Math 1C – Multivariate Calculus

My Notes

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SUBSECTION 1.1

Points and Vectors

You are probably used to the idea of points. There is not much difference between points in $\mathbb{R}^2$ (the cartesian plane) and $\mathbb{R}^3$, or even $\mathbb{R}^n$ They are all represented by an ordered pair of size n. But points are not the only way to think about things, we can also think of them as a vector. For example, in $\mathbb{R}^2$ the location $x = 2$, $y = 3$ could be thought of as the point $(3, 2)$ or it could be thought of as the direction "go 3 units to the right and 2 units up". Vectors have a direction and a magnitude/length, and in $\mathbb{R}^3$ can be represented as

$$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \langle a_1, a_2, a_3\rangle,$$

where $\hat{i}, \hat{j}, \hat{k}$ are vectors of length one in $\mathbb{R}^3$, and $\hat{i}$ is along the x-axis, $\hat{j}$ is along the y-axis and $\hat{j}$ is along the x-axis. This way any vector $\vec{A}$ in $\mathbb{R}^3$ can be thought of as scaling the hats and adding them up. To better understand this notation, we need to understand vector addition and scalar multiplication.

| Section ??. Vectors

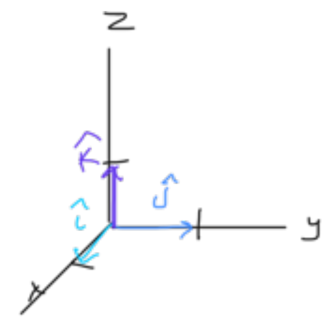


Figure 1. $\hat{i}, \hat{j}, \hat{k}$ components of $\mathbb{R}^3$

Let $\vec{A}, \vec{B}$ be two vectors. Let's say you want to perform the operation $\vec{A} + \vec{B}$. What does this mean? Let's first think of this in geometric terms. Because vectors don't have a set starting point, we can think of $\vec{A} + \vec{B}$ as moving $\vec{B}$ to the end, or head of $\vec{A}$. As you can see in figure two, adding $\vec{A}$ to $\vec{B}$ has the same effect as adding $\vec{B}$ to $\vec{A}$ since it forms a prallelagram. Thus $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. Thinking of vector addition arithmetically, we have the follwing definition.

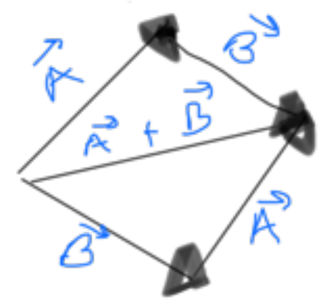


Figure 2. $\vec{A} + \vec{B}$ forms a parallelagram.

Definition 1 Given two vectors $\vec{A}$ and $\vec{B}$ where $\vec{A} = \langle a_1, a_2, \ldots, a_n\rangle$ and $\vec{B} = \langle b_1, b_2, \ldots b_n\rangle$

$$\vec{A} + \vec{B} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle.$$

This brings us to one big destinction between vectors and points. Because vectors can be thought of in terms of change, be it a change in temperature, stock price, or position, it makes sense to add vectors, whearas it does not make sense to add points. For example, adding the daily change in sea-level for a week will give the weeks change in sea level, but adding the sea level heights for each day, does not give a meaningful result.

First thinking about scalar multiplication geometrically, if we perform the operation $2\vec{A}$, we get a vector with the same direction, but whose magnitude is twice as large. In this way, the original vector will be stretched or compressed by a constant factor. The distance of the vector will remain the same though. More formally, we have the following definition.

Definition 2 Let $n \in \mathbb{Z}^*, c \in \mathbb{R}$ and $\vec{X} \in \mathbb{R}^n$ with $\vec{X} = \langle x_1, x_2, \ldots, x_n\rangle$. Then

$$\begin{aligned} c \cdot \vec{X} &= c \cdot \langle x_1, x_2, \ldots, x_n\rangle \\ &= \langle c \cdot x_1, c \cdot x_2, \ldots, c \cdot x_n\rangle. \end{aligned}$$

Now that we have this idea of scalar multiplication, what does it mean for two vectors to have the same direction?

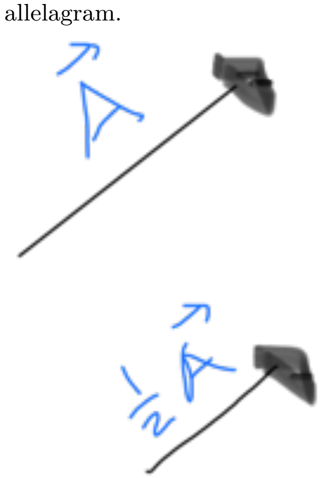


Figure 3. Effect of scaling $\vec{A}$ by $\frac{1}{2}$

Theorem 1 For vectors $\vec{x}, \vec{y}$ with initial point at origin, two vectors are in the same direction if and only if $\vec{x} = c \cdot \vec{y}$ for some scalar $c \in \mathbb{R}$.

Given a vector, we might want to find it's magnitude. This is actually just as simple in $\mathbb{R}^n$ as it is in $\mathbb{R}^2$. This is because in all examples we can are finding the distance of a straight line, which can be represented by a straight line on a plane. This we already know how to find; we can use the pythagorean theorem. For example, let's say we are in $\mathbb{R}^3$ and we want to find the distance of vector $\vec{A}$ where $\vec{A} = \langle a_1, a_2, a_3 \rangle$. As you can see in figure 4, to do this, we first need to find the distance x. To do this, we use the pythagorean theorem:

$$x = \sqrt{a_1^2 + a_2^2}.$$

Thus, the distance of $\vec{A}$ written $||\vec{A}||_2$ is:

$$\begin{aligned} ||\vec{A}||_2 &= \sqrt{x^2 + a_3^2} \\ &= \sqrt{\left(\sqrt{a_1^2 + a_2^2}\right)^2 + a_3^2} \\ &= \sqrt{a_1^2 + a_2^2 + a_3^2}. \end{aligned}$$

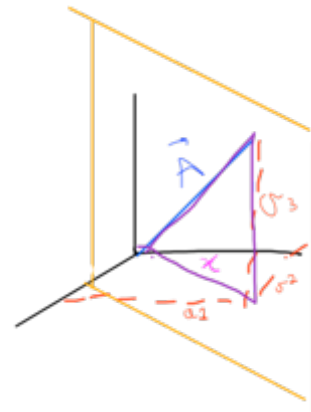


Figure 4. To find thie distance of $\vec{A}$, we must first find length x and then we have a triangle (purple) on a 2d plane (yellow).

Subsection 1.2

Dot Product

Let's begin with a formal definition of the dot product.

Definition 3 Given two vectors $\vec{A}, \vec{B}$, where $\vec{A} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\vec{B} = \langle b_1, b_2, \ldots, b_n \rangle$, then

$$\vec{A} \cdot \vec{B} = \sum a_i b_i = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \rangle.$$

One very important thing to notice about this definition is that the result of $\vec{A} \cdot \vec{B}$ is a SCALAR $\in \mathbb{R}$, not a vector. What is the use of such a weird operation. First, let's think about the operation geometrically.

Theorem 2 Given two vectors $\vec{A}, \vec{B}$ then

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos(\theta).$$

Does this theorem hold up if we take the dot product of one vector with itself? We se that

$$\begin{aligned} \vec{A} \cdot \vec{A} &= |\vec{A}||\vec{A}| \cdot \cos(0) \\ &= |\vec{A}|^2 \cdot 1 \\ &= \left(\sqrt{(a_1^2 + a_2^2 + \cdots + a_n^2)}\right)^2 \\ &= a_1 a_1 + a_2 a_2 + \cdots + a_n a_n \end{aligned}$$

which indeed lines up with our definition of the dot product. This discovery leads us to our first lemma, which we will use to help prove Theorem 2.

Lemma 1 Given vector $\vec{A}$, then

$$\vec{A} \cdot \vec{A} = ||\vec{A}||_2 \cdot ||\vec{A}||_2 = ||\vec{A}||_2^2$$

We will also use the law of cosines in our proof. If you recall, the law of cosines states

$$c^2 = a^2 + b^2 - 2ab\cos\gamma,$$

where γ denotes the angle contained between sides of length a and b and opposite the side of length c. We can use the law of cosines with vectors by taking the two norm of the vector which will give us the length. The idea behind the proof of theorem 2 is that we will form a vector triangle by connecting the heads of vectors $\vec{A}$ and $\vec{B}$ with a third vector $\vec{C}$. Then, by using Lemma 1, we can convert the magnitudes of vectors $\vec{A}, \vec{B}, \vec{C}$ and then through algebraic manipulation, prove theorem 2.

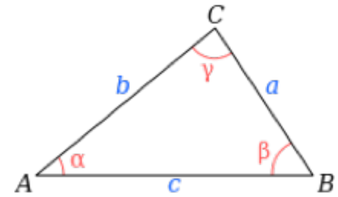


Figure 5. A non right triangle. In our example, we can use the law of cosines to calculate the length of C.

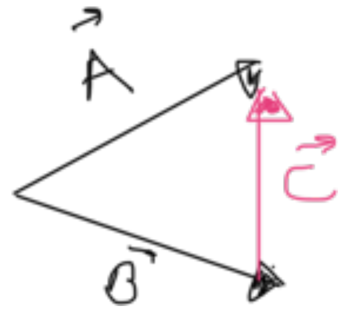


Figure 6. Note that $\vec{B} + \vec{C} = \vec{A}$ and so $\vec{C} = \vec{A} - \vec{B}$.

Proof of Theorem 2: Let $\vec{A}$, $\vec{B}$ be any vectors. Arrange the vectors so they are both starting from the origin. Let $\vec{C}$ be a vector going from the head of $\vec{B}$ to the head of $\vec{A}$. Therefore $\vec{C} = \vec{A} - \vec{B}$. Thus, by the law of cosines, we have:

$$\begin{aligned}
\|\vec{C}\|_2^2 &= \|\vec{A}\|_2^2 + \|\vec{B}\|_2^2 - 2\|\vec{A}\|_2\|\vec{B}\|_2\cos\theta \\
\|\vec{A} - \vec{B}\|_2^2 &= \|\vec{A}\|_2^2 + \|\vec{B}\|_2^2 - 2\|\vec{A}\|_2\|\vec{B}\|_2\cos\theta && \textit{By Substitution} \\
(\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) &= \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} - 2\|\vec{A}\|_2\|\vec{B}\|_2\cos\theta && \textit{By Lemma 1} \\
\vec{A} \cdot \vec{A} - 2(\vec{A} \cdot \vec{B}) + \vec{B} \cdot \vec{B} &= \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} - 2\|\vec{A}\|_2\|\vec{B}\|_2\cos\theta \\
-2(\vec{A} \cdot \vec{B}) &= -2\|\vec{A}\|_2\|\vec{B}\|_2\cos\theta \\
\vec{A} \cdot \vec{B} &= \|\vec{A}\|_2\|\vec{B}\|_2\cos\theta.
\end{aligned}$$

Thus, $\vec{A} \cdot \vec{B} = ||\vec{A}||_2||\vec{B}||_2\cos(\theta)$ □

One thing you might be wondering is "why is the dot product distributive"? Well, if you look back at the dot product, you see that it is just the sum of the multiples of the components. Thus, when applying the definition of the dot product, you are really just performing multiplication and addition under the covers so it follows mostly the same properties. This fact is the root of the distributive property proof.

This is all well and good, but we still haven't really gotten to what this weird and thus far useless opperation called the dot product is good for.

To Be Continued: Another way to think about the dot product is that $\mathbf{A} \cdot \mathbf{B} = ||\mathbf{A}||_2 \cdot ||\text{proj}_{\mathbf{A}}\mathbf{B}||_2 = ||\mathbf{B}||_2 \cdot ||\text{proj}_{\mathbf{B}}\mathbf{A}||_2$. That is the dot product $\vec{A} \cdot \vec{B}$ is equal to the magnitude of $\vec{A}$ times the magnitude of the projection of $\vec{B}$ onto $\vec{A}$ and is positive when θ, the angle between the two vectors, is less than 90 degrees and negative when it is greater than.

1.2.1 Applications of the Dot Product

Dot-Product, h'uh. Yeah! What is it good for? Absolutely somethin uh-huh, uh-huh. There are three main uses for which we can use the dot product. Now that we know that the dot product can be expressed in terms of cosine, we can use that fact to determine the angle between two vectors. The second way we can use the dot product is to test for orthoganality of two vectors, and the third way we can use the dot product is to find the length of a vector projected onto another.

1.2.2 Angle Between Vectors

Whenever we have angles, one thing we often might want to know is how big or small an angle is. Recall that we proved that

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos(\theta).$$

Thus, multiplying both sides by $\frac{1}{|\vec{A}||\vec{B}|}$ we have $\frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \cos(\theta)$. Since we know how to find the length of both $\vec{A}$ and $\vec{B}$, by substitution we have $\frac{\vec{A} \cdot \vec{B}}{\sqrt{a_1^2+a_2^2+\cdots+a_n^2} \cdot \sqrt{b_1^2+b_2^2+\cdots+b_n^2}} = \cos(\theta)$ and thus:

$$\theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{\sqrt{a_1^2+a_2^2+\cdots+a_n^2} \cdot \sqrt{b_1^2+b_2^2+\cdots+b_n^2}}\right)$$

which we can calculate with our good ol' ti-85 plus.

1.2.3 Testing for Orthoganality

One thing have noticed about vectors is that when they are pointed in generally the same direction, the dot product is positive, when they are ppointed in generally opposite directions, the dot product is negative, and when they are perpendicular (orthoganal) to eachother, the dot product is zero. Why is this? By will of god? Well maybe, but also because the sign is determined by cosine, since lengths of vectors never negative.

Let's say we have the equation

$$x + 2y + 3z = 0.$$

What does this equation describe? A point? A line? A black hole? A plane? Well, it actually describes a plane. Why is this? This is because it is actually the dot product between two orthogonal vectors. To see this let $\vec{P} = \langle x, y, z\rangle$ and $\vec{A} = \langle 1, 2, 3\rangle$. Thus, we see that since $x+2y+3z = 0$, and $\vec{A} \cdot \vec{B} = x+2y+3z$, then $\vec{A}$ and $\vec{P}$ are orthogonal to each other (hint hint, $\cos(\pi) = 0$). Thus, using the follwoing, we can easily test for orthoganality:

$$\begin{aligned} \vec{A} \cdot \vec{B} = 0 &\iff \cos\theta = 0 \\ &\iff \theta = \pi \\ &\iff \vec{A} \perp \vec{B}. \end{aligned}$$

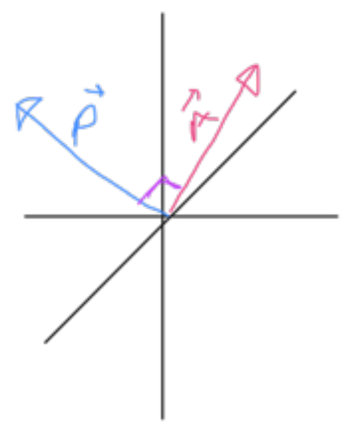


Figure 7. $\vec{P}$ and $\vec{A}$ are orthogonal, and thus theta is 90 deg and the dot product is zero.

1.2.4 Length of Projection

Let's say we want to find the length of a projection of vector $\vec{A}$ onto $\hat{u}$, where $\hat{u}$ is the component vector (length 1) in the direction of $\hat{u}$. Then, the length of projection is $|\vec{A}|\cos(\theta)$. To see this, note that:

$$\begin{aligned} \cos(\theta) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{|\hat{u} \cdot c|}{|\vec{A}|} \\ |\vec{A}|\cos(\theta) &= c \cdot |\hat{u}| \\ &= 1c \\ &= c \end{aligned}$$

where c is a scalar of $\hat{u}$, and the length of the projection of $\vec{A}$ onto $\hat{u}$. But since $\hat{u}$ is a unit vector, $|\vec{A}|\cos(\theta) = |\vec{A}||\hat{u}|\cos(\theta) = \vec{A}\cdot\hat{u} = c$. But let's say that we want to find the length of the projection of $\vec{A}$ onto $\vec{B}$ written mathematically as $\text{proj}_{\mathbf{B}}\mathbf{A}$. Well then, we just need to scale things down by the length of $\vec{B}$.

Theorem 3 The length of the projection of $\vec{A}$ onto $\vec{B}$ is

$$\text{proj}_{\mathbf{B}}\mathbf{A} = \frac{\vec{A}\cdot\vec{B}}{|\vec{B}|}.$$

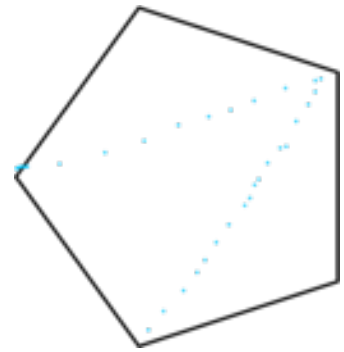


Figure 8. We can find the area of a pentagon by dividing it into triangles.

Section 2

Lecture 2 – Cross Product

In this lecture, we will take a look at this thing called the determinant which can be used to calculate the area of the parallelogram formed by two vectors in 2d space, or the volume formed by three vectors in 3d space and how that relates to the cross product.

Subsection 2.1

Determinant

2.1.1 An overview of the Determinant

Recall that we can represent a vector by scaling and adding together the unit basis vectors $\hat{\imath}, \hat{\jmath}$ (when in 2D space). So for example the vector $\langle 2, 0\rangle$ is the result of $2\hat{\imath} + 0\hat{\jmath}$. When we calculate the determinant, what we are really doing is calculating the area of the unit square after a linear transformation. This allows you to calculate the area of an object after a linear transformation was done, as the determinant is the scalar of the grid after a linear transformation. But, the area can be negative, what does this mean? It just means that $\hat{\imath}$ and $\hat{\jmath}$ switched sides ($\hat{\imath}$ is now to left of $\hat{\jmath}$). This is multivariable calculus though, and we don't have to worry about linear transformations, so for now, we can just think of the determinant as the area of the parallelogram formed by two vectors.

2.1.2 Calculating Area/Volume

Let's say we have some shape (a pentagon for example) in a plane we would like the area of. Can we do this given some vectors? Yes, of course. But since a pentagon is a weird shape, we can divide it into triangles, and thus all we need to know is how to find the area of a triangle. If you remember from trigonometry class, we can find the area of a triangle with the following formula:

$$\text{Area} = \frac{1}{2}\ \text{base}\ \cdot\ \text{height}\ = \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta.$$

There's one issue though, we can find $|\vec{A}||\vec{B}|\cos\theta$ really easily, (it's the dot product), but that's in terms of cosine. What is one to do? Well, if we rotate $\vec{A}$ 90 degrees and call that new vector $\vec{A}'$, then the angle between $\vec{A}'$ and $\vec{B}$, (θ'), will be $\theta' = 90 - \theta$. Thus $\cos(\theta') = \sin(\theta)$.

Coneceptually, this can help us find the area of our triangle, but we still do not have a way to find the vector $\vec{A}'$. We can use the fact that $\vec{A}'\cdot\vec{A} = 0$ and have $\vec{A}' = \langle -a_2, a_1\rangle$. Thus, returning back to the area of our triangle, we have:

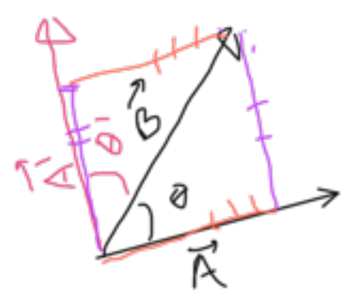
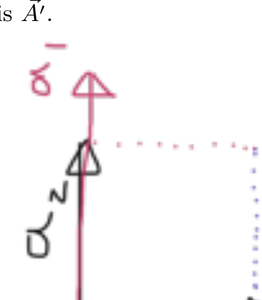


Figure 9. $\vec{A}$ rotated 90 degrees is $\vec{A}'$.



$$\begin{aligned}\text{Area} &= \frac{1}{2}\text{ base } \cdot \text{ height} &&= \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta \\ & &&= \frac{1}{2}\left|\vec{A}'\|\vec{B}\right|\cos\theta \\ & &&= \frac{1}{2}\vec{A}\cdot\vec{B} \\ & &&= \frac{1}{2}\left(a_1b_2 - a_2b_1\right)\end{aligned}$$

What we actually just learned is this operation called the determinant.

Definition 4 The determinant of $\vec{A}$ and $\vec{B}$ is:

$$\begin{aligned}\det(\vec{A},\vec{B}) &= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= a_1b_2 - a_2b_1\end{aligned}$$

where $\vec{A}$ and $\vec{B}$ are two vectors in a plane.

What this operation does, since there is no division by two, is calculate the area of the parallelogram formed by vectors $\vec{A}$ and $\vec{B}$. We don't have to use trigonometry to get the area of a parallelogram; we can also use geometry.

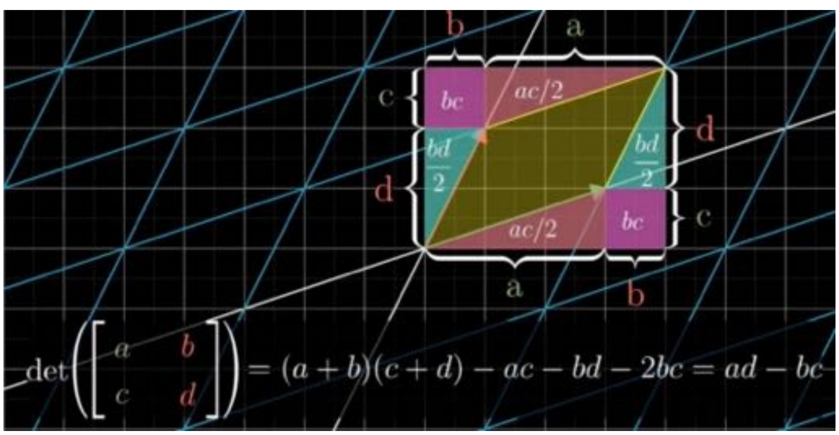


Figure 11. Deriving determinate geometrically.

One thing to Though, the area is actually the absolute value of the determinant since there is nothing forcing the determinant to be positive. There also exists a determinant in 3d space.

Definition 5 The determinant of $\vec{A},\vec{B},\vec{C}$ is:

$$\begin{aligned}\det(\vec{A},\vec{B},\vec{C}) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\end{aligned}$$

What does this determinant actually do? Well, it calculates the volume of a parallelpiped in 3d space. A parallelpiped is just the box created out of parallelograms formed by three vectors. This value will either be nonzero or zero. What does it mean vor the volume of an object in 3d space to be zero? It means that the vectors all lie on the same plane, are all in the same direction (so formm a line), or are all zero (so form a dot).

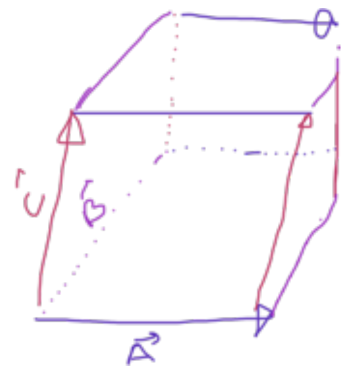


Figure 12. Three vectors $\vec{A}, \vec{B}, \vec{C}$ form a parallelpiped.

2.1.3 Properties of the Determinant

If you scale one of the vectors, the area of the parallelogram will scale by a similar amount.

Subsection 2.2

Cross Product

2.2.1 A Standard Introduction

To make things simple, we will start in two dimensions. Let's say we have two vectors $\vec{v}$ and $\vec{w}$. If we take a copy of $\vec{v}$ and move its tail to the tip of $\vec{w}$ and take a copy of $\vec{w}$ and move its tail to the tip of $\vec{v}$ the four vectors form a parallelogram. As we have shown, the area of this parallelogram is the determinant. Well, what is the cross product? The cross product is a vector perpendicular to the plane formed by two vectors in 3D space whose length is the area of the parallelogram formed by those two vectors. To compute this vector, we compute the following determinant:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \det\left(\begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} \begin{matrix} v_1 & w_1 \\ v_3 & w_2 \\ v_3 & w_3 \end{matrix} \right).$$

Given that we can use a determinant to calculate the area of a parallelogram in 2D space, it seems reasonable that the determinant is used. This ultimately gives us the following definition:

Definition 6 The cross product of two vectors in 3D space is:

$$\begin{aligned} \vec{A} \times \vec{B} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k} \\ &= \hat{\imath}\underbrace{(a_2b_3 - a_3b_2)}_{\text{Some number}} + \hat{\jmath}\underbrace{(a_3b_1 - a_1b_3)}_{\text{Some number}} + \hat{k}\underbrace{(a_1b_2 - a_2b_1)}_{\text{Some number}} \end{aligned}$$

Theorem 4 Given two vectors in 3D space, $\vec{A}, \vec{B}$, then

$$|\vec{A} \times \vec{B}| = \text{ Area of parallelogram formed by the vectors } \vec{A}, \vec{B}.$$

Theorem 5 Given two vectors in 3D space, $\vec{A}, \vec{B}$, then $\text{dir}(\vec{A} \times \vec{B})$ is perpendicular (obeying right hand rule) to the plane of the parallelogram formed by vectors $\vec{A}, \vec{B}$

We could just leave it here, most classes do, but we haven't really gone into why this definition gives us a vector that is perpendicular to the parallelogram whose length is the area of the parallelogram and that obeys the right hand rule.

The right hand rule says that the right hand points in direction of $\vec{A}$, the fingers point in the direction of $\vec{B}$, and the thumb points in the direction of $\vec{A} \times \vec{B}$. But, it is worth a deeper look at the parallelogram.

2.2.2 Duality

To fully understand the cross product, and how it is related to the determinant, we first need to understand duality. The idea behind duality is that whenever there is a linear transformation from some space to the number line, it is associated with a unique vector in that space, called the dual vector. Thus, performing a linear transformation to the number line is the same as taking the dot product with that vector. The one row transformation matrix tells where each basis vector will land on the number line after the transformation is performed.

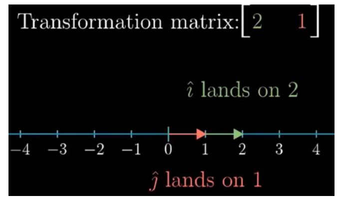


Figure 13. A transformation matrix describes where the basis vectors will land after the transformation.

2.2.3 Upon Closer Inspection

Computing the cross product $\vec{v} \times \vec{w}$ is this weird operation that involves creating a matrix whose second column is $\vec{v}$, whose third column is $\vec{w}$ and whose first column is the basis vectors. Then, you compute the determinant of the matrix:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \det\left(\begin{bmatrix} \hat{\imath} & v_1 & w_1 \\ \hat{\jmath} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{bmatrix}\right)$$

To fully determine what is happening we will define a 3d-to-1d linear transformation in terms of $\vec{v}$ and $\vec{w}$, find its dual vector and then show that this dual vector IS $\vec{v} \times \vec{w}$. This will make clear the relationship between computation of the cross product, and it's geometric clear.

Let's begin with a reasonable false start; since the "cross product" in 2 dimensions involves taking the determinate of 2 vectors, it would be reasonalbe for the cross product in 3 dimensions to be the determinate of three vectors. Although wrong, this actually gets us really close to the determinate. Since we know that the cross product takes in two vectors, let's replace the first vector with three variables x, y, z. Thus, we have a function from three dimensions to the number line:

$$\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}} = \det\left(\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\right)$$

Figure 14. A reasonable, albeit wrong guess at the cross product in 3d.

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right).$$

Geometrically we have a function that for any x, y, z we find the volume of the parallelpiped created by three vectors (with a plus or minus sign depending upon orientation). This is actually a linear transformation. Why is this so? Well, my intuition would be that since we are taking the determinant of three vectors, we are calculating its volume. Thus, if we scale our variable vector by n, the volume and thus the determinant will be scaled by n. I am not sure if this is true or not.

$$\begin{bmatrix} ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

Figure 15. The 1×3 matrix encoding the 3d-to-1d linear transformation.

Because this function is linear, there is a 1×3 matrix that encodes this 3d to 1d linear transformation (representing the locations of the basis vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ after the transformation). Thus, becuase of duality, there exists a vector ($\vec{p}$) such that

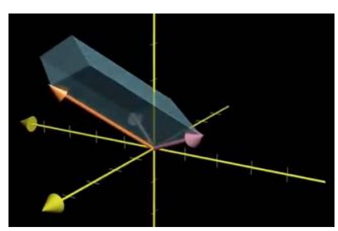


Figure 16. The blue box represents the volume of parellelpiped.

Subsection 2.3
Another Look at Volume

What if we don't want to use determinants to calculate the volume?

$$\begin{aligned}
&\text{Volume} = \text{area of base - heignt} \\
&= |\vec{B} \times \vec{C}| \cdot (\vec{A} \cdot \hat{n}) \\
&= |\vec{B} \times \vec{C}| \cdot \vec{A} \cdot \frac{(\vec{B} \times \vec{C})}{|\vec{B} \times \vec{C}|} \qquad voffset \\
&= A \cdot (\vec{B} \times \vec{C}) \\
&= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}
\end{aligned}$$

Subsection 2.4

Algebraic Properties of the Cross Product

2.4.1 Anticommutative

Recall that if the commutative property states

$$\mathrm{x} \circ \mathrm{y} = \mathrm{y} \circ \mathrm{x}$$

where $\circ$ is an opperation. The cross product is not commutative but actually anticommutative. This means that

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}.$$

The geometric reasoning behind this is that the resulting vector is perpendicular to the other side of the plane that vectors $\vec{A}$ and $\vec{B}$ lie on; the right hand rule forces you to point your thumb in the opposite direction.

Algebraically, this is due to how determinants are calculated; what was

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} = (a_2b_3 - a_3b_2)\hat{\imath}$$

is now

$$\begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix} \hat{\imath} = (a_3b_2 - a_2b_3)\hat{\imath}.$$

2.4.2 Self Cross Product

When you take the cross product with itself, the result is the zero vector:

$$\vec{A} \times \vec{A} = \vec{0}.$$

Geometrically, this is because the area of the parallelogram formed by one line is 0, and so the magnitude of the cross product will be 0. Algebraically because

$$\begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix} \hat{\imath} = (a_2a_3 - a_3a_2)\hat{\imath} = 0\hat{\imath}.$$

Subsection 2.5

Is a point on a plane?

Let's say we have three points P_1, P_2, and P_3. With these points you can create both a plane and two vectors.

Let's say we also have another point P and we want to see if P is in the plane formed by our three points. How can we do this?

The first way we can do this is by finding the volume of the parallelpiped formed by the vectors $\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \overrightarrow{P_2P}$ by calculating $\det\left(\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \overrightarrow{P_2P}\right)$.

The second way is by finding a normal vector $\vec{N}$ to our two vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ and then checking to see if this normal vector is perpendicular to $\overrightarrow{P_1P}$. These two methods are actually equivalent:

$$
\begin{aligned}
P \text{ is in Plane } &\Leftrightarrow \overrightarrow{P_1P} \perp \vec{N} \\
&\Leftrightarrow \overrightarrow{P_1P} \cdot \vec{N} = 0 && \textit{By def of dot product} \\
&\Leftrightarrow \overrightarrow{P_1P} \cdot \left(\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right) = 0 && \textit{By geometric def of cross product} \\
&\Leftrightarrow \det\left(\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}\right) = 0 && \textit{By def of triple product}
\end{aligned}
$$

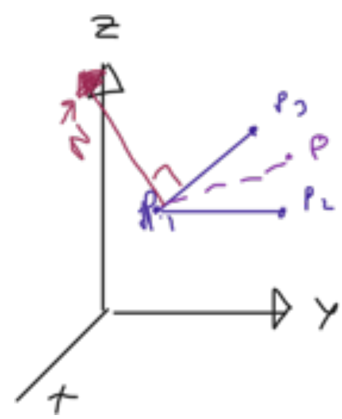


Figure 17. Points P_1, P_2, P_3 create a plane and we can use the normal vector $\vec{N}$ to see if point P is in plane.

Section 3

Lecture 3 – Matrices

Subsection 3.1

Linear Transformations

Matrices are really convenient tools for simplifying equations, performing matrix operations and solving a system of equations etc... but they are not just an algebraic convenience. They also represent linear transformations. Because these linear transformations are linear they can be described using just the coordinates of where each basis vector will land $(\hat{\imath}, \hat{\jmath})$. Matrices allow us to store this information, where the columns represent where the columns represent the coordinates of where each basis vector lands. So for example

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

just says that after the transformation $\hat{\imath}$ will now go from the origin to point (a, c) and $\hat{\jmath}$ will now go from the origin to the point (b, d) and that the vector $\langle x, y \rangle$ will now go from the origin to the point $(ax + by, cx + dy)$.

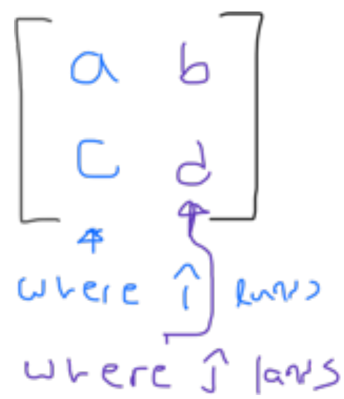


Figure 18. Matrices represent a linear transformation by storing how the basis vectors $\hat{\imath}$ and $\hat{\jmath}$ are transformed.

Subsection 3.2

Matrix Multiplication

3.2.1 A Geometric Interpretation

Multiplying two matrices has the geometric meaning of applying one transformation and then another where the resulting product represents the composition of those two transformations. For example the following multiplication

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{Shear}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Composition}}$$

represents first rotating the grid 90 degrees and then shearing it, and the result represents where $\hat{\imath}$ and $\hat{\jmath}$ will eventually end up.

Note that the order in which the transformations takes place is from left and right. So the above can be seen as first performing the rotation and then the shear. Not the other way around. So:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{\text{Shear}} \left(\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}} \begin{bmatrix} x \\ y \end{bmatrix} \right) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Composition}}.$$

3.2.2 Calculating Product

Lets say you are transforming the vector $\vec{X}$ in the following way:

$$\begin{cases} u_1 = 2x_1 + 3x_2 + 3x_3 \\ u_2 = 2x_1 + 4x_2 + 5x_3 \\ u_3 = x_1 + x_2 + 2x_3 \end{cases}$$

then this transformation can be expressed using matrix multiplication:

$$\underbrace{\begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{X}} = \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\vec{U}}.$$

To perform matrix multiplication you take the dot product between the rows of A and the columns of $\vec{X}$. Thus, $u_1 = 2x_1 + 3x_2 + 3x_3$ which is exactly what we were expencting.

More generally, when we multiply an $l \times m$ matrix with and $m \times n$ matrix we will get and $l \times l$ matrix. If multiplying two matrices A, B together, you can place B next to and above A so as to help determine the size of the resulting matrix as well as which columns and rows you will be calculating the dot product of.

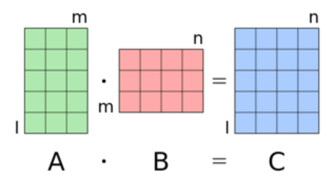
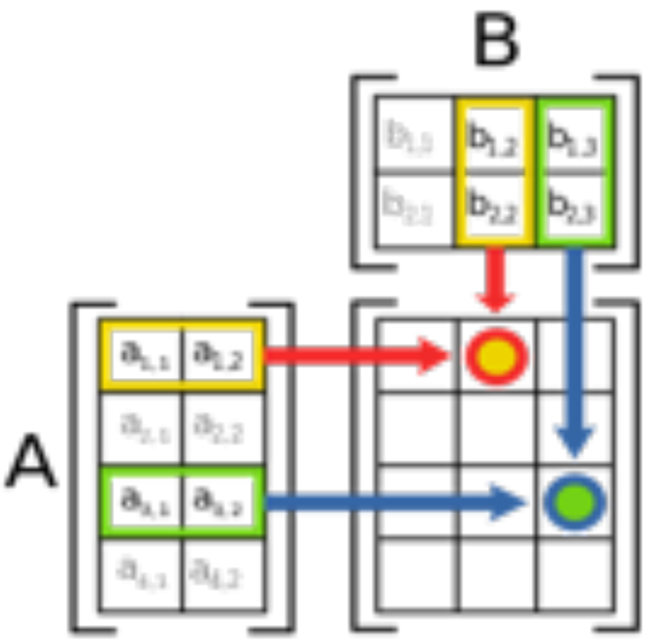


Figure 19. For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The resulting matrix has the number of rows of the first and the number of columns of the second matrix.



3.2.3 Non-Commutativity

Note, that when multiplying two matrices together $AB \neq BA$. Geometrically, this makes sense because performing a rotation and then a shear is often different then performing a shear and then a rotation for example. Also, there is no guarantee that matrix multiplication would even be possible; the number of rows of the first matrix does not have to equal the number of columns in the second. So while AB could be possible, BA might not be.

3.2.4 Identity Matrix

The identity matrix represents a transformation from something to itself. So I is an idenity matrix if

$$IX = X.$$

What does the identity matrix look like? Well, thinking about matrices as linear transformations, it becomes clear that in the $\hat{\imath}$ column, the first spot will be set to 1, in the $\hat{\jmath}$ column, the second spot will be set to 1 and so forth. Thus, for an $n \times n$ matrix, the identity matrix

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with n 1s in the diagonal position and everything else zero.

Subsection 3.3

Inverse Matrix

One of the main ways we can use linear algebra is to help us solve systems of linear equations. As a reminder, you have a system of equations when you have a list of variables, lets say x, y, z and a set of equations relating them:

$$\underbrace{\begin{gathered} 6x - 3y + 2z = 7 \\ x + 2y + 5z = 0 \\ 2x - 8y - z = -2 \end{gathered}}_{\text{Equations}}.$$

When these equations take on a special form such that there are only constants infront of the variables and all the variables are only being added (no xy or $\sin(x)$ for example) then you have a linear system of equations. Because we have a system of linear equations we can use a matrix to represent them:

$$\begin{aligned} 2x + 5y + 3z &= -3 \\ 4x + 0y + 8z &= 0 \\ 1x + 3y + 0z &= 2 \end{aligned} \rightarrow \overbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}^{\vec{x}} = \overbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}^{\vec{v}}.$$

3.3.1 Geometric Interpretation

This is not just a notational trick or an algebraic convenience. The matrix A corresponds with some linear transformation such that

$$A\vec{x} = \vec{v}.$$

Thus, to solve, we are looking for a vector $\vec{x}$ who after being transformed by A equals vector $\vec{v}$.

