Real time fluid dynamics

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Outline

1. Notaion

1.1 Differential operators

$$\nabla = (\partial_1, \partial_2, ..., \partial_n) \qquad \text{(Nabla operator)}$$

$$u \cdot \nabla = \sum_{i=1}^{n} u_i \partial_i$$

$$(u \cdot \nabla)u = \sum_{i=1}^{n} u_i \partial_i u.$$

$$\nabla \cdot \nabla u = \Delta u = \sum_{i=1}^{n} (\partial_i u)^2 \qquad \text{(Laplace operator)}$$

2. Equations of fluids

2.1 Navier–Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \Delta \boldsymbol{u} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\boldsymbol{f}$$

$$\nabla \cdot u = 0$$

- $\boldsymbol{u}$ is the velocity vector field.
- $\boldsymbol{f}$ is the external forces.
- ρ is the scalar density field.
- p is the pressure field.
- ν is the kinematic viscosity.

2.1 Navier–Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \overbrace{(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}}^{\text{Advection}} = \underbrace{\nu \Delta \boldsymbol{u}}_{\text{Diffusion}} \quad \overbrace{-\frac{1}{\rho}\nabla p}^{\text{Internal source}} \quad + \quad \underbrace{\frac{1}{\rho}\boldsymbol{f}}_{\text{External source}} \quad .$$

1. Advection – How the velocity moves.
2. Diffusion – How the velocity spreads out.
3. Internal source – How the velocity points towards parts of lesser pressure.
4. External source – How the velocity is changed subject to external intervention, like a fan blowing air.

3. Equations for fluid simulations

3.1 Navier–Stokes equations 2

$$\frac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nu \Delta \boldsymbol{u} + \frac{1}{\rho}\boldsymbol{f}$$

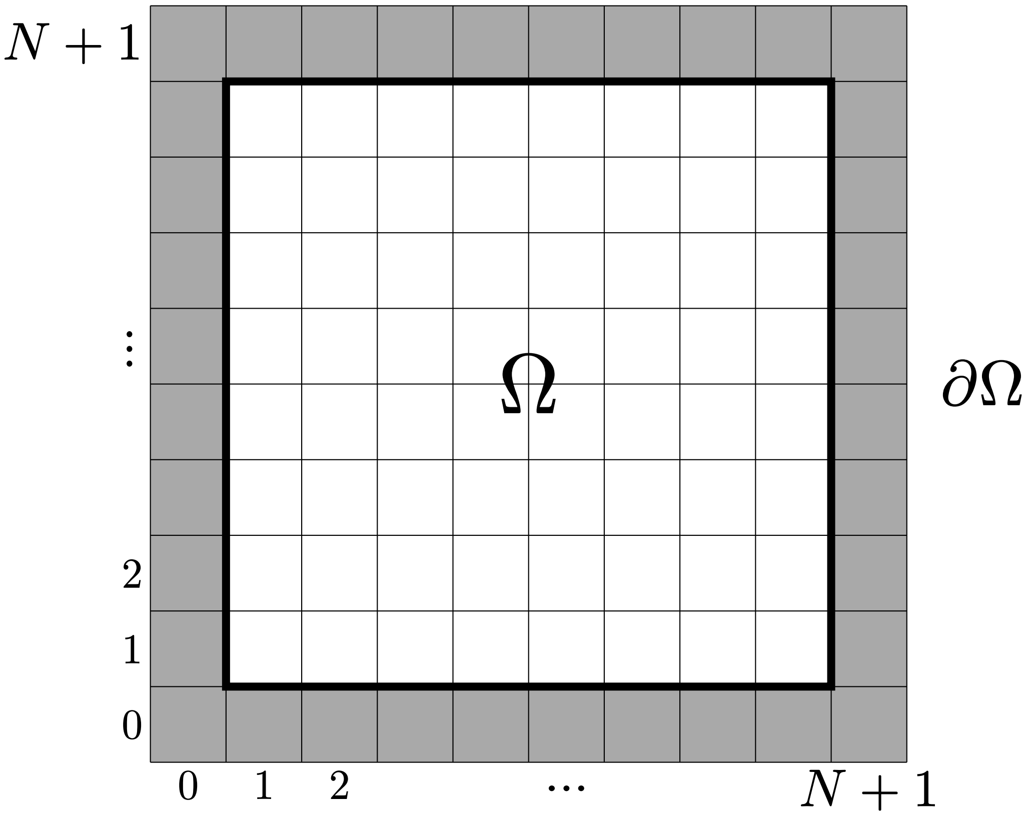
$$\frac{\partial \rho}{\partial t} = -(\boldsymbol{u} \cdot \nabla)\rho + \kappa \Delta \rho + S$$

