CS 330 : Discrete Computational Structures

Spring Semester, 2019
ASSIGNMENT #1
Due Date: Monday, January 28

Suggested Reading: Rosen Sections 1.1 - 1.3; LLM Sections 1.1, 3.1 - 3.4

These are the problems that you need to hand in for grading. Always explain your answers and show your reasoning.

1. [**6 Pts**] Let p, q and r be the propositions "you have the flu", "you miss the final", and "you pass the course" respectively. Express (a) $p \vee \neg q$, (b) $(\neg p \wedge \neg q) \rightarrow r$ and (c) $(p \wedge \neg r) \vee (q \wedge r)$ as English sentences.

2. [**6 Pts**] Translate the following English sentences into logic. First, define your basic propositions and use logical operations to connect them.
(a) Being sunny is sufficient for us to play tennis.
(b) To play tennis, it is necessary that it be sunny and not windy.
(c) It is sunny only if we play tennis, unless it is windy.

3. [**10 Pts**] Use logical reasoning or truth tables to solve the following puzzle:
Five friends can't agree on whether they want Chinese or Italian food for dinner. It is not true that both David and Ellen want Italian. Ben and Ellen want the same cuisine. Ben and Cathy don't want the same cuisine. If Cindy wants Italian, then so does Ann. If David wants Chinese, then so do Ann and Ellen. Who wants what cuisine?

4. [**4 Pts**] Determine whether $(\neg q \wedge (\neg p \vee q)) \rightarrow \neg p$ is a tautology. Prove your answer using a truth table.

5. [**4 Pts**] Construct the truth table for $(p \wedge q) \rightarrow (q \vee r))$.

6. [**10 Pts**] Prove that $(p \rightarrow r) \vee (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$ are logically equivalent, (a) by truth tables, and (b) by deduction using a series of logical equivalences studied in class.

7. [**10 Pts**] Assume that $p \,\mathrm{NAND}\, q$ is logically equivalent to $\neg(p \wedge q)$. Then, (a) prove that {NAND} is functionally complete, i.e., any propositional formula is equivalent to one whose only connective is NAND. Now, (b) prove that any propositional formula is equivalent to one whose only connectives are XOR and AND, along with the constant TRUE. Prove these using a series of logical equivalences.

For more practice, you are encouraged to work on the problems given at the end of Rosen, Sections 1.1 - 1.2.

Wei Xian Lim 901579699

1. (a) It's either you have the flu, or you will not miss the final.
   (b) If you don't have the flu and you don't miss the final, you pass the course.
   (c) You have the flu and you don't pass the course, or you miss the final and you pass the course.

2. P = Sunny
   Q = Play tennis
   R = Windy

   (a) $P \rightarrow Q$
   (b) $(P \land \neg R) \rightarrow Q$
   (c) $R \rightarrow \neg(Q \rightarrow P)$

3.
   (a) Not true that both David and Ellen want Italian
   $\neg(D \land E) = \neg D \lor \neg E$
   (b) Ben and Ellen want same cuisine.
   $B = E$
   (c) Ben and Cathy don't want the same cuisine.
   $B \neq C$
   (d) If Cathy wants Italian, then so does Ann
   $C \rightarrow A$
   (e) If David wants Chinese, then so does Ann and Ellen.
   $D \rightarrow (A \land E)$

   Assuming David wants Italian, then Ellen wants Chinese because (a). If Ellen wants Chinese, then Ben wants Chinese because (b). If Ben wants Chinese, then Cathy wants Italian because (c).

   David = Italian
   Ann = Italian
   Ellen = Chinese
   Ben = Chinese
   Cathy = Italian

4.

P	Q	$\neg P$	$\neg Q$	$\neg P \lor Q$	$\neg Q \land (\neg P \lor Q)$	$\neg Q \land (\neg P \lor Q) \rightarrow \neg P$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Yes, it is a tautology because the outcomes are all true.

5.

P	Q	R	P ∧ Q	Q V R	(P ∧ Q) → (Q V R)
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	F	T	T
F	T	F	F	T	T
F	F	T	F	T	T
F	F	F	F	F	T

6. (a)

P	Q	R	P → R	Q → R	(P → R) V (Q → R)	P ∧ Q	(P ∧ Q) → R
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	F	T
T	F	F	F	T	T	F	T
F	T	T	T	T	T	F	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

(b) (P → R) V (Q → R)

≡ (¬P V R) V (¬Q V R)

≡ (¬P V ¬Q) V R

≡ ¬ (P ∧ Q) V R

≡ ¬ (¬ (P ∧ Q)) → R

≡ (P ∧ Q) → R

7. (a) P NAND Q

≡ ¬ (¬P ∧ ¬Q)	DeMorgan's Law
≡ ¬P NAND ¬Q	definition of NAND
≡ (¬P ∧ ¬P) NAND (¬Q ∧ ¬Q)	Idempotent Law
≡ ¬ (P ∧ P) NAND ¬ (Q ∧ Q)	DeMorgan's Law
≡ (P NAND P) NAND (Q NAND Q)	definition of NAND

(b) P XOR Q ≡ (¬P AND Q) OR (P AND ¬Q)

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Spring Semester, 2019
ASSIGNMENT #2
Due Date: Monday, February 4

Suggested Reading: Rosen Sections 1.4 - 1.6; LLM Chapter 3

These are the problems that you need to turn in. For more practice, you are encouraged to work on other problems. Always explain your answers and show your reasoning.

1. [**4 Pts**] Translate the sentence "All bears can swim and catch fish." using three predicates. Then, state the negation of the statement with no negation to the left of the quantifier. Last, translate this back into English.
2. [**6 Pts**] Are $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ logically equivalent? If yes, give a proof. If no, give a counterexample.
3. [**6 Pts**] For the following problems, let $S(x)$, $F(x)$ and $A(x, y)$ be the statements "x is a student", "x is a faculty member" and "x has asked y a question". Let the domain be all people at ISU. Translate the following sentences into logic.
   (a) There are at least two faculty members who have not asked questions to any students.
   (b) Everyone has asked a question to at least one student and at least one faculty member.
4. [**9 Pts**] LLM p.91: Problem 3.36 (a), (b), (c)
5. [**12 Pts**] LLM p.92: Problem 3.37 (a), (b), (d), (e)
6. [**13 Pts**] Define propositions or predicates and prove the following using the appropriate rules of inference:
   (a) [**4 Pts**] If there is ice or snow, then school will be closed and practice will be canceled. If school is closed, then exams will be postponed. Exams were not postponed. Therefore, it did not snow.
   (b) [**4 Pts**] Mary, a student in class, is from Minnesota. All Minnesotans know how to ice fish. Therefore, someone in class knows how to ice fish.
   (c) [**5 Pts**] All bears are good swimmers. If you can catch fish, you will not go hungry. If you can't catch fish, you are not a good swimmer. Therefore, no bears go hungry.

For more practice, you are encouraged to work on the problems given in Rosen, Sections 1.4 - 1.6 and in LLM Chapter 3.

Wei Xian, Lim 901579699

1. $S(X) = X$ can swim
   $F(X) = X$ can catch fish
   $X =$ all bears

   $\forall x\ (S(x) \wedge F(x))$

   Negation $= \exists x\ \neg(S(x) \wedge F(x))$

   English: There exist bears that can't swim and catch fish.

2. $\forall x(P(x) \rightarrow Q(x))$ is not equivalent to $\forall x\ P(x) \rightarrow \forall x\ Q(x)$
   $X =$ all people
   $P(x) = x$ is comedian
   $Q(x) = x$ is funny
   $\forall x(P(x) \rightarrow Q(x)) =$ all comedians are funny. But not everyone is comedian.
   $\forall x\ P(x) \rightarrow \forall x\ Q(x) =$ If everyone is comedian, then everyone is funny.

3. $S(x) = x$ is a student
   $F(x) = x$ is a faculty member
   $A(x, y)$ x has asked y a question

   (a) $\exists x \forall y\ F(x) \wedge (S(y) \rightarrow \neg A(x, y))$
   (b) $\forall x \exists y\ (F(y) \wedge S(y)) \rightarrow A(x, y)$

4. PREFIX $(x, y) = x$ is a prefix of y
   SUBSTRING $(x, y) = x$ is a substring of y
   NO – 1S(x) = x is empty or a string of 0's

   (a) $\exists y\ (x = yyy)$
   (b) NO – 1S(x) $\wedge \exists y\ (x = yy)$
   (c) $\neg$ (SUBSTRING $(0, x) \wedge$ SUBSTRING $(1, x)$)

5.
   (a) Equal (m, n) = Zero(k) AND A(m, n, k)
   (b) One (n) = $\forall x$ M(x, x, n) $x = x \,.\, n$
   (d) Prime (p) = $\forall x$ NOT(One(x)) AND NOT(Equal(x, p) AND Greater(p, x) AND NOT(M(Zero(k), p, 1/x))

(e) Two(n) = ∀x One(x) AND A(n, x, x)

6.