When a transformation is performed, there are two cases: either we remain in the same dimension, for example going from 3D space to 3D space, or we transform into a lower dimension, for example going from 3D space to a plane (2D) or from 2D space to a point or a line. Because our matrix A stores the coordinates of $\hat{\imath}, \hat{\jmath}, \hat{k}$ we can calculate the determinant of A and if that returns 0 (correlating to no area or no volume) then an inverse matrix does not exist, otherwise one does exist.

What is an inverse matrix and how can it be used to solve the problem of finding what vector $\vec{x}$ after being transformed by A equals vector $\vec{v}$ (solving a set of linear equations)?

Definition 7 The inverse of a square matrix $(n \times n)$ A is a matrix M such that the identity matrix I

$$\begin{aligned} I &= AM \\ &= MA. \end{aligned}$$

We can set $M = A^{-1}$ to denote the inverse matrix. That is, the inverse matrix A^{-1} reverses the transformation that A performed.

We can use this inverse matrix to find $\vec{x}$.

$$\begin{aligned} A\vec{x} &= \vec{v} \\ A^{-1}A\vec{x} &= A^{-1}\vec{v} \\ \vec{x} &= A^{-1}\vec{v}. \end{aligned}$$

3.3.2 Computing Inverse Matrix

To compute the inverse matrix we use the following formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

where $\operatorname{adj}(A)$ is the adjoint matrix of A. We already know how to calculate the determinant, and to calculate the adjoint matrix we perform the following steps. We will start with matrix

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}.$$

1. Calculate the matrix of minors.
   We calculate the smaller 2×2 determinants just like we did with the cross product and place the results in another matrix. Thus, we have the following:

$$\begin{bmatrix} 3 & -1 & -2 \\ 3 & 1 & -1 \\ 3 & 4 & 2 \end{bmatrix}.$$

2. Flip the signs in a checkerboard pattern. This will give us the following:

$$\begin{bmatrix} 3 & 1 & -2 \\ -3 & 1 & 1 \\ 3 & -4 & 2 \end{bmatrix}.$$

3. Transpose. We switch rows and columns:

$$\begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}.$$

From here we can just calculate the determinant of A, plug in the adjoint of A and we have the inverse of A.

Section 4

Lecture 4 – Square Systems and Equations of Planes

Subsection 4.1

Equation of a Line

Before we get into forming the equation of a plane, let's first review the equation of a line and use the intuition gaine from that to inform the information we need for a plane.

For an equation of a line we need two pieces of information: the slope of our line and a point on our line. With these two pieces of information we can determine where our line will be at a given x or y coordinate. Recall the famous formula

$$y = mx + b$$

where m is the slope, b is the the y-intercept and y and x are the coordinates of an arbitrary point on our line. This equation uses the slope of the line m, a point $(0, b)$ on the line allowing us to determine the y coordinate our line will take given an x coordinate.

The other equation of a line you might have come across is

$$y - y_0 = m(x - x_0).$$

This equation gives us a lot more flexibility; it allows us to define a line not just by the point on the y-intercept, but rather by any point on the line. This equation works because it uses the slope (which tells us how much the line will move for a change in x) and then multiplies that by the actual change in x $(x - x_0)$.

Subsection 4.2

Vector Equation of a Line

When describing a line in 3D space we need a point on the line and a direction that the line takes. Let's say that we have a known point on the line $P_0(x_0, y_0, z_0)$. Well, this point can also be described by the vector $\overrightarrow{OP_0}$ where O is the origin. Let's let $\vec{r}_0 = \overrightarrow{OP_0}$. Now that we have a point on the line ($\vec{r}_0$) we need a direction for the line. We can describe this direction in terms of a vector $\vec{v}$. Since we might want say that another generic point $P(x, y, z)$ is also on the line, we don't quite have enought information; we need to scale $\vec{v}$ so that it reaches this new point $P(x, y, z)$. We wil let $\vec{r} = \overrightarrow{OP}$. In this way we can define this new point P as the sum of $\vec{r}_0 + t\vec{v}$,

$$\vec{r} = r_0 + t\vec{v}.$$

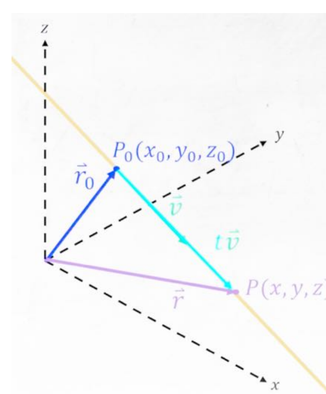


Figure 20. We can define a point on our line by $\vec{r} = \vec{r}_0 + t\vec{v}$.

Subsection 4.3

Equation of a Plane

To define a plane we need a point on the plane and an orientation of the plane. Let $P_0(x_0, y_0, z_0)$ be a known point on our plane. To describe the orientation of our plane we will have a vector $\vec{n}$ normal to the plane. This tells us a point on the plane as well as its orientation. We also want a way to be able to describe all of the points on the plane. Let $P(x, y, z)$ be any point on our plane. Since we know that this point is on the plane, we know that the vector $\overrightarrow{P_0P}$ will be orthogonal to the normal vector $\vec{n}$ This gives us an equation of a plane:

$$\vec{n} \cdot \overrightarrow{P_oP} = 0.$$

Let's look at a specific example. Let $\vec{n} = \langle 4, 5, 6\rangle$, $P_0 = (1, 2, 3)$. Thus

$$\begin{aligned}
\vec{n} \cdot \overrightarrow{P_oP} &= 0 \\
\langle 4, 5, 6\rangle \cdot \langle x - 1, y - 2, z - 3\rangle &= 0 \\
4(x - 1) + 5(y - 2) + 6(z - 3) &= 0 \\
4x - 5 + 5y - 10 + 6z - 18 &= 0 \\
4x + 5y + 6z - 32 &= 0.
\end{aligned}$$

Or mor generally

$$ax + by + cz = d.$$

This is called the component form of an equation of a plane. Note that $\langle a, b, c\rangle$ is a vector normal to the plane and that d can be calculated by plugging in the normal vector for a, b, c and setting x, y, z to the known point on the plane.

Subsection 4.4

Planes, Trains, and Systems of Equations

Recall that in the last chapter we learned how to solve systems of equations by computing the inverse matrix, but now with our knowledge of planes, we can take a more geometric look at systems of equations. Let

$$A = \begin{cases} P_1 = x + 0y + 1z = 1 \\ P_2 = x + 1y + 0z = 2 \\ P_3 = x + 2y + 3z = 3 \end{cases}.$$

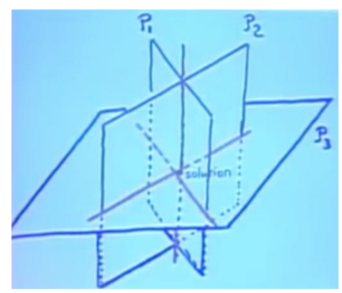


Figure 21. The solution to a 3×3 system of equations is the point at which all three planes intersect*.

The solution to A is the point (x, y, z) that satisfies all three equations. But, theres a much more interesting geometric interpretation of this. Notice how all equations take the form of the equation of a plane. So geometrically, the solution* is the point at which all three planes intersect. Note that we can also think about the solution as first taking the line formed by the intersection of P_1 and P_2 and then finding the point at which that line intersects P_3. But what if the third plane is parallel to the line formed by the intersection of P_1 and P_2? Well then there are two possible cases: either the line lies on the plane of P_3 in which case there are infinitely many solutions or the line does not, in which case there are no solutions. More broadly given a 3×3 system of equations there are 4 possible cases:

- Case 1: Solution is a point,
- Case 2: Solution is a line,
- Case 3: Solution is a plane ($P_1 = P_2 = P_3$),
- Case 4: There is no solution.

4.4.1 Detecting Solution Type

Trying to draw your three planes accurately is impossible, so how can we detect what case we have? Recall that we can find the the inverse of a matrix with the following equation:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A).$$

We are always able to find the adjoint matrix of A, $\text{adj}(A)$, but the there is the posibility that $\det(A) = 0$, and we can't divide by zero. There are two possible cases:

- Case 1: $\det(A) \neq 0$. In this case there is a unique solution
- Case 2: $\det(A) = 0$. Recall that taking the determinant of a 3×3 matrix returns the volume of the parallelpiped formed by three vectors. Thus, a determinant of 0 means that space has been squished onto a plane, a line or a point. There are two subcases: we have a homogeneous system or a general system.
  - **Homogeneous System.** A homogeneous system has the equation $A \cdot \vec{x} = \vec{0}$. That is, all three planes in our system of linear equations pass through the origin. You can find the equation of the line by taking the cross product of any of the two normal vectors (they are all on the same plane.) *Needs Work.*
  - **General System.** There are either infinitley many solutions or no solutions. If you find a solution then there are infinitely many, but if you find conflicting solutions there are none.

Section 5

Lecture 5 – Parametric Equations

We have seen that when two planes intersect, they form a line. Thus, we could describe a line in $\mathbb{R}^3$ by the equation of two intersecting planes. This is not the most useful way to represent a line though; a much better way to think about a line is that path that a moving point takes. These are called parametric equations.

Subsection 5.1

A Parametric Example

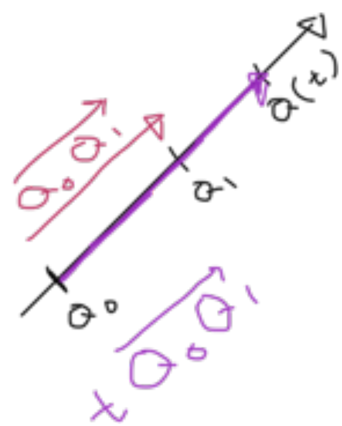


Figure 22. To find $Q(t)$ we just scale $\overrightarrow{Q_0Q_1}$ by t.

Let's say that we have a straight line that travels through the following two points:

$$Q_0 = (-1, 2, 2) \quad \text{and} \quad Q_1 = (1, 3, -1).$$

Let's assume the point is moving at a constant speed and that at $t = 0$ the point is at position Q_0 and at $t = 1$ the point is at position Q_1. What if we want to know the position $Q(t)$ at time t? Well, since the point is moving at a constant speed then

$$\begin{aligned} \overrightarrow{Q_0Q(t)} &= t\overrightarrow{Q_0Q_1} \\ &= t\langle 2, 1, -3\rangle \\ \vec{Q}(t) &= t\langle 2, 1, -3\rangle + \vec{Q}_0. \end{aligned}$$

You can probably see that we have defined this line using the following information:

- A point on the line: $\vec{Q}_0$
- A direction of the line: $\overrightarrow{Q_0Q_1}$.

Thus, to define a line in $\mathbb{R}^3$, we need these two pieces of information. Thus, more generally, the equation for a line $\vec{r}(t)$ is:

$$\vec{r}(t) = \vec{p} + t\vec{v}$$

where $\vec{p}$ is a pont on the line and $\vec{v}$ is in the direction of the line. Note that we can break this equation up into its vector components such that:

$$\vec{r}(t) = \begin{cases} x(t) = p_0 + tv_0 \\ y(t) = p_1 + tv_1 \\ z(t) = p_2 + tv_2 \end{cases}.$$

Subsection 5.2
Applications

5.2.1 Line and Plane Intersection

Let's say that you have the equation of a plane

$$ax + by + cz = d$$

and you want to find the point at which a line passes through that plane. Then you just find t by substituting in your parametric equation of a line for x, y, z as follows:

$$ax(t) + by(t) + cz(t) = d$$

solve for t and then find the point by substituting in the found value of t. If you find that there is no solution then the line is parallel to and outside the plane. If there is a solution no matter the value of t then the line lies inside the plane.

5.2.2 Cycloid

Parametric equations aren't restricted to describing straight lines. More generaly, parametric equations can be used for arbitrary motion in the plane or in space.

How can we use parametric equations to describe a cycloid? As a reminder, a cycloid is formed if we track the path a point on a bicycle tire takes in space as the bike is moving forward (with no slippage of wheels).

Given a wheel of radus a rolling on floor (x-axis) and a point P on the rim of wheel that starts at origin, can we track the trajectory that wheel takes on a plane. That is can we determine the position $(x(t), y(t))$ of the point P?

First of all, is time really the best thing to follow? Well, if we speeded up the rate the bike is travelling the same path would be drawn. Instead, what can follow the position of P as a function of angle θ by which the wheel has rotated. Since we have been using vectors a lot, can vectors help us in this problem?

The point P is the vector $\overrightarrow{OP}$ whose head lies at point P. This is a difficult vector to be found, but we can break it down into simpler vectors such that:

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}.$$

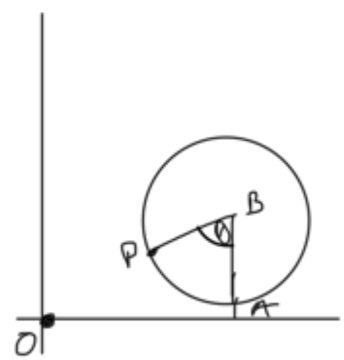
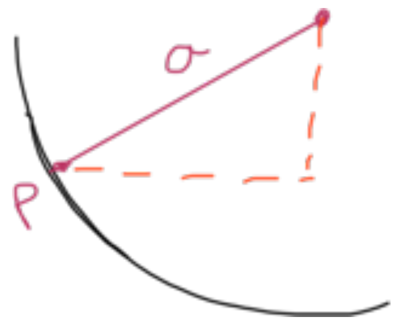


Figure 23. A wheel that draws a cycloid. Point A is the point at which the wheel is touching the floor, B is the center and P is the point we want the coordinates of. Also, θ is the amount the wheel has rotated.

The values of these three vectors are:

- $\overrightarrow{OA}$. The y component will obviously be 0. The x component will be $a\theta$ where a is the radius of the wheel. The reason for this is that the distance of the section of wheel is the same as the distance the wheel has travelled on the road since there is no slip. Thus $\overrightarrow{OA} = \langle a\theta, 0\rangle$.
- $\overrightarrow{AB}$. The vector is pointing straight up, so $\overrightarrow{AB} = \langle 0, a\rangle$.
- $\overrightarrow{BP}$. Creating a triangle:



we see that $\overrightarrow{BP} = \langle -a\sin\theta, -a\cos\theta\rangle$.

Thus, we have

$$\overrightarrow{OP} = \langle a\theta - a\sin\theta, a - a\cos\theta\rangle.$$

Another question is what happens near the bottom as the wheel is tracing our cycloid? First, let's make the problem as simple as possible; let's divide everything by a so that the unit of length is equal to the length of the radius of our wheel. Thus everything will be measured in terms of wheel radius. Therefore $a = 1$ and so we have:

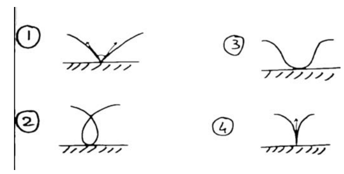


Figure 24. What path does the cycloid take near the bottom?

$$\begin{cases} x(\theta) = \theta - \sin\theta \\ y(\theta) = 1 - \cos\theta. \end{cases}$$

What we can do is try to get an idea of what is happening when θ is close to 0. The easiest way to do this is to just plug in 0 into our equation. Doing this we see that

$$\begin{aligned} \overrightarrow{OP} &= \langle \theta - \sin\theta, 1 - \cos\theta\rangle \\ &= \langle 0 - \sin(0), 1 - \cos(0)\rangle \\ &= \langle 0, 0\rangle. \end{aligned}$$

This isn't really helpfull, can we get a better approximation? Yes, we can use something called taylor expansion. Taylor approximation says that if t is close to zero (small) then

$$f(t) \approx f(0) + tf'(0) + \frac{t^2}{2}f''(0) + \frac{t^3}{6}f'''(0)\cdots.$$

Therefore the taylor series expansion for $x(\theta)$ is:

$$\begin{aligned} &f(a) + \frac{f'(a)}{1}(\theta - a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3\cdot 2}(x-a)^3 \\ =&0 + \frac{1-\cos(0)}{1}(\theta - 0) + \frac{\sin(0)}{2}(\theta - 0)^2 + \frac{\cos(0)}{6}(\theta)^3 \\ =&0 + \frac{1-1}{1}(\theta) + \frac{0}{2}(\theta)^2 + \frac{1}{6}(\theta^3) \\ =&\frac{\theta^3}{6}. \end{aligned}$$

Simalarily the taylor series expansion for $y(\theta)$ is:

$$\begin{aligned}
&f(a)+\frac{f'(a)}{1}(\theta-a)+\frac{f''(a)}{2}(x-a)^2+\frac{f'''(a)}{3\cdot 2}(x-a)^3\\
=&0+\frac{\sin(0)}{1}(\theta)+\frac{\cos(0)}{2}(\theta)^2+\frac{-\sin(0)}{6}(\theta)^3\\
=&0+0+\frac{1}{2}\theta^2-0\\
=&\frac{\theta^2}{2}.
\end{aligned}$$

Therefore, for "small" values of theta, the path that point on wheel takes resembles:

$$\begin{aligned}
x(\theta)&\approx\frac{\theta^3}{6}\\
y(\theta)&\approx\frac{\theta^2}{2}.
\end{aligned}$$

Let's think about what the slope of the path will be as $\theta\to 0$. Well,

$$\frac{\text{rise}}{\text{run}}=\frac{y}{x}\approx\frac{\frac{\theta^2}{2}}{\frac{\theta^3}{6}}=\frac{3}{\theta}.$$

But as $\theta\to 0$, $\frac{3}{\theta}\to\infty$ and so the path has a vertical slope at the origin.

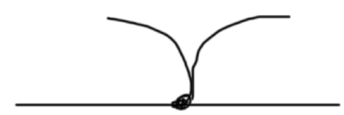


Figure 25. Path near origin has a vertical tangent.

Section 6

Lecture 6 – Velocity and Acceleration; Kepler's Second Law

In lectur 5 we found looked at a cycloid with a wheel radius of 1 and described it using the parametric equation

$$\begin{cases} x(\theta)=\theta-\sin\theta\\ y(\theta)=1-\cos\theta\end{cases}.$$

We could also describe this using the position vector $\vec{r}(t)=\overrightarrow{OP}=\langle t-\sin t, 1-\cos t\rangle$. Note that before we described the position in terms of angle θ but now we are describing it in terms of time t because we say that the wheel is moving at unit speed (time and angle are the same thing). More generally, the position vector describes the position of a moving point and takes the form

$$\vec{r}(t)=\langle x(t), y(t), z(t)\rangle.$$

Subsection 6.1

Velocity Vector

Now that we know the position of a point as a function of time, we might want to know things like speed of the point, acceleration of the point, and direction that point is travelling.

Whereas a cars spedometer will only tell you how fast the car is moving and not in which direction it is moving, the velocity vector will tell you both. In mathematical speak, the speed at which a point is moving is the scalar magnitude of its velocity vector.

The velocity vector

$$\text{Velocity Vector: } \vec{v} = \frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle.$$

6.1.1 Cycloid Example

Returning to our cycloid, we see that the velocity vector is:

$$\vec{v} = \langle 1 - \cos t, \sin t \rangle.$$

What does this this tell us about our point on the wheel at the origin ($t = 0$)? We see that at $t = 0$ $\vec{v} = \vec{0}$! Although it is moving right before and right after, rather counterintuitively, the point is not when $t = 0$.

6.1.2 Speed

As we said before the spped (scalar) is just the magnitude of the velocity vector.

$$\begin{aligned} |\vec{v}| &= \sqrt{(1 - \cos t)^2 + \sin^2 t} \\ &= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\ &= \sqrt{2 - 2\cos t} \end{aligned}$$

We can then determnine when our point is moving its fastest and when is it moving its slowest. When $t = 0$, $|\vec{v}| = 0$ and so this is its slowest. When $t = \pi$, $|\vec{v}| = 2$ and this is its fastest speed. This is when the point is at the top of the wheel. It is moving twice as fast as the wheel because the wheel is moving to the right at unit speed and the wheel is rotating as well so the two effects add up giving you a speed of two.

Subsection 6.2

Acceleration

You might think that acceleration just has to do with change in speed. But actually, lets say you are travelling at 55 mph and make a tight turn (while maintaining speed) you will feel something, this something is acceleration. There is a sideways acceleration at this point. The acceleration vector is

$$\vec{a} = \frac{d\vec{v}}{dt}.$$

6.2.1 Cycloid Example

Our cycloid will have the acceleration vector

$$\vec{a} = \langle \sin t, \cos t.$$

So at $t = 0 : \vec{a} = \langle 0, 1 \rangle$. What this means is that at the origin the acceleration vector is pointing vertically. This is another way to understand that the trajectory at the origin has a vertical tendency.

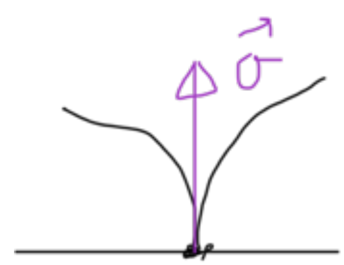


Figure 26. Acceleration of point at $t = 0$ is vertical.

Subsection 6.3

Arg Length

We weirdly enough have s represent the arc length. Thus,

$$s = \text{ distance travelled along trajectory.}$$

How do you relate arc length (s) and time (t)? Well, the rate of change of arc length over time is speed. Thus:

$$\left|\frac{ds}{dt}\right| = \text{ speed } = |\vec{v}|.$$

Note that we have the absolute value of $\frac{ds}{dt}$ because we could be moving backwards.

If you want to know the length of an arch of cycloid is, the only way to really do it is integrate speed from 0 to 2π:

$$\int_0^{2\pi} \sqrt{2 - 2\cos t}\,dt.$$

This is really difficult to integrate, but we will learn a cool trick soon.

Subsection 6.4

Unit Tangent Vector

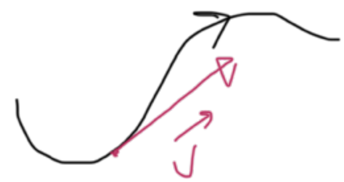
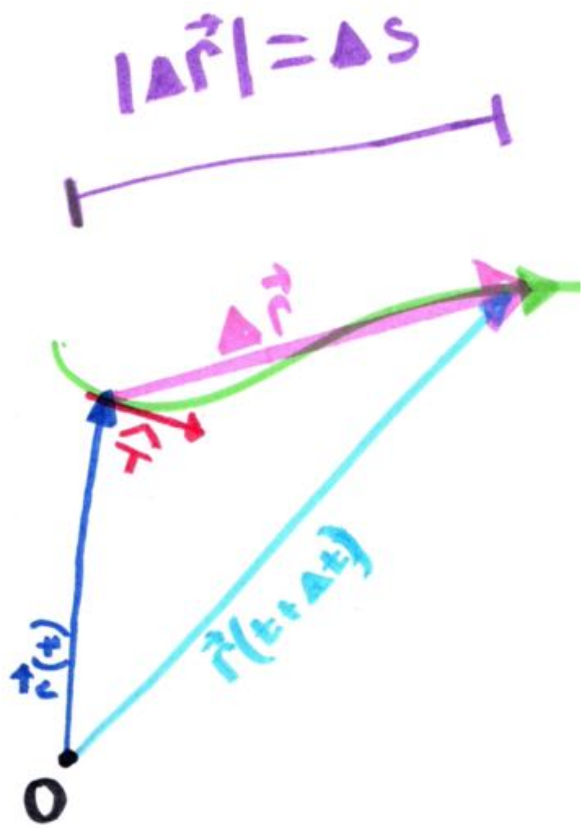


Figure 27. The velocity vector is tangent to path.

When you are moving along some tragectory, the velocity vector is tangent to the path. Therefore, to get the unit tangent vector, you just have to divide $\vec{v}$ by the magnitude of v:

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|}.$$

It might seem like speed, velocity, position and direction are all related in some way. This is the case, but how are they related? A Picture might help.



Notice that right now $\Delta\vec{r}$ and $\hat{T}$ are not in the same direction, but as $\lim_{\Delta t \to 0}$ the vector $\Delta\vec{r}$ will be in the same direction as $\hat{T}$ scaled by the scalar Δs. Also note that similarly as $\lim_{\Delta t \to 0}$ the unit tangent vector $\hat{T} = \frac{\Delta \vec{r}}{\Delta s}$. Also note that

$$\text{Velocity has } \begin{cases} \text{direction: tangent to traj. } \hat{T} \\ \text{length equal to speed. } \frac{ds}{dt} \end{cases}$$

Therefore we have velocity vector equal to

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \hat{T}\frac{ds}{dt}$$

where $\frac{ds}{dt} = |\vec{v}|$ and $\frac{d\vec{r}}{ds} = \hat{T}$.

Subsection 6.5

Kepler's Second Law (1609)

One question you might be wondering is "why is using vectors helpful when most of what we have been doing we can do just as easily using coordinates"? One place where vectors come in handy is with Kepler's second law of celestial mechanics. Kepler was observing the motion of the planets and wanted to find a way to explain the path that the planets took. Keplers three laws state:

1. Planets move in elliptical orbits with the Sun as a focus
2. A planet's motion is in a plane and covers the same area of space in the same amount of time no matter where it is in its orbit. Another way to say this is that the motion of planets is in a plane and the area is swept out by the line from sun to planet at a constant rate.
3. A planet's orbital period is proportional to the size of its orbit (its semi-major axis).

Figure 28. Due to Kepler's second law, the planet will move faster the closer it is to the sun and move slower the farther it is so that it covers the same amount of area per unit of time.

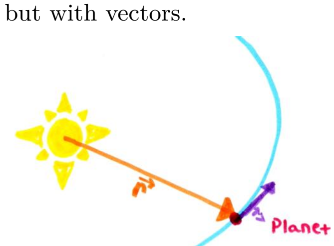
The second law is interesting because it tells you that once you know what a planet's orbit will look like you know how fast that planet will at a given point along its path; the closer it is to the sun the faster it will move and the farther it is away the slower it will move so that the same amount of area is covered.

Kepler probably came about this prediction by making a lot of observations and plotting points to determine what was true and what wasn't. Newton later explained this using his formula for gravitational attraction. How can we look at Kepler's second law in terms of vectors?

Let's show that Kepler's second law holds, one piece at a time.

Figure 29. Planetary motion, but with vectors.

1. The first part says that as a planet moves it creates a constant amount of area per time.
   Using vectors, how can we calculate the area? Well since for small changes in time, the swept area resembles a triangle and the area of a triangle is half that of a parallelogram and the area of a parallelogram is given by the magnitude of the cross product. Thus:
   $$\text{Area } \approx \frac{1}{2}|\vec{r} \times \Delta\vec{r}| \approx \frac{1}{2}|\vec{r} \times \Delta\vec{r}|\Delta t$$
   over time Δt. Because the area swept is only proportional to time, and nothing like how far away the planet is, Kepler's second law states that $|\vec{r} \times \vec{v}| =$ constant.
2. A planet stays on the same plane while orbiting the sun.
   What does this fact mean in terms of vectors?



Well we can use vectors to further understand Kepler's second law.

$$
\begin{aligned}
\text{Kepler's } 2^{\text{nd}} \text{ law} &\Leftrightarrow \vec{r} \times \vec{v} = \text{ constant vector} \\
&\Leftrightarrow \frac{d}{dt}(\vec{r} \times \vec{v}) = \vec{0} \\
&\Leftrightarrow \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = 0 \\
&\Leftrightarrow \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = 0 \\
&\Leftrightarrow 0 + \vec{r} \times \vec{a} = 0 \\
&\Leftrightarrow \vec{a} \parallel \vec{r} \\
&\Leftrightarrow \text{ gravitational force is } \parallel \vec{r}.
\end{aligned}
$$

Section 8

Lecture 8 – Level Curves, Partial Derivatives, Tangent Plane

Subsection 8.1

Functions

First, how should we think about functions? Recall functions of 1 variable, this means you have some quantity that depends upon one variable. For example the function $f(x) = \sin x$, f depends upon x. To graph this we need one axis for the input (x axis) and one axis for the output (y axis).

A function of two variables is very similar, but in this case the function f depends upon two variables (x, y). To graph this we need two axis for the input (x, y axis) and one axis for the output z axis. Let's graph the function

$$f(x, y) = 1 - x^2 - y^2.$$

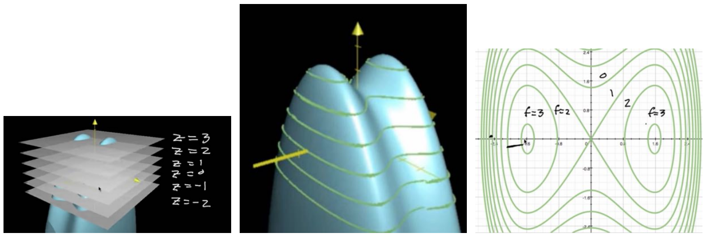
Let's first graph the function in the yz-plane. To do this we let $x = 0$. Therfore, we have $z = 1 - y^2$. Similarily setting $y = 0$ we have $z = 1 - x^2$. But what about the xy-plane? We can figure out how our surface interacts with the xy-plane by setting $z = 0$. Thus, we have $x^2 + y^2 = 1$. This is the unit circle.

As you can see, it is fairly difficult to draw a 3d object on a 2d plane. Instead we can creat a contour plot. Given the function $f(x, y) = z$ a contour line is a contour line $f(x, y)$

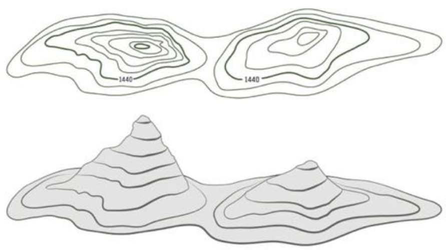
Subsection 8.2

Contour Plots and Level Curves

Given a function with two inputs and one output, we could graph this in $\mathbb{R}^3$, but that is not always the most convenient. It is really hard to draw and so a computer is needed, and even then it is not always clear what is hapenning. So one way to draw this three dimensional graph in two dimensions is to take a slice of the graph along the $z = z_0$ plane where z_0 is some constant. Thus, if find multiple contour curves, you can draw those on a 2d graph, and if the curves are closer together that means that the graph is steeper at that point.



You might have encountered a contour plot if you've looked at a topographical map of the mountains



Or, mathematically speaking:

$$C_{z_0}(f) = \{(x, y, z_0) : f(x, y) = z_0 \text{ for } (x, y) \in D\}.$$

To be a little more clear, the contour curve is the curve in three dimensional space (the green lines in image 2), and that squishing all those together onto the xy plane we have a graph of level curves. The level curve is the contour curve placed onto the xy plane, or mathematically

$$L_{z_0}(f) = \{(x, y) \in D : f(x, y) = z_0.\}$$

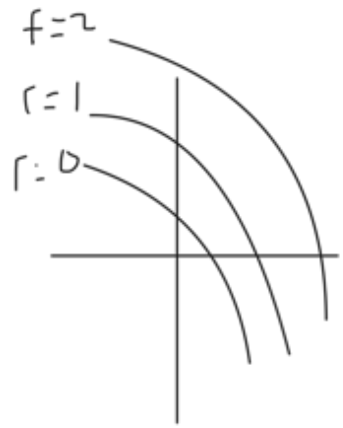


Figure 31. See that as x increases f increases and as y increases f increases.

Subsection 8.3

Partial Derivatives

8.3.1 Derivatives of Single Variable Functions

Given a function of one variable, $f(x)$, it's derivative can be calulated as

$$\begin{aligned} f'(x) &= \frac{df}{dx} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \end{aligned}$$

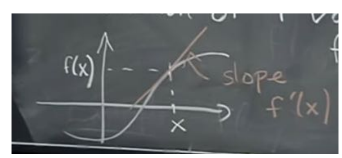


Figure 32. Derivative of $f(x)$ is the slope of tangent line.

We can also use this derivative to approximate values:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

This is just the equation of our tangent line using the point $(x_0, f(x_0)$ and the slope $f'(x_0)$.

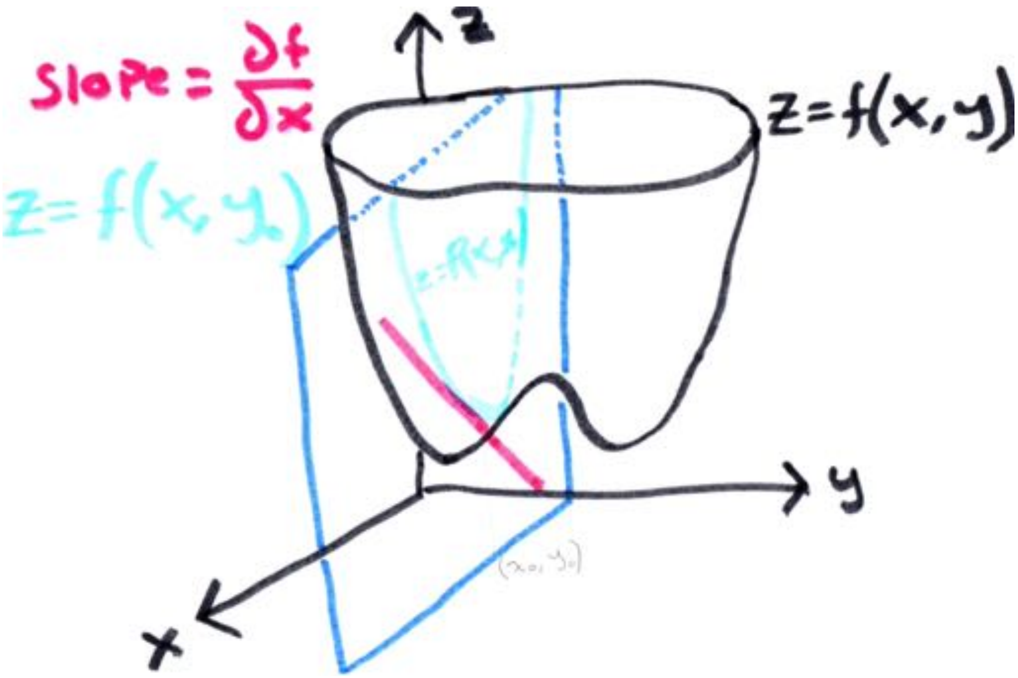
8.3.2 Derivatives of Multi Variable Functions

To see how a function $f(x, y)$ changes as we take a slight nudge in the x direction, or a slight nudge in the y direction we can set y or x constant and calculate the partial derivatives:

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}.$$

Geometrically this works by taking a slice of our graph on the $y = y_0$ plane to get $\frac{\partial f}{\partial x}$ and seeing how a change in x effects f. The same applies for $\frac{\partial f}{\partial y}$.



How do we compute a partial derivative? Well, if we were taking $\frac{\partial f}{\partial x}$ we would let y be a constant and we would take derivative as normal. Similarily if we were to compute $\frac{\partial f}{\partial y}$ we would let x be a constant and take the derivative as normal.

8.3.3 Approximation Formula

With partial derivatives we saw what happens if we vary x and keep y a constant, or we vary y and keep x a constant, but how does $f(x, y)$ change as we vary both x and y? Let's first look at a simpler problem. In one variable calculus, we can approximate $f(x)$ for points near $(x_0, f(x_0))$ with the tangent line passing through $(x_0, f(x_0))$. This line has the formula:

$$y - f(x_0) = \frac{d}{dx}(f(x_0))(x - x_0)$$
$$y = \frac{d}{dx}(f(x_0))(x - x_0) + f(x_0).$$

With two inputs, we take a similar approach; instead of finding the tangent line approximation at a point, we will find the tangent plane approximation at a point. Recall, that to define a plane we need two things: a point on the plane, and a direction of the plane. Assume that we have the function $z = f(x, y)$ and we want to find the tangent plane at the point $()x_0, y_0, f(x_0, y_0))$. The point on the plane we already have, this is the point

$(x_0, y_0, f(x_0, y_0))$. The direction of the plane will be a vector normal to the plane? But how can we get this? Well, we will do a little trick. Let's create a three input function from our two input function. Then let's find the gradient vector of that function at the point $(x_0, y_0, f(x_0, y_0))$. This will give us the normal vector to our plane at that point.

Let $F(x, y, z) = f(x, y) - z = 0$. The gradient vector of $F(x, y, z)$ is

$$\nabla F(x_0, y_0, z_0) = \langle \frac{\partial}{\partial x} f(x_0, y_0), \frac{\partial}{\partial y} f(x_0, y_0), -1 \rangle.$$

Therefore, since the equation of a plane is $\vec{n} \cdot \overrightarrow{P_0P}$ we have:

$$\begin{aligned}
0 &= \vec{n} \cdot \overrightarrow{P_0P} \\
&= \langle \frac{\partial}{\partial x} f(x_0, y_0), \frac{\partial}{\partial y} f(x_0, y_0), -1 \rangle \cdot \overrightarrow{\langle x_0, y_0, z_0 \rangle \cdot \langle x, y, z \rangle} \\
&= \langle \frac{\partial}{\partial x} f(x_0, y_0), \frac{\partial}{\partial y} f(x_0, y_0), -1 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\
&= \frac{\partial}{\partial x} f(x_0, y_0)(x - x_0) + \frac{\partial}{\partial y} f(x_0, y_0)(y - y_0) - (z - z_0) \\
z &= \frac{\partial}{\partial x} f(x_0, y_0)(x - x_0) + \frac{\partial}{\partial y} f(x_0, y_0)(y - y_0) + z_0 \\
z &= \frac{\partial}{\partial x} f(x_0, y_0)(x - x_0) + \frac{\partial}{\partial y} f(x_0, y_0)(y - y_0) + f(x_0, y_0).
\end{aligned}$$

We can use this to approximate values of $f(x, y)$ close to some point (x_0, y_0):

$$f(x, y) \approx \frac{\partial}{\partial x} f(x_0, y_0)(x - x_0) + \frac{\partial}{\partial y} f(x_0, y_0)(y - y_0) + f(x_0, y_0).$$

Section 9

Lecture 9: Max-Min Problems and Least Squares

A very important application of calculus is in maximizing and minimizing functions. You may want to maximize profit, minimize cost, or minimize distance travelled. Recall that if you have a function of one variable, to find its minimum or its maximum you first need to find the point(s) where the derivative of that function is zero. When dealing with a multivariable function, the approach is very similar. Given some function $f(x, y) = z$, the local minimums or local maximums will be when both

$$f_x = 0 \quad \text{and} \quad f_y = 0.$$

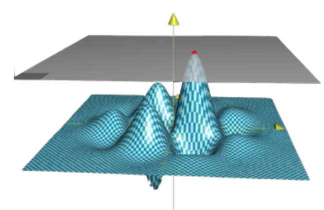


Figure 33. If we are at a local minimum or maximum then the tangent plane is flat.

