UNIVERSITY OF LONDON

Mathematics for business
Volume 1

R. Shipsey

CO0001

2006

Undergraduate study in
Computing and related programmes

Goldsmiths
UNIVERSITY OF LONDON

This guide was prepared for the University of London by:

R. Shipsey

This is one of a series of subject guides published by the University. We regret that due to pressure of work the author is unable to enter into any correspondence relating to, or arising from, the guide. If you have any comments on this subject guide, favourable or unfavourable, please use the form at the back of this guide.

In this and other publications you may see references to the 'University of London International Programmes', which was the name of the University's flexible and distance learning arm until 2018. It is now known simply as the 'University of London', which better reflects the academic award our students are working towards. The change in name will be incorporated into our materials as they are revised.

University of London
Publications Office
32 Russell Square
London WC1B 5DN
United Kingdom

london.ac.uk

Published by: University of London

Contents

Introduction

0.1 Preface

This is a full unit course for the Diploma in Computing and Information Systems. The aim of the course is to build upon school mathematics to give students further skills in algebra, functions, calculus and geometry which can then be applied to solve practical problems in Business and Economics.

This subject guide is split into two volumes. The first volume begins with a chapter containing review material with which you should already be familiar from school. It then goes on to discuss linear equations, functions, simultaneous equations, matrices and linear programming problems. Each chapter begins by descibing a particular branch of mathematics and then goes on to explain how this mathematics may be applied to particular problems in Business and Economics.

Volume 2 of the subject guide is a continuation of this volume and includes chapters on differential calculus, applications of differentiation, exponential and logarithmic functions, series and integration.

0.2 Learning outcomes

On completion of this course students should be able to:

- Use directed numbers, fractions and exponents accurately and appropriately using a calculator where necessary.
- Work with algebraic, logarithmic, exponential and simple trigonometric functions.
- Recognise and create graphs of basic functions and their transformations.
- Manipulate matrices and formulate and solve applied problems by matrix methods.
- Differentiate and integrate basic functions and apply the relevant rules to more complex functions.
- Understand ideas of series, particularly geometric series and thier application to problems of finance.

0.3 Method of assessment

There will be one examination lasting three hours at the end of the academic year. This examination consists of 10 compulsory questions each worth 10 marks, giving a total of 100 marks. The questions are designed to test your knowledge of the material over the complete range of the subject. Your mark for the examination will account for 80% of your final mark for the course.

In addition there will be three tests each lasting one hour which will be sat at intervals throughout the academic year. Your marks from these tests will account for a total of 20% of your final mark for the course.

You will find examples of two tests at the end of Volume 1 of the subject guide. There are two sample examination papers at the end of Volume 2. All of these test and examination questions have solutions included at the back of the relevant volume so that as part of your revision you can practise solving questions and then check your answers.

Throughout the subject guide there are numerous worked examples and at the end of every chapter (excluding Chapter 1) there are sample examination questions. Again these examination questions have full solutions at the back of the guide so that you can check your answers.

0.4 How to use the guide

The subject guide is divided into eleven chapters - the first six chapters are in this Volume and next five are in Volume 2. Depending on your mathematical experience, you may need to spend various amounts of time on each chapter. However it is likely that you will spend approximately two weeks working on the material in each chapter.

Chapter 1 contains review material. If you are confident that you can do all of the mathematics covered in this chapter then you can move on. On the other hand, if it has been a while since you last did any mathematics then you may need to spend more time on this chapter.

You cannot read a mathematics book like a novel. You will need to have pen, paper, ruler, graph paper and calculator at the ready and be prepared to use them.

Throughout the subject guide there are *learning activities* which are designed to test your understanding of the mathematics. Solutions for these activities are given at the back of the guide. You should always attempt to do the activity questions yourself first before you look up the solutions. If your solution is correct then you can move on, if not then try and work through the given solution. If you still feel you need more practice then you should attempt the relevant exercises in the recommended book.

At the end of each chapter there are sample examination questions which you should attempt to make sure that you have fully understood the material covered in that chapter. Again try the question yourself before you look at the solution.

0.5 Calculators

Throughout the course you will be developing and using your mental arithmetic skills. However you will also be solving more complex problems and for these you are expected to use a calculator. It is therefore important that you own and know how to use a calculator - this should be a scientific calculator which has $[a\frac{b}{c}]$, $[\surd]$, $[x^y]$, $[sin]$, $[cos]$, $[tan]$, $[ln]$ and $[e^x]$ buttons as well as the usual $[+]$, $[-]$, [x] and $[\div]$.

In Chapter 1 some hints are given on how to use your calculator, but you should refer to the manual that came with the calculator to learn how to use your own model efficiently.

Since different calculators vary in the way that input is entered, it is best if you use the same calculator throughout the course and for the examination. In this way you will be quite comfortable and familiar with your calculator in the examination.

