Com S 230 Discrete Computational Structures

Spring Semester 2017

Exam 2

Friday, March 31, 2017

Time: 50 minutes

Name: ______________________________

ID: ____________________

The exam is **closed book. No notes or calculators are allowed.** Please go over all the questions in the exam before you start working on it. Attempt the questions that seem easier first. The exam has a total of 120 points but you will only be scored over 100 points. This means that you can get a total of 20 **extra credit** points. If you see yourself getting stuck on one question, continue on and come back to it later if you have time. Try to stay as brief and to the point as possible. Good luck!

1	2	3	4	5	6	Total
18	30	16	18	18	20	100

1. Short Answers [**18 Points**] Answer these questions briefly. No justification is required.

   (a) [**6 Pts**] Give an *inductive definition* for a_i of the sequence $5, 12, 26, 54, 110, \ldots$, where a_0 is the first term.

   (b) [**6 Pts**] Give an *inductive definition* for S, the set of ordered pairs (a, b) where a and b are integers, and $a = b$.

   (c) [**6 Pts**] Consider this inductive definition for a set of integers A.

   Base Case: $7 \in A$.

   Inductive Step: if m is in A and n is in A, then mn is in A.

   What are the elements in A? Define A using set-builder notation.

2. Equivalence Relations and Partial Orders [**30 Points**]

   Consider the following relations on the set $\mathcal{R}^+$.

   $R_1 = \{(a, b) \mid a - b \in \mathcal{Z}\}$ and $R_2 = \{(a, b) \mid a/b \in \mathcal{Z}\}$

   (a) [**16 Pts**] State whether the relation is reflexive, anti-reflexive, symmetric, anti-symmetric and transitive. Justify your answer with a proof or a counter-example.

      i. [**8 Pts**] R_1

      ii. [**8 Pts**] R_2

(b) **[4 Pts]** One of these relations is an equivalence relation. The other is either a partial order or a strict partial order. Which is which?

i. R_1

ii. R_2

(c) **[10 Pts]** For the equivalence relation, answer the following.

i. Define the equivalence class containing 1.

ii. Define the equivalence class containing 1/2.

iii. Describe all the different equivalence classes using set-builder notation. How many equivalence classes are there in all? Are there a finite or infinite number of members in each equivalence class?

3. Mathematical Induction I [**16 Points**]

   Consider the predicate $P(n) : n^2 + n + 1$ is even.

   (a) Prove the inductive step; *i.e.* for all $k \in \mathcal{N}$, if $P(k)$ then $P(k+1)$.

   (b) For what values of n is $P(n)$ actually true? What is wrong with your proof?

4. Mathematical Induction II [**18 Points**]

Let F(n) define the Fibonacci numbers, where $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for all $n \geq 2$.

Consider the statement

$$F(1) + F(3) + \cdots + F(2n-1) = F(2n)$$

Now, prove the statement for all $n \geq 1$ using mathematical induction.

(a) [**6 Pts**] State the base case and prove it.

(b) [**4 Pts**] State what you assume and what you prove in the inductive step.

(c) [**8 Pts**] Now, prove the inductive step.

5. Strong Induction [**18 Points**]

Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent and 11-cent stamps. Prove that $P(n)$ is true for all $n \geq 21$, using the steps below.

(a) Prove $P(n)$ for all $n \geq 21$ by *regular* induction.

i. [**3 Pts**] Prove $P(21)$ to complete the basis step.

ii. [**6 Pts**] Prove $P(k) \to P(k+1)$ for all $k \geq 21$.

(b) Now, prove $P(n)$ for all $n \geq 21$ by *strong* induction.

i. [**3 Pts**] Prove that $P(21)$, $P(22)$ and $P(23)$ to complete the basis step.

ii. [**6 Pts**] Prove that for all $k \geq 23$, if $P(j)$ is true for $21 \leq j \leq k$, then $P(k+1)$ is true.

6. Structural Induction [**20 Points**]

Consider this inductive definition for a set of strings S over the alphabet $\{a, b\}$.

Base Case: ab is in S.

Inductive Case: if x is in S, then axb is in S.

Let $A = \{a^i b^i \mid i \in \mathcal{Z}^+\}$. We prove that $S = A$.

Note: a^i is short form for a string of i a's. So, $aabb$ is also written as a^2b^2, and $aaabbb$ is also written as a^3b^3. The strings ab, $aabb$ and $aaabbb$ are all in A.

(a) [**10 Pts**] Prove that $S \subseteq A$, *using structural induction.* In other words, prove that if $x \in S$, then $x = a^i b^i$ for some positive integer i. Prove the *base case* and the *inductive step*.

(b) [**10 Pts**] Prove that $A \subseteq S$, *using mathematical induction.* In other words, prove that if $x = a^i b^i$ for some positive integer i, then $x \in S$. Prove the *base case* and the *inductive step*.