Differential Equations Project
Two-Body Problem

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Introduction

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Abstract

The two body problem is a problem that has fascinated mathemeticians, astronomers, and physicists for millenia, from Newton to Fynman. It states that given the initial positions, masses, and velocities of two moving bodies, can you determine the position of the two bodies in the future? For this project we will be looking at a simplified version of the problem, the one body problem, where you assume one body is stationary, and the other is orbition around it. This has basis in the real world; a comet orbiting about the sun, or a satellite orbiting about the earth. We will also only be looking at the objects position on a plane, in our case the cartesian plane. For this problem we make use of the inverse square law to obtain our equation of motion. This equation, depending upon the initial condition describes a set of curves, a circle, an ellipse, a hypoerbola, or parabola, which are known as the conic sections.

Subsection 1.2

History of the Problem

The path that celestial bodies take has fascinated scientists and astronomers for thousands of years. It was the ancient greeks who first hypothesised the geocentric model, which places the earth at the center of the universe, and all other bodies orbiting around it (taking a circular orbit). The heliocentric model which places the sun at the center of the universe was first presented by greek astronomer and mathematician Aristarchus of Samos at around 280BC.Source: Wikipedia Many centuries later in 1609 Johannes Kepler published Astronomia Nova in which he concluded that all planets move in ellipses with the Sun at one focus. This idea came after struggling for years to reconcile deviations in Mars' orbit from a circle. In this publication, Kepler introduced the idea that planetary orbits are the result of physical causes. Source: Wikipedia Some seventy years later, Newton in 1687 published his *Philosophiae Naturalis Principia Mathematica* in which he "demonstrated that an inverse-square law of gravity, together with some basic principles of dynamics, would account for not only elliptical orbits but Kepler's other laws of planetary motion as well, and more besides" (David Goodstein, Feynman's Lost Lecture).

Subsection 1.3

Summary of ODE Techniques

In this paper we do a lot of work to arrive at a second-order, linear, non-homogeneous ODE. This is easily solved, by first solving the homogoneous form of the ODE and then adding the constant. To solve the initial value problem, we let $\theta = 0$ (where θ is the independent variable), and then make use of both $q(0)$ and $q'(0)$ (where $q(\theta)$ is the general solution) to find our specific solution.

The inverse-square law states that the force of gravity between two objects is inversely porportional to the square of the distance between them. That is,

$$F = G\frac{M_1 M_2}{r^2}$$

where F is the force, G is the gravitational constant, M_1 and M_2 are the masses of the two objects and r is the distance between them.

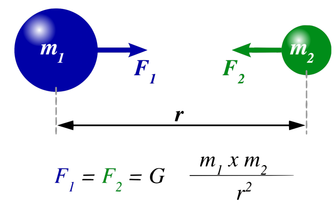

Figure 1. Source: Wikipedia

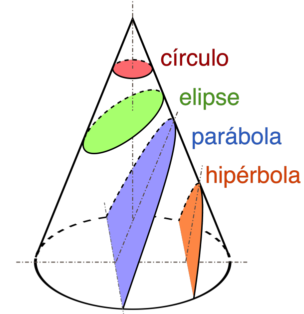

Figure 2. The conic sections, thus named because they are the four curves which can be obtained when you intersect a plane with the surface of a cone. As we will show, these are the paths traced by any body travelling through space under the force of gravity. Source: Wikipedia

Section 2

One Moving Body Problem

How will we be able to derive the equation of motion, given the masses of two objects, the distance between them, and the velocities of the orbiting object? Well, first of all, instead of trying to derive an equation the tells us where our orbiting body is at a given time, we will derive an equation that describes the path our object takes in terms of θ (the angle between the orbiting object and the major axis).