$$\nabla \cdot u = 0$$

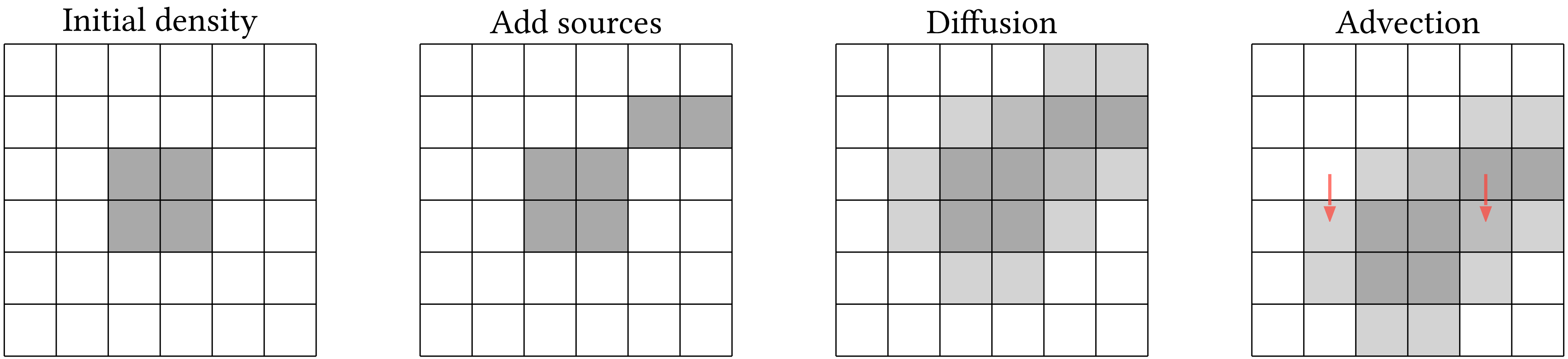
$$\partial_\nu u|_{\partial\Omega} = 0$$

