CO1102 Mathematics for computing
Examiners' commentary 2015–16
Zone A

General remarks

The overall performance on this paper was good, with the majority of students coping well with most of the subject. There were a number of weak candidates but also some candidates who gained high marks.

Several candidates have problems with basic algebra and arithmetic, and many more struggle with precise mathematical notation and exact definitions. Thus many did poorly or left blank large parts of Questions 4 and 8, which require an understanding of the notation and terminology concerning relations and substantial calculations respectively. It also happens frequently that a candidate knows the answer to a question, but is unable to phrase it using the proper mathematical terms. Candidates are advised to pay attention to the notation used in the subject guide and to strive to adopt this notation in their own work. When you have solved an exercise, always compare your notation as well as your answers to the model answers provided in the subject guide or by your lecturer. If your notation is substantially different, try to improve it.

When revising for the examination, you should attempt the sample examination questions in the subject guide. After you have solved them, compare your solutions to the ones given in the guide. It is hard to give a general rule for how you can mark your own answers, as exercises may have more than one correct solution method. However, if you find that you have the right answer, but that your method differs substantially from or is less theoretical than the one offered in the subject guide, you might want to reconsider your approach to the solution.

A major mistake made by a large number of candidates is not to show their working and calculations and not to explain the reasoning behind an answer. Especially in questions involving arithmetic, failure to show calculations generally results in the loss of marks. Also questions requiring a short answer are best answered by initially giving a short answer to the question followed by your *reason* for this particular answer. Remember that examiners are more interested in how you arrived at an answer than in the answer itself!

What follows are some answers, hints, solutions and comments on the examination questions that may help you, as you revise for your examination.

Comments on specific questions

Question 1

(a) The answer is $(1000101)_2$.
Some candidates failed to show the binary working here. It is important to show all carries.

(b) $s = 2^0 + 2^2 + 2^4 + 2^6 + 2^8 + 2^{10} + 2^{12}$, so in binary and hex $s = (1010101010101)_2$ and $(1555)_{16}$ respectively, and $2s = (10101010101010)_2$ and $(2AAA)_{16}$ respectively.

(c) Let $y = 0.75757575\ldots$. Then $100y = 75.75757575\ldots$.
Subtraction yields $99y = 100y - y = 75$, and hence $y = 75/99 = 25/33$.

Question 2

(a) $|B| = 2^7 = 128$.

(b) (i) $\varphi(1010000) = 2$.

(ii) The range of φ is the set $\{0, 1, 2, 3, \ldots, 7\}$. Many candidates used sloppy notation here and thus did not get full marks. Note that the range of a function is a set, and the notation used in a good answer must reflect this.

(iii) $\binom{7}{2} = 21$ strings have bitsum 2 as these have seven 0s and two 1s. For φ to be one-to-one, 2 should have a unique preimage in B, but it has 21.

(c) (i) $\binom{7}{4} = 35$ strings have precisely four 0s, so $P(\text{four 0s}) = 35/128$.

(ii) 7 is odd, so no string has an equal number of 0s and 1s. Thus $P(\text{equal 0s and 1s}) = 0$. Note that a probability is a number between 0 and 1. Many candidates gave a vague answer like ‘no chance’ here and thus did not get full credit for this part.

(iii) $P(\text{more 0s than 1s}) = 1/2$. There is more than one possible argument why this is so, but for full marks the answer must be justified. One way to argue is to set up a one-to-one correspondence between the strings that have more 0s than 1s and the ones that have more 1s than 0s by simply swapping 0s and 1s. This shows exactly half of the strings are of one type and half the strings are of the other type because no strings have an equal number of 0s and 1s, and thus $P(\text{more 0s than 1s}) = P(\text{more 1s than 0s}) = 1/2$. Alternatively you could just compute the number of strings that have more 0s than 1s as
$$\binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7} = \binom{7}{3} + \binom{7}{2} + \binom{7}{1} + \binom{7}{0} = 64,$$
and thus $P(\text{more 0s than 1s}) = P(\text{more 1s than 0s}) = 64/128 = 1/2$.

Question 3

(a) A positive integer p is a prime, if $p > 1$ and p has no other factors than $\pm p$ and ± 1. Note that the number 1 is not a prime, but it is divisible by ± 1 and ± 1, which is why it is very important to exclude 1 specifically in this definition.

(b) $5880 = 2^3 \cdot 3 \cdot 5 \cdot 7^2$.

(c) (i) This is true, for if $x = n^2 + 2n = n(n+2)$ then n and $n+2$ are factors of x.

(ii) This is false as $41^2 + 41 + 41 = 43 \cdot 41$ by part (i).

(d) If $p+1$ or $p-1$ is a prime then p is even or $p \leq 3$. This part was poorly answered by many candidates. For full marks the implication must be reversed, both statements must be negated, and De Morgan's Law must be used correctly when doing so.