(a) I = ice
S = snow
C = school is open
E = exam postponed

(1) I ∨ S → ¬C	hyp 1
(2) ¬C → E	hyp 2
(3) I ∨ S → E	hypothetical syllogism 1,2
(4) ¬E	
¬ (I ∨ S)	Modus tollens 3, 4]

(b) X = people
M = Mary
M(x) = x is Minnesotan
I(x) = x knows how to ice fish
C(x) = x in the class

(1) C(m) ∧ M(m)	hyp 1
(2) ∀x M(x) → I(x)	hyp 2
(3) M(m) → I(m)	universal instantiation 2
(4) M(m)	simplification 1
(5) I(m)	modes ponens
(6) C(m)	simplification 1
(7) C(m) ∧ I(m)	conjunction 5, 6
(8) ∃x C(x) ∧ I(x)	existential generalization 7

(c) X = all bear
S(x) = x is good swimmer
C(x) = x can catch fish
H(x) = x goes hungry

(1) ∀x S(x)	hyp 1
(2)S(c)	universal instantiation 1
(3) C(x) → ¬ H(x)	hyp 2
(4) ¬ C(x) → ¬ S(x)	hyp 3
(5) S(x) → C(x)	Modus tollens 4
(6) S(c) → C(c)	universal instantiation 5
(7) C(c) → ¬ H(c)	universal instantiation 3
(8) S(c) →¬ H(c)	hypothetical syllogism 7, 8

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ASSIGNMENT #3

Due Date: Monday, February 11

Suggested Reading: Rosen Sections 1.7 - 1.8; Lehman et al. Chapter 1

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. Always explain your answers and show your reasoning.

1. [**5 Pts**] Prove that the sum of three consecutive integers is divisible by 3.

2. [**8 Pts**] Prove or disprove that for all integers p, p is even if and only if p^3 is even.

3. [**6 Pts**] Prove, using a direct proof that $x + yz$ is rational if x, y and z are all rational numbers.

4. [**6 Pts**] Let x and y be non-zero rational numbers and let z be an irrational number. Prove that xyz is irrational. Can you use a direct proof? Why or why not?

5. [**6 Pts**] Let n be an integer. Prove, by contrapositive, that if $3n + 11$ is even, then n is odd.

6. [**6 Pts**] Suppose a piano teacher schedules 40 lessons over the week. Prove that she will have to schedule at least 6 on some day.

7. [**7 Pts**] Prove that the square root of 7 is irrational.

8. [**6 Pts**] Prove that there exist x and y where x is irrational and y is a non-zero rational, but x^y is rational. Is your proof constructive or non-constructive? Explain.

For more practice, you are encouraged to work on the problems given in Rosen, Sections 1.7 - 1.8 and in LLM Chapter 1.

Wei Xian, Lim 901579699

1. Let k = first integer, k + 1 = second integer, k + 2 = third integer
   $k + (k + 1) + (k + 2) = 3k + 3 = 3(k + 1)$
   Therefore $3(k + 1)$ is always divisible by 3

2. Let k = all integer
   Even = 2k
   Odd = 2k + 1

   $P = 2k + 1$ (1)
   $p^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1$ (2)
   $p^3 = (4k^2 + 4k + 1)(2k + 1) = 8k^3 + 12k^2 + 6k + 1$
   $= 2(4k^3 + 6k^2 + 3k) + 1$ (3)

   $P = 2k$ (1)
   $p^2 = 4k^2$ (2)
   $p^3 = 8k^3 = 2(4k^3)$ (3)

   Therefore p^3 will be odd if and only if p is odd. And p^3 is even if and only if p is even.

3. Let x, y, z be rational and let a, b, c, d, e and f be integers.
   $X = \frac{a}{b}, \quad y = \frac{c}{d}, \quad z = \frac{e}{f}$

   For b, d , f ≠ 0

   $$x + yz = \frac{a}{b} + \frac{c}{d}\left(\frac{e}{f}\right) = \frac{a}{b} + \frac{ce}{df}$$
   $$= \frac{adf + bce}{bdf}$$
   Since a, b, c, e, d, f is integer and d ≠ 0, f ≠ 0, b ≠ 0
   So x + yz is rational.

4. Let xyz = rational and x, y is rational
   $$z = \frac{xyz}{xy} \quad Let\ x = \frac{a}{b}, \quad y = \frac{c}{d}, \quad xyz = \frac{e}{f}$$

   $$z = xyz \times \frac{1}{xy}$$
   $$z = \frac{e}{f} \times \frac{bd}{ac}$$

$z = \dfrac{ebd}{fac}$

Since a, b, c, d, e, f is integer, and

Therefore, if xyz is rational, then z is rational. So z is irrational when xyz is irrational.

You cannot use direct proof because there is no way of showing irrational number.

5. If 3n + 11 is even, then n is odd

   Let n be even.

   n = 2k

   3n + 11 = 3(2k) + 11

   = 6k + 11

   = 2(3k +5) + 1

   Therefore, if 3n + 11 is even, then n is odd.

6. Proof by contradiction: she has to schedule at most 6 on same day.
   So each day gets 6 lessons, and there are 7 days in a week. So 6 x 7 = 42
   So the statement is proven because she only have 40 lessons over the week.

7. Proof by contradiction: Assuming square root of 7 is rational.

   $\sqrt{7} = \dfrac{a}{b}$

   For a, b is integer, and in reduced form with no common factors. B ≠ 0

   $7 = \dfrac{a^2}{b^2}$

   $7b^2 = a^2$

   Since left side is divisible by 7, so right side will also be divisible by 7. So, we can let a = 7k for k ≠ 0.

   $7b^2 = (7k)^2$

   $7b^2 = 49k^2$

   $b^2 = 7k^2$

   B^2 is multiple of 7, which means b is also multiple of 7

   But a and b should not have common factor. Therefore, it contradicts with the assumption and square root of 7 is irrational

8. X is irrational and y is non-zero rational but x^y is rational

   Let x = $\sqrt{2}$ because we know that $\sqrt{2}$ is irrational and y = 2

   $$x^y = \sqrt{2}^2$$

   $$x^y = (2^{\frac{1}{2}})^2$$

   $$x^y = 2$$

   Therefore, it is proven that when x is irrational and y is non-zero, x^y is rational.

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Spring Semester, 2019

ASSIGNMENT #4

Due Date: Wednesday, February 20

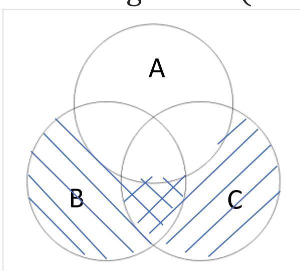
Suggested Reading: Rosen Sections 2.1 - 2.3; Lehman et al. Chapter 4.1, 4.3, 4.4

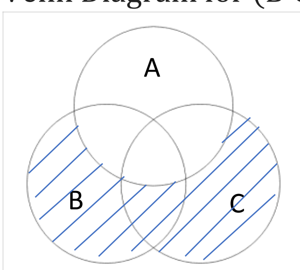
These are the problems that you need to hand in for grading. Always explain your answers and show your reasoning.

1. [**6 Pts**] Prove that if $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.
2. [**6 Pts**] Use Venn diagrams to show that $(B - A) \cup (C - A) = (B \cup C) - A$.
3. [**6 Pts**] Use an iff argument to prove that $(B - A) \cup (C - A) = (B \cup C) - A$. Use logical equivalences in your proof.
4. [**8 Pts**] Disprove the statements below.
   (a) If $A - C = B - C$ then $A = B$.
   (b) If $A \cap C = B \cap C$ then $A = B$.
5. [**8 Pts**] Prove that if $A - C = B - C$ and $A \cap C = B \cap C$ then $A = B$.
6. [**8 Pts**] Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. To prove $S = T$, where S and T are sets, prove that each set is a subset of the other. To prove $S \subseteq T$, prove that if any element x is in S then it is in T. You *may not* use logical equivalences in your proof. Use general proof techniques like 'proof by contradiction' and 'proof by cases'.
7. [**4 Pts**] Prove that $f(n) = 5n + 12$ is one-to-one, where the domain and co-domain of f is $\mathcal{Z}^+$. Show that f is not onto.
8. [**4 Pts**] Prove that $f(m, n) = 3mn$ is onto, where the domain of f is $\mathcal{R} \times \mathcal{R}$ and the co-domain of f is $\mathcal{R}$. Show that f is not one-to-one.

For more practice, you are encouraged to work on the problems given in Rosen Sections 2.1 - 2.3 and Lehman et al. Chapter 4.1, 4.3, 4.4.