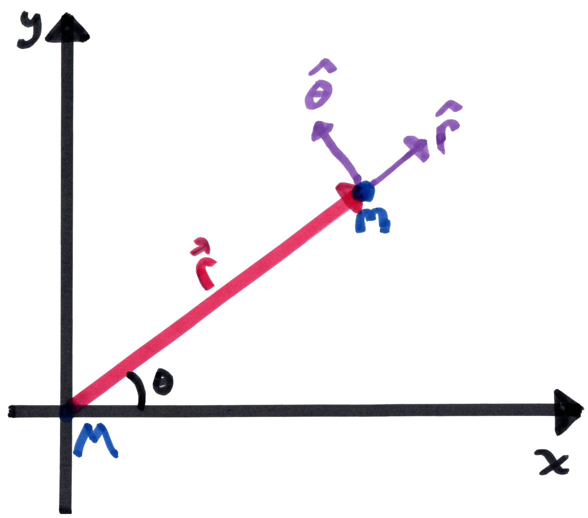

Figure 3. A visual representation of our problem, where M is the large mass, m is the small mass that is orbiting around the larger one, $\vec{r}$ is the position vector of m, θ is the angle of rotation with respect the the major axis, $\hat{\theta}$ is the unit angle vector, and $\hat{r}$ is the unit position vector.

2.0.1 Variables and Important Identities

Although there is only one moving part, there are a lot of things to keep track of. So first defining our variables we have:

$$\begin{aligned}
M &= \text{ Mass of larger object} \\
m &= \text{ Mass of smaller object} \\
r &= \text{ distance between planets} \\
\vec{r} &= \text{ Position vector of } m \ = \langle x_m, y_m \rangle \\
\hat{r} &= \text{ Direction you would go if you increase } r \text{ but hold } \theta \\
\hat{\theta} &= \text{ Direction if you hold } r \text{ but increase theta.} \\
e &= \text{ eccentricity of ellipse}
\end{aligned}$$

We will make extensive use of identities, and so let's keep a list of those used here:

$$\begin{aligned}
\vec{r} &= r\hat{r} \\
\frac{d\hat{r}}{d\theta} &= \hat{\theta} \\
\vec{v} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\
\frac{d\hat{\theta}}{d\theta} &= -\hat{r} \\
\dot{\theta} &= \frac{\vec{L}}{mr^2} \\
\vec{h} &= \frac{\vec{L}}{m} \\
v_0 &= \frac{h}{r_0} \\
q &= \frac{1}{r} \\
K &= \frac{1}{a\left(1-e^2\right)} \\
K &= \frac{GM}{h^2}
\end{aligned}$$

The meaning behind these variables will be explained further on, this part is mostly just a reference to go back to.

2.0.2 Initial Calculations

Because we will be deriving the eqation of motion in terms of θ, let's put the position vector $\vec{r}$ in terms of θ as well:

$$\hat{r} = \hat{x}\cos(\theta) + \hat{y}\sin(\theta).$$

As the planet moves around then what causes $\hat{r}$ to change? Well, $\hat{r}$ changes as time changes, but this is only because as time changes the angle changes. Thus calculating we have

$$\frac{d\hat{r}}{d\theta} = \hat{x}(-\sin\theta) + \hat{y}\cos\theta.$$

Note that his is perpendicular to $\hat{r}$ because the dot product

$$\begin{aligned}
\hat{r}\cdot\frac{d\hat{r}}{d\theta} &= -\sin(\theta)\cos\theta + \cos\theta\sin\theta \\
&= 0.
\end{aligned}$$

This is in fact the unit theta vector (if we hold r constant) and so

$$\hat{\theta} = \hat{x}(-\sin\theta) + \hat{y}\cos\theta.$$

Also note that

$$\frac{d\hat{\theta}}{d\theta} = \hat{x}(-\cos\theta) + \hat{y}(-\sin\theta) = -\hat{r}.$$

2.0.3 Velocity in Polar Coordinates

We want the acceleration vector of our planet, but in polar coordinates. Building up to this, let's first calculate the velocity vector. Velocity is the derivative of position vector with respect to time,

$$\vec{v} = \frac{d\vec{r}}{dt}.$$

But what is the position vector $\vec{r}$ in polar coordinates? It is just $r \cdot \hat{r}$. Thus we have

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(r\hat{r}\right).$$

Note that although the radius r is just a function of time, the radial unit vector $\hat{r}$ only changes when θ changes, but theta only changes when time changes so in addition to the product rule, we need to use the chain rule:

$$\vec{v} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}.$$

A note on notation. For this paper we will be using dot notation (also called Newton's notation) to represent change of the dependent variable with respect to time. For example

$$\dot{\theta} = \frac{d\theta}{dt} = \theta'$$

are all equivalent.

