University of London International Programmes

CO3352 Operations research and combinatorial optimisation
Coursework Assignments – 2013-14

Important notes: please read before you start work on these assignments

1. Coursework assignments are to be done in conjunction with the subject guide for CO3352, recommended books, and any other source(s) that you can obtain.

2. Coursework assignments 1 and 2 carry equal weight.

3. You are expected to submit your work electronically. You are encouraged to identify appropriate software to help you typeset mathematical text and prepare diagrams; however, if this is not possible, handwritten submissions in **scanned pdf format** will be acceptable.

4. You must acknowledge any use of the published or unpublished works of other people (including material obtained from websites). Note that few if any marks are given for reproducing material from other sources. If your teacher gives you guidelines, please acknowledge the extent of this in your submission.

5. Plagiarism detection software may be used on the submitted coursework. Last year, there were again instances where plagiarism was detected and the offenders were penalised. Please be warned that the markers are vigilant in attempting to detect plagiarism. Plagiarism includes, but is not limited to, the submission of work, parts of which are essentially identical to other submissions. Plagiarism also includes copying from the Internet without acknowledgement, copying from theses, copying from books or from the subject guide, etc. Remember, if you can find it on the Internet so can the markers.

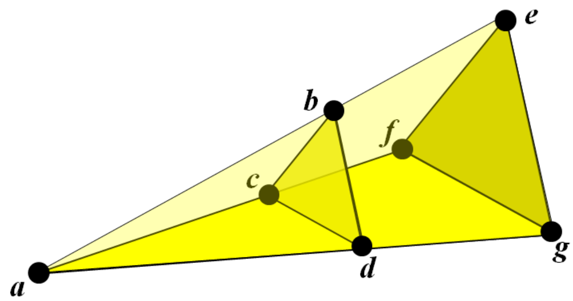
CO3352 Coursework Assignment 1: Matroids and their Representations

Please note:

- Chapters 1 and 2 of the subject guide for this course will help you in completing this assignment. Square-bracketed references, e.g. **[1.3.2]**, are to relevant sections of the subject guide.
- You are expected to spend between 15 and 30 hours completing this assignment.

The aim of this assignment is to appreciate a classic theorem of Hazel Perfect. She was one of the pioneers of matroid theory in the 1960s and 70s and was jointly responsible for discovering the class of transversal matroids.

But first we equip ourselves with an example matroid to study. To give a broader understanding of matroids, we use a geometrical (pictorial) representation which is very useful for small matroids. Consider the diagram below. It is a tetrahedron having four triangular faces: aef, aeg, afg, and efg, to which has been added an internal triangular face bcd parallel to efg.



The diagram represents a matroid M on the ground set $A = \{a,b,c,d,e,f,g\}$ as follows: any subset of A is **independent** if and only if it contains no **three** points in a line; and no **four** points in a plane. For example, abe (short for $\{a,b,e\}$) is not independent since a,b and e lie in a straight line; $cdfg$ is not independent since c,d,f and g lie in the same plane (the 'floor' of the tetrahedron); but cdf and $abcd$ are both independent.

Question 1 **[Total marks: 30]** This question explores the matroid M in terms of its independent sets. (See **[1.2.2]**)

(a) List the **eight** subsets of A which contain a and are independent in M. **[4]**

(b) List the **twelve** subsets of A which contain a and are not independent in M. **[6]**

(c) Given that cdf and $abcd$ are independent in M, what does the **Exchange Axiom** tell us about this pair of subsets, and what further independent set of M can be written down as a result? **[4]**

(d) How may it be determined from the tetrahedron diagram above that M has rank 4? **[4]**

(e) Suppose the elements of A are given the following weightings:

element	a	b	c	d	e	f	g
weight	3	4	1	3	4	2	3

(i) What are the **two** possible independent sets of M which the Greedy Algorithm might choose as having maximum weight? In what order? **[1.3.4]** **[4]**

(ii) In what order are the elements of these two sets selected by the Greedy Algorithm? **[4]**

(iii) Which two elements of A cannot be chosen by the Greedy Algorithm, and why? **[4]**

Question 2 **[Total marks: 30]** Perfect's theorem deals with an interaction between **two** matroids defined on the same set with one being a **transversal matroid** (see the last two pages of **[1.3.4]**).

Thus, suppose that six subsets of $A = \{a, b, c, d, e, f, g\}$ are defined as represented by the incidence table shown on the right, where a '1' in row i, column j, indicates that A_i contains element j of A (so $A_1 = \{a, g\}$, etc).

	a	b	c	d	e	f	g
A_1	1						1
A_2		1	1		1		
A_3				1			
A_4						1	
A_5						1	1
A_6						1	

(a) Explain what it means to say that abd is a **transversal** for the three sets A_1, A_2, A_3 (note that, as in question 1, we use abd as shorthand for $\{a, b, d\}$). **[4]**

(b) Write down the **six** (complete) transversals for the four sets A_1, A_2, A_3, A_4. **[6]**