0.6 Reading

There are various books published on Mathematics for Business and Statistics which you might find useful. In particular it is recommended that you get a copy of the following book as it includes many examples and exercises similar to those in the subject guide which you can use for further practice if necessary.

Edward T. Dowling *Mathematical Methods for Business and Economics* (McGraw-Hill, 1993). ISBN 0-07-017697-3.

0.6.1 Further reading

If you have not studied mathematics recently then you might find it helpful to get a school mathematics book. For example the following revision guide covers most of the material in Chapter 1 as well as linear equations and graphs, simultaneous equations and inequalities:

Letts Revise GCSE Mathematics ISBN 1-85805-434-6 or ISBN 1-85805-932-1 (revised edition).

For a clear explanation of basic techniques in arithmetic, algebra and graphs, the following two volumes are also recommended:

Lynne Graham & David Sargent *Countdown to Mathematics volumes 1 and 2* (Addison-Wesley, 1981) ISBN 0-201-13703-5 and ISBN 0-201-13731-3.

I hope that you enjoy this course and find the subject guide useful. A lot of people are needlessly scared of mathematics and think that any course with *mathematics* in the title must be very hard. It's really not that difficult if you put in the necessary work.

Chapter 1

A review of some basic mathematics

In this chapter, we will cover some mathematics with which you should already be familiar. Some of you may not have studied mathematics for a while and you should try all of the exercises in this chapter to ensure that you can do them. It is assumed in the rest of the subject guide that all of the material in this chapter can be used without further explanation. It is therefore essential that you understand all of the mathematics in this chapter before you proceed with the course.

Additional reading

Any G.C.S.E. or equivalent school mathematics book should cover the material in this chapter.
See Modules 1 and 2 of *Sargent: Volume 1* for a step by step guide on rounding, negative numbers, brackets, decimals, fractions, percentages and using a calculator. Factorisation of quadratic equations is covered in Module 6 *Sargent: Volume 2*.
See also Chapter 1 of *Dowling* for further examples and exercises.

1.1 Positive and negative numbers

Numbers which are greater than zero, are called *positive numbers*. The following numbers are all positive:

$$1, 2, 3, 5.5, 3.08, 745.2, 0.04, 7695402$$

A number which is less than zero, is called a *negative number*. The following numbers are all negative:

$$-1, -2, -3, -5.8, -0.74, -0.02, -9746367$$

If you add two positive numbers together, the answer is another positive number. For example:

$$\begin{aligned} 7 + 4 &= 11 \\ 4.6 + 7.95 &= 12.55 \\ 0.02 + 0.45 &= 0.47 \end{aligned}$$

We can also subtract one positive number from another. If you subtract a small positive number from a bigger positive number, then the answer will be another positive number. For example:

$$7 - 4 = 3$$

However, if you subtract a positive number from a smaller positive number, then the answer will be a negative number. For example:

$$4 - 7 = -3$$

1.1.1 The number line

It can be helpful to draw a *number line* as below.

$$-8 \quad -7 \quad -6 \quad -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$$

To add two numbers together, start at the first number and move along the line to the **right** however many you are adding on (the second number). Where you end up on the line is the answer.

For example, to calculate $-2 + 5$ using the number line, start at -2 and move 5 places to the right. You should end up at 3 and this is the answer.

To subtract two numbers using the number line, start at the first number, and move along the line to the **left** however many you are subtracting (the second number). Where you end up on the line is the answer.

For example, to calculate $5 - 7$ using the number line, start at 5 and move 7 places along to the left. You should end up at -2 and this is the answer.

1.1.2 Addition with negatives

We have seen that if you add together two positive numbers, then the result will be a positive number. Similarly, if you add together two negative numbers, then the result will be a negative number. For example:

$$(-7) + (-4) = (-11)$$

Note that the brackets () used in the expression above are not strictly necessary, but it is sometimes useful to use brackets with negative numbers, so that it is clear that the minus sign (-) goes with the number.

To enter a negative number on your calculator use the [+/-] button. For example, to do the sum above on your calculator follow the sequence [+/-][7][+][+/-][4][=].

1.1.3 Subtraction with negatives

Subtracting a negative number is the same as adding a positive number. This is because a double negative makes a positive. For example:

$$\begin{aligned} 7 - (-3) &= 7 + 3 = 10 \\ -7 - (-3) &= -7 + 3 = -4 \\ 3 - (-7) &= 3 + 7 = 10 \\ -3 - (-7) &= -3 + 7 = 4 \end{aligned}$$

Remember this important rule as

Two minuses make a plus.

[1]

[1] It may help to think of this rule working in spoken English. The two negatives "do not" and "dislike" in the sentence "I do not dislike chocolate" combine to make a positive. The meaning of the sentence is really "I do like chocolate".

Learning activity

Work out the following sums using a number line. You can check your answers using your calculator.

1. $-2 + 4$
2. $-3 + (-4)$
3. $-16 - (-4)$
4. $8 + (-4)$
5. $4 - 7$
6. $28 - (-4)$
7. $-15 - (-3)$
8. $42 - (-6)$
9. $-7 + 8$

1.1.4 Multiplication with negatives

When you multiply numbers there is a simple rule for what happens to the signs.