1. .$A \subseteq B \leftrightarrow \{ x \in A \rightarrow x \in B\}$
   So, $C \subseteq D \leftrightarrow \{ x \in C \rightarrow x \in D\}$
   We want to show A x $C \subseteq B$ x D.
   So, $(y, z) \in A$ x $C \rightarrow (y, z) \in B$ x D
   $(y, z) \in A$ x C implies that $\{y \in A\} \wedge \{y \in C\}$
   Which means $\{y \in B\} \wedge \{y \in D\}$ because $A \subseteq B$ and $C \subseteq D$
   Therefore, it's proved that A x $C \subseteq B$ x D

2. Venn Diagram of $(B - A) \cup (C - A)$
   

   Venn Diagram for $(B \cup C)$ - A
   

3. $(B - A) \cup (C - A) = (B \cup C) - A$
   <u>Iff argument</u>
   Iff $(B - A) \cup (C - A)$
   Iff $x \in (B - A) \vee x \in (C - A)$ — Definition of Union
   Iff $(x \in B \cap x \in \bar{A}) \vee (x \in C \cap x \in \bar{A})$ — Definition of intersection
   Iff $(x \in B \vee x \in C) \cap x \in \bar{A}$
   Iff $(x \in (B \vee C)) \cap x \in \bar{A}$
   Iff $(B \cup C) - A$

   <u>Logical equivalences</u>
   $(B \cup C) - A$
   $(B \cup C) \cap \bar{A}$
   $(B \cap \bar{A}) \cap (C \cap \bar{A})$
   $(B - A) \cup (C - A)$

4. (a) if A – C = B – C then A = B

   A = {1, 2, 3}
   B = {2, 3}
   C = {1}
   A – C = {2, 3}
   B – C = {2, 3}

   Therefore, the statement is wrong because A – C and B – C both share the same elements but A ≠ B.

   (b) if A ∩ C = B ∩ C then A = B

   A = {1, 2, 3}
   B = {2, 3, 4}
   C = {2, 3}
   A ∩ C = {2, 3}
   B ∩ C = {2, 3}

   Therefore, the statement is wrong because A ∩ C and B ∩ C both share the same elements but A ≠ B.

5. Prove that if A − C = B − C and A ∩ C = B ∩ C then A = B.

   To proof that A - C = B - C and A ∩ C = B ∩ C then A = B. Let A and B be sets. Let's try to prove that if $x \in B$, then $x \in A$. We can assume that $x \in B$, and to conclude that $x \in A$. There are two cases now which are either $x \in C\ or\ x \notin C$.

   <u>Case 1</u>

   If $x \in$ C, we assume that $x \in B$, $x \in B \cap C$. Since A ∩ C = B ∩ C we can conclude that $x \in A \cap C$. Hence $x \in A\ and\ x \in C$. Therefore, $x \in A$.

   <u>Case 2</u>

   If $x \notin C$, we assume that $x \in B$, $x \in B - C$. Since A − C = B − C we can conclude that $x \in A - C$. Hence $x \in A\ and\ x \notin C$. Therefore, $x \in A$.

   Both of these cases show that x ∈ B → x ∈ C. Hence, $B \subseteq A$

   Let's assume the other way, if $x \in A$, then $x \in B$. With the two cases now, which are either $x \in C\ or\ x \notin C$.

   <u>Case 1</u>

   If $x \in$ C, we assume that $x \in A$, $x \in A \cap C$. Since A ∩ C = B ∩ C we can conclude that $x \in B \cap C$. Hence $x \in B\ and\ x \in C$. Therefore, $x \in B$.

   <u>Case 2</u>

If $x \notin C$, we assume that $x \in A, x \in A - C$. Since A − C = B − C we can conclude that $x \in B - C$. Hence $x \in B \ and \ x \notin C$. Therefore, $x \in B$.

Both of these cases show that x ∈ A → x ∈ B. Hence, $A \subseteq B$

We have shown that $A = B$, by the definition of equivalency which is If x ∈ B → x ∈ C. (This proves B ⊆ A). If x ∈ A → x ∈ B . (This proves A ⊆ B).

6. $\overline{A \cup \mathrm{B}} = \bar{A} \cap \bar{B}$

$A \cup (\mathrm{B} \cap C) = (A \cup \mathrm{B}) \cap (A \cup \mathrm{C})$

$Assume\ \exists x(\mathrm{B} \cap \mathrm{C})$

$= x \in A\ V\ (x \in B\ \Lambda\ x \in C)$

$= (x \in A\ V\ x \in B)\ \Lambda(x \in A\ V\ x \in C)$

$=\ x \in (A\ V\ B)\ \Lambda\ x \in (A\ V\ C)$

$= x \in (A\ \cup\ B)\ \cap\ x \in (A\ \cup\ C)$

$=\ x \in\ A\ \cup (\mathrm{B}\ \cap\ C)$

$= A \cup (B \cap C) \subseteq (\mathrm{A} \cup \mathrm{B}) \cap (\mathrm{A} \cup \mathrm{C})$ (1)

$Assume\ \exists x\big((\mathrm{A} \cup \mathrm{B}) \cap (\mathrm{A} \cup \mathrm{C})\big)$

$= x \in (\mathrm{A} \cup \mathrm{B}) \cap (\mathrm{A} \cup \mathrm{C})$

$= x \in (\mathrm{A}\ V\ \mathrm{B})\ \Lambda\ x \in (\mathrm{A}\ V\ \mathrm{C})$

$= \big((x \in A)\ V\ (x \in B)\big)\Lambda\ \big((x \in A)V\ (x \in C)\big)$

$= (\ x \in A)\ V\ (x \in B\ \Lambda\ C)$

$= (\ x \in A)\ \cup\ (x \in B\ \cap\ C)$

$= x \in \big(A\ \cup (B\ \cap\ C)\big)$

$= x \in (A \cup B) \cap (A \cup C)$

$= (A \cup B) \cap (A \cup C)\ \subseteq\ \big(A\ \cup (B\ \cap\ C)\big)$ (2)

$=\ A \cup (B \cap C) \subseteq (\mathrm{A} \cup \mathrm{B}) \cap (\mathrm{A} \cup \mathrm{C})$ (1) (2)

$\overline{A \cup \mathrm{B}} = \bar{A} \cap \bar{B}$

Q.E.D

7. f: $Z^+ \rightarrow Z^+$

f(n) = 5n + 12

f(y) = 5y + 12

f(n) = f(y)
5n + 12 = 5y + 12
5n = 5y
n = y
Therefore, it is one-to-one

f(n) = 5n + 12
Let 5n + 12 = 0
5n = -12
n = $-\frac{12}{5}$
Therefore, f is not onto because domain and co-domain supposed to be positive integers and n is not positive nor integer in this case.

8. f(m, n) = 3mn for f: R x R → R
$a = 3mn$
$m = \frac{a}{3n}$
$a = 3\left(\frac{a}{3n}\right)n$
$a = a$
Therefore, it is onto because inverse of it gives 1 value only

f(m, n) = 3mn
f(-2, -2) = 3(-2)(-2) = 12
f(2, 2) = 3(2)(2) = 12

f(-2, -2) and f(2, 2) both have different values of m and n but both the equations have the same result. There, it is not one-to-one.

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Spring Semester, 2019

ASSIGNMENT #5

Due Date: Wednesday, March 6

Suggested Reading: Rosen Sections 2.3, 9.1 and 9.5; Lehman et al. Chapter 4.3, 4.4, 10.6 and 10.10

These are the problems that you need to hand in for grading. Always explain your answers and show your reasoning.

1. [**10 Pts**] Let g be a total function from A to B and f be a total function from B to C.
   (a) If $f \circ g$ is one-to-one, then is f one-to-one? Prove or give a counter-example.
   (b) If $f \circ g$ is one-to-one, then is g one-to-one? Prove or give a counter-example.

2. [**10 Pts**] For each of these relations decide whether it is reflexive, anti-reflexive, symmetric, anti-symmetric and transitive. Justify your answers. R_1 and R_2 are over the set of real numbers.
   (a) $(x, y) \in R_1$ if and only if $xy \geq 0$
   (b) $(x, y) \in R_2$ if and only if $x = 2y$

3. [**8 Pts**] Let R_3 be the relation on $\mathcal{Z}^+ \times \mathcal{Z}^+$ where $((a, b), (c, d)) \in R_3$ if and only if $a + d = b + c$.
   (a) Prove that R_3 is an equivalence relation.
   (b) Define a function f such that $f(a, b) = f(c, d)$ if and only if $((a, b), (c, d)) \in R_3$.
   (c) Define the equivalence class containing $(1, 1)$.
   (d) Describe the equivalence classes. How many classes are there and how many elements in each class?

