Real time fluid dynamics

Leonardo Toffalini

2025-12-01

Outline

1. Notaion

1.1 Differential operators

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$$\nabla \cdot \nabla u = \Delta u = \sum_{i=1}^{n} (\partial_i u)^2 \qquad \text{(Laplace operator)}$$

2. Equations of fluids

2.1 Navier–Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \Delta \boldsymbol{u} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\boldsymbol{f}$$

$$\nabla \cdot u = 0$$

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- p is the pressure field.
- ν is the kinematic viscosity.

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2. Diffusion – How the velocity spreads out.

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1. Advection – How the velocity moves.
2. Diffusion – How the velocity spreads out.
3. Internal source – How the velocity points towards parts of lesser pressure.
4. External source – How the velocity is changed subject to external intervention, like a fan blowing air.

3. Equations for fluid simulations

3.1 Navier–Stokes equations 2

$$\frac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nu \Delta \boldsymbol{u} + \frac{1}{\rho}\boldsymbol{f}$$

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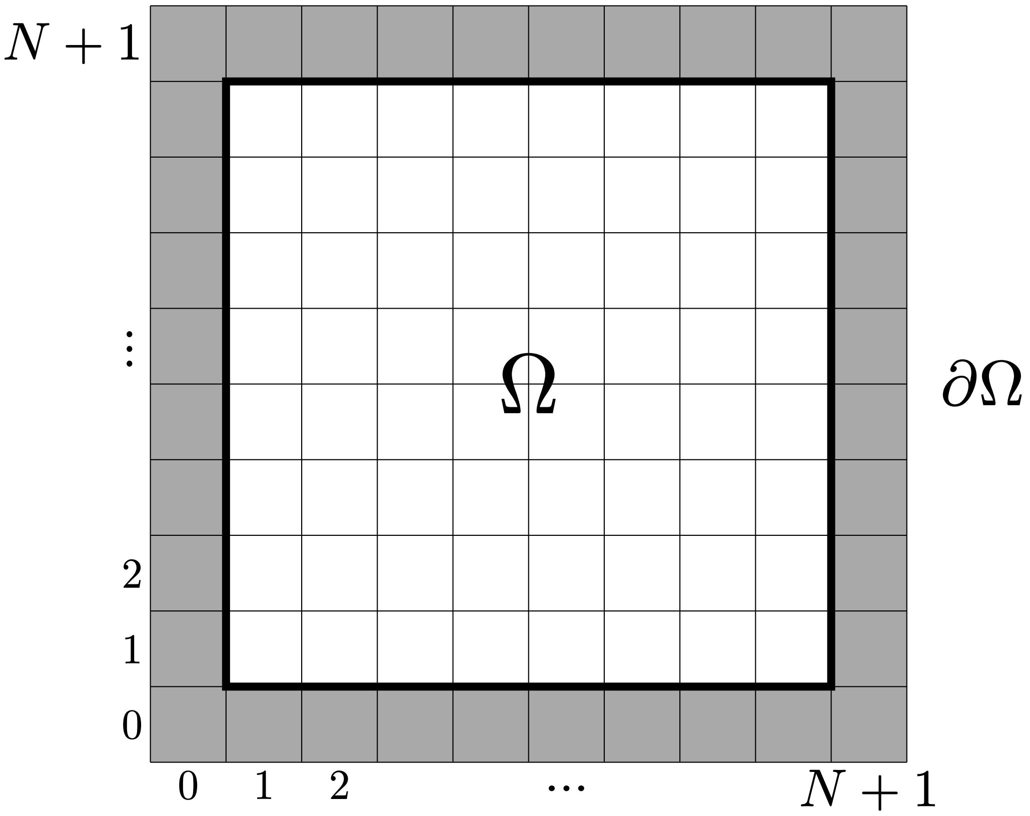
$$\nabla \cdot u = 0$$

$$\partial_\nu u|_{\partial \Omega} = 0$$

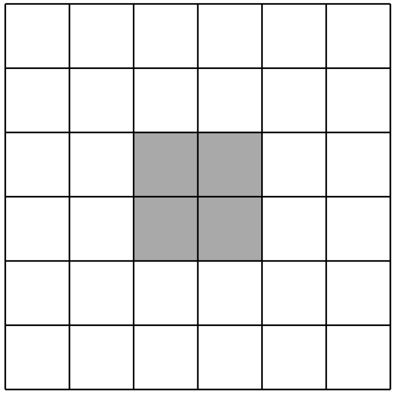
4. Simulating fluids

4.1 Fluid in a box



4.2 Moving densities

Initial density



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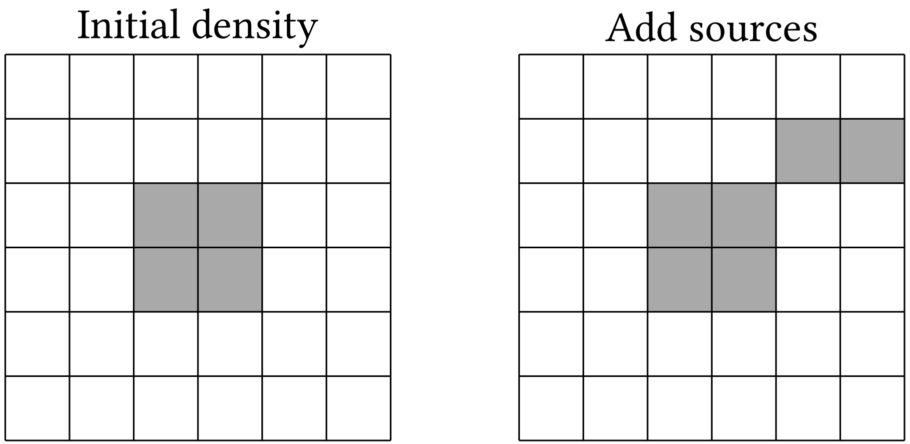

