DD2447 Statistical Methods in Applied Computer Science
Assignment 3

Leonardo De Clara, Riccardo Sena

December 20, 2023

Exercise 1

Since we are dealing with a Hidden Markov Model we know that $\{x_n\}_{n\geq 1}$ is a discrete-time Markov process, therefore the Markov property holds: $x_n|x_{n-1}, x_{n-2}, .., x_1 = x_n|x_{n-1}$. Moreover the following assumptions hold:

- y_i is independent of $y_{1:i-1}$ when conditioned on x_i
- y_i is independent of $x_{1:i-1}$ when conditioned on x_i
- y_i and x_i are independent of $x_{i+1:n}$

The assumptions stated above and the Markov property allow us to state that:

- $q(x_{1:n}) = q(x_1) \prod_{i=2}^{n} q(x_i|x_{i-1})$
- $p(y_{1:n}|x_{1:n}) = p(y_n|x_n) \prod_{i=1}^{n-1} p(y_i|x_i)$
- $p(x_{1:n}) = p(x_n|x_{n-1})p(x_1) \prod_{i=2}^{n-1} p(x_i|x_{i-1})$

We can use these equations to expand w_n's definition:

$$
\begin{aligned}
w_n = \frac{p(y_{1:n}|x_{1:n})p(x_{1:n})}{q(x_{1:n})} = \qquad (1)\\
= \frac{p(y_n|x_n)\prod_{i=1}^{n-1} p(y_i|x_i)p(x_n|x_{n-1})p(x_1)\prod_{i=2}^{n-1} p(x_i|x_{i-1})}{q(x_n|x_{n-1})q(x_1)\prod_{i=2}^{n-1} q(x_i|x_{i-1})} = \\
= \frac{p(y_n|x_n)p(x_n|x_{n-1})}{q(x_n|x_{n-1})} \frac{\prod_{i=1}^{n-1} p(y_i|x_i)p(x_1)\prod_{i=2}^{n-1} p(x_i|x_{i-1})}{q(x_1)\prod_{i=2}^{n-1} q(x_i|x_{i-1})} = \\
= \frac{p(y_n|x_n)p(x_n|x_{n-1})}{q(x_n|x_{n-1})} \frac{p(y_{1:n-1}|x_{1:n-1})p(x_{1:n-1})}{q(x_{1:n-1})}
\end{aligned}
$$

Knowing that

$$\alpha(x_n) = \frac{p(y_n|x_n)p(x_n|x_{n-1})}{q(x_n|x_{n-1})}$$

and

$$w_{n-1} = \frac{p(y_{1:n-1}|x_{1:n-1})p(x_{1:n-1})}{q(x_{1:n-1})}$$

We can then write

$$w_n = \alpha(x_n)w_{n-1}$$