4. Simulating fluids

4.1 Fluid in a box



4.2 Moving densities



5. Diffusion

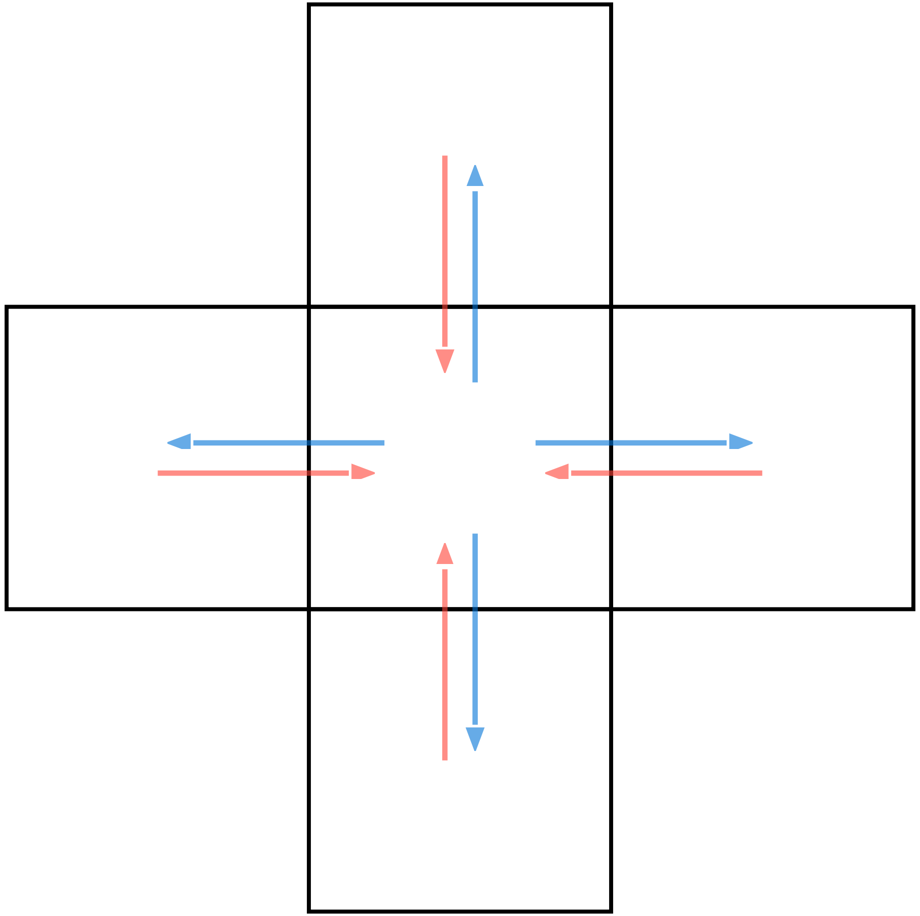
5.1 Diffusion equation

$$\frac{\partial \rho}{\partial t} = \kappa \Delta \rho$$

$$\frac{\rho_{\text{next}} - \rho_{\text{prev}}}{\Delta t} = \kappa \Delta \rho_{\text{prev}} \quad \text{(Forward difference)}$$

$$\rho_{\text{next}} = \rho_{\text{prev}} + (\Delta t) \kappa \Delta \rho_{\text{prev}} \qquad \text{(Helmholtz eq.)}$$

5.2 Duffusion



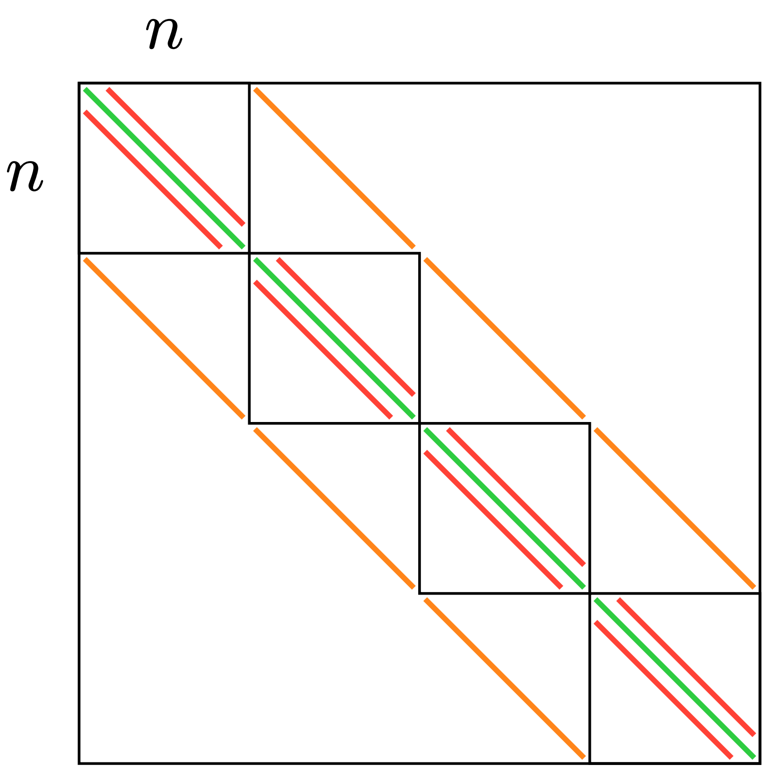
5.3 Finite difference method

$$\partial_1^2 \rho \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$\partial_2^2 \rho \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$$

$$(\Delta_h \rho_h)_{i,j} = \frac{\rho_{i+1,j} + \rho_{i-1,j} + \rho_{i,j+1} + \rho_{i,j-1} - 4\rho_{i,j}}{h^2}$$