Why is this? This is because if only one of the partial derivatives was zero then you could still move along some path such that the function would either increase or decrease. Basically, by having both partial derivatives equal to zero we are ensuring that the tangent plane is flat.

Definition 8 We say that (x_0, y_0) is a critical point of f if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

For example, let's say we want to minimize or maximize the function $f(x, y) = x^2 - 2xy + 3y^2 + 2x - 2y$. Taing the partial derivatives we see that

$$\begin{cases} f_x = 2x - 2y + 2 = 0 \\ f_y = -2x + 6y - 2 = 0 \end{cases}$$

Thus we have two linear equations we want to solve for. One thing to note is that it is often helpful to add the two equations together if that will help us eliminate a variable. Doing this we see that $4y = 0$ and thus $y = 0$. Plugging $y = 0$ back in we see that $x = -1$. Thus our critical point is at $(-1, 0)$.

The question remains though, is this a local minimum or a local maximum? Looking back at our original equation, we see that we can actually simplify it:

$$\begin{aligned} f(x,y) &= x^2 - 2xy + 3y^2 + 2x - 2y \\ &= (x-y)^2 + 2y^2 + 2x - 2y \\ &= ((x-y)+1)^2 + 2y^2 - 1. \end{aligned}$$

Thus we have the sum of two squares minus one. The two squares will never be negative, and since the critical point causes them to zero out, that is the smallest value, and thus a minimum.

Subsection 9.1
Saddle Points

Note that there are actually three possible options for this critical point. It is either: a local minimum, a local maximum, or a saddle point. A saddle point is formed when looking at the graph from one direction it looks like a local minimum, and from another it looks like a local maximum. An intuition for why this can happen is that the partial derivative f_x might be concave up indicating that it is a local minimum, whereas the partial derivative f_y might be concave down indicating that it is a local maximum. Thuse these two opposing forces create a saddle point. Note, that this isn't the complete picture, but it gives a good idea of the intuition behind saddle points. The question arises though, is this a maximum or a minimum?

Subsection 9.2
Least-Squares Interpolation

Let's say we are recording annual sales data and we notice that sales have been going up. Given this data, we want to find the line that best approximates our data. What does it mean for this line to best fit our data?

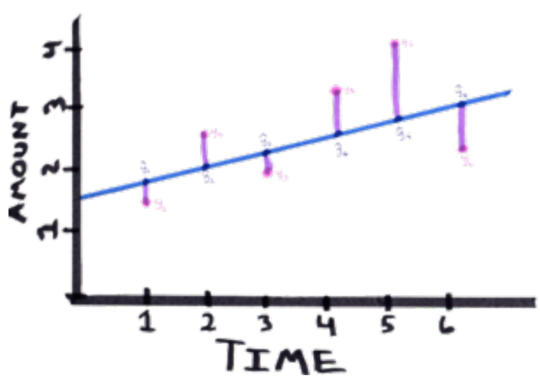


Figure 34. We can see that our best fit line will be the one such that the sum of the lengths of purple lines is minimized.

Looking at the graph, we see that the best fit line will be that such that the sum of the lengths of the purple lines is minimized. How can we express this mathematically?

We want to minimize the distance

$$\sum_{i=1}^{n}(y_i - \hat{y}_i)$$

where y_i is the y value of our data point when $x = x_i$ and $\hat{y}_i$ is the y value of our line when $x = x_i$. But an issue arises, $(y_i - \hat{y}_i)$ could be either negative or positive and so the sum might not be accurate. We could take the absolute value, but that is harder to optimize, so we will take the square of $(y_i - \hat{y}_i)$. Thus, we want to find the line such that

$$\sum_{i=1}^{n}(y_i - \hat{y}_i)^2.$$

is minimized. Recall that a line takes the form $y = ax + b$. Thus changing the values of a and b will change our line. Thus we can replace $\hat{y}_i = mx_i + b$ and so we want to minimize the multivariable function $D(a, b)$ where

$$D(a,b) = \sum_{i=1}^{n}(y_i - (ax_i + b))^2.$$

9.2.1 Linear Best Fit

To minimize this function, we must first find a critical point. Thus, we need to find the point where the partial derivatives are zero. Calculating the partial derivatives we have:

$$\frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2 \cdot (y_i - (ax_i + b)) \cdot (-x_i) = 0$$

$$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2\,(y_i - (ax_i + b)) \cdot (-1) = 0.$$

Dividing by two and expanding out we get:

$$\sum_{i=1} \left(x_i^2 a + x_i b - x_i y_i\right) = 0$$
$$\sum_{i=1}^{n} (x_i a + b - y_i) = 0$$

Rewriting this we get

$$\left(\sum_{i=1}^{n} x_i^2\right) a + \left(\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} x_i y_i$$
$$\left(\sum_{i=1}^{n} x_i\right) a + nb = \sum_{i=1}^{n} y_i.$$

Thus we have a 2×2 linear system. This allows us to solve for a and b. We can show that this is a minimum by the second partial derivative test.

9.2.2 Exponential Best Fit

Moores law states that the number of transitors will double every 2 years (or 18 months depending). Given that we have data points stating the number of transistors in a chip for each year, can we create a line that best fits this exponential trend? Well we could use $y = ce^{ax}$, but that would make our best fit line impossibly difficult to solve. Instead, we can plot the log of y as a function of x, and thus we get a linear relation. Thus we can get the linear best fit by

$$\ln(y) = \ln(c) + ax.$$

Thus all we do is look for the best straight line fit of the log of y.

9.2.3 Quadratic Best Fit

Recall the quadratic law: $y = ax^2 + bx + c$. Thus, to find our best fit line we need to minimize $D(a, b, c)$ where

$$D(a, b, c) = \sum_{i=1}^{n} \left(y_i - \left(ax_i^2 + bx_i + c\right)\right)^2 .$$

Section 10

Lecture 16: Double Integrals

Subsection 10.1

Single Integral Review

Remember that when you have a function of one variable $(f(x))$ and you take its integral $\int_a^b f(x)dx$, the integral corresponds to the area below the graph of $f(x)$ between the interval $[a, b]$. How is this done? Well, we approximate the area by creating rectangles so that the top of the rectangle touches the graph of $y = f(x)$. In that way we can approximate the area under the graph.

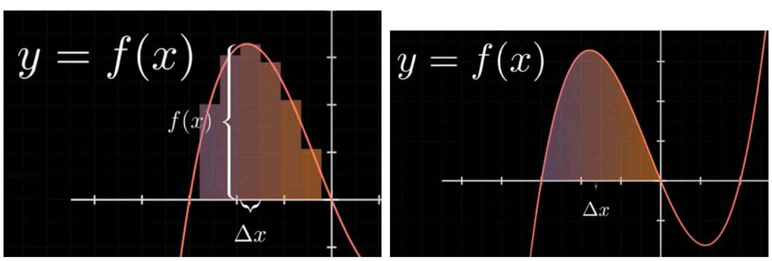


Figure 35. Notice that as we create smaller and smaller rectangles we get better and better approximations of the area, until we get the exact area under our function.

A quick reminder about notation. The integral symbol $\int$ is just an elongated s. This is to indicate that we are just taking the sum of these rectangles. Also Δx changes to dx to indicate that the change in x is finite and very very small. Thus, the integral is just an infinite sum of base times height.

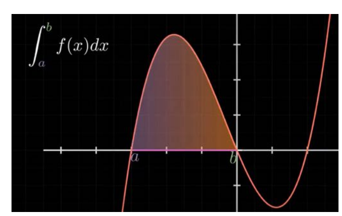


Figure 36. Integral calculates area under graph.

Subsection 10.2

Double Integral Intuition

Now let's say that we have a function of two two variables $z = f(x, y)$. This function creates a 3d surface. Similar to how the single integral finds the area under a graph between two points, the double integral calculates the volume under a surface over some region. Since this region now lies on the xy plane, what shape does it take? Well, it could really take any shape. To find the volume under this surface, over some region D, we can use a very similar principle to single variable calculus. We create a lot of rectangle prisms whose height is $f(x, y)$, width is dx and length is dy. We then sum up all the prisms, and as dx and dy get smaller and smaller we get closer and closer to the actual volume. Thus, the volume of the surface $f(x, y)$ over the region D is the double integral

$$\iint_D = f(x, y)dA$$

where ΔA is the area of the base of each small prism. Thus to calculate the volume we take the sum of the base of the prism, times its height for each prism:

$$\text{Volum } = \sum_i V_i = \sum_i f(x_i, y_i) \cdot \Delta A.$$

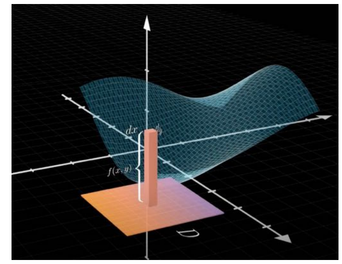


Figure 37. We can use rectangular prisms to approximate volume under a surface.

To get the double integral we take the limit as $\Delta A \to 0$. Just like in single variable calculus where we don't actually chop up our function into small rectangles and count their sums, there is a better way in multivariable calculus.

To compute $\iint_D f(x, y)dA$ we will scan a plane parallel to yz plane and take slices of our surface. That is will will take some plane $x = x_0$ giving us a one variable function $f(x_0, y)$ (x_0 is a constant) for which we can compute the area under.

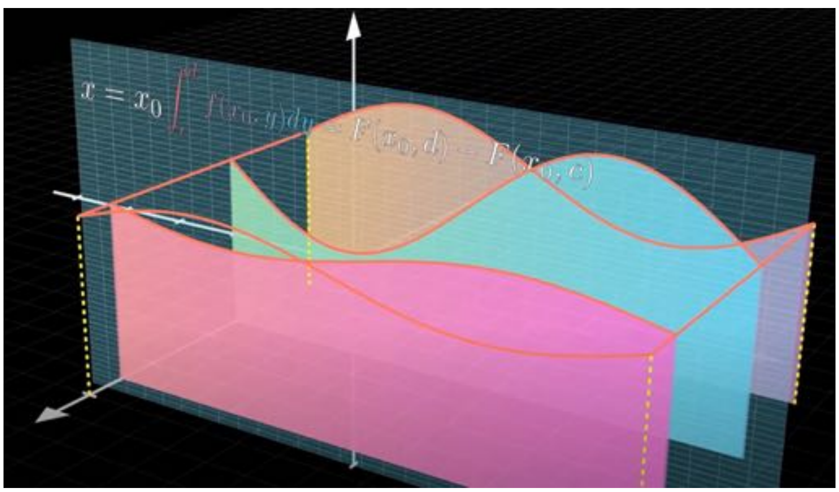


Figure 38. Area under surface for multiple values of x_0.

Then, we will add up the areas for all values of x_0 giving us our volume.

Let $S(x)$ = area of slice by plane parallel to yz plane. Then

$$\text{Volume } = \int S(x)dx.$$

What should the range of x be? Well, our range is from the very smallest value of x that we have all the way to the very largest value of x in our region. Thus we have

$$\text{Volume } = \int_{x_{\min}}^{x_{\max}} S(x)dx.$$

But what is the value of $S(x)$? Well, it is an itegral itself: $S(x) = \int f(x, y)dy$ where x is some constant and y is the variable of integration. What is the range of y? Well, it is the smallest value of y to the largest value of y. But we actually have to be very careful. Often times these values will depend upon x themselves.

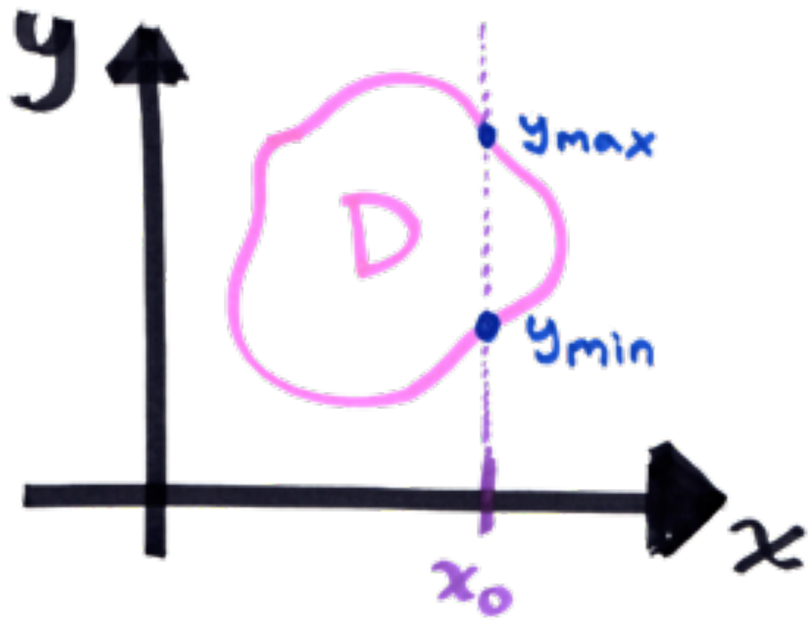


Figure 39. Let D be the region that we want to calculate the volume of our surface. Notice that when we let $x = x_0$ the values $y_{\min}$ and $y_{\max}$ depend upon the constant x_0.

Thus we have:

$$\iint_R f(x,y)dA = \int_{x_{\min}}^{x_{\max}} \left[\int_{y_{\max}(x)}^{y_{\max}(x)} f(x,y)dy \right] dx.$$

This is called an iterated integral because iterate twice the process of taking an integral. The key point to remember though is that we first write the integral for the independent variable (in this case $\int_{x_{\min}}^{x_{\max}}$) and then that of the dependent variable (in this case $\int_{y_{\max}(x)}^{y_{\max}(x)}$).

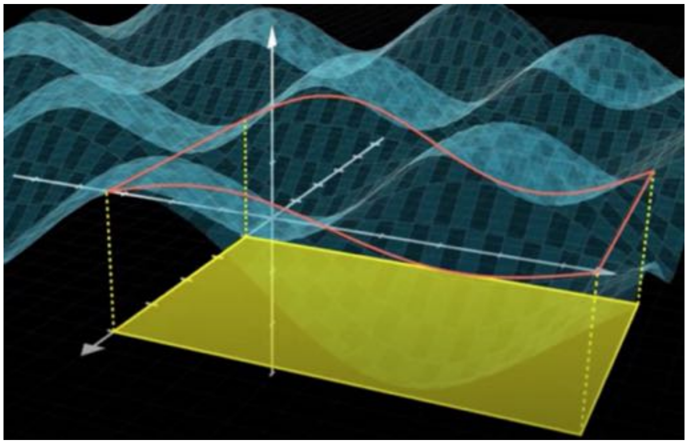
Subsection 10.3

Double Integration Domain

Recall that in single variable calculus, when taking the integral the domain was just a straight line; we calculated the integral starting from some point along the x axis, and finished along another point along the x axis. Thus the domain that we took the integral was $[a, b]$. But things get a little bit more complicated with double integrals. Our domain could really be any shape on the xy plane. The domain could be a simple rectangle, it could be the area between two one variable functions, or it could be a circle or an ellipse (thus a change to polar coordinates).

10.3.1 Rectangular Domain

Let's start by looking at a picture of the volume we want to calculate.



To calculate this volume, we will can break the process into two steps.

1. We will let x be some constant x_0. That is, we will take a slice of our graph along a plane parallel to the xy plane. This way we take a 2d section of our graph, allowing us to then calculate the single variable integral.

2. The second step is to sum up all the single variable integrals for all x values. In this way we can calculate the total volume.

In this way a double integral is an integral of integrals.

$$\int_a^b \left(\int_c^d f(x,y)dy \right) dx.$$

Furthermore, the result is unchanged if we swap the order of integration.

Lets take a look at a quick example. Suppose we want to calculate the double integral

$$\iint_D \cos(x+2y)dxdy$$

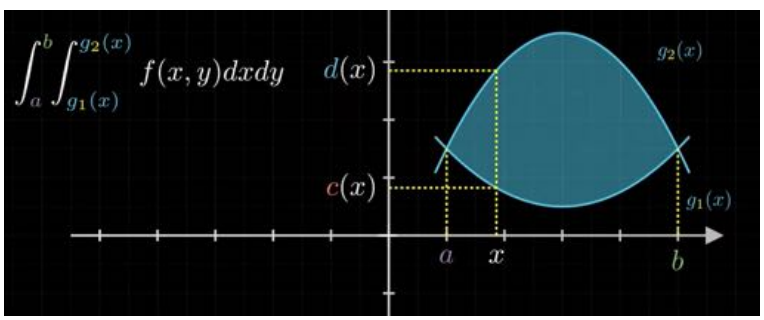
where D is the domain $D := \left\{(x,y) \in [0,\pi] \times \left[\frac{\pi}{2}, \pi\right]\right\}$. That notation may look funny, but all it is saying is the set of coordinates (x,y) such that they lie in the rectangle created such that their x coordinates are between $[0,\pi]$ and whose y coordinates are between $[\frac{\pi}{2}, \pi]$.

First we will integrate over the y variable, considering the x as a constant, and then we will sum up all the integrals for each value of x:

$$\begin{aligned}
\int_0^\pi \left(\int_{\frac{\pi}{2}}^\pi \cos(x+2y)dy\right) dx &= \int_0^\pi \left[\frac{1}{2}\sin(x+2y)\right]_{\frac{\pi}{2}}^\pi dx \\
&= \int_0^\pi \frac{1}{2}\left(\sin(x+2\pi) - \sin\left(x+2\frac{\pi}{2}\right)\right) dx \\
&= \int_0^\pi \frac{1}{2}(\sin(x+2\pi) - \sin(x+\pi))dx \\
&= \int_0^\pi \frac{1}{2}(\sin x - \sin(x+\pi))dx \\
&= \int_0^\pi \frac{1}{2}(\sin x - \sin(x+\pi))dx \\
&= \int_0^\pi \frac{1}{2}(\sin x + \sin x)dx \\
&= \int_0^\pi \sin x dx \\
&= -\cos(\pi) - (-\cos(0)) \\
&= 2.
\end{aligned}$$

10.3.2 Bounded Domain

What happens then if our domain is bounded by two functions? Well let's say our range is determined by the boundry created by functions $g_1(x)$ and $g_2(x)$ for x values between a, b.



We see that x can be chosen freely, but that y is dependent upon x. That is the range of y values depends upon which x coordinate we are in. How does this affect our search for the volume below a surface? Now, when we take a slice of our function and compute the first integral, the areas will depend not only on the height determined by our two variable function $f(x,y)$, but also by the width determined by the functions $g_1(x)$ and $g_2(x)$. Because the y range depends upon x we must integrate the y range first and then the x range to obtain our volume.
Let's say we want to find the volume below the surface xy for the domain D, where

$D := \{(x,y) \in [0,1] \times [x^2, x]\}$. Thus, we have

$$\begin{aligned}
\iint_D xydxdy &= \int_0^1 \int_{x^2}^x xydxdy \\
&= \int_0^1 \left(\int_{x^2}^x xydy \right) dx \\
&= \int_0^1 x \left(\int_{x^2}^x ydy \right) dx \\
&= \int_0^1 x \left[\frac{y^2}{2} \right]_{x^2}^x dx \\
&= \int_0^1 x \left(\frac{(x)^2}{2} - \frac{\left(x^2\right)^2}{2} \right) dx \\
&= \int_0^1 x \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx \\
&= \int_0^1 \left(\frac{x^3}{2} - \frac{x^5}{2} \right) dx \\
&= \frac{1^4}{8} - \frac{0^4}{8} - \frac{1^6}{12} + \frac{0^6}{12} \\
&= \frac{1}{8} - \frac{1}{12} \\
&= \frac{1}{24}.
\end{aligned}$$

SUBSECTION 10.4

Examples

10.4.1 Example 1

Let $z = f(x,y) = 1 - x^2 - y^2$. Find the volume under the surface between the region $0 \leq x \leq 1$ and $0 \leq y \leq 1$. First let's think about our region. We see that x and y are actually independent of eachother and our region is a square of width 1. Thus, it doesn't actually matter what is our inner and what is our outer integral. Thus, let's set up our integral as:

$$\int_0^1 \int_0^1 f(x,y)dydx.$$

Calculating the inner integral:

$$\begin{aligned}
\int_0^1 f(x,y)dy &= \int_0^1 1 - x^2 - y^2 dy \\
&= \left(y - yx^2 - \frac{1}{3}y^3 \right)\Big|_0^1 \\
&= \left(1 - x^2 - \frac{1}{3} \right) - (0).
\end{aligned}$$

We can now calculate our outer integral:

$$\begin{aligned}\int_0^1 \left(1-x^2-\frac{1}{3}\right)dx &= \left(x-\frac{1}{3}x^3-\frac{1}{3}x\right)\Big|_0^1\\ &= \left(1-\frac{1}{3}-\frac{1}{3}\right)-(0)\\ &= 1-\frac{2}{3}\\ &= \frac{1}{3}.\end{aligned}$$

10.4.2 Example 2

Let's find the volume under the same surface as before ($f(x,y)=1-x^2-y^2$) but this time let D be the quarter disk such that

$$\begin{aligned}x^2+y^2 &\le 1,\\ x &\ge 0,\\ y &\ge 0.\end{aligned}$$

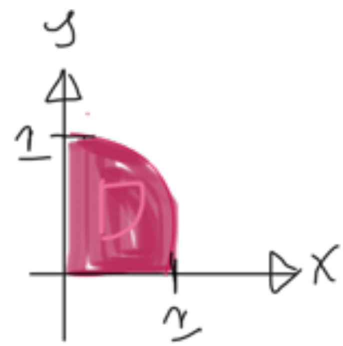


Figure 40. Region D for which we will calculate volume.

If our scanning plane is parallel to the yz plane (x is some constant x_0) then what will our bounds be? Well we see that y will range from 0 to $\sqrt{1-x^2}$ and that x will range from 0 to 1. Thus, we have our integral

$$\int_0^1\int_0^{\sqrt{1-x^2}} \left(1-x^2-y^2\right)dydx.$$

First we calculate the inner integral:

$$\begin{aligned}\int_0^{\sqrt{1-x^2}} \left(1-x^2-y^2\right)dy &= \left(y-yx^2-\frac{1}{3}y^3\right)\Big|_0^{\sqrt{1-x^2}}\\ &= \left(\sqrt{1-x^2}-\sqrt{1-x^2}\left(x^2\right)-\frac{1}{3}\left(\sqrt{1-x^2}\right)^3\right)\\ &= \sqrt{1-x^2}-\sqrt{1-x^2}\left(x^2\right)-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}\\ &= \frac{2}{3}\left(1-x^2\right)^{\frac{3}{2}}.\end{aligned}$$

Calculating the outer integral:

$$\int_0^1 \frac{2}{3}\left(1-x^2\right)^{\frac{3}{2}}dx = \frac{\pi}{8}.$$

I left out the steps because they get rather nasty and you need to use the double angle formula and a table of integrals. But what this means is that this isn't actually the best way to solve this problem. The better way would be via a change of coordinates (to polar coordinates) which we will learn in the next lecture.

10.4.3 Change of order of Integration

Calculate the double integral

$$\int_0^1\int_x^{\sqrt{x}} \frac{e^y}{y}dydx.$$

This is an interesting example, because it's not really possible to integrate

$$\int \frac{e^y}{y} dy.$$

Thus, to solve it, we can make our life easier by changing the order of integration. To help yourself visualize how our ranges of integration change, we can flip the graph on its side, so that y is in place of the x axis and x is in place of the y axis. In this way we can see that as y goes from 0 to 1, x will range from y^2 to y, giving us our new integral

$$\int_0^1 \int_{y^2}^{y} \frac{e^y}{y} dx dy.$$

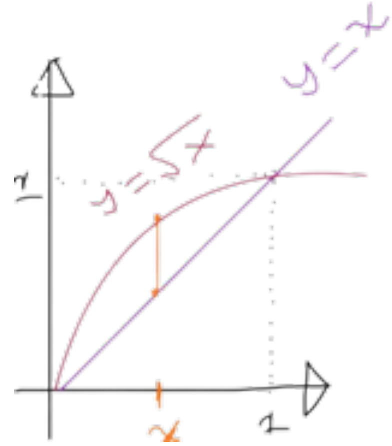


Figure 41. We see that for any given x between 0 and 1, y will range from $y = x$ to $y = \sqrt{x}$.

First calculating the inner integral:

$$\begin{aligned}\int_{y^2}^{y} \frac{e^y}{y} dx &= \frac{e^y}{y} x \Big|_{x=y^2}^{y} \\ &= e^y - e^y y.\end{aligned}$$

Now we can calculate the outer integral:

$$\int_0^1 e^y - e^y y = -ye^y + 2e^y.$$

Note that $\frac{d}{dy} - ye^y = -e^y - e^y y$, thus all we need to do is add $2e^y$.

Section 11

Lecture 17: Double Integrals in Polar Coordinates

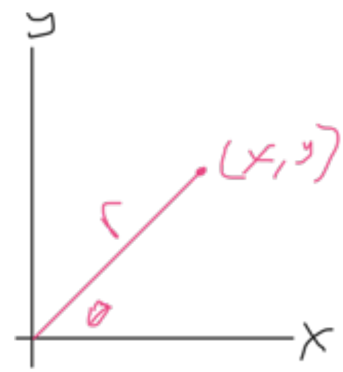


Figure 42. Polar coordinates replaces (x, y) with some angle θ and some distance r.

Recall from the last lecture where we integrated a function $f(x, y)$ over the region D where D was the quarter disk of radius 1. See 10.4.2. Using xy coordinates made calculating the integral really fairly difficult, and not quite appropriate when using a square grid to calculate integral of a circular region. Instead, a much better approach is to convert to polar coordinates. Also recall from trigonometry that

$$\begin{aligned}x &= r\cos\theta \\ y &= r\sin\theta.\end{aligned}$$

When integrating over a region that lies on the xy plane, we broke up the region into lots of little squares, each with an area of $dA = dx \cdot dy$. Things are a little bit more complicated in polar coordinates because the areas are not uniform.

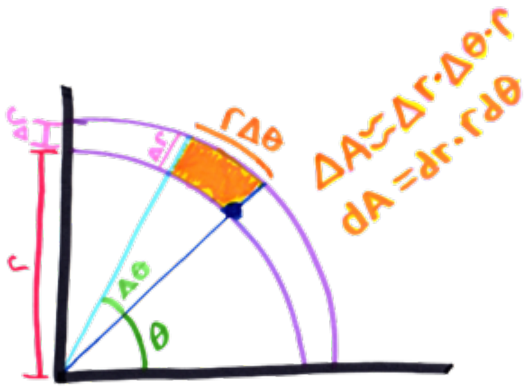


Figure 43. Note that the change in area depends upon which grid box you are in.

From the image above, you can see that we have to account for the width of grid box changing as r changes. Thus, we have

$$dA = r \cdot dr \cdot d\theta.$$

Subsection 11.1

A Polar Example

Let's return to yesterday's problem. See 10.4.2. We want to calculate the volume under the surface

$$f(x, y) = f(x, y) = 1 - x^2 - y^2$$

over the region D where

$$\begin{aligned} x^2 + y^2 &\leq 1, \\ x &\geq 0, \\ y &\geq 0. \end{aligned}$$

Since we are integrating with respect to r and θ now, what will be our ranges of integration? Well, since r ranges from 0 to 1, and θ ranges from 0 to $\frac{\pi}{2}$, we can set up our integral as follows:

$$\int_0^{\frac{\pi}{2}} \int_0^1 r \cdot dr \cdot d\theta.$$

Our function is still in terms of x and y though. To change from $f(x, y) \longrightarrow f(\theta, r)$ we could use substitution ($x = r\cos\theta$, $y = r\sin\theta$), but if we rewrite our equation:

$$\begin{aligned} f(x, y) &= 1 - x^2 - y^2 \\ &= 1 - (x^2 + y^2) \end{aligned}$$

we see that $x^2 + y^2$ is actually just the square of a distance from the origin, so it is just r^2. Thus, in polar coordinates we have $f = 1 - r^2$. Thus, our integral is

$$\int_0^{\frac{\pi}{2}} \int_0^1 (1 - r^2) r dr d\theta.$$

We can now solve our integral:

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_0^2 \left(1-r^2\right) r dr d\theta &= \int_0^{\frac{\pi}{7}} \left[\left(1-r^2\right)^2 \cdot \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right]_0^1 d\theta \\ &= \int_0^{\frac{\pi}{2}} 0 + \frac{1}{4} d\theta \\ &= \frac{1}{4}\theta\bigg|_0^{\frac{\pi}{2}} \\ &= \frac{1}{4}\left(\frac{\pi}{2}\right) \\ &= \frac{\pi}{8}.\end{aligned}$$

Subsection 11.2

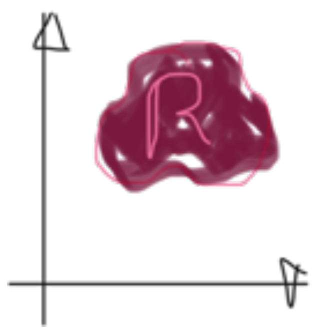
Applications

Double integrals aren't just used for finding the volume under a surface, they have many other uses as well.

11.2.1 Find the area of region R.

$$\text{Area}(R) = \iint_R 1 dA.$$

Why is this the case? Well, let's say you have some region R.



What is the area of this region? Well, it is the sum of all the tiny little areas. If you really want to think about it in terms of volume, you can say it is the volume under a flat plane ($z = 1$), where height equals 1 over the region R. Since volume is base $\times$ height, and height is 1, that is the area of the base.

11.2.2 Find the mass of (flat) object

Let's say an object has density $\delta =$ mass per unity area. What will the mass of each section of our object be? Well, we see that

$$\Delta m = \delta \cdot \Delta A$$

where A is area. Thus, the total mass will be the sum of the mass of all the tiny sections. Thus

$$\text{Mass } = \iint_R \delta \cdot dA.$$

If the density is always the same (it is a constant), then the you could of course take the density out of the integral and multiply it by the area: $\delta \iint_R 1dA$. But, on the other hand, if the object has varying density, then you can get the mass by integrating the density. The reason why we are only able to deal with flat objects right now is because we are dealing with double integrals. We will be able to calculate the mass of non flat objects when we learn triple integrals.

11.2.3 Average value of f in R.

We know how to calculate the average of a finite set. For example, if you want to calculate your average score on your problem sets, you just compute the total score and divide by the number of psets. What if instead you want to calculate the average over an infinite set? For example let's say you want to calculate the average temperature in a room? Well, there are an infinite number of points at which you could collect your data.

To compute the average value, you sum up all the values of our function f over region R, and then divide by the area of the region. Thus,

$$\text{Average of } f = \bar{f} = \frac{1}{\text{Area }(R)} \iint_R f dA.$$

11.2.4 Weighted average value of f in R.

Just like we can calculate the mass of an object with variable density, we can calculate the average value of f, giving more weight to some value than others. Let's say for example one of the problem sets is worth twice the others. Then, what you will do is count double that score to your sum, and then count it as two when it is time to divide.

The the weighted average is the sum of the values, but each value weighted by a certain coefficient. Then you will divide by the sum of the weight. Thus to calculate the weighted average we take the integral

$$\frac{1}{\text{Mass}(R)} \iint_R f\delta \, dA.$$

Notice that the density function tells us the weight of f, and the mass is the sum of all the densities over the region R.

11.2.5 Find center of mass

In physics, one often cares about finding the center of mass. Often times you might say that this point is in the "middle" of the object, such as is the case for a flat circular object of homogenous density. But if the object takes a very irregular shape, or has varying densities, then it is not always so cut and dry. Strictly speaking, the center of mass is the point where you would have to concentrate all of the mass if you want the object to behave equivalently from the point of view of mechanics. Another way to think about it, is if you have a flat object, the center of mass is that objects balance point.

If you have a flat object, and you have that object in the xy plane, then the coordinates of the center of mass will just be the weighted averages of x and y. Thus,

$$\text{Center of Mass } = (\bar{x}, \bar{y})$$

where $\bar{x}$ is the weighted average with respect to x and $\bar{y}$ is the weighted average with respect to y. Thus

$$\bar{x} = \frac{1}{\text{Mass}} \iint x\delta dA \quad \text{and} \quad \bar{y} = \frac{1}{\text{Mass}} \iint y\delta dA.$$

11.2.6 Moment of inertia around Origin

Moment of inertia is to rotation of the solid what mass is to translation. The mass of a solid is what makes it hard to push. On the other hand, how hard it is to spin something is given by it's moment of enertia. That is, it is how hard it is to import rotation motion of an object about an axis. But how can we calculate an objects moment of inertia?

To do this, we will use the concept of kinetic energy. Kinetic energy just says that an object with both mass and velocity has energy. The amount of this energy is given by the formula:

$$\text{Kinetic Energy } = \frac{1}{2}mv^2,$$

where m is mass and v is velocity. What if instead of pushing an object in a straight line, you make it go around the origin in a circle at a particular angular velocity. What is the velocity of your of our mass m? Well, let's say it has some angular velocity ω. Recall angular velocity is just change in theta over change in time or $\omega = \frac{d\theta}{dt}$.

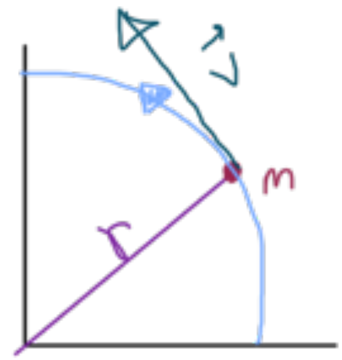


Figure 44. Kinetic energy of mass rotating around origin.

To calculate the velocity of our mass, we first need the speed. Well, the distance the mass will travel will be $\Delta\theta \times r$, and thus its speed will be $\frac{\Delta\theta \times r}{\Delta t} = r \times \omega$. Because linear velocity is just speed, we have $v = r \times \omega$. Thus, the kinetic energy of our rotating mass is

$$\begin{aligned}\text{Kinetic Energy } &= \frac{1}{2}mv^2\\ &= \frac{1}{2}m(r\cdot\omega)^2\\ &= \frac{1}{2}mr^2\omega^2.\end{aligned}$$

Thus,

$$\text{Moment of Inertia } = mr^2.$$

We have to be careful here though. That is only for point mass. That is, where our mass is concentrated at a single point. as fun as it is to swing a small ball around a string, sometimes you might want to swing something a little bit bigger. The key idea here is that the moment of inertia of our bigger object will just be the sum of the moments of inertia of our little pieces. Thus, we will cut our solid into little chunks and compute the moment of inertia for each piece.

For a solid with density δ

$$\Delta m = \delta\ \Delta A$$

and so the moment of inertia of each little piece will be

$$\Delta m \cdot r^2 = r^2 \cdot \delta\Delta A.$$

Thus, finally, summing these all together we get our moment of inertia for our object:

$$\text{Moment of Inertia } = I_O = \iint_R r^2 \cdot \delta \cdot dA.$$

Remember, if you are in the x-coordinates, then $r^2 = x^2 + y^2$ (distance formula). This gives us

$$\text{Rotational Kinetic Energy } = \frac{1}{2}I_O\omega^2.$$

11.2.7 Moment of Inertia around Axis

Let's say for example that you want to compute the moment of inertia, not around the object, but around some axis.

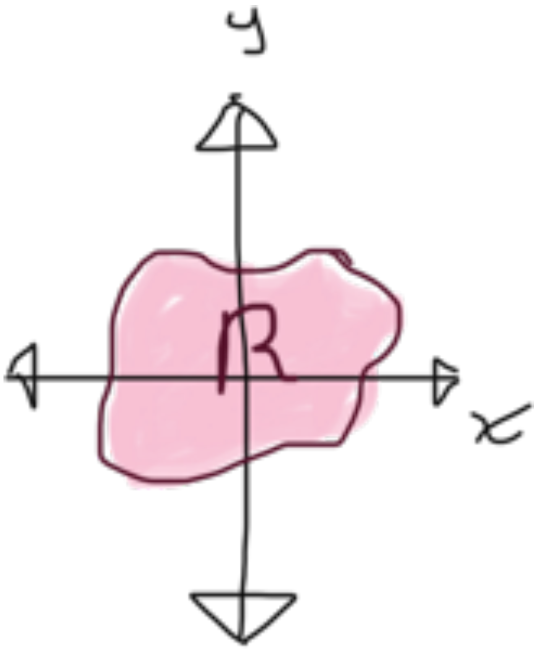


Figure 45. Let's say you skewer your object at the x-axis, and you want to rotate it about the x axis. How can you calculate the moment of inertia?

Well, in reality, nothing has changed. The moment of inertia for any given point is just the square of distance from x-axis times the mass of the point. Distance is $|y|^2 = y^2$ and so

$$I_X = \iint y^2 \cdot \delta \cdot dA.$$

Example
Let's say we have a disk of radius a centered about the origin and of uniform density that we want to rotate about origin. What is moment of inertia of disk? Well,

$$\begin{aligned} I_O &= \iint r^2 dA \\ &= \iint r^2 \cdot r \cdot dr d\theta && \textit{We want to integrate in polar coordinates} \\ &= \int_0^{2\pi} \int_0^a r^2 \cdot r \cdot dr d\theta && \textit{distance from point to origin is between 0 and a.} \\ &= 2\pi \frac{a^4}{4} = \frac{\pi a^4}{2}. \end{aligned}$$

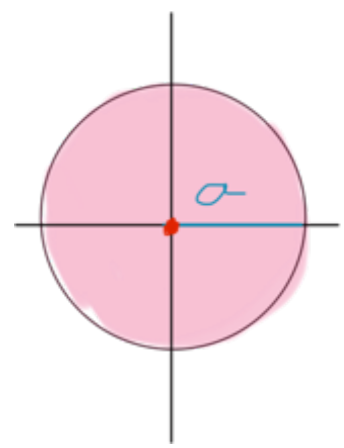


Figure 46. Disk of radius a centered at origin.

