1. Base test: $P(1) = 1^3 = \left(\frac{1(1+1)}{2}\right)^2 = 1$

   So, P(1) is true.

   Induction test: $P(k) = 1^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2 = \left(\frac{(k^2+k)}{2}\right)^2$

   $$P(k+1) = 1^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2 = \left(\frac{k^2+3k+2}{2}\right)^2$$

   So, by inductive hypothesis,

   $P(k+1) = P(k) + (k+1)^3$

   $= \left(\dfrac{(k^2+k)}{2}\right)^2 + (k+1)^3$

   $= \dfrac{1}{4}(k^2+k)^2 + (k+1)^3$

   $= \dfrac{1}{4}k^4 + \dfrac{1}{2}k^3 + \dfrac{1}{4}k^2 + k^3 + 3k^2 + 3k + 1$

   $= \dfrac{1}{4}(k^4 + 6k^3 + 13k^2 + 12k + 4)$

   $= \dfrac{1}{4}(k^2 + 3k + 2)^2$

   $= \left(\dfrac{k^2 + 3k + 2}{2}\right)^2 = P(k+1)$

2. Base test: $P(1) = 2 = \frac{1(1+1)(1+2)}{3} = 2$

   So, P(1) is true.

   Induction Test: $P(k) = 1 \cdot 2 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$

   $$P(k+1)1 \cdot 2 + \cdots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

   So, by inductive hypothesis,

   $P(k+1) = P(k) + (k+1)(k+2)$

   $= \dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2)$

   $= \dfrac{1}{3}(k^3 + 3k^2 + 2k) + k^2 + 3k + 2$

   $= \dfrac{1}{3}(k^3 + 6k^2 + 11k + 6)$

   $= \dfrac{1}{3}\big((k+1)(k+2)(k+3)\big) = P(k+1)$

3. Base test: $P(0) = 2$

$$= \frac{(1-(-7)^{0+1})}{4} = 2$$

Therefore, P(0) is true.

Inductive Test: $P(k) = 2 - 2 \cdot 7 + \cdots + 2(-7)^k = \frac{(1-(-7)^{k+1})}{4}$

$$P(k+1) = 2 - 2 \cdot 7 + \cdots + 2(-7)^k + 2(-7)^{k+1} = \frac{(1-(-7)^{k+2})}{4}$$

So, by inductive hypothesis,

$$P(k) + 2(-7)^{k+1} = \frac{(1-(-7)^{k+1})}{4} + 2(-7)^{k+1}$$

$$= \frac{1}{4}((1-(-7)^{k+1}) + 8(-7)^{k+1})$$

$$= \frac{1}{4}(1 + 7(-7)^k - 56(-7)^k)$$

$$= \frac{1}{4}(1 - 49(-7)^k)$$

$$= \frac{(1-(-7)^{k+2})}{4} = P(k+1)$$

4. Base Test: $P(1)$: $1/2\mathrm{n} \leq [1 \cdot 3 \cdot 5 \cdots (2\mathrm{n}-1)] / [2 \cdot 4 \cdot 6 \cdots 2\mathrm{n}]$,

$$\frac{1}{2} \leq \frac{1}{2}$$

Therefore P(1) is true.

Inductive test: Assume $P(k) = \frac{1}{2k} \leq \frac{2k-1}{2\mathrm{k}}$

$$P(k+1) = \frac{1}{2(k+1)} \leq \frac{2k}{2\mathrm{k}+1}$$

So, by assumption,

$$\frac{1}{2k} \leq \frac{2k-1}{2\mathrm{k}}$$

$$1 \leq 2k - 1$$

$$\frac{1}{2(k+1)} \leq \frac{2k-1}{2(k+1)}$$

$$\frac{1}{2(k+1)} \leq \frac{2k}{2k+2} - \frac{1}{2(k+1)}$$

Therefore, if $\frac{1}{2(k+1)} \leq \frac{2k}{2k+2} - \frac{1}{2(k+1)}$. Then $P(k+1) = \frac{1}{2(k+1)} \leq \frac{2k}{2\mathrm{k}+1}$ will always be true also.

5. Base Test: $P(0) = \frac{6}{0} = 0$ which is true as $0 \in N$
   Induction Test: $P(k) = k^3 - k$
   $$P(k+1) = (k+1)^3 - (k+1)$$
   $= k^3 + 3k^2 + 3k + 1 - k - 1$
   $= k^3 + 3k^2 + 2k$
   $= (k^3 - k) + (3k^2 + 3k)$
   $= P(k) + 3k(k+1)$
   Therefore, if P(k) is divisible by 6, P(k+1) is also divisible by 6 because $3k(k+1)$ is also divisible by 6.

6.
   a. Regular Induction
      Base test: $p(18) = 7cents + 7cents + 4cents$
      Induction test: Assume we have a postage for k cents
      Come up with a postage for k+1 cents.

      Case 1: k contains a 7-cent stamp
      We add two 4-cent stamps and remove the 7-cent stamp.
      $$k + 4 + 4 - 7 = k + 1$$

      Case 2: k contains no 7-cent stamps (only 4-cent stamps)
      If k has no 7-cent stamp, then k is multiple of 4. Since $k \geq 18$, then k must be at least 20 because lowest multiple of 4 is 20 in this case. Therefore, at least five 4-cent stamps are used. If we add another 4-cent and remove three 7-cent stamps, we will get k

   b. Strong Induction
      Basis steps:
      P(18): 18 cents can be made of two 7-cent stamps and one 4-cent stamp as 2(7)+4 $= 18$.
      P(19): 19 cents can be made from three 4-cent stamps and one 7-cent stamp as $3(4) + 7 = 19$.
      P(20): 20 cents can be made from five 4-cent stamps as $5(4) = 20$.

      We assume $p(20) \land p(21) \land \cdots \land p(k)$ for $k \geq 20$

      Prove $p(k+1)$
      $$20, 21, \cdots, k-3, k-2, k-1, k, k+1$$
      If $p(k-3)$= true, then $p(k-3+4)$is also true. Therefore, $p(k+1)$ is true.

7. Base Test: $P(1) = 1 = 2^0$
   Therefore, P(1) is true.
   Inductive Test: Assume $P(1) \Lambda\, P(2)\, \Lambda \cdots\, \Lambda\, P(k)$ is true. We want to prove that $P(k+1)$ is true.
   Case 1: $k+1$ is even.
   So, by inductive hypothesis, k+1 is divisible by 2
   $\frac{k+1}{2} = 2^{x1} + 2^{x2} + \cdots + 2^{xz}$ where x1, x2, …, xz are distinct.
   $k+1 = 2(2^{x1} + 2^{x2} + \cdots + 2^{xz})$
   $k+1 = 2^{x1+1} + 2^{x2+1} + \cdots + 2^{xz+1}$
   If x1, x2, …, xz are distinct power of 2, then x1 + 1, x2 + 1, … , xz + 1 are also distinct power of 2

   Case 2: $k+1$ is odd
   So, by inductive hypothesis,
   $k = 2^{x1} + 2^{x2} + \cdots + 2^{xz}$ where x1, x2, …, xz are distinct.
   If we add both side by 1, which 1 is also equals to 2^0
   $k+1 = 2^0 + 2^{x1} + 2^{x2} + \cdots + 2^{xz}$
   0, x1, x2, …, xz are still distinct power of 2.