4. [**10 Pts**] Prove that these relations on the set of all ISU students are equivalence relations. Describe the equivalence classes. Now, describe new equivalence relations which are refinements of the relations given.
   (a) $(a, b) \in R_4$ if and only if a and b live in the same building
   (b) $(a, b) \in R_5$ if and only if a and b graduated from the same high school

5. [**12 Pts**] Consider the following relations on the set of positive real numbers. One is an equivalence relation and the other is a partial order. Which is which? For the equivalence relation, describe the equivalence classes. What is the equivalence class of 2? of π? Justify your answers.
   (a) $(x, y) \in R_6$ if and only if $x/y \in \mathcal{Z}$
   (b) $(x, y) \in R_7$ if and only if $x - y \in \mathcal{Z}$

Wei Xian Lim 901579699

1. (a) Proof by contrapositive: $F \circ G$ and G one-to-one don't ensure F is.
As a counter-example, let $A = \{1\}$, $B = \{1, 2\}$, $C = \{1\}$, and $g : A \rightarrow B$ where $g(1) = 1$ and $f : B \rightarrow C$ where $f(1) = f(2) = 1$. Then $F \circ G : A \rightarrow C$ is defined by $(F \circ G)(1) = 1$. This map is a bijection from A = f1g to C = f1g, so is one-to-one. However, F is not one-to-one, since $F(1) = F(2) = 1$.

   (b) Suppose $x, y \in A$ and $g(x) = g(y)$. Therefore $F \circ G (x) = F \circ G (y) \rightarrow x = y$ because $F \circ G$ is one-to-one

2. (a) it is anti-reflexive. We cannot have (1, -1) because $(1, -1) \ngeq 0$
it is symmetric. $xy = yx \in R_1$
it is transitive. We have $(x, y) \in R_1$ and $(y, z) \in R_1$
if $xy \geq 0$, both x and $y \geq 0$. So all x, y, z are ≥ 0.
Therefore, it follows that $(x, z) \in R_1$

   (b) It is anti-reflexive. We cannot have (2, 2) because it doesn't satisfy the statement $x = 2y$.
It is anti-symmetric. we can have (4, 2) but we cannot have (2, 4)
It is non-transitive. If we have $(x, y) = (4, 2)$ and $(y, z) = (2, 1)$, $(x, z) = (4, 1)$ but it doesn't satisfy the statement $x = 2y$ as $(4) \neq 2(1)$

3. (a) It is reflexive. Because $0 = 0 \rightarrow a - a = b - b \rightarrow ((a, b),(a, b)) \in R$
It is symmetric. $((a, b),(c, d)) \rightarrow a - c = b - d \rightarrow c - a = d - b \rightarrow ((c, d),(a, b))$
It is transitive. Have $((a, b),(c, d)) \rightarrow a - c = b - d$, $((c, d),(e, f)) \rightarrow c - e = d - f$.
Then, $(a - c) + (c - e) = (b - d)+(d - f) \rightarrow a - e = b - f \rightarrow ((a, b),(e, f))$

   (b) $f(x, y) = x - y$. Then we have: $f(a, b) = f(c, d) \rightarrow a - b = c - d$
$\rightarrow a - c = b - d$
$\rightarrow ((a, b),(c, d))$

   (c) $((a, b), (1, 1)) \rightarrow a - 1 = b - 1 \rightarrow a = b \rightarrow$ The class is $\{(a, a)\}$
$\rightarrow$ 2 elements: (2, 2) and (3, 3)

   (d) There are infinite classes with infinite number of elements. Each class is the set of tuples $\{(a, b)\}$ where the difference between a and b is constant.

4. (a) It is reflexive because $(a, a) \in R_4$.
   It is symmetric because $(a, b) \in R_4 = (b, a) \in R_4$
   It is transitive because if a and b is in the same building, b and c is in the same building, then a and c is in the same building.
   Therefore, it is an equivalent relation.

   $R_4 = \{(a, b) \mid \text{lives in the same building}\}$

   If R is the refinement of S. Subset or refinement of same building or same unit or same floor.

   (b) It is reflexive because $(a, a) \in R_5$.
   It is symmetric because $(a, b) \in R_5 = (b, a) \in R_5$
   It is transitive because $(a = b) = (b = c)$. a and b graduated from the same high school, b and c graduated from the same high school, so a and c graduated from the same high school.
   Therefore, it is an equivalent relation.

   $R_5 = \{(a, b) \mid \text{graduated from the same high school}\}$

   If R is the refinement of S. Subset or refinement of same high school, same college, or same year.

5. (a) it is reflexive → $x/x = 1 \in Z$
   It is anti-symmetric → x/y does not always mean y/x unless $x = y$
   It is transitive → if we have $(x, y) = (1, 2)$ and $(y, z) = (2, 3)$ then we have $(x, z) = (1, 3)$. $x/z = 1/3$ which is $\in Z$.

   $(x, y) \in R_6$ if and only if $x/z \in Z$ is partial ordered set.

   (b)It is reflexive → $x - x = 0 \in Z$
   It is symmetric → $x - y \in Z$ → $-(x-y) \in Z$ → $y - x \in Z$
   It is transitive → have $x - y \in Z$ & $y - z \in Z$, $x - z$

   $$= x - y + y - z$$
   $$=(x - y) + (y - z) \in Z$$

   $(x, y) \in R_7$ if and only if $x - y \in Z$ is equivalent relation
   $[2]_{R7} = Z^+$ and $[\pi]_{R7} = \{ \pi - 3 + n \mid n \in N\}$

CS 230 : Discrete Computational Structures
Spring Semester, 2019
ASSIGNMENT #6
Due Date: Wednesday, March 13

Suggested Reading: Rosen Section 5.1 - 5.2; Lehman et al. Chapter 5.1 - 5.3

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

For Problems 1-5, prove the statements by mathematical induction. Clearly state your basis step and prove it. What is your inductive hypothesis? Prove the inductive step and show clearly where you used the inductive hypothesis.

1. **[6 Pts]** $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2))^2$, for all $n \in \mathcal{Z}^+$.
2. **[6 Pts]** $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3$, for all $n \in \mathcal{Z}^+$.
3. **[6 Pts]** $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1})/4$, for all $n \in \mathcal{N}$.
4. **[6 Pts]** $1/2n \leq [1 \cdot 3 \cdot 5 \cdots (2n-1)]/[2 \cdot 4 \cdot 6 \cdots 2n]$, for all $n \in \mathcal{Z}^+$.
5. **[6 Pts]** 6 divides $n^3 - n$, for all $n \in \mathcal{N}$.
6. **[12 Pts]** Let $P(n)$ be the statement that n-cent postage can be formed using just 4-cent and 7-cent stamps. Prove that $P(n)$ is true for all $n \geq 18$, using the steps below.
   (a) First, prove $P(n)$ by regular induction. State your basis step and inductive step clearly and prove them.
   (b) Now, prove $P(n)$ by strong induction. Again, state and prove your basis step and inductive step. Your basis step should have multiple cases.
7. **[8 Pts]** Use strong induction to prove that every positive integer n can be expressed as the sum of distinct powers of 2. *Hint: Separately consider the cases when k is odd or even.*

For more practice, you are encouraged to work on other problems, like the ones below.

1. Rosen Section 5.1 Problem 4
2. Rosen Section 5.1 Problem 12
3. Rosen Section 5.1 Problem 31
4. Rosen Section 5.2 Problem 26

1. Base test: $P(1) = 1^3 = \left(\frac{1(1+1)}{2}\right)^2 = 1$
   So, P(1) is true.
   Induction test: $P(k) = 1^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2 = \left(\frac{(k^2+k)}{2}\right)^2$

   $$P(k+1) = 1^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2 = \left(\frac{k^2+3k+2}{2}\right)^2$$

   So, by inductive hypothesis,
   $P(k+1) = P(k) + (k+1)^3$
   $= \left(\frac{(k^2+k)}{2}\right)^2 + (k+1)^3$
   $= \frac{1}{4}(k^2+k)^2 + (k+1)^3$
   $= \frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2 + k^3 + 3k^2 + 3k + 1$
   $= \frac{1}{4}(k^4 + 6k^3 + 13k^2 + 12k + 4)$
   $= \frac{1}{4}(k^2 + 3k + 2)^2$
   $= \left(\frac{k^2+3k+2}{2}\right)^2 = P(k+1)$

2. Base test: $P(1) = 2 = \frac{1(1+1)(1+2)}{3} = 2$
   So, P(1) is true.
   Induction Test: $P(k) = 1 \cdot 2 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$

   $$P(k+1) 1 \cdot 2 + \cdots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

   So, by inductive hypothesis,
   $P(k+1) = P(k) + (k+1)(k+2)$
   $= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$
   $= \frac{1}{3}(k^3 + 3k^2 + 2k) + k^2 + 3k + 2$
   $= \frac{1}{3}(k^3 + 6k^2 + 11k + 6)$
   $= \frac{1}{3}\big((k+1)(k+2)(k+3)\big) = P(k+1)$

3. Base test: $P(0) = 2$

$$= \frac{(1-(-7)^{0+1})}{4} = 2$$

Therefore, P(0) is true.