If the signs are the same then the answer is positive.
If the signs are different then the answer is negative.

So multiplying two positive numbers together results in a positive answer:

$$7 \,.\, 5 = 35$$

[2]

[2] Note that we will always use a dot . instead of a x sign for multiplication. This is to avoid confusion later when we use the letter x in algebraic expressions.

Multiplying two negative numbers together also results in a positive answer:

$$(-6) \,.\, (-9) = 63$$

But multiplying a positive number by a negative number, or a negative number by a positive number, results in a negative answer:

$$\begin{aligned} 5 \,.\, (-10) &= (-50) \\ (-4) \,.\, 2 &= (-8) \end{aligned}$$

The same rules apply when dividing numbers. So for example:

$$\begin{aligned} 15 \div 3 &= 5 \\ (-15) \div 3 &= (-5) \\ 15 \div (-3) &= (-5) \\ (-15) \div (-3) &= 5 \end{aligned}$$

Learning activity

Work out the following in your head and then use your calculator to check your answers.

1. $(-2) \cdot 4$
2. $(-3) \cdot (-4)$
3. $25 \div (-5)$
4. $7 \cdot 8$
5. $(-8) \div (-4)$
6. $72 \div 9$
7. $3 \cdot (-7)$
8. $(-40) \div 8$

1.2 BODMAS

BODMAS stands for

- Brackets
- pOwers
- Division
- Multiplication
- Addition
- Subtraction

This is the *order of precedence* of mathematical operations i.e., the order in which the operations must be performed. It is very important to follow the BODMAS rules when working out the value of a mathematical expression. To see the importance of BODMAS, consider the following expression:

$$5 \cdot 4 - 8 \div 2$$

Following the rules of BODMAS we do the division and multiplication steps first and finally the subtraction. We can put brackets into the expression to make it clear which steps we are performing first.

$$\begin{aligned} (5 \cdot 4) &\quad - \quad (8 \div 2) \\ = 20 &\quad - \quad 4 \\ = 16 & \end{aligned}$$

Following is another example of a complicated expression, which can be evaluated without ambiguity if the rules of BODMAS are followed.

$$4 + 5^2 \div (3 - 2^3)$$

We evaluate the Bracket first, calculating the pOwer and then the Subtraction within the Bracket. Next we evaluate the pOwer and then the Division and finally the Addition.

$$\begin{aligned} & 4+5^2 \div (3-2^3) \\ = \; & 4+5^2 \div (3-8) \\ = \; & 4+5^2 \div -5 \\ = \; & 4+25 \div -5 \\ = \; & 4+-5 \\ = \; & -1 \end{aligned}$$

Learning activity

Use the rules of BODMAS to evaluate the following mathematical expressions.

1. $10-7 \,.\, 2+3^2$
2. $(3+2)^2$
3. $15 \div 3+2$
4. $(12+8) \div 5 \,.\, 2$

Put brackets into the following equations so that the given answers are correct.

5. $17+13 \div 2 \,.\, 3=5$
6. $10 \,.\, 4 \,.\, 5-20=0$

1.3 Sets of numbers

There are many different types of numbers and we need to define these different types.

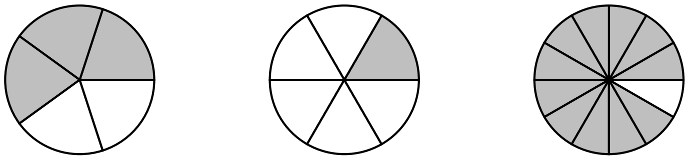
- An **integer** is a whole number such as $...-3,-2,-1,0,1,2,3,...$
- A **decimal number** is of the form $a.b$ where a and b are integers. Examples of decimal numbers are $3.5, 4.72, -5.0321.1000.587653$
- A **rational number** is a number which can be written as $\frac{a}{b}$ where a and b $(b \neq 0)$ are both integers. If a rational number is written as a decimal number then the decimal can be written exactly or is *recurring*. For example, $\frac{3}{4}=0.75$ and $\frac{5}{8}=0.625$ are rational numbers with finite decimal representations which can be written exactly. The rational number $\frac{1}{3}$ has decimal representation 0.33333333333....... This is a recurring decimal because the 3 keeps repeating, and this is usually written as $0.'3'$ the ' ' signs show that the 3 is recurring. Another example of a recurring decimal is $0.4545454545 45... = 0.'45'$ which is the decimal representation of the rational number $\frac{5}{11}$. [3]
- An **irrational number** is a number which is not rational i.e., it cannot be written as $\frac{a}{b}$ or as a finite or recurring decimal. Examples of irrational numbers are $\sqrt{2}, \sqrt{3}, \sqrt{7}$ and π.
- The set of **real numbers** is the set of all integers, rational numbers and irrational numbers. Unless otherwise specified when we talk about *a number x* we are usually assuming that x is a real number.

