THIS PAPER IS NOT TO BE REMOVED FROM THE EXAMINATION HALLS

UNIVERSITY OF LONDON

CO3352 ZA

BSc Examination

COMPUTING AND INFORMATION SYSTEMS AND CREATIVE COMPUTING

Operations Research and Combinatorial Optimisation

Wednesday 21 May 2014 : 10.00 – 12.15

Duration: 2 hours 15 minutes

There are FIVE questions in this paper. Candidates should answer **FOUR** questions. All questions carry equal marks, and full marks can be obtained for complete answers to a total of **FOUR** questions. The marks for each part of a question are indicated at the end of the part in [.] brackets.

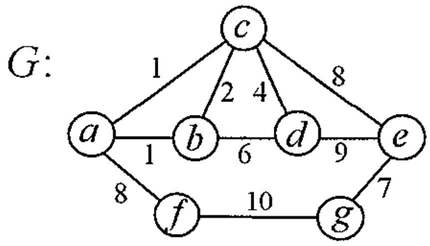
Only your first FOUR answers, in the order that they appear in your answer book, will be marked.

There are 100 marks available on this paper.

A hand held calculator may be used when answering questions on this paper but it must not be pre-programmed or able to display graphics text or algebraic equations. The make and type of machine must be stated clearly on the front cover of the answer book.

Question 1

A graph G with weighted edges is given diagrammatically as shown below:



(a) For the graph G, specify:

(i) a walk from a to c of total edge weight 3; [1]

(ii) all closed walks at b having total edge weight 4. [4]

(iii) a walk from d to e of length 4 edges and having minimum total weight; [4]

(b) For the graph G show, by means of a series of diagrams or otherwise, the steps that would be taken by the greedy algorithm to construct a minimum-weight spanning tree in G. Give the final output of the algorithm and calculate the weight of the tree it finds. [8]

(c) (i) Say what is meant by a **matching** in an undirected graph. [2]

(ii) Show that the Greedy Algorithm will not succeed in finding a maximum-weight matching in the graph G. [6]

Question 2

A matroid M on ground set $A = \{a, b, c, d\}$ is represented over the real numbers by the following matrix X:

$$X = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \end{array}.$$

(a) (i) List the independent sets of M. [3]

(ii) What does it mean to say that M has rank 3, and how is this rank determined from matrix X? [2]

(iii) By applying Gaussian elimination and column reordering to X, produce a matrix in the form $[I_3 \,|\, Y]$, where I_3 is the 3×3 identity matrix and Y is a 3×1 column vector. [5]

(iv) By using your answer to part (iii), or otherwise, write down a matrix representing the dual matroid M^* of M, and list the independent sets of M^*. [4]

(b) Suppose that the entry in row 2, column 2 of X is changed from 1 to -1, giving a new matrix X'.

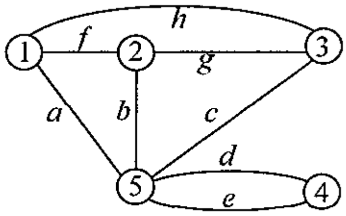
(i) Draw an undirected graph G whose cycle matroid $M(G)$ is represented by the matrix X'. [4]

(ii) Explain why the matrix X' from part (i) can be replaced by a 2×4 matrix having the same rank as X' and again representing the matroid $M(G)$. [3]

(iii) By constructing the matrix representing the dual matroid of $M(G)$, or otherwise, explain why $M(G)$ can be said to be **self-dual**. [4]

Question 3

An undirected graph G with vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set $E = \{a, b, c, d, e, f, g, h\}$ is specified by the following drawing:

G:


The application of cycle and cocycle matroids to finding maximum-length paths in G will be investigated in this question.

(a) Write a subset of E which is a **spanning tree** of G but which fails to be a **non-cut** (i.e., deleting the edges of this subset will cut the graph into two or more connected components.) [3]

(b) Write down a subset of E of size 4 which is a non-cut but which fails to be a spanning tree. [3]

(c) Write down a subset of E of which is simultaneously a spanning tree and a non-cut. [3]

(d) Explain briefly why a spanning tree of G which is also a non-cut and in which the degree of vertex 5 is at most 2 must be a path. Give an example of such a path. [5]

(e) Matrices B and B^* representing the cycle matroid and the cocycle matroids of G, respectively, are given below:

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad B^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Let D be the matrix diag(a, b, c, d, e, f, g, h) whose only nonzero elements are the diagonal elements which are assigned the names of the edges of G.

