

Homework 7

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Exercise 1

The perfect matching polytope of a graph G is given by

$$P_{PM} = \{x \in \mathbb{R}_+^E \mid x(\delta(v)) = 1 \text{ for } v \in V, \quad x(\delta(U)) \geq 1 \text{ for } U \subseteq V, \quad |U| \text{ odd}\}.$$

Separating over $P_{PM}(G)$ means that for a vector $x \in \mathbb{R}^E$, we want to decide if $x \in P_{PM}(G)$ or not.

1. Prove that separating over $P_{PM}(G)$ reduces to the following **odd minimum cut problem**:
give $x \in \mathbb{Q}^E$, find

$$\min_{S \subseteq V, |S| \text{ odd}} \sum_{u \in S, v \notin S, uv \in E} x_{uv}.$$

2. Prove that the odd minimum cut problem is polynomially solvable.

Solution: TODO

□

Exercise 2

Consider an affine map $\varphi(x) = Ax + b$, where $A \in \mathbb{R}^{n \times n}$ is an invertible matrix. Show that $\varphi(E(x_0, M)) = E(Ax_0 + b, AMA^T)$.

Solution: We need to show that $x \in \varphi(E(x_0, M)) \iff x \in E(Ax_0 + b, AMA^T)$.

$$\begin{aligned} x \in \varphi(E(x_0, M)) &\iff \varphi^{-1}(x) \in E(x_0, M) \\ &\stackrel{\text{def}}{\iff} (\varphi^{-1}(x) - x_0)^T M^{-1} (\varphi^{-1}(x) - x_0) \leq 1 \\ &\iff (A^{-1}(x - b) - x_0)^T M^{-1} (A^{-1}(x - b) - x_0) \leq 1 \\ &\iff (A^{-1}(x - b - Ax_0))^T M^{-1} (A^{-1}(x - b - Ax_0)) \leq 1 \\ &\iff (x - b - Ax_0)^T A^{-T} M^{-1} A^{-1} (x - b - Ax_0) \leq 1 \\ &\iff (x - (Ax_0 + b))(AMA^T)^{-1} (x - (Ax_0 + b)) \leq 1 \\ &\stackrel{\text{def}}{\iff} x \in E(Ax_0 + b, AMA^T). \end{aligned}$$

□

Exercise 3

Let $E(x_0, M) \subseteq \mathbb{R}^2$ be an ellipsoid where

$$x_0 = (1, 2) \text{ and } M = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}.$$

Let $\varphi(x) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}x + (3, 5)$. Show that M is positive definite. Write up the ellipsoid $\varphi(E(x_0, M))$.

Solution: Clearly M is positive definite, as its two leading minors are 9 and 36, both of which are positive, thus by Sylvester's theorem M is positive definite.

By the previous exercise, $\varphi(E(x_0, M)) = E(Ax_0 + b, AMA^T)$ with $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $b = (3, 5)$ if A is invertible, which indeed it is. Giving us the following

$$\begin{aligned} \varphi(E(x_0, M)) &= E(Ax_0 + b, AMA^T) \\ &= E\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix}, AMA^T\right) \\ &= E\left(\begin{pmatrix} 2 \cdot 1 + 1 \cdot 2 + 3 \\ 1 \cdot 1 + 2 \cdot 2 + 5 \end{pmatrix}, AMA^T\right) \\ &= E\left(\begin{pmatrix} 7 \\ 7 \end{pmatrix}, AMA^T\right) \\ &= E\left(\begin{pmatrix} 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 40 & 26 \\ 26 & 25 \end{pmatrix}\right). \end{aligned}$$

□

Exercise 4

Let both B_1 be the unit ball with the origin as a center and B_2 be the unit ball with the point $(1, 0)$ as a center in \mathbb{R}^2 . Let

$$x_1 = (1, 1), \quad x_2 = (2, 2), \quad M_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad M_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Prove that there exists no affine mapping φ such that $\varphi(B_1) = E(x_1, M_1)$ and $\varphi(B_2) = E(x_2, M_2)$.

Solution: Notice that a ball can be defined as an ellipse as follows:

$$B_1 = E\left((0, 0), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$
$$B_2 = E\left((1, 0), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

Let $\varphi(x) = Ax + b$, where $A \in \mathbb{R}^{2 \times 2}$ and $b \in \mathbb{R}^2$.

From Exercise 2 we know that

$$\varphi(B_1) = \varphi\left(E\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right) = E\left(A\begin{pmatrix} 0 \\ 0 \end{pmatrix} + b, A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}A^T\right) = E(b, AA^T),$$

and likewise

$$\varphi(B_2) = E\left(A\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b, AA^T\right).$$

Since the problem asks for $\varphi(B_1) = E_1$ it means that $AA^T = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and because $\varphi(B_2) = E_2$ it means that $AA^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

This is a contradiction as A is some fixed matrix, thus AA^T cannot take two different values. \square

Exercise 5

Let $C = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$ and let $a = (3, 3)$. Find all hyperplanes that separates C and a .

Solution: A hyperplane P is defined as $x \cdot v = c$ where $v \in \mathbb{R}^n$ is the normal of the plane. A plane separates two points $a, b \in \mathbb{R}^n$ if $a \cdot v \leq c$ and $b \cdot v \geq c$.

It is enough to show that a plane P separates the four corners of the square, thus it separates their convex hull.

So we need to find $v \in \mathbb{R}^2$ and $c \in \mathbb{R}$ such that

$$(0, 0) \cdot (v_1, v_2) \leq c$$

$$(0, 1) \cdot (v_1, v_2) \leq c$$

$$(1, 0) \cdot (v_1, v_2) \leq c$$

$$(1, 1) \cdot (v_1, v_2) \leq c$$

and $(3, 3) \cdot (v_1, v_2) \geq c$.

This results in the following five simple inequalities

$$0 \leq c, \quad v_1 \leq c, \quad v_2 \leq c,$$

$$v_1 + v_2 \leq c, \quad v_1 + v_2 \geq \frac{c}{3}.$$

Any plane $x \cdot v = c$ that satisfies the above five condition separates C and a .

Example: $v = (1, 1)$ and $c = 3$.

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