[3] The set of all rational numbers includes the set of all integers. This is because the integer 5 for example can be written as the rational number $\frac{5}{1}$ and is equal to the finite decimal number 5.0.

1.4 Fractions

A *fraction* is a number of the form $\frac{a}{b}$ where a and b are both *integers* and b is non-zero. The number on the top is called the *numerator* and the number on the bottom is called the *denominator*.

It can be helpful when dealing with a fraction $\frac{a}{b}$, to visualise a cake cut into b equal sections with a of these sections a different colour from the rest. For example the fractions $\frac{3}{5}$, $\frac{1}{6}$ and $\frac{11}{12}$ are illustrated below.



Proper fractions and top heavy fractions

If the numerator of a fraction is smaller than its denominator, the fraction is called a *proper fraction* and its value is less than 1.

If the numerator of a fraction is larger than the denominator, then the fraction is called *top heavy*. Such a fraction represents a whole number plus a proper fraction. For example, the top heavy fraction $\frac{17}{5}$ represents the number $3\frac{2}{5}$.

To convert $\frac{41}{7}$ into an integer and proper fraction, we calculate $41 \div 7 = 5$ remainder 6. Thus $\frac{41}{7} = 5\frac{6}{7}$.

Conversely, to convert $3\frac{7}{8}$ into a top heavy fraction, we perform the following calculation: $3\frac{7}{8} = \frac{(3\ .\ 8)+7}{8} = \frac{31}{8}$.

Equivalent fractions

If the numerator and denominator of a fraction have a factor in common i.e. there is a number which divides into both the numerator and denominator, then we can divide both the numerator and denominator by this factor to get a new fraction. For example,

$$\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

A fraction is said to be *in its lowest terms* if the numerator and denominator have no factors in common.

Two fractions are called *equivalent* if they are the same when they are both written in their lowest terms. For example, the fractions $\frac{6}{8}, \frac{3}{4}, \frac{15}{20}, \frac{75}{100}$ are all equivalent.

Learning activity

Which of the following fractions are equivalent: $\frac{2}{3}, \frac{24}{36}, \frac{7}{10}, \frac{40}{60}, \frac{18}{27}, \frac{29}{45}$?

1.4.1 Addition, subtraction, multiplication and division of fractions

Addition of fractions

Two fractions can only be added together if they have the same denominator. To add together two fractions which have the same denominator simply add together the numerators and leave the denominator unchanged. For example,

$$\frac{3}{4}+\frac{2}{4}=\frac{(3+2)}{4}=\frac{5}{4}=1\frac{1}{4}$$

If the two fractions to be added do not have the same denominator then we have to change each one into an equivalent fraction. This can be done by multiplying the numerator and denominator of each fraction by the denominator of the other fraction. For example:

$$\frac{3}{7}+\frac{2}{5}=\frac{3\,.\,5}{7\,.\,5}+\frac{2\,.\,7}{5\,.\,7}=\frac{15}{35}+\frac{14}{35}=\frac{(15+14)}{35}=\frac{29}{35}$$

[4]

[4]Note that $\frac{3}{7}$ is equivalent to $\frac{15}{35}$ and $\frac{2}{5}$ is equivalent to $\frac{14}{35}$.

To add numbers which have an integer and fractional part, we can simply add together the integer parts and then the fractional parts as in the following example:

$$2\frac{3}{5}+7\frac{1}{8}=2+7+\frac{3}{5}+\frac{1}{8}=9+\frac{24}{40}+\frac{5}{40}=9+\frac{24+5}{40}=9\frac{29}{40}$$

Subtraction of fractions

Subtraction of fractions is very similar to addition. We must first make the denominators of the two fractions the same, then we subtract the numerators. For example:

$$\frac{7}{8}-\frac{3}{5}=\frac{7\,.\,5}{8\,.\,5}-\frac{3\,.\,8}{5\,.\,8}=\frac{35}{40}-\frac{24}{40}=\frac{35-24}{40}=\frac{11}{40}$$

To subtract numbers which have an integer and fractional part, we could do the integer subtraction first and then the fractional subtraction. However, if the second fraction is bigger than the first fraction this will leave us with a negative fractional part. In this case, it is better to turn the numbers into top heavy fractions first and then perform the subtraction. For example:

$$3\frac{1}{4}-2\frac{3}{5}=\frac{13}{4}-\frac{13}{5}=\frac{65}{20}-\frac{52}{20}=\frac{13}{20}$$

Multiplication of fractions

Multiplying two fractions together is very easy. We simply multiply the numerators together, and multiply the denominators together. For example:

$$\frac{2}{3}\cdot\frac{4}{7}=\frac{2\,.\,4}{3\,.\,7}=\frac{8}{21}$$

Note that if we want to multiply numbers which have an integer and a fractional part, then we **must** change the numbers into top heavy fractions before multiplying. For example:

$$1\frac{1}{2} \;.\; 2\frac{2}{3} = \frac{3}{2} \;.\; \frac{8}{3} = \frac{3\;.\;8}{2\;.\;3} = \frac{24}{6} = 4$$