Recall that $\frac{d\hat{r}}{d\theta} = \hat{\theta}$. We will also use dot notation to represent the nth derivative with respect to time, so by substitution and with the use of dot notation we have

$$\begin{aligned}\vec{v} &= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} \\ &= \dot{r}\hat{r} + r\hat{\theta}\dot{\theta}.\end{aligned}$$

2.0.4 Acceleration in Polar Coordinates

We can now calculate the acceleration vector:

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} \\ &= \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) \\ &= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} \\ &= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{d\theta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{d\theta}\dot{\theta}.\end{aligned}$$

Recall though that $\frac{d\hat{\theta}}{d\theta} = -\hat{r}$ and $\frac{d\hat{r}}{d\theta} = \hat{\theta}$ so by substitution we have

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\hat{\theta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\hat{r})\dot{\theta}.$$

Combining the $\hat{r}$ and $\hat{\theta}$ terms we have:

$$\begin{aligned}\vec{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(\dot{r}\dot{\theta} + \dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta} \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta},\end{aligned}$$

where $(\ddot{r} - r\dot{\theta}^2)$ is the acceleration in the radial direction and $2\dot{r}\dot{\theta} + r\ddot{\theta}$ is the acceleration in the tangential direction.

2.0.5 Using Inverse Square Law

We will now make use of the inverse square law to get a differential equation of the radius with respect to time. Note though that gravity only imparts a force in the radial direction, and so we have

$$\begin{aligned}\vec{F} &= m\vec{a} \\ \frac{-GMm}{r^2}\hat{r} &= m(\ddot{r} - r\dot{\theta}^2)\hat{r}.\end{aligned}$$

We can simplify this further with the use of angular momentum. Angular momentum is defined as

$$\vec{L} = I\vec{\omega}$$

where I is moment of inertia and $\vec{\omega}$ is the angular velocity vector. Note that for a point mass m then

$$I = r^2 m$$

where r is the radius of the point mass from the center of rotation. Source: Wikipedia Also recall that the angular velocity vector is just $\dot{\theta}$ and so we have

$$\begin{aligned}\vec{L} &= I\vec{w} \\ &= r^2 m\dot{\theta}.\end{aligned}$$

In terms of $\dot{\theta}$ we have:

$$\dot{\theta} = \frac{\vec{L}}{mr^2},$$

which gives us a way to get rid of our dependence on $\dot{\theta}$. Also note that in a closed system, angular momentum remains constant, and since we have a closed system, $\vec{L}$ is constant. We derive this from the fact that tangential acceleration is zero ($2\dot{r}\dot{\theta}+r\ddot{\theta} = 0$). Note that $\frac{d}{dt}r^2 = 2r\dot{r}$ and so by substitution we have

$$\begin{aligned}2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0 \\ 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} &= 0 \\ \frac{d}{dt}(r^2)\dot{\theta} + r^2\frac{d}{dt}(\dot{\theta}) &= 0 \\ \frac{d}{dt}(r^2\dot{\theta}) &= 0.\end{aligned}$$

Thus, since the derivative is zero, $r^2\dot{\theta}$ is a constant and therefore $\vec{L} = r^2 m\dot{\theta}$ is also a constant (m is clearly a constant as well). Before substituting for $\dot{\theta}$, let $\vec{h} = \frac{\vec{L}}{m}$, where $\vec{h}$ is the specific angular momentum. This gives us

$$\dot{\theta} = \frac{\vec{h}}{r^2}.$$

Substituting this into our equation for force we have

$$\begin{aligned}
\frac{-GMm}{r^2}\hat{r} &= m\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} \\
\frac{-GMm}{r^2} &= m\left(\ddot{r} - r\dot{\theta}^2\right) \\
\frac{GM}{r^2} + \left(\ddot{r} - r\dot{\theta}^2\right) &= 0 \\
\frac{GM}{r^2} + \left(\ddot{r} - r\left(\frac{\vec{h}}{r^2}\right)^2\right) &= 0 \\
\frac{GM}{r^2} + (\ddot{r} - \frac{\vec{h}^2}{r^3}) &= 0 \\
\ddot{r} - \left(\frac{\vec{h}^2}{r^3}\right) + \frac{GM}{r^2} &= 0.
\end{aligned}$$

2.0.6 Obtaining our Differential Equation

We just found a differential equation of the radius, but with respect to time, but what we really want is an equation the form $r(\theta)$, which will give us the position of our object with respect to θ. Firstly, we let $q = \frac{1}{r}$ which will aide us in converting the derivatives of $r(t)$ into derivatives of $q(\theta)$.

