations on the set $\mathcal{R}^+$.

   $R_1 = \{(a, b) \mid a - b \in \mathcal{Z}\}$ and $R_2 = \{(a, b) \mid a/b \in \mathcal{Z}\}$

   (a) [**16 Pts**] State whether the relation is reflexive, anti-reflexive, symmetric, anti-symmetric and transitive. Justify your answer with a proof or a counter-example.

      i. [**8 Pts**] R_1

      $\forall a \in \mathcal{R}^+, a - a = 0 \in \mathcal{Z},$ *so* $(a, a) \in R_1$.

      Therefore, reflexive, but not anti-reflexive.

      $\forall a, b \in \mathcal{R}^+,\ (a, b) \in R_1 \ \Rightarrow a - b \in \mathcal{Z}$
      $\Rightarrow b - a = -(a - b) \in \mathcal{Z}$
      $\Rightarrow (b, a) \in R_1$.

      Therefore, symmetric, but not anti-symmetric.

      $\forall a, b, c \in \mathcal{R}^+,\ (a, b) \in R_1 \land (b, c) \in R_1$
      $\Rightarrow a - b \in \mathcal{Z} \land b - c \in \mathcal{Z}$
      $\Rightarrow a - c = (a - b) + (b - c) \in \mathcal{Z}$
      $\Rightarrow (a, c) \in R_1$.

      Therefore, transitive.

      ii. [**8 Pts**] R_2

      $\forall a \in \mathcal{R}^+, a/a = 1 \in \mathcal{Z},$ *so* $(a, a) \in R_2$.

      Therefore, reflexive, but not anti-reflexive.

      $2/1 \in \mathcal{Z}$, *which implies* $(2, 1) \in R_2$, *but* $1/2 \notin \mathcal{Z}$, *which implies* $(1, 2) \notin R_2$.

      Therefore not symmetric.

      $\forall a, b \in \mathcal{R}^+,\ (a, b) \in R_2 \land (b, a) \in R_2 \ \Rightarrow a/b \in \mathcal{Z} \land b/a \in \mathcal{Z} \ \Rightarrow a = b$

      Therefore, anti-symmetric.

      $\forall a, b, c \in \mathcal{R}^+,\ (a, b) \in R_2 \land (b, c) \in R_2$
      $\Rightarrow a/b \in \mathcal{Z} \land b/c \in \mathcal{Z}$
      $\Rightarrow a/c = (a/b)(b/c) \in \mathcal{Z} \ \Rightarrow (a, c) \in R_2$.

      Therefore, transitive.

(b) [**4 Pts**] One of these relations is an equivalence relation. The other is either a partial order or a strict partial order. Which is which?

i. R_1

equivalence relation since reflexive, symmetric and transitive

ii. R_2

partial order since reflexive, anti-symmetric and transitive

(c) [**10 Pts**] For the equivalence relation, answer the following.

i. Define the equivalence class containing 1.

$[1] = \{a \mid a - 1 \in \mathcal{Z},\ a \in \mathcal{R}^+\} = \mathcal{Z} \cap \mathcal{R}^+ = \mathcal{Z}^+$

ii. Define the equivalence class containing 1/2.

$[1/2] = \{a \mid a - 1/2 \in \mathcal{Z},\ a \in \mathcal{R}^+\} = \{a + 1/2 \mid a \in \mathcal{N}\}$

$= \{0.5, 1.5, 2.5, \ldots\}.$

iii. Describe all the different equivalence classes using set-builder notation. How many equivalence classes are there in all? Are there a finite or infinite number of members in each equivalence class?

The equivalence classes are $[i] = \{i + n \mid n \in \mathcal{N}\}$ *for all* $i \in (0, 1]$.
There are uncountably infinite number of equivalence classes since $(0, 1]$ *is uncountably infinite.*
There are a countably infinite number of elements in each class $[i]$ *since there is a bijection* $f(n) = i + n$ *from* $\mathcal{N}$ *to* $[i]$*, and* $\mathcal{N}$ *is countably infinite.*

3. Mathematical Induction I [**16 Points**]

Consider the predicate $P(n) : n^2 + n + 1$ is even.

(a) Prove the inductive step; *i.e.* for all $k \geq 0$, if $P(k)$ then $P(k+1)$.

Assume $P(k)$, *i.e.*, $k^2 + k + 1$ *is even. So,* $k^2 + k + 1 = 2a$ *for some* $a \in \mathcal{Z}$. *Now,*

$(k+1)^2 + (k+1) + 1$

$= k^2 + 2k + 1 + k + 1 + 1$
$= (k^2 + k + 1) + 2(k+1)$
$= 2(a + k + 1)$

Since $a + k + 1 \in \mathcal{Z}$, *it follows that* $(k+1)^2 + (k+1) + 1$ *is even.*

So, $P(k+1)$ *is true.*

(b) For what values of n is $P(n)$ actually true? What is wrong with your proof?

$P(n)$ *is false for all values of* n. *The proof does not hold since the base case is false and cannot be proven, even though the inductive step can be proven.*

4. Mathematical Induction II [**18 Points**]

Let F(n) define the Fibonacci numbers, where $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for all $n \geq 2$.

Consider the statement

$$F(1) + F(3) + \cdots + F(2n-1) = F(2n)$$

Now, prove the statement for all $n \geq 1$ using mathematical induction.