Division of fractions

To divide one fraction by another, follow this simple rule "turn the second fraction upside down and then multiply". For example:

$$\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \;.\; \frac{2}{1} = \frac{2\;.\;2}{3\;.\;1} = \frac{4}{3} = 1\frac{1}{3}$$

As with multiplying, to divide numbers which include an integer and fractional part, first make all of the numbers into top heavy fractions as in the following example.

$$2\frac{1}{5} \div 3\frac{2}{9} = \frac{11}{5} \div \frac{29}{9} = \frac{11}{5} \;.\; \frac{9}{29} = \frac{11\;.\;9}{5\;.\;29} = \frac{99}{145}$$

Note that when calculating with a mixture of whole numbers and fractions, it can be helpful to think of the whole numbers as fractions with numerator 1. This is particularly useful when dividing a fraction by a whole number. For example:

$$\frac{4}{7} \div 3 = \frac{4}{7} \div \frac{3}{1} = \frac{4}{7} \;.\; \frac{1}{3} = \frac{4\;.\;1}{7\;.\;3} = \frac{4}{21}$$

Most calculators have a fraction button $[a\frac{b}{c}]$. To input the number $2\frac{3}{7}$ on your calculator follow the sequence $[2][a\frac{b}{c}][3][a\frac{b}{c}][7]$.

Learning activity

Calculate the following fractions by hand. Check your answers using your calculator if it has an $[a\frac{b}{c}]$ button.

1. $\frac{1}{2} + \frac{1}{3}$
2. $\frac{3}{5} + \frac{2}{3}$
3. $1\frac{1}{2} + \frac{4}{7}$
4. $\frac{4}{5} - \frac{2}{3}$
5. $2\frac{1}{2} - 1\frac{3}{4}$
6. $\frac{2}{3} \cdot \frac{2}{5}$
7. $\frac{3}{4} \cdot \frac{7}{10}$
8. $1\frac{1}{2} \;.\; 2\frac{1}{3}$
9. $\frac{1}{2} \div \frac{1}{4}$
10. $\frac{4}{5} \div \frac{3}{4}$
11. $1\frac{1}{2} \div \frac{3}{8}$
12. $4\frac{7}{8} \div 3$

1.5 Indices and roots

The mathematical expression x^n means x is multiplied by itself n times. Thus $3^4 = 3 \,.\, 3 \,.\, 3 \,.\, 3 = 81$. The number n in x^n may be called an *index* (plural *indices*) or a *power* or an *exponent*. The expression x^n is usually read as " x to the power n".

Rules of indices

If we multiply together two terms of the form x^n then we **add** the exponents. For example:

$$x^2 \,.\, x^3 = x^{2+3} = x^5$$

You can see why this is if we write the expressions out in full:

$$x^2 \,.\, x^3 = x \,.\, x \,.\, x \,.\, x \,.\, x = x^5$$

If we divide one term of the form x^n by another term of the same form, then we **subtract** the second exponent from the first. For example:

$$x^7 \div x^4 = x^{7-4} = x^3$$

Again it is clear why this happens if we write the expressions out in full:

$$x^7 \div x^4 = \frac{x^7}{x^4} = \frac{x \,.\, x \,.\, x \,.\, x \,.\, x \,.\, x \,.\, x}{x \,.\, x \,.\, x \,.\, x}$$

We can cancel all of the x's in the denominator with 4 of the x's in the numerator. This will leave us with $(7-4) = 3$ x's in the numerator.

$$\frac{x \,.\, x \,.\, x \,.\, \cancel{x} \,.\, \cancel{x} \,.\, \cancel{x} \,.\, \cancel{x}}{\cancel{x} \,.\, \cancel{x} \,.\, \cancel{x} \,.\, \cancel{x}} = \frac{x \,.\, x \,.\, x}{1} = x^3$$

We can also raise an expression of the form x^n to a power to find $(x^n)^m$. In this case, we multiply together the two exponents n and m to find the new exponent. For example:

$$(x^3)^4 = x^{(3 \,\cdot\, 4)} = x^{12}$$

Once again, writing the expression out in full makes it clear why this rule applies.

$$(x^3)^4 = (x \,.\, x \,.\, x)^4 = (x \,.\, x \,.\, x) \,.\, (x \,.\, x \,.\, x) \,.\, (x \,.\, x \,.\, x) \,.\, (x \,.\, x \,.\, x) = x^{12}$$

Negative indices

Suppose we wanted to simplify the expression:

$$x^3 \div x^5$$

Following the rules of indices, this would give us

$$x^{3-5} = x^{-2}$$

The exponent is a negative number. Writing the expression out in full shows us what x^{-2} means.

$$x^3 \div x^5 = \frac{x^3}{x^5} = \frac{x \, . \, x \, . \, x}{x \, . \, x \, . \, x \, . \, x \, . \, x} = \frac{1}{x \, . \, x} = \frac{1}{x^2}$$

A negative exponent means "one over" the expression.