(c) What is the **rank** of the transversal matroid for the family of subsets $A_1, \ldots, A_6$? **[4]**

(d) Write down the **four** maximal (partial) transversals for the family of subsets $A_1, \ldots, A_6$? **[8]**

(e) An alternative way of viewing the subset system $A_1, \ldots, A_6$ is as a bipartite graph, in which the vertices the sets $\{A_1, \ldots, A_6\}$ and the elements $\{a, b, c, d, e, f, g\}$, with A_i being adjacent to an element of A if and only if the element belongs to A_i. Draw this bipartite graph and indicate a subset of edges corresponding to a complete transversal of $\{1, 2, 3, 4, 5\}$. **[8]**

Question 3 **[Total marks: 40]** Now we come to Perfect's theorem. Let $B = \{1, 2, 3, 4, 5, 6\}$. Suppose that we say that a subset of B is independent if and only if:

1. the corresponding subsets from the collection $A_1, \ldots, A_6$ of question 2 have a complete transversal in $A = \{a, b, c, d, e, f, g\}$; and
2. this transversal is contained in an independent set of the matroid M of question 1.

For example, the subset $\{1, 2, 5\}$ is independent in B because abf is a transversal for A_1, A_2, A_5 **and** abf is independent in the tetrahedron matroid M. But $\{1, 3, 4, 5\}$ is not independent in B: although the subset $adfg$ is a transversal for A_1, A_3, A_4, A_5, it is not independent in the tetrahedron matroid M and there is no other transversal for these subsets which can be chosen instead.

In fact, we are looking for subsets of A which are simultaneously independent in the transversal matroid for the A_i and in the matroid M. Such subsets form the **intersection** of the two matroids (see the first few pages of Chapter 3 of the subject guide; it does not matter if you have not yet reached this point in your study: it is a good idea to read ahead a little!) The subsets in a matroid intersection normally fail to obey the matroid axioms themselves. However, Perfect's theorem says that, if one of the matroids is the transversal matroid of a subset family, then we can still get a matroid from the intersection, but it is found in the set of indices of the subsets (our set B).

(a) Find the **seven** maximal independent subsets of $B = \{1, 2, 3, 4, 5, 6\}$, with respect to rules 1 and 2 above. **[10]**

(b) Find two subsets of $A = \{a, b, c, d, e, f, g\}$ which are independent in the tetrahedron matroid M of question 1 and also in the transversal matroid of question 2 but which fail to obey the Exchange Axiom for matroids with respect to these two properties (confirming that this matroid intersection is not itself a matroid). **[10]**

(c) The independent subsets of B form a matroid which is represented by the following matrix:

$$X = \begin{array}{c} \\ v \\ w \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & & 1 & & & \\ & & -1 & & 1 & \\ & & & 1 & -1 & 1 \\ -1 & 1 & & -1 & & -1 \end{pmatrix} \end{array}$$

where all blank entries should be taken to be zeros. (See **[2.3]**).

(a) Choose **two** of the subsets that you listed in part (a) and show that the corresponding columns of X are linearly independent. **[5]**

(b) Choose any subset of four elements of B which are **not** independent and show that the corresponding columns of X are not linearly independent. **[5]**

(c) By interpreting X as the incidence matrix of a suitable graph, explain how the independent subsets of B constitute the cycle matroid of this graph. **[10]**

End of Coursework Assignment 1

CO3352 Coursework Assignment 2: Linear Programming and Game Theory

Please note:

- Chapter 4 of the subject guide for this course will help you in completing this assignment. Square-bracketed references, e.g. **[4.3.3]**, are to relevant sections of the subject guide.
- You are expected to spend between 15 and 30 hours completing this assignment.
- Questions 1 and 2 are intended to be relatively straightforward: you need to be comfortable with the material involved in order to meet the demands of the examination in this subject area; Question 3 is more demanding and should allow you to assess your ability to aim for high second class or first class marks.

The theory of games, in its simplest versions, is almost synonymous with linear programming. Thus, two competitors A and B each choose an action (called a **strategy**); according to which of a fixed set of strategies they choose, either B pays A or A pays B, recorded as a negative payment from B to A. A wants to maximise her expected payout from B; B wants to minimise this payout.

We shall use the following example: A and B both choose a number from the set $\{1, 2\}$. If they choose the same number then B pays A \$3; if they choose different numbers then A pays B 2×her number, recorded as a negative payout for B. This is recorded in a **game matrix**:

$$\begin{array}{rc|cc|}
 & & \multicolumn{2}{c}{B\text{'s choice}} \\
 & & 1 & 2 \\
\cline{3-4}
A\text{'s choice} & 1 & 3 & -2 \\
 & 2 & -4 & 3 \\
\cline{3-4}
\end{array}$$

The game matrix makes clear the strategic options: B would like to win the \$4 (row two, column 1, a negative payout to A) but his choice of '1' risks his paying A the \$3, etc. It is assumed that the game is to be played repeatedly. This means that B cannot *always* choose '1', otherwise A will respond by always choosing '1' as well. So B must sometimes choose '1' and sometime choose '2': this is called a **mixed strategy**. Of course A will respond with her own mixed strategy.

