Model Predictive Control

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November 8th, 2024

1. Linear-quadratic regulator
    - Cost function
    - State-space representation of a system
    - LQR design

2. Model Predictive Control
    - Principles and characteristics
    - Non-linear MPC
    - Adaptive MPC
    - Applications, advantages, and limitations

- set of LDE with a quadratic cost function

Student	**A**	Time	**B**	Money	**C**	Experience (1-5)	**J**
Cooking at home	**1**	60min	**5**	10€	**2**	2	**114**
Ordering a takeout	**1**	30min	**5**	20€	**2**	3	**136**
Going to a restaurant	**1**	45min	**5**	30€	**2**	1	**197**

CEO	**A**	Time	**B**	Money	**C**	Experience (1-5)	**J**
Cooking at home	**1**	60min	**0.1**	10€	**10**	2	**81**
Ordering a takeout	**1**	30min	**0.1**	20€	**10**	3	**62**
Going to a restaurant	**1**	45min	**0.1**	30€	**10**	1	**58**

$$J = \int_0^\infty \left(x^T Q x + u^T R u \right) dt \rightarrow \min \tag{1}$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- $\dot{x}$ - time derivative of the state vector
- x - state vector
- y - output vector
- u - input vector
- A - system matrix
- B - input matrix
- C - output matrix
- D - feedforward matrix

Linear-quadratic regulator

State-space representation of a system

$$\begin{bmatrix} \frac{di_a(t)}{dt} \\ \frac{d\omega(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{c}{L_a} \\ \frac{c}{J} & -\frac{b}{J} \end{bmatrix} \begin{bmatrix} i_a(t) \\ \omega(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \end{bmatrix} u_a(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i_a(t) \\ \omega(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k}{m} & -\frac{b}{m} & 0 \\ 0 & 0 & -\frac{1}{\tau} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ \frac{K}{\tau} \end{bmatrix} \begin{bmatrix} 0 \\ f(t) \\ u(t) \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ q(t) \end{bmatrix}$$

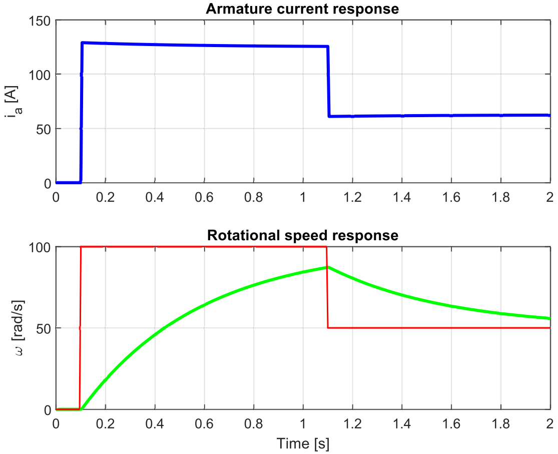


Figure: State variables responses with a low Q_{22}

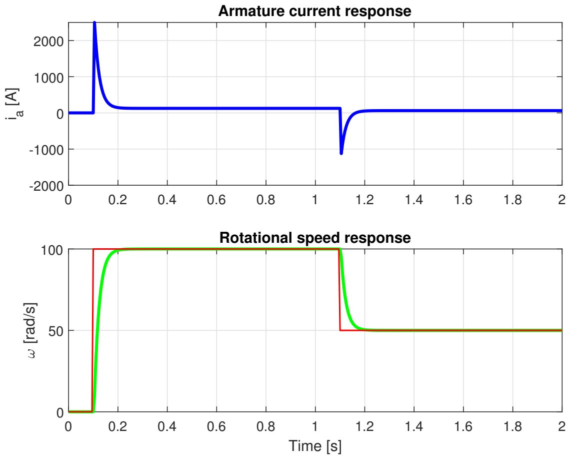


Figure: State variables responses with a high Q_{22}

Model Predictive Control

- MIMO systems
- effectively handling cross-coupled input-output influence
- real-time optimization
- digital signal processors

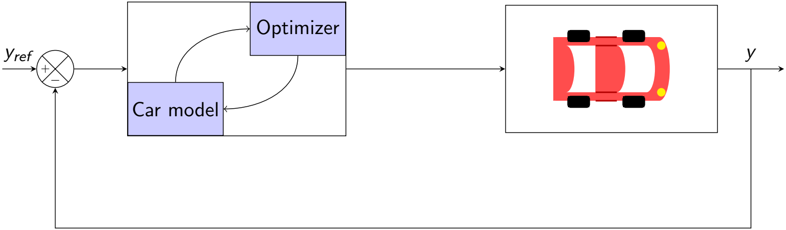


Figure: A principle scheme of a MPC-driven car

- too big $\rightarrow$ system dynamics; aliasing
- too small $\rightarrow$ expensiveness in both computation and MCU price
- Nyquist criterion
- open-loop system response

$$\frac{T_r}{20} \leq T_s \leq \frac{T_r}{10}$$

Model Predictive Control

Prediction horizon

- denoted as p
- a measure of how many time steps MPC is able to predict the output
- iterative optimization in each time step (following figure)
- 'discarding' optimized actions from previous time steps due to possible disturbances
- minimizing the error and ensuring smooth changes in actuation
- too small $p \rightarrow$ unable to stop the car in front of a red traffic light
- too big $p \rightarrow$ excessive wasting of computations