We can use the rules of indices to simplify expressions by combining the powers. For example

$$\left(\frac{x^7 \, . \, x^3}{x^5}\right)^2 = x^{(7+3-5) \, . \, 2} = x^{10}$$

Zero indices

What is the result if we calculate $x^5 \div x^5$?
Following the rules of indices we get $x^{5-5} = x^0$. But dividing a number by itself gives the answer 1. Therefore $x^5 \div x^5 = 1$. This illustrates another rule of indices:

For all values of x, $x^0 = 1$

Notes

We can only use these rules of indices if the base value x is the same in all the terms. Thus $2^3 \, . \, 2^4 = 2^7$. But we cannot simplify $2^3 \, . \, 3^4$ in this way.

It is also important to note that expressions of the form $x^n + x^m$ can only be simplified by factorisation (see 1.8.2).

Learning activity

Write these as a single power of 7:

1. $7^4 \, . \, 7^6$
2. $(7^2)^3$
3. $7^{-3} \, . \, 7^6$
4. $7^5 \div 7^4$
5. $\frac{7^3}{7^8}$

1.5.1 Roots

The *square root* of a number x is written $\sqrt{x}$ and is the number which when multiplied by itself gives the result x. Thus $\sqrt{x} \, . \, \sqrt{x} = x$ or $(\sqrt{x})^2 = x$. For example, the square root of 9 is 3 because $3 \, . \, 3 = 9$. This can be written as $\sqrt{9} = 3$. In fact, -3 is also the square root of 9 because $-3 \, . \, -3 = 9$ as well. Therefore we should write $\sqrt{9} = \pm 3$.

Instead of using a $\sqrt{}$ symbol, we can express the square root of a number using the exponent $\frac{1}{2}$. This is because $(\sqrt{x})^2 = x$ and using the rules of indices $(x^{\frac{1}{2}})^2 = x^{\frac{1}{2}.2} = x^1 = x$.

We can also find the *cubed root* of a number $^3\sqrt{x}$. This is the number which when raised to the power 3 equals x. So $(^3\sqrt{x})^3 = x$. For example, the cubed root of 8 is 2 because $2 \,.\, 2 \,.\, 2 = 8$. Thus $^3\sqrt{8} = 2$. Note that this time we do not need the $\pm$ sign because $-2 \,.\, -2 \,.\, -2 = -8$ and therefore -2 is a cubed root of -8 but not of 8.

We can find any root of a number. The "n^{th} root of x" can be written as $^n\sqrt{x}$ or $x^{\frac{1}{n}}$. In either case, this is the number which when raised to the power n gives the answer x. So $(^n\sqrt{x})^n = x$ and $(x^{\frac{1}{n}})^n = x$.

Writing roots in the form $x^{\frac{1}{n}}$ can be helpful because then we can simplify expressions by adding, subtracting and multiplying the indices as described in section 1.5.

For example, $x^{\frac{1}{2}} \,.\, x^3 = x^{3+\frac{1}{2}} = x^{3\frac{1}{2}} = x^{\frac{7}{2}} = (\sqrt{x})^7$.

Notice how the expression $x^{\frac{7}{2}}$ has been rewritten as $(\sqrt{x})^7$. In general we have

$$x^{\frac{a}{b}} = (^b\sqrt{x})^a =^b \sqrt{(x^a)}$$

Using a calculator to find roots and powers

Most calculators have a $[\sqrt{}]$ button and an $[x^2]$ button which can be used to find the square root of a number and the square of a number respectively.

Some calculators also have an $[x^y]$ button which allows you to calculate any power. For example to calculate 6^4 input $[6][SHIFT][x^y][4][=]$. You should get the answer 1296. Note that the order of input may be different on different calculators so you may need to experiment until you can get the correct answer.

Similarly, on some calculators you can calculate roots using the $[x^{\frac{1}{y}}]$ button. For example to calculate $16^{\frac{1}{4}}$ input $[16][SHIFT][x^{\frac{1}{y}}][4][=]$. You should get the answer 2 but again you may need to experiment, or consult your calculator manual, to find the correct input sequence for your own calculator.

Note that you cannot take the square root of a negative number. For example, if you try to find $\sqrt{(-9)}$ using your calculator you will get an error message. This is because there is no real number which can be multiplied by itself to give a negative answer.[5]

[5] Mathematicians use a system of numbers called *complex numbers* which include the square roots of negative numbers using i to represent $\sqrt{(-1)}$ but this is beyond the scope of this course.