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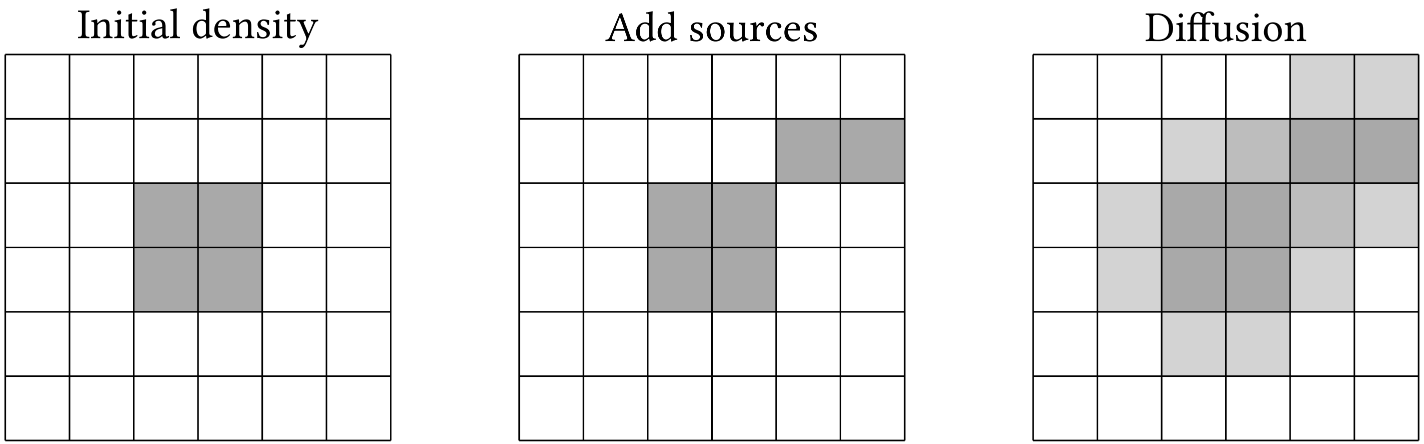
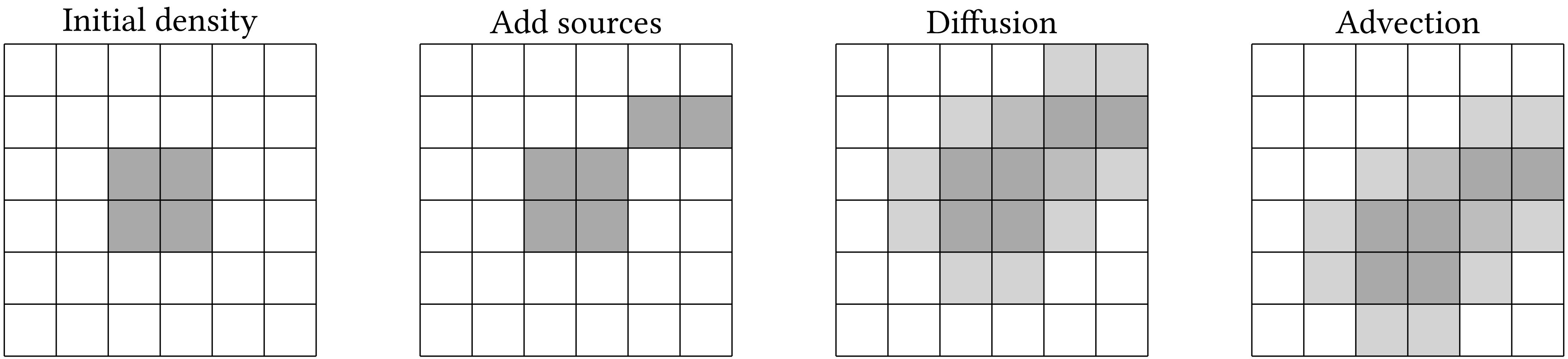


4.2 Moving densities



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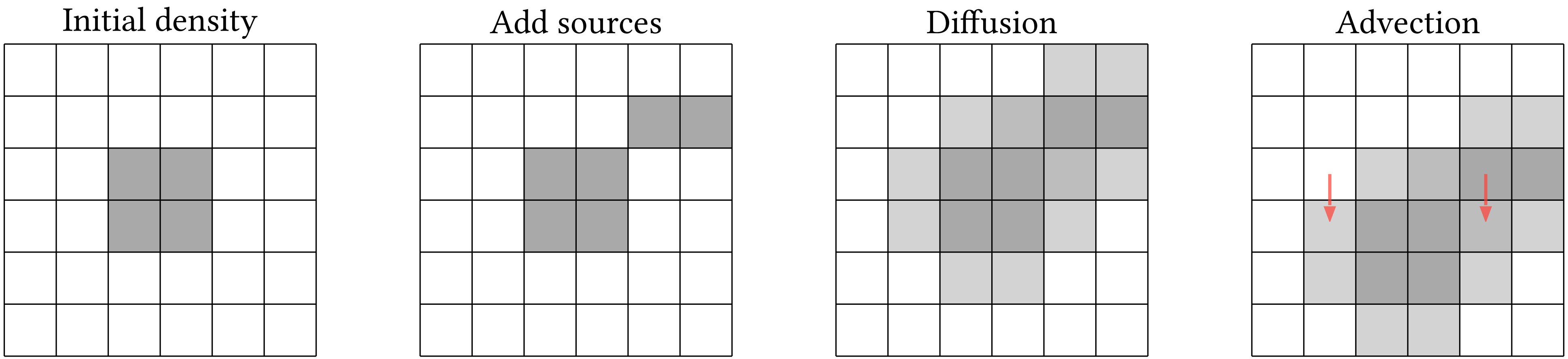

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$$\rho_{\text{next}} = \rho_{\text{prev}} + (\Delta t) \kappa \Delta \rho_{\text{prev}}$$

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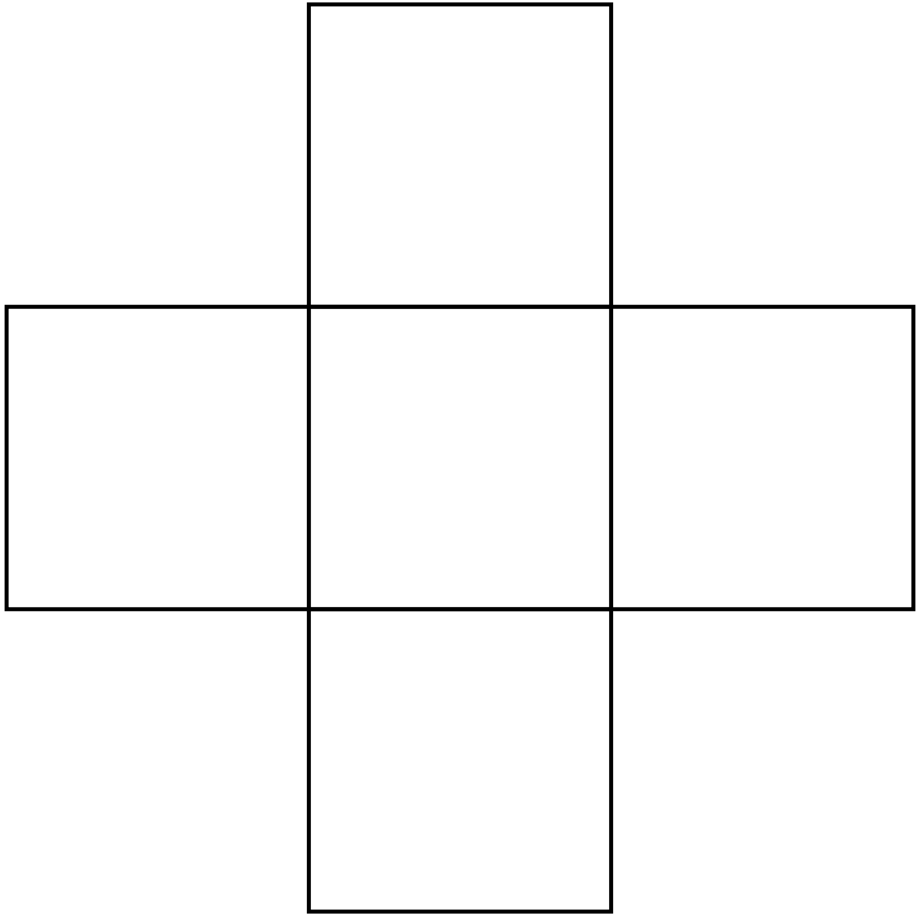
$$\rho_{\text{next}} = \rho_{\text{prev}} + (\Delta t) \kappa \Delta \rho_{\text{prev}} \qquad \text{(Helmholtz eq.)}$$