$$\frac{T_{set}}{30} \leq p \leq \frac{T_{set}}{20}$$

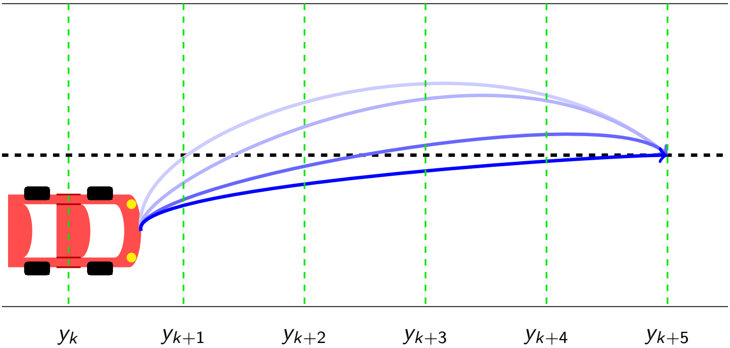


Figure: Illustration of the prediction horizon on a road lane, case $p = 5$

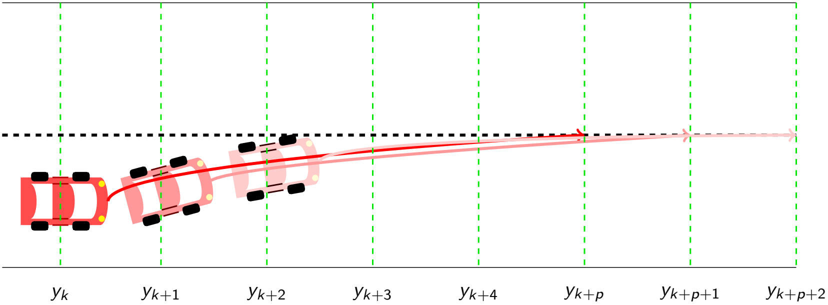


Figure: Illustration of the iterative nature of the prediction horizon on a road lane

- denoted as m
- a measure of how many time steps MPC is able to predict the control actions
- if $m \neq p$, the rest of the control actions are held constant
- a similar trade-off as per prediction horizon
- upon the setpoint change, only a first few steps hold a significant effect on the actuation

$$0.1p \leq m \leq 0.2p, \qquad m \geq 2 \tag{2}$$

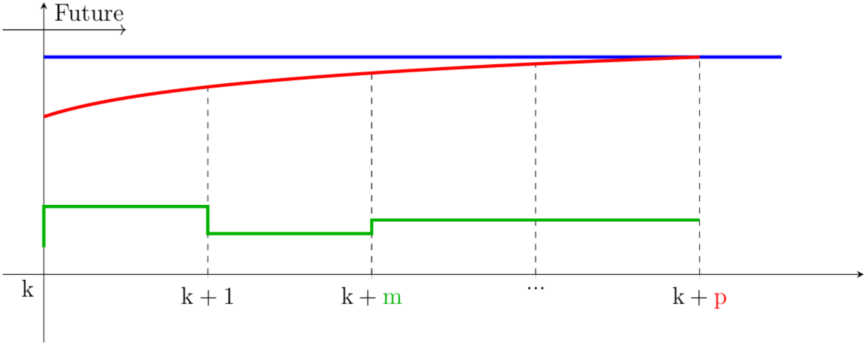
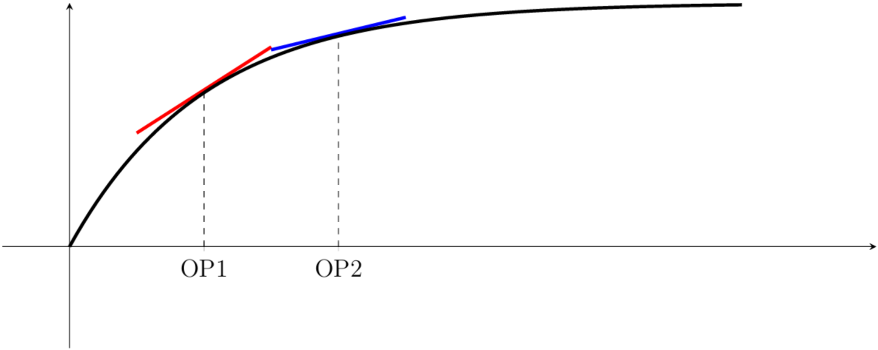


Figure: Illustration of the control horizon compared to the prediction horizon

- soft constraints $\rightarrow$ can be violated briefly
- hard constraints $\rightarrow$ cannot be violated by any means
- applicable to the inputs, the rate of change of inputs, and on the outputs
- conflict example $\rightarrow$ hard constraint on minimal speed limit on a highway and obstacle avoidance; setting the minimal speed limit to soft constraint due to a possible snowstorm

