# Linear algebra

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# 1 Introduction

In linear algebra we study object, also a point is a linear object. Usually we study this object from an geometry point of view, in this course we see the algebrian point of view. When we do **operations** we are doing algebra. Vector are a kind of linear object, we can see a vector like a point.

In $\mathbb{R}^3$ we have more kind of linear objects. In $\mathbb{R}^4$ we start to have problem whit the geometrical point of view. Whit equations i can describe objects in every dimension. Vector spaces is very important in linear algebra. Matrix are very important because we also have functions for transform object.

# 2 Sets

We denote by $\emptyset$ the empty set.

$$\mathbb{N} = \{0, 1, 2, \cdots\} \text{ Natural numbers}$$
$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\} \text{ Integers}$$
$$\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$$

$\mathbb{R}$ the set of all real numbers, Every real number is the limit of a sequence of rational numbers.

## 2.1 Examples

$$A = \{x \in \mathbb{R} : -5 \leq x \leq 2\}\, 11 \qquad B = \{x \in \mathbb{Z} : -5 \leq x \leq 2\}$$
$$C = \{x \in \mathbb{R} : x > 1\}$$

## 2.2 Operation between sets

Given two set $A$ and $B$ we define the following operation:

- Intersection: $A \cap B = \{x : x \in A \cap x \in B\}$
- Union $A \cup B = \{x : x \in A \vee X \in B\}$
- Difference $A \setminus B = \{x : x \in A \wedge x \notin B\}$
- Cartesian product $A \times B = \{(x, y) : x \in A \wedge y \in B\}$

# 3 Cartesian plane

The cartesian plane is a 2 dimensional object. All the $\mathbb{R}$ can be represented on an ≪oriented line≫ where we fixed a special point, namly the origin O. In a similar way we can describe all the elements of $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$ with the cartesian plane. We take

2 oriented lines which are orthogonal and we call the origin O the interaction between these lines (representing the element $(0,0) \in \mathbb{R}^2$.

The horizontal line is called the axis of abscises or the x-axis. The vertical line is called of ordinate or the y-axis.

Evrey point $P$ descibe an element of $\mathbb{R}^2$ since it is identified by two set number $X_P$ and $X_P$ which are its cordinated, i. e.,

$$\begin{aligned} \text{The y-axis is the following set } &= \left\{(0,y) \in \mathbb{R}^2\right\} = \{(0,y) : y \in \mathbb{R}\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : x = 0 \wedge y \in \mathbb{R}\right\} \end{aligned}$$

$$\begin{aligned} \text{The x-axis is the following set } &= \left\{(x,0) \in \mathbb{R}^2\right\} = \{(z,0) : z \in \mathbb{R}\} \\ &= \left\{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0\right\} \end{aligned}$$

Given two point $A$ and $B \in \mathbb{R}^2$, we can evaluate theri distance (euclidean distance) which is the lengh of the segment $\overline{AB}$:

$$\begin{aligned} d(A,B) = ||\overline{AB}|| &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\ &= \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} \end{aligned}$$

## 3.1 Vector in the cartesian plane

In general a vector is an arrow. Given a point $P$ in the cartesian plane, i. e., a point of two numbers $(x_P, y_P) \in \mathbb{R}^2$ it also indentified a vector of the plane. The point $P$ identifies the vector $\vec{OP} = (x_P, y_p)$. Sometimes we will also use the notion $\vec{\mathbf{v}}$ or $\mathbf{v}$ for a vector.

$$\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{R}^2$$

We have that, from this algebric point of view, point and vector are equivalent, they are the same object described by a point or set.

## 3.2 Operations in the cartesian plane

### 3.2.1 Scalar multiplication

Multiplication by a sclar, i. e., by a real number. Givern a point $A = (x_A, y_A)(\Leftrightarrow Q = (x_A, y_A) = \vec{OA})$ and a scalr $t \in \mathbb{R}$ we define the multiplication by a scalar $tA = (tx_A, ty_A)$.

$$\mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}^2$$

$$(A \times t) \longrightarrow tA$$

The effect of the multiplication by a scalar on a vector $\vec{\mathbf{a}}$ ub fact the vertor $\eth$ is stretched

### 3.2.2 Sum

Given two points $A = (x_A, y_A)(\Leftrightarrow a = (x_A, y_A))$ and $B = (x_B, y_B)$ the sum is defined as

$$A + B = (x_A + x_B, y_A + y_B)$$

$$\begin{aligned} \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (A, B) &\longrightarrow A + B \end{aligned}$$

$a + b(\Leftrightarrow A + B)$ is the diagonal of the parallelogram whith $a$ and $b$
the following proprieties hold:

$$\forall\ a, b, c \in \mathbb{R}^2, \forall\ t \in \mathbb{R} \text{ we have}$$

- $a + b = b + a$
- $(a + b) + c = a + (b + c)$
- $t(a + b) = ta + ta$
- $a + 0 = a$ when $0 = (0, 0) =$ origin
- $(-1)a = -a$ when $a + (-a) = 0$

## 3.3 Lines in the cartesian plane

Given a point $A = (x_A, y_A) \in \mathbb{R}^2$, there exist one and only one line $\ell$ through the origin O and the point A. Now we went to describe the coordinates of all point.

Let us take a generic point $P = (x_P, y_P) \in \ell$ and let us find its coordinate in term of $A$. The triangles $OAH$ and $OPC$ are similar (they have the same angles) and consequently the side are proportion (by the same factors):

$$||\overline{OK}|| = t||\overline{OH}|| \quad \text{and} \quad ||\overline{PK}|| = t||\overline{AH}|| \text{ for a certain } t \in \mathbb{R}$$

that is

$$\begin{cases} x'_p = tx_A \\ y'_p = ty_A \end{cases} \quad \forall\ t \in \mathbb{R} \qquad \begin{cases} x = tx_a \\ y = ty_A \end{cases} \quad \forall\ t \in \mathbb{R}$$

These are the parametric equations of a line through the origin and given a point $A$.

$$A = (2, 3) \qquad \begin{cases} x = 2t \\ y = 3t \end{cases} \quad \forall\ t \in \mathbb{R}$$

$$\text{In other words} \quad \ell = \left\{(x, y) \in \mathbb{R}^2 : x = tx_A, y = ty_A, \forall\ t \in \mathbb{R}\right\} \quad (1)$$

The cartesian equation of a line through the origin is

$$y = mx \qquad \text{where} \qquad m = \frac{y_A}{x_A} \quad \text{slope}$$

$$\ell \left\{ (x,y) \in \mathbb{R}^2 : y = mx, m = \frac{y_A}{x_A} \right\} \quad (2)$$

Let us Considering the set (1) and check that $O$ ans $A$ belong to this set:

$$O \in \ell \text{ because if we take } t = 0, \text{ we get } x = 0 \cdot X_A = 0, y = 0 \cdot y_A = 0$$

$$\Rightarrow (0,0) \in \ell$$

$A \in \ell$ because if we take $t = 1$, we have $x = x_A$, $y = y_A \Rightarrow A \in \ell$

Let us check that $O$ and $A$ belong to $\ell$ exploiting the set defined in (2):

$$O = (0,0) \in \ell \quad \text{is true, because} \quad 0 = m \cdot 0$$
$$A = (x_A, y_A) \in \ell \quad \text{is true, because} \quad y_A = mx_A \Leftrightarrow y_A = \frac{y_A}{\cancel{x}_A} \cdot \cancel{x}_A$$

We can check that a point provided by the parametric equations satisfies the cartesian equation.

Let us consider a point $(tx_A, ty_A)$ and check if it satisfies the cartesian equation $y = mx$:

$$ty_A = mtx_A \Leftrightarrow ty_A = \frac{y_A}{\cancel{x}_A} t \cancel{x}_a \Leftrightarrow ty_A = ty_A x$$

**Example**

Consider the line through the origin and $A = (5,2)$.

$$\text{The parametric equation are} \quad \begin{cases} x = 5t \\ y = 2t \end{cases} \quad \forall t \in \mathbb{R}$$

$$t = 0 \quad (0,0), \qquad t = 1 \quad (5,2), \qquad t = -2 \quad (-10,-4)\,,$$
$$t = \frac{1}{3} \quad \left(\frac{5}{3}, \frac{2}{3}\right) \qquad \ldots \qquad t = \sqrt{3} \quad \left(5\sqrt{3}, 2\sqrt{3}\right)$$

All belong to the line.