(a) [**6 Pts**] State the base case and prove it.

$n = 1 :$ *Prove that* $F(1) = F(2)$.

$F(1) = 1$ *and* $F(2) = F(0) + F(1) = 0 + 1 = 1$.

Thus, $F(1) = F(2)$.

(b) [**4 Pts**] State what you assume and what you prove in the inductive step.

Assume $F(1) + F(3) + \cdots + F(2k-1) = F(2k)$

Prove $F(1) + F(3) + \cdots + F(2k+1) = F(2k+2)$

(c) [**8 Pts**] Now, prove the inductive step.

$F(1) + F(3) + \cdots + F(2k+1)$

$= [F(1) + F(3) + \cdots + F(2k-1)] + F(2k+1)$

$= F(2k) + F(2k+1)$, *by inductive hypothesis*

$= F(2k+2)$, *by inductive definition of Fibonacci numbers.*

5. Strong Induction [**18 Points**]

Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent and 11-cent stamps. Prove that $P(n)$ is true for all $n \geq 21$, using the steps below.

(a) Prove $P(n)$ for all $n \geq 21$ by *regular* induction.

i. [**3 Pts**] Prove $P(21)$ to complete the basis step.

21 *cents can be obtained using seven* 3*-cent stamps.*

ii. [**6 Pts**] Prove $P(k) \to P(k+1)$ for all $k \geq 21$.

Assume $P(k)$, *i.e.*, k*-cent postage can be formed. There are 2 cases; (A) either some* 11*-cent stamp was used, or (B) no* 11*-cent stamps were used.*

A. If an 11*-cent stamp was used in* k*-cent postage, replace an* 11*-cent stamp with four* 3*-cent stamps to form* $k - 11 + 4(3) = k + 1$ *cent postage.*

B. If no 11*-cent stamp was used, then only* 3*-cent stamps were used, so* k *is a multiple of* 3*. Since* $k \geq 21$*, at least seven* 3*-cent stamps were used. Replace seven* 3*-cent stamps with two* 11*-cent stamps to form* $k - 7(3) + 2(11) = k + 1$ *cent postage.*

(b) Now, prove $P(n)$ for all $n \geq 21$ by *strong* induction.

i. [**3 Pts**] Prove that $P(21)$, $P(22)$ and $P(23)$ to complete the basis step.

21 *cents can be obtained using seven* 3*-cent stamps.*
22 *cents can be obtained using two 11-cent stamps.*
23 *cents can be obtained using four 3-cent stamps and one 11-cent stamp.*

ii. [**6 Pts**] Prove that for all $k \geq 23$, if $P(j)$ is true for $21 \leq j \leq k$, then $P(k+1)$ is true.

Assume $P(j)$ *for* $21 \leq j \leq k$*. Since* $k \geq 23$*, we know that* $P(k-2)$*. So* $k-2$ *cent postage can be formed using* 3*-cent and* 11*-cent stamps. Add a* 3*-cent stamp to form* $(k-2)+3 = k+1$ *cent postage.*

6. Structural Induction [**20 Points**]

Consider this inductive definition for a set of strings S over the alphabet $\{a, b\}$.

Base Case: ab is in S.

Inductive Case: if x is in S, then axb is in S.

Let $A = \{a^i b^i \mid i \in \mathcal{Z}^+\}$. We prove that $S = A$.

Note: a^i is short form for a string of i a's. So, $aabb$ is also written as a^2b^2, and $aaabbb$ is also written as a^3b^3. The strings ab, $aabb$ and $aaabbb$ are all in A.

(a) [**10 Pts**] Prove that $S \subseteq A$, *using structural induction.* In other words, prove that if $x \in S$, then $x = a^i b^i$ for some positive integer i. Prove the *base case* and the *inductive step.*

Base: *Since $ab \in S$ by base case of inductive definition of S, we prove that $ab = a^i b^i$ for some $i \in \mathcal{Z}^+$. True, since $ab = a^1 b^1$ and $i = 1 \in \mathcal{Z}^+$.*

Ind: *If $x \in S$ then by induction hypothesis, $x = a^i b^i$ for some $i \in \mathcal{Z}^+$. Now $axb \in S$, by the inductive case of the definition of S. Prove that $axb = a^j b^j$ for some $j \in \mathcal{Z}^+$.*

Let $x \in S$ where $x = a^i b^i$, $i \in \mathcal{Z}^+$. Now, $axb = aa^i b^i b = a^{i+1} b^{i+1}$, and $i+1 \in \mathcal{Z}^+$. So, $axb = a^j b^j$ for some $j \in \mathcal{Z}^+$.

(b) [**10 Pts**] Prove that $A \subseteq S$, *using mathematical induction.* In other words, prove that if $x = a^i b^i$ for some positive integer i, then $x \in S$. Prove the *base case* and the *inductive step.*

Base: *$(i = 1)$ Prove $a^1 b^1 \in S$.*

$a^1 b^1 = ab \in S$ by base case of inductive definition of S.

Ind: *Assume $a^i b^i \in S$. Prove that $a^{i+1} b^{i+1} \in S$.*

By inductive step of definition of S, $x \in S$ implies $axb \in S$. So, if $a^i b^i \in S$, then $aa^i b^i b = a^{i+1} b^{i+1} \in S$.