Inductive Test: $P(k) = 2 - 2 \cdot 7 + \cdots + 2(-7)^k = \frac{(1-(-7)^{k+1})}{4}$

$$P(k+1) = 2 - 2 \cdot 7 + \cdots + 2(-7)^k + 2(-7)^{k+1} = \frac{(1-(-7)^{k+2})}{4}$$

So, by inductive hypothesis,

$$P(k) + 2(-7)^{k+1} = \frac{(1-(-7)^{k+1})}{4} + 2(-7)^{k+1}$$

$$= \frac{1}{4}((1-(-7)^{k+1}) + 8(-7)^{k+1})$$

$$= \frac{1}{4}(1 + 7(-7)^k - 56(-7)^k)$$

$$= \frac{1}{4}(1 - 49(-7)^k)$$

$$= \frac{(1-(-7)^{k+2})}{4} = P(k+1)$$

4. Base Test: $P(1)$: $1/2\mathrm{n} \leq [1 \cdot 3 \cdot 5 \cdots (2\mathrm{n}-1)]/ [2 \cdot 4 \cdot 6 \cdots 2\mathrm{n}]$,

$$\frac{1}{2} \leq \frac{1}{2}$$

Therefore P(1) is true.

Inductive test: Assume $P(k) = \frac{1}{2k} \leq \frac{2k-1}{2\mathrm{k}}$

$$P(k+1) = \frac{1}{2(k+1)} \leq \frac{2k}{2\mathrm{k}+1}$$

So, by assumption,

$$\frac{1}{2k} \leq \frac{2k-1}{2\mathrm{k}}$$

$$1 \leq 2k - 1$$

$$\frac{1}{2(k+1)} \leq \frac{2k-1}{2(k+1)}$$

$$\frac{1}{2(k+1)} \leq \frac{2k}{2k+2} - \frac{1}{2(k+1)}$$

Therefore, if $\frac{1}{2(k+1)} \leq \frac{2k}{2k+2} - \frac{1}{2(k+1)}$. Then $P(k+1) = \frac{1}{2(k+1)} \leq \frac{2k}{2\mathrm{k}+1}$ will always be true also.

5. Base Test: $P(0) = \frac{6}{0} = 0$ which is true as $0 \in N$
   Induction Test: $P(k) = k^3 - k$
   $$P(k+1) = (k+1)^3 - (k+1)$$
   $= k^3 + 3k^2 + 3k + 1 - k - 1$
   $= k^3 + 3k^2 + 2k$
   $= (k^3 - k) + (3k^2 + 3k)$
   $= P(k) + 3k(k+1)$
   Therefore, if P(k) is divisible by 6, P(k+1) is also divisible by 6 because $3k(k+1)$ is also divisible by 6.

6.
   a. Regular Induction
      Base test: $p(18) = 7cents + 7cents + 4cents$
      Induction test: Assume we have a postage for k cents
      Come up with a postage for k+1 cents.

      Case 1: k contains a 7-cent stamp
      We add two 4-cent stamps and remove the 7-cent stamp.
      $$k + 4 + 4 - 7 = k + 1$$

      Case 2: k contains no 7-cent stamps (only 4-cent stamps)
      If k has no 7-cent stamp, then k is multiple of 4. Since $k \geq 18$, then k must be at least 20 because lowest multiple of 4 is 20 in this case. Therefore, at least five 4-cent stamps are used. If we add another 4-cent and remove three 7-cent stamps, we will get k

   b. Strong Induction
      Basis steps:
      P(18): 18 cents can be made of two 7-cent stamps and one 4-cent stamp as 2(7)+4 $= 18$.
      P(19): 19 cents can be made from three 4-cent stamps and one 7-cent stamp as $3(4) + 7 = 19$.
      P(20): 20 cents can be made from five 4-cent stamps as $5(4) = 20$.

      We assume $p(20) \wedge p(21) \wedge \cdots \wedge p(k)$ for $k \geq 20$

      Prove $p(k+1)$
      $$20, 21, \cdots, k-3, k-2, k-1, k, k+1$$
      If $p(k-3)$= true, then $p(k-3+4)$is also true. Therefore, $p(k+1)$ is true.

7. Base Test: $P(1) = 1 = 2^0$
   Therefore, P(1) is true.
   Inductive Test: Assume $P(1) \Lambda\ P(2)\ \Lambda \cdots\ \Lambda\ P(k)$ is true. We want to prove that $P(k+1)$ is true.
   Case 1: $k+1$ is even.
   So, by inductive hypothesis, k+1 is divisible by 2
   $\frac{k+1}{2} = 2^{x1} + 2^{x2} + \cdots + 2^{xz}$ where x1, x2, …, xz are distinct.
   $k+1 = 2(2^{x1} + 2^{x2} + \cdots + 2^{xz})$
   $k+1 = 2^{x1+1} + 2^{x2+1} + \cdots + 2^{xz+1}$
   If x1, x2, …, xz are distinct power of 2, then x1 + 1, x2 + 1, … , xz + 1 are also distinct power of 2

   Case 2: $k+1$ is odd
   So, by inductive hypothesis,
   $k = 2^{x1} + 2^{x2} + \cdots + 2^{xz}$ where x1, x2, …, xz are distinct.
   If we add both side by 1, which 1 is also equals to 2^0
   $k+1 = 2^0 + 2^{x1} + 2^{x2} + \cdots + 2^{xz}$
   0, x1, x2, …, xz are still distinct power of 2.

CS 230 : Discrete Computational Structures
Spring Semester, 2019
ASSIGNMENT #7
Due Date: Thursday, March 28

Suggested Reading: Rosen Sections 5.2 - 5.3; Lehman et al. Chapters 5, 6.1 - 6.3

These are the problems that you need to turn in. For more practice, you are encouraged to work on other problems. **Always explain your answers and show your reasoning.**

1. [**10 Pts**] Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, for all positive integers n, where f_i are the Fibonacci numbers.

2. [**12 Pts**] Consider the following state machine with five states, labeled 0, 1, 2, 3, 4 and 5. The start state is 0. The transitions are $0 \to 1$, $0 \to 2$, $1 \to 3$, $2 \to 3$, $3 \to 4$, $3 \to 5$, $4 \to 0$ and $5 \to 0$.

   Prove that if we take n steps in the state machine we will end up in state 0 if and only if n is divisible by 3. Argue that to prove the statement above by induction, we first have to *strengthen the induction hypothesis*. State the strengthened hypothesis and prove it.

3. [**8 Pts**] Suppose $P(1)$ and $P(2)$ are true. Determine for what values of n, $P(n)$ is true if

   (a) for every positive integer k, if $P(k)$ is true then $P(k+3)$ is true.

   (b) for every positive integer k, if $P(k)$ is true then $P(k+2)$ is true.

4. [**12 Pts**] A robot wanders around a 2-dimensional grid. He starts out at (0,0) and can take the following steps: (-1,+3), (+2,-2) and (+4,0). Define a state machine for this problem. Then, define a Preserved Invariant and prove that the robot will never get to (2,0).

5. [**8 Pts**] Suppose $P(n)$ is true for every positive integer n that is a power of 2. Also, suppose that $P(k+1) \to P(k)$ for all positive integers k. Now, prove that $P(n)$ is true for all positive integers.

For more practice, you are encouraged to work on other problems, like the ones in the textbook.

1. Rosen, Section 5.2: Exercise 16

2. Rosen, Section 5.3: Exercise 16

3. LLM Problem 6.3

4. LLM Problem 6.4

1. Fibonacci numbers = 0, 1, 1, 2, 3, …
   Base: $F_1^2 = 0$
   Inductive step:
   Assume $f_1^2+f_2^2+\ldots+f_n^2 = f_nf_{n+1}$
   Prove $f_1^2+f_2^2+\ldots+f_n^2 + f_{n+1}^2 = f_{n+1}f_{n+2}$
   LHS = $f_1^2+f_2^2+\ldots+f_n^2 + f_{n+1}^2$ By IH
   $= f_nf_{n+1} + f_{n+1}^2$
   $= f_{n+1}(f_n + f_{n+1})$
   $= f_{n+1}f_{n+2}$ by Fibonacci definition
   = RHS

2. Base n = 0
   State Machine is in state 0 after 0 steps.

   Induction step:
   Assume that k is divisible by 4 if and only if state machine is in state 0 after k steps.
   Prove that (k+1) is divisible by 4 if and only if state machine is in state 0 k steps.

   K mod 4 ≡ 0
   State machine is in state 0 after k steps, by IH. So, it will be in state 1 or 2 after k+1 step. Since k is divisible by 4, k+1 would have a remainder when divided by 4, so (k+1) assertion holds.

3. P(1) and P(2) are true
   a. $p(k) \to p(k+3)$
      $p(1+3) \to p(4)$
      $p(2+3) \to p(5)$
      $p(4+3) \to p(7)$
      $p(5+3) \to p(8)$

      Therefore, p(k) is true when k is 1, 2, 4, 5, 7 and any other number that is not multiple of 3.

   b. $p(k) \to p(k+2)$
      $p(1+2) \to p(3)$
      $p(2+2) \to p(4)$
      $p(3+2) \to p(5)$
      $p(4+2) \to p(6)$
      $p(5+2) \to p(7)$
      $p(7+2) \to p(9)$

      Therefore, p(k) is true when k is 1, 2, 3, 4, and any other positive integers.