Second Example
What if instead, we wanted to rotate disk about a point on its circumferance? We can

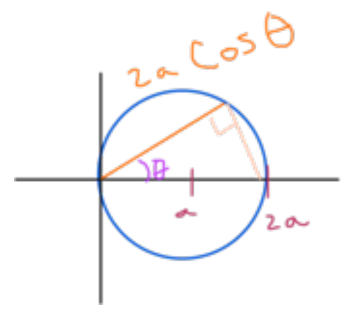


Figure 47. We can shift disk so origin is a point on circumference. Note that the maximum distance a point can be from origin is $r = 2a\cos\theta$.

set up our integral as:

$$\begin{aligned}
I_O &= \iint r^2 dA \\
&= \iint_0^{2a\cos\theta} r^2 r dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^2 r dr d\theta \\
&= \frac{\pi a^4}{2}.
\end{aligned}$$

Section 12

Lecture 18: Change of Variables

Subsection 12.1

A General Idea

Let's say that we want to find the area of an ellipse with semiaxes a, b with equation

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1.$$

How can we approach this problem? Well, we could set up the integral as

$$\iint_{(\frac{x}{a})^2+(\frac{y}{b})^2<1} 1 \; dxdy.$$

This doesn't look too fun. Often times in mathematics, we want to try and solve a hard problem, by converting it into an easier problem to solve. We will do this here.

Notice that if we were dealing with a circle, and not an ellipse, we would just convert this to polar coordinates and everything would be really easy. So, what we will do is rescale our x and y coordinates by a and b, "unsquishing" our circle, so that we can integrate over the much easier circle. Thus we will let

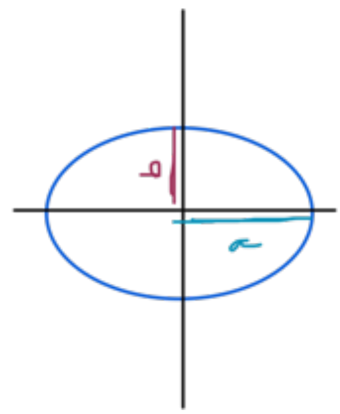


Figure 48. An ellipse with semiaxes a, b.

$$u = \frac{x}{a} \quad \text{and} \quad v = \frac{y}{b}$$

and integrate over u and v (after calculating du and dv in terms of dx and dy repspectively). First calculating $\frac{du}{dx}$:

$$\begin{aligned}
\frac{du}{dx} &= \frac{d}{dx}\frac{x}{a} \\
&= \frac{1}{a} \\
a \cdot du &= dx.
\end{aligned}$$

Calculating $\frac{dv}{dy}$:

$$\begin{aligned}
\frac{dv}{dy} &= \frac{d}{dy}\left(\frac{y}{b}\right) \\
&= \frac{1}{b} \\
b \cdot dv &= dy.
\end{aligned}$$

Now we can set up our integral:

$$
\begin{aligned}
\iint_{u^2+v^2<1} 1\,dxdy &= \iint_{u^2+v^2<1} 1(a\cdot du)(b\cdot dv) && \textit{By substitution} \\
&= \iint_{u^2+v^2<1} ab\cdot dudv && \textit{Reorganizing} \\
&= ab\iint_{u^2+v^2<1} 1\cdot dudv && a, b \textit{ are constants} \\
&= ab\cdot(\text{area of unit disk}) \\
&= \pi ab.
\end{aligned}
$$

This was a fairly simple example, but the general idea is that we want to figure out the scale factor. That is, the relationship between $dxdy$ and $dudv$.

Subsection 12.2

A Linear Example

Let's say that we want to compute some integral $\iint dxdy$, but for whatever reason we let

$$u = 3x - 2y \quad \text{and} \quad v = x + y.$$

Well, we need to find some relationship between

$$dA = dxdy \quad \text{and} \quad dA' = dudv.$$

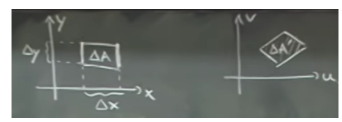


Figure 49. The area $\Delta A'$ differs from ΔA by some scaling factor.

We should note that in this case the scaling factor is the same for any choice of rectangle because we are doing a linear change of variables (linear transformation). Thus, because we can pick any rectangle, let's see how the unit square in xy coordinates will get transformed when placed on the uv coordinate grid.

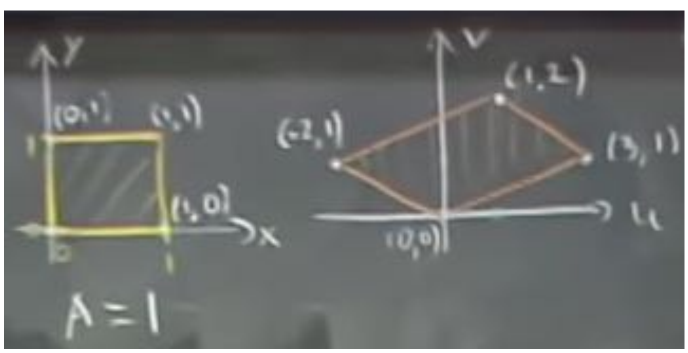


Figure 50. We can determine the coordinates of our new parallelogram by plugging in the xy coordinates into our equations for u and v above.

We can calculate the area of this parallelogram by calculating the determinant of vectors $\langle 3, 1\rangle$ and $\langle -2, 1\rangle$. Thus:

$$
\begin{aligned}
\text{Area of } \Delta A' &= \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} \\
&= 5.
\end{aligned}
$$

Therefore

$$dA' = 5dA$$
$$dudv = 5dxdy.$$

Our integral then is

$$\iint \cdots dxdy = \iint \cdots \cdot \frac{1}{5} dudv.$$

SUBSECTION 12.3

Not-Necessarily Linear Transformations

Let's say that instead of a linear transformation, we have the general case where

$$u = u(x, y) \quad \text{and} \quad v = v(x, y).$$

Then we have some not necessarily linear transformation.

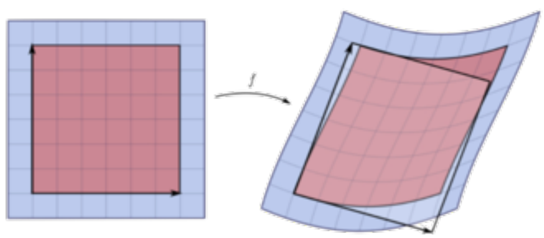


Figure 51. A non linear transformation. If we want to calculate the area of the the red region in the uv coordinate grid (right), we run into trouble because the area of each square is different. But, at any point in the red region, the jacobian can approximate local behavior really well.

The idea is that at any given point in the uv grid, and we want to calculate the area of a small box, if we let that box be really small, then a parallelogram is formed. Thus, we just need to know the vectors from that point tangent to the u and v grid lines.

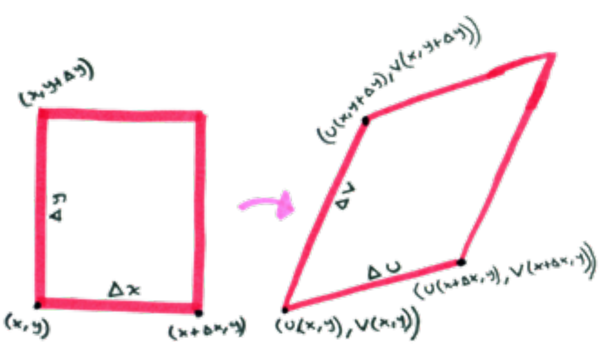


Figure 52. We see that as $\Delta x, \Delta y \to 0$, the respective box formed by $\Delta u, \Delta v$ becomes a parellelogram.

Let's say that we have some box in our region that we want to calculate the area of, with lower left coordinates (x, y). If the width and height of that box are $\Delta x, \Delta y$ respectively then the area is $\Delta x \cdot \Delta y$. Let's say that instead the transformation from

xy coordinates to uv coordinates via a linear transformation, we have a non-linear transformation. That is, a point (x, y) becomes the point $(u(x, y), v(x, y))$. Well, recall that when we had a linear transformation, we wanted to find the scalar s such that the area of each little box $dxdy = s \cdot dudv$. The problem we have though, is that since we no longer have a linear transformation, there is no longer a guarantee that grid lines are evenly spaced together, or that they have remained parallel. This means that we no longer have some constant scalar.

You can see in figure 51 that that the grid boxes are close to being parallelograms. The key insight then is that as those grid boxes become smaller and smaller, the become more and more like parellelograms. Thus, in the limiting process inherent with integrals, they can be treated like paraellelograms.

Can we determine the area of any one of these little boxes in our uv coordinate grid? If so, then we can determine the area $dxdy$ in different terms. Well, the key here is that because these curved lines are now straight lines, we can use the linear approximation of $u(x, y)$ and $v(x, y)$. Recall the linear approximation formula:

$$\begin{aligned} f(x, y) &\approx \frac{\partial f}{\partial x}(x, y)(x - x_0) + \frac{\partial f}{\partial y}(x, y)(y - y_0) + f(x_0, y_0) \\ f(x, y) - f(x_0, y_0) &\approx \frac{\partial f}{\partial x}(x, y)\Delta x + \frac{\partial f}{\partial y}(x, y)\Delta y \\ \Delta f &\approx \frac{\partial f}{\partial x}(x, y)\Delta x + \frac{\partial f}{\partial y}(x, y)\Delta y. \end{aligned}$$

We will approach this problem with two different approaches:

1. **Matrices represent linear transformations**
   We can use our linear approximation formula above to calculate Δu and Δv:

$$\begin{aligned} \Delta u &\approx \frac{\partial u}{\partial x}(x, y)\Delta x + \frac{\partial u}{\partial y}(x, y)\Delta y \\ \Delta v &\approx \frac{\partial v}{\partial x}(x, y)\Delta x + \frac{\partial v}{\partial y}(x, y)\Delta y. \end{aligned}$$

   In matrix form this is written as:

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} \frac{\partial u}{\partial x}(x, y) & \frac{\partial u}{\partial y}(x, y) \\ \frac{\partial v}{\partial x}(x, y) & \frac{\partial v}{\partial y}(x, y) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

   This change to matrices, is not just a stylistic choice, it is now representative of some linear transformation. Thus, for any point (x, y) on on our xy plane, after performing our change of coordinates, $\hat{\imath}$ will land at $(\frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y))$ (blue column) and $\hat{\jmath}$ will land at the coordinates represented by the red column. We can use this information to then determine where any vector will land in our new coordinate system. Because the matrix

$$\begin{bmatrix} \frac{\partial u}{\partial x}(x, y) & \frac{\partial u}{\partial y}(x, y) \\ \frac{\partial v}{\partial x}(x, y) & \frac{\partial v}{\partial y}(x, y) \end{bmatrix}$$

   describes a linear transformation, we can determine the amount that that transformation scales the area of that object. That is, the determinant will determine how much bigger or smaller $dudv$ is than $dxdy$. Thus,

$$dudv = \det\left(\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\right) dxdy.$$

Note, that we must take the absolute value of the determinant, because the determinant can be negative.

2. **A vector approach**
   Now, let's try to find the affect that the new coordinate system has on the area of $dxdy$ by composing vectors of the edges of $dudv$ and then calculating the area of the parallelogram. We will find out where our vectors $\langle \Delta x, 0\rangle$ and $\langle 0, \Delta y\rangle$ land, so that we can then calculate the change in area (scaling factor) after the change of coordinates, for our particular small box. To do this, recall that matrix-vector multiplication is just a way to compute what the transformation does to a given vector. Thus, we see that after the transformation

$$\langle \Delta x, 0\rangle \to \langle \frac{\partial u}{\partial x}\Delta x, \frac{\partial u}{\partial y}\Delta x\rangle$$

   and that

$$\langle 0, \Delta y\rangle \to \langle \frac{\partial v}{\partial x}\Delta y, \frac{\partial v}{\partial y}\Delta y\rangle.$$

   To calculate the area of this, we get

$$\begin{aligned}\text{Area of } dudv &= \frac{\partial u}{\partial x}\Delta x \cdot \frac{\partial v}{\partial y}\Delta y - \frac{\partial u}{\partial y}\Delta x \cdot \frac{\partial v}{\partial x}\Delta y \\ &= \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)\Delta x\Delta y.\end{aligned}$$

   This is the same result as above.

Subsection 12.4

The Jacobian Matrix

We call this matrix of partial derivatives the Jacobian matrix.

$$\text{Jacobian } = J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

A very helpful property of the Jacobian matrix is that

$$dudv = |\det(J)| dxdy = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dxdy.$$

12.4.1 Change to Polar Coordinates

For many reasons, we might want to work in polar coordinates instead. We have done this before, but let's use the Jacobian to do it this time. We want to go from

$$\iint_R \cdots dxdy \to \iint_G \cdots dr\theta.$$

But remember, we have to find the scaling factor of $dxdy$. We know that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Thus, calculating the jacobian we have

$$\begin{aligned} J = \frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} x_0 & x_0 \\ y_r & y_\theta \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r\left(\cos^2\theta + \sin^2\theta\right) \\ &= r(1) \\ &= r. \end{aligned}$$

Thus $dxdy = r \cdot drd\theta$ and our integral will take the form

$$\iint_G \cdots r(drd\theta).$$

12.4.2 A Simple Example

Let's say we want to compute

$$\int_0^1 \int_0^1 x^2 y dxdy$$

and for some evil reason, we want to compute it by changing variables, such that

$$u = x \quad \text{and} \quad v = xy.$$

1. **Compute area element**

$$\begin{aligned} J = \frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} \\ &= x. \end{aligned}$$

   Therefore

$$\begin{aligned} dudv &= x \cdot dxdy \\ \frac{1}{x} \cdot dudv &= dxdy. \end{aligned}$$

   Note that $x \geq 0$ so there is no need to take absolute value.

2. **Integrand in terms of** u, v

$$\begin{aligned} x^2 y dxdy &= x^2 y \frac{1}{x} dudv \\ &= xydudv \\ &= vdudv. \end{aligned}$$

   Thus, we will be computing

$$\iint_{???} vdudv.$$

3. **Compute bounds** In the xy coordinate system the graph of our bounds looked like

066.png

Figure 53. The graph of bounds of of integral in xy coordinates.

Because we are going to first integrete $\int vdu$, we will be keeping v a constant as we change u. Thus, going back to xy, we have $v = xy =$ a constant. Thus, our graphs will take on the following form:

067.png

Figure 54. The graph of v on xy grid.

We first need to compute the bounds of u. That is, we need to find the u for which $y = 1$. Well, since we know that $y = 1$, we see that $v = xy$, and $v = x$. Thus, since both $u = x$ and $v = x$, therefore $u = v$. When graph of $v =$ constant enters our region, we see that $u = v$. The graph of $v =$ constant exits region when $x = 1$ and since $x = u$, when $u = 1$.

That is the first part. Now on to the range of v. Well since $v = xy$, the smallest v is zero and the largest is one. Thus our integral will be

$$\int_0^1 \int_{u=v}^1 ???vdudv.$$

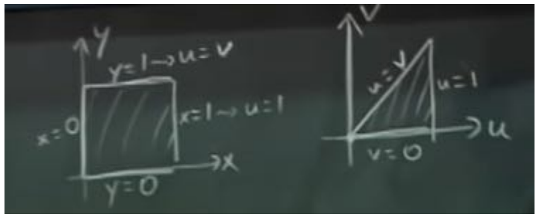


Figure 55. We could just draw our bounds on the uv plane instead

SECTION 13

Lecture 19: Vector Fields and Line Integrals

SUBSECTION 13.1

Vector Fields

What is a vector field?

Definition 9 A vector field is a vector valued function

$$\vec{F} = M\hat{\imath} + N\hat{\jmath}$$

where M and N are functions of x, y.

From this definition, you see that every point in the plane you have a vector. Just like in a corn field everywhere you have corn, in a vector field everywhere you have vectors. Thus, at each point (x, y) we have a vector $\vec{F}$ that depends upon x, y. Examples of vector fields include

- Velocity in a fluid
- Force field $\vec{F}$. For example, field of gravitational forces.

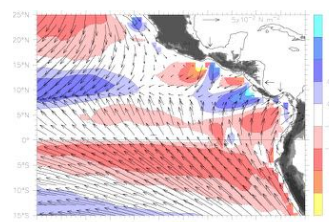
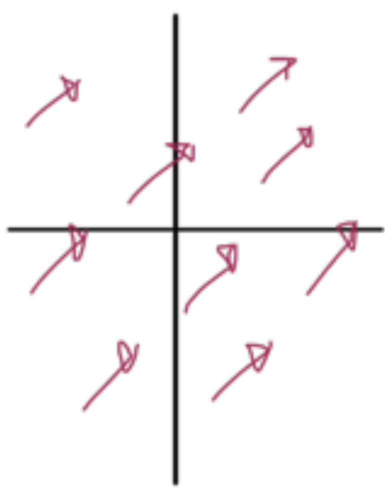


Figure 56. Vector fields can be used to represent wind direction and wind speed at given points.

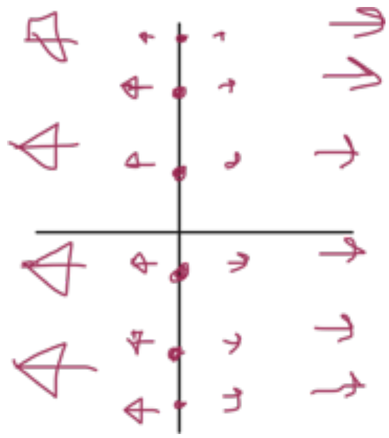
13.1.1 Example 1

Let's say we have the vector field $\vec{F} = 2\hat{\imath} + \hat{\jmath}$. Then we would draw the vector $\langle 2, 1\rangle$ at every point (x, y). This is a rather silly vector field since it doesn't depend upon x or y.



13.1.2 Example 2

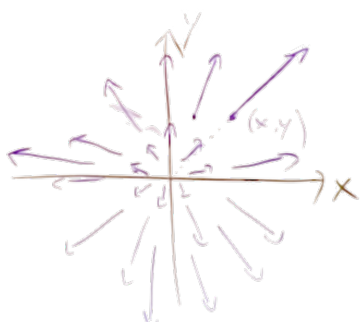
Let's say $\vec{F} = x\hat{\imath}$. Since there is no $\hat{\jmath}$ component, at every location, the vector is in $\hat{\imath}$ direction. The magnitude is dependent upon x position, so we get the following vector field.



Note, that in these drawings, we are not going for exact accuracy, as a computer can do that much better than we can, but more to get a feeling of what our vector field looks like.

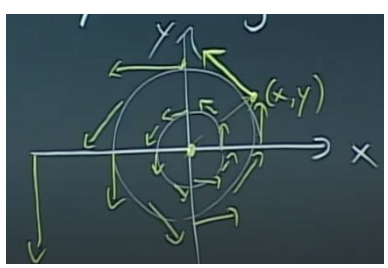
13.1.3 Example 3

Vector field $\vec{F} = x\hat{\imath} + y\hat{\jmath}$. This is the vector field such that each vector points radially away from the origin whose magnitude is distance away from origin.



13.1.4 Example 4

Vector $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$. How is the vector $\langle -y, x\rangle$ related to the vector $\langle x, y\rangle$? Well, they have the same magnitude and are orthogonal to eachother. Thus, we get the vector field as shown below.



If this were describing a fluid, it would correspond to a fluid rotating around the origin at uniform speed. This is the velocity field for uniform rotation.

SUBSECTION 13.2
Work and Line Integrals

13.2.1 What is work?

You may recall that work is often expressed as

$$\text{work} \;=\; \text{force} \;\times\; \text{distance}.$$

So for example if you apply a lot of force, for a long distance, then you have done a lot of work. But what about if the force being applied is not in the same direction of movement? For example, let's say we have a block on a frictionless plane that is at an incline. The block is under the force of gravity pushing down, but the movement is in the direction of incline plane.

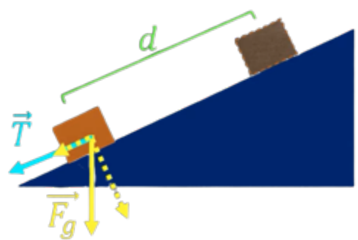


Figure 57. The force of gravity can be decomposed into a force tangential to the incline plane ($\vec{T}$) and another that is normal to the incline plane.

Because our block is moving in direction tangential to the incline plane, all the work that is being done is in that direction. Thus, the amount of work being done is

$$W = \left(\vec{F}_g \cdot \vec{T}\right) d.$$

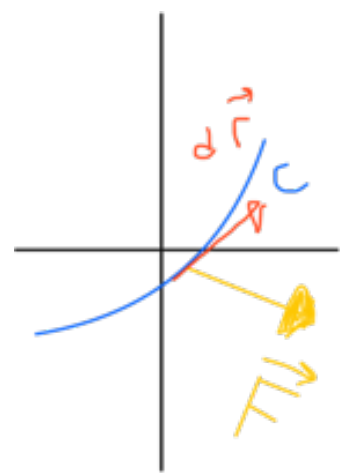
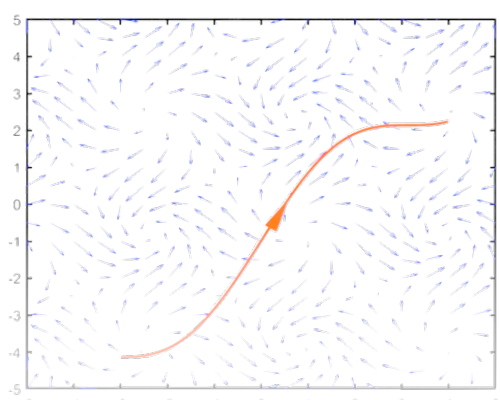


Figure 58.

13.2.2 Work, line integrals and vector fields.

Let's consider a field $\vec{F}$ and a curve C.



If we want to calculate the amout of work done by our field on a particle moving along the curve c, we sum up all the tiny movements along path (a small vector tangent to c) dotted with our field $\vec{F}$:

$$W = \int_c \vec{F} \cdot d\vec{r},$$

where $d\vec{r}$ is a small movement tangent to c.

This gives us a good way to think about the wour the field is importing on the partical moving along a path, but to compute this integral we actually need it in a different form. Recall from 6.4 that the velocity vector $\vec{v} = \frac{d\vec{r}}{dt}$. Thus, we can rewrite our integral as

$$\begin{aligned} W &= \int_c \vec{F} \cdot d\vec{r} \\ &= \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt} dt. && \textit{$\frac{d\vec{r}}{dt}$ is velocity vector} \end{aligned}$$

Subsection 13.3

Examples

13.3.1 Example 1

Let's say we have a vector field

$$\vec{F} = -y\hat{\imath} + x\hat{\jmath}$$

and we want to calculate the work that field emparts on a particle moving along the curve

$$c(t) = \begin{cases} x = t \\ y = t^2 \end{cases}$$

for $0 \leq t \leq 1$. Let's compute this:

$$\begin{aligned} W &= \int_c \vec{F} \cdot d\vec{r} \\ &= \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt} dt. \end{aligned}$$

What is $\vec{F}$? Well, we have

$$\begin{aligned} \vec{F} &= -y\hat{\imath} + x\hat{\jmath} \\ &= \langle -y, x \rangle && \textit{In vector form} \\ &= \langle -t^2, t \rangle. && x = t, y = t^2 \end{aligned}$$

Thus, we have

$$W = \int_0^1 \langle -t^2, t \rangle \cdot \frac{d\vec{r}}{dt} dt.$$

What is the velocity vector at a given t? Well, looking at our curve, we see that $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 2t$ and so the velocity vector is $\langle 1, 2t \rangle$. Thus, we have

$$\begin{aligned} W &= \int_0^1 \langle -t^2, t \rangle \cdot \langle 1, 2t \rangle dt \\ &= \int_0^1 \left(-t^2 + 2t^2\right) dt \\ &= \left(-\frac{1}{3}t^3 + \frac{2}{3}t^3\right)\Bigg|_0^1 \\ &= \frac{-1}{3} + \frac{2}{3} \\ &= \frac{1}{3}. \end{aligned}$$

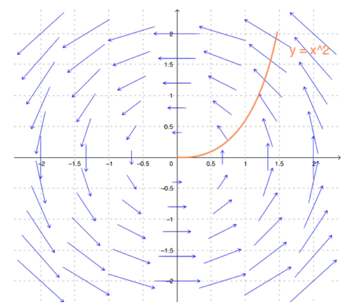


Figure 59. Vector field $\vec{F}$ and the path $y = x^2$ which we will be calculating work along.

13.3.2 Example 2

Let's say that we have $\vec{F} = \langle M, N \rangle$. We can express $d\vec{r} = \langle dx, dy \rangle$. What does this notation mean? This is a weird way to write things, but it is actually rather convenient. It gives us a way to calculate our integral in terms of $d\vec{r}$ instead of converting to x and y first. Thus we have

$$\int_c \vec{F} \cdot d\vec{r} = \int_c M dx + N dy.$$

How do we compute this type of integral? Well, it is a little bit tricky because both M and N depend on x and y. So we can't just say that $\cancel{\int_c M dx + N dy = \int_c M dx + \int_c N dy}$.

This is because integrating M with respect to dx doesn't take into the fact that M is in terms of bot x AND y, and so you would be left with y's in our equation, and we don't want that, we just want a number. Therefore, what we will do is use the fact that x and y are bothe expressed in terms of our parameter, and substitute.

Returning to our example above, we have

$$\begin{aligned}
\int_c \vec{F} \cdot d\vec{r} &= \int_c -y dx + x dy \\
&= \int_c -t^2 dx + t dy && \textit{x = t, y = t}^2 \\
&= \int_c -t^2(dt) + t(2t dt) && dx = dt \textit{ and } dy = 2t dt \\
&= \int_0^1 -t^2(dt) + t(2t dt) && 0 \leq t \leq 1 \\
&= \int_0^1 (-t^2 + 2t^2) dt && \textit{Factoring out dt} \\
&= \int_0^1 t^2 dt && \textit{Factoring out dt} \\
&= \frac{1}{3}.
\end{aligned}$$

We should note that $\int_c \vec{F} \cdot d\vec{r}$ depends on the trajectory c, but NOT on the parameterization. For example we could have said that

$$\begin{cases} x = \sin\theta \\ y = \sin^2\theta \end{cases} \quad 0 \leq \theta \leq \frac{\pi}{2}$$

and integrated that way. This would make our life overly complicated though, so we want to find the simplest parameterization. A shortcut that you might want to do, is keep one of the variables x and then write y in terms of x and dy in terms of dx:

$$
\begin{aligned}
\int_c \vec{F}\cdot d\vec{r} &= \int_c -y dx + x dy \\
&= \int_c -x^2 dx + x dy && \color{blue}{y = x^2} \\
&= \int_c -x^2 dx + x(2x dx) && \color{blue}{dy = 2x dx} \\
&= \int_c \left(2x^2 - x^2\right) dx \\
&= \int_c x^2 dx \\
&= \int_0^2 x^2 dx \\
&= \frac{1}{3}x^3\bigg|_0^1 && \color{blue}{x = t} \\
&= \frac{1}{3}
\end{aligned}
$$

Subsection 13.4

Geometric Approach

Let's say we have some curve $c = \vec{r}(t)$. Then we can draw the following diagram.

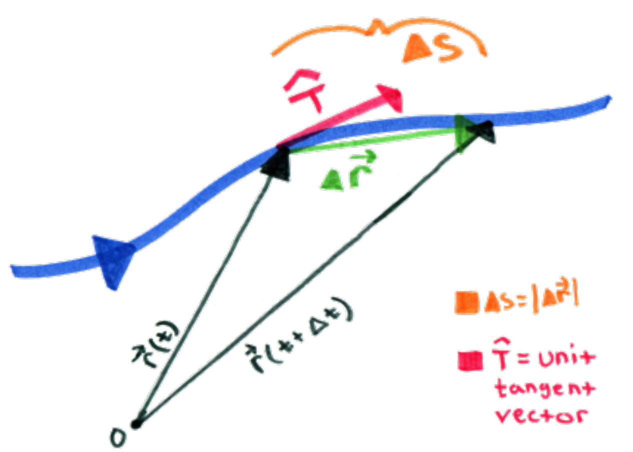


Figure 60. We can see that $d\vec{r} = \langle dx, dy\rangle = \hat{T}ds$. That is, $d\vec{r}$ is a vector in direction of $\hat{T}$ with length ds.

Thus, we say that

$$\int_c \vec{F}\cdot d\vec{r} = \int_c M dx + N dy = \int_c \vec{F}\cdot\hat{T} ds$$

where $\vec{F}\cdot\hat{T}$ is a scalar quantity. How does this help us? Well, we can use this geometric intuition to solve line integrals where there is some geometric relationship between the curve and the field. Let's look at some examples.

13.4.1 Example 1

Let's say that our trajectory is the curve

$$c: \text{ circle of radius } a \text{ centered at origin, moving counterclockwise,}$$

and the vector field is

$$\vec{F} = x\hat{\imath} + y\hat{\jmath}.$$

Without doing any calculation, you can see that the work done by the field on a particle moving along the curve is zero because the field is perpendicular to curve at all points along curve.

We see that $\vec{F} \perp \hat{T}$, so $\vec{F} \cdot \hat{T} = 0$, therefore $\int_c \vec{F} \cdot \hat{T} ds = 0$.

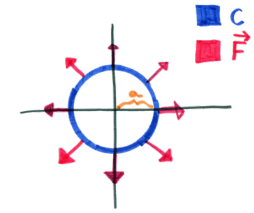


Figure 61. Field $\vec{F}$ and curve c with radius a.

13.4.2 Example 2

Let's say we have the same c as above, but this time

$$\vec{F} = -y\hat{\imath} + x\hat{\jmath}.$$

We see that the field is tangent to curve, and thus parallel to unit tangent vector. Therefore, since $\vec{F} \cdot \hat{T}$ is the component of $\vec{F}$ in direction of $\vec{T}$ and since they are in the same direction, then

$$\vec{F} \cdot \hat{T} = \|\vec{F}\|_2 = a.$$

Therefore we can calculate the integral as follows:

$$\begin{aligned}
\int_c \vec{F} \cdot \hat{T} ds &= \int_c a ds && \textit{as shown above} \\
&= a \int_c ds && \textit{a is a constant} \\
&= a2\pi(a) && \int_c \textit{ds is length of curve} \\
&= 2\pi a^2
\end{aligned}$$

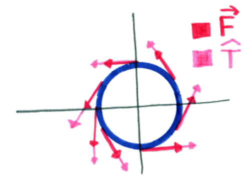


Figure 62. Field $\vec{F}$ and curve c.

13.4.3 Example 3

Let's say we have the field

$$\vec{F} = \langle y, x \rangle$$

and the three curves shown. What is the work being done? To solve this, we need to set up three integrals, one for each curve.

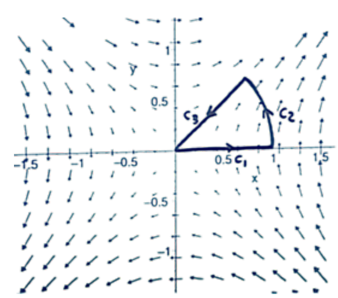


Figure 63. Vector field $\vec{F} = \langle y, x \rangle$ and curves c_1, c_2, c_3.

- c_1 : x-axis.
  Since we are moving along the $x-$axis, $y = 0$. Therefore, the $\hat{\imath}$ component of field is always zero, and the $\hat{\jmath}$ component of our field will be increasing as x increases. Thus, curve c_1 is perpendicular to field, so no work is being done. Thus,

  $$W_{c_1} = \int_{c_1} \vec{F} \cdot \hat{T} ds = 0.$$

  We could also solve this by saying

  $$W_{c_1} = \int_{c_1} \cdot d\vec{r} = \int_{c_1} y dx + x dy.$$

  But, $y = 0 \Rightarrow dy = 0 \Rightarrow \int_{c_1} y dx + x dy = 0$.

- c_2: Portion of unit circle.
  We can parameterize x, y in terms of θ:

$$\begin{array}{ll} x \equiv \cos\theta & dx = -\sin\theta d\theta \\ y = \sin\theta & dy = \cos\theta d\theta \\ 0 \le \theta \le \frac{\pi}{4} & \end{array}$$

  Therefore we have

$$\begin{aligned} \int_{c_2} \vec{F} \cdot d\vec{r} &= \int_{c_2} M dx + N dy \\ &= \int_{c_2} y dx + x dy \\ &= \int_0^{\frac{\pi}{4}} \sin\theta(-\sin\theta) d\theta + \cos\theta\cos\theta d\theta \\ &= \int_0^{\frac{\pi}{4}} \left(\cos^2\theta - \sin^2\theta\right) d\theta \\ &= \int_0^{\frac{\pi}{4}} \cos(2\theta) d\theta \\ &= \left[\frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2} - 0 \\ &= \frac{1}{2}. \end{aligned}$$

- c_3: Along line $y = x$.
  Note that we start from $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and go to $(0,0)$. Also, since $x = y$, $dx = dy$, and so by substitution we have:

$$\begin{aligned} \int_{c_3} y dx + x dy = \int_{\frac{1}{\sqrt{2}}}^0 x dx + x dx = \int_{\frac{1}{\sqrt{2}}}^0 2x dx &= \left[x^2\right]_{\frac{1}{\sqrt{2}}}^0 \\ &= 0 - \frac{1}{2} \\ &= -\frac{1}{2}. \end{aligned}$$

Therefore

$$W = \int_c = \int_{c_1} + \int_{c_2} + \int_{c_3} = 0 + \frac{1}{2} - \frac{1}{2} = 0.$$

SECTION 14

Path Independence and Conservative Fields

Let's say we have a field $\vec{F}$ where $\vec{F}$ represents gravity, and we have a path from point A to B. If we want to calculate work, then it turns out that $\int_C \vec{F} \cdot d\vec{r}$ only depends on endpoints. That is, that

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A).$$

Why is this? What is it about the gravitational field $\vec{F}$ that makes this true?

Recall, earlier we learned about gradient vectors of $f(x, y)$, where

$$\nabla f = \langle f_x, f_y \rangle.$$

When we talked about them earlier, we only mentioned them in terms of it being a single variable. But in reality, it is a vector dependent upon x and y, so it is actually a vector field.

This brings us to the special case where $\vec{F} = \nabla f$. In this case

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A).$$

This is called the fundemental theorem of line integrals.

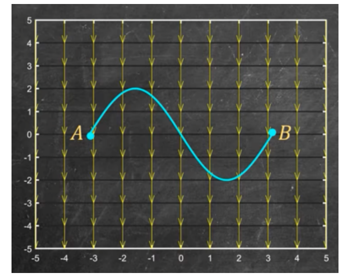


Figure 64. Field $\vec{F}$ representing gravity, and a path in that field.

SUBSECTION 14.1

Fundemental Theorem of Line Integrals

Let's first recall the fundemental theorem of calculus. It states that if f is integrable on $[a, b]$, and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a).$$

The fundemental theorem of line integrals is as follows:

Theorem 6 Let C be a smooth curve parameterized by $\vec{r}(t)$ from $\vec{r}(a) = A$ to $\vec{r}(b) = B$. Then for continuous $\vec{F} = \nabla f$

$$\int_C = \vec{F} \cdot d\vec{r} = f(B) - f(A).$$

Proof: Let $\vec{F}$ be a gradient field such that $\vec{F} = \nabla f$. To compute $\int_C \nabla f \cdot d\vec{r}$ we must parameterize x and y. Let $x = x(t)$ and $y = y(t)$. Then $dx = x'(t)dt$ and $dy = y'(t)dt$.

Therefore

$$
\begin{aligned}
\int_C \nabla f \cdot d\vec{r} &= \int_C f_x dx + f_y dy && \textit{component form} \\
&= \int_C \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \right) dt && \textit{by substitution} \\
&= \int_{t_0}^{t_1} \frac{df}{dt} dt && \textit{by chain rule} \\
&= [f(x(t), y(t))] \int_{t_0}^{t_1}] && \textit{by FToC}
\end{aligned}
$$

$$f(B) - f(A)$$

NEED TO THINK ABOUT THIS MORE.

14.1.1 A quick example

Returning to our earlier problem, see 13.4.3, is this vector field in fact a gradient field? Well, the vector field is $\vec{F} = \langle y, x \rangle$. This is in fact a gradient field. To see this, let $f(x, y) = xy$. Then $\nabla f = \langle dx, dy \rangle = \langle y, x \rangle$. Thus, an easier way to solve this we would just subtract the starting point from the end point. Because we start and end at $f(0, 0)$, the work done is zero.

Subsection 14.2

Consequences of Fundemental Theorem of Line Integrals

WARNING: This only applies if $\vec{F}$ is a gradient field. That is $\vec{F} = \nabla f$ for some function $f(x, y)$.

14.2.1 Path Independence

Given two paths (C_1, C_2) with the same starting and ending points (P_0, P_1), then

$$\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}.$$

That is, it doesn't matter what path you take, the result is exactly the same.

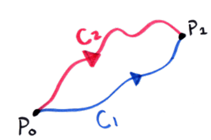


Figure 65. Paths C_1 and C_2 with same starting and ending points.

14.2.2 Conservative Fields

If $\vec{F} = \nabla f$ and C is a *closed* curve, then $\int_C \vec{F} \cdot d\vec{r} = 0$. The word conservative comes from the idea of conservation of energy. It tells you that you cannot get energy for free. If you find a closed curve whose work is zero, that is not enough to show that the field is conservative. You must be able to show this for any closed curve.

Note, that not all fields are conservative. For example, see 13.4.2, where we have a closed curve, but the work done is note zero, therefore it is not conservative. Also, it is not path-independent and is not a gradient.

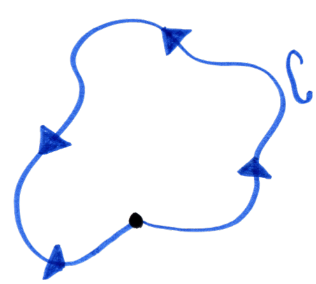


Figure 66. Curve C is a closed curve.

Subsection 14.3

Equivalent Properties

- $\vec{F}$ is conservative $\Leftrightarrow$ path independent.
  $\Leftarrow$ If path independent and we have any closed curve, then start and end points are the same and so $f(B) = f(A)$. Therefore, $f(B) - f(A) = 0$. Another way to think about this is that instead of moving, you could stay where you are, and so work is equal to zero.