The point $(10,3)$ belong to the line? $\quad t = 2 \quad (10,4)$

In order to answer to this question we must solve the following linear system

$$\begin{cases} 5t = 10 \\ 2t = 3 \end{cases}$$

where $t$ is the unknown, but this linear system doesn't have solutions!

This $(10,3)$ doesn't belong to the line The cartesian equation of this line is $y = \frac{2}{5}x \to 3 = \frac{2}{5} \cdot 10$ is not true! is not an identity

Now, let us describe the parametric equations of a line (not necessary through the origin). Given two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ we want to describe the coordinates of a generic point $P$ over the line through $A$ and $B$. We can consider the line $\ell'$ parallel to $\ell$ and passing through the origin. The parametric equations of $\ell'$ are:

$$\begin{cases} x = t(x_B - x_A) \\ y = t(y_B - y_A) \end{cases} \qquad \forall\ t \in \mathbb{R}$$

The parametric equations of $\ell$ are

$$\begin{cases} x = x_A + t(x_B - x_A) \\ y = y_A + t(y_B - y_A) \end{cases} \qquad \forall\ t \in \mathbb{R}$$

this

$$\ell \left\{(x, y) \in \mathbb{R}^2 : x = x_A + t(x_B - x_A), y = y_A + t(y_B - y_A),\ \forall\ t \in \mathbb{R}\right\}$$

Similarly, tha cartesian equation is

$$y = mx + q$$

where $m = \frac{y_B - y_A}{x_B - x_A}$ slope and $q = \frac{y_A x_B - x_A y_B}{x_B - x_A}$ constant term

# 4 Linear independence (linear dependence)

If two vectos $\vec{v}_1$ and $\vec{v}_2$ lie on the same line (throgh the origin) then they are linearly dependent (from a geometrical point of view).

For an algebraic point of view this means that

$$\vec{v}_1 = t\vec{v}_2 \quad t \in \mathbb{R}$$

Two vectors are linearly independent if and only if they are not linearly dependent.

In general if $\vec{w}_1$ and $\vec{w}_2$ are linearly independent vectors than all the vectors $\vec{v} \in \mathbb{R}$ can be written as a linear combination of $\vec{w}_1$ and $\vec{w}_2$, this means that

$$\forall\ \vec{v} \in \mathbb{R}^2 \qquad \vec{v} = a\vec{w}_1 + b\vec{w}_2 \qquad \text{fore some } a, b \in \mathbb{R}$$

and we say that $\{\vec{w}_1, \vec{w}_2\}$ is a basis of $\mathbb{R}^2$.

A special basis is $\{\vec{e}_1, \vec{e}_2\}$ where $\vec{e}_1 = (1, 0)$, $\vec{e}_2 = (0, 1)$ and it is called tha canonical basis.

$\{\vec{e}_1, \vec{e}_2\}$ is also an orthogonal basis because $\vec{e}_1$ and $\vec{e}_2$ ore orthogonal vectors.

# 5 Cartesian space

The cartesian space is $\mathbb{R}^3 = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ and its elements are triple of real number $(x, y, z) \quad \forall\, x, y, z \in \mathbb{R}$.

$O$ is a fixed point of the space, called the origin and we also have three orthogonal lines intersecting in the origin.

A point $A \in \mathbb{R}^3$ is identified by three real numbers $(x_A, y_A, z_A)$ where:

$x_a$ is the distance from the plane $Oyz$

$y_a$ is the distance from the plane $Oxz$

$z_a$ is the distance from the plane $Oxy$

Also in the cartesian space, we can compute the Euclidean distance between the origin $||\overline{OA}|| = d(O, A)$. Similarly the distance between $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ is $||\overline{AB}|| = d(A, B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$.

Similarly to $\mathbb{R}^2$, also in $\mathbb{R}^3$ the vectors are arrows starting from the origin and they are describes by triples of real numbers.

**Operations in $\mathbb{R}^3$**

Let us define some operations in $\mathbb{R}^3$:

- Scalar multiplication:

$$\begin{aligned} &\mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ &(t, \vec{v}) \longmapsto t\vec{v} = (tv_1, tv_2, tv_3) \\ &(t, V) \longmapsto tV = (tv_1, tv_2, tv_3) \end{aligned}$$

  where $\vec{v} = V = (v_1, v_2, v_3)$

- Sum (componentwise):

$$\begin{aligned} &\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ &(\vec{v}, \vec{w}) \longmapsto \vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3) \\ &(V, W) \longmapsto V + W = (v_1 + w_1, v_2 + w_2, v_3 + w_3) \end{aligned}$$

  where $\vec{v} = (v_1, v_2, v_3) \quad \vec{w} = (w_1, w_2, w_3)$

Given three vectors in $\mathbb{R}^3$, $\vec{u}, \vec{v}, \vec{w}$, we say that they are linearly dependent if there exist three real numbers $a, b, c \in \mathbb{R}$ not all zeros such that

$$a\vec{u} + b\vec{v} + c\vec{w} = 0 = (0, 0, 0)$$

$$\vec{u} = \alpha\vec{v} + \beta\vec{w}$$

$$a\vec{u} + b\vec{v} + c\vec{w} = 0 \Leftrightarrow a\vec{u} = -b\vec{v} - c\vec{w}$$

If the vectors $\vec{u}, \vec{v}, \vec{w}$, are not linearly dependent, then we sat that they are linearly independent. In general, given $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ we say that they are linearly dependent if and only if there exist $a_1, a_2, \dots, a_n \in \mathbb{R}$ not all zeros such that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = 0$$

If $\vec{v}_1, \dots, \vec{v}_n$ are not linearly dependent, then we say that they are linearly independent.

In the space, if we have there linearly independent vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$, then all the vectors $\vec{v} \in \mathbb{R}^3$ can be written ad a linear combination of $\vec{w}_1, \vec{w}_2, \vec{w}_3$. And we say the $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is a basis of $\mathbb{R}^3$.

The canonical basis of $\mathbb{R}^3$ is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ where $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$.

Let us check that $\vec{e}_1, \vec{e}_2, \vec{e}_3$ ale linearly independent, we must find if there exist $a, b, c$ not all zeros such that

$$a\vec{e}_1 + b\vec{e}_2 + v\vec{e}_3 = \vec{0}$$
$$(a,0,0) + (0,b,0) + (0,0,c) = \vec{0}$$
$$(a,b,c) = (0,0,0)$$
$$\begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases} \iff \vec{e}_1, \vec{e}_2, \vec{e}_3 \text{ are l.i.}$$

$\forall\ \vec{v} \in \mathbb{R}^3$, $\vec{v} = (v_1, v_2, v_3)$, then $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3$.

**Example**

$$\vec{v} = (5,3,2) = 5(1,0,0) + 3(0,1,0) + 2(0,0,1)$$

## 5.1 Equation of a plane in $\mathbb{R}^3$

We wold like to characterize all the point belonging to some plane in $\mathbb{R}^3$

$$\begin{aligned} \text{The plane} \qquad Oxy &= \{(x,y,0) : x, y \in \mathbb{R}\} \\ Oyz &= \{(0,y,z) : y, z \in \mathbb{R}\} \\ Oxz &= \{(x,0,z) : x, z \in \mathbb{R}\} \end{aligned}$$

Given a generic plane in the space, it is uniquely determined by:

- Two interacting lines
- Three non collinear points
- two linearly independent vectors (identify uniquely a plane through the origin) and a point in $\mathbb{R}^3$

We start considering a plane described by two linearly independent vectors, i.e., we are describing all the planes through the origin.

We want to describe any point $P = (x_P, y_P, z_P)$ belonging to the plane identified by two linearly independent vectors $\vec{v}$ and $\vec{w}$.

