Analysis test

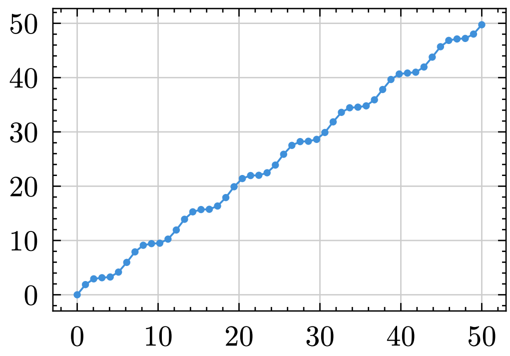
Exercise 1

Determine the logical relation between these two statements:
1. **P:** The function f is strictly monotonically increasing.
2. **Q:** The function f converges to infinity.

Solution: $P \implies Q$ but not the other way around.

If a function f is strictly monotonically increasing it means that for all $x > y$ we have $f(x) > f(y)$. Notice the strict inequality! This implies that if we take greater and greater values $x \in \mathbb{R}$ we get greater and greater values $f(x)$, thus the function converges to infinity.

The other way around is not necessarily true, take for example $f(x) = x + \sin(x)$, which tends to infinity but is not monotonically increasing.



□

Exercise 2

a) Determine the limit of the following

$$\lim_{n\to\infty}\left(1+\frac{n}{n^2}\right)^{2n}$$

b) Determine the limit of the following

$$\lim_{x\to\infty}\frac{4x^2+9x-3}{5x^2+4x-13}$$

Solution:
a)

$$\lim_{n\to\infty}\left(1+\frac{n}{n^2}\right)^{2n} = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{n}\right)^{2} = e^2$$

b)

$$\lim_{x\to\infty}\frac{4x^2+9x-3}{5x^2+4x-13} = \lim_{x\to\infty}\frac{4\frac{x^2}{x^2}+9\frac{x}{x^2}-\frac{3}{x^2}}{5\frac{x^2}{x^2}+4\frac{x}{x^2}-\frac{13}{x^2}} = \lim_{x\to\infty}\frac{4+\frac{9}{x}-\frac{3}{x^2}}{5+\frac{4}{x}-\frac{13}{x^2}} = \frac{4}{5}$$

□

Exercise 3

a) Determine the limit of the following

$$\lim_{n\to\infty} \frac{4^n + 2}{6^n}$$

b) Determine if the following limit is convergent

$$\lim_{n\to\infty} \frac{n^2 + 3^n}{n!}$$

Solution:

a)

$$\lim_{n\to\infty} \frac{4^n + 2}{6^n} = \lim_{n\to\infty} \frac{4^n}{6^n} + \frac{2}{6^n} = \lim_{n\to\infty} \left(\frac{4}{6}\right)^n + \frac{2}{6^n} = 0$$

b)

$$\lim_{n\to\infty} \frac{n^2 + 3^n}{n!} = \lim_{n\to\infty} \frac{n^2}{n!} + \frac{3^n}{n!} = \lim_{n\to\infty} \frac{3^n}{n!} = \lim_{n\to\infty} \frac{\overbrace{3 \cdot 3 \cdot 3 \cdot \ldots \cdot 3}^{n}}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} = 0$$

□

Exercise 4

a) Determine the following limit

$$\lim_{x\to 0^+} \frac{x^{-3} + 5x^{-2}}{2x^{-2} + x^{-1}}$$

b) Determine the following limit

$$\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

Solution:

a)

$$\lim_{x\to 0^+} \frac{x^{-3} + 5x^{-2}}{2x^{-2} + x^{-1}} = \lim_{x\to 0^+} \frac{\frac{x^{-3}}{x^{-3}} + 5\frac{x^{-2}}{x^{-3}}}{2\frac{x^{-2}}{x^{-3}} + \frac{x^{-1}}{x^{-3}}} = \lim_{x\to 0^+} \frac{1+5x}{2x + x^2} = +\infty$$

Because it is always positive, and as $x \to 0^+$ the denominator tends to 0, thus the ratio tends to $+\infty$.

b) Notice that the problem is asking for $f'(x)$, where $f(x) = x^3$, so the answer should be $3x^2$.

Since $(a+b)^3 = a^3 + 3ab^2 + 3a^2b + b^3$ we have the following

$$\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3xh^2 + 3x^2h + h^3 - x^3}{h}$$

$$= \lim_{h\to 0} = \frac{3xh^2 + 3x^2h + h^3}{h} = \lim_{h\to 0} (3xh + 3x^2 + h^2) = 0 + 3x^2 + 0 = 3x^2.$$

□

Exercise 5

Determine at which point is the tangent of the following function horizontal

$$f(x) = 3x^3 - 2x^2 + 5$$

Solution: The tangent line being horizontal means that $f'(x) = 0$, because the derivative of f means the steepness of the tangent line, so if the tangent is horizontal it means that is has 0 steepness.

So we need to find an x at which $f'(x) = 0$.

$$0 = f'(x) = 3 \cdot 3x^2 - 2 \cdot 2x = 9x^2 - 4x$$
$$0 = 9x^2 - 4x = x \cdot (9x - 4)$$

From this, we get that $f'(x) = 0$ at $x = 0$ and $x = 4/9$. □