$\Rightarrow$. If conservative than path independent. Let say we have two points, connected by two curves. We must show that $W_{c_1} = W_{c_2}$. Imagine we have a closed curve that traces from start point along c_1 and returns along c_2. Then this is a closed loop. Therefore, due to conservative property, work is zero. This means that $W_{c_1} + W_{-c_2} = 0$. Therefore $W_{c_1} = W_{c_2}$.

- $\Leftrightarrow \vec{F}$ is a gradient field.
  $\Leftarrow$ If $\vec{F}$ is a gradient field then $\vec{F}$ is path independent and conservative. We proved this using the fundemental theorem of calculus.
  $\Rightarrow$ If $\vec{F}$ is path indpendent and conservative then it is a gradient field. We will show this tomorrow.
- $\Leftrightarrow Mdx + Ndy$ is an exact differential. This means that it can be put in the form df for some function f. NEEDS WORK.

Section 15

Lecture 21: Gradient Fields and Potential Functions

Subsection 15.1

Is $\vec{F}$ a gradient field?

Recall that if a vector field is a gradient field, $\vec{V} = \nabla f$, then line integral can be computed by taking the chane in potential from endpoint to starting point of the curve: $\int_c \vec{F} \cdot d\vec{r} = f(P_1) - f(P_0)$.

The question remains though, how can we test if a given vector field, $\vec{F} = \langle M, N \rangle$ is a gradient field? Let's first make a couple of observations. If $\vec{F} = \nabla f$ then $M = f_x$ and $N = f_y$. Recall an interesting property of the second partial derivatives, that:

$$f_{xy} = f_{yx}.$$

Theorem 7 If $\vec{F} = \langle M, N \rangle$ is defined and differentiable everywhere, and $M_y = N_x$ then $\vec{F}$ is a gradient field.

15.1.1 Example 1

Is the field $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ a gradient field? If you recall from 13.4.2, this is not a gradient field because if we calculate the work done over the unit circle (a closed curve) then the result is 2π, and thus $\vec{F}$ is not conservative. We can also determine this using theorem 7:

$$M_y = -1$$
$$N_y = 1$$

and $-1 \neq 1$.

15.1.2 Example 2

Given vector field

$$\vec{F} = (4x^2 + axy)\hat{\imath} + (3y^2 + 4x^2)\hat{\jmath}$$

for what value(s) of a, if any, will $\vec{F}$ be a gradient field? Well, we see that

$$\begin{aligned}\vec{F} &= \langle M, N \rangle \\ &= \langle 4x^2 + axy, 3y^2 + 4x^2 \rangle\end{aligned}$$

and so

$$M_y = ax \quad \text{and} \quad N_x = 8x.$$

Therefore, if $a = 8$, then $\vec{F}$ is a gradient field.

Subsection 15.2

Finding f for which $\vec{F} = \nabla f$.

Note, that we can only do this if $\vec{F}$ is a gradient field, so first check that $M_y = N_x$. Guessing is often a decent way to find the function f, but is there a systematic approach that will work? Yes, we will explore two ways to determine the function f.

15.2.1 Line Integrals

To calculate f, let's take some curve c in our field $\vec{F}$ going from the origin to a point (x_1, y_1). But because our our field is a gradient field we have

$$\begin{aligned}\int_c \vec{F} \cdot d\vec{r} &= f(x_1, y_1) - f(0,0) \\ f(x_1, y_1) &= \int_c \vec{F} \cdot d\vec{r} + f(0,0)\end{aligned}$$

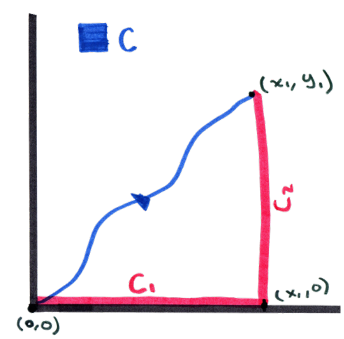


Figure 67. Because $\vec{F}$ is a gradient field and thus path independent, instead of computing the integral for path c, we can compute the integral for the path along the x-axis and parallel to y-axis (red).

where $f(0,0)$ is just a constant. Recall from single variable calculus that

$$f = \int df + \text{ constant}.$$

Also, recall that $\vec{F} = \nabla f$ (partial derivatives of f).

15.2.2 Example 3

Let's return to our previous example, see 15.1.2. We know that

$$\begin{aligned}\vec{F} &= (4x^2 + 8xy)\hat{\imath} + (3y^2 + 4x^2)\hat{\jmath} \\ &= \langle 4x^2 + 8xy, 3y^2 + 4x^2 \rangle\end{aligned}$$

is a gradient field. We want to find the function $f(x, y)$ such that

$$\vec{F} = \nabla f.$$

To find $f(x_1, y_1)$ we have

$$f(x_1, y_1) = \int_c \vec{F} \cdot d\vec{r} + f(0,0).$$

Splitting up the integral into $\int_c = \int_{c_1} + \int_{c_2}$. First calculating $\int_{c_1}$ we have

$$\begin{aligned}
\int_{c_1} \vec{F} \cdot d\vec{r} &= \int_{c_1} M dx + N dy \\
&= \int_{c_1} \left(4x^2 + 8xy\right) dx + \left(3y^2 + 4x^2\right) dy \\
&= \int_{c_1} \left(4x^2 + 8xy\right) dx + 0 \left(3y^2 + 4x^2\right) && \textit{$y = 0$, thus $dy = 0$} \\
&= \int_0^{x_1} \left(4x^2 + 8xy\right) dx + 0 \left(3y^2 + 4x^2\right) && \textit{x ranges from $0 \to x_1$.} \\
&= \left[\frac{4}{3}x^3 + \frac{8}{2}x^2 y\right]_0^{x_1} \\
&= \frac{4}{3}\left(x_1\right)^3 + 4\left(x_1\right)^2 y \\
&= \frac{4}{3}\left(x_1\right)^3 && y = 0\ .
\end{aligned}$$

Now calculating $\int_{c_2}$ we have

$$\begin{aligned}
\int_{c_2} \vec{F} \cdot d\vec{r} &= \int_{c_2} M dx + N_{dy} \\
&= \int_{c_2} \left(3x^2 + 8x_y\right) dx + \left(3y^2 + 4x^2\right) dy \\
&= \int_{c_2} \left(3y^2 + 4x_1^2\right) dy && x = x_1, dx = 0 \\
&= \int_0^{y_1} \left(3y^2 + 4\left(x_1\right)^2\right) dy && y : 0 \to y_1 \\
&= \left[y^3 + 4\left(x_1\right)^2 y\right]_0^{y_1} \\
&= y_1^3 + 4x_1^2 y_1
\end{aligned}$$

Thus,

$$\begin{aligned}
f(x_1, y_1) &= \int_c \vec{F} \cdot d\vec{r} + f(0,0) \\
&= \int_{c_1} + \int_{c_2} + f(0,0) \\
&= \frac{4}{3}\left(x_1\right)^3 + y_1^3 + 4x_1^2 y_1 + f(0,0) \\
&= \frac{4}{3}\left(x_1\right)^3 + y_1^3 + 4x_1^2 y_1 + c.
\end{aligned}$$

Because x_1 and y_1 are any x and y, we can drop the subscripts and thus

$$f(x,y) = \frac{4}{3}x^3 + y^3 + 4x^2 y + c$$

where c is a constant.

15.2.3 Antiderivatives

Instead of calculating the integral by hand, we might say “hey, I already know how to take antiderivatives from calc 1, can we use those skills”? Yes, we can. Our goal is the same: we want to find the function f such that $f_x = M$ and $f_y = N$. We will take the following steps:

1. Find f by integrating M with respect to x. This function f will include an integration constant. This integration constant will not be a true constant, as it will be a function of y, $g(y)$.
2. Differentiat f with respect to y to find f_y.
3. Solve for $g(y)$ by comparing found f_y with known f_y, then replace $g(y)$ with its value in our function f.

Let's look at an example.

15.2.4 Example 4

Looking at the same field $\vec{F}$ that we looked at in example 3, we have

$$\vec{F} = \langle 4x^2 + 8xy, 3y^2 + 4x^2 \langle.$$

Taking the antiderivative of $f_x = 4x^2 + 8xy$ we have

$$f = \frac{4}{3}x^3 + 4x^2y + \text{ integration constant.}$$

Note, that all the integration constant means is that the value does not depend upon x. Thus, it is not a true constant and depends upon y. Thus,

$$f = \frac{4}{3}x^3 + 4x^2y + g(y).$$

Now, what we can do is use f to compute fy (which we already know), and then compare the found value with our known value to give us the value of $g(y)$. We see that $f_y = 4x^2 + g'(y)$. Thus, using our known f_y we see that

$$\begin{aligned} 4x^2 + g'(y) &= 3y^2 + 4x^2 \\ g'(y) &= 3y^2 + 4x^2 - 4x^2 \\ g'(y) &= 3y^2. \end{aligned}$$

Thus, we see that $g(y) = y^3 + c$, where c is a true constant that does not depend upon any variable. Thus, by substitution we have

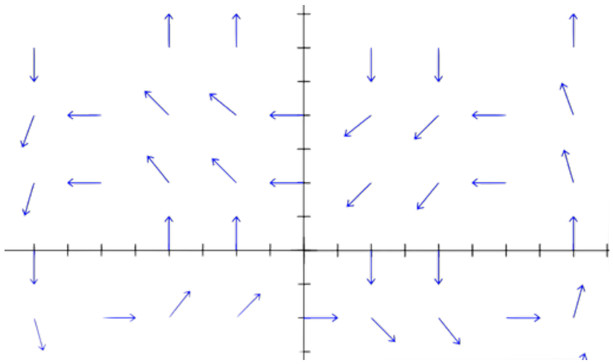
$$\begin{aligned} f &= \frac{4}{3}x^3 + 4x^2y + g(y) \\ &= \frac{4}{3}x^3 + 4x^2y + y^3 + c. \end{aligned}$$

Subsection 15.3

Two Dimensional Curl

15.3.1 Intuition

Let's say we have some vector field $\vec{F}$:



Curl has to do with the fluid flow interpretation of vector fields. That is, you can imagine that each point in space is a particle. Since that particle lies in a vector field, that particle is associated with with a vector. That particle is moving in such that the vector attached to it is its velocity vector. In this way, as the particle moves over time, that it's velocity vector is changing in a way determined by the vector field. As time progresses our particle could be turning, accelerating, going in a straight line or doing any number of things.

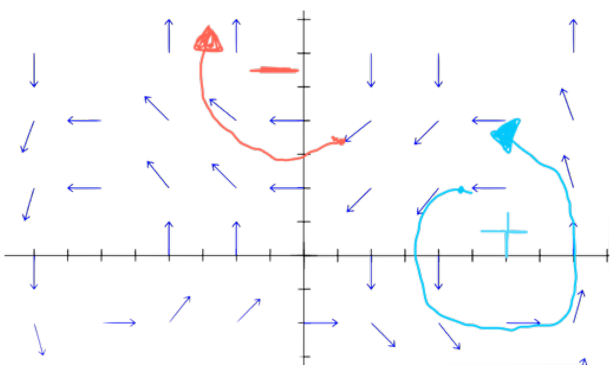


Figure 68. In a region where there is counterclockwise rotation of our particle, curl is positive, and where there is clockwise rotation, curl is negative.

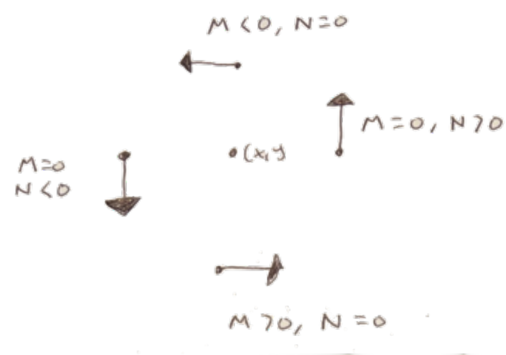
The idea of curl isn't restricted to fluids, it just gives a really good visual idea of what is going on.

15.3.2 Formula

A vector field function $\vec{F}(x, y)$ has two inputs and two outputs. We often express the components of the output as functions $M(x, y)$ and $N(x, y)$. For example, the vector field $\vec{F}(x, y) = (4x^2 + 8xy)\hat{\imath} + (3y^2 + 4x^2)\hat{\jmath}$ has components

$$M(x, y) = 4x^2 + 8xy \quad \text{and} \quad N(x, y) = 3y^2 + 4x^2.$$

Let's look at a fairly contrived example. Let's take some point (x, y) which is associated with the zero vector. We will then place vectors on all four "sides" of this point in the following manner:



We know that curl($\vec{F}(x, y)$) should be positive (there is counterclockwise rotation), thus let's take a look at the partial derivatives of M and N to see if we can use those as a way to quantify curl. Looking at M, we see that as y increases, M goes from positive to negative. Thus $M_y < 0$.

Looking at N, we see that as x increases, N increases (goes from negative to positive), thus $N_x > 0$.

Definition 10 We define the curle of a field to be

$$\text{curl}(\vec{F}) = N_x - M_y.$$

From this definition we see that when there is counterclockwise rotation, and when curl is negative we have clockwise rotation. Also, we see that our field is conservative (is a gradient field) if curl($\vec{F}$) $= N_x - M_y = 0$.

15.3.3 A Formal Definition

We have seen that curl measures the rotational aspect of our field about a given point; positive curl and we have counterclockwise rotation, negative curl and we have clockwise rotation, and zero curl there is no rotation. We can calcuate curl by $N_x - M_y$, but how do we actually define it formally?

To determine the how much our field is rotating about a point, (the origin for example) it would make sense to draw a closed curve (in this case a circle oriented counterclockwise) around the origin, and calculate the line integral

$$\oint_C \vec{F} \cdot d\vec{r}.$$

This will tally up the extent to which the field is in the direction of our curve, and since our curve is oriented counterclockwise and circular, will give a way to measure the rotational aspect of our field about that point.

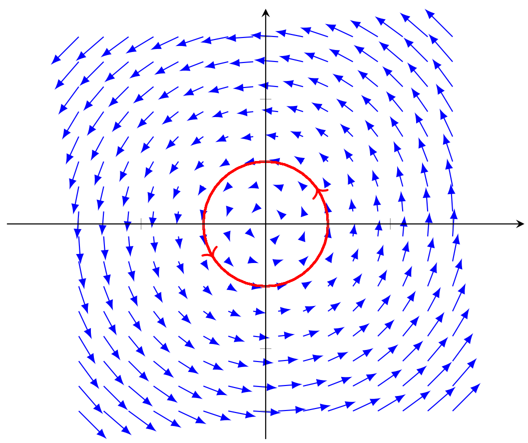


Figure 69. We want to calculate the counterclockwise rotation of our field around a particular point (in this case the origin). To do this it would make sense to compute the line integral of a closed curve centered about the origin and then let the radius that curve approach zero. *Note this is almost, but not quite correct.*

Let $\vec{F} = \langle -y, x\rangle$ (counterclockwise rotating field), let's calculate the line integral (amount of counterclockwise rotation) of a circle with radius r and see what happens as r approaches zero:

$$\begin{aligned}
\oint_C \vec{F}\cdot d\vec{r} &= \int_C M dx + N dy \\
&= \int_C (-y)dx + xdy \\
&= \int_C (-r\sin\theta)dx + r\cos\theta dy \\
&= \int_0^{2\pi} (-r\sin\theta)(-r\sin\theta)d\theta + r\cos\theta r\cos\theta d\theta \\
&= r^2\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)d\theta \\
&= r^2\int_0^{2\pi} 1 d\theta \\
&= 2\pi r^2.
\end{aligned}$$

We encounter a problem, if we let $r \to 0$, then that means the amount our field is "rotating" right around the origin is zero. But this isn't correct, there is obviously rotation happening. The problem is though, that as we make the closed curve smaller and smaller, the line integral approaches zero because the length of the curve approaches zero, and thus the aprea approaches zero.

The solution then is to divide the line integral of the closed curve by it's area, giving us *average rotation per unit area.* We then let the area of the closed curve enclosing our point approach zero, giving us curl.

Definition 11 **2D Curl: Formal Definition**

$$2\text{D}\,\text{curl}(\vec{F}) = \lim_{|A_{(x,y)}|\to 0} \underbrace{\left(\frac{1}{|A_{(x,y)}|}\oint_C \vec{F}\cdot d\vec{r}\right)}_{\text{Average rotation per unit area}}$$

where

- **F** is a two-dimensional vector field.
- (x, y) is some specific point in the plane.
- $A_{(x,y)}$ represents some region around the point (x, y). For instance, a circle centered at (x, y).
- $|A_{(x,y)}|$ indicates the area of $A_{(x,y)}$.
- $\lim_{|A_{(x,y)}|\to 0}$ indicates we are considering the limit as the area of $A_{(x,y)}$ goes to 0 , meaning this region is shrinking around (x, y).
- C is the boundary of $A_{(x,y)}$, oriented counterclockwise.
- $\oint_C$ is the line integral around C, written as $\oint$ instead of $\int$ to emphasize that C is a closed curve.

15.3.4 Applications

What does the curl measure?

- For a velocity field:
  Curl measures rotation component of function. For example, a constant velocity field $\vec{F} = \langle a, b\rangle$ for constants a, b. We clearly see that there is no rotational aspect to a constant velocity field, and so $\text{curl}(\vec{F}) = 0$.

  Looking at the radial vector field $\vec{F} = \langle x, y\rangle$ where everything flows away from the origin, we see that

$$\begin{aligned}\text{curl}(\vec{F}) &= N_x - M_y\\ &= \frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\\ &= 0 - 0\\ &= 0.\end{aligned}$$

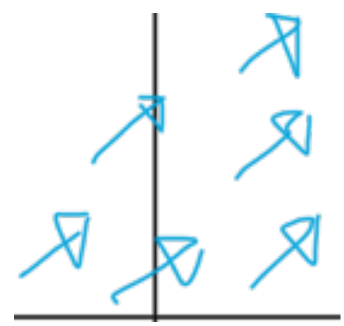


Figure 70. A constant velocity field, $\text{curl}(\vec{F}) = 0$.

  On the other hand, if you consider our favorite rotation vector field $\vec{F} = \langle -y, x\rangle$ then

$$\begin{aligned}\text{curl}(\vec{F}) &= \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\\ &= 2.\end{aligned}$$

  This corresponds to the fact that we are rotating. Note that we are rotating at unit angular speed, and so curl actually measures twice angular speed of rotation component at any given point. Note also that the sign of the curl tells you whether you are going clockwise or counter-clockwise.

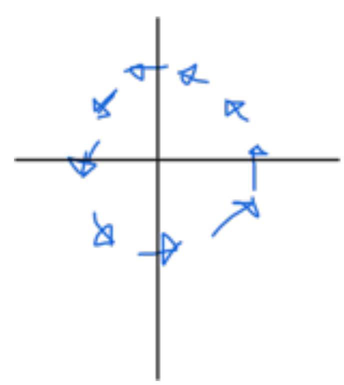


Figure 71. Field $\vec{F} = \langle -y, x\rangle$.

Theorem 8 Curl measures twice $(2x)$ angular velocity of rotation component of velocity field.

- Force field:
  The curl of a force field measures the torque exerted on a test object that you put at any point in the field. Just like how $\frac{\text{force}}{\text{mass}}$ is what causes acceleration (derivative of velocity)

$$\frac{\text{torque}}{\text{moment of inertia}} = \frac{d}{dt} \cdot \text{ angular velocity } = \text{ angular acceleration}$$

In summary, if you are in a velocity field the curl will tell you how fast that thing is spinning at any given time(by a factor of two), and if you are in a force field, the curl tells you how quickly the angular velocity will increase or decrease.

Section 16

Lecture 22: Green's Theorem

Subsection 16.1

Introduction

We saw that the $\text{curl}(\vec{F}) = N_x - M_y$ where $\vec{F} = \langle M, N \rangle$. This measures how far that vector field is from being conservative. That is, if the curl is zero and that field is defined everywhere then that field is conservative. That means that if you have a line integral over a closed curve, you don't have to compute it, it's going to be zero.

But now, let's say you have a general vector field, and you want compute the line integral over a closed curve. Well, you could compute it directly, or you could use green's theorem.

Theorem 9 **Green's Theorem**:
If C is a closed curve going counterclockwise, enclosing a region R, and you have a vector field $\vec{F}$ defined and differentiable in R, then

$$\oint_c \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F} dA$$

or in component form

$$\oint_c M dx + N dy = \iint_R (N_x - M_y)\, dA.$$

This is a really strange statement. The left hand side is a line integral and thus is computed by parameterizing the curve c (expressing x and y in terms of some variable t for example) and computing the one variable integral over t. The right hand side is a double integral, so you slice up the region R into little squares and integrate with respect to $dxdy$ or $dydx$ after setting up the bounds carefully. Thus you can see the two are completely different. The left side lives on the curve, while the right hand side lives inside the curve on the region R.

We will explore how this theorem works, what it says, what are the consequences and try to convince ourselves that it is true. WARNING: This only applies for closed curves.

16.1.1 Example 1

Let $C =$ circle of radius 1 centered at point $(2,0)$, counterclockwise. Compute

$$\oint_C \underbrace{ye^{-x}}_{M}\,dx + \underbrace{\left(\frac{1}{2}x^2 - e^{-x}\right)}_{N}dy.$$

If we were to compute this directly we would need to parameterize the curve in terms of θ:

$$\begin{array}{c} x = 2+\cos\theta \\ y = \sin\theta \end{array} \quad \text{and} \quad \begin{array}{c} dx = -\sin\theta d\theta \\ dy = \cos\theta \end{array}.$$

This does not look like fun, we would end up integrating something involving e^{-x}.

Instead of doing this we can use green's theorem and instead compute a double integral:

$$\begin{aligned} \iint_R (N_x - M_y)\,dA &= \iint_R \underbrace{\left(x + e^{-x}\right)}_{N_x} - \underbrace{e^{-x}}_{M_y}dA \\ &= \iint_R xdA. \end{aligned}$$

How can we compute this integral? Well, if we recall

$$\bar{x} = \frac{1}{\text{Mass}}\iint x\delta dA$$

we see that

$$\iint_R xdA = \bar{x}\cdot\ \text{area}.$$

By symmetry $\bar{x} = 2$ and the area of unit circle is π, so

$$\iint_R xdA = \bar{x}\cdot\ \text{area}\ = 2\pi.$$

16.1.2 Example 1, Alternative

What if you didn't think of using the center of mass formula to solve this integral? Well, we probably want to solve this integral in polar coordinates, but we run into a problem. We want to calculate the integral over unit circle shifted right 2. This makes determining the range of θ and $r =$ radius difficult. Therefore we can shift our region R left two, giving us the integral:

$$\int_0^{2\pi}\int_0^1 dA.$$

But note, that we shifted the region R, we also need to shift our surface. To do this we have

$$\int_0^{2\pi}\int_0^1 (x+2)dA = \int_0^{2\pi}\int_0^1 (r\cos\theta + 2)rdrd\theta.$$

Inner integral:

$$\begin{aligned}\int_0^1 (r\cos\theta+2)r dr &= \int_0^1 \left(r^2\cos\theta+2r\right) dr \\ &= \left[\frac{1}{3}r^3\cos\theta+r^2\right]_0^1 \\ &= \frac{1}{3}\cos\theta+1\end{aligned}$$

Outer integral:

$$\begin{aligned}\int_0^{2\pi}\left(\frac{1}{3}\cos\theta+2\right)d\theta &= \left[\frac{1}{3}\sin\theta+\theta\right]_0^{2\pi} \\ &= (0+2\pi)-(0+0) \\ &= 2\pi\end{aligned}$$

Subsection 16.2

Proving Green's Theorem

Let's say we have some general field $\vec{F}=M(x,y)\hat{\imath}+N(x,y)\hat{\jmath}$, and some closed curve C we need to show that

$$\oint_c M dx + N dy = \iint_R (N_x - M_y)\, dA.$$

Well, doing this outright is rather difficult, but we can make our life a lot easier.

1. We only need to prove the special case where $N=0$ and therefore $N_x=0$. Why is this good enough? Well if we can prove that

   $$\oint_C M dx = \iint_R -M y dA$$

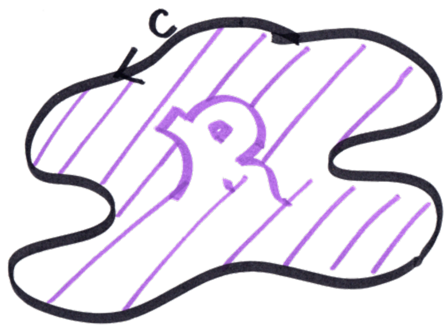
   we can use a very similar argument to show that

   $$\oint_C N dy = \iint_R N_x dA$$

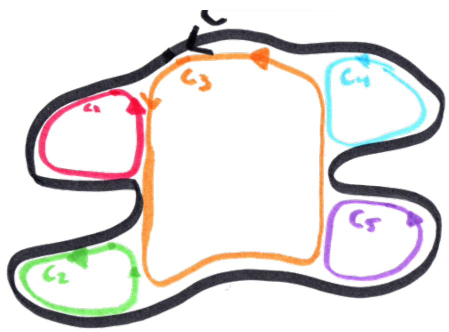
   and thus we can combine the results and show that

   $$\begin{aligned}\oint_C M dx + \oint_C N dy &= -\iint_R M_y dA + \iint_R N_x dA \\ \oint_C (M dx + N dy) &= \iint_R (N_x - M_y) dA.\end{aligned}$$

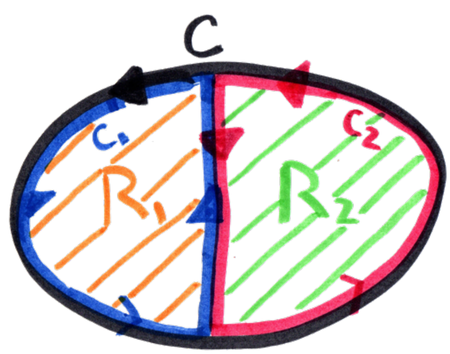
2. We only need to prove for a "vertically simple" region.
   Why is this? Let's say we have a region that isn't vertically simple:



We are able to split up the curve and the region into smaller regions that are vertically simple:



Why are we able to do this? You might se something fishy going on; there are overlapping curves. Well let's look at a simpler example:



We we see that the the overlapping part of curves c_1 and c_2 are going in opposite directions and thus cancel eachother out, and thus $C = C_1 + C_2$.

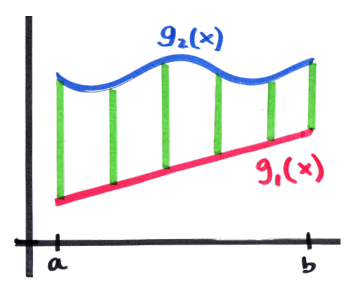


Figure 72. A vertically simple region.

Definition 12 Vertically Simple Region

A vertically simple region is a region where every vertical line drawn share the same upper function ($g_2(x)$) and bottom function ($g_1(x)$). This region can be represented as such

$$R = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$

We have thus taken the problem of computing the line integral over a over a non simple region D and broken it into vertically simple pieces. Thus, if we can show that

$$\oint_C M dx = \iint_R -M_y dA$$

we have related the line integral over a closed curve with the double integral over the region it encloses.

16.2.1 Proof

Let curve C enclose some region R. Assume region R is vertically simple (type I region) and therefore can be characterized by

$$R = \{(x,y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

We must show that

$$\oint_C M dx = \iint_D -M_y dA. \tag{16.1}$$

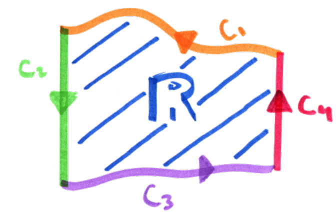


Figure 73. A vertically simple region R enclosed by curve $C = C_1 + C_2 + C_3 + C_4$.

Let's first compute the line integral from equation 16.1. We will compute C_1, C_2, C_3, C_4 seperately. Starting with $\oint_C M dx$. Because our curve C bounds a vertically simple region we can parameterize C_3 as follows:

$$C_3 = \begin{cases} x = x \\ y = g_1(x). \end{cases}$$

Thus, by substituion we have

$$\oint_{C_3} M(x,y) dx = \int_a^b M(x, g_1(x)) dx.$$

Computing $\oint_{C_1}$:
We will use a similar technique as above but our bounds will go from $b \to a$ since that is the direction of the curve C_1. Thus we have

$$\begin{aligned} \oint_{C_1} M(x,y) dx &= \int_b^a M(x, g_2(x)) dx \\ &= -\int_a^b M(x, g_2(x)) dx. \end{aligned}$$

Computing $\oint_{C_2}$ and $\oint_{C_4}$:
Note that for C_2 $x = a$ and for C_4 $x = b$, where both a, b are constants. Thus in both cases $dx = 0$ and so

$$\oint_{C_2} = \oint_{C_4} = 0.$$

Therefore we have

$$\begin{aligned} \oint_C M(x,y) dx &= \oint_{C_1} M(x,y) dx + \oint_{C_2} M(x,y) dx + \oint_{C_3} M(x,y) dx + \oint_{C_4} M(x,y) dx \\ &= -\int_a^b M(x, g_2(x)) dx + 0 + \int_a^b M(x, g_1(x)) dx + 0 \\ &= \int_a^b M(x, g_1(x)) dx - \int_a^b M(x, g_2(x)) dx. \end{aligned}$$

Now let's compute the double integral and see if we can get it to match the line integral. We will let $dA = dydx$ since that is easier, giving us the double integral

$$\iint_D -M_y dA = -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial M(x,y)}{\partial y} dydx.$$

Inner integral:

$$\int_{g_1(x)}^{g_2(x)} \frac{\partial M(x,y)}{\partial y} dy = M\left(x, g_2(x)\right) - M\left(x, g_1(x)\right).$$

Outer integral:

$$\int_a^b \left(M\left(x, g_2(x)\right) - M\left(x, y_1(x)\right)\right) dx = \int_a^b M(x, g_2(x))dx - \int_a^b M(x, g_1(x))dx.$$

Thus we have

$$\iint_D -M_y dA = \int_a^b M(x, g_1(x)) - \int_a^b M(x, g_2(x)).\square$$

QUESTION: When proving green's theorem we just have to prove for

$$\begin{aligned} \text{Vertically simple (Type I regions)} &\quad \text{and } N = 0 \\ \text{Horizontally simple (Type II regions)} &\quad \text{and } M = 0 \end{aligned}$$

where M and N represent the scalar for $\hat{\imath}$ and $\hat{\jmath}$ respectively: $\vec{F} = M\hat{\imath} + N\hat{\jmath}$. What is the connection between the type of region and the special case?

Subsection 16.3

Historical Applications

Green's theorem used to be incredibly useful in computing the area surrounded by some curve. It used to because computers have taken over.

Let's say you were an experimental scientist, you would plot a curve, and you want to know the area under that curve. The problem though is that you don't know the function of this curve, so you can't integrate. The other option is to count the squares of the graph paper to give you a rough estimate, but you might be counding for a long while! Well, people invented a device called a planimeter. What this would do is trace along some curve, and compute the line integral

$$\oint_C xdy = \iint_R 1dA$$

giving you the area inside the cruve.

Section 17

Lecture 23: Flux Integrals

What is a flux integral? Well, recall that the line integral

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{T} dS = \int_C Mdx + Ndy$$

computes the amount of work done by the field on a particle in the direction of the curve C. Or said another way, it is the amount that the curve moves in the direction of the field.

Flux is the antithesis to this. That is, it measures the extent to which the curve is normal to the field.

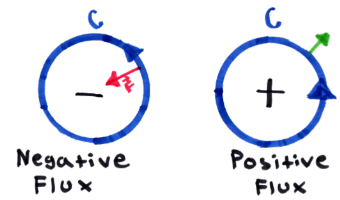


Figure 74. We say that the unit normal vector $\hat{n}$ points outward of a closed loop (to the right of curve).

Definition 13 **2D Flux**

Flux is defined as

$$\int_C \vec{F} \cdot \hat{n} ds$$

where $\hat{n}$ is the unit normal vector to C oriented 90^{deg} clockwise to the unit tangent vector $\hat{T}$.

Subsection 17.1

Interpretation of Flux

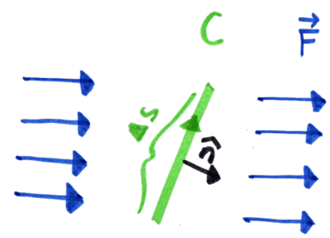


Figure 75. Zooming in on our curve, our field becomes more and more similar and our curve becomes more and more like a straight line.

Why do we have this formula anyways? Well, thinking of $\vec{F}$ as a velocity vector is really beneficial here. Let's say C is a permeable membrane, then flux measures how much fluid passes through C per unit of time.

Why is this true? Let's say that we have some curve C and a vector field $\vec{F}$.

Looking at a small section of length Δs of our curve, we see that as time passes the fluid passes through curve, creating a parallelogram. Note that since we have a velocity vector, each vector represents how far and in what direction the fluid moves per unit of time. Thus, the area of this parallelogram is the amount of fluid that has passed thorugh this small section of curve per unit of time. How can we find this area? Well, drawing a picture helps.

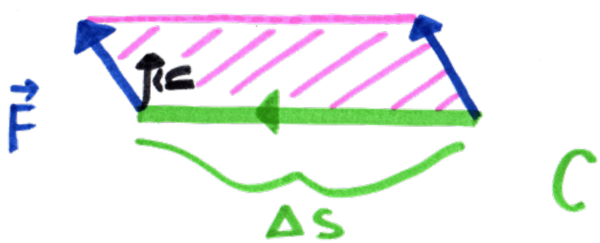


Figure 77. The area of a parallelogram is base $\times$ height. We see that the base is Δs and height is $\vec{F} \cdot \hat{n}$.

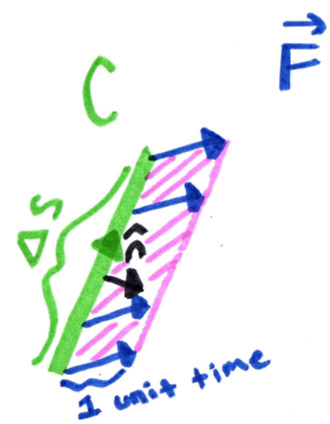


Figure 76. We see that the amount of fluid that has passed through this section of curve is the area of the parallelogram.

We know that our base is Δs and height is the length of the projection of $\vec{F}$ on the vector $\hat{n}$. Because $\hat{n}$ is a unit vector height $= \vec{F} \cdot \hat{n}$. We have just calculated the area for one small section of our curve, we have to sum up all the small parallelograms for each part of our curve and let the distance of each section approach zero. This is just the integral, thus:

$$\frac{\text{Amount of “stuff” passing over curve}}{\text{unit time}} = \int_C \vec{F} \cdot \hat{n} ds$$

which is just 2D flux. Implicit in this explanation is the fact that we are counting positively all the stuff that flows across C in the direction of $\hat{n}$ (left to right) and negatively all the stuff that flows across C opposite the direction of $\hat{n}$ (right to left). Thus, flux computes the net flow across C not total flow.

17.1.1 Example 1

Let C = circle of radius a centered at origin going counterclokwise and $\vec{F} = x\hat{\imath} + y\hat{\jmath}$. We want to calculate

$$\int_C \vec{F} \cdot \hat{n} ds.$$

Well since $\vec{F}$ is in same direction as $\hat{n}$then

$$\begin{aligned} \vec{F} \cdot \hat{n} &= \|\vec{F}\|_2 \\ &= \sqrt{x^2 + y^2} \\ &= \sqrt{r^2} \\ &= \sqrt{a^2} \\ &= a. \end{aligned}$$

Therefore

$$\begin{aligned} a \int_C \vec{F} \cdot \hat{n} ds &= a \int_0^{2\pi} ds \\ &= 2\pi a^2. \end{aligned}$$

Note that $\int_0^{2\pi} ds$ is just length of curve between bounds.

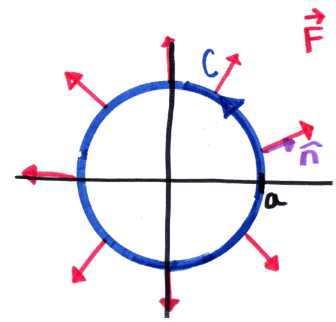


Figure 78. Circle C with radius of a centered at origin and vector field $\vec{F}$.

Subsection 17.2

Computing Flux Integrals

We can't always rely on a geometric approach to solving flux integrals, most of the time we just integrate $Mdx + Ndy$ in coordinates. How do we do this? Well recall that when calculating work we saw that

$$d\vec{r} = \hat{T} ds = \langle dx, dy \rangle$$

and thus can calculate the integral

$$\int_C M dx + N dy.$$

Well, since $\hat{n}$ is $\hat{T}$ rotated 90 deg clockwise

$$\hat{n} ds = \langle dy, -dx \rangle$$

and thus given a field $\vec{F} = \langle P, Q \rangle$ (changed letters to make clear that we are solving a flux integral) and a curve C we can compute flux as

$$\begin{aligned} \int_C \vec{F} \cdot \hat{n} ds &= \int_C \underbrace{\langle P, Q \rangle}_{\vec{F}} \cdot \underbrace{\langle dy, -dx \rangle}_{\hat{n} ds} \\ &= \int_C P dy - Q dx. \end{aligned}$$

Why is this? This might not be perfectly clear still, so let's draw some pictures to get a more visual representation. Let's ignore our field for now and and let's look at a small section of some curve C defined parametrically such that $C = \vec{r}(t)$ (see 60 for review).

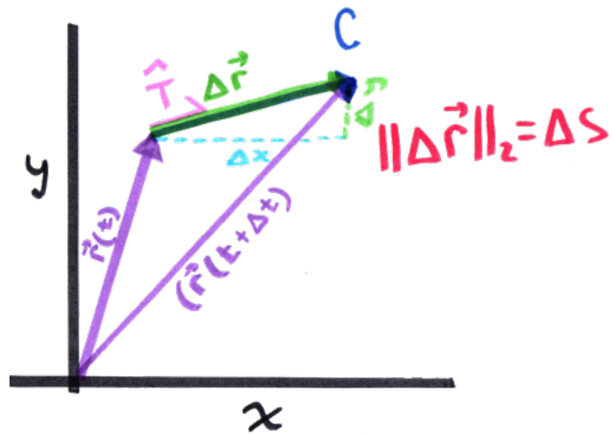


Figure 79. Looking at a small section of curve C we see that $\Delta\vec{r} = \langle \Delta x, \Delta y \rangle = \hat{T}\Delta s$.