Remember that two linearly independent vectors can be used for constructing all the vectors belonging to the same plane (think, e.g., to the special case of the cartesian plane). All the points $P = (x_P, y_P, z_P)$ belonging to the plane containing the vectors $\vec{v}$ and $\vec{w}$ are linear combinations of $\vec{v}$ and $\vec{w}$:

$$P = \overline{OP} = (x_P, y_P, z_P) = s\vec{v} + t\vec{w} \qquad \forall\ s, t \in \mathbb{R}$$

The parametric equations of the plane $\Pi$ through the origin and containg $\vec{v}$ ans $\vec{w}$ are

$$\begin{cases} x = sv_1 + tw_1 \\ y = sv_2 + tw_2 \\ z = sv_3 + tw_3 \end{cases} \qquad \forall\ s,t \in \mathbb{R}$$

In other words

$$\Pi = \left\{(x,y,z) \in \mathbb{R}^3 : x = sv_1 + tw_1, y = sv_2 + tw_2, z = sv_3 + tw_3, \forall\ s,t \in \mathbb{R}\right\}$$

The parametric equations of a plane $\Pi$ identified by two linearly independent vectors $\vec{v}$ and $\vec{w}$ and through a point $A = (x_A, y_A, z_A)$ are

$$\begin{cases} x = x_A + sv_1 + tw_1 \\ y = y_A + sv_2 + tw_2 \\ z = z_A + sv_3 + tw_3 \end{cases} \qquad \forall\ s,t \in \mathbb{R}$$

$$s = 0,\ t = 0 \qquad \begin{cases} x = x_A \\ y = y_A \\ z = z_A \end{cases}$$

We can obtain the cartesian equation of a plane. If we focus on the equations $\begin{cases} x = x_A + sv_1 + tw_1 \\ y = y_A + sv_2 + tw_2 \end{cases}$ where $s$ and $t$ are unknowns. We can solve this linear system and we obtain the solutions:

$$\begin{cases} s = ax + by + f & a,b,f \in \mathbb{R} \\ t = cx + dy + g & c,d,g \in \mathbb{R} \end{cases}$$

Now we substitute these values of $s$ and $t$ in the third equations:

$$z = z_A + sv_3 + tw_3$$

and we obtain something like $z = m + nx + py \quad m,n,p \in \mathbb{R}$ Cartesian equation of a plane in the space

$$\alpha x + \beta y + \gamma z + \delta = 0$$

Cartesian equation, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ depending on $\begin{matrix} x_A, & y_A, & z_A \\ v_1, & v_2, & v_3 \\ w_1, & w_2, & w_3 \end{matrix}$

## 5.2 Equations of a line in $\mathbb{R}^3$

A line is always described by two points.

Let us start with a line through the origin and a point $V = (v_1, v_2, v_3)$, i.e., we are considering the line describes by the vector $\vec{v} = \overline{OV} = (v_1, v_2, v_3)$ and so all the points in this line $\ell$ are described by

$$(x, y, z) = t\vec{v} = (tv_1, tv_2, tv_3) \quad \forall\ t \in \mathbb{R} \iff \begin{cases} x = tv_1 \\ y = tv_2 \\ z = tv_3 \end{cases}$$

A general line in the space is then described by a vector $\vec{v}$ (giving the direction ) and a point $C = (x_C, y_C, z_C)$ and the parametric equations are:

$$\begin{cases} x = x_C + tv_1 \\ y = y_C + tv_2 \\ z = z_C + tv_3 \end{cases} \qquad \forall\ t \in \mathbb{R}$$

The above description of a line is equivalent to describe it giving two point $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$

$$\begin{cases} x = x_A + t(x_B - x_A) \\ y = y_A + t(y_B - y_A) \\ z = z_A + t(z_B - z_A) \end{cases} \iff \begin{cases} x = x_B + t(x_B - x_A) \\ y = y_B + t(y_B - y_A) \\ z = z_B + t(z_B - z_A) \end{cases} \qquad \forall\ t \in \mathbb{R}$$

A line in the space is identified by the intersection between two (non-parallel) plane:

$$\begin{cases} \alpha x + \beta y + \gamma z + \delta = 0 \leftarrow \text{Cartesian equation of a plane} \\ \alpha' x + \beta' y + \gamma' z + \delta' = 0 \leftarrow \text{Cartesian equation of a plane} \end{cases} \qquad \alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' \in \mathbb{R}$$

# 6 Vector space

- 0 - dimensional object $\rightarrow \mathbb{R}^0$
- 1 - dimensional object $\rightarrow \mathbb{R}^1$
- 2 - dimensional object $\rightarrow \mathbb{R}^2$
  - 0 - dimensional object (points)
  - 1 - dimensional object (lines)
- 3 - dimensional object $\rightarrow \mathbb{R}^3$
  - 0 - dimensional object (points)
  - 1 - dimensional object (lines)
  - 2 - dimensional object (planes)

Take now 2 cubes move one of them along to a new (the $4^{th}$) direction, connect the vertexes and we get a 4-dimensional object and if we extend it at the infinity in all the 4 direction, then we get the 4 - dimensional space $\to \mathbb{R}^4$. A $n$-dimensional space is described by $\mathbb{R}^n$.

In $\mathbb{R}^n$, a $k$-dimensional object (where $0 \leq k \leq b-1$ is described by parametric equations with $k$ different parameter, it has $k$ degrees of freedom. Given a space of dimension $n$ (i.e., $\mathbb{R}^n$) a k-dimensional object is a subspace of $\mathbb{R}^n$ of dimension $k$.

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

Similarly to $\mathbb{R}^2$ and $\mathbb{R}^3$, also in $\mathbb{R}^n$ points and vectors are the same objects described by $n$-tuple of real numbers.

- Multiplication by a scalar

$$\begin{aligned} \mathbb{R} \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (t, \vec{v}) &\longmapsto t\vec{v} = (tv_1, tv_2, \ldots, tv_n) \end{aligned}$$

- Sum (componentwise)

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (\vec{v}, \vec{w}) &\longmapsto \vec{v} + \vec{w} = (v_1 + w_1, \ldots, v_n + w_n) \end{aligned}$$

$\mathbb{R}^n$ equipped with these two operations has nice proprieties! Consider a set $V$ equipped with two operations:

$$\begin{aligned} * : V \times V &\longrightarrow V \\ \odot : \mathbb{R} \times V &\longrightarrow V \end{aligned}$$

where this operations satisfy the following proprieties:

1. $*$ is associative: $\forall\ u, v, w \in V \quad (u * v) * w = u * (v * w)$
2. $*$ is commutative: $\forall\ u, v \in V \quad u * v = v * u$
3. extistence of identity: $\exists e \in V$ s.t. $\forall\ v \in V \quad e * v = v$
4. existence of inverses: $\forall\ v \in V,\ \exists w \in V$ s.t. $v * w = e$
5. $\forall\ v \in V$, $1 \odot v = v$
6. $\forall\ s, t \in \mathbb{R},\ \forall\ v \in V, \quad (s + t) \odot v = (s \odot v) * (t \odot v)$
7. $\forall\ s \in \mathbb{R}, \forall\ v, w \in V \quad t \odot (v * w) = (t \odot v) * (t \odot w)$

Then we say that $V$ is a vector space over $\mathbb{R}$ (real vector space)

**Remark**

$\mathbb{R}^n$ with our operation is a vector space over $\mathbb{R}$

In a vector space $V$ we always have a basis that is a set of elements such that all the elements of $V$ are linear combination of the elements of the basis.

In $\mathbb{R}^n$, a basis contains always $n$ elements, indeed a set of $n$ linear independent vectors generates all the elements of $\mathbb{R}^n$.

In other words if $\left\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\right\}$ is a basis of $\mathbb{R}^n$, then $\forall\ \vec{v} \in \mathbb{R}^n$ we have $\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + \dots + a_n\vec{b}_n$ for $a_1, \dots, a_n \in \mathbb{R}$.

A special basis is the canonical basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$:

$$\vec{e}_1 = (1, 0, \dots, 0) \qquad \vec{e}_2 = (0, 1, \dots, 0) \qquad \cdots \qquad \vec{e}_{n-1} = (0, \dots, 1, 0) \qquad \vec{e}_n = (0, \dots, 0, 1)$$

In $\mathbb{R}^n$, we say that $m$ vectors $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent when

$$c_1\vec{v}_1 + \dots, c_w\vec{v}_m = \vec{0} \iff c_1 = \dots = c_m = 0$$

We say that they are linearly dependent when there exist $c_1, \dots, c_m \in \mathbb{R}$ not all zeros such that $c_1\vec{v}_1 + \dots, c_w\vec{v}_m = \vec{0}$

In $\mathbb{R}^n$, we can construct a basis if we take $n$ linearly independent vectors.