Learning activity

1. Without using a calculator, evaluate the following:
   (a) $4^{\frac{1}{2}}$
   (b) $81^{\frac{1}{4}}$
   (c) $16^{\frac{3}{4}}$
   (d) 5^0
   (e) 3^{-3}

(f) $25^{-\frac{1}{2}}$
(g) $8^{-\frac{2}{3}}$
(h) $\left(\frac{1}{2}\right)^{-1}$
(i) $\left(\frac{1}{4}\right)^{\frac{1}{2}}$
(j) $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$

2. Find x such that:
   (a) $x^{\frac{2}{3}} = 4$
   (b) $10^x = 1$
   (c) $x^{-\frac{1}{2}} = 10$
   (d) $x^{-\frac{5}{8}} = 1$
   (e) $x^{\frac{1}{4}} = -3$

1.5.2 Surds

Some numbers have square roots which are integers. For example, $\sqrt{4} = 2$ and $\sqrt{9} = 3$, such numbers are called *square numbers*. In most cases however, the square root of a number is not an integer. For example, if you use the [$\sqrt{}$] button on your calculator to work out $\sqrt{7}$ you get the answer 2.645751311. It is often easier and more accurate to leave $\sqrt{7}$ in an expression rather than write out 2.645751311.

For example:

$$\sqrt{28} \,.\, \sqrt{25} = \sqrt{(7 \,.\, 4)} \,.\, \sqrt{25} = \sqrt{7} \,.\, \sqrt{4} \,.\, \sqrt{25} = \sqrt{7} \,.\, 2 \,.\, 5 = 10\sqrt{7}$$

A mathematical expression which includes a $\sqrt{}$ symbol is called a *surd*.

Surds can be combined using the following rules where a, b, c and d are integers:

$$a\sqrt{b} \,.\, c\sqrt{d} = ac\sqrt{(bd)}$$

$$\frac{a\sqrt{(b)}}{c\sqrt{d}} = \frac{a}{c}\sqrt{\left(\frac{b}{d}\right)}$$

$$a\sqrt{b} + c\sqrt{b} = (a+c)\sqrt{b}$$

$$a\sqrt{b} - c\sqrt{b} = (a-c)\sqrt{b}$$

Note that $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$ can only be simplified if $\sqrt{b}$ and $\sqrt{d}$ are multiples of a common root. For example:

$$\begin{aligned} & 7\sqrt{3} - 2\sqrt{12} \\ &= 7\sqrt{3} - 2\sqrt{(4 \,.\, 3)} \\ &= 7\sqrt{3} - 2\sqrt{4}\sqrt{3} \\ &= 7\sqrt{3} - 2 \,.\, 2\sqrt{3} \\ &= 7\sqrt{3} - 4\sqrt{3} \\ &= 3\sqrt{3} \end{aligned}$$

When simplifying expressions which include $\surd$ symbols, we make the surd as small as possible by dividing out as many square numbers as possible. For example:

$$\begin{aligned} &\surd 396 \\ &= \surd(4 \,.\, 99) \\ &= 2 \,.\, \surd 99 \\ &= 2 \,.\, \surd(9 \,.\, 11) \\ &= 2 \,.\, 3 \,.\, \surd 11 \\ &= 6\surd 11 \end{aligned}$$

We do not usually leave $\surd x$ in the denominator of a fraction. To move the $\surd x$ into the numerator we can multiply the numerator and denominator by $\surd x$. For example:

$$\frac{5\surd 2}{\surd 7} = \frac{5\surd 2 \,.\, \surd 7}{\surd 7 \,.\, \surd 7} = \frac{5\surd 14}{7}$$

Learning activity

Simplify the following leaving your answers in surd form with integer denominators.

1. $2\surd 3 \,.\, 5\surd 7$
2. $4\surd 15 \div 2\surd 5$
3. $8\surd 2 + 5\surd 2$
4. $9\surd 5 - 4\surd 20$
5. $\surd 50$
6. $\frac{\surd 80}{2}$
7. $\surd\left(\frac{1}{8}\right)$
8. $\frac{\surd 300}{\surd 50}$

1.6 Percentages

A *percentage* can be thought of as a fraction of 100.

To convert a percentage into a fraction, simply express the percentage as the numerator over a denominator of 100 and then cancel if possible. For example, 70% is the same as $\frac{70}{100} = \frac{7}{10}$.

To convert a fraction into a percentage, divide the numerator by the denominator and then multiply the answer by 100. For example $\frac{2}{5} = 2 \div 5 \,.\, 100 = 40\%$.

To find a percentage of a total, divide the total by 100 to find the amount that is represented by 1%; now multiply this by the value that you wish to find. For example, 56% of 80 is calculated as follows:

$$\begin{aligned} 100\% &= 80 \\ 1\% &= 80 \div 100 = 0.8 \\ 56\% &= 0.8 \,.\, 56 = 44.8 \end{aligned}$$

Learning activity

In a survey 420 people were asked whether they preferred drink A, drink B or drink C.

1. 84 people preferred drink A. What percentage of people preferred drink A?
2. 15% of the people preferred drink C. How many people preferred drink C?
3. Assuming that all of the people surveyed expressed a preference, what percentage of the people preferred drink B?