Because we use $\vec{F} \cdot \hat{T}\Delta s$ to compute the extent to which the curve is in direction of of C, to compute the extent the field flows through the curve we rotate $\hat{T}$ 90 deg.

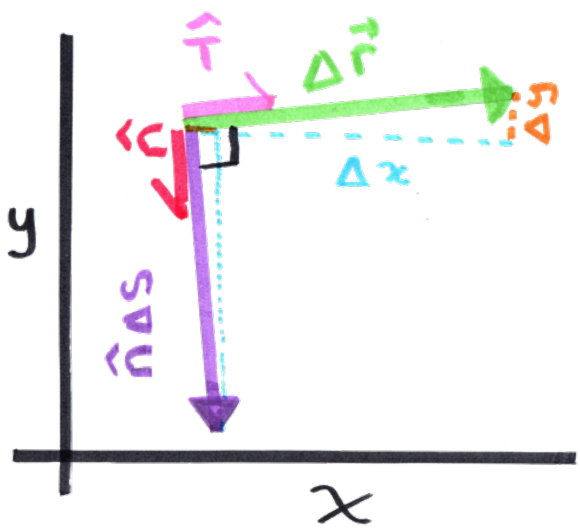


Figure 80. Rotating vector $\hat{T}\Delta s$ 90 deg we have $\hat{n}\Delta s$.

Thus, as you can see if $\vec{F} = \langle P, Q \rangle$ and we have some curve C, to compute flux we compute the integral

$$\int_C \vec{F} \cdot \hat{n} ds = \int_C P dy - Q dx.$$

Subsection 17.3

Green's Theorem for Flux

Let's say you have some curve C enclosing some region R counterclockwise and you would rather not compute the line integral $\oint_C \vec{F} \cdot \hat{n} ds$ directly. We can actually use green's theorem to relate the line integral for flux with the region it encloses.

Theorem 10 **Normal Form of Green's Theorem**

If C encloses a region R counterclockwise and $\vec{F} = \langle P, Q\rangle$ where $\vec{F}$ is defined and differentiable at all points in R then

$$\oint_C \vec{F} \cdot \hat{n} ds = \iint_R \operatorname{div}\left(\vec{F}\right) dA$$

where function

$$\operatorname{div}\left(\langle P, Q\rangle\right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Therefore

$$\oint_C \vec{F} \cdot \hat{n} ds = \oint_C \langle P, Q\rangle \cdot \hat{n} ds = \iint_R P_x + Q_y.$$

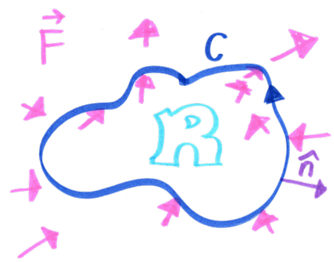


Figure 81. The flux of a closed curve in some field $\vec{F}$ measures the flux out of R through C (if C is oriented counterclockwise).

17.3.1 Proof

We want to show that

$$\oint_C -Q dx + P dy = \iint_R (P_x + Q_y) dA.$$

That is, we want to relate the closed curve to the region that it contains. To prove this let's not worry about what it represents and just focus on the math. To prove this we will just see if we can get our integral to look like our regular green theorem from before. To this end let $M = -Q$ and $N = P$. Thus we have

$$\oint_C -Q dx + P dy = \oint_C M dx + N dy.$$

Well, by green's theorem that means that

$$\oint_C M dx + N dy = \iint_R (N_x - M_y) dA.$$

Thus by substitution we have

$$\iint_R (N_x - M_y) dA = \iint_R (P_x - -Q_y) dA = \iint_R (P_x + Q_y) dA. \square$$

17.3.2 Example 2

In example 17.1.1 we were able to calculate the flux using geometry. This time let's use green theorem.
First calculating the divergence of $\vec{F}$:

$$\operatorname{div}(\vec{F}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2.$$

Therefore by normal form of green's theorem we have:

$$\begin{aligned}\oint_C \vec{F}\cdot\hat{n}ds &= \iint_R \text{div}(\vec{F})dA \\ &= \iint_R 2dA \\ &= 2\iint_R dA \qquad \qquad \iint_R dA \textit{ equals area of region.} \\ &= 2\cdot\pi a^2\end{aligned}$$

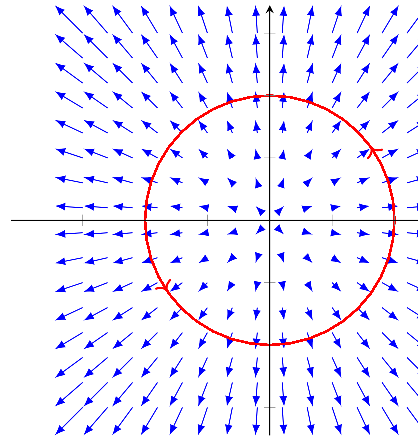


Figure 82. Curve C is a circle with radius a oriented clockwise, and $\vec{F} = x\hat{\imath} + y\hat{\jmath}$.

17.3.3 Example 3

What if this time our circle isn't centered at the origin? Well the vectors exiting the circle (in direction of $\hat{n}$) are bigger than those entering the circle. So maybe we should have a different result. Well, by green's theorem, we see that there is nothing about the fact that the circle is centered at the origin; therefore, the circle can be centered about any point and the result will be the same.

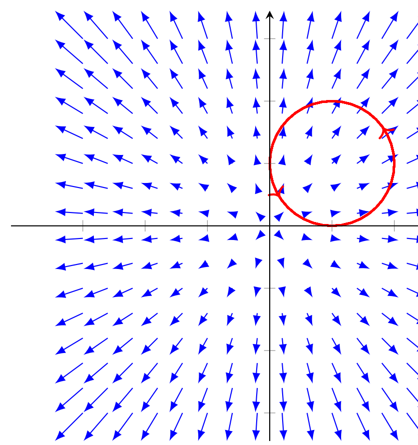


Figure 83. Our curve C is a circle not centered at the origin.

17.3.4 Divergence

What is divergence and what does it measure? Well, divergence measures how much things are diverging. For example divergence will be zero if the vector field is such that all vectors have the same direction (translation) as well as with the vector field where everythings is rotating about the origin (rotation). That is to say that divergence is not sensitive to translation or rotation but to expansion. There are two ways to think about this:

1. Measures how much the flow is "expanding". Looking at our example, things are moving away from the origin at an ever increasing rate and so the area that is being occupied is more and more. This could be representative of gas expanding.

2. Water on the other hand doesn't shrink or expand so the fact that it is taking more and more space means that there is more and more water being added to the system. Looking at the field $\vec{F}$ this means that we have water being pumped into the system at all points. That is, divergence measures the "source rate", how much fluid is being pumped into the system per unit time per unit area. Note, for a different field, we might have water being taken out of the system at some points (sink), so divergence over some region R measures the total source rate which is just (sources - sinks).

Section 18

Lecture 24: Simply Connected Regions

Subsection 18.1

Problem Statement

Given curve enclosing a region we can use green's theorem to compute work or flux along that curve by computing a double integral of the region the curve encloses. One condition though is that $\vec{F}$ (and derivatives) must be defined and differentiable everywhere in R. But what if there are hole(s) in the region R? For example the vector field

$$\vec{F} = \frac{-y\hat{\imath} + x\hat{\jmath}}{x^2+y^2}$$

is defined everywhere except at the origin.

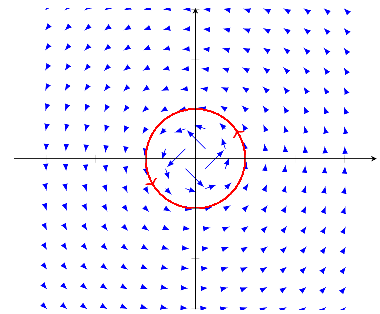


Figure 84. The field $\vec{F}$ is not defined at origin, so we are unable to use green's theorem to compute line integral.

18.1.1 What We Know

We know that given any curve that doesn't enclose the origin that the work done along that curve is zero, and that for the unit circle centered at origin, the work done is 2π. Let's look at why this is the case for each point seperately:

- **Work along any path that doesn't enclose origin is zero**

  Because the curve does not enclose the origin, and thus the region it encloses is defined and differentiable everywhere, we are able to use green's theorem, thus we have

$$\begin{aligned} W = \int_C \vec{F} \cdot d\vec{r} &= \iint_R \operatorname{curl}(\vec{F}) dA \\ &= \iint_R (N_x - M_y) dA. \end{aligned}$$

  Computing N_x we have:

$$\begin{aligned} N_x &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) \\ &= \frac{(1)\left(x^2+y^2\right) - x(2x)}{\left(x^2+y^2\right)^2} \\ &= \frac{y^2+x^2-2x^2}{\left(x^2+y^2\right)^2} \\ &= \frac{y^2-x^2}{\left(x^2+y^2\right)^2}. \end{aligned}$$

  Computing M_y:

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \\ &= \frac{(-1)\left(x^2+y^2\right) - (-y)(2y)}{\left(x^2+y^2\right)^2} \\ &= \frac{2y^2-y^2-x^2}{\left(x^2+y^2\right)^2} \\ &= \frac{y^2-x^2}{\left(x^2+y^2\right)^2}. \end{aligned}$$

Therefore by substitution

$$\begin{aligned}
W &= \iint_R (N_x - M_y) dA \\
&= \iint_R \left(\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2} \right) dA \\
&= \iint_R \left(\frac{y^2 - y^2 + x^2 - x^2}{(x^2+y^2)^2} \right) dA \\
&= \iint_R 0 dA \\
&= 0.
\end{aligned}$$

- **Work done along unit circle centered at origin is 2π**

  Because the unit circle encloses the origin we cannot use green's theorem, therefore we have to compute the work done along curve manually. We will parameterize the integral with regards to θ:

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_C M dx + N dy \\
&= \int_C \frac{-y}{(x^2+y^2)} dx + \frac{x}{x^2+y^2} dy \\
&= \int_C \frac{-y}{r^2} dx + \frac{x}{r^2} dy && x^2 + y^2 = r^2 \\
&= \int_C -y dx + x dy && r = 1 \\
&= \int_C -\sin\theta dx + \cos\theta dy && x = \cos\theta, y = \sin\theta \\
&= \int_C -\sin\theta(-\sin\theta) d\theta + \cos\theta \cos\theta d\theta && dx = -\sin d\theta, dy = \cos\theta d\theta \\
&= \int_0^{2\pi} \left(\cos^2\theta + \sin^2\theta\right) d\theta \\
&= \int_0^{2\pi} 1 d\theta \\
&= 2\pi.
\end{aligned}$$

18.1.2 The Question

We were able to fairly easily parameterize the unit circle centered at the origin, but what about a different curve centered around the origin? Well, that could get a lot more difficult to solve, therefore, what can we do?

Subsection 18.2

The Solution

What we can do is create two closed curves, c' and c'', both of which contain the origin (undefined point) of our vector field $\vec{F}$.

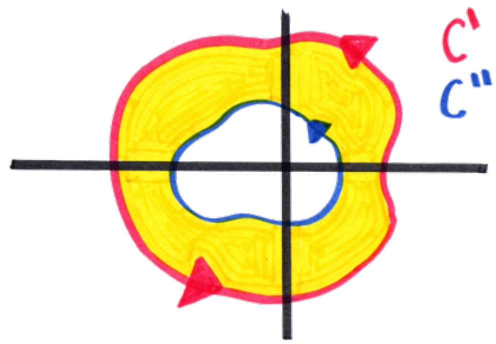


Figure 85. We are able to relate the curves c' and c'' to the yellow area.

Because the region R is defined and differentiable everywhere, we can relate the region R with the two closed curves C' and C'' in the following way:

$$\begin{aligned} \oint_{C'} \vec{F} \cdot d\vec{r} - \oint_{C''} &= \iint_R \operatorname{curl}(\vec{F}) dA \\ &= 0 && \textit{In our case}. \end{aligned}$$

Thus we can see that

$$\oint_{C'} \vec{F} \cdot d\vec{r} = \oint_{C''}$$

and so any closed curve that contains origin, the work done along curve is 2π. Why are we able to do this?

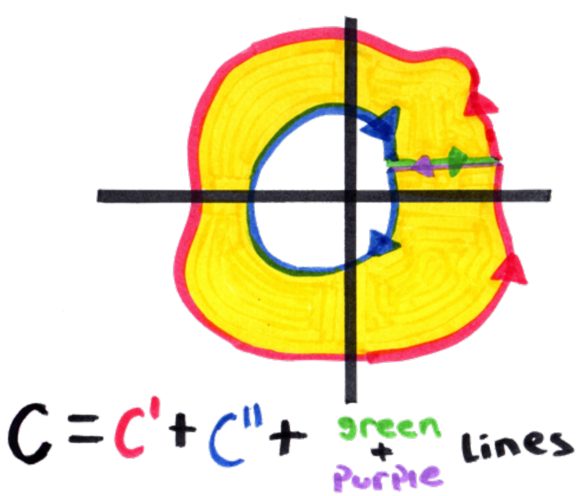


Figure 86. We can connect curves C' and C'' thus avoiding undefined origin and creating a single closed curve C surrounding a well defined and differentiable region R.

Well, as seen above we can connect the two curves C' and C'' creating a single closed curve C, and thus we are able to use green's theorem. Also note that the green line and purple line are in opposite directions, thus the work done by each line cancels eachother out. Therefore by green's theorem we have

$$\cdot d\vec{r} = \iint_R \text{curl}(\vec{F})dA.$$

Recall though that in 84 both curves are going counterclockwise, whereas in figure 85 C' is going counterclockwise, but C'' is going clockwise. Therefore, switching the direction of C'' we have

$$\oint_{Cprime} \vec{F}\cdot d\vec{r} - \oint_{C;\prime\prime} \vec{F}\cdot d\vec{r} = \iint_R \text{curl}(\vec{F})dA.$$

Subsection 18.3

An Informal Definition

Informally an object is simply connected if it consists of one piece and does not have any holes in it. Thus a line is simply connected, but a circle isn't. Similarly a simply connected region is a region that is connected (not two disconnected regions) and does not have any holes in it. A region is simply connected if you can take any loop in the region, contract it to a point and that point will remain in the region.

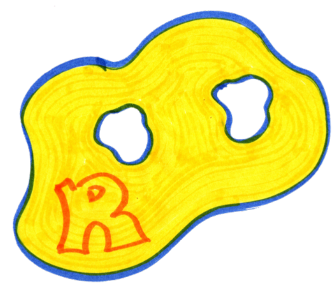


Figure 87. Region R is not simply connected because a loop that encircles one of the holes could not be contracted to a point without exiting the region R.

Definition 14 **Simply Connected Region**

A connected region R in the plane is simply connected if the interior of any closed curve in R is also also contained in R.

Definition 15 **Gradient Field**

If curl $\vec{F} = 0$ and domain where $\vec{F}$ is defined is simply connected, then $\vec{F}$ is conservative and a gradient field.

Section 19

Lecture 25: Triple Integrals in Rectangular and Cylindrical

We have learned about double integrals in the plane but now we are going to learn about triple integrals in space. We will use many of the same techniques we learned in the plane and apply them in space. With double integrals we took some region in the plane and split it up into small areas. With triple integrals we will take some region R in space and split it up into small volumes and integrate some function f:

$$\iiint_R f dV.$$

This just means that we take every little small volume, multiply it by the value of function f at that point and then return the sum of all these little operations. The volume element dV in xyz coordinates will take the form $dV = dxdydz$ (or any permutation of the set $\{dx, dydz\}$).

SUBSECTION 19.1
Examples

19.1.1 Example 1: Volume of region between parabaloids

We will use triple integrals to find the region R between the prabaloids

$$z = x^2 + y^2$$
$$z = 4 - x^2 y^2$$

by computing the integral

$$\text{Volume } = \iiint_R 1dV.$$

We will integrate first with respect to dz giving us the integral

$$\iiint_R 1dzdA$$

where dA represents the small area $dxdy$ or $dydx$. Why will we first compute the integral dz? This is because we are given our equations in the form $z = f(x, y)$ and thus given an x and a y we can easily find the upper and lower bounds of z; the lower bound of z will be given by the equation $z = x^2 + y^2$ and the upper bound of z will be given by the equation $z = 4 - x^2 - y^2$. Thus, so far our integral is

$$\text{Volume } = \int_?^? \int_?^? \int_{x^2+y^2}^{4-x^2-y^2} 1dzdA.$$

How can we find the bounds for xy and should we have $dA = dxdy$ or $dA = dydx$? Well what we will do is place a light source directly above our region R in space and take note of the shadow our region creates on the xy plane. This shadow will contain all the (x, y) values used to create our region in space R.

How can we find the equation of this shadow in the xy-plane? Well we know that the shadow is formed whenever

$$z_{\text{lower bound}} \leq z_{\text{upper bound}}$$
$$x^2 + y^2 \leq 2. \qquad \textit{Disk of radius } \sqrt{2}$$

Therefore it doesn't really matter if we let $dA = dxdy$ or $dA = dydx$. We will let $dA = dydx$. Therefore we can compute our bounds just like we did for double integrals and get the integral

$$\text{Volume } = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} 1dzdydx.$$

Computing the inner integral:

$$\int_{x^2+y^2}^{4-x^2-y^2} 1dz = [z]_{x^2+y^2}^{4-x^2-y^2} = 4 - 2x^2 - 2y^2.$$

We then have the double integral

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} 4 - 2x^2 - 2y^2 dydx$$

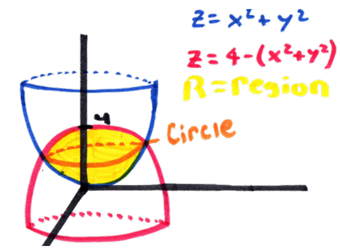


Figure 88. Drawing the two paraboloids we see that a circle is formed where they intersect. Also note that this circle takes place at largest point of greatest width of our region.

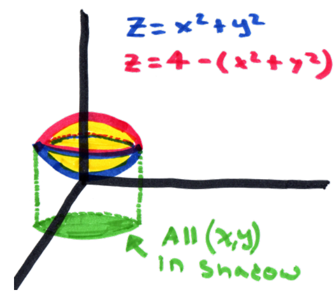


Figure 89. We see that all (x, y) values used to create our region R will be in green shadow. Therefore, to find the x and y bounds of our integral we need to find the equation of the shadow in the xy plane.

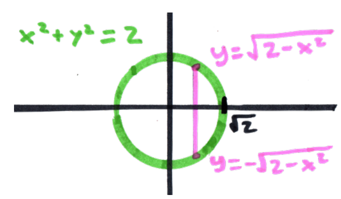


Figure 90. We see that for a given x value the upper limit of y is $\sqrt{2-x^2}$ and the lower limit of y is $-\sqrt{2-x^2}$.

which we would solve by changing to polar coordinates.

19.1.2 Example 1, but in polar coordinates

We want to convert the integral

$$\text{Volume } = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} 1dzdydx$$

to polar coordinates. We will do this by letting $r^2 = x^2 + y^2$ and with the knowledge that the radius of our circle will range from 0 to $\sqrt{2}$. Therefore by substitution we have

$$\begin{aligned} \text{Volume } &= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} 1dzdydx \\ &= \iiint_R dz \cdot rdrd\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} 1 \cdot dzrdrd\theta. \end{aligned}$$

Subsection 19.2

Cylindrical Coordinates

In our second example 1 we were using this thing called cylindrical coordinates. What are cyclindrical coordinates? In cylindrical coordinates we define a point in space by the distance of that point projected onto the xy-plane from the origin, the angle θ of that projected point and the height difference z between the point and the projected point.

Agian recall that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Why are they called cylindrical coordinates? Recall that in the xy-plane the equation

$$r = a$$

defines a circle; the set of points (x, y) such that $r = a$. But in space, the set of points (x, y, z) that meet the condition that $r = a$ defines a cylinder.

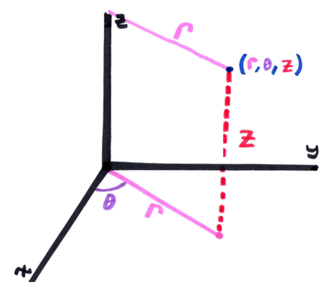


Figure 91. A point in cylindrical coordinates is defined by (r, θ, z)

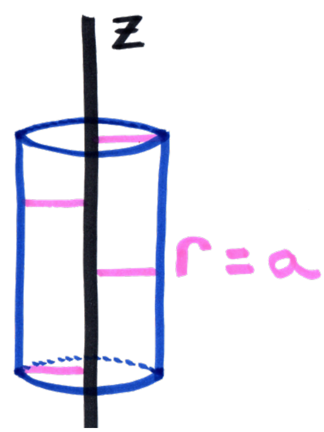


Figure 92. When in xyz coordinates (space) the equation $r = a$ defines a cylinder with radius a.

Subsection 19.3

Applications

The core idea does not change with the addition of an axis, so see 11.2 for a more in depth exploration of these applications.

19.3.1 Volume of region R

As we have seen the volume of region R is given by the triple integral

$$\text{Volume } = \iiint_R 1dV.$$

19.3.2 Mass of region R

Well we know that

$$\text{density } \delta(x, y, z) = \frac{\Delta\text{mass}}{\Delta \text{ volume}}$$

thus to find the mass of a tiny volume in our region we have

$$dm = \delta \cdot dV.$$

Therefore to find the mass of the entire region, we sum up the masses of all our little volumes (given some density function $\delta(x, y, z)$):

$$\text{Mass } = \iiint_R \delta(x, y, z) dV.$$

19.3.3 Average value of $f(x, y, z)$ in R

This would be the sum of $f(x, y, z)$ over the region R divided by the volume of the region R. Therefore we have

$$\bar{f} = \frac{1}{\text{Vol}(R)} \iiint_R f dV$$

where $\bar{f}$ is the average value of f.

19.3.4 Weighted average value of $f(x, y, z)$ in R

If we want to give the value of f more weight at particular positions (x, y, z) than others than

$$\text{Weighted Avg. } = \frac{1}{\text{Mass}(R)} \iiint_R f \delta dV$$

where δ represents the weight we give f at a given position.

19.3.5 Center of Mass

The center of mass is $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{\text{Mass}} \iiint_R x \delta dV.$$

Note that $\bar{y}$ and $\bar{z}$ have similar functions. When calculating this, we can try and calculate fewer dimensions by taking symetry into acount.

19.3.6 Moment of Inertia

Recall that moment of inertia is how hard it is to spin something. This is similar to what mass is to translation, where the higher mass something has, the harder it is to move. Recall that in the xy-plane we found that the moment of inertia about the origin is

$$I_O = \iint_R r^2 \cdot \delta \cdot dA.$$

Well, notice that when we were rotating the object about the origin we were secretly rotating it about the z-axis. Thus, in space, the moment of inertia of an object about the z-axis is:

$$I_z = \iiint_R r^2 \cdot \delta dV = \iiint_R (x^2 + y^2) \delta dV.$$

Thinking more generally, the moment of inertia about an axis is just

$$I_{\text{axis}} = (\text{distance from axis })^2 \cdot \delta dV.$$

Applying this generalization to the x and y axis, we can find the moment of inertia about those axis as well:

$$I_x = \iiint_R (y^2 + z^2) \delta dV \quad \text{ and } \quad I_y = (x^2 + z^2) \delta dV.$$

Subsection 19.4

More Examples

19.4.1 Example 2

Calculate the moment of inertia of a solid cone between

$$z = ar \quad \text{and} \quad z = b$$

with density $\delta(x, y, z) = 1$.

Firstly, why is this a cone?. Let's first think about the function $z = r$. Just thinking about the yz-plane ($x = 0$) we have $z = r = \sqrt{x^2 + y^2}$. But since $x = 0$ we have $z = \sqrt{y^2} = y$. This is just a straight line with slope 1. Then with the addition of x we just rotate that line about the z-axis and so we have a cone. Therefore, the equation

$$z = ar$$

is just a cone with slope a. The equation $z = b$ is just a plane parallel to the xy plane with height z, so it just sets the height of our cone.

Let $dV = dzrdrd\theta$, giving us the integral

$$\iiint_R r^2 dzrdrd\theta = \iiint_R r^3 dzdrd\theta.$$

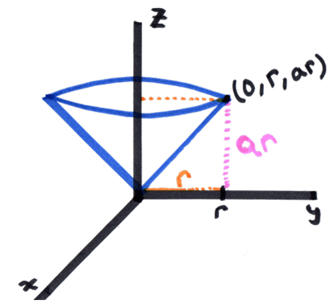


Figure 93. Graph of solid cone lying between curves $z = ar$ and $z = b$. Cone has slope of $\frac{ar}{r} = a$.

What are the bounds of integration? Letting θ vary we want to know for a particular θ what the range of r values will be such that we remain in region R, and then for a given r value what the range of z values will be such that we remain in R.

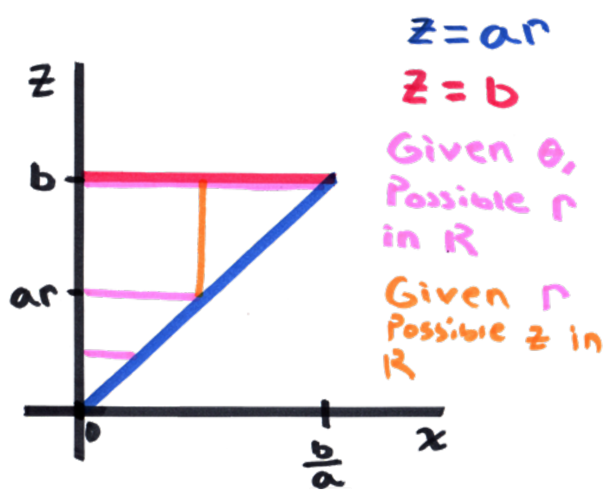


Figure 94. If we lock theta (in this case $\theta = 0$), then we see that the possible values of r such that we remain in our region R are $0 \le r \le \frac{b}{a}$. For a given one of these r, we see that the range of z values for which we are still in region R are $ar \le z \le b$.

We can thus set up our integral as follows:

$$\begin{aligned} \int_0^{2\pi} \int_0^{\frac{b}{a}} \int_{ar}^{b} r^2 \cdot dzrdrd\theta &= \int_0^{2\pi} \int_0^{\frac{b}{a}} \int_{ar}^{b} r^3 \cdot dzdrd\theta \\ &= \frac{\pi}{10} \cdot \frac{b^5}{a^4} && \textit{By Inspection} \end{aligned}$$

19.4.2 Example 3

Setup triple integral $\iiint_R 1dV$ where R is the region such that $z > 1-y$ and inside unit ball centered at origin. Since we are given two functions in terms of z (recall that equation of unit circle has equation $x^2+y^2+z^2=1$ which we can put in terms of z) it is easist to let $dV = dzdxdy$. We see that y varies between 0 and 1, thus so far our bounds of integration are:

$$\int_0^1 \int_?^? \int_?^? 1dzdxdy.$$

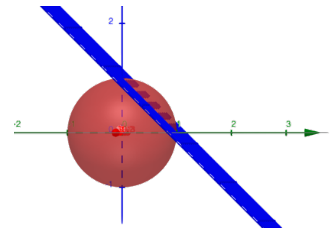


Figure 95. Graph of unit ball centered at origin and plane $z = 1-y$.

Since our region in terms of z is $z > 1-y$ and $z < \sqrt{1-x^2-y^2}$ we have the bounds for z:

$$\int_0^1 \int_?^? \int_{1-y}^{\sqrt{1-x^2-y^2}} 1dzdxdy.$$

Finding the bounds for x is the difficult part. To do this we look at the shadow created on the xy plane by our region R. To find this equation we note that the shadow is defined when the plane and ball have equal height z, when $z_{\text{lower bound}} = z_{\text{upper bound}}$. Thus setting these equal we are able to find the equation of the shadow (which in this case is an ellipse):

$$\begin{aligned} z_{\text{lower bound}} &= z_{\text{upper bound}} \\ 1-y &= \sqrt{1-x^2-y^2} \\ (1-y)^2 &= 1-x^2-y^2 \\ 1-2y+y^2 &= 1-x^2-y^2 \\ x^2 &= 2y-2y^2 \\ x &= \pm\sqrt{2y-2y^2}. \end{aligned}$$

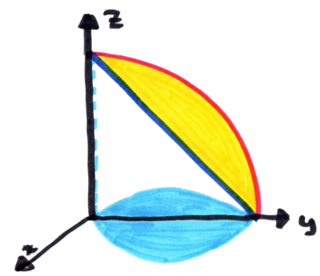


Figure 96. To find the bounds for x and y we find the equation of the shadow on the xy-plane created by our region R.

Threfore $x_{\text{lower bound}} = -\sqrt{2y-2y^2}$ and $x_{\text{upper bound}} = \sqrt{2y-2y^2}$ and our integral is

$$\int_0^1 \int_{-\sqrt{2y-2y^2}}^{\sqrt{2y-2y^2}} \int_{1-y}^{\sqrt{1-x^2-y^2}} 1dzdxdy.$$

Section 20

Lecture 26: Spherical Coordinates and Surface Area

With spherical coordinates we will represent a point in space using its distance to the origin and two angles. How do we do this? We describe a point in space using the three coordinates:

- $\rho =$ rho. This is the distance from origin to point.
- $\varphi = \phi = ph_i$. This is the angle going down from the positive z-axis. Note that $0 \le \phi \le \pi$.
- θ. This is angle starting from x-axis going counterclockwise. The same θ we know and love where $0 \le \theta \le 2\pi$.

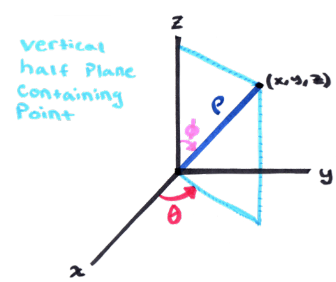


Figure 97. We can describe a point in space in terms of the two angles θ and ϕ, and the distance from origin to point ρ.

Subsection 20.1

Ways of Thinking About

20.1.1 A Geographical Approach

If you let ρ be a constant and have angles θ and ϕ vary, then you have a sphere. How do the values of θ and ϕ affect your position on the sphere?

- ϕ measures how far south you are on the sphere, or in other terms, how far you are from the north pole. In this way ϕ can be thought of as latitude, but instead of 0 being the equator, it is the north pole.
- θ measures how far east or west you are. So starting at the x-axis and going counterclockwise (east).

Using these conventions, you can get a pretty good idea of where your point is in space given it's spherical coordinates.

Figure 98. In spherical coordinates ϕ can be thought of as latitude and θ as longitude.

20.1.2 Cylindrical coordinates, Twice

Recall that in cylindrical coordinates 19.2 we described some point in space, but put x and y in terms of θ and r (distance from z axis). Well, let's do the same thing here. Given some point in space let's describe it in cylindrical coordinates. Therefore our point will have some distance r from the z-axis (from the origin if we are in xy-plane) and some angle θ from the x-axis. Let's look at the specific vertical half-plane zr that includes our point. The idea here is that we will describe that point in terms of distance from the origin ρ, and some angle ϕ from the positive z-axis. This understanding of spherical coordinates is the key to convert between cartesian and spherical coordinates.

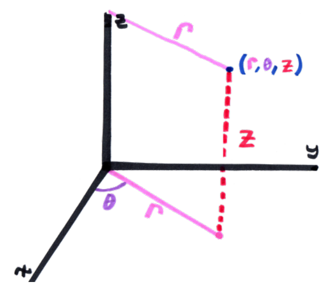


Figure 99. First placing our point in cylindrical coordinates, we can describe its point on the xy-plane in terms of r and θ.

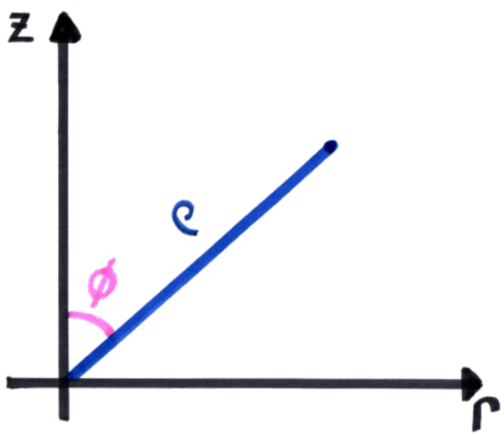


Figure 100. Looking at the zr-plane, we are able to put z and r in terms of ϕ and ρ. Recall that we had already made xy in terms of θ and r. Thus, we can describe point in space in terms of ρ, θ and ϕ.

We see that

$$z = \rho\cos\phi \quad \text{and} \quad r = \rho\sin\phi.$$

If you remember x and y in terms of $\sin\theta$ and $\cos\theta$ then by substitution we have:

$$\begin{aligned} x &= r\cos\theta = \rho\sin\phi\cos\theta \\ y &= r\sin\theta = \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

Also, if you ever need to switch back we have:

$$\rho = \sqrt{r^2 + z^2} = \sqrt{x^2 + y^2 + z^2}.$$

SUBSECTION 20.2

Example Surfaces

20.2.1 $\rho = a$

This is a sphere of radius a centered at origin.

20.2.2 $\phi = \frac{\pi}{4}$

This is the shape where any point on surface can be reached by the vector of length r in direction of $\phi = \frac{\pi}{4}$ with relation to the z-axis. This is a cone. Drawing a picture will help us see this. Note that when ϕ is $\frac{\pi}{4}$ the $\cos\phi = \sin\phi$ thus, since

$$z = \rho\cos\phi \quad \text{and} \quad r = \rho\sin\phi$$

then when $\phi = \frac{\pi}{4}$, $z = r$.

Note a special case, what is the shape formed by $\phi = \frac{\pi}{2}$. This is the flattest of all cones, just the xy-plane. In general if $\phi < \frac{\pi}{2}$ then you are in the upper half space, and if $\phi > \frac{\pi}{2}$ then you are in the lower half space.

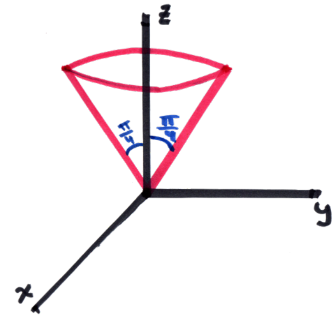


Figure 101. Cone formed by the equation $\phi = \frac{\pi}{4}$.

SUBSECTION 20.3

Triple Integrals in Spherical

20.3.1 What is the value of the volume element dV?

Just like when we went from cartesian to polar coordinates we saw that $dA = rdrd\theta$, we need to figure out what dV is in spherical coordinates. We know that it will involve the small changes $d\rho, d\phi$,and $d\theta$, but what what else will it involve? We will start by first calculating the area on our surface and then multiply by $\Delta\rho$ since that is simple multiplication.

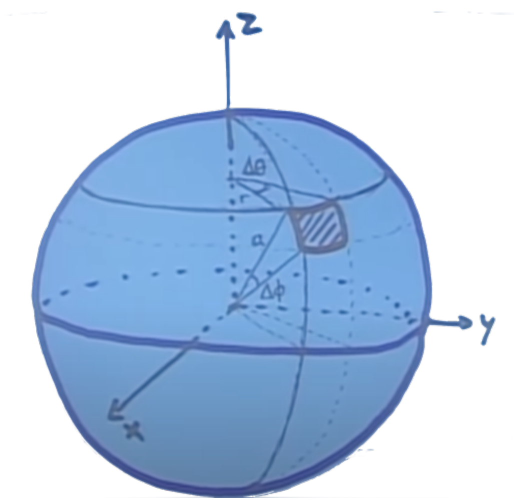


Figure 102. We want to figure out the area of this surface area element.

Although the lines of our "rectangle" on the surface are curvy, as that the changes get smaller and smaller it will look more and more like a rectangle, so we can just try and figure out the lengths of the sides of our rectangle.

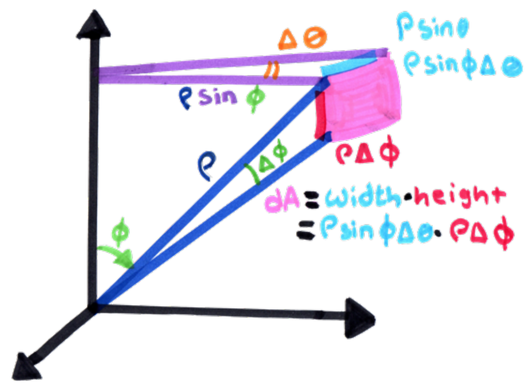


Figure 103. We want to find the area of this small surface area element of a sphere.

We will find the width, height and depth seperately, and then just multiply them together to get dV:

- **Width of surface area elemnt**
  Well, to find the width of our surface area element we can use the arc angle formula:
  $$\text{Length of Arc} \;=\; \text{angle} \;\cdot\; \text{radius}\;.$$
  Note that a sphere is just composed of circles of varying radiuses stacked on top of eachother. We want to know the radius of the particular circle that the blue line is composed from. Well, creating a right triangle with ρ we see that the radius is $\rho \sin \phi$. The angle is just $\Delta\theta$ so therefore:
  $$\begin{aligned} \text{width} \;&=\; \text{Angle} \;\cdot\; \text{radius} \\ &= \Delta\theta \cdot \rho \sin \phi. \end{aligned}$$

- **Height of surface area elemnt**
  Because the red curve (height) is created from rotating ρ about the origin along a plane, the radius of the circle is simple ρ. Therefore the length of height element is simply
  $$\text{height} \;= \Delta\phi \cdot \rho.$$

- **Depth of elemnt**
  Well the depth of element is just the change in ρ so
  $$\text{depth} \;= \Delta\rho.$$

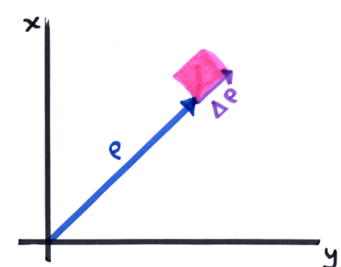


Figure 104. Looking at a top down view of our volume element to visualize its thickness (depth). Thus we see that depth of volume element is $\Delta\rho$.