5.4 FDM matrix

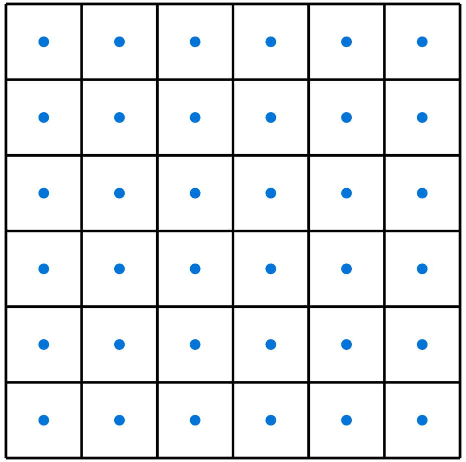


$$\text{green} = 4/h^2, \text{red} = -1/h^2, \text{orange} = -1/h^2$$

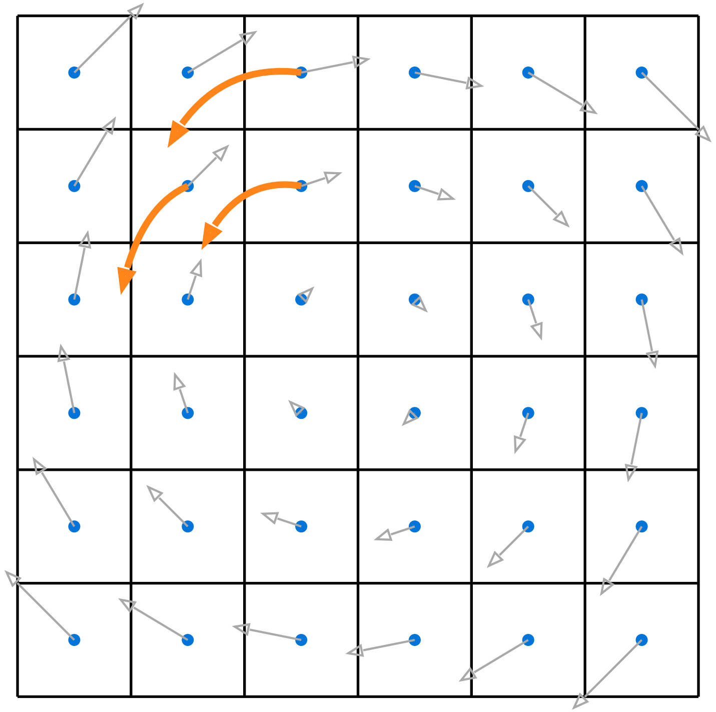
6. Advection

6.1 Advection equation

$$\frac{\partial \rho}{\partial t} = -(\boldsymbol{u} \cdot \nabla)\rho$$

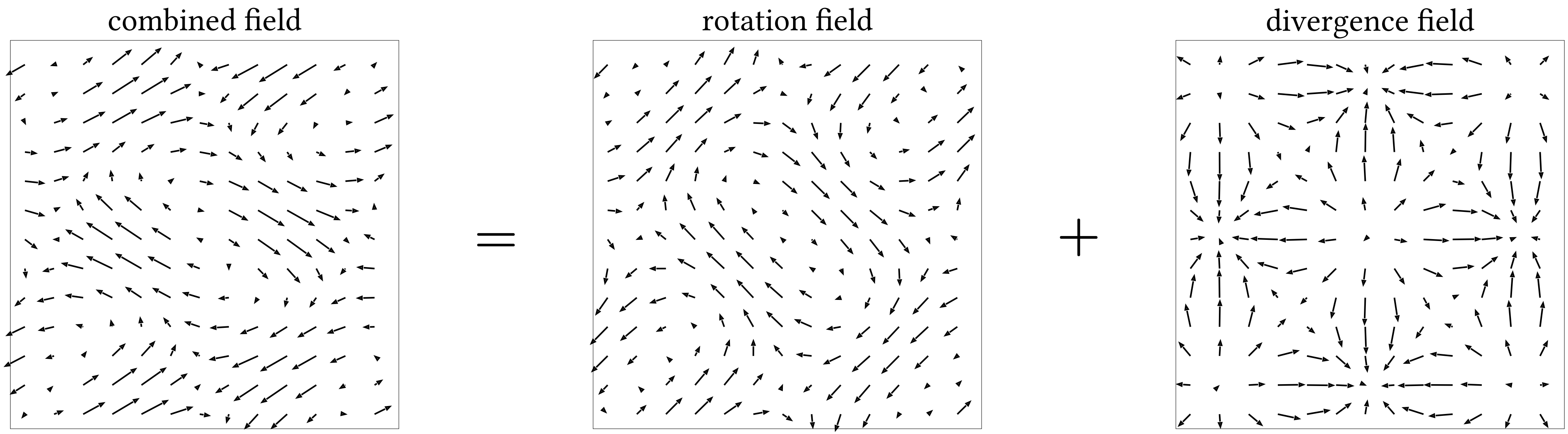