## 6.1 Euclidean norm

Given a vector $\vec{v} \in \mathbb{R}^n$m the Euclidean norm of $\vec{v}$ is

$$\begin{aligned} ||\vec{v}|| = d(\vec{v}, \vec{0}) &= \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} \\ d(V, O) & \end{aligned}$$

The Euclidean norm satisfies the triangle inequality:

$$\forall\ \vec{v}, \vec{w} \in \mathbb{R}^n \qquad ||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$$

## 6.2 Dot product

$$\begin{aligned} \bullet :& \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \\ & (\vec{v}, \vec{w}) \longmapsto \vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n \end{aligned}$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$ (in general in $\mathbb{R}^n$ two vectors $\vec{v}$ and $\vec{w}$ are orthogonal if and only if $\vec{v} \bullet \vec{w} = 0$

## 6.3 Equation of a plane given an orthogonal vector and a point

Considering a vector $\vec{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$, the plane through the origin and orthogonal to $\vec{n}$ is the set of points $(x, y, z)$ such that the vector $(x, y, z)$ is orthogonal to $\vec{n}$, i.e., they must satisfy

$$(x, y, z) \bullet (n_1, n_2, n_3) = 0$$

$$\Updownarrow$$

$$n_1 x + n_2 y + n_3 z = 0$$

Cartesian equation of the plane throught the origin and orthogonal to $\vec{n}$.

In general, the equation of a plane through a point $A = (x_A, y_A, z_A) \in \mathbb{R}^3$ and orthogonal to $\vec{n}$ in the set of points $(x, y, z) \in \mathbb{R}^3$ such that the vector $(x - x_A, y - y_A, z - z_A)$ is orthogonal to $\vec{n}$ and so

$$(x - x_A, y - y_A, z - z_A) \bullet (n_1, n_2, n_3) = 0$$
$$\Updownarrow$$
$$n_1(x - x_A) + n_2(y - y_A) + n_3(z - z_A) = 0$$

Cartesian equation of a plane through the point $A$ and orthogonal to $\vec{v}$.

**Remark**

Given $X \subseteq \mathbb{R}^n$, in order to check if $X$ is a vector space, it is sufficient to check if $(X \neq \emptyset)$ it is closed under addition and multiplication by a scalar, i.e,

$$\forall\ x, y \in X,\ \text{check if } x + y \in X$$

$$\forall\ x \in X,\ \forall\ \lambda \in \mathbb{R},\ \text{check if } \lambda x \in X$$

# 7 Matrices

Given $m, n \in \mathbb{N} \setminus \{0\}$ an $m \times n$ matrix ($m$ rows and $n$ columns). Is and object of this kind

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

where the elements of $A$ are $a_{ij} \in \mathbb{R} \quad \forall\ i, j\ 1 \leq i \leq m \ \ 1 \leq j \leq n$.

Sometimes we will also write $\overline{A} = (a_{ij})$. We also call the elements $a_{ij}$ as element of the matrix. A matrix is an element of $\mathbb{R}^{m \times n} = \mathbb{R}^{m \cdot n} = \mathbb{R}^{m,n}$

If a matrix $A \in \mathbb{R}^{m \times m}$ is called a square matrix

A matrix in $\mathbb{R}^{1 \times n}$ is called a row vector

A matrix in $\mathbb{R}^{m \times 1}$ is called a column vector

Since $\mathbb{R}^{m \times n}$ (the set of all matrices with $m$ rows and $n$ columns) is a set of the kind $\mathbb{R}^K$, we know that $\mathbb{R}^{m \times n}$ is a vector space where the operations are

- Addition (componentwise)

$$\forall\ A, B \in \mathbb{R}^{m \times n} \qquad \text{we have } A + B = (a_{ij} + b_{ij}) \in \mathbb{R}^{m \times n}$$

$$\begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1m} + b_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

- Multiplication by a scalar

$$\forall\ A \in \mathbb{R}^{m\times n} \quad \forall\ \lambda \in \mathbb{R} \quad \text{we have } \lambda A = (\lambda a_{ij}) \in \mathbb{R}^{m\times n}$$

$$\begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \dots & \lambda a_{mn} \end{bmatrix}$$

In general a linear system is a set of equation with a certain number of unknowns that must satisfy these equations simultaneously:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & +a_{1n}x_n = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & +a_{2n}x_n = b_2 \\ \vdots & & \vdots & & \ddots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & +a_{mn}x_n = b_m \end{cases}$$

where $x_1, x_2, \dots, x_n$ are unknowns

$a_{ij} \in \mathbb{R}$ given real numbers, coefficients of the linear system where $1 \leq i \leq m$ and $1 \leq j \leq n$

$b_k \in \mathbb{R}$ given real numbers, constant term of the linear system where $1 \leq k \leq m$

Given the above linear system, we can associate to it some matrices:

The matrix of coefficients $A = (a_{ij}) \in \mathbb{R}^{m\times n}$

The vector (matrix) of unknowns $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n\times 1}$

The vector (matrix) of constant tems $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^{m\times 1}$

Linear system with $m$ equations and $n$ unknowns.

The definition of product became, thinking a linear system with

matrix of coefficients $A \in \mathbb{R}^{m\times n}$

vector of unknowns $\vec{x} \in \mathbb{R}^{n\times 1}$

vector of constant terms $\vec{b} \in \mathbb{R}^{m\times 1}$

is equivalent to

$$A\vec{x} = \vec{b}$$

$$\mathbb{R}^{m\times n} \times \mathbb{R}^{n\times 1} \to \mathbb{R}^{m\times 1}$$

**Remark**

The dot product between two vectors $\vec{v} = (v_1, \dots, v_n)$ and $\vec{w} = (w_1, \dots, w_n)$ can be written also

$$\begin{bmatrix} v_1, \dots, v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \vec{v} \bullet \vec{w} \to \mathbb{R}$$

**Definition**

The transpose of a matrix $A \in \mathbb{R}^{m\times n}$ is a new matrix

$$A^T \in \mathbb{R}^{n\times m}$$

where the rows of $A^T$ are the columns of $A$ or equivalently the column of $A^T$ are the rows of $A$

$$A = \begin{bmatrix} 1 & \pi & -\sqrt{2} & 3 \\ 0 & -1 & 0 & 3 \\ \sqrt{3} & \frac{1+\sqrt{5}}{2} & 2\pi & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 0 & \sqrt{3} \\ \pi & -1 & \frac{1+\sqrt{5}}{2} \\ -\sqrt{2} & 0 & 2\pi \\ 3 & 3 & -1 \end{bmatrix}$$

**Remark**

The product between matrices is non commutative!

When $A$ and $B$ are square matrices, e.g., in $\mathbb{R}^{n\times n}$ then

$$AB \in \mathbb{R}^{n\times n} \quad \text{and} \quad BA \in \mathbb{R}^{n\times n} \qquad \text{but usually}$$

$$AB \neq BA$$

**Example**

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In the example $AB = 0$ product is 0 even if $A, B \neq 0$

$$A + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A$$

in general if $A \in \mathbb{R}^{m\times n}$, then $0 \in \mathbb{R}^{m\times n}$ whole element are all zeros is the identity.

**Definition**

The matrix $I_n \in \mathbb{R}^{n\times n}$ with $I_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$

$$I_n = (I_{ij}) \qquad \begin{aligned} &I_{ij} = 0 \text{ if } i \neq j \\ &I_{ii} = 1 \text{ for all } i = 1, \dots, n \end{aligned}$$

is the **identity matrix**. Indeed $\forall\ A \in \mathbb{R}^{n\times n}$ we have $AI_n = I_nA = A$

# 8 Solving linear systems

**Definition**

Two linear system are equivalent if they have the same solutions

$$\begin{cases} x + 2y + z = 1 \\ 3x - 4y + z = 0 \end{cases} \quad \text{is equivalent to} \quad \begin{cases} x + 2y + z = 1 \\ 2x - y + z = \frac{1}{2} \end{cases}$$

First of all, we focus on homogeneous linear system, which are linear system where the constant term are all zeros:

$$A\vec{x} = \vec{0}$$

Our goal is to introduce some «transportation» which are called elementary operations on the matrix $A$ such that the new matrix $B$ where

$$B\vec{x} = \vec{0} \quad \text{is equivalent to} \quad A\vec{x} = \vec{0}$$

Let us define there elementary operations $\quad A \in \mathbb{R}^{m\times n}$

1. **Row switching**: switch two rows of $A$ (if we switch two rows of $A$ this is equivalent to switch two equations of the linear system $A\vec{x} = \vec{0}$; we get an equivalent new linear system).