5.2 Duffusion



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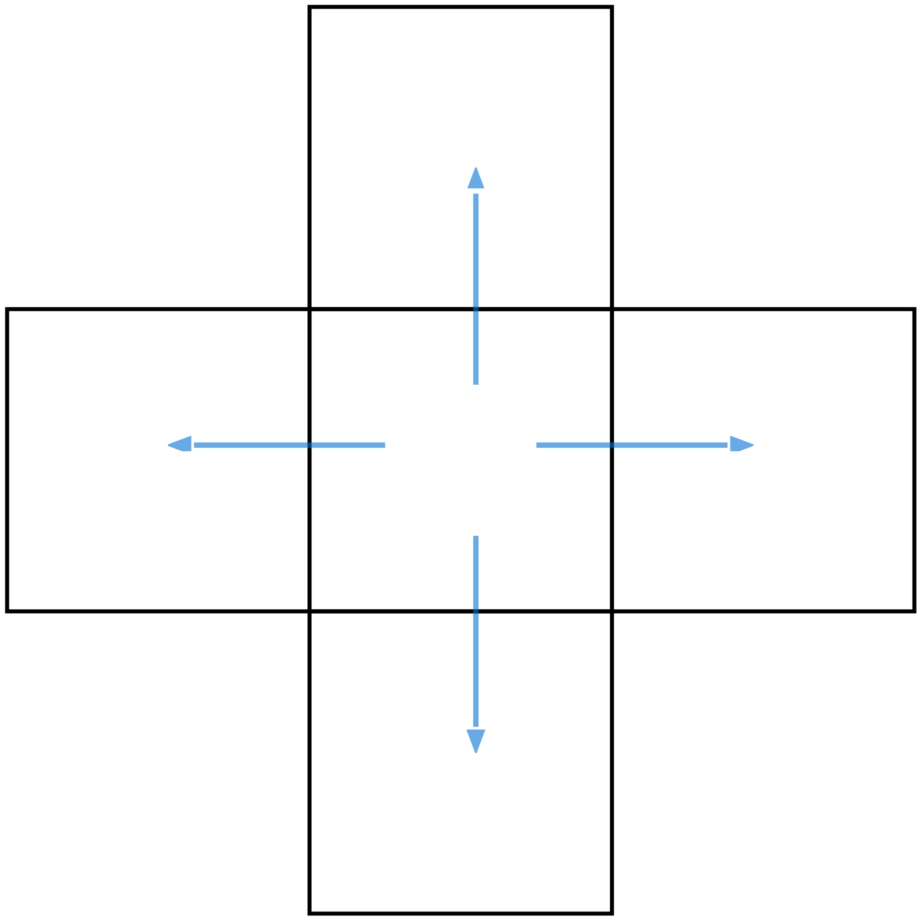

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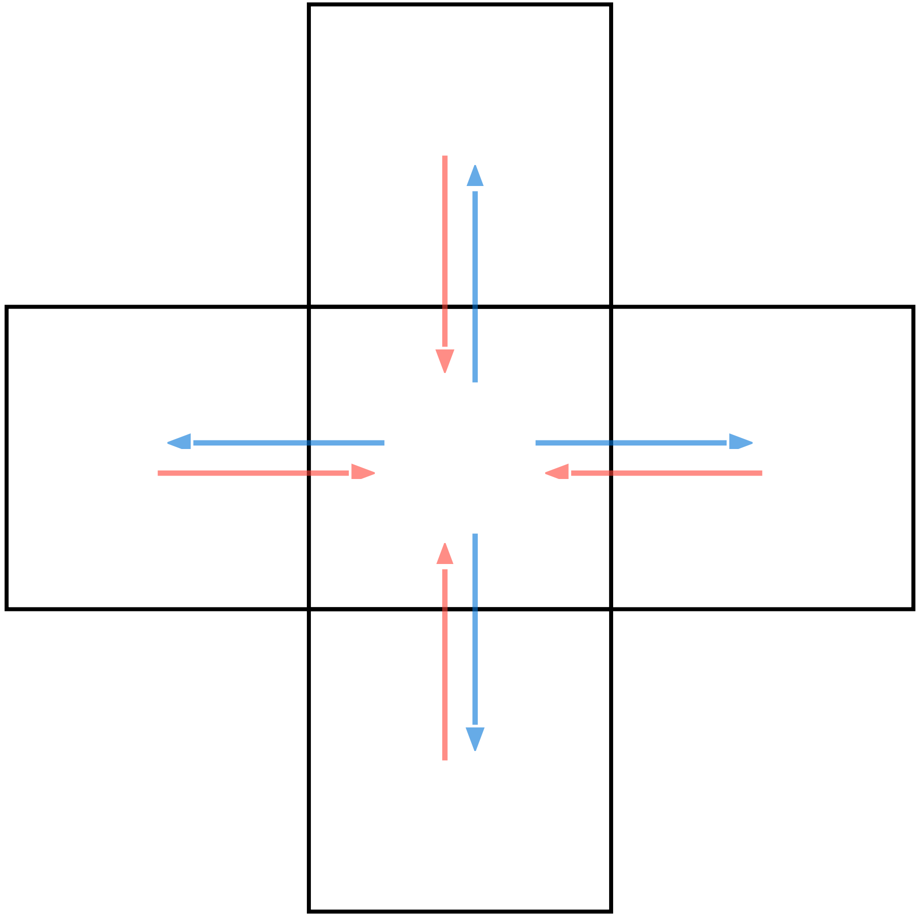

5.3 Finite difference method

$$\partial_1^2 \rho \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

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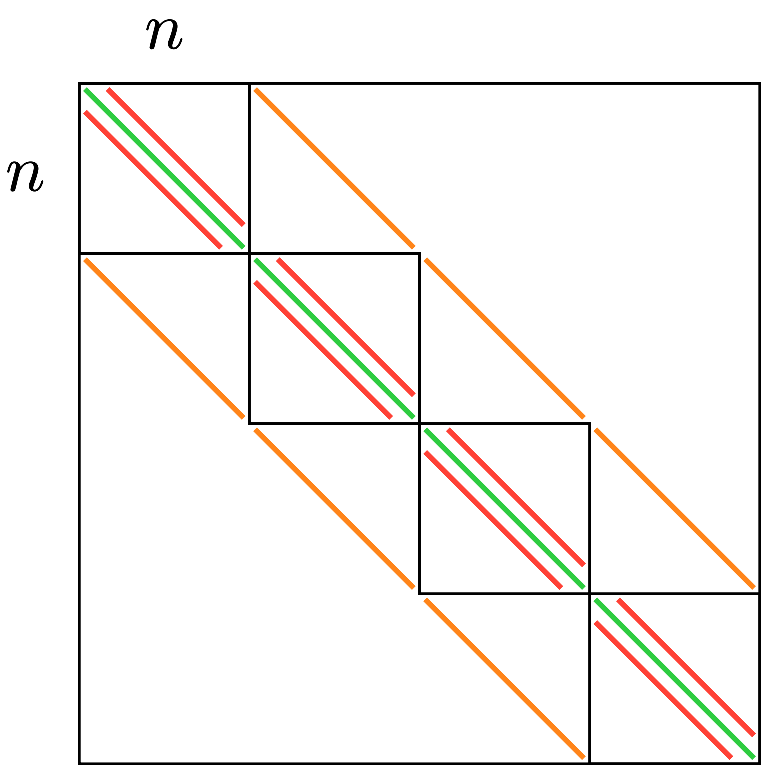
$$(\Delta_h \rho_h)_{i,j} = \frac{\rho_{i+1,j} + \rho_{i-1,j} + \rho_{i,j+1} + \rho_{i,j-1} - 4\rho_{i,j}}{h^2}$$