Let's first exapand $\dot{r}$. Recall that radius changes with respect to θ, but θ only changes because time is changing and so we can use the chain rule:

$$\begin{aligned}
\dot{r} &= \frac{dr}{dt} \\
&= \frac{dr}{d\theta}\frac{d\theta}{dt} \\
&= \frac{d}{d\theta}\left(\frac{1}{q}\right)\dot{\theta} \\
&= -\frac{1}{q^2}\dot{\theta} \\
&= -\frac{1}{q^2}\frac{h}{r^2}\frac{dq}{d\theta} \qquad\qquad \dot{\theta} = \frac{\vec{h}}{r^2} \\
&= -h\frac{1}{q^2}q^2\frac{dq}{d\theta} \\
&= -h\frac{dq}{d\theta}.
\end{aligned}$$

Similarily, we have

$$\begin{aligned}
\ddot{r} &= \frac{d\dot{r}}{dt} \\
&= \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} \\
&= \frac{d}{d\theta}\left(-h\frac{dq}{d\theta}\right)\dot{\theta} \\
&= -h\frac{d^2q}{d\theta^2}\dot{\theta} \\
&= -h\frac{d^2q}{d\theta^2}\frac{h}{r^2} \\
&= -h^2q^2\frac{d^2q}{d\theta^2}.
\end{aligned}$$

Plugging these values of $\dot{r}$ and $\ddot{r}$ into our equation of motion we have

$$\begin{aligned}
\ddot{r} - \left(\frac{\vec{h}^2}{r^3}\right) + \frac{GM}{r^2} &= 0 \\
-h^2q^2\frac{d^2q}{d\theta^2} - h^2q^3 + GMq^2 &= 0 \\
-h^2\frac{d^2q}{d\theta^2} - h^2q + GM &= 0 \\
h^2(\frac{d^2q}{d\theta^2} + q) &= GM \\
\frac{d^2q}{d\theta^2} + q &= \frac{GM}{h^2}.
\end{aligned}$$

This is just a second order, linear, non-homogeneous, ordinary differential equation, which we know how to solve.

2.0.7 Solving the ODE

Simplifying our notation we have

$$q'' + 0q' + q = 0.$$

Finding the roots of $r^2 + 0r + r = 0$ we see that $r = \pm i$. Therefore the general solution to our homogeneous equation is

$$\begin{aligned} q^*(\theta) &= e^{\alpha\theta}(c_1 \cos(\beta\theta) + c_2 \sin(\beta x)) \\ &= e^{0\theta}(c_1 \cos\theta + c_2 \sin\theta) \\ &= c_1 \cos\theta + c_2 \sin\theta. \end{aligned}$$

Note that $q(\theta) = \frac{GM}{h^2}$ is a particular solution to our ODE, and so the general solution is

$$q(\theta) = A\cos\theta + B\sin\theta + \frac{GM}{h^2}.$$

2.0.8 Initial Value Problem

Now, what are the values for A, B, and h? To find this let's assume some initial conditions. Let r_0 be the initial distance from orbiting body, v_0 be the initial tangential velocity, and u_0 be the inital outward radial velocity (rate at which radius is increasing). Without a loss of generality, let's assume that that at $t = 0$, $\theta = 0$. Let's first look at h, because it is the easiest. Recall that

$$\begin{aligned} \dot{\theta} &= \frac{h}{r^2} \\ h &= \dot{\theta} r_0^2 \\ &= \frac{v_0}{r_0} r_0^2 && \dot{\theta} = \frac{v}{r} \\ &= v_0 r_0. \end{aligned}$$

Now what about A? Well note that

$$\begin{aligned} q(\theta) &= A\cos\theta + B\sin\theta + \frac{GM}{h^2} \\ q(0) &= A + \frac{GM}{h^2} \\ \frac{1}{r_0} &= A + \frac{GM}{h^2} \\ A &= \frac{1}{r_0} - \frac{GM}{(v_0 r_0)^2}. \end{aligned}$$