   We can switch two rows of $A$ performing the following product:

   $$S_{ij}A$$

   where $S_{ij}$ is a matrix in $\mathbb{R}^{m\times m}$ obtained from the identity matrix $I_m$ switching the $i^{th}$ row of $I_m$ with the $j^{th}$ row of $I_m$

   $$A\vec{x} = \vec{0}$$
   $$\Updownarrow$$
   $$S_{ij}A\vec{x} = S_{ij}\vec{0}$$

$$\Updownarrow$$
$$B\vec{b} = \vec{0}$$

where $B$ is obtained from $A$ switching the $i^{th}$ row with the $j^{th}$ row

2. **Row multiplication**: Multiply a row of $A$ by a scalar $\lambda \in \mathbb{R}$ (if you multiply an equation by a scalar you get an equivalent equation)

   We can obtain this operation, considering the following product

   $$D_i(\lambda)A$$

   Where $D_i(\lambda)$ is a diagona matrix $m \times m$ obtained from $I_m$ replacing $I_{ii}$ with $\lambda$

   $$A\vec{x} = \vec{0} \iff D_i(\lambda)A\vec{x} = \vec{0} \iff B\vec{x} = \vec{0}$$

3. **Row addition**: add to a row of $A$ anther row of $A$ multiplied by a scalar $\lambda \in \mathbb{R}$. We can obtain this elementary operation, considering the following product.

   $$E_{ij}(\lambda)A$$

   Where $E_{ij}(\lambda)$ is a $m \times m$ matrix obtained from $I_m$ replacing the element $I_{ij}$ with $\lambda \in \mathbb{R}$. This product replaces the $i^{th}$ row of $A$ with the sum of $i^{th}$ row of $A$ with the $j^{th}$ row of $A$ multiplied by $\lambda$.

   $$A\vec{x} = \vec{0} \iff E_{ij}(\lambda)A\vec{x} = \vec{0} \iff B\vec{x} = \vec{0}$$

**Definition**

A square matrix $A \in \mathbb{R}^{n \times m}$ is inverible (w.r.t. the products) if there exist a matrix $B \in \mathbb{R}^{n \times m}$ such that $AB = BA = I_n$
In this case we write $B = A^{-1}$. The inverse of $A$ is denoted my $A^{-1}$

Since when we start from a linear system with constant vector $\vec{b}$ we get an equivalent linear system with a new constant vector, it is useful to def with the argumented matrix $(A|b) \in \mathbb{R}^{m \times n+1}$

In general, in order to solve a linear system $A\vec{x} = \vec{b}$ we want to obtain an equivalent linear system $B\vec{x} = \vec{c}$ where

$$(B|\vec{c}) \quad \text{(obtained from } (A|\vec{b}) \text{ using elementary operations)}$$

is in the so-callet **Row-echelon form**.

**Definition**

A matrix $M \in \mathbb{R}^{a \times b}$ is in **row-echelon form (REF)** if:

1. The leading (first nonzero) entry of each nonzero row (called a *pivot*) is strictly to the right of the pivot of the row above it.

2. All zero rows (if any) are at the bottom of the matrix.

The rows of a matrix, denoted by rank($M$), is the number of pivot of any REF for $M$. A column of $M$ is called pivot column if it contains one pivot.

A matrix $M$ is in reduced row-echelon form RREF if

1. in in REF
2. all the pivot are 1
3. the only non-zero element of any pivot column is the pivot itself (REF is not unique. RREF is unique)

Let consider $A\vec{x} = \vec{b}$ where $A \in \mathbb{R}^{m\times n}$ and focus on the augmented matrix $\left(A|\vec{b}\right)$, then we transform it into a matrix $\left(A'|\vec{b'}\right)$ that is REF and $A'\vec{x} = \vec{b'}$ is equivalent to $A\vec{x} = \vec{b}$

1. If rank$\left(A'|\vec{b'}\right) \neq$ rank$(A')$, then $A'\vec{x} = \vec{b}$ and $A\vec{x} = \vec{b}$ have no solutions

$$A' = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rank}(A') = 3$$

$$\left(A'|\vec{b'}\right) = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 5 & 3 & -1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right] \qquad \text{rank}\left(A'|\vec{b'}\right) = 4 \neq \text{rank}(A')$$

$$\begin{cases} x - \phantom{5}y + \phantom{-}2z = 0 \\ \phantom{x -{}} 5y + \phantom{-}3z = -1 \\ \phantom{x - 5y +{}} -2z = 2 \\ \phantom{x - 5y + -2{}} 0 = 5 \end{cases}$$

Indeed, this means that the column $\vec{b'}$ in the augmented matrix must contain a pivot, i.e., a non-zero element $k$ in correspondence to a zero row of $A'$ and this yelds to a fals identity of the kind

$$0 = k \qquad k \in \mathbb{R} \qquad k \neq 0$$

2. If rank $\left(A'|\vec{b'}\right)$ = rank$(A')$ = $n$ number of unknowns, them there exists a unique

solution

$$A' = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad A' \in \mathbb{R}^{4\times 3} \Rightarrow \text{ 4 equations, 3 uknowns}$$

$$\text{rank}(A') = 3$$

$$\left(A'|\vec{b}'\right) = \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{rank}\left(A'|\vec{b}'\right) = 3$$

$$\begin{cases} x - \ \ y + \ \ 2z = -1 \\ \quad 5y + \ \ 3z = \ \ 0 \\ \qquad\quad -2z = \ \ 3 \\ \qquad\qquad\ \ 0 = \ \ 0 \end{cases}$$

3. If $\text{rank}\left(A'|\vec{b}'\right) = \text{rank}(A') < n$ ($n$ = number of unknowns) the linear system has infinitely many solutions.

   Consider for example

$$A' = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rank}(A') = 2 < 3 = n$$

$$\left(A'|\vec{b}'\right) = \left[\begin{array}{ccc|c} 1 & -1 & 2 & \sqrt{2} \\ 0 & 5 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{rank}\left(A'|\vec{b}'\right) = 2$$

$$\Updownarrow$$

$$\begin{cases} x - \ \ y + 2z = \sqrt{2} \\ \qquad 5y + 3z = -1 \\ \qquad\qquad\ \ 0 = \ \ 0 \\ \qquad\qquad\ \ 0 = \ \ 0 \end{cases} \qquad \forall\ z \in \mathbb{R}$$

we have infinitely many solutions, because $z$ is a free variable, it can assume any value in $\mathbb{R}$

**Rouché–Capelli Theorem**

Let $A\vec{x}=\vec{b}$ be a linear system, with $A \in \mathbb{R}^{m\times n}$ $\binom{n \text{ equations}}{m \text{ unknowns}}$ and let $\left(A'|\vec{b}'\right)$ be in REF obrained from $\left(A|\vec{b}\right)$, then (this $A'\vec{x}'=\vec{b}'$ is equivalent to $A\vec{x}=\vec{b}$)

1. if rank $\left(A|\vec{b}\right) \neq \operatorname{rank}(A')$, then $A'\vec{x}=\vec{b}$ doesn't have solutions
2. if rank $\left(A|\vec{b}\right) = \operatorname{rank}(A') = n$ then $A'\vec{x}=\vec{b}$ has a unique solution
3. if rank $\left(A|\vec{b}\right) = \operatorname{rank}(A') < n$ then $A'\vec{x}=\vec{b}$ has infinitely many solutions and $n$-rank$(A')$ is the number of free variables

# 9 Vector subspaces

Given a vector space $V$ and $W \subseteq V$, we say that $W$ is a vector subspace of $V$ if it is a vector space.