5.4 FDM matrix



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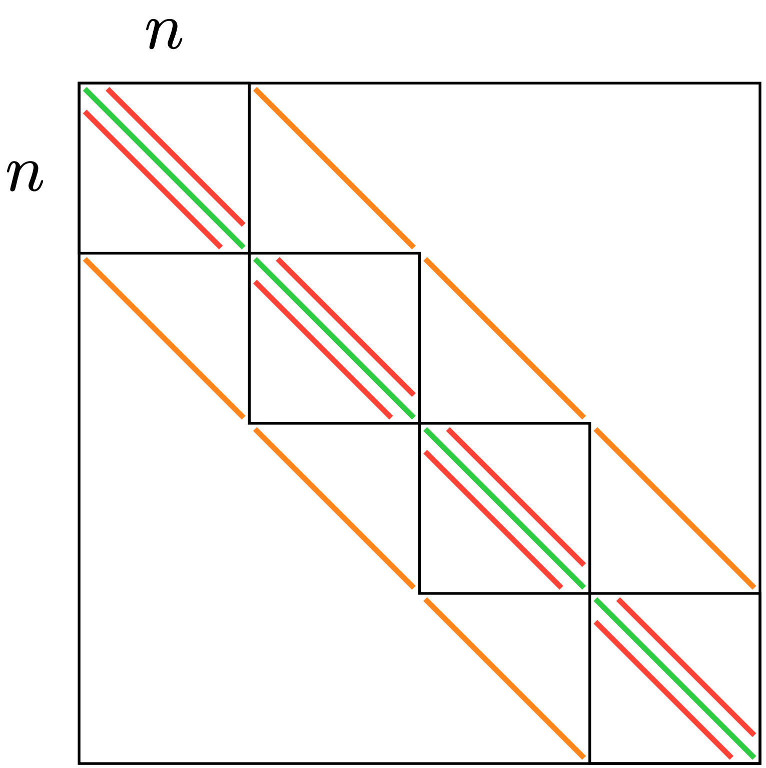

$\text{green} = 4/h^2$, $\text{red} = -1/h^2$, $\text{orange} = -1/h^2$

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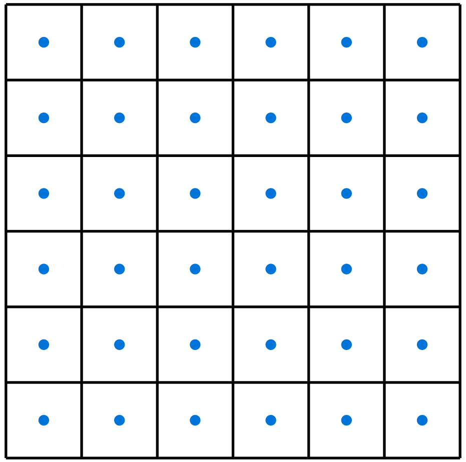
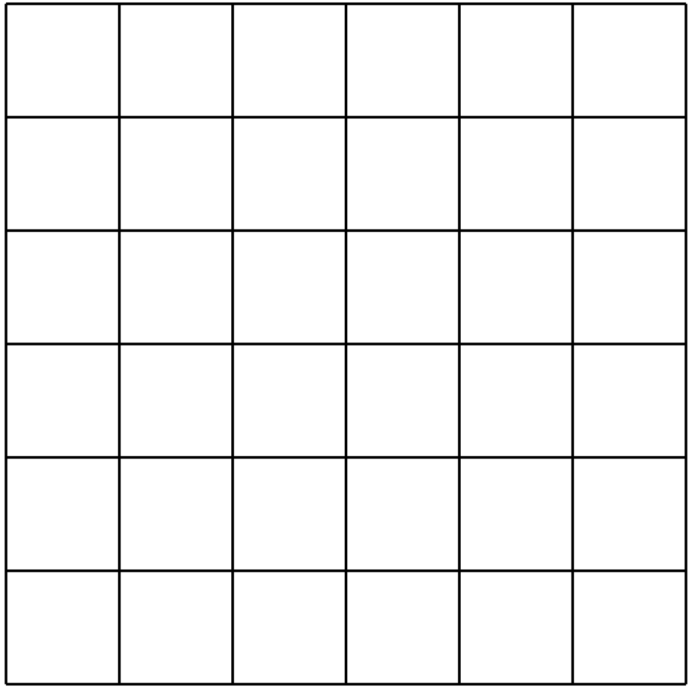


6.2 Semi-Lagrange



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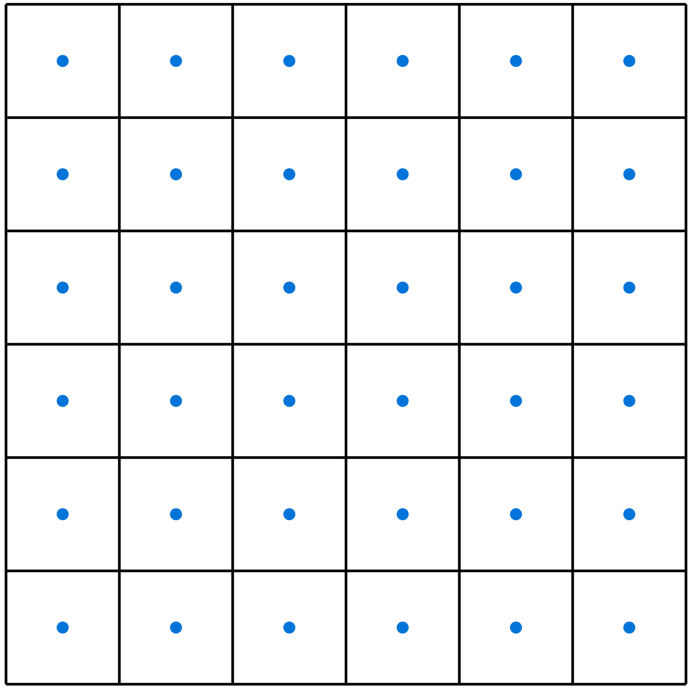

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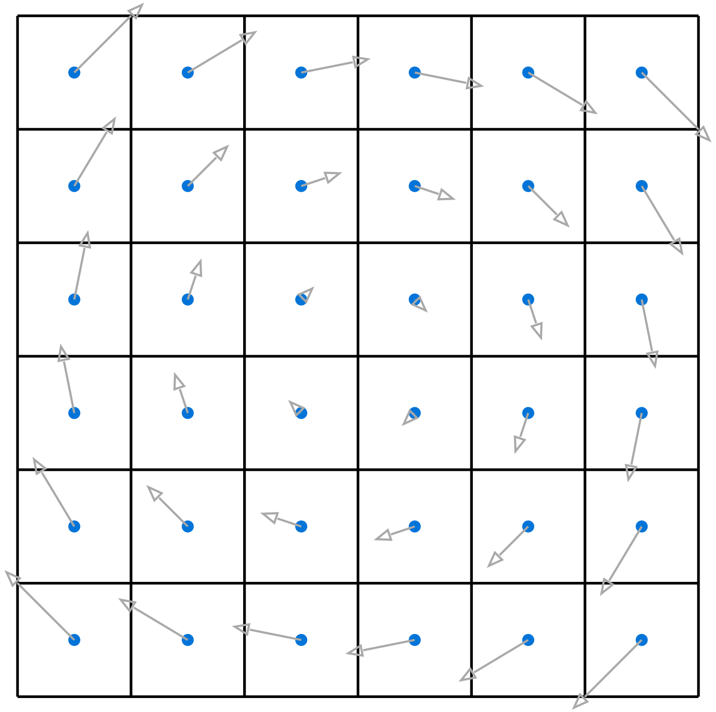