4. Start (0,0)
   Steps (-1,+3), (+2,-2) and (+4,0)
   So, robot can only be at position (x, y) which $|x| + |y|$ will always divisible by 4

   Base case:
   $\mathrm{P}(0) = 0$ which is divisible by 4

   Induction step:
   Case 1: (-1, +3)
   $|-1| + |3| = 4$ and it is divisible by 4
   Case 2: (+2, +2)
   $|2| + |2| = 4$ and it is divisible by 4
   Case 3: (+4, 0)
   $|4| + |0| = 4$ and it is divisible by 4

   (2,0)
   $|2| + |0| = 2$ and it is not divisible by 4

   Therefore, robot will never get to (2, 0) position.

5. Assume $S = \{1,2,3,4, \dots, k, k+1\}$
   Let k be an element of S.
   P(k) is true if k is a power of 2 and false otherwise.
   Let $\mathrm{k} + 1$ be an element of S too.
   We know that k+1 will always be greater than k and p(k+1) is true because all integers of S are true.

CS 230 : Discrete Computational Structures
Spring Semester, 2019
ASSIGNMENT #8
Due Date: Wednesday, April 3

Suggested Reading: Rosen Sections 5.3; Lehman et al. Chapters 5, 6.1 - 6.2, 7

These are the problems that you need to hand in for grading. Always explain your answers and show your reasoning.

1. **[12 Pts]** Let $S \subseteq \mathcal{Z}^+$ be defined recursively by (1) $4 \in S$ and (2) if $x \in S$ and $y \in S$, then $x \cdot y \in S$. Let $A = \{4^i \mid i \in \mathcal{Z}^+\}$. Prove that
   (a) **[6 Pts]** $A \subseteq S$ by mathematical induction.
   (b) **[6 Pts]** $S \subseteq A$ by structural induction.

2. **[8 Pts]** Let BP, the set of balanced parentheses, be defined recursively by (B) $\epsilon \in BP$, (I1) if $x \in BP$ then $\{x\} \in BP$, and (I2) if $x \in BP$ and $y \in BP$ then $xy \in BP$. For example, $\{\{\}\}\{\} \in BP$. Prove that if $x \in BP$ then x has the same number of $\{$ and $\}$.

3. **[10 Pts]** Given the inductive definition of full binary trees (FBTs), define $I(T)$, the set of internal vertices in tree T, and $L(T)$, the set of leaves in tree T, inductively. Then, use structural induction to prove that for all FBTs T, $|L(T)| = |I(T)| + 1$.

4. **[20 Pts]**
   (a) **[8 Pts]** Let $L = \{(a, b) \mid a, b \in \mathcal{Z}, (a-b) \bmod 4 = 0\}$. Give an inductive definition of L. Let L' be the set obtained by your inductive definition.
   (b) **[6 Pts]** Prove that $L' \subseteq L$.
   (c) **[6 Pts]** Prove that $L \subseteq L'$.
   (d) **Extra Credit [5 Pts]** Is your inductive definition unambiguous? In other words, for any $x \in L$, is there a unique way you can produce x using your rules?

For more practice, you are encouraged to work on other problems, like the ones below.

1. Rosen, Section 5.3: Exercise 8

2. Rosen, Section 5.3: Exercise 11-15

3. Let S be defined by (1) $(0, 0) \in S$, and (2) if $(a, b) \in S$, then $(a, b+5) \in S$, $(a+1, b+4) \in S$ and $(a + 2, b + 3) \in S$.
   (a) Use structural induction to prove that if $(a, b) \in S$ then 5 divides $a + b$.
   (b) Disprove the converse of the statement above, *i.e.*, show that if $a, b \in \mathcal{N}$, and $a + b$ is divisible by 5, it does not follow that $(a, b) \in S$. Modify the recursive definition of S to make the converse true.

4. Rosen, Section 5.3: Exercise 27-42

1.

a. Base case: $i = 1$
   $4^i = 4$ and $4 \in S$
   Inductive step:
   Assume $4^k \in S\ for\ k \in Z^+$
   Prove $4^{k+1} \in S$
   $$4^{k+1} = 4\ x\ 4^k$$
   $4\ and\ 4^k$ both are $\in S$, therefore, $4^{k+1} \in S$ and $A \subseteq S$

b. Base case: $4 \in S$ is proven by inductive definition(1).
   $4 = 4^1\ and\ 1\ is\ \in Z^+$ Therefore, $4 \in A$ is true too.
   Inductive Step:
   Assume x $\in S$ and y $\in S$, and x and y are $\in A$
   Prove xy $\in A$
   $$x = 4^a, y = 4^b$$
   $$xy = (4^a)(4^b)$$
   $$= 4^{a+b}$$
   So xy $\in A$. Therefore $S \subseteq A$

2. (I1) if x ∈ BP then {x} ∈ BP , and (I2) if x ∈ BP and y ∈ BP then xy ∈ BP .
   X = {x}
   If x ∈ BP, which means {x} ∈ BP, By (I1).
   {x} has same amount of { and } which is one for each.
   Therefore, we can say that if x ∈ BP, then x has the same number of { and }.

3. $|l(T)| = |i(T)| + 1$
   Base case: T has only a single vertex. So, there's only 1 root and no vertex.
   Therefore, it satisfies $|l(T)| = |i(T)| + 1$
   Recursive step:
   Let T_1 and T_2 be left root and right root.
   So $|l(T_1)| = |i(T_1)| + 1$ and $|l(T_2)| = |i(T_2)| + 1$
   Vertex $|i(T)| = |i(T_1)| + |i(T_2)| + 1$
   Leaves $|l(T)| = |l(T_1)| + |l(T_2)|$
   $|l(T)| = |i(T_1)| + 1 + |i(T_2)| + 1$
   $|l(T)| = (|i(T_1)| + |i(T_2)| + 1) + 1$
   $|l(T)| = |i(T)| + 1$

4.

a. Base case: $(0, 0) \in L'$
   Inductive definition: if (a, b) $\in L'$, then $(a + 1, b + 1) \in L'$, $(a - 1, b - 1) \in L'$, $(a, b + 4) \in L'$ and $(a, b - 4) \in L'$

b. Base case: (a, b) = (0, 0)
   $a - b = 0 - 0 = 0$ which $\in L$
   if (a, b) $\in L'$, then $(a + 1, b + 1) \in L'$, $(a - 1, b - 1) \in L'$, $(a, b + 4) \in L'$ and $(a, b - 4) \in L'$
   Since (a, b) $\in L$ and a – b is multiple of 4. So,
   $(a + 1) - (b + 1) = a + 1 - b - 1$
   $= a - b = multiple\ of\ 4$
   $(a - 1) - (b - 1) = a - 1 - b + 1$
   $= a - b = multiple\ of\ 4$
   $(a) - (b + 4) = a - b - 4$
   $= a - b - 4 = multiple\ of\ 4$
   $(a) - (b - 4) = a - b + 4$
   $= a - b + 4 = multiple\ of\ 4$
   All sets we added to L' will always be in L, therefore, $L' \subseteq L$

c. Let (s, t) be an arbitrary element of L, which means (s, t) $\in L$ where x – y is multiple of 4. t can be written as s – 4k for k any positive integer. $(s, s - 4k)$

   Case 1: s and k are positive. If move $(a + 1, b + 1)$ for s times and $(a, b - 4)$ for k times, then we will get $(s, s - 4k)$.
   Case 2: s and k are negative. . If move $(a - 1, b - 1)$ for -s times and $(a, b + 4)$ for -k times, then we will get $(s, s - 4k)$.
   Case 3: s is positive, and k is negative. If move $(a + 1, b + 1)$ for s times and $(a, b + 4)$ for -k times, then we will get $(s, s - 4k)$.
   Case 4: s is negative, and k is positive If move $(a - 1, b - 1)$ for -s times and $(a, b - 4)$ for k times, then we will get $(s, s - 4k)$.

   Therefore, we can say that $L \subseteq L'$

d. No, I think my inductive definition is not unambiguous. There is other unique way to produce x using my rules.

CS 230 : Discrete Computational Structures
Spring Semester, 2019
ASSIGNMENT #9
Due Date: Thursday, April 18

Suggested Reading: Rosen Section 2.5; LLM Chapter 7.1

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

1. **[12 Pts]** Show that the following sets are countably infinite, by defining a bijection between $\mathcal{N}$ (or $\mathcal{Z}^+$) and that set. You do not need to prove that your function is bijective.
   (a) **[6 Pts]** the set of integers divisible by 7
   (b) **[6 Pts]** $A \times \mathcal{Z}^+$ where $A = \{2, 3\}$
2. **[7 Pts]** Argue that if $A \subseteq B$, and A is uncountable then B is uncountable.
3. **[8 Pts]** Prove that the set of functions from $\mathcal{N}$ to $\mathcal{N}$ is uncountable, by using a diagonalization argument.
4. **[14 Pts]** Determine whether the following sets are countable or uncountable. Prove your answer.
   (a) the set of real numbers with decimal representation consisting of all 9's (9.99 and 99.999... are such numbers).
   (b) the set of real numbers with decimal representation consisting of 8's and 9's
5. **[9 Pts]** Give an example of two uncountable sets A and B (along with a justification) such that $A - B$ is
   (a) finite (b) countably infinite (c) uncountably infinite

For more practice, you are encouraged to work on other problems like the ones below. You can find more problems in the textbook.