Given a vector space $V$ and $W \subseteq V$ then $W$ is a vector subspace if and only if

a) said $e \in V$ the identity w.r.t. the «addition», we have $e \in W$

b) $\forall\, w_1, w_2 \in W$, we have $w_1 * w_2 \in W$

c) $\forall\, w \in W$, $\forall\, \lambda \in \mathbb{R}$, we have $\lambda \odot w \in W$

Given a homogeneous linear system, the set of solutions (if it is not the empty set) is a vector space (in particular is a vector subspace of $\mathbb{R}^n$ where $n$ is the number of unknowns)

Consider a linear system $A\vec{x}=\vec{0}$ where $A \in \mathbb{R}^{m\times n}$, let $S$ be the set of solutions. We can have that $S=\{(0,\dots,0)\in\mathbb{R}^n\} \subseteq \mathbb{R}^n$ and it is a vector space (the trivial one). In general we have that $S=\left\{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n = \mathbb{R}^{n\times 1} : A\vec{x}=\vec{0}\right\}$

a) We can check tat $\vec{0} \in S$

b) $\forall\, \vec{v}, \vec{w} \in S$, i.e., $A\vec{v}=\vec{0}$ and $A\vec{w}=\vec{0}$ we want to check that $\vec{v}+\vec{w}\in S$, i.e., we want to check that
$$A\left(\vec{v}+\vec{w}\right)=\vec{0}$$
indeed we can observe that $A\left(\vec{v}+\vec{w}\right)=A\vec{v}+A\vec{w}=\vec{0}+\vec{0}=\vec{0}$

c) $\forall\, \vec{v}\in S$, $\forall\, \lambda\in\mathbb{R}$ we check that $\lambda\vec{v}\in S$, i.e., $A\left(\lambda\vec{v}\right)=\vec{0}$

Given a non-homogeneous linear system, the set of its solutions is not a vector space.

Let us consider $A\vec{x} = \vec{b}$ where $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \neq \vec{0}$ then $\vec{0} \notin S = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n : A\vec{x} = \vec{b} \right\}$ because $A\vec{0} = \vec{0} \neq \vec{b}$

**Definition**

Let $V$ be a vector space and $v_1, v_2, \ldots, v_k \in V$ we define the set generated by $v_1, \ldots, v_k$, and we call it the spanning set of $v_1, \ldots, v_n$ the following set

$$\mathrm{span}(v_1, \ldots, v_k) = \langle v_1, \ldots, v_k \rangle = \{c_1 v_1 + \cdots + c_k v_k : \forall\ c_1, \ldots, c_k \in \mathbb{R}\}$$

**Example**

In $\mathbb{R}^3$, given two vectors $\vec{v}_1, \vec{v}_2$ linearly independent, their spamming set is a plane through the origin

$$\mathrm{span}(\vec{v}_1, \vec{v}_2) = \{s\vec{v}_1 + t\vec{v}_2 : \forall\ s, t \in \mathbb{R}\}$$

**Definition**

Let $V$ be a vector space, we say that $\{v_1, \ldots, v_k\}$ with $v1, \ldots, v_k \in V$ is set of generators of $V$ if $V = \mathrm{span}(v_1, \ldots, v_k)$

**Definition**

Let $V$ be a vector space, we say that $\{b_1, \ldots, b_n\}$ with $b_1, \ldots, b_n \in V$ is a basis of $V$ if

- $V = \mathrm{span}(b_1, \ldots, d_n)$
- $b_1, \ldots, b_n$ are linearly independent

**Definition**

Let $V$ be a vector space, we sat that $w_1, \ldots, w_h \in V$ are linearly independent if (given $c_1, \ldots, c_h \in \mathbb{R}$)

$$c_1 w_1 + \cdots + c_h w_h = 0 \quad (1)$$

$$\Updownarrow$$

$$c_1 = \cdots = c_h = 0$$

if (1) has a solution $c_1, \ldots, c_h$ not all zeros, then $w_1, \ldots, w_h$ are linearly dependent.

Given a vector $\vec{v} \in \mathbb{R}^4$, the component w.r.t the canonical bases are $(v_1, v_2, v_3, v_4)$ because

$$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3 + v_4\vec{e}_4$$

that is $v_1, v_2, v_3, v_4$ are the coefficent og the linear combination of $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ from which we obtain $\vec{v}$.

If we have another basis $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4$, then the compoent og $\vec{v}$ are $(w_1, w_2, w_3, w_4)$ such that

$$\vec{v} = w_1\vec{b}_1 + w_2\vec{b}_2 + w_3\vec{b}_3 + w_4\vec{b}_4$$

In the previous example the component of $(1, 0, 3, 5)$ w.r.t. the canonical basis are $(1, 0, 3, 5)$, but the component w.r.t. the basis $(1, 2, 3, 5), (2, 1, 1, 0), (1, 1, 1, 0), (0, 1, 0, 0)$ are

$$(1, 0, 3, 5) = 1 \cdot (1, 2, 3, 5) + 0 \cdot (2, 1, 1, 0) + 0 \cdot (1, 1, 1, 0) + (-2)(0, 1, 0, 0)$$

Two linear system are equivalent if the sets of solutions are the same

Let $S_1$ be the set of solutions of $\begin{cases} x_1 - x_2 + 7x_3 = 0 \\ x_1 = x_2 - 7x_3 \end{cases}$

$$S_1 = \{(x_2 - 7x_3, x_2, x_3, x_4) : x_2, x_3, x_4 \in \mathbb{R}\}$$

$(-6, 1, 1, 5)$ is a solution of the linear system

Let $S_2$ be the set of solutions of $\begin{cases} -2x_1 + 2x_2 - 14x_3 + 5x_4 = 0 \\ x_4 = 0 \end{cases}$

$$S_2 = \{(x_2 - 7x_3, x_2, x_3, 0) : x_2, x_3 \in \mathbb{R}\}$$

$$S_2 \subseteq S_1 \qquad \text{but} \qquad S_2 \neq S_1$$

$$S_1, S_2 \subseteq \mathbb{R}^4 \qquad \begin{array}{l} S_1 \text{ has 3 free variables} \\ S_2 \text{ has 2 free variables} \end{array}$$

$$c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{v}$$

Let $V$ be a vector setspace

**Definition**

We say that $\{v_1, \ldots, v_k\}$, whith $v_1, \ldots, v_k \in V$ is a set of generators of $V$ if

$$V = \text{span}(v_1, \ldots, v_k) = \{a_1v_1 + \cdots + a_kv_k : a_1, \ldots, a_k \in \mathbb{R}\}$$

That is $\forall\ vinV$, we have $v = \vec{c}_1v_1 + \cdots + c_kv_k$ for some $c_1, \ldots, c_k \in \mathbb{R}$

**Definition**

We say that $\{b_1, \ldots, b_h\}$, with $b_1, \ldots, b_h \in V$ is a basis of $V$ if

1. $V = \text{span}(b_1, \ldots, b_h)$ (i.e., $b_1, b_h$ is a set of generators)
2. $b_1, \ldots, b_h$ are linearly idependet

Example

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a basis of $\mathbb{R}^3$, $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, (1, 2, 3)\}$ is a set of generators of $\mathbb{R}^3$ but not a basis

Definition

The number of elements is any basis of $V$ is called the dimension of $V$

The dimension of $\mathbb{R}^n$ is $n$

Given $W \subseteq \mathbb{R}^n$ s.t. $W$ is a vector space, i.e., a vector subspace of $\mathbb{R}^n$, then the dimension $W$ can be $1, 2, \ldots, n-1$ (where $W$ is a proper subset of $\mathbb{R}^n$). If $W = \left\{\vec{0}\right\}$ then it is a trivial vector subspace of $\mathbb{R}^n$ with dimension 0.