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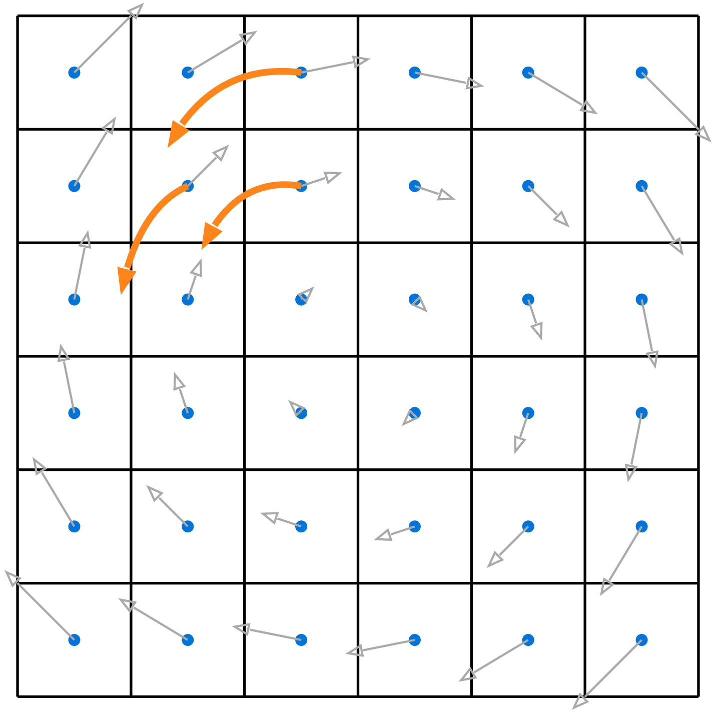
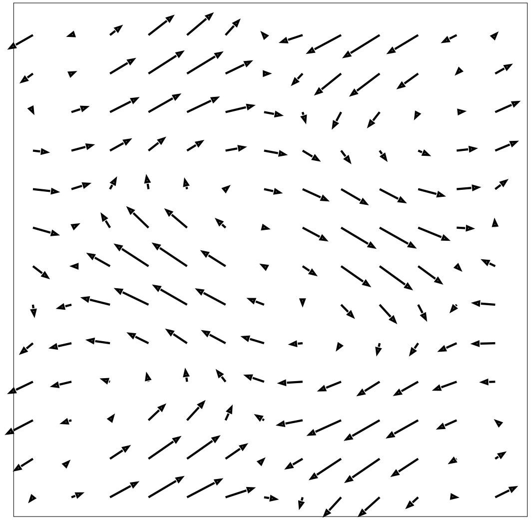


7. Evolving velocites

7.1 Helmholtz–Hodge decomposition

combined field



$$=$$

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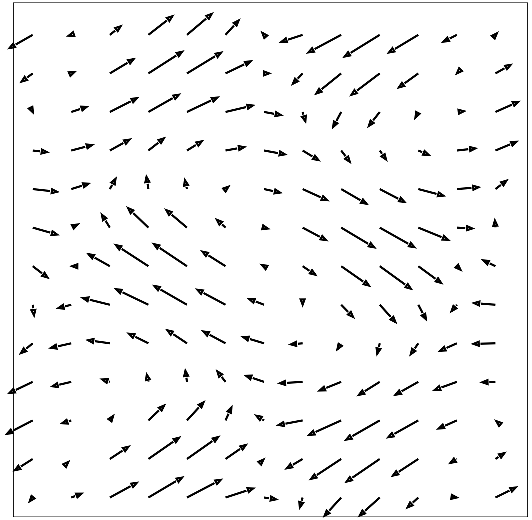
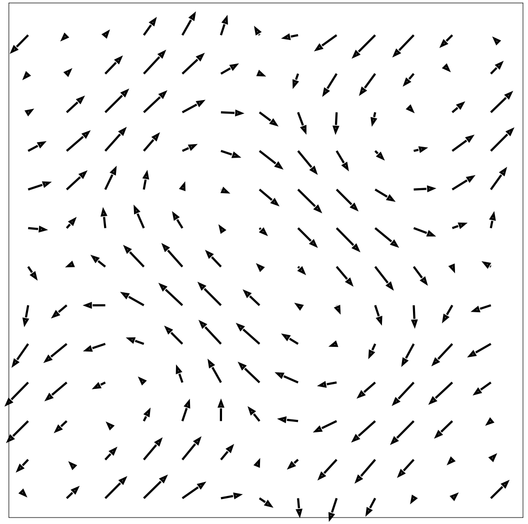


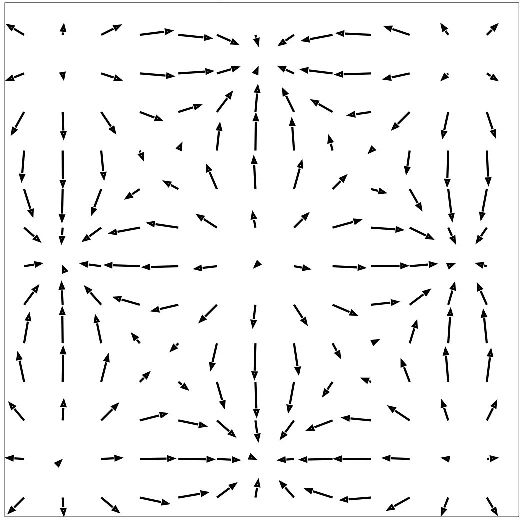
=

rotation field



+

divergence field



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$$\boldsymbol{w} = \boldsymbol{u} + \nabla q$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad q : \mathbb{R}^n \to \mathbb{R}$$

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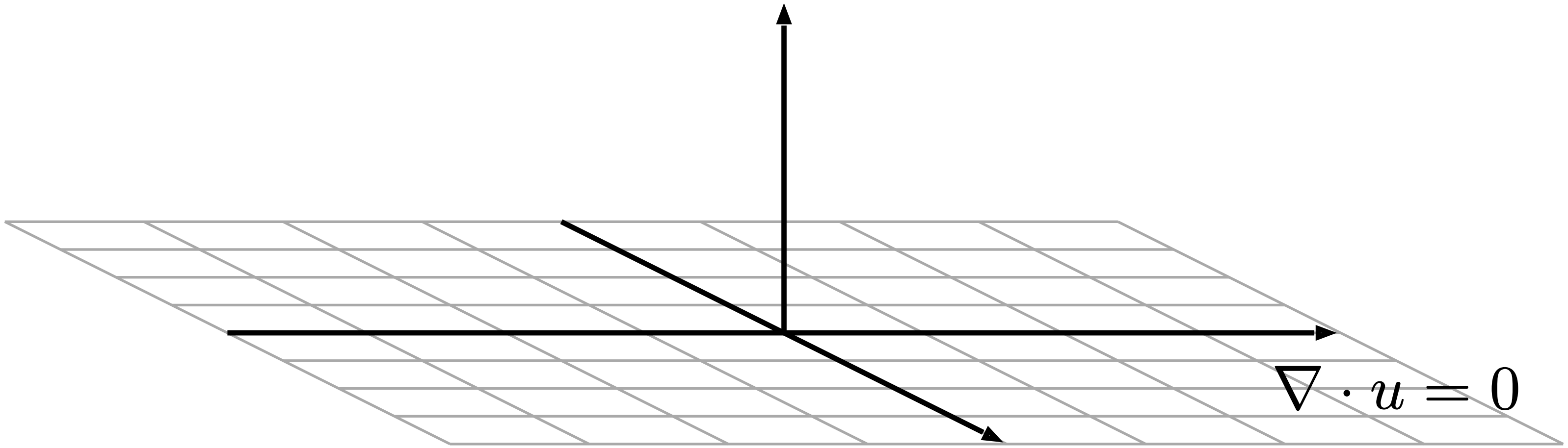
$$\nabla \cdot \boldsymbol{w} = \Delta q \qquad \text{(Poisson eq.)}$$

$$\boldsymbol{u} = \boldsymbol{w} - \nabla q$$

7.2 Simulation steps



$$u_1 \xrightarrow{\text{add source}} u_2 \xrightarrow{\text{diffusion}} u_3 \xrightarrow{\text{advection}} u_4 \xrightarrow{\text{projection}} u_5$$