1. Argue that the set of all finite strings over the alphabet Σ, where $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, /\}$, is countable. Use this to argue that the set of positive rationals is countable.
   Hint: Represent any positive rational as a finite string.
2. Show that a countably infinite union of countably infinite sets is countably infinite.

1.

a. $S = \{ 7k \mid k \in N\}$
So S is subset of N and S is infinite.
Since N is countably infinite and S is infinite subset of N, and we know that infinite subset of a countably infinite set is countably infinite.
So S is also countably infinite.

b. $A = \{ 2, 3\}$
$B = A * Z^+$
$= B_1 \cup B_2$ where $B_1 = \{2\} * Z^+$ and $B_2 = \{3\} * Z^+$
We know that Z^+ is countably infinite and the function $B_1 \rightarrow Z$ is a bijection.
So B_1 is also countably infinite. Same thing goes for B_2.
We know that countable U countable is also countable.
$B_1 \cup B_2$ is countable and Z^+ is countable infinite,
Therefore, B is countable infinite.

2. Assuming that B is not uncountable, then B is finite set. And since A is a subset of B, therefore A is also a finite set. Because element in A must be in B. So, if B is countable, A is also countable. By contrapositive, it's true that if A is uncountable, then B is uncountable.

3. $F_{N\rightarrow N}$ is uncountable.
Suppose for contradiction, $F_{N\rightarrow N}$ is countable.
Where , $F_{N\rightarrow N} = \{ F_1, F_2, F_3, \dots \}$
Let $B = \{ i \mid i \notin F_{N\rightarrow N}\}$
Some $B \in F_{N\rightarrow N}$, so $\exists j$ where $\mathrm{B} = F_{N\rightarrow N}$
There are 2 cases:
Case 1: $j \in B$
$j \in B => j \notin F_{N\rightarrow N}$
$=> j \notin \mathrm{B}$ by definition of B

Case 2: $j \notin \mathrm{B}$
$j \notin \mathrm{B} => j \in F_{N\rightarrow N}$
$=> j \in B$
So $F_{N\rightarrow N}$ is uncountable.

4.

a.

	0	1	2	3	And so on
0		0.9	0.99	0.999	...
1	9	9.9	9.99	9.999	...
2	99	99.9	99.99	99.999	...
3	999	999.9	999.99	999.999	...
And so on	...	...	...	...	...

Therefore, this set is countable as it can be counting off diagonally.

b. Suppose S is uncountable set of real numbers [0, 1] where decimal representation consisting of 8's and 9's
0.888 where there's infinite number of 8 after the decimals, same goes to 0.999
By contradiction, suppose S is countable where S it has element s1, s2, s3 and so on.
//not complete

5.

a. Let A = set of R numbers of (0, 1] and B = set of R numbers of (0, 1)
A – B = {1}
Therefore A-B is a finite set.

b. Let A = set of R number (0, 1) union Z^+ and B = set of R number of (0, 1)
A – B = Z^+
Therefore, A – B is countably infinite.

c. Let A = set of R number (0, 1) and B = set of R numbers (1, 2)
A – B = A as there are no common elements.
Therefore, A – B is uncountably infinite.

CS 330 : Discrete Computational Structures
Spring Semester, 2019
ASSIGNMENT #10
Due Date: Thursday, April 25

Suggested Reading: Rosen Sections 6.1 - 6.3.

These are the problems that you need to turn in. Always explain your answers and show your reasoning. **Spend time giving a complete solution. You will be graded based on how well you explain your answers. Just correct answers will not be enough!**

1. [**4 Pts**] An ISU Computer Science shirt is sold in 9 colors, 5 sizes, striped or solid, and long sleeve or short sleeve. (a) How many different shirts are being sold? (b) What if the red and gold shirts only come in short-sleeve and solid?
2. [**6 Pts**] (a) How many different four-letter codes can there be? (b) What if letters cannot be repeated, and one of the letters is K? (c) What if letters can be repeated and at least one of the initials is K?
3. [**4 Pts**] How many integers between 10000 and 99999, inclusive, are divisible by 5 or 7?
4. [**4 Pts**] How many ways can 10 friends line up if Ann, Beth and Chris have to stand next to each other (a) if Ann is ahead of Beth and Beth is ahead of Chris? (b) if Ann, Beth and Chris can be in any order?
5. [**8 Pts**] Let A and B be sets of 9 elements and 10 elements, respectively. (a) How many different functions possible from A to B? from B to A? (b) How many different relations possible from A to B? (c) How many of the functions from A to B are one-to-one? (d) How many of the functions from B to A are onto?
6. [**4 Pts**] In how many ways can a photographer arrange 7 people in a row from a family of 10 people, if (a) Mom and Dad are in the photo, (b) either Mom or Dad is in the photo, not both.
7. [**6 Pts**] A sack contains 50 movie tickets, 5 for each of 10 different movies. Five friends want to go to a movie. (a) How many tickets would you have to remove from the sack to guarantee that everyone will be able to watch the same movie? (b) How many tickets would you have to remove from the sack if everyone wants to go to 'Avengers'? (c) Which problem required the use of the Pigeonhole Principle?
8. [**3 Pts**] Suppose that all the students in this class of 85 are CS majors, SE majors and Math majors. Show that there are at least 30 CS majors, at least 30 SE majors or at least 27 math majors in the class.
9. [**5 Pts**] How many bit strings of length 7 contain (a) exactly three 1s? (b) at most three 1's? (c) at least three 1's?
10. [**6 Pts**] 12 women and 10 men are on the faculty. How many ways are there to pick a committee of 8 if (a) Claire and Jane will not serve together, (b) at least one woman must be chosen, (c) at least one man and one woman must be chosen. Are there multiple ways to solve these problems? Explain.

1. 9 colors, 5 sizes, striped or solid, long or short sleeve
   (a) $9 * 5 * 2 * 2$
   (b) (Red, gold) + the rest of the color
   $(2 * 5 * 1 * 1) + (7 * 5 * 2 * 2)$

2.
   (a) $26 * 26 * 26 * 26$ because the letter can be reused after each letter and there's 26 different letters.
   (b) $4 * 25 * 24 * 23$, 4 because K can be in any positions, and $25 * 24 * 23$ because the letter used before cannot be used again.
   (c) $4 * 26 * 26 * 26$, 4 because K can be in any position, and $26 * 26 * 26$ because K can be reused.

3. D_5 = divisible by 5 and D_7 = divisible by 7, S = {1, … , 99999}, T = {1, … , 9999}
   T – S = {1000, …, 99999}
   $|D_5 \cup D_7| = |D_5| + |D_7| - |D_5 \cap D_7|$
   $= (19999 - 1999) + (14285 - 1428) - (2857 - 285)$

4.
   (a) 8!
   Because we can consider Ann, Beth, and Chris as one because the order must be Ann-Beth-Chris order. So $(7 + 1)!$
   (b) $8! * 3!$
   3! Because Ann, Beth and Chris can be in any order among 3 of them. And 8! Because we can take Ann, Beth and Chris as a whole because they have to stand next to each other. So $(7 + 1)!$

5.
   (a) 9^{10} for function A to B and 10^9 for function B to A
   (b) 2^{9*10}
   because each element in set A can link to any of the element in set B and theirs is 9 elements in set A and 10 elements in set B.
   (c) $\frac{10!}{(10-9)!}$ functions
   because if all the element from A that is linked must be excluded
   (d) $\frac{C(10,2)*C(9,2)*9!}{2!} + C(10,3) * 9!$ functions
   because onto functions must have all element in B linked to A.

6.
   (a) $C(8,5) * 7!$,
   $C(8,5)$ because only choose 5 out of 8 people as mom and dad is chosen already. And 7! to shuffle all 7 people in all position.
   (b) $(mum\ in) + (dad\ in)$

$C(8,6) * 7! + C(8,6) * 7!$
If mom or dad is in already, then we need to choose 6 out of 8 people because if mom is in, dad is not and vice versa.