Question 4

(a) Most candidates drew the relationship digraph here and tried to justify the answers from it with varying degree of success. Some forgot that all vertices should have loops and some missed out arcs also. Further, the relation here is actually an equivalence relation, but the components corresponding to each equivalence class were not always distinguishable in the graphs provided, and consequently wrong conclusions were made. Finally, it is particularly hard to prove transitivity by merely looking at a graph, as many cases are too easily missed, so an algebraic proof is preferable in this case.

(i) R is symmetric as for all integers a and b we have $a-b=-(b-a)$, so if 6 is a factor of $(a-b)$ then 6 is a factor of $(b-a)$ also.

(ii) R is reflexive as for all integers a we have $a-a=0$, and 6 is a factor of 0. Many candidates confused the terms 'multiple' and 'factor' here and stated wrongly that R is not reflexive as 0 is not a factor of 6.

(iii) R is transitive as for all integers a, b and c we have that if 6 is a factor of $(a-b)$ and 6 is a factor of $(b-c)$ then 6 is a factor of $(a-c)$. This is so because $a-c=(a-b)+(b-c)$.

(b) R' is not reflexive as *e.g.* 6 is not a factor of $(2+2)$.
R' is not transitive as *e.g.* 6 is a factor of $(2+4)$ and 6 is a factor of $(4+2)$, but 6 is not a factor of $(2+2)$.

Question 5

(a) (i) Some candidates confused the two symbols $\cap$ and $\cup$. Try to remember that $\cup$ stands for $\cup$nion. Other marks were lost here due to unclear shading. Remember always to explain which shading represents which set.

(ii) The membership table asked for is:

A	B	C	$A \cup C$	$B \cup C$	X	$X - C$	$Y = A \cap B$
0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	0	0
1	0	0	1	0	0	0	0
1	0	1	1	1	1	0	0
1	1	0	1	1	1	1	1
1	1	1	1	1	1	0	1

Remember to justify why your table shows that $X - C \subseteq Y$, *e.g.* point out that the column for Y has 1s whenever the column for $X - C$ has 1s.

(b) $A = \{-1, 8\}$ as the third root of the polynomial is not an integer.

(c) *E.g.* $B = \{5n + 2 : n \in \mathbb{Z}, -3 \leq n \leq 12\}$.

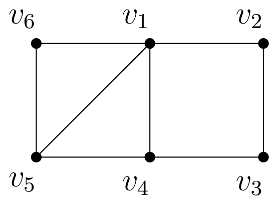
Question 6

(a) (i) Non-isomorphic graphs with the same degree sequence exist, *cf.* part (b)(i) and (b)(iii).

(ii) G has 6 vertices, one for each term in the degree sequence.

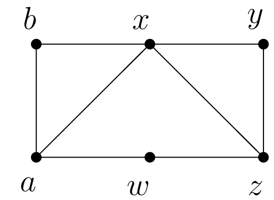
(iii) G has $(4 + 3 + 3 + 2 + 2 + 2)/2 = 8$ edges by the Handshaking Lemma.

(b) (i) The graph G:



(ii) $(v_1, v_2, v_3, v_4, v_5, v_6, v_1)$ is a 6-cycle.

(iii) Let H be the following graph.



This graph has a 6-cycle, 6 vertices and 8 edges and the same degree sequence as G, but *e.g.* the two vertices of degree 3 are adjacent in G while here in H they are not, so H is not isomorphic to G.

Question 7

(a) Each edge counts 1 towards the degree of each of its end vertices, hence it counts two towards the sum of all the vertex degrees, and the sum of all vertex degrees is thus twice the number of edges in the graph.

(b) It must be connected and have no cycles.

(c) A tree on n vertices has $n - 1$ edges.

(d) By part (a) the sum of the degrees of the vertices must be even, but a sum of 17 odd numbers is odd, so this tree cannot exist.

(e) (i) The root has record number $\lfloor \frac{19+1}{2} \rfloor = 10$.

(ii) In the left subtree we have records 1–9, so its root is $\lfloor \frac{9+1}{2} \rfloor = 5$.
In the right subtree we have records 11–19, so its root is $\lfloor \frac{19+11}{2} \rfloor = 15$.

Question 8

(a) Many candidates scored very poorly in this part of the exercise. Basic algebra and arithmetic needs to be practised regularly, otherwise you will forget the rules.

(i) $4 \cdot 2^n + 2^{n+2} = 2^{n+2} + 2^{n+2} = 2^{n+3}$;

(ii) $\log_2(\sqrt{2^x}) = \frac{1}{2}\log_2(2^x) = \frac{x}{2}$.