Now let's find B by taking the derivative of q:

$$\begin{aligned} q'(0) &= B\cos(0) - A\sin(0) \\ &= B. \end{aligned}$$

But because q is an expression of r and r is in terms of θ we can use the chain rule, giving us

$$\begin{aligned} q' &= \frac{dq}{d\theta} \\ &= \frac{dq}{dr}\frac{dr}{d\theta} \\ &= \frac{d}{dr}\left(\frac{1}{r}\right) \cdot \frac{dr}{d\theta} \\ &= -\frac{1}{r^2}\frac{dr}{d\theta}. \end{aligned}$$

. We can similarilty use the chain rule for $\frac{dr}{d\theta}$ giving us

$$\begin{aligned} q' &= -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} \\ &= -\frac{1}{r^2}\frac{1}{\dot{\theta}}\frac{dr}{dt} \\ &= -\frac{1}{r^2}\frac{r^2}{h}\frac{dr}{dt} \\ &= -\frac{1}{h}\frac{dr}{dt}. \end{aligned}$$

But note that $\frac{dr}{dt}$ is just our initial (outward) radial velocity, giving us

$$\begin{aligned} q'(0) &= -\frac{1}{v_0 r_0} u_0 \\ B &= -\frac{u_0}{v_0 r_0}. \end{aligned}$$

Putting everything together, we thus have

$$\begin{aligned} q(\theta) &= A\cos\theta + B\sin\theta + \frac{GM}{h^2} \\ &= \underbrace{\left(\frac{1}{r_0} - \frac{GM}{(v_0 r_0)^2}\right)}_{A}\cos\theta - \underbrace{\left(\frac{u_0}{v_0 r_0}\right)}_{B}\sin\theta + \frac{GM}{\underbrace{(v_0 r_0)}_{h}{}^2} \\ &= \frac{1}{r_0}\left(\left(1 - \frac{GM}{v_0^2 r_0}\right)\cos\theta - \left(\frac{u_0}{v_0}\right)\sin\theta + \frac{GM}{v_0^2 r_0}\right). \end{aligned}$$

But since $r = \frac{1}{q}$ we have

$$r(\theta) = \frac{r_0}{\left(1 - \frac{GM}{v_0^2 r_0}\right)\cos\theta - \left(\frac{u_0}{v_0}\right)\sin\theta + \frac{GM}{v_0^2 r_0}}.$$

This describes the orbit of our object about a much more massive object. Depending upon the initial conditions the “orbit” will be one of the conic sections.

2.0.9 Standard Form of Conic Section

A problem arises though, we are not usually given r_0, v_0 or u_0, so to get past this difficulty, let’s put our equation into a form more resembling that of an eclipse in polar coordinates:

$$r(\varphi) = \frac{a(1-e^2)}{1-e\cos\varphi}$$

where $a = \frac{\text{perihelion + aphelion}}{2}$ and e is not Euler's constant but instead the eccentricity of the ellipse, and where φ goes counterclockwise and starts from the major axis (max/min$(r(\varphi))$ occurs when $\varphi = 0$). We will use the trigonometric forumula

$$\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta)$$

to rewrite our equation of motion to better resemble that of an ellipse. Using the formula we see that

$$\underbrace{-C\cos\theta_0}_{A}\cos\theta \underbrace{-C\sin\theta_0}_{B}\sin\theta = -C\cos(\theta_0 - \theta)$$

where θ_0 and C are constants that allow us to replace the variables A and B. Note also that

$$\begin{aligned} A^2 + B^2 &= (-C\cos\theta_0)^2 + (-C\sin\theta_0)^2 \\ &= C^2\cos\theta_0^2 + C_2\sin\theta_0^2 \\ &= C^2(\cos\theta_0^2 + \sin\theta_0^2) \\ &= C^2. \end{aligned}$$

Clearly $r(\theta) > 0$. Note that because $-1 \leq \theta - \theta_0 \leq 1$ and when describing an ellipse $\theta - \theta_0$ is unbounded, for $r(\theta)$ to describe an ellipse $k > C$. If $K < C$ then a hyperbolic path is described and the angle of $\theta - \theta_0$ is bounded with r going to infinity as θ approaches its bounds.