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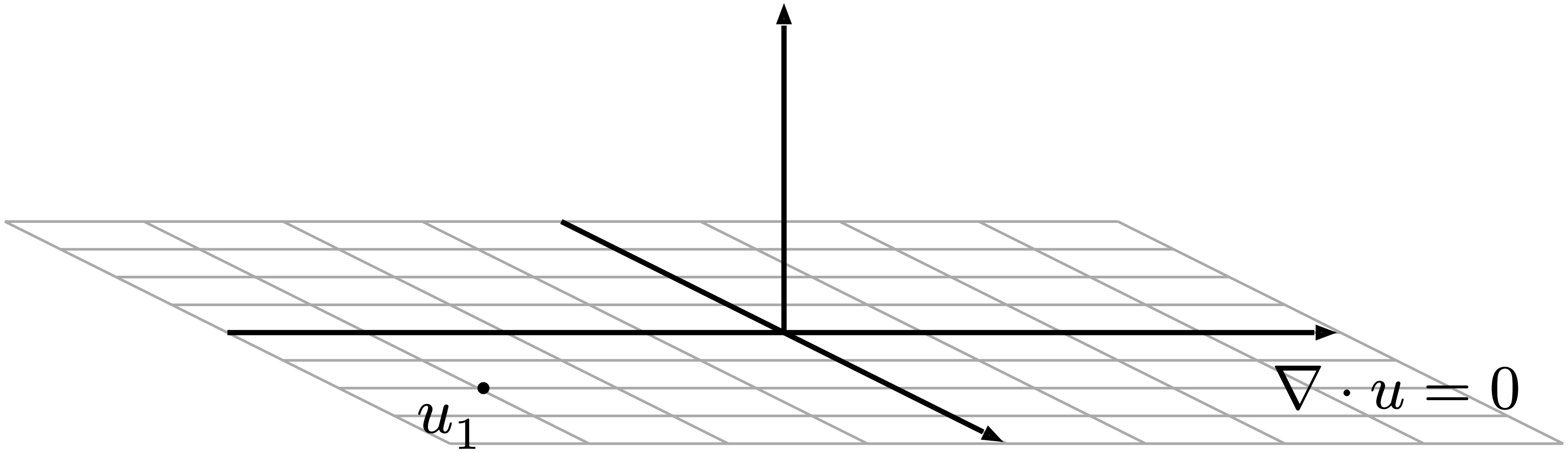

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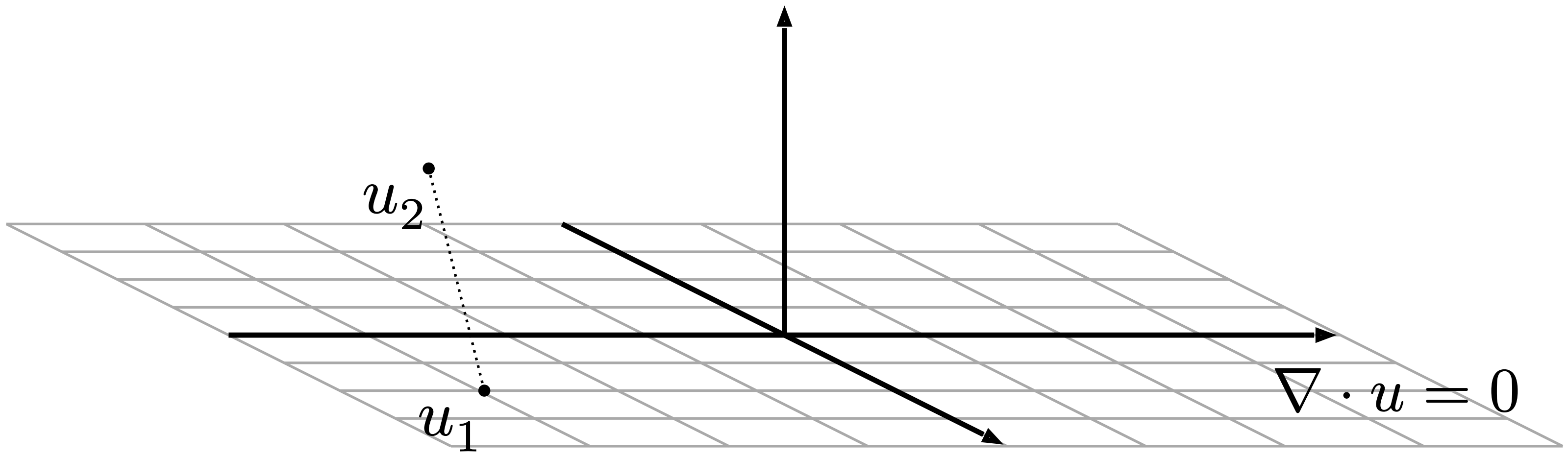

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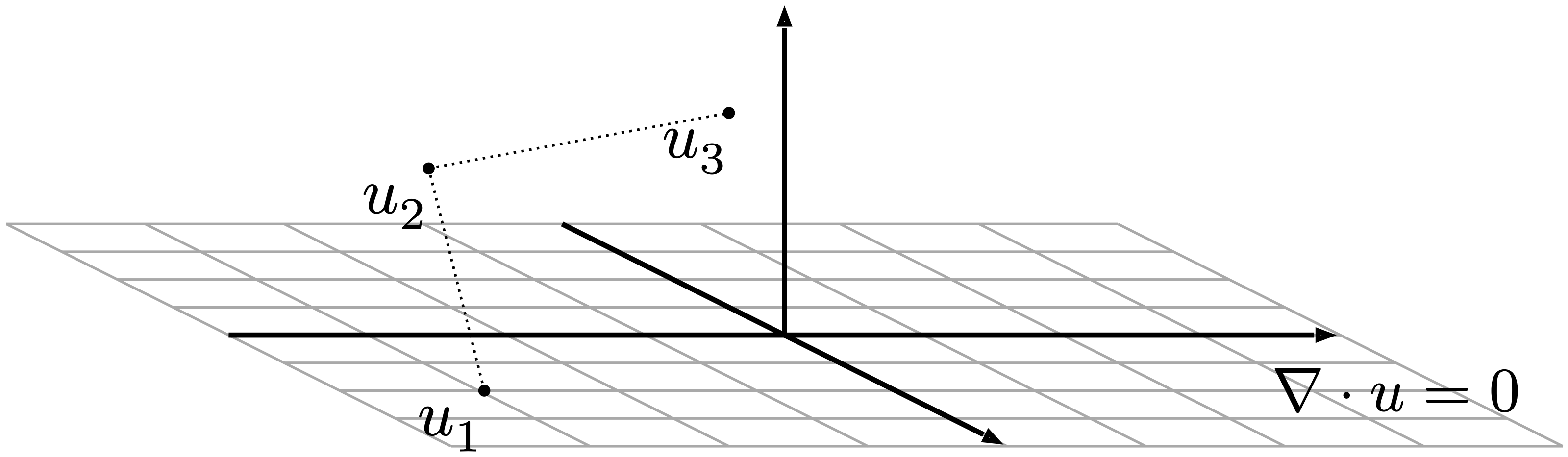