(i) Construct the Binet-Cauchy product $\Phi = B \times D \times (B^*)^T$. [5]

(ii) Explain how $\det \Phi$ can be used to identify **six** paths of length 4 in G which contain edge d, and write down these paths. [6]

Question 4

The **bin packing** problem in combinatorial optimisation is the following:

> Given a set of items, each having a given size, and a 'bin capacity' C, to distribute the items into n subsets ('bins') each of total size at most C, with n as small as possible.

For example, suppose $C = 10$ and that W, X, Y and Z have the sizes and values shown on the right. Then the subsets $\{W\}$, $\{X\}$, $\{Y, Z\}$ each have total size less than 10; and no distribution into two subsets can achieve this, so $n = 3$ is an optimal solution to this instance of the bin packing problem.

item	W	X	Y	Z
size	6	6	3	5

(a) Explain why $n = 3$ is also an optimal solution to the bin packing problem instance specified below, given the limit $C = 9$; and find a distribution of the items $V, \ldots, Z$ into three bins of capacity $C = 9$. [5]

item	V	W	X	Y	Z
size	5	3	5	2	3

(b) Suppose we try to solve the bin packing problem instance in part (a), using two bins, using the following integer linear programme:

$$
\begin{array}{llllll}
5b_{11} & +3b_{12} & +5b_{13} & +2b_{14} & +3b_{15} & \leq \ 9 \\
5b_{21} & +3b_{22} & +5b_{23} & +2b_{24} & +3b_{25} & \leq \ 9
\end{array}
$$
$$b_{11} + b_{12} = 1, b_{12} + b_{22} = 1, b_{13} + b_{23} = 1, b_{14} + b_{24} = 1, b_{15} + b_{25} = 1$$

where the b_{ij} are constrained to take values in the set $\{0, 1\}$.

(i) Explain briefly what it would mean to take $b_{11} = 1$ and $b_{21} = 0$. [3]

(ii) If bin capacity is increased to $C = 10$ how would this change be represented in the above programme, and what values of the b_{ij} might be found to solve this amended programme? [3]

(iii) Explain briefly why no objective function needs to be maximised or minimised in this programme. [3]

(iv) The bin packing problem is NP-Complete. Explain briefly why this suggests that integer linear programming will not offer a polynomial-time algorithm for solving the problem. [3]

(v) A linear relaxation of the above programme may be solved rapidly using the simplex method. Find such a solution and explain why it fails to solve the instance of the bin packing problem which the programme models. [8]

Question 5

(a) A convex polyhedron in $\mathbb{R}^2$ has three vertices:

$$v_1 = (1,2), \;\; v_2 = (2,4), \;\; v_3 = (5,1).$$

(i) Sketch on the xy-axes, in the range $0 \leq y \leq 6$, the three straight lines which define the edges of this polyhedron, showing their y-intercepts. [4]

(ii) Write down the equations of the straight lines which you sketched in part (i) and hence give three inequalities which define the polyhedron. [6]

(iii) Suppose that the equalities from part (ii) are combined with an objective function $f(x,y) = x - 7y$. By using your sketch from part (i), or otherwise, solve the linear programme which minimises $f(x,y)$ subject to the inequalities. [5]

(b) A linear program with three basic variables x_1, x_2 and x_3 is given as:

$$\begin{array}{rrcl}
\textbf{maximise} & 2x_1 + 3x_2 + 3x_3 & & \\
\textbf{subject to} & x_1 + x_2 + x_3 & \leq & 10 \\
 & x_1 + 2x_2 + x_3 & \leq & 11 \\
 & 2x_1 + x_2 + x_3 & \leq & 8 \\
 & x_1, x_2, x_3 & \geq & 0
\end{array}$$

(i) State the Duality Theorem of linear programming. [2]

(ii) An optimal solution to this linear programme is given by

$$x_1 = \frac{5}{3}, \;\; x_2 = \frac{11}{3}, \;\; x_3 = 1.$$

Show that these values satisfy the constraints of the linear programme. [2]

(iii) Give the dual of the given linear programme and give the value of an optimal solution of this dual. [6]

END OF PAPER