Therefore

$$C = \sqrt{A^2 + B^2}.$$

For simplicity, let $K = \frac{GM}{h^2}$. Therefore

$$\begin{aligned} r(\theta) &= \frac{1}{A\cos\theta + B\sin\theta + K} \\ &= \frac{1}{K - C\cos(\theta_0 - \theta)} \\ &= \frac{1}{K - C\cos(\theta - \theta_0)} \\ &= \frac{\frac{1}{K}}{1 - \frac{C}{K}\cos(\theta - \theta_0)}. \end{aligned}$$

From this and the standard form of the llipse we see that

$$K = \frac{1}{a(1-e^2).}$$

We have shown how to calculate C in terms of A and B, but how do we calculate θ_0? Note that $B = -C\sin\theta_0$ and therefore

$$\begin{aligned} B &= -C\sin\theta_0 \\ -\frac{B}{C} &= \sin\theta_0 \\ \theta_0 &= \arcsin\left(-\frac{B}{C}\right). \end{aligned}$$

2.0.10 An Applications Problem

Let's put everything together and calculate the velocity of Halley's comet at aphelion given the following information:

$$\begin{aligned} \text{Aphelion } &= 5.27 \times 10^9 \text{ km} \\ \text{Perihelion } &= 87.8 \times 10^6 \text{ km} \\ \text{Eccentricity } &= 0.967. \end{aligned}$$

Given this information we want to solve for v_0, u_0 and h. Recall that

$$K = \frac{1}{a(1-e^2)}.$$

First calculating a we have

$$\begin{aligned} a &= \frac{5.27 \times 10^9 + 87.8 \times 10^6}{2} \\ &= 2.6789 \times 10^9 \text{ km.} \end{aligned}$$

Therfore

$$K = \frac{1}{2.6789 \times 10^9(1-0.967^2)}.$$

From NASA we have

$$GM = 132,712 \times 10^6 \frac{km^3}{s^2}.$$

Also recall that $K = \frac{GM}{h^2}$ and so

$$\begin{aligned} h^2 &= \frac{GM}{K} \\ &= \frac{GM}{\frac{1}{a(1-e^2)}} \\ &= GMa(1-e^2) \\ h &= \sqrt{GMa(1-e^2)} \\ &= 4.8038838473 \times 10^9. \end{aligned}$$

Recall that $v_0 = \frac{h}{r_0}$ and so we now have enough information to calculate v_0 at aphelion:

$$\begin{aligned} v_0 &= \frac{4.8038838473 \times 10^9}{5.27 \times 10^9} \\ &= 0.911552912201 \frac{\text{km}}{\text{second}}. \end{aligned}$$

This is the same velocity that NASA gives. Also note that at both perihelion and aphelion radial velocity is zero and so $u_0 = 0$, and thus we don't have to accound for radial velocity. This gives us enough information to form the equation of motion:

$$\begin{aligned} r_0 &= 5.27 \times 10^9 \\ v_0 &= 0.911552912201 \\ u_0 &= 0 \\ h &= 4.8038838473 \times 10^9. \end{aligned}$$

Leaving out the actual numerical values in our equation (for simplicity) we have:

$$
\begin{aligned}
r(\theta) &= \frac{r_0}{\left(1 - \frac{GM}{v_0^2 r_0}\right)\cos\theta - \left(\frac{u_0}{v_0}\right)\sin\theta + \frac{GM}{v_0^2 r_0}} \\
&= \frac{r_0}{\left(1 - \frac{GM}{v_0^2 r_0}\right)\cos\theta + \frac{GM}{v_0^2 r_0}}.
\end{aligned}
$$

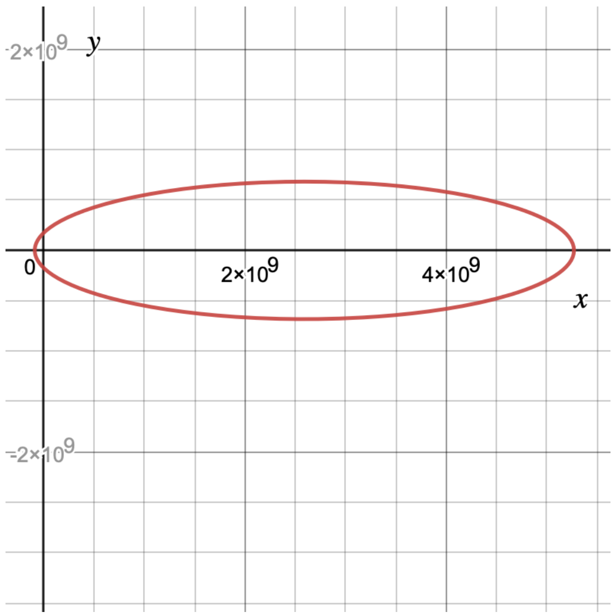

Figure 4. The path that Halley's comet takes, with the sun located at the origin. Thu numbers represent distance from sun in km. This is the same graph that we get if we just plug in values a and e into the standard equation for an ellipse.