Thus the approximate volume of our element is

$$\begin{aligned} \Delta V &\approx \Delta\text{width} \;\cdot \Delta\text{height} \cdot \Delta\text{depth} \\ &= (\Delta\theta \cdot \rho \sin \phi) \cdot (\Delta\phi \cdot \rho) \cdot (\Delta\rho) \\ &= \rho^2 \sin \phi \Delta\rho\Delta\phi\Delta\theta \end{aligned}$$

and by letting the changes in width, height, and depth trend towards zero we have

$$dV = \rho^2 \sin \phi d\rho d\phi d\theta.$$

Subsection 20.4
Examples

20.4.1 Example 1

Recall in 19.4.2 we calculated the volume of a piece of a sphere sliced by a slanted plane. Since all we care about is finding the volume we will be a little bit smarter this time and rotate the sphere so that our piece of interest is centered by the z-axis. To calculate the volume of this shaded region we calculate the triple integral

$$V(R) = \iiint_R 1 dV.$$

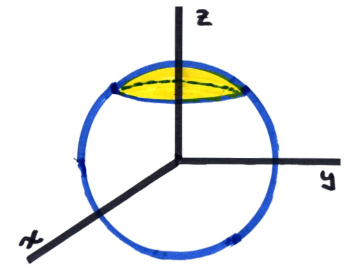


Figure 105. Shaded region is that of the unit sphere above plane $z = \frac{1}{\sqrt{2}}$.

Since we are using spherical coordinates we will let $dV = \rho^2 \sin\phi d\rho d\phi d\theta$. We must find the bounds of θ, ϕ, and ρ.

- **Bounds of θ**
  Because we are letting θ vary, we see that θ ranges from 0 to 2π. Thus so far we have the integral
  $$\int_0^{2\pi} \int_?^? \int_?^? 1 \cdot \rho^2 \sin\phi d\rho d\phi d\theta.$$
- **Bounds of ϕ**
  Well starting at the north pole, how what is the range of ϕ. Well since the north pole is in region the lower bound of ϕ is 0. To find the upper bound recall that
  $$z = \rho\cos\phi.$$
  We want to find ϕ when $z = \frac{1}{\sqrt{2}}$ and when $\rho = 1$ (unit sphere, distance is 1). Thus by substitution we have
  $$\begin{aligned} z &= \rho\cos\phi \\ \frac{1}{\sqrt{2}} &= 1\cos\phi \\ \phi &= \frac{\pi}{4}. \end{aligned}$$
  We see this geometrically by taking a slice of region along the zr-plane. Therefore, so far we have the integral
  $$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_?^? 1 \cdot \rho^2 \sin\phi d\rho d\phi d\theta.$$

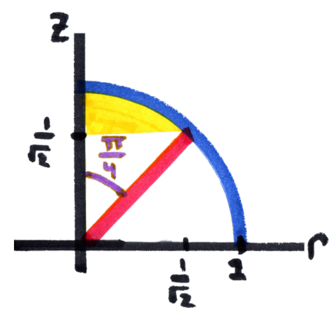


Figure 106. Taking a slice of region along zr-plane we see that $\phi = \frac{\pi}{4}$. Recall unit circle and note that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

- **Bounds of ρ**
  The upper bound of ρ is clearly 1, but what is the lower bound? Well, it changes depending upon ϕ. Recall that
  $$z = \rho\cos\phi.$$
  Since we know that z must equal $\frac{1}{\sqrt{2}}$ we have $\rho = \frac{\frac{1}{\sqrt{2}}}{\cos\phi} = \frac{1}{\sqrt{2}\cos\phi}$. Thus, the final integral is
  $$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_{\frac{1}{\sqrt{2}\cos\phi}}^1 1 \cdot \rho^2 \sin\phi d\rho d\phi d\theta.$$

By inspection

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\sqrt{2}\cos\phi}}^{1} 1\cdot\rho^2\sin\phi\, d\rho d\phi d\theta = \frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}}.$$

Subsection 20.5

Applications

We of course have the same applications as before: finding volumes, masses, and average value of functions. Now though, if for example we want to find the average distance from the solid to the origin then we no longer need to integrate with respect to x, y, z we can just integrate with respect to ρ. Using spherical coordinates to calculate moment of inertia is most useful when you have rotation about z-axis. This is because if you have to involve x or y then you will have that really bad $x = \rho\sin\phi\cos\theta$ for example.

20.5.1 Gravitational Attraction

Gravitational attraction is what causes apples to fall and we humans to not float away. Physics says that if you have two masses than they attract each other with a force that's directed towards eachother and in intensity it's proportional to the two masses and inversely proportional to the square of the distance between them:

$$F = G\frac{m_1 m_2}{r^2}$$

where F is the force, G is the gravitational constant, m_1 is mass of object 1, m_2 is mass of object 2, and r is the distance between the centers of the masses.

If we want to find the gravitational force between ourself and some body (the earth for example), it's not so simple. The inclination would be to treat the body as a point mass, but that's not so simple if our body isn't of homogenous density or is some weird shape. Therefore, what we have to do is sum up all the gravitational forces acted upon you by all small parts of the body.

For our example we will treat ourself as a point mass situated at the origin m, and the body for which we want to calculate gravitational force some object in space. The magnitude of the force vector between ourself m and the mass of a small piece ΔM is

Figure 107. Placing ourselves m as a point mass situated at the origin, we want to calculate both the magnitude and direction of gravitational force vector between m and ΔM.

$$\|\vec{F}\| = \frac{G\cdot\Delta M\cdot m}{\rho^2}.$$

The direction unit vector is simply $\langle x, y, z\rangle$ divided by distance:

$$\mathrm{dir}(\vec{F}) = \frac{\langle x, y, z\rangle}{\rho}.$$

Therefore the complete force vector is given by

$$\vec{F} = \frac{G\Delta M m}{\rho^3}\langle x, y, z\rangle.$$

To get the total force acted between the two objects we have to sum up all the small forces, giving us the triple integral

$$\begin{aligned}\text{Total } \vec{F} &= \iiint_R \frac{G\cdot m\cdot\langle x, y, z\rangle}{\rho^3}dM \\ &= \iiint_R \frac{G\cdot m\cdot\langle x, y, z\rangle}{\rho^3}\delta dV && \Delta M = \delta\Delta V\end{aligned}$$

To compute this integral, we have a problem. Computing the integral in xyz coordinates would be fine except for the inclusion of ρ^3 which would mean the addition of $\rho^3 = (x^2+y^2+z^2)^{\frac{3}{2}}$ which would make life difficult. The problem with computing it in spherical coordinates though is x and y, which when converted to spherical coordinates become difficult. The solution then is to rotate our object so that it is centered about the z-axis. This means that the gravitational force will be along the z-axis, and so we just have to figure out the z comonent of $\vec{F}$. Thereforem, to compute the force for the z-component of vector ($\vec{F}_z$) we have:

$$\begin{aligned}
\vec{F}_z &= \iiint_R \frac{G\cdot m\cdot z}{\rho^3}\delta dV \\
&= Gm\iiint_R \frac{z}{\rho^3}\delta dV && \textit{Gm are constants} \\
&= Gm\iiint_R \frac{\rho\cos\phi}{\rho^3}\delta\cdot\rho^2\sin\phi d\rho d\phi d\theta && \textit{spherical coordinates} \\
&= Gm\iiint_R \cos\phi\sin\phi\delta\cdot d\rho d\phi d\theta.
\end{aligned}$$

We can use this to prove Newton's theorem which states that gravitational attraction of a spherical planit with uniform density is equal to that of a point mass (with same total mass) at its center.

Section 21

Lecture 27: Vector Fields in 3D, Surface Integrals and Flux

We saw how to set up triple integrals when dealing with space, now we will learn about vector fields in space. We have previously learned about vector fields in the plane, see 13.

Definition 16 A vector field in space is a vector valued function

$$\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k} = \langle P, Q, R\rangle$$

where P, Q, R are functions of x, y, z.

Subsection 21.1

Examples of Vector Fields

21.1.1 Force Fields

An example force field would be the gravitational attraction of a solid mass at the origin $(0,0,0)$ on a mass m at point (x,y,z). This would be given by a vector field $\vec{F}$ directed towards origin and whose magnitude is given by $\frac{c}{\rho^2}$, where c is some constant and ρ is distance to the origin. The formula for this vector field would be:

$$\vec{F} = -c\frac{\langle x,y,z\rangle}{\rho\cdot\rho^2} = -c\frac{\langle x,y,z\rangle}{\rho^3}.$$

The vector $\langle x,y,z\rangle$ is just the vector going from origin to our point. Dividing that by distance ρ we have the unit vector in direction. Multiplying that by $-c$ we point that vector in direction of origin multiplied by some constant and then divide by distance squared (ρ^2).

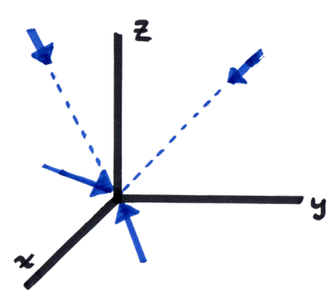


Figure 108. The vector field of gravitational attraction would look something like this where the vectors are directed towards origin and magnitude is inversely proportional to the square of the distance to origin.

21.1.2 Gradient Field

We have seen in $2D$ that gradient fields are examples of vector fields. The same applies in $3D$. Let $u = u(x, y, z)$ then $\nabla u = \langle u_x, u_y, u_z \rangle$.

Subsection 21.2

Flux 3D

Recall that we learned about flux in 2 dimensions, see 17. We saw that if we had some velocity field and a curve on the plane, then we could compute compute flux to determine the amount of stuff passing over the curve per unit time. When talking about flux in 3 dimensions we cant really think of flux as a line integral. This is because if you have a curve in space, you can't really ask how much stuff (air, water…) is passing through the curve; what is more applicable is asking how much stuffing is passing through some surface. So for example if you had a net in the ocean, you might want to ask how much water is passing through the net, instead of asking how much water is passing through a fishing line.

Therefore, when calculating flux in 3 dimensions we calculate a surface integral and not a line integral. Let's say we have a vevctor field $\vec{F}$ and a surface S, and we want to compute the flux (amount of stuff passing through surface per unit of time), how do we do this? Well, let's focus on computing the "flux" at a particular point on our surface. To compute the "flux" at this point we just want to calculate the amount of $\vec{F}$ in direction of $\hat{n}$, which is just $\vec{F} \cdot \hat{n}$. This is just for a particular point though, we want to calculate over the entire surface S.

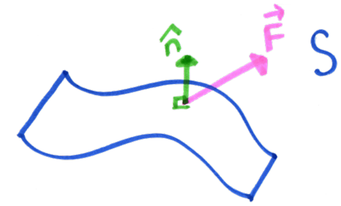


Figure 109. Computing the "flux" at a particular point on our surface it becomes clear that we want to calculate the amount of $\vec{F}$ in direction of $\hat{n}$, $\vec{F} \cdot \hat{n}$.

Definition 17

Three Dimensional Flux

Given some surface S, a vector field $\vec{F}$ and the vector normal (see note below) to the surface $\hat{n}$ then

$$\text{Flux}(S) = \iint_S \vec{F} \cdot \hat{n} dS$$

where dS is a surface area element.

Note that we tend to reserve dA for coordinate plane areas.

Note that there are two vectors normal to a surface (since a surface has two sides), on what side is our vector $\hat{n}$? Recall that in $2D$ flux we ran into a similar problem and chose the vector $\hat{T}$ rotated 90 deg to be our vector $\hat{n}$, the effect of which was that given a closed curve $\hat{n}$ was pointing outside curve (assuming counterclockwise rotation of curve). Well, there is not a good equivalent to rotating the tangent vector 90 deg in space, but the general tendency is, given a closed surface to choose the normal vector pointing outside that surface. In this way we calculate the amount of stuff flowing from inside to outsie surface.

21.2.1 Calculating Flux

If we have a surface S where $\vec{F}$ is tangent to surface at all points then of course flux will be zero. What about all the other cases, how can we calculate flux? We will let $\overrightarrow{dS} = \hat{n} \cdot dS$. We have this because it is often easier to calculate $\overrightarrow{dS}$ then it is to seperate $\hat{n} dS$ seperately. Recall that when computing flux integrals in 2 dimensions we did a similar thing letting $\hat{n} ds = \langle dy, -dx \rangle$. See 17.2.

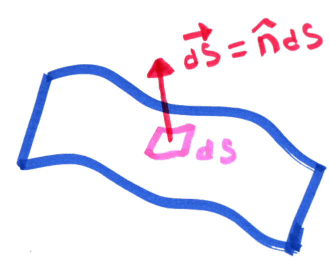


Figure 110. We will let the vector $\overrightarrow{dS} = \hat{n} \cdot dS$, or in words, the vector tangent to surface element dS with magnitude dS.

Subsection 21.3

Examples

21.3.1 Example 1

Let's say we have $\vec{F} = \langle x, y, z\rangle$ and S is the sphere centered at origin with radius a, and we want to calculate flux (where normal vector is directed outside of sphere). We need to calculate

$$\iint_S \vec{F} \cdot \hat{n} dS.$$

We will use the geometric interpretation that $\vec{F}$ is in same direction as $\hat{n}$. Therefore

$$\begin{aligned}\vec{F} \cdot \hat{n} &= \|\vec{F}\|_2 \\ &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{\rho^2} \\ &= \rho \\ &= a.\end{aligned}$$

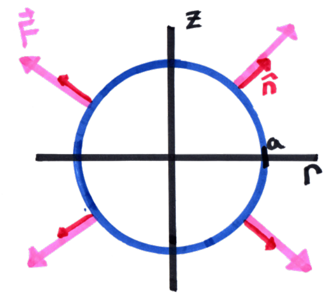


Figure 111. Taking a slice of our sphere along the zr axis, we see that $\vec{F}$ is normal to surface S.

Thus

$$\begin{aligned}\iint_S \vec{F} \cdot \hat{n} dS &= a \iint_S 1 dS \\ &= a \cdot 4\pi a^2 & \iint_S 1 dS = Area(S) \\ &= 4\pi a^3.\end{aligned}$$

21.3.2 Example 2

Again we will have the sphere of radius a centered at origin but this time with the field $\vec{H} = z\hat{k}$. We know that $\hat{n} = \frac{\langle x,y,z\rangle}{a}$. Thus we have the integral:

$$\begin{aligned}\iint_S \vec{F} \cdot \hat{n} dS &= \iint_S \langle 0, 0, z\rangle \cdot \frac{\langle x, y, z\rangle}{a} dS \\ &= \frac{1}{a} \iint_S z^2 dS.\end{aligned}$$

We will use spherical coordinates. Thus, $z = a\cos\phi$ and $dS = a^2 \sin\phi d\phi d\theta$, see 20.3.1. Therefore by substitution:

$$\begin{aligned}\frac{1}{a} \iint_S z^2 dS &= \iint_S (a\cos\phi)^2 a^2 \sin\phi d\phi d\theta \\ &= \frac{1}{a} \iint_S a^4 \cos^2\phi \sin\phi d\phi d\theta \\ &= a^3 \iint_S \cos^2\phi \sin\phi d\phi d\theta.\end{aligned}$$

Calculating the inner integral:

$$\begin{aligned}\int_0^\pi \cos^2\phi \sin\phi d\phi &= \left[-\frac{1}{3}\cos^3\phi\right]_0^\pi \\ &= -\frac{1}{3}(-1-1) \\ &= \frac{2}{3}.\end{aligned}$$

Calculating the outer integral:

$$\int_0^{2\pi} \frac{2}{3} d\theta = \frac{2}{3} \int_0^{2\pi} 1 d\theta$$
$$= \frac{4\pi}{3}$$

Therefore

$$\iint_S \vec{F} \cdot \hat{n} dS = a^3 \cdot \frac{4\pi}{3}.$$

SUBSECTION 21.4

Calculating Flux, Various Surfaces

For the follwoing surfaces, we will see how we can set up the integral

$$\iint_S \vec{F} \cdot \hat{n} dS$$

given some $\vec{F}$.

21.4.1 Plane $S : z = a$

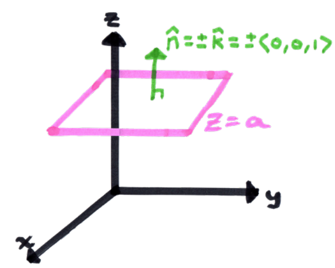


Figure 112. Given a plane $z = a$ we see that $\hat{n} = \pm \hat{k}$.

Given a horizontal plane $z = a$ we know of course that the normal vector will be $\hat{n} = \pm \hat{k}$, $\pm$ because it depends upon the orientation of plane. What is the value of dS? Well, since we are on a horizontal plane, z plays no part in dS, so we have $dS = dxdy = dydx$. Thus if we want to calculate flux some region R where R is a region of the plane $z = a$ then we calculate the double integral

$$\iint_R \vec{F} \cdot \hat{n} dS = \iint_R \vec{F} \cdot \pm \langle 0, 0, 1 \rangle dA$$

where $dA = dxdy = dydx$.

21.4.2 Plane $S : x = a$

Here, we have the same exact reasoning as for the plane $z = a$, but now we have the vertical plane zy. Therefore $\hat{n} = \pm \hat{\imath}$ and $dS = dydz = dzdy$.

21.4.3 S : sphere of radius a centered at origin

We have already done this. The normal vector is $\hat{n} = \pm \frac{\langle x,y,z \rangle}{a}$ and $dS = a^2 \sin \phi d\phi d\theta$. We can then convert x, y, z to spherical coordinates, but it is actually best to do this after calculating $\vec{F} \cdot \langle x, y, z \rangle$, because if there is some symmetry, then things might cancel or convert to something much simpler in spherical coordinates.

21.4.4 Cylinder of radius a centered about z-axis

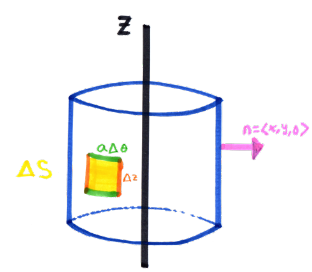
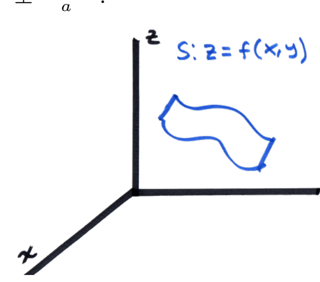


Figure 113. We see that the area $dS = adzd\theta$ and $\hat{n} = \pm \frac{\langle x,y,0 \rangle}{a}$.

We see that $ds = adzd\theta$ and $\hat{n} = \pm \frac{\langle x,y,0 \rangle}{a}$.

21.4.5 Surface $S : z = f(x, y)$



To compute flux, we need to find $\hat{n}$ and dS. Instead of finding $\hat{n}$ and dS seperately we will find $\hat{n}dS$. To do thi we want to find some result in terms of x and y. What about z? Well, since we are given $z = f(x, y)$ any z that exists we can replace with x and y.

Theorem 11 Given some surface $S : z = f(x, y)$ then

$$\hat{n}dS = \pm \underbrace{\langle -f_x, -f_y, 1\rangle}_{\text{not } \hat{n}} \underbrace{dxdy}_{\text{not } dS}.$$

Therefore, setting up our integral we have

$$\iint_R \vec{F} \cdot \pm\langle -f_x, -f_y, 1\rangle dxdy$$

and to find the bounds of our region, we look at the shadow our surface creates on the xy-plane.

SECTION 22

Lecture 28: Divergence Theorem

SUBSECTION 22.1

Why does $\hat{n}dS = \pm\langle -f_x, -f_y, 1\rangle dxdy$?

Recall that in yesterday's lecture, when we are able to express our surface in terms of $z = f(x, y)$, then we can express our surface in terms of x and y, and can compute flux as:

$$\iint_S \vec{F} \cdot \hat{n}dS = \iint_R \vec{F} \cdot \pm\langle -f_x, -f_y, 1\rangle dxdy.$$

The question remains though, why does $\hat{n}dS = \pm\langle -f_x, -f_y, 1\rangle dxdy$? Well, after looking the surface, ΔS and and the vector $\hat{n}$, a thought occurs. As ΔS gets smaller and smaller it will become more and more like a parallelogram, and so all we need to do is find the area of the parallelogram ΔS formed by two vectors $\vec{u}, \vec{v}$ which make up two sides of ΔS. To do this we take the cross product $\vec{u} \times \vec{v}$ which will return a vector in the normal direction whose magnitude is the area of ΔS. In this way we kill two birds with one stone; not only do we find the area of dS, but we find $\vec{n}dS$!

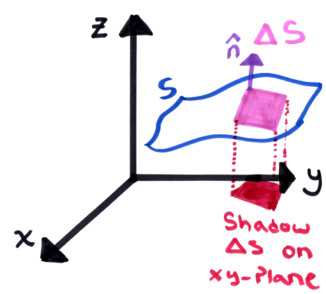


Figure 115. We see that we want to find the area of ΔS and the vector $\hat{n}$ normal to surface.

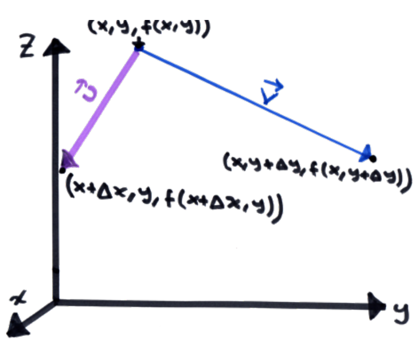


Figure 116. Zooming in on our parallelogram ΔS, we see the values of the head and tail of $\vec{u}$ and $\vec{v}$, which we can use to compute cross product $\vec{u} \times \vec{v} = \hat{n}\Delta S$.

We see that

$$\begin{aligned} &\vec{u} \text{ from } (x, y, f(x, y)) \\ &\qquad \text{to } (x + \Delta x, y, \underbrace{f(x + \Delta x, y)}_{\approx f(x,y)+f_x\Delta x}) \end{aligned}$$

and

$$\vec{v} \text{ from } (x, y, f(x,y)) \\ \text{to } (x, y+\Delta y, \underbrace{f(x, y+\Delta y)}_{\approx f(x,y)+f_y \Delta y}).$$

Therefore

$$\vec{u} = \langle dx, 0, f_x dx\rangle \quad \text{and} \quad \vec{v} = \langle 0, dy, f_y dy\rangle.$$

Computing the cross product $\vec{u} \times \vec{v}$:

$$\begin{aligned}
\vec{u} \times \vec{v} &= \langle dx, 0, f_x dx\rangle \times \langle 0, dy, f_y dy\rangle \\
&= \begin{vmatrix} \hat{\jmath} & \hat{\jmath} & \hat{k} \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} \\
&= (0 - f_x dxdy)\,\hat{\imath} - (f_y dxdy)\,\hat{\jmath} + (dxdy)\hat{k} \\
&= \langle -f_x dxdy, -f_y dxdy, dxdy\rangle \\
&= \langle -f_x, -f_y, 1\rangle\, dxdy.
\end{aligned}$$

This is how we get $\hat{n} dS = \pm\langle -f_x, -f_y, 1\rangle dxdy$. Note that we have $\pm$ because either the normal vector could be pointing up or down. Note that it is pointing up if $+$ (as z component will be 1), and pointing down if $-$ (as z component will be -1).

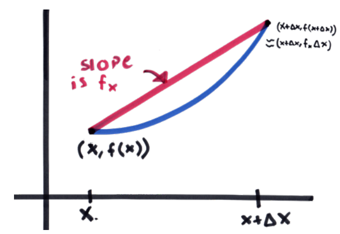


Figure 117. In this simple example we can approximate point $(x+\Delta x, f(x+\Delta x))$ as $(x+\Delta x, f(x,y)+f_x\Delta x)$. This is because $f_x \frac{\text{rise}}{\text{run}}$ and multiplying by Δx (run), we end up with $f(x,y)+$ rise.

22.1.1 Example 1

$\vec{F} = z\hat{k}$ through portion of paraboloid $z = x^2 + y^2$ above unit disk, where normal vector is directed upwards. Calculate flux. Computing flux:

$$\begin{aligned}
\iint_S \vec{F} \cdot \hat{n} dS &= \iint_R \langle 0, 0, z\rangle \cdot \langle -f_x, -f_y, 1\rangle dxdy \\
&= \iint_R \langle 0, 0, z\rangle \cdot \langle -2x, -2y, 1\rangle dxdy \\
&= \iint_R zdxdy \\
&= \iint_R (x^2 + y^2) dxdy \\
&= \int_0^{2\pi} \int_0^1 r^3 dr d\theta \\
&= \frac{1}{2}\pi.
\end{aligned}$$

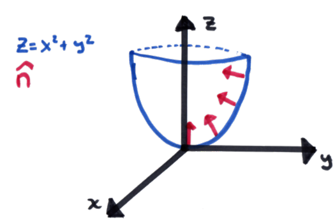


Figure 118. Our parabaloid $z = f(x,y) = x^2+y^2$ with normal vector directed upwards.

SUBSECTION 22.2

Parametric Surfaces

When calculating amount of stuff flowing through our surface (flux), it was necessary that our surface was defined $z = f(x,y)$, this allowed us to describe all points in terms of x, y $(x, y, f(x,y))$. We can also describe a surface via parametrization.

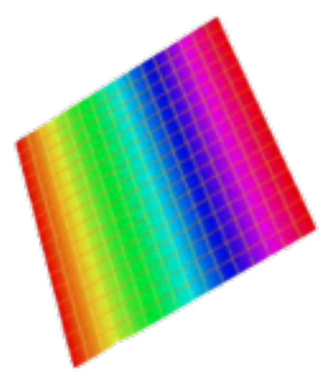


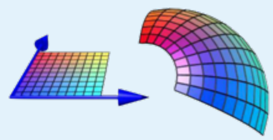
Figure 119. A plane

Definition 18 **Parametrized Surface**

A parametrization of a surface is a vector-valued function

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle,$$

where $x(u, v), y(u, v), z(u, v)$ are three functions of two variables. A parametrized surface is the image of the uv-map. That is the map $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ maps a point (u, v) on the flat uv-coordinate system to a point (x, y, z) on the 3D coordinate system xyz.



Note that if we keep the first parameter u constant, then $v \mapsto \vec{r}(u, v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u, v)$ traces a curve the surface. These curves are called grid curves. A computer draws surfaces using grid curves.

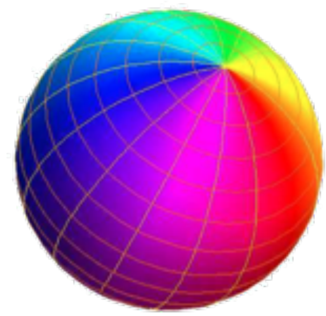


Figure 120. A sphere

22.2.1 Examples of Parametrized Surfaces

There are four very important examples of parametrized surfaces:

1. **Planes**
   Parametric: $\vec{r}(s, t) = \overrightarrow{OP} + s\vec{v} + t\vec{w}$
   Implicit: $ax + by + cz = d$.
   Parametric to Implicit: find the normal vector $\vec{n} = \vec{v} \times \vec{w}$.
   Implicit to Parametric: find two vectors $\vec{v}$, $\vec{w}$ normal to the vector $\vec{n}$. For example, find three points P, Q, R on the surface and forming $\vec{u} = \overrightarrow{PQ}, \vec{v} = \overrightarrow{PR}$.

2. **Spheres**
   Parametric: $\vec{r}(u, v) = \langle a, b, c\rangle + \langle \rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v)\rangle$.
   Implicit: $(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2$.
   Parametric to implicit: reading off the radius.
   Implicit to parametric: find the center (a, b, c) and the radius r possibly by completing the square.

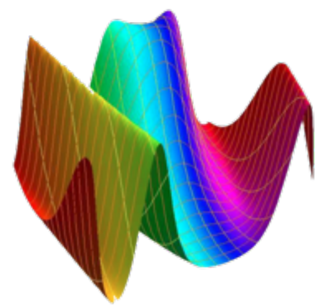


Figure 121. A surface of form $z = f(x, y)$.

3. **Graphs**
   Parametric: $\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$
   Implicit: $z - f(x, y) = 0$. Parametric to Implicit: think about $z = f(x, y)$
   Implicit to Parametric: use x and y as the parameterizations.

4. **Surfaces of Revolution**
   Parametric: $r(u, v) = (g(v)\cos(u), g(v)\sin(u), v)$
   Implicit: $\sqrt{x^2 + y^2} = r = g(z)$ can be written as $x^2 + y^2 = g(z)^2$.
   Parametric to Implicit: read off the function $g(z)$ the distance to the z-axis.
   Implicit to Parametric: use the function g.

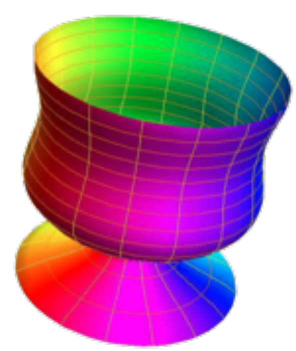


Figure 122. A surface of Revolution

Subsection 22.3

Flux of Parametric Surface

What if we are unable to express our surface in terms of $f(x, y)$, how can we compute flux? Well, firstly, why might we not be able to express our surface in terms of $z =$

$f(x, y)$? Well, maybe it is defined parametrically:

$$S : \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$$

which in vector form is $\langle x, y, z\rangle = \vec{r} = \vec{r}(u, v)$.

We thus have some surface which is described in terms of u and v, and since our goal is to put $\hat{n}dS$ in a more usable form, we want to have

$$\hat{n}dS = \underbrace{\text{something}} \cdot dudv.$$

To put $\hat{n}dS$ in a more usabble format, we go about the same process as earlier, we take the cross products of the sides of dS to get a vevctor normal to our surface S whose magnitude is the area of ΔS.

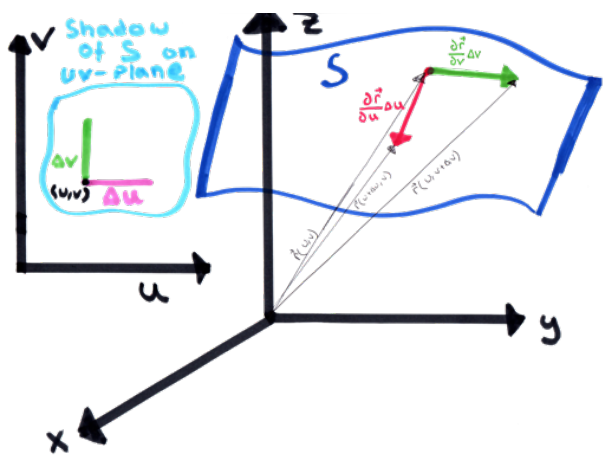


Figure 123. By taking the cross product of the red and green vectors we get a vector normal to surface S whose magnitude is the area of ΔS (which is equal to $\hat{n}dS$). Also note, that in our earlier we had $\vec{r}(u, v) = \langle x, y, f(x, y)\rangle$, so the uv-plane was just the xy-plane (a special case).

We therefore have:

$$\begin{aligned} \hat{n} \cdot dS &\approx \frac{\partial \vec{r}}{\partial u}\Delta u \times \frac{\partial \vec{r}}{\partial v}\Delta v \\ &= \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)\Delta u \Delta v \end{aligned}$$

and letting $\Delta u, \Delta v \to 0$, we have:

$$\hat{n} \cdot dS = \pm\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) dudv.$$

Remember we have $\pm$ because it depends upon direction of normal vector.

Question: Why are the sides of parallogram $\frac{\partial \vec{r}}{\partial u}\Delta u$ and $\frac{\partial \vec{r}}{\partial v}\Delta v$?

22.3.1 Example 2

Let's redo the problem from example 1 22.1.1, but using the more general form of $\hat{n} \cdot dS$. We want to comput flux of $\vec{F} = z\hat{k}$going through the portion of paraboloid

$$z = x^2 + y^2$$

above unit disk, where normal vector is directed upwards. Parametrizing our surface we have

$$S = \begin{cases} x = x(u,v) = u \\ y = y(u,v) = v \\ z = z(u,v) = u^2 + v^2 \end{cases}$$

and so $\vec{r}(u,v) = \langle u, v, u^2 + v^2 \rangle$. Calculating $\hat{n} \cdot dS$:

$$\begin{aligned} \hat{n} \cdot dS &= \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{dv} \right) dudv \\ &= \langle 1, 0, 2u \rangle \times \langle 0, 1, 2v \rangle dudv \\ &= \langle -2u, -2v, 1 \rangle dudv. \end{aligned}$$

Caclulating flux then we have:

$$\begin{aligned} \iint_S \vec{F} \cdot \hat{n} dS &= \iint_S \langle 0, 0, z \rangle \cdot \langle -2u, -2v, 1 \rangle dudv \\ &= \iint_S \langle 0, 0, u^2 + v^2 \rangle \cdot \langle -2u, -2v, 1 \rangle dudv \\ &= \iint_S (u^2 + v^2) dudv. \end{aligned}$$

At this point we subsitute $u = x$ and $v = y$ and convert to polar coordinates, getting same ansswer as before.

Subsection 22.4

Flux of Surface Given Normal Vector

Let's say that instead of describing our surface S in terms of $z = f(x,y)$ or parametrically, we describe it in terms of a vector normal to the surface $\vec{N}$ (does not have to be the unit normal vector), how can we compute flux? First of all, why would we be presented this form of a surface? Well, some example surfaces that we could see in this form would be:

1. A plane $ax + by + cz = d$: where $\vec{N} = \langle a, b, c \rangle$.
2. S given by equation $g(x,y,z) = 0$: $\vec{N} = \nabla g$.

22.4.1 Value of $\hat{n} \cdot dS$

As you have seen, the key to computing the flux ($\iint_S \vec{F} \cdot \hat{n} dS$) has been to get $\hat{n} dS$ in a more usable format. To do this, let's think geometrically about $\hat{n} dS$.

Let our surface S be a slanted plane.

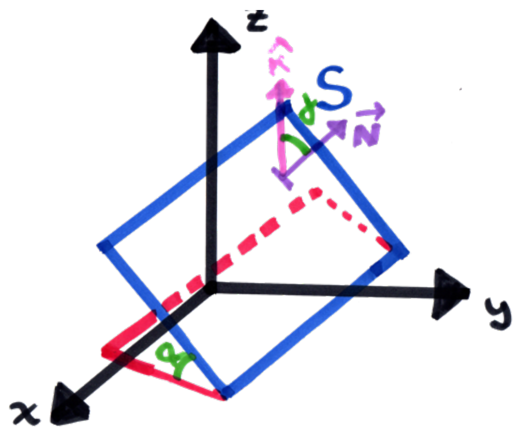


Figure 124. Drawing surface S in the xyz grid. Let α be the angle between our slanted plane S, and the xy-plane.

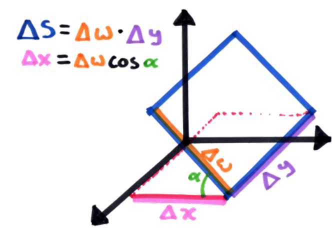


Figure 125. Looking at enlarged slices of ΔS and ΔA, we see that $\Delta S, \Delta A$ share an edge (purple). Thus, we see that $\Delta A = \Delta x \Delta y = \Delta S \cos\alpha$.

Since we have our plane in the xyz grid, we want to have dS in terms of $dxdy$. To this end we look at the horizontal projection of our plane on the xy-plane. We see that because our plane is slanted $\Delta S \neq \Delta A$, and in fact, as our plane becomes more and more slanted the more $\Delta A < \Delta S$. To this end, note that

$$\Delta A = \Delta S \cos\alpha.$$

It is not very convenient to have the $\cos\alpha$, so how can we replace it? Well, recall the geometric definition of the dot product:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos(\theta)$$

where θ is the angle between the two vectors $\vec{A}, \vec{B}$. Then we have

$$\begin{aligned} \vec{N} \cdot \hat{k} &= \|\vec{N}\|_2 \cdot \|\hat{k}\|_2 \cos\alpha \\ &= \|\vec{N}\|_2 \cos\alpha && \|\hat{k}\|_2 = 1 \\ \cos\alpha &= \frac{\vec{N} \cdot \hat{k}}{\|\vec{N}\|_2}. \end{aligned}$$

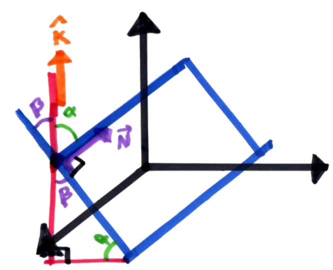


Figure 126. We see that the angles in our red triangle are $90\,\text{deg} + \alpha + \beta = 180\,\text{deg}$. Since we have the angles $\beta + 90\,\text{deg}$ on a line, the angle between $\hat{k}$ and $\vec{N}$ must be α.

Putting everything together, we have:

$$\begin{aligned} \Delta A &= \Delta S \cos\alpha \\ &= \Delta S \cdot \frac{\vec{N}\hat{k}}{\|\vec{N}\|_2} \\ \Delta S &= \Delta A \cdot \frac{\|\vec{N}\|_2}{\vec{N}\hat{k}}. \end{aligned}$$

But what we really care about is not ΔS, but $\hat{n}\Delta S$. So multiplying everything by $\hat{n}$, we have:

$$\begin{aligned}
\hat{n}\Delta S &= \hat{n} \cdot \Delta A \cdot \frac{\|\vec{N}\|_2}{\vec{N}\hat{k}} \\
&= \Delta A \cdot \frac{\hat{n}\|\vec{N}\|_2}{\vec{N}\hat{k}} \\
&= \Delta A \cdot \frac{\pm\vec{N}}{\vec{N}\hat{k}} && \|\vec{N}\|_2\hat{n} = \pm\vec{N} \\
&= \pm\frac{\vec{N}}{\vec{N}\hat{k}}\Delta A
\end{aligned}$$

Recall that we have $\pm$ because it depends if $\hat{n}$ is in direction of $\vec{N}$ or in opposite direction. Also, if instead we wanted to project our surface S onto a different plane, the zy-plane for example, then we have

$$\hat{n}\Delta S = \pm\frac{\vec{N}}{\vec{N}\hat{\imath}}\Delta A$$

22.4.2 Example 3

Let's say that we have

$$S : g(x, y, z) = z - f(x, y) = 0.$$

That is a really strange way to write $z = f(x, y)$, but now we know that a vector normal to the surface is $\vec{N} = \nabla g$. We want to compute the amount of stuff flowing through our surface S given some vector field $\vec{F}$:

$$\begin{aligned}
\iint_S \vec{F} \cdot \hat{n}\,dS &= \iint_R \vec{F}\frac{\nabla g}{\nabla g\hat{k}}dxdy \\
&= \iint_R \vec{F}\frac{\langle -f_x, -f_y, 1\rangle}{\langle -f_x, -f_y, 1\rangle \cdot \langle 0, 0, 1\rangle}dxdy \\
&= \iint_R \vec{F}\langle -f_x, -f_y, 1\rangle dxdy.
\end{aligned}$$

Subsection 22.5

Divergence Theorem

We have been spending a lot of time on how to compute flux integrals, but now let's look at a way to avoid computing flux integrals, the divergence theorem. This is also known as the "Gauss-Green Theorem" and is the 3D analogue of Green's theorem for flux.