Question 1 [Total marks: 20] The first question just explores the possible outcomes of our example game.

(a) If B's mixed strategy is to alternate between '1' and '2', what will A's mixed strategy be? **[2]**

(b) Suppose A and B both adopt the mixed strategy of each tossing a fair coin, choosing '1' if the outcome is Heads and choosing '2' otherwise. Simulate this:

(i) draw up a table with 20 columns, labelled 1 to 20, and 3 rows, labelled A and B and 'payout'. Take a coin and toss it 20 times for A, writing down the outcomes in the first row of the table. Then toss the coin another 20 times, writing down the outcomes in the second row. **[5]**

(ii) Now fill in the third row of the table: for example, if column 1 has Heads for A and Tails for B then record -2 in row 3 (A chooses '1', B chooses '2', A's payout to B is 2×1\$, recorded as a negative payout of -2). **[5]**

(iii) Add up the entries in the third row and state who benefits from the Heads-Tails mixed strategy (negative total is a win for B). **[5]**

(c) Our analysis of this game in Questions 2 and 3 will use a version of the game matrix which has been made *positive* by adding the same amount to each entry of the matrix so that no entries remain ≤ 0. We can add 5 for our matrix:

$$\begin{array}{rc|cc|}
 & & \multicolumn{2}{c}{B\text{'s choice}} \\
 & & 1 & 2 \\
\cline{3-4}
A\text{'s choice} & 1 & 3+5 & -2+5 \\
 & 2 & -4+5 & 3+5 \\
\cline{3-4}
\end{array}
\quad \text{giving} \quad
\begin{array}{rc|cc|}
 & & \multicolumn{2}{c}{B\text{'s choice}} \\
 & & 1 & 2 \\
\cline{3-4}
A\text{'s choice} & 1 & 8 & 3 \\
 & 2 & 1 & 8 \\
\cline{3-4}
\end{array}$$

Explain briefly how using the positive game matrix will change the entries in row 3 of the table you constructed in part (b), and how the original entries may be recovered. **[3]**

Question 2 [Total marks: 40] We will now transform our game matrix into a linear programme **[4.3.1]** which aims to minimise the worst possible average payout for B. We use P_B to denote this worst average payout. We assume that B has a mixed strategy which chooses '1' with probability q_1 and chooses '2' with probability q_2, where $q_1 + q_2 = 1$ (thus in question 1(b), B used $q_1 = \frac{1}{2}$ and $q_2 = \frac{1}{2}$). Recall that the average of two quantities x and y chosen with probabilities p_1 and p_2 is given by $p_1 \times x + p_2 \times y$.

(a) Given the positive game matrix from question 1(c) explain why B's attempt to use the mixed strategy q_1, q_2 to minimise P_B can be expressed as the two inequalities:

$$8q_1 + 3q_2 \leq P_B \qquad (1)$$

$$q_1 + 8q_2 \leq P_B. \qquad (2)$$

[5]

(b) Use $x_1 = q_1/P_B$ and $x_2 = q_2/P_B$. Divide by P_B on both sides of equations (1) and (2) in part (a) to get two new inequalities in the variables x_1 and x_2. **[5]**

(c) The equations in part (b) will be our linear programming constraints. The objective function will be $x_1 + x_2$.

(i) What is the value of the objective function in terms of P_B. **[4]**

(ii) Does B wish to minimise or maximise his objective function? (Remember that $P_B \geq 1$ because we made our game matrix positive). **[4]**

(d) Replace $\leq$ with $=$ in your inequalities from part (b) to obtain two straight line equations. Plot these equations, with x_1 on the horizontal axis and x_2 on the vertical axis. The plot should show x_1 and x_2 in the range 0 to 1. (You may wish to use a spreadsheet or graphing package for your plot, but a manually prepared plot on graph paper, scanned into your document, is acceptable.) **[8]**

(e) Explain how to determine approximately from your plot in part (c) the optimum value of the linear programme derived in part (c) and the values of x_1 and x_2 which achieve this optimum value. **[8]**

(f) What values of q_1 and q_2 should B use for his mixed strategy in order to minimise his worst average payout? **[3]**

(g) What is the value of the minimised worst average payout for B for the *original* game matrix (as studied in question 1, parts (a) and (b))? **[3]**

Question 3 [Total marks: 30] [4.3.4].

(a) Use a suitable software package to solve the linear programme in question 2, part (c). You may use the QSopt software provided on the CD-ROM accompanying the Subject Guide for this module, or one of the many packages which can be used for free on the web (e.g. www.phpsimplex.com/ *but note that you use such free-ware at your own risk!*). Your submission should include the output of the software you used (e.g. as a screen shot or output file). **[15]**

(b) Write down the dual linear programme for the linear programme in question 2, part (c). Discuss briefly how the theory of linear programming duality **[4.3.3]** allows us to interpret the dual linear programme and its solution in terms of the game being played by A and B. **[15]**

End of Coursework Assignment 2