7.
(a) K = 10, r = 5
$N \geq (5-1)10 + 1$
At least 41 tickets to get 5 same movie ticket.
(b) 5 to 50 tickets because assuming the you took the first 45 tickets and all of it is not Avengers ticket or you are lucky and first 5 tickets are Avengers ticket.
(c) Problem A requires Pigeonhole Principle

8. F

9.
(a) $C(7,3) = 35$
(b) $C(7,0) + C(7,1) + C(7,2) + C(7,3) = 1 + 7 + 21 + 35 = 64$
(c) $2^7 - C(7,0) - C(7,1) - C(7,2) = 128 - 1 - 7 - 21 = 99$

10.
(a) $2 * C(20,7)$
(b) $Total - only\ man$
$C(22,8) - C(10,8) =$
(c) $Total - only\ man - only\ woman$
$C(22,8) - C(10,8) - C(12,8) =$

CS 230 : Discrete Computational Structures
Spring Semester, 2019
ASSIGNMENT #11 [**Extra Credit**]
Due Date: Friday, May 3

Suggested Reading: Rosen Sections 6.4 - 6.5, CLRS Chapter on Graphs

These are the problems that you need to turn in. Always explain your answers and show your reasoning. **Spend time giving a complete solution. You will be graded based on how well you explain your answers. Just correct answers will not be enough!**

1. [**5 Pts**] Prove, using a combinatorial argument, that $C(m+n,2) = C(m,2) + C(n,2) + mn$, where $m, n \geq 2$. To make your combinatorial argument, describe a problem that both the lhs and rhs expressions count.

2. [**8 Pts**] Prove, (a) using a combinatorial argument, and (b) using an algebraic proof, that $C(n,2)C(n-2,k-2) = C(n,k)C(k,2)$.

3. [**6 Pts**] A cookie shop sells 5 different kinds of cookies. How many different ways are there to choose 16 cookies if (a) you pick at least two of each? (b) you pick at least 4 oatmeal cookies and at most 4 chocolate chip cookies?

4. [**9 Pts**] How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 20$, where x_i is a non-negative integer, for all i, if (a) there are no restrictions? (b) $x_1 > 1$, $x_2 > 2$, $x_3 > 3$, $x_4 > 4$? (c) $x_1 > 4$ and $x_3 < 5$?

5. [**9 Pts**] How many ways are there to split 30 people into three committees of 5 people each and five committees of 3 people each if (a) all eight committees have different tasks? (b) all eight committees have the same task? (c) the three 5-member committees and two of the 3-member committees are all given the same task while the remaining three 3-member committees are not given any task yet?

6. [**6 Pts**] How many ways are there to pack 6 different books into 6 identical boxes with no restrictions placed on how many can go in a box (some boxes can be empty)? What if the books are identical?

7. [**6 Pts**] How many ways can we place 12 books on a bookcase with 5 shelves if the books are (a) indistinguishable copies (b) all distinct? Note that the position of the books on the shelves matter.

8. [**6 Pts**] Consider a graph G that has 7 vertices with degrees of 5, 4, 3, 3, 2, 2, 1. How many edges does G have? Explain.

For more practice, you are encouraged to work on other problems, like the ones below. Remember that there will be graph problems in the exam, so make sure you go over problems on graphs!

1. How many integers between 1000 and 9999 inclusive contain (a) at least one 0 and at least one 1, (b) at least one 0, at least one 1 and at least one 2? Solve part (a) using the Inclusion-Exclusion Principle for two sets, and part (b) using the Inclusion-Exclusion Principle for three sets: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

2. Prove that a graph is a tree if and only if it is acyclic but adding any edge will create a cycle.

3. Prove by induction that a complete binary tree of height h has 2^h leaves. Use the inductive definition of complete binary trees.

4. If G is a simple graph with n vertices and $n - 1$ edges, (a) is G connected? (b) is G acyclic? For each question, if *yes*, give a short justification. If *no*, give a counterexample.

1. F

2.

   a. F
   b. $\mathrm{C(n, 2)C(n-2, k-2) = C(n, k)C(k, 2)}$

      $LHS = \frac{n!}{2!\,(n-2)!} * \frac{(n-2)!}{(k-2)!\,(n-2-k+2)!}$

      $= \frac{n!}{2!\,(k-2)!\,(n-k)!}$

      $= \frac{n!}{k!\,(n-k)!} * \frac{k!}{2!\,(k-2)!}$

      $= \mathrm{C(n, k)C(k, 2)} = \mathrm{RHS}$

3.

   a. Must take at least two of each means, 10 cookies are fixed, because $5 * 2$
      And we left with 6 cookies to choose from.

      $$\binom{6+5-1}{6} = \frac{10!}{4!\,6!}$$

   b. Sum of all cases of chocolate chip cookies.

      (1) 0 chocolate $16 - 4\ oatmeal = 12$

      $\binom{12+4-1}{12} = \frac{15!}{3!\,12!}$

      (2) 1 chocolate $16 - 4\ oatmeal - 1\ chocolate = 11$

      $\binom{11+4-1}{11} = \frac{14!}{3!\,11!}$

      (3) 2 chocolate $16 - 4\ oatmeal - 2\ chocolate = 10$

      $\binom{10+4-1}{10} = \frac{13!}{3!\,10!}$

      (4) 3 chocolate $16 - 4\ oatmeal - 3\ chocolate = 9$

      $\binom{11+4-1}{11} = \frac{12!}{3!\,9!}$

      (5) 4 chocolate $16 - 4\ oatmeal - 4\ chocolate = 8$

      $\binom{11+4-1}{11} = \frac{11!}{3!\,8!}$

      $Ans = \frac{15!}{3!\,12!} + \frac{14!}{3!\,11!} + \frac{13!}{3!\,10!} + \frac{12!}{3!\,9!} + \frac{11!}{3!\,8!}$

4.

a. $\binom{20+4-1}{20} = \frac{23!}{3!20!}$

b. $x1 + x2 + x3 + x4 = 20$

X1 is reduced by 1 because possibility to get 1 is removed.
X2 is reduced by 2 because possibility to get 1 and 2 are removed
X3 is reduced by 3 because possibility to get 1, 2, 3 are removed
X4 is reduced by 4 because possibility to get 1, 2, 3, 4 are removed

$20 - 1 - 2 - 3 - 4 = 10$

$\binom{10 + 4 - 1}{10} = \frac{13!}{3!\,10!}$

c. $20 - 4 = 16$ Because we let x3 > 4. Then we reduce the possibility of x3 > 4 to get x3 < 5

$$\frac{23!}{3!\,20!} - \binom{16 + 4 - 1}{16} = \frac{23!}{3!\,20!} - \frac{19!}{3!\,6!}$$

Then we add it by x1 > 4

$$\frac{23!}{3!\,20!} - \frac{19!}{3!\,6!} + \binom{16 + 4 - 1}{16} = \frac{23!}{3!\,20!}$$

5.

a. $\frac{30!}{5!5!5!3!3!3!3!3!}$ Because all of them are distinct.

b. $\frac{30!}{5!5!5!3!3!3!3!3!*5!*3!}$ It needs to divide by 3! and 5! Because the 3 groups of 5 and 5 groups of 3 is not distinct.

c. $\frac{30!}{5!5!5!3!3!3!3!3!*3!*2!*3!}$ It needs to divide by 3!, 2!, 3! Because first 3! is there are 3 groups of 5 doing the same task, 2! Because there are 2 groups of 3 doing the same task, and last 3! Because the 3 groups of 3 is not doing any task.

6. 6-0-0-0-0-0 – 1 way
   5-1-0-0-0-0 – $\binom{6}{5} = 6$
   4-2-0-0-0-0 – $\binom{6}{4}\binom{4}{2} = 15 * 6$
   4-1-1-0-0-0 – $\binom{6}{4} = 15$
   3-3-0-0-0-0 – $\binom{6}{3} * \frac{1}{2} = \frac{20}{2}$
   3-2-1-0-0-0 – $\binom{6}{3}\binom{3}{2} = 20 * 3$
   3-1-1-1-0-0 – $\binom{6}{3}\binom{3}{1}\binom{2}{1} * \frac{1}{3!} = 20 * 3 * 2 * \frac{1}{3*2}$
   2-2-2-0-0-0 – $\binom{6}{2}\binom{4}{2} * \frac{1}{3!} = 15 * 6 * \frac{1}{3*2}$
   2-2-1-1-0-0 – $\binom{6}{2}\binom{4}{2}\binom{2}{1} * \frac{1}{2} * \frac{1}{2} = 15 * 6 * 2 * \frac{1}{2} * \frac{1}{2}$
   2-1-1-1-1-0 – $\binom{6}{2}\binom{4}{1}\binom{3}{1}\binom{2}{1} * \frac{1}{4!} = 15 * 4 * 3 * 2 * \frac{1}{4!}$
   1-1-1-1-1-1 – 1 way
   If books are different, there's $1 + 6 + 90 + 15 + 10 + 60 + 20 + 15 + 45 + 15 + 1$ ways $= 278$ ways

   If books are identical, then there's only 11 ways to put because there's only 11 combinations

7.
   a. $\binom{12+5-1}{12} = \frac{16!}{4!12!}$
   b. $\binom{12+5-1}{12}12! = \frac{16!}{4!}$ need to times by 12! Because the 12 books can be put in every different position.

8. Adding all degrees $= 5 + 4 + 3 + 3 + 2 + 2 + 1 = 20$
   Edge $= \frac{20}{2} = 10$