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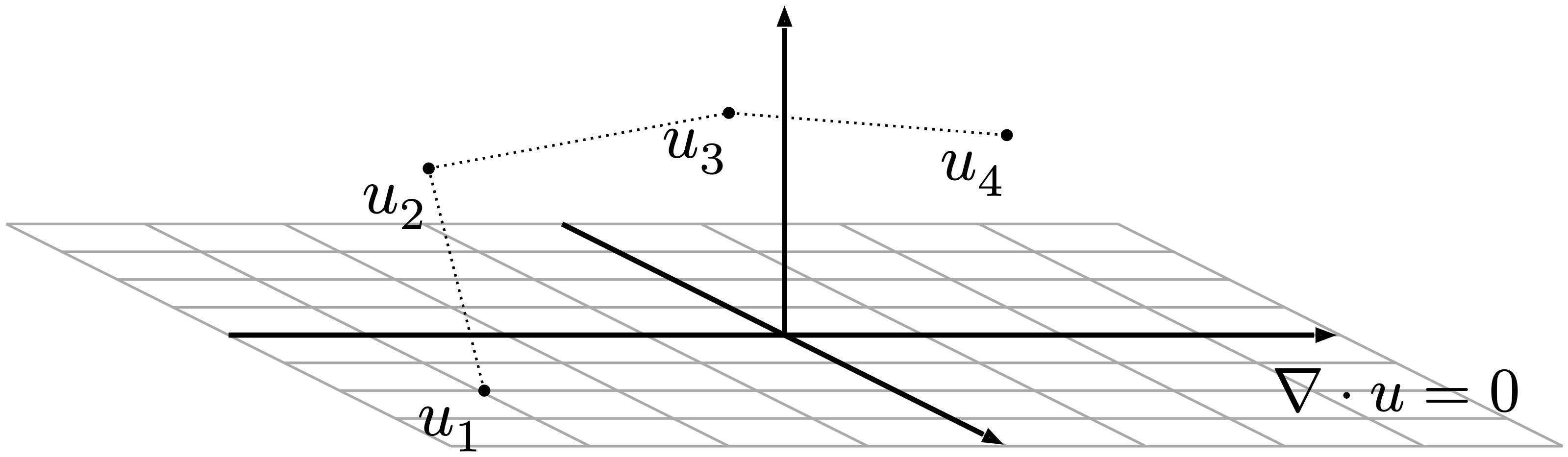

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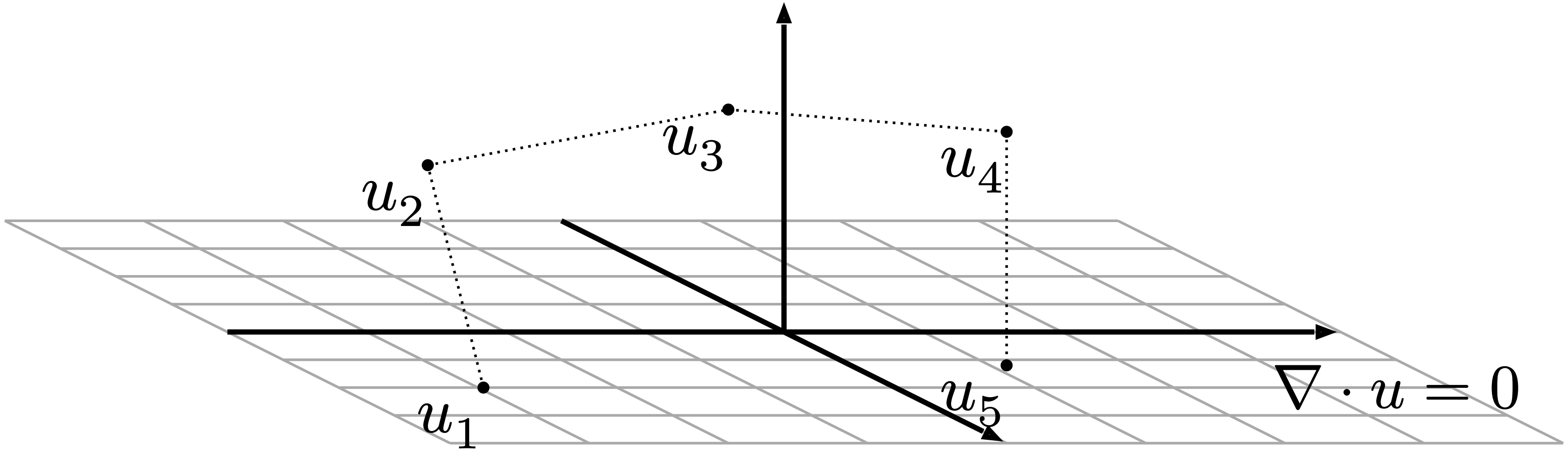

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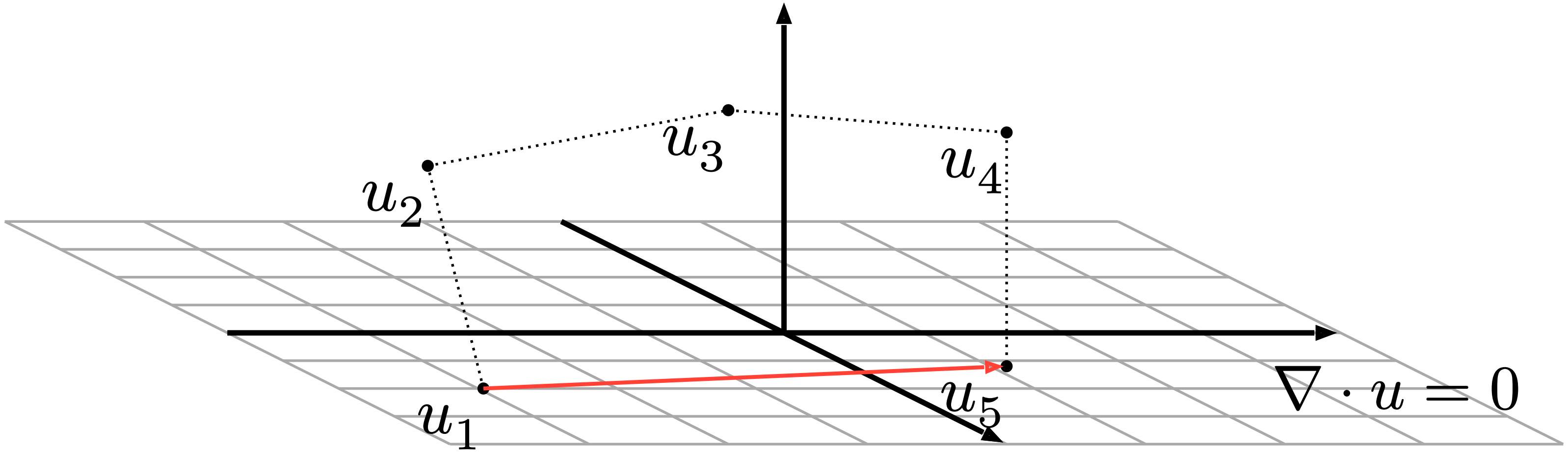