In $\mathbb{R}^2$ we can have

- dimension $W = 0$, $W = \{(0, 0)\}$
- dimension $W = 1$, then a basis is $\{\vec{v}\}$ and $W = \text{span}(\vec{v}) = \{t\vec{v} : t \in \mathbb{R}\}$ that is $W$ is a line in $\mathbb{R}^2$ contaning the origin

In $\mathbb{R}^3$, we can have

- dimension $W = 0$, $W = \{(0, 0, 0)\}$
- dimension $W = 1$, then a basis is $\{\vec{v}\}$ and $W = \text{span}(\vec{v}, \vec{w})$ that is $W$ is a line in $\mathbb{R}^3$ containg the origin
- dimension $W = 2$, then a basis is $\{\vec{v}, \vec{w}\}$ and $W = \text{span}(\vec{v}, \vec{w})$ that is $W$ is a plane in $\mathbb{R}^3$ containg the origin.
- dimension $W = h$, for $1 \leq h \leq n-1$, $W$ is a $h$-dim. space through the origin dercibed by a basis $\left\{\vec{b}, \ldots, \vec{b}_h\right\}$, i.e. $W = \text{span}(\vec{b}_1, \ldots, \vec{b}_h)$

Given two vector spaces $V$ and $W$, we say that they are isomorphic if there exist a function (called isomorphism)

$$f : V \to W$$

s.t.

1. $f$ is surjective: every element of $W$ has pre-image, i.e., $\forall\ w \in W$ there exists $v \in V$ s.t. $f(v) = w$
2. $f$ is injective: $\forall\ v_1, v_2 \in V$ if $v_1 \neq v_2$ then $f(v_1) \neq f(v_2)$ equivalently we can say that $\forall\ v_1, v_2 \in V$

   $$f(v_1) = f(v_2) \Leftrightarrow v_1 = v_2$$

   (A function $f$ is both injective and surjective is called a bijection and it is a one-to-one correspondence)

3. Let $+_v$ and $+_w$ be the operation between elements of $V$ and $W$, respectively, $\forall\ v_1, v2 \in V\ f(v_1 +_v v_2) = f(v_1) +_w f(v_2)$

   $\forall\ k \in \mathbb{R}, \forall\ v \in V, f(kv) = kf(v)$

A function $f$ only satisfying 3. is called a linear map (a bijective liear map is an isomorphian).

A vector sapce $V$ of dimension $n$ is isomorphic to $\mathbb{R}$ indeed, let $\{b_1, \dots, b_n\}$ be a basis of $V$, this $\forall\ v \in V$

$$v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n \quad \text{for} \quad x_1, x_2, \dots, x_n \in \mathbb{R}$$

then

$$\begin{aligned} f : V &\to \mathbb{R}^n \\ v &\to (x_1, x_2, \dots, x_n) \end{aligned}$$

is an isomorphian.

## 9.1 Example of vector space with infinite dimesnion

Let $\mathbb{R}[x]$ be the set of polynomials with coefficents in $\mathbb{R}$

$$\mathbb{R}[x] = \left\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n : a_0, \dots, a_n \in \mathbb{R} \quad \forall n \in \mathbb{N}\right\}$$

Given $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ where $a_n \neq 0$ and we say that the degree $\deg P(x)$ is $n$

**Example**

$$P_1(x) = 1 + 2x + 3x^3 + x^4 \qquad P_2(x) = \frac{1}{2} - x + 2x^2$$

$$P_1(x) + P_2(x) = \frac{3}{2} + x + 2x^2 + 3x^3 + x^4$$

**Example**

$$P(x) = -3 + x + \sqrt{2}x^2 + x^5 - 7x^6, \qquad k = 2$$

$$kP(x) = -6 + 2x + 2\sqrt{2}x^2 + 2x^5 - 14x^6$$

The set $\mathbb{R}[x]$ with these operations is a vector space.

There is a one to one corresponcence (an injective and surjective map) between $\mathbb{R}[x]$ and $\mathbb{R}^\infty$ what is the set of vectors of infinite lenght

$$\begin{aligned} f : \mathbb{R}[x] &\to \mathbb{R}^\infty \\ P(x) &\mapsto (a_0, a_1, \dots, a_n, 0, 0, 0, \dots) \end{aligned}$$

This map is also a linear map (meaning that $f(P_1(x)+P_2(x)) = f(P_1(x))+f(P_2(x))$, $f(kP(x)) = kf(P(x))$)

This $f$ is an isomorphian and $\mathbb{R}[x]$ is «equal» (up to isomorphian) to $\mathbb{R}^\infty$.

A basis of $\mathbb{R}^\infty$ is

$$\{(1,0,0,\dots),(0,1,0,\dots),(0,0,1,0,0,\dots),\dots\}$$

and the dimension of $\mathbb{R}^\infty$ is $\infty$

$$P(x) = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n + 0 \cdot x^{n+1} + \dots$$

# 10 Determinant of a matrix

Consider $A\vec{x} = \vec{b}$ for solving it we transorm using elementary operations the augmented matrix $\left(A|\vec{b}\right)$ in RREF.

If $A \in \mathbb{R}^{n\times n}$, we can solve $A\vec{x} = \vec{b}$ also in another way

$$5x = 3$$
$$x = \frac{1}{5} \cdot 3$$

if $A \in \mathbb{R}^{n\times n}$ is an invertible matrix, that is there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I_n$ and we denote $B$ by $A^{-1}$ and we call it the inverse of $A$. This if $A$ is invertible and the inverse in $A^{-1}$ then

$$A\vec{x} = \vec{b} \Leftrightarrow A^{-1} \cdot A\vec{x} = A^{-1}\vec{b} \Leftrightarrow I_n \cdot \vec{x} = A^{-1}\vec{b} \Leftrightarrow \vec{x} = A^{-1}\vec{b}$$

For studying the invertibility of a square matrix we introduce the notion of determinat.

The determinat is a real number associated to a square matrix. Let us start with a matrix $A \in \mathbb{R}^{2\times 2}$ which is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad a,b,c,d \in \mathbb{R}$$
$$\det(A) = ad - bc$$

Consider now $A \in \mathbb{R}^{3\times 3}$ which is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

In general given $A \in \mathbb{R}^{n\times n}$, $A = (a_{ij})$

$$\det A = a_{11} \cdot \det(A_{11}) - a_{12} \cdot \det(A_{12}) + a_{13} \cdot \det(A_{13}) - a_{14} \cdot \det(A_{14}) + \cdots + (-1)^n a_{1n} \det(A_{1n})$$

where $A_{11} \in \mathbb{R}^{n-1\times n-1}$ obtained from $A$ removing first row and first column
where $A_{12} \in \mathbb{R}^{n-1\times n-1}$ obtained from $A$ removing first row and second column

where $A_{1j} \in \mathbb{R}^{n-1 \times n-1}$ obtained from $A$ removing first row and $j^{th}$ column for $j = 1, \dots, n$
This is the so-called Laplace expansion of the determinat.

**Remark**

If $A \in \mathbb{R}^{2\times 2}$, then $\det(A)$ is the area of the parallelogram with sides the columns of $A$. If $A \in \mathbb{R}^{3\times 3}$, then $\det(A)$ is the volume of the parallelepiped with sides the columns of $A$. Similarly this works if $A \in \mathbb{R}^{n\times n}$, $\det(A)$ is the volume of the parallelepiped with dimension $n$ and sides the columns of $A$.

**Remark**

A matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if $\det(A) \neq 0$

**Remark**

Consider $A\vec{x} = \vec{b}$ $A \in \mathbb{R}^{n\times n}$, the solution is unique if and only if $\det(A) \neq 0$ and the solution is given by $x_i = \frac{\det(A_i)}{\det(A)}$ where $A_i \in \mathbb{R}^{n\times n}$ obtained from $A$ replacing the $i^{th}$ column of $A$ with $\vec{b}$
This is the Cramer rule.

**Example**

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & 7 \end{bmatrix} - 0 \det \begin{bmatrix} 0 & 4 \\ 0 & 7 \end{bmatrix} + 2 \det \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \\ &= 1 \times 14 - 0 + 2 \times 0 = 14 \\ \det(B) &= -1 \cdot \det \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} + 2 \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = 4 - 2 = 2 \end{aligned}$$

**Remark**

For computing the determinat we can apply the definition exploiting any row (not necessarly the firt one).
This, if a matrix has a row with many zeros is more convenient to use.