6.2 Semi-Lagrange



7. Evolving velocites

7.1 Helmholtz–Hodge decomposition



7.1 Helmholtz–Hodge decomposition

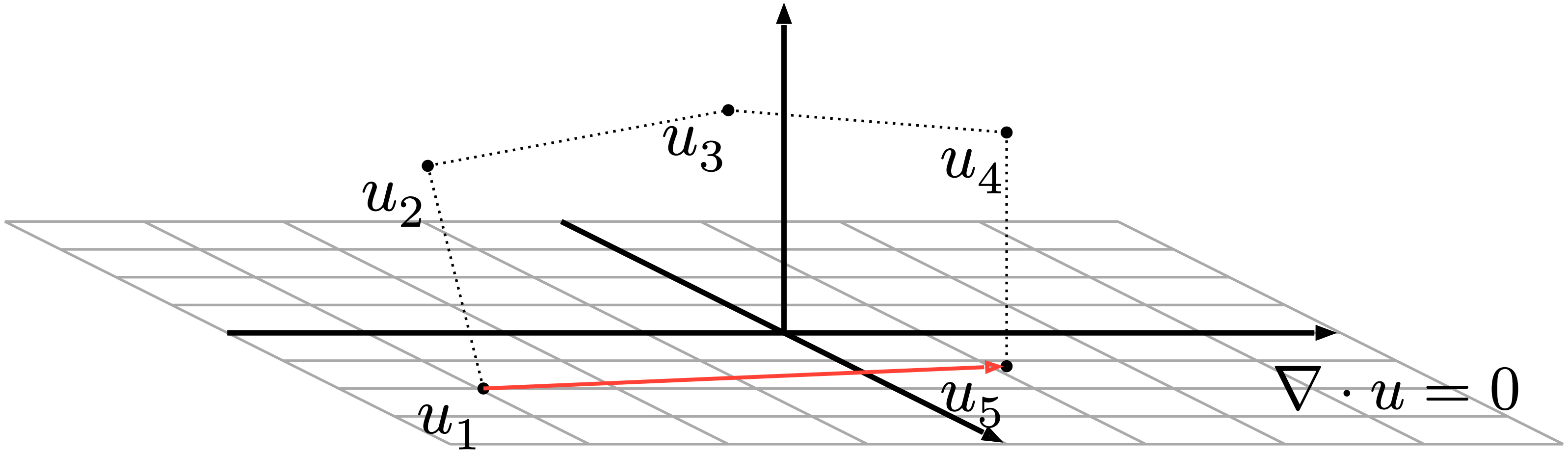
$$\boldsymbol{w} = \boldsymbol{u} + \nabla q$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad q : \mathbb{R}^n \to \mathbb{R}$$