(b) (i) $f(2) = 3\cdot(-1)^2 - 4\cdot(-1) + 1 = 8$ and $f(f(-1)) = f(8) = 3\cdot 8^2 - 4\cdot 8 + 1 = 161$.

(ii) $3x^2 - 4x + 1 = 0$ if and only if $(3x-1)(x-1) = 0$, that is when either $(3x-1) = 0$ or $(x-1) = 0$, and so the preimages of 0 are $\frac{1}{3}$ and 1.

(iii) A short calculation shows $f(f(x)) = 27x^4 - 72x^3 + 54x^2 - 8x$, so for all $x \geq 1$ we have

$$|f(f(x))| \leq (27 + 72 + 54 + 8)x^4 = 161|x^4|,$$

showing that $f(f(x))$ is $O(x^4)$.

Question 9

(a) A number of candidates failed to use this recurrence relation correctly. Note first that $a_0 = 2$ here, not 0. Further, the right hand side is $3a_n + 1$, not $3n + 1$.

$$\begin{aligned}
a_0 &= 2; \\
a_1 &= 3a_0 + 1 = 3\cdot 2 + 1 = 7; \\
a_2 &= 3a_1 + 1 = 3\cdot 7 + 1 = 22; \\
a_3 &= 3a_2 + 1 = 3\cdot 22 + 1 = 67; \\
a_4 &= 3a_3 + 1 = 3\cdot 67 + 1 = 202.
\end{aligned}$$

(b) Some candidates gave very good proofs, but many others did not attempt this part of the question or did poorly. Many attempts at the proof offer a good base and inductive hypothesis, but then go wrong in the induction step because they mix up what they know and what they still need to prove. One good piece of advice that will save many errors is that you should never write down an =-symbol (or a >-symbol) without specifically checking that you are absolutely sure you know that what is on the left hand side of it is equal to (or greater than) what is on the right hand side of it.

When proving an inequality (or an identity) by induction the proof has 4 steps: the base, the inductive hypothesis, the induction step and a final remark stating why the identity holds by induction.

When proving the base case and the induction step, you must demonstrate that the left hand side (LHS) of the inequality is bigger than the right hand side (RHS) of the inequality for the case in question. It is usually best to keep the two computations completely separate in order not to confuse what you know and what you have not proven yet.

Here we want to prove the inequality $a_n > 3^n$ for all $n \geq 0$:

Base case: When $n = 0$, $LHS = a_0 = 2$ and $RHS = 3^0 = 1$, and so the inequality thus holds for $n = 0$.

Inductive hypothesis: Suppose the inequality holds for some $k \geq 0$, *i.e.*

$$a_k > 3^k.$$

Induction step: We must prove that the inequality also holds for $n = k + 1$, *i.e.* that $a_{k+1} > 3^{k+1}$. But

$$\begin{aligned} LHS &= a_{k+1} \\ &= 3a_k + 1 \text{ by the recurrence relation} \\ &> 3 \cdot 3^k + 1 \text{ by the inductive hypothesis} \\ &= 3^{k+1} + 1 \\ &> 3^{k+1} \\ &= RHS. \end{aligned}$$

So $a_n > 3^n$ for $n = k + 1$ and thus for all $n \geq 0$ by induction.

Question 10

(a) Multiplying two matrices is a skill that a substantial number of candidates do not seem to master. Many left this question blank. There were also a lot of basic calculation errors made by those who attempted it; some were multiplying the matrices wrongly. Part (ii) shows that you should get in (i) that N and M are inverses of each other, so the product should be the identity matrix, which indeed it is:

(i)

$$NM = \begin{pmatrix} 4 & -9 & -2 \\ -1 & 2 & 1 \\ -2 & 5 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 5 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 12-9-2 & 4+0-4 & 20-18-2 \\ -3+2+1 & -1+0+2 & -5+4+1 \\ -6+5+1 & -2+0+2 & -10+10+1 \end{pmatrix},$$

giving us that

$$NM = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) From part (i) we have that N is the inverse of M, as we found $NM = I$. Now, $M\mathbf{x} = \mathbf{y}$ first gives us that $NM\mathbf{x} = N\mathbf{y}$ and hence by using that N is the inverse of M we get $I\mathbf{x} = N\mathbf{y}$, which reduces to $\mathbf{x} = N\mathbf{y}$.

(b) Many candidates do not know the difference between the augmented matrix for a system of equations and writing the system as a matrix equation, and they thus lost a mark here. The matrix equation for the system is

$$\begin{pmatrix} 3 & 1 & 5 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

(c) The system can be solved by Gaussian Elimination, and full marks would be given for this solution method. However, in this case we already know the inverse of the matrix of coefficients from (a)(i), so by using (a)(ii) we get an easier solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & -9 & -2 \\ -1 & 2 & 1 \\ -2 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -11 \\ 4 \\ 6 \end{pmatrix}.$$