Example

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\det(B) = 1 \cdot \det \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} + 0 + 0 = 2$$

$$C = \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ -7 & 5 & 1 & 2 \\ 3 & 1 & -5 & 1 \end{bmatrix} \det(C) = 0$$

## 10.1 Properties

1. If $A \in \mathbb{R}^{n \times n}$ is in REF and the number of pivots is $n$, then $\det(A)$ is equal to the product of pivots

2. If $A \in \mathbb{R}^{n \times n}$ is in REF and the number of pivots is $< n$, then $\det(A) = 0$

3. Given $A \in \mathbb{R}^{n \times n}$ and let $B$ be a matrix obtained from $A$ switching two rows, then $\det(B) = -\det(A)$ $(\det(A) = -\det(B))$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

   check that $\det(A) = -\det(B)$

4. Given $A \in \mathbb{R}^{n \times n}$ and let $B$ be a matrix obtained from $A$ multiplying a row of $A$ by a scalar $k$, then $\det(B) = k\det(A)$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

   check that $\det(B) = \frac{1}{7} \cdot \det(A)$

5. Given $A \in \mathbb{R}^{n \times n}$ and let $B$ be a matrix obtained from $A$ adding to a row of $A$ a multiple of another row, then $\det(B) = \det(A)$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix}$$

   check that $\det(B) = \det(A)$

**Remark**

Pay attention! If you have a matrix $A$ ans transorm it replacing, e.g., the first row $R_1$ with the row $2R_1 + 3R_2$ then we obrain a new matrix $B$ and $\det B = 2 \det A$

## 10.2 Proprieties

Given $A, B \in \mathbb{R}^{n\times n}$, $\det(AB) = \det(A)\det(B)$ let us see a method for computing the inverse of a matrix $A \in \mathbb{R}^{n\times n}$ such that $\det A \neq 0$.

We want to compute an unknown matrix $A^{-1} \in \mathbb{R}^{n\times n}$ such that

$$A \cdot A^{-1} = I_n$$

that is

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

$a_{ij}$ are known real numbers

$x_{ij}$ are unknown real numbers

$$\begin{cases} a_{11}x_{11} + \cdots + a_{1n}x_{n1} = 1 \\ a_{21}x_{11} + \cdots + a_{2n}x_{n1} = 0 \\ \quad \cdots \\ a_{n1}x_{11} + \cdots + a_{nn}x_{n1} = 0 \end{cases} \qquad \text{Linear system with } n \text{ unknown and } n \text{ equations}$$

$$\Updownarrow$$

Linear system whose augmented matrix is $(A|\vec{e}_1)$ where $\vec{e}_1 = (1, 0, \ldots, 0)$

Similarly, if we focus on the seond column of $A^{-1}$, we get a linear system whose unknone are $(x_{12}, x_{22}, x_{32}, \ldots, x_{n2})$ and the augmented matrix is

$$(A|\vec{e}_2) \qquad \text{where} \qquad \vec{e}_2 = (0, 1, 0, \ldots, 0)$$

This, in order to find $A^{-1}$ we need to solve $n$ linear system whose augmented matrices are

$$(A|\vec{e}_1), (A|\vec{e}_2), \ldots, (A|\vec{e}_n)$$

We can solve all these linear systems considering

$$(A|I_n) \quad \leftarrow \quad \det A \neq 0$$

and transforming it into RREF we obtain

$$\left(I_n|A^{-1}\right)$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \qquad \det A = -6 \neq 0 \Rightarrow A \text{ inverible}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 6 & 3 & -1 \end{array}\right] \begin{array}{l} R_1 \\ 3R_1 - R_2 \end{array}$$

$$\longrightarrow \left[\begin{array}{cc|cc} 3 & 0 & 0 & 1 \\ 0 & 6 & 3 & -1 \end{array}\right] \begin{array}{l} 3R_1 - R_2 \\ R_2 \end{array} \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{6} \end{array}\right] \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{6}R_2 \end{array} \qquad \text{RREF}$$

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$$

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Check that $A^{-1} \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

# 11 Linear function

Definition

Given two vector spaces $V$ and $W$, with finite dimension. $V \simeq \mathbb{R}^n$ and $W \simeq \mathbb{R}^m$ a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a map such that

1. $\forall\ \vec{v}_1, \vec{v}_2 \in \mathbb{R}^n, f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$
2. $\forall\ \vec{v} \in \mathbb{R}^n, \forall\ k \in \mathbb{R}, f(k\vec{v}) = kf(\vec{v})$

$$f\left(\vec{0}\right) = \vec{0}$$

$$\vec{0} \in \mathbb{R}^n \qquad \vec{0} \in \mathbb{R}^m$$

By 1. we know that $\forall\ \vec{v} \in \mathbb{R}^n$, then $f\left(\vec{0} + \vec{v}\right) = f\left(\vec{0}\right) + f\left(\vec{v}\right) = f\left(\vec{v}\right) \Rightarrow f(\vec{v}) = f\left(\vec{0}\right) + f\left(\vec{v}\right) \Rightarrow f\left(\vec{0}\right) = \vec{0}$

If we know the image of the vectors of a basis of $\mathbb{R}^n$, then we are able to compute $f$ an all the vectors of $\mathbb{R}^n$.

If $B = \left\{\vec{b}_1, \ldots, \vec{b}_n\right\} \subseteq \mathbb{R}^n$ is a basis of $\mathbb{R}^n$ then, $\forall\ \vec{v} \in \mathbb{R}^n$, we have that $\vec{v} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n$ fore some $x_1, \ldots, x_n \in \mathbb{R}$ and

$$\begin{aligned} f(\vec{v}) &= f\left(x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n\right) \\ &= x_1 f\left(\vec{b}_1\right) + x_2 f\left(\vec{b}_2\right) + \cdots + x_n f\left(\vec{b}_n\right) \end{aligned}$$

Let $C = \{\vec{c}_1, \ldots, \vec{c}_m\} \subseteq \mathbb{R}^m$ be a basis of $\mathbb{R}^m$, if we know that

$$f\left(\vec{b}_1\right) = \vec{w}_1 = a_{11}\vec{c}_1 + a_{21}\vec{c}_2 + \cdots + a_{m1}\vec{c}_m$$

$$f\left(\vec{b}_2\right) = \vec{w}_2 = a_{12}\vec{c}_1 + a_{22}\vec{c}_2 + \cdots + a_{m2}\vec{c}_m$$

$$\cdots$$

$$f\left(\vec{b}_n\right) = \vec{w}_n = a_{1n}\vec{c}_1 + a_{2n}\vec{c}_2 + \cdots + a_{mn}\vec{c}_m$$

We define

$$A = \left[f\left(\vec{b}_1\right)\ f\left(\vec{b}_2\right)\ \cdots\ f\left(\vec{b}_n\right)\right]$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

This is the matrix whose columns are the component of $f(\vec{b}_1)$ w.r.t. the basis $C$ of $\mathbb{R}^m$. This matrix $A$ is the matrix associated to $f$ and «it works like $f$»

Take $\vec{v} \in \mathbb{R}^n$, $\vec{v} = v_1\vec{b}_1 + \cdots + v_n\vec{b}_n$

$$\begin{aligned} f(\vec{v}) &= v_1 f\left(\vec{b}\right)_1 + \cdots + v_n f\left(\vec{b}_n\right) = v_1\vec{w}_1 + \cdots + v_n\vec{w}_n = \\ &= v_1\left(a_{11}\vec{c}_1 + \cdots + a_{m1}\vec{c}_m\right) + \cdots + v_n\left(a_{1n}\vec{c}_1 + \cdots + a_{mn}\vec{c}_m\right) \\ &= \left(v_1a_{11} + \cdots + v_na_{1n}\right)\vec{c}_1 + \cdots + \left(v_1a_{m1} + \cdots + v_na_{mn}\right)\vec{c}_m \\ &= W \in \mathbb{R}^m \end{aligned}$$

where $\vec{w} = A\vec{v} \Leftrightarrow \vec{w} = f\left(\vec{v}\right)$

$$\underset{\mathbb{R}^{m\times n}}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}} \underset{\mathbb{R}^{n\times 1}}{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}} = \underset{\mathbb{R}^{m\times 1}}{\begin{bmatrix} a_{11}v_1 + \cdots + a_{1n}v_n \\ \vdots \\ a_{m1}v_1 + \cdots + a_{mn}v_n \end{bmatrix}}$$