$$\begin{aligned} \nabla \cdot \boldsymbol{w} &= \nabla \cdot \boldsymbol{u} + \nabla \cdot \nabla q \\ \nabla \cdot \boldsymbol{w} &= 0 + \nabla \cdot \nabla q \\ \nabla \cdot \boldsymbol{w} &= \Delta q \qquad \text{(Poisson eq.)} \end{aligned}$$

$$\boldsymbol{u} = \boldsymbol{w} - \nabla q$$

7.2 Simulation steps



$$u_1 \xrightarrow{\text{add source}} u_2 \xrightarrow{\text{diffusion}} u_3 \xrightarrow{\text{advection}} u_4 \xrightarrow{\text{projection}} u_5$$

8. Appendix

8.1 Appendix

- https://github.com/leonardo-toffalini/viscous
- https://github.com/leonardo-toffalini/fishy

8.1 Appendix

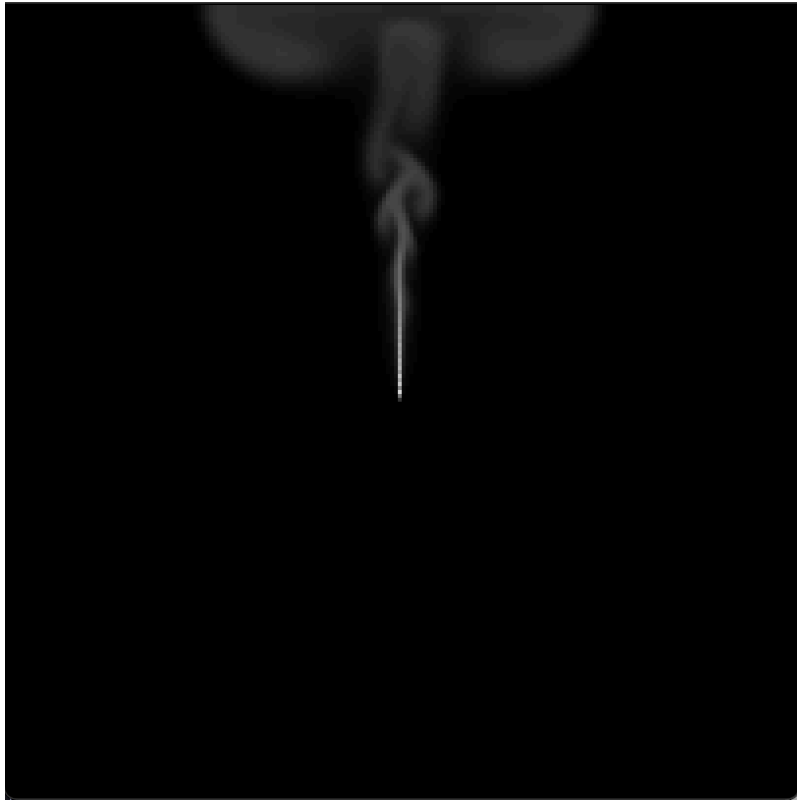


Figure 1: Smoke emitting from the tip of a cigarette

8.1 Appendix

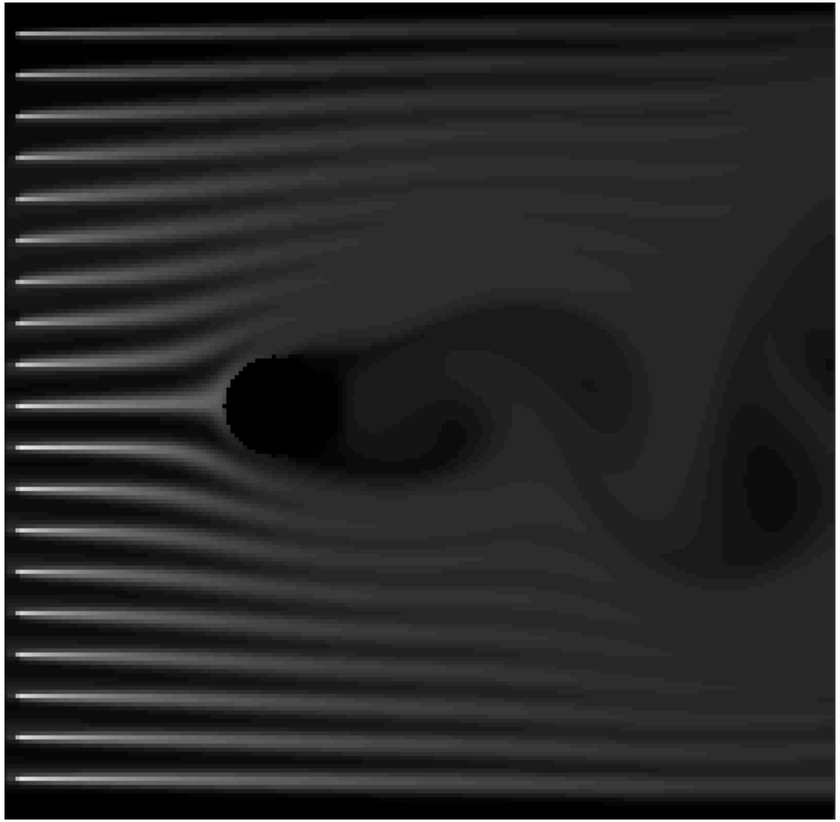


Figure 2: Vortex shedding

8.1 Appendix

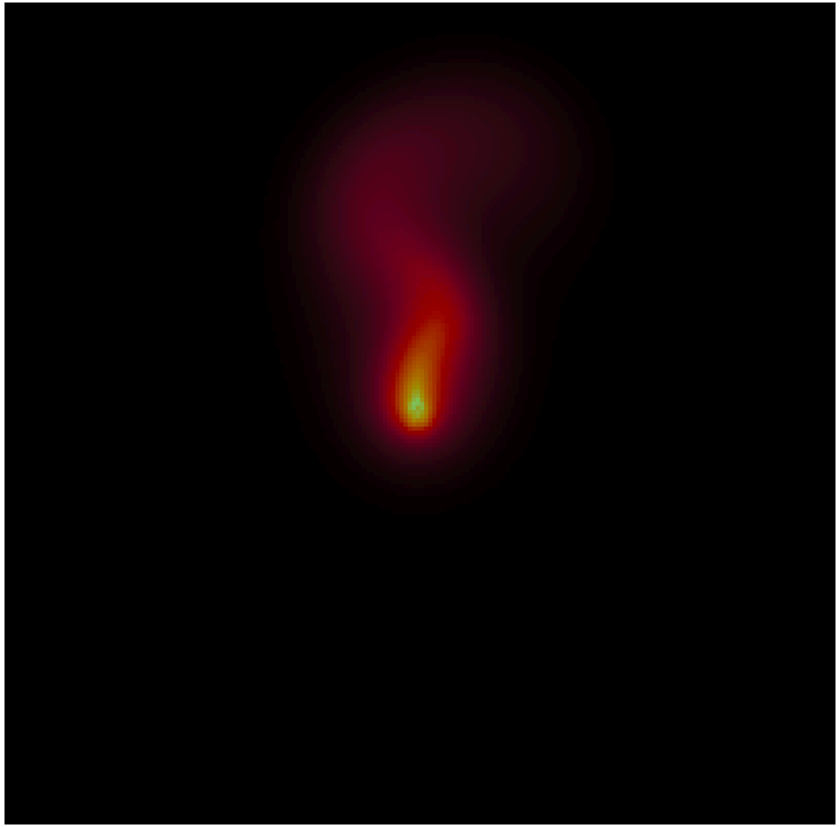


Figure 3: Flickering fire

Bibliography

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