- nonlinear systems
- online linearization
- plant model updating in each time step
- number of states and constraints remains the same $\rightarrow$ the optimization problem is the same for all operating points

Model Predictive Control
Adaptive MPC



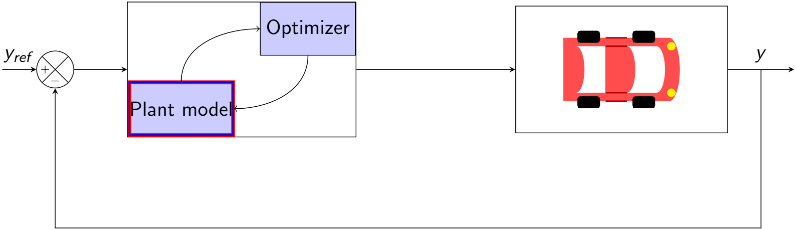


Figure: A principle scheme of adaptive MPC-driven car

- nonlinear systems
- offline linearization
- number of states and constraints possibly change for different operating points
- multiple independent MPC controllers
- switching algorithm

Model Predictive Control

Gain-scheduled MPC

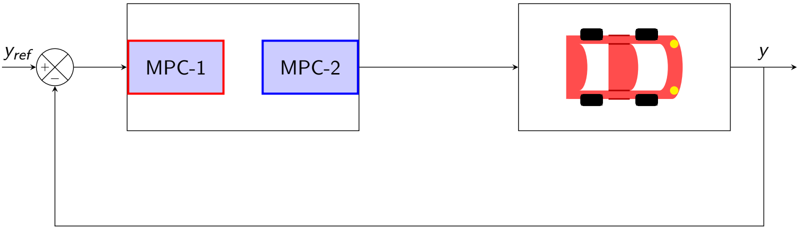


Figure: A principle scheme of gain-scheduled MPC-driven car

- nonlinear system, nonlinear constraints, nonlinear cost function $\rightarrow$ non-convex optimization problem

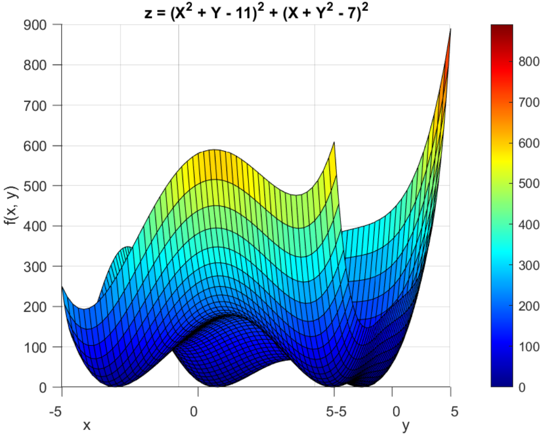


Figure: 3D plot of a function with multiple local optima

Model Predictive Control

Koopman operator

- lifting/embedding the nonlinear dynamics into a higher-dimensional space where its evolution is approximately linear
- linear predictors

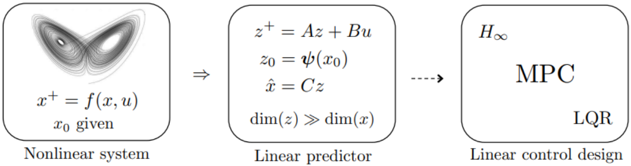


Figure: Linear predictor for a nonlinear controlled dynamical system [7]

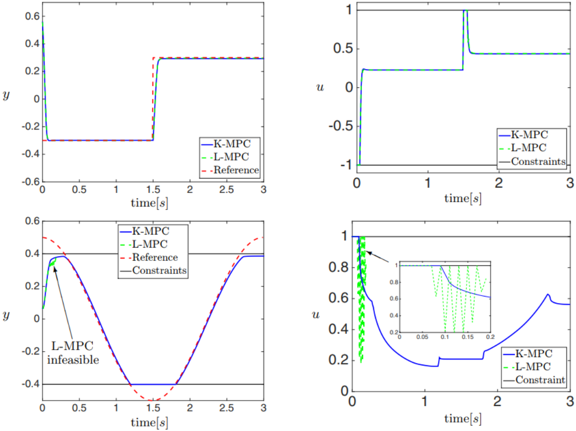


Figure: Feedback control of a bilinear motor [7]

References I

N. S. Nise
Control Systems Engineering
7th edition, Wiley, 2014.

M. Tóthová, J. Pitel, A. Hošovský, J. Sárosi
Numerical Approximation of Static Characteristics of McKibben Pneumatic Artificial Muscle
International Journal of Mathematics and Computers in Simulation, 2015.

João P. Hespanha
Linear Systems Theory
Princeton Univeristy Press, 2018.

William S. Levine (editor)
Control System Advanced Methods - The Control Handbook
CRC Press, Taylor & Francis Group, 2011.

References II

- MATLAB Documentation
  https://www.mathworks.com/help.
- David A. Haggerty, Michael J. Banks, E. Kamenar, et. al.
  Control of soft robots with inertial dynamics
  Science Robotics, 2023.
- M. Korda, I. Mezić
  Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control
  Science Direct, University of California, Santa Barbara, 2018.