Section 3

Conclusion

In this paper we looked at a simplified version of the two body problem, the one-body problem, where a much smaller mass is orbiting around a much bigger mass. We also took for granted that the path that our smaller object takes lies on a plane. The question remains, what happens when we have a our two-body problem? Well, for that problem the idea is to find the center of mass, and calculate the "reduced-mass" of the system. From there you convert the problem into a one-body problem.

Section 4

Infographic

Title: Why are planetary orbits elliptical?

Group Members: Leonard Mohr

Summary: One of the subjects that has fascinated astronomers, mathemeticians, and physicists, from Kepler to Feynman has been the two body problem; that is, given two bodies (the earth and sun for example), can we describe their motions using an explicit equation? For this project, I look at a simpler version of this problem, the one body problem, where instead of two moving bodies, there is only one, orbiting around the other. This problem is grounded in the physical world, think the Earth orbiting about the Sun, or a satellite orbitting around the earth.

First off, we will be modelling the motion in polar coordinates; that is, instead of finding $r(t)$ we will find $r(\theta)$. We eventually find the general solution to a second order, linear, non-homogeneous differential equation:

$$r(\theta) = \frac{1}{A\cos\theta + B\cos\theta}.$$

Depending upon the the values for A and B we can generate all of the conic sections (circle, ellipse, hyperbola and parabola). We then show that this gives us our equation for an ellipse

$$r(\theta) = \frac{a\left(1-e^2\right)}{1-e\cos\theta}.$$

Then given the following for Halley's comet:

$$\begin{aligned} \text{Perihelion :}&8.9263 \times 10^7 \text{ km} \\ \text{Aphelion :}&5.2733 \times 10^9 \text{ km} \\ \text{Eccentricity :}&0.967. \end{aligned}$$

Note that perihelion is just when the object is closest to the sun, and aphelion is when it is farthest. Eccentricity is just a number that represents how noncircular the orbit is 0 is circular. Then solving for a we have

$$\begin{aligned} a &= \frac{\text{perihelion } + \text{ aphelion}}{2} \\ &= \frac{8.9263 \times 10^7 + 5.2733 \times 10^9}{2} \\ &= 2681281500.0 \text{ km} \\ &\approx 17.923 \text{ AU }. \end{aligned}$$

This gives us our equation for Halley's comet:

$$r(\theta) = \frac{17.923(1-0.967^2)}{1-0.967\cos\theta}.$$

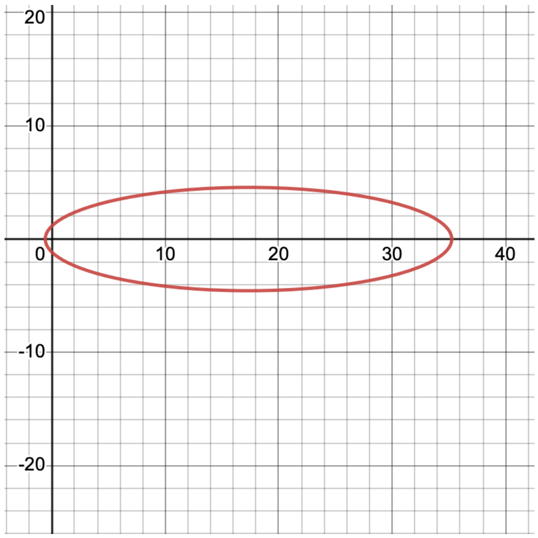

Figure 5. Graphical representation of Halley's Comet

Subsection 4.1
Sources

- Three Solutions to two body problem
- Elliptical Orbit Based on Initial Conditions
- Deriving Kepler's First Law
- Deriving Kepler's First Law Part Two
- Feynman's Last Lecture
- NASA – Halley's Comet
- NASA sun facts