Theorem 12 **Divergence Theorem**

If S is a *closed* surface enclosing a region D, oriented with $\hat{n}$ directed outwards and $\vec{F}$ is defined and differentiable everywhere in D, then

$$\oiint_S \vec{F} \cdot \hat{n}\,dS = \iiint_D \operatorname{div}(\vec{F})dV$$

where $\operatorname{div}(\vec{F}) = \operatorname{div}(P\hat{\imath} + Q\hat{\jmath} + R\hat{k}) = P_x + Q_y + R_z$.

Why is this? We will look at a formal proof in the next lecture, but less formally, why does this make sense? Well, $\oiint_S \vec{F} \cdot dS$ calculates the amount of stuff flowing out of our surface S. Also recall that that when we talked about 2D divergence, see 17.3.4, we mentioned how divergence over some region R, calculates how much stuff is being added to the system per unit time. The same applies for 3D divergence. Thus, it makes sense that to calculate the amount of stuff flowing through a closed surface, we just take the sum of (sources-sinks) for the region that it encloses.

22.5.1 Example 4

Let's revisit 21.3.2, but this time we will use divergence theorem so that we don't have to calculate the flux integral. We want to compute flux of $z\hat{k}$ through a sphere of radius a centered at the origin. Computing flux:

$$\begin{aligned}
\oiint_S \vec{F} \cdot \hat{n} dS &= \iiint_D \operatorname{div}(\vec{F}) dV \\
&= \iiint_D (0+0+1) dV \\
&= \iiint_D 1 dV \\
&= \operatorname{vol}(D) \\
&= \frac{4}{3}\pi a^3.
\end{aligned}$$

Section 23

Lecture 29: Divergence Theorem, Applications and Proof

Subsection 23.1

Del Operator, ∇

The ∇ operator, pronounced "del" is convenient mathematical notation, where del can be interpreted as a vector of partial derivative operators:

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle.$$

The del operator is mainly used to remember the following:

1. Gradient:

$$\begin{aligned}
\nabla f &= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f \\
&= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle
\end{aligned}$$

2. Divergence of $\vec{F}$:

$$\begin{aligned}
\nabla \cdot \vec{F} &= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle \\
&= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\end{aligned}$$

where P, Q, R are functions of x, y, z.

3. Curl of $\vec{F}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\partial} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\widehat{k}$$

Subsection 23.2
Physical Interpretation

Recall that $\text{div}(\vec{F})$ calculates the amount of flux generated at infitessimal region about point per unit volume. Therefore, if we have a velocity field $\vec{F}$ of an incompressible fluid then there are three possibilites at each point:

- $\text{div}(\vec{F}) = 0$. Then fluid is simply moving from one place to another.
- $\text{div}(\vec{F}) > 0$. Because we have an incompressible fluid, then fluid is being added to system at that given point, and we have a sink.
- $\text{div}(\vec{F}) < 0$. Because our fluid is incompressible, the fluid can't compress, and thus must be sucked out of region (sink).

Thus it becomes really clear why we are relating the surface integral of a closed surface with the region that it encloses:

$$\oiint_S \vec{F} \cdot \hat{n} dS = \iiint_D \text{div}(\vec{F}) dV.$$

That is we calculate the amount of stuff leaving closed surface by summing all the sources and subtracting all the sinks from the region it encloses.

Subsection 23.3
Proof of Divergence Theorem

23.3.1 Idea

The proof of the divergence theorem is very similar to that of Green's theorem (see 16.2.1):

1. Given some vector field $\vec{F} = \langle P, Q, R\rangle$ (where P, Q, R are functions of x, y, z), we will break $\vec{F}$ into it's components, such that

$$\vec{F} = \langle P, 0, 0\rangle + \langle 0, Q, 0\rangle + \langle 0, 0, R\rangle.$$

   Thus, to prove that

$$\oiint_S \vec{F} \cdot \hat{n} dS = \iiint_D \text{div}(\vec{F}) dV$$

   will will simplify this proof by letting $\vec{F} = \langle 0, 0, R\rangle$. The reason we are able to do this is that proving for the other two components of $\vec{F}$ follows so closely, that proving a part, proves the whole.

2. We will put a condition on our surface that the surface can broken up into vertically simple pieces, horizontally simple pieces and longitudally simple surfaces. This allows us to just prove for a vertically simple surface when $\vec{F} = \langle 0, 0, R\rangle$, a

horizontally simple surface when $\vec{F} = \langle 0, Q, 0\rangle$ and a longitudally simple surface when $\vec{F} = \langle P, 0, 0\rangle$.

Definition 19 **Vertically Simple Region**

A region R is *vertically simple* (type I) if it consists of all (x, y, z) that satisfy $(x, y) \in U$, where U is the projection of R on the xy-plane and $f_1(x, y) \le z \le f_2(x, y)$ for some continuous function $f_1(x, y)$ and $f_2(x, y)$ where, for all $(x, y) \in U$, we have $f_1(x, y) \le f_2(x, y)$. In set notation:

$$\{(x, y, z) \mid (x, y) \in D, \quad f_1(x, y) \le z \le f_2(x, y)\}.$$

23.3.2 Theorem

Using the above simplifications, we can prove the divergence theorem by proving the much simpler statement:

Let $\vec{F} = \langle 0, 0, R\rangle$ and S be any vertically simple surface, then

$$\oiint_S \vec{F} \cdot \hat{n} dS = \iiint_D \operatorname{div}(\vec{F}) dV.$$

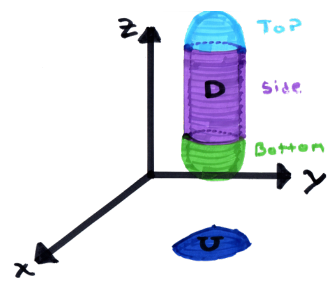


Figure 127. A vertically simple surface can be divided into three surfaces: a top, a bottom, and a side. Note, that in some cases there will be no side, for example in a sphere. We use U for the region on xy-plane since R is already being used by the vector field.

23.3.3 Proof

For this proof, we will simplify the right side of the equation, and then the left side of the equation showing that they are equal.

1. **Right-hand side of equation**:
   Because $\vec{F} = \langle 0, 0, R\rangle$, $\operatorname{div}(\vec{F}) = 0_x + 0_y + R_z = R_z$. Therefore

   $$\iiint_D \operatorname{div}(\vec{F}) dV = \iiint_D R_z dV.$$

   Let $dV = dzdydx$ and U be the region created by the shadow of S on the xy-plane. Also recall that because our surface is vertically simple the lower and upper bounds of z are:

   $$z_{\text{lower}} = f_1(x, y) \quad \text{and} \quad z_{\text{upper}} = f_2(x, y)$$

   for some continuous functions f_1 and f_2. Thus by letting $dV = dzdydx$ we have:

   $$\iiint_D R_z dV = \iint_U \left(\int_{f_1(x,y)}^{f_2(x,y)} R_z dz \right) dydx.$$

   But wait, we are taking the definite integral (with respect to z) of the partial derivative $\frac{\partial R(x,y,z)}{\partial z}$. Thus $\int_{f_1(x,y)}^{f_2(x,y)} R_z dz = [R(x, y, f_2(x, y)) - R(x, y, f_1(x, y))]$. Thus in summary we have:

$$\begin{aligned}
\oiint_S \vec{F}\cdot\hat{n}dS &= \iiint_D \operatorname{div}(\vec{F})dV \\
&= \iiint_D R_z dV && \vec{F} = \langle 0,0,R\rangle \\
&= \iiint_D R_z dzdydx && dV = dzdydx \\
&= \iint_U \left(\int_{f_1(x,y)}^{f_2(x,y)} R_z dz\right) dydx && S \textit{ is vertically simple} \\
&= \iint_U \left(R(x,y,f_2(x,y)) - R(x,y,f_1(x,y))\right) dydx. && \textit{Integral of derivative}
\end{aligned}$$

2. **Left-hand side of equation**
   Because we have a vertically simple surface we can break S into three distinct surfaces:
   $$S = S_{\text{top}} + S_{\text{middle}} + S_{\text{bottom}}.$$
   Furthermore, we actually know that S_{top} and S_{bottom} are described by the functions $z = f_2$ and $z = f_1$ respectively:
   $$S_{\text{top}} : z = f_2(x,y) \quad \text{and} \quad S_{\text{bottom}} : z = f_1(x,y).$$
   We will compute $\oiint_S \vec{F}\cdot\hat{n}dS$ by computing the parts seperately:
   $$\oiint_S \vec{F}\cdot\hat{n}dS = \iint_{S_{\text{top}}} + \iint_{S_{\text{middle}}} + \iint_{S_{\text{bottom}}}.$$
   Key to this will be recalling the fact that
   $$\iint_S \vec{F}\cdot\hat{n}dS = \iint_R \vec{F}\cdot \pm\langle -f_x, -f_y, 1\rangle\, dxdy$$
   (see 22.1).

   a) **Top**:
      Computing $\iint_{S_{\text{top}}}$ we have:
      $$\begin{aligned}
      \iint_{S_{\text{top}}} \vec{F}\cdot\hat{n}dS &= \iint_U \vec{F}\cdot\pm\langle -\frac{\partial f_2(x,y)}{\partial x}, -\frac{\partial f_2(x,y)}{\partial y}, 1\rangle dA \\
      &= \iint_U \langle 0,0,R(x,y,f_2(x,y))\rangle \cdot \pm\langle -\frac{\partial f_2(x,y)}{\partial x}, -\frac{\partial f_2(x,y)}{\partial y}, 1\rangle dA \\
      &= \iint_U \pm R(x,y,f_2(x,y))dA.
      \end{aligned}$$
      Note that we have $\pm$, what is it? Well, since this is the top surface, the normal vector will be pointing up such that is directed outsie of closed surface S. Therefore
      $$\iint_{S_{\text{top}}} \vec{F}\cdot\hat{n}dS = \iint_U R(x,y,f_2(x,y))dA.$$

   b) **Middle**:
      For the middle surface, we will use geometric intuition; because the middle surface just connects the top and bottom sections, the surface is straight

up and down, and so normal vector is parallel to xy-plane. Recall that $\vec{F} = \langle 0, 0, R \rangle$ (and has no $\hat{\imath}$ or $\hat{\jmath}$ component) and so $\hat{n}$ is orthogonal to $\vec{F}$. Therefore, flux is zero at all points on surface and so:

$$\iint_{S_{\text{middle}}} \vec{F} \cdot \hat{n} dS = 0.$$

c) **Bottom**:
We will use the same process as for the top surface:

$$\begin{aligned}
\iint_{S_{\text{bottom}}} \vec{F} \cdot \hat{n} dS &= \iint_U \vec{F} \cdot \pm \langle -\frac{\partial f_1(x,y)}{\partial x}, -\frac{\partial f_1(x,y)}{\partial y}, 1 \rangle dA \\
&= \iint_U \langle 0, 0, R(x,y,f_1(x,y)) \rangle \cdot \pm \langle -\frac{\partial f_1(x,y)}{\partial x}, -\frac{\partial f_1(x,y)}{\partial y}, 1 \rangle dA \\
&= \iint_U \pm R(x,y,f_1(x,y)) dA.
\end{aligned}$$

In this case though, because we are looking at the bottom surface, and the normal vector is directed outside of closed surface, we have:

$$\iint_{S_{\text{bottom}}} \vec{F} \cdot \hat{n} dS = \iint_U -R(x,y,f_1(x,y)) dA.$$

Putting this all together we have:

$$\begin{aligned}
\oiint_S \vec{F} \cdot \hat{n} dS &= \iint_{S_{\text{top}}} + \iint_{S_{\text{middle}}} + \iint_{S_{\text{bottom}}} \\
&= \iint_U R(x,y,f_2(x,y)) dA + 0 + \iint_U -R(x,y,f_1(x,y)) dA \\
&= \iint_U R(x,y,f_2(x,y)) dA - \iint_U R(x,y,f_1(x,y)) dA \\
&= \iint_U \left[R(x,y,f_2(x,y)) - R(x,y,f_1(x,y)) \right] dA
\end{aligned}$$

which is equal to the right hand side of the equation. □

Subsection 23.4
Heat Equation

Let's start with the most simple case, a metal rod whose temperature varies along the rod. We want to find an equation that describes how a slice of our rod will change temperature with respect to a change in time.

Figure 128. Looking at a snapshot of a metal rod at a given time where blue represents cold and red represents hot. At a given point on our rod, how will the temperature at that point change as time changes? Well, we see that our selected point is surounded by hotter temperatures that as time progresses that point should get hotter.

We thus see that the change in temperature of our rod with respect to time must be related somehow to the change in temperature with respect to change in position.

Theorem 13 **Heat Equation (One Dimension)**

We can express how a temperature changes at a given point with respect to a change in time as a partial differential equation that relates change in temperature with respect to time ($\frac{\partial T}{\partial t}$) to change in temperature with respect to change in position ($\frac{\partial^2 T}{\partial x^2}$):

$$\frac{\partial T}{\partial t}(x,t) = \alpha \cdot \frac{\partial^2 T}{\partial x^2}(x,t)$$

where $T(x,t)$ is temperature at a given position at a given time, t is time, x is position along metal rod, and α is some constant.

23.4.1 Intuition

To understand where this equation comes from let's look at a discrete set of temperatures along our metal rod (at some time t).

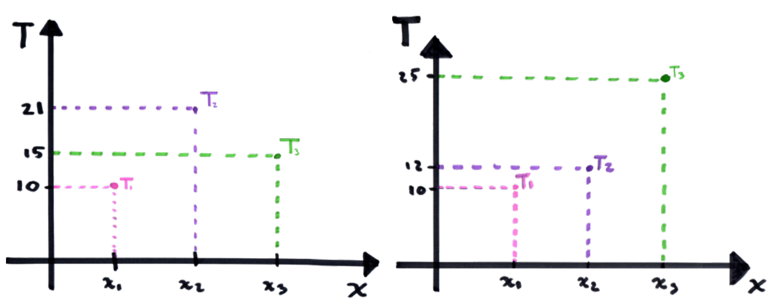


Figure 129. We see that on the left the neighbours of x_2 (x_1 and x_3) are both colder than x_2, thus it makes sense that ast time changes T_2 will bevcome colder. On the right, although T_1 is colder than T_2 and T_3 is warmer than T_2 it makes sense that as time changes T_2 will become warmer since the right hand neigbour is warmer than the left is cold.

Note that $\frac{T_1+T_3}{2}$ is the average neighbour temperature. We can think about the change in temperature T_2 as time passes as

$$\frac{dT_2}{dt} = (\frac{T_1+T_3}{2} - T_2).$$

This makes sense because it means that if the average neighbour temperature is greater than T_2, then T_2 will increase and vice versa. We also see that the higher the discrepency is, the faster the temperature change will be. Rewriting this equation to more closely resemble the heat equation we have:

$$\begin{aligned}
\frac{dT_2}{dt} &= \alpha\left(\frac{T_1+T_3}{2} - T_2\right) \\
&= \alpha\left(\frac{T_1+T_3-T_2-T_2}{2}\right) \\
&= \frac{\alpha}{2}\underbrace{\left(\underbrace{(T_3-T_2)}_{\Delta T_2} - \underbrace{(T_2-T_1)}_{\Delta T_1}\right)}_{\text{Difference of differences}} \\
&= \alpha \quad \underbrace{\Delta\Delta T_1}_{\text{Second Difference}} && \textit{\alpha is just some constant}
\end{aligned}$$

Note that Δ is the difference operator. Also, note that $\Delta\Delta T_1$ is just a compact way of writing how much T_2 differs from the average of its neighbours. The second derivative is just the continuious equivalent of the second difference, so we have:

$$\frac{\partial T}{\partial t}(x,t) = \alpha\left(\frac{\partial^2 T}{\partial x^2}(x,t)\right).$$

This gives us further intuition about the second derivative that can be used well beyond the heat equation: the second derivative gives us a measure of how a value compares to the average of its neighbours. So for example, when the second derivative is positive, the average value of a points neighbours is greater than that of the point, and slope will be increasing and curve will be concave up. Similarily, when the second derivative is negative, the average value of its neighbours at a point will be less than the point, and so curve will be concave down.

23.4.2 Transition to 3D

When in 3D space, the partial differential equation that describes how temperature at a point changes with respect to time is very similar to the one-dimensional case; all we have to do is add the partial derivatives for the extra spacial coordinates.

Theorem 14 **Heat Equation (Three Dimensions)**

We can express how a temperature changes at a given point with respect to a change in time as a partial differential equation that relates change in temperature with respect to time ($\frac{\partial T}{\partial t}$) to change in temperature with respect to change in position ($\frac{\partial^2 T}{\partial xr} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$):

$$\frac{\partial T}{\partial t}(x,y,z,t) = \alpha\left(\underbrace{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}}_{\text{Laplacian}}\right)$$

where $T(x,t)$ is temperature at a given position at a given time, t is time, (x,y,z) is position in space, and α is some constant.

23.4.3 What is the Laplacian?

This process of adding the partial derivatives together is so common that we have given it a name, the laplacian. But what is it exactly? Well, the basic idea is that it gives us a measure of how different a point is from the average of its neighbours. Why is this? Well, how is the laplacian defined?

Definition 20

Laplacian

The Laplacian of f, denoted Δf, $\nabla \cdot \nabla f$, or $\nabla^2 f$ is a differential operator (accepts a function and returns another function) given by the divergence of the gradient of a scalar valued function on Euclidean space:

$$\Delta f = \operatorname{div}(\operatorname{grad}(f)).$$

Let's say that we have some function

$$z = f(x, y).$$

Recall that the gradient of f, ∇f is a vector valued function that gives the direction of steepest ascent. That is, it points in the direction on the xy-plane the direction you should walk to increase most rapidly the value of $f(x, y)$.

What does it mean to take the divergence of the gradient? Let's think about what Δf will be at a local maximum and a local minimum. Notice when we are at a local maximum, then the gradient field right around that point will be directed towards the (x, y) coordinates of the top of the mountain or hill. Thus, at a local maximum $\nabla \cdot \nabla f$ will be negative. This makes sense because the average of all neihbouring points will be less than that of the top of the mountain.

We thus get a general idea of why the Laplacian operator gives a measure of how different a point is from the average of its neighbours.

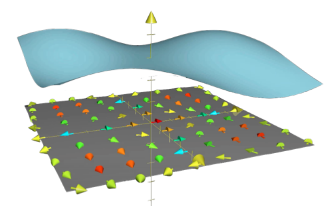


Figure 130. Here we have some function $z = f(x, y)$ and the gradient field ∇f which points in the direction of steepest ascent.

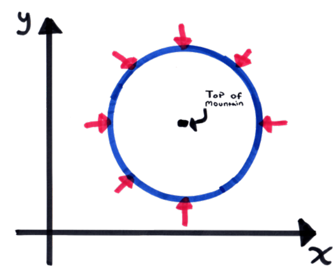


Figure 131. Looking at the top down view of a mountain, we see that the gradient field is pointing towards the (x, y) coordinates of the top of the mountain. Thus, the divergence at this point will be negative.

23.4.4 Temperature flow

We have seen how temperature changes at a point as time passes, but is there a vector field $\vec{F}$ that describes how temperature is flowing at a given time t?

We know from physics as well as common sense that heat will flow from high temperatures to low temperatures, but actually this statement can be made stronger, heat will flow in the direction in which temperature decreases the fastest. Thus heat flows in the direction opposite that of the gradient of temperture:

$$\text{Direction of } \vec{F} = -\nabla T.$$

How fast will this temperature flow? Well, in fact it is proportional to how great the temperature difference is. Thus, temperature will change more quickly the greater the difference between temperatures is. Because the magnitude of the gradient vector describes the maximum rate of change at a point, we just have to add a proportionality constant α and we get

$$\vec{F} = -\alpha \nabla T.$$

The value of this proportionality constant depends upon the medium for which temperature is being transmitted and how readily it transmits temperature.

23.4.5 Temperature flow and temperature change relation

How does this temperature flow affect the temperature? In other words how can we relate

$$\vec{F} \quad \text{and} \quad \frac{\partial T}{\partial t}?$$

To understand this let's take some region D bounded by a surface S. We want to relate the the flow of heat into and out of the region to the change in temperature of the region. By summing up the inflow of heat and subtracting the outflow of heat we can quantify the change in temperature that the region faces. We can do this using flux:

$$-\oiint_S \vec{F} \cdot \hat{n} dS.$$

We have negative flux because the normal vector is directed outside the region and we actually want to calculate inflow of heat. That is when

$$-\oiint_S \vec{F} \cdot \hat{n} dS > 0$$

temperature will go up and when

$$-\oiint_S \vec{F} \cdot \hat{n} dS < 0$$

temperature will go down. Let's look at two ways of thinking of this:

1. We can think of the amount of heat coming into region minus the amount of heat flowing out in terms of change of temperature to the overal region in terms of change of time:

$$-\oiint_S \vec{F} \cdot \hat{n} dS = \frac{d}{dt} \iiint_D T dV \;).$$

   That is, we look at how the sum of heat in our region changes with respect to time, and we know that this will be equal to amount of heat going into system - amount of heat leaving system. Also, we are able to put $\frac{d}{dt}$ inside the integral because t and T are different variables. Thus we have

$$\begin{aligned} -\oiint_S \vec{F} \cdot \hat{n} dS &= \frac{d}{dt} \iiint_D T dV \\ &= \iiint_D \frac{\partial T}{\partial t} dV. \end{aligned}$$

2. We can also use the divergence theorem to calculate the amount of heat added to region given the flow of heat $\vec{F}$:

$$\begin{aligned} -\oiint_S \vec{F} \cdot \hat{n} dS &= -\iiint_D \operatorname{div}(\vec{F}) dV \\ &= -\iiint_D \operatorname{div}(-\alpha \operatorname{grad}(T)) dV \\ &= \iiint_D \alpha \nabla^2 T dV. \end{aligned}$$

Putting 1 and 2 together we see that:

$$\iiint_D \alpha \nabla^2 T dV = -\oiint_S \vec{F} \cdot \hat{n} dS = \iiint_D \frac{\partial T}{\partial t} dV.$$

But that means that

$$\iiint_D \alpha \nabla^2 T dV = \iiint_D \frac{\partial T}{\partial t} dV.$$

Since D can by any region, that means that

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T.$$

Section 24

Lecture 30: Line Integrals in Space

Recall in lecture 19, 20 and 21, and 22 we covered line integrals, computing them, and how we could avoid the computation if we had a gradient field $\vec{F} = \nabla f$ for some function f. Well, now that we are in space, computing line integrals remains fairly similar, but checking to see if vector field $\vec{F}$ is a gradient field changes a lot.

Subsection 24.1

Computing Line Integrals in Space

Let's say we have some curve

$$d\vec{r} = \langle dx, dy, dz \rangle$$

and some vector field

$$\vec{F} = \langle P, Q, R \rangle$$

where P, Q, R are functions of x, y, z. Then to compute the work done along curve we have

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz.$$

We then parameterize x, y, z in terms of one variable and compute the integral with respect to that variable.

Steps to evaluate $\int_C P dx + Q dy + R dz$:

1. Parameterize C

2. Express x, y, z and dx, dy, dz in terms of parameter

3. Compute single integral with respect to parameter.

24.1.1 Example 1

Let

$$\vec{F} = \langle yz, xz, xy \rangle$$

and

$$C : x = t^3, y = t^2, z = t, 0 \leq t \leq 1.$$

Computing dx, dy, dz:

$$dx = 3t^2 dt, dy = 2t dt, dz = dt.$$

We can now compute line integral:

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_C yzdx + xzdy + xydz \\ &= \int_C t^3 3t^2 + t^4 2tdt + t^5 dt \\ &= \int_0^1 6t^5 dt \\ &= [t^6]_0^1 \\ &= 1.\end{aligned}$$

24.1.2 Example 2

We will have the same vector field

$$\vec{F} = \langle yz, xz, xy\rangle$$

but have curve $C' = C_1 + C_2 + C_3$ be composed of lines connecting the points $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$. Note that for curves C_1 and C_2, $z = 0$. Therefore for those curves we have

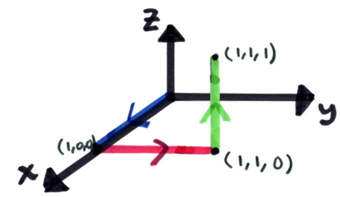


Figure 132. Curve C' composed of C_1, C_2, and C_3.

$$\begin{aligned}\int_{C_1,C_2} \vec{F}\cdot d\vec{r} &= \int_C yzdx + xzdy + xydz \\ &= \int_C y(0)dx + x(0)dy + xy(0) \\ &= 0.\end{aligned}$$

Computing work for curve C_3:

$$\begin{aligned}\int_{C_3} \vec{F}\cdot d\vec{r} &= \int_{C_3} yzdx + xzdy + xydz \\ &= \int_{C_3} 1zdx + 1zdy + 1\cdot 1dz \\ &= \int_0^1 1dz \\ &= 1.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{C'} \vec{F}\cdot d\vec{r} &= \int_{C_1} + \int_{C_2} + \int_{C_3} \\ &= 0 + 0 + 1 \\ &= 1.\end{aligned}$$

Note that in both example 1 and example 2 we took two different paths from $(0,0,0) \to (1,1,1)$ and got same result. This turns out to be the case because $\vec{F}$ is a gradient field and thus is path independent. In fact, we see that the function f for which $\vec{F} = \nabla f$ is

$$f(x,y,z) = x\cdot y\cdot z.$$

Subsection 24.2

Is $\vec{F}$ a gradient field?

Well recall that $\vec{F}$ is a gradient field if

$$\vec{F} = \langle P, Q, R \rangle = \langle f_x, f_y, f_z \rangle$$

for some function $f(x, y, z)$. When we were dealing with functions of two variables of the form $f(x, y)$ we saw that $\vec{F}$ is a gradient field if for some vector field

$$\vec{F} = N(x, y), M(x, y) \rangle$$

then $\vec{F}$ is a gradient field if $M_y - N_x = 0$. This is due to the symmetry of second derivatives, that is

$$f_{xy} = f_{yx}.$$

See Schwarz's therem for more information on the symmetry of second derivatives.
Well, it terms that this property is not constrained to functions of two variables. Thus given some $f(x, y, z)$ then at some point P

$$f(P)_{xy} = f(P)_{yx} \quad \text{and} \quad f(P)_{xz} = f(P)_{zx} \quad \text{and} \quad f(P)_{yz} = f(P)_{zy}$$

if the second partial derivatives of f are continuous in the neighborhood of P.

Thus we can test if $\vec{F} = \nabla f$ by testing for symmetry of the second derivatives. Because $\nabla f = \langle f_x, f_y, f_z \rangle$ then $\vec{F} = \langle P, Q, R \rangle$ is a gradient field if

- $P_y = Q_x$
- $P_z = R_x$
- $Q_z = R_y$.

Theorem 15

Gradient Field
If $\vec{F} = \langle P, Q, R \rangle$ is defined in a simply connected space and

$$P_y = Q_x \quad \text{and} \quad P_z = R_x \quad \text{and} \quad Q_z = R_y$$

then $\vec{F} = \nabla f$ for some function f.

24.2.1 Example 3

For which a and b is

$$axydx + (x^2 + z^3)dy + (byz^2 - 4z^3)dz$$

exact? Let's compare

- P_y and Q_x:

  $$P_y - Q_x = ax - 2x.$$

  Thus $a = 2$.

- P_z and R_x:

  $$P_z - R_x = 0 - 0.$$

- Q_z and R_y:

$$Q_z - R_y = 3z^2 - bz^2.$$

  Thus $b = 3$.

Therefore

$$axydx + (x^2 + z^3)dy + (byz^2 - 4z^3)dz$$

is an exact when $a = 2$ and $b = 3$.

Subsection 24.3

Finding potential function f

Recall for gradient fields in $\mathbb{R}^2$ we learned how to find the potential function in two different manners, using line integrals and using antiderivatives, we will look at both.

24.3.1 With Line Integrals

For a refresher on using line integrals to find a potential function in $\mathbb{R}^2$ see 15.2. In $\mathbb{R}^3$ the process is exactly the same, just with one extra step for the extra axis. That is we use the fact that being a gradient field is equivalent to being path independent to find f. To do this we choose an arbitrary point (x_1, y_1, z_1) and calculate the line integral to that point from the origin, moving parallel to the axis. Thus we have

$$\int_c \vec{F} \cdot d\vec{r} = f(x_1, y_1, z_1) - f(0,0,0)$$
$$f(x_1, y_1, z_1) = \int_c \vec{F} \cdot d\vec{r} + f(0,0,0)$$

where $f(0,0,0)$ is just a constant.

24.3.2 Example 4

We have already shown that

$$\vec{F} = \langle 2xy, x^2 + z^3, 3yz^2 - 4z^3 \rangle$$

is a gradient function. Using the line integral method, determine f. Splitting c into three curves $c = c_1 + c_2 + c_3$ we compute line integral for each section of c:

- c_1 is line from $(0,0,0) \to (x_1, 0, 0)$. Calculating line integral we have:

$$\begin{aligned}\int_{c_1} \vec{F} \cdot d\vec{r} &= \int_0^{x_1} Pdx \\ &= \int_0^{x_1} 2xydx \\ &= \int_0^{x_1} 2x(0)dx \\ &= 0\end{aligned}$$

- c_2 is line from $(x_1, 0, 0) \to (x_1, y_1, 0)$. Computing line integral we have:

$$\begin{aligned}\int_{c_2} \vec{F} \cdot d\vec{r} &= \int_0^{y_1} Q dy \\ &= \int_0^{y_1} (x^2 + z^3) dy \\ &= \int_0^{y_1} (x_1^2 + 0) dy \\ &= y_1 x_1^2.\end{aligned}$$

- c_3 is line from $(x_1, y_1, 0) \to (x_1, y_1, z_1)$. Computing line integral we have:

$$\begin{aligned}\int_{c_3} \vec{F} \cdot d\vec{r} &= \int_0^{z_1} R dz \\ &= \int_0^{z_1} (3yz^2 - 4z^3) dz \\ &= \int_0^{z_1} (3y_1 z^2 - 4z^3) dz \\ &= y_1 z_1^3 - z_1^4.\end{aligned}$$

Putting all this together we have

$$\begin{aligned}f(x, y) &= \int_c \vec{F} d\vec{r} + f(0, 0, 0) \\ &= \int_{c_1} + \int_{c_2} + \int_{c_3} + f(0, 0) \\ &= 0 + y_1 x_1^2 + y_1 z_1^3 - z_1^4 + f(0, 0, 0) \\ &= x^2 y + yz^3 - z^4 + C\end{aligned}$$

where C is some constant.

24.3.3 Using antiderivatives

The process to using antiderivatives is a little bit more involved than it was for gradient field $\vec{F} = \langle f_x, f_y \rangle$ now that

$$\vec{F} = \langle P, Q, R \rangle = \langle f_x, f_y, f_z \rangle$$

and we want to find f. The steps we take are:

1. Integrate $P = f_x$ with respect to x which will give us $f(x, y, z) = \cdots + g(y, z)$.

2. Differentiate this f with respect to y and compare with $Q = f_y$. This will give us $g_y = \cdots$.

3. Integrate g_y with respect to y giving us $g = \cdots + h(z)$.

4. Plug $g = \cdots + h(z)$ into $f = \cdots + g(x, y)$.

5. Differentiate f with respect to z and compare with $R = f_z$.

6. Determine value of $h(z)$ and plug back in giving us the equation for f.

24.3.4 Example 5

Find the potential function f for

$$\vec{F} = \langle 2xy, x^2 + z^3, 3yz^2 - 4z^3 \rangle.$$

We have already shown that $\vec{F}$ is a gradient field, so using the steps above:

1. Integrate $P = f_x$ with respect to x which will give us $f(x, y, z) = \cdots + g(y, z)$:

$$\begin{aligned} f(x,y,z) &= \int 2xy dx \\ &= x^2 y + g(y,z). \end{aligned}$$

2. Differentiate this f with respect to y and compare with $Q = f_y$. This will give us $g_y = \cdots$:

$$\begin{aligned} f_y &= x^2 + g_y \\ x^2 + z^3 &= x^2 + g_y \\ g_y &= z^3 \end{aligned}$$

3. Integrate g_y with respect to y giving us $g = \cdots + h(z)$:

$$\begin{aligned} \int g_y &= \int z^3 \\ g(y,z) &= z^3 y + h(z) \end{aligned}$$

4. Plug $g = \cdots + h(z)$ into $f = \cdots + g(x, y)$:

$$\begin{aligned} f(x,y,z) &= x^2 y + g(y,z) \\ &= x^2 y + z^3 y + h(z). \end{aligned}$$

5. Differentiate f with respect to z and compare with $R = f_z$:

$$\begin{aligned} f_z &= 3z^2 y + h'(z) \\ 3yz^2 - 4z^3 &= 3yz^2 + h'(z) \\ h'(z) &= -4z^3 \\ h(z) &= -z^4 + C \end{aligned}$$

6. Determine value of $h(z)$ and plug back in giving us the equation for f:

$$\begin{aligned} f(x,y,z) &= x^2 y + z^3 y + h(z) \\ &= x^2 y + z^3 y - z^4 + C \end{aligned}$$

   where C is a true constant of integration.

Subsection 24.4

Curl in 3D

Theorem 16 **Computing Curl in 3D**
$3D$ curl can be computing in the following way:

$$\underbrace{\nabla \times \overrightarrow{\mathbf{F}}}_{\text{Notation for curl}} = (R_y - Q_z)\,\hat{\imath} + (P_z - R_x)\,\hat{\jmath} + (Q_x - P_y)\,\widehat{k}.$$

What the hell is going on here?!? This is really different from 2D curl, see 15.3, where in two dimensions we represented rotationional aspect about a single point using a single number, and now we have a $3D$ vector to represent rotation about a single point in three dimensions. Why this discrepency?

Well, when in two dimensions we are able to represent rotation with a single value, angular velocity, and use the positive or negative sign to determine the direction of rotation: positive for counterclockwise rotation and negative for clockwise rotation. In three dimensions though we can't just use sign to represent direction, we need a vector to indicate direction, and the magnitude of that vector to indicate angular speed.

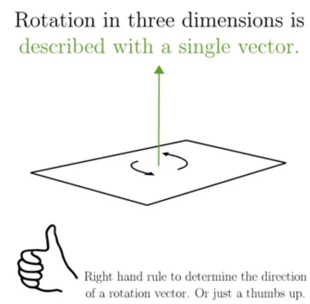


Figure 133. Curling your fingers in the direction of rotation then the vector representing the rotational aspect of flow is oriented in direction of your thumb.

24.4.1 Right Hand Rule

Somewhat counterintuitively the vector indicating direction of flow does not point in the direction of flow, but orthogonal to the plane of rotation. To visualize this, curl the fingers of your right hand in the direction of rotation and stick up your thumb. The vector representing this flow takes the same orientation as your thumb. That is, your thumb points along the axis of rotation. We adopt a similar idea from 2D curl, pointing "up" is counterclockwise rotation, pointing "down" represents clockwise rotation. If this still seems counterintuitive, see that to compute 3D curl by taking the cross product of the del operator (∇, see 23.1 for more information) and the vector field, and the result of taking the cross product is a vector orthogonal to the plane.

24.4.2 Visualizing Rotation in $3D$

How can you visualize rotation in three dimensions? Well, because the direction of rotation can be along any plane, it can be tough to visualize. Let's say you are outside (in 3D space) and playing with a paper windmill and you are moving it around so you try various orientations, well then the plane of rotation will be that when the windmill is rotating the fastest and the vector will point along the axis of the windmill in direction such that it obeys the right hand rule. In this way just like the gradient vector indicates direction of steepest ascent, the curl vector indicates direction of greatest rotation.

Technically what we have just described is how to find the direction of greatest rotation about a region and not a point. Well, if we apply a limiting process to our windmill, we will get the direction of greatest rotation about that point.

24.4.3 Computing Three Dimensional Curl

We have seen that to compute three dimensional curl we have some crazy looking formula. How can we remember this? Well we just take the cross product between the del operator and our vetor field vector $\vec{F} = \langle P, Q, R\rangle$:

$$\begin{aligned}
\text{curl}\,\vec{\mathbf{v}} &= \nabla \times \vec{\mathbf{v}}\nabla \times \vec{F} \\
&= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \times \langle P, Q, R\rangle \\
&= \det\left(\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}\right) \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}} \\
&= (R_y - Q_z)\hat{\imath} + (P_z - R_x)\hat{\jmath} + (Q_x - P_y)\hat{k}.
\end{aligned}$$

Section 25

Lecture 31: Stokes' Theorem

In the last lecture we looked at curl, a way to measure the rotational aspect of the flow about a point. The question is though, apart from getting a better sense of the motion of our fluid, what is the use of measuring curl? Well there are two main uses:

- Determine if vector field is conservative. Recall that field is conservative if and only if its curl is zero. If this is the case, we can then find a potential function, and us the fundemental theorem of calculus to calculate line integral.
- In two dimensions we related the line integral of a closed curve with the double integral of curl of the region it encloses. In three dimensions we can do the same thing, we can relate a clouse curve with a three dimensional object it bounds. This is Stokes' theorem.

Theorem 17

Stokes' Theorem
Stokes' theorem states that

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \underbrace{(\nabla \times \vec{F})}_{\text{Curl of } \vec{F}} \cdot \hat{n} dS.$$