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8. Appendix

8.1 Appendix

- https://github.com/leonardo-toffalini/viscous
- https://github.com/leonardo-toffalini/fishy

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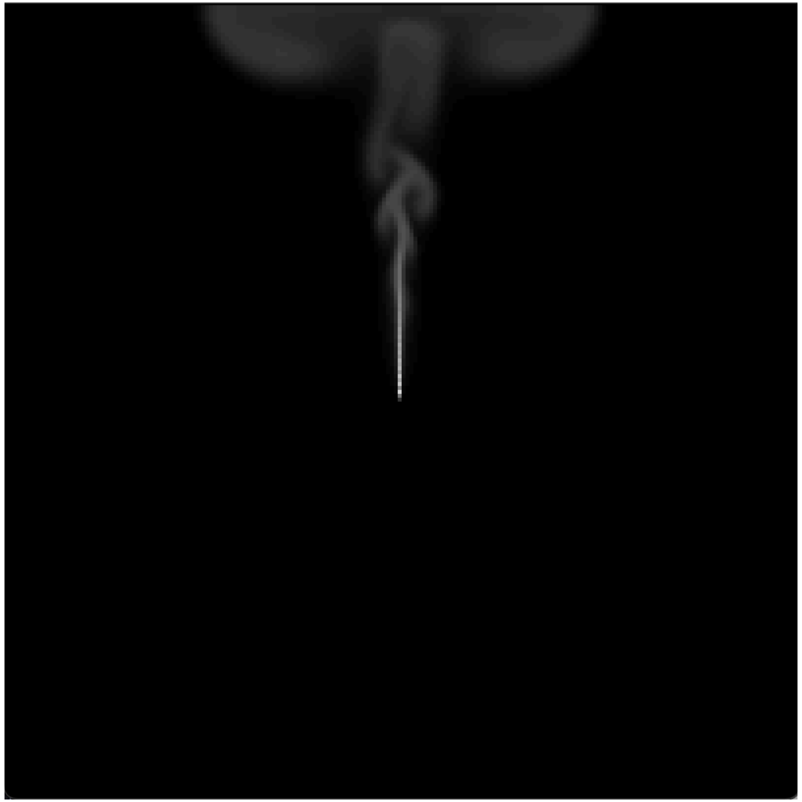

Figure 1: Smoke emitting from the tip of a cigarette

8.1 Appendix

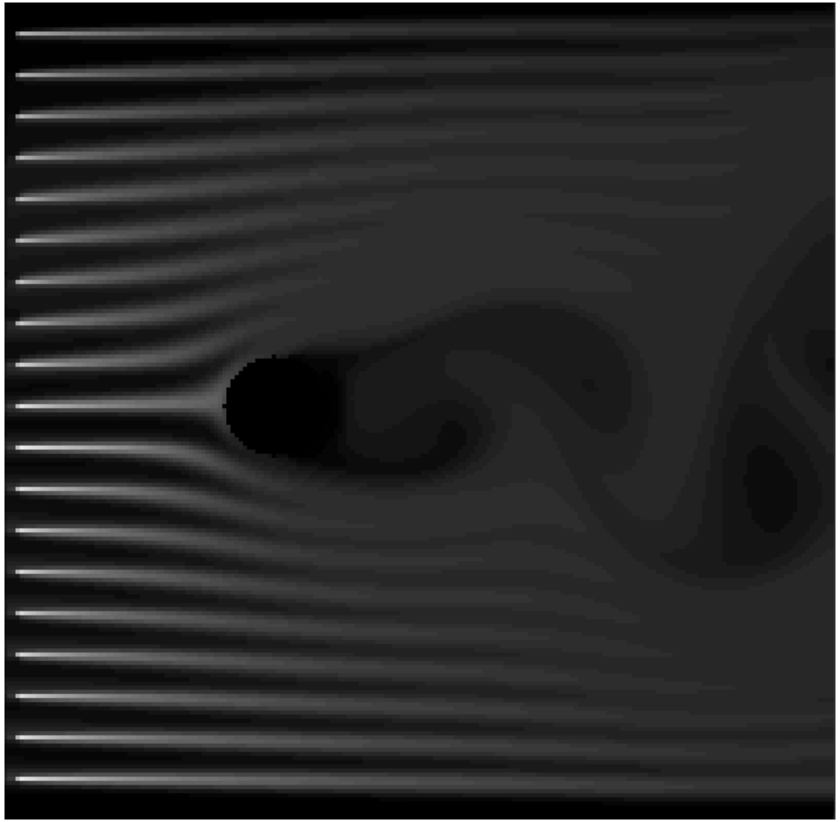

Figure 2: Vortex shedding

8.1 Appendix

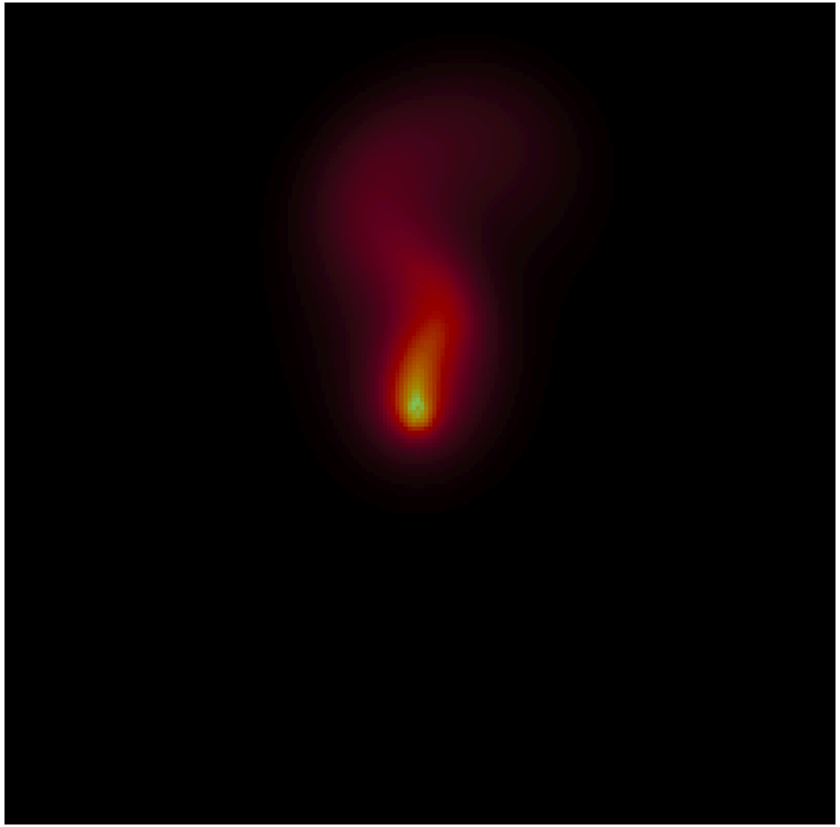

Figure 3: Flickering fire

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