1.6.1 Calculating percentage increase and decrease

We often need to increase or decrease an amount by a given percentage. For example, suppose we want to increase 45 by 35%. We could first calculate 35% of 45 as above.

$$45 \div 100\ .\ 35 = 15.75$$

We now add 15.75 onto the original 45 to get 60.75.

Alternatively, we can perform this calculation in one step by realising that what we are calculating is 135% of 45 . The one step calculation is as follows:

$$45 \div 100\ .\ 135 = 60.75$$

Similarly we can decrease an amount by a given percentage. For example suppose we want to decrease 70 by 20%. We can first calculate 20% of 70 = 14 and then subtract 14 from 70 to get 56.

Alternatively, we can realise that when we decrease something by 20%, we will be left with 80% of the original total. In this example, 80% of 70 is given by

$$70 \div 100\ .\ 80 = 56$$

Learning activity

Calculate the values of the following after the % increase or decrease:

1. $250 after an increase of 12%.
2. A 16% increase on 80m.
3. A 26% increase on an attendance of 6500.
4. A 9% increase on sales of $2400.
5. The cost of a $1500 computer after VAT at 17.5% is added.
6. $150 after a decrease of 8%.
7. A decrease of 12% on $250.
8. The price of an $11500 car after a discount of 15%.
9. A 6% drop in an attendance of 850

Calculating 100%

Suppose we are told that a computer costs $3200 including VAT of 17.5% and we wish to find out the cost of the computer before the VAT was added?

Now we know that 117.5% = $3200 and we have to find 100%. We divide 3200 by 117.5 to find 1% and then multiply this answer by 100 to find 100%.

$$\begin{aligned} 117.5\% &= 3200 \\ 1\% &= 3200 \div 117.5 = 27.234 \\ 100\% &= 27.234 \,.\, 100 = 2723.4 \end{aligned}$$

Therefore the computer costs $2723.40 before VAT is added.

The following example shows how to calculate the original cost of an item before a percentage decrease is applied.

A dress costs $24 in a sale after a price reduction of 20%. How much did the dress originally cost?

The dress now costs $100 - 20 = 80\%$ of its original value. We have:

$$\begin{aligned} 80\% &= 24 \\ 1\% &= 24 \div 80 = 0.3 \\ 100\% &= 0.3 \,.\, 100 = 30 \end{aligned}$$

Therefore the dress originally cost $30.

Learning activity

1. Bill's salary is increased by 5% to $44100. How much was Bill's salary before the increase?
2. In a sale, a television is reduced in price by 30%. The sale price of the television is $315. How much did the television originally cost?

1.7 Prime numbers

A *prime number* is a positive integer which has no factors except for 1 and itself. i.e., no integer, greater than 1 and less than the prime number, divides exactly into the prime number. The number 2 is the smallest prime number. Note that 1 is not a prime number, and 2 is the only even prime number (because 2 is a factor of every other even number). The next prime numbers are $3, 5, 7, 11, 13, \ldots$[6]

A number which is not prime is called *composite*.

[6] There are infinitely many prime numbers. The largest prime number known so far has 7816230 digits. There are big prizes for finding unknown primes so the hunt for the next prime continues.

Learning activity

Find all of the prime numbers up to 100.

One way to do this is to list all of the numbers from 2 to 100. Now cross out all the multiples of 2 (except for 2) and all the multiples of 3 (except for 3). Now 4 has already been crossed out and so is not a prime number but 5 has not been crossed out and is the next prime. Cross out all multiples of 5 and then continue to move up through the list in the same manner. Each time you come to a number which has not been crossed out it is a prime - cross out all of its multiples. This method of finding primes is called *The Sieve of Eratosthenes*.

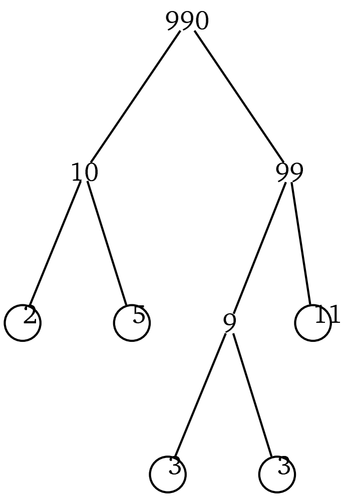
Products of prime powers

Every integer can be written as a unique product of its prime powers. This means that we can write the number as a multiplication of all of its prime factors. For every different number, the list of prime factors will be different. To write a number as the product of its prime powers, we have to find out which primes divide into the number, and how often. For example, suppose we want to write 990 as a product of its prime powers. We can do this systematically by working up though the prime numbers 2,3,5,7,11,... trying each in turn to see if it is a divisor of 990 (and if so how many times 990 can be divided by the prime)

$$990 = 2 \,.\, 495$$
$$495 = 3 \,.\, 165$$
$$165 = 3 \,.\, 55$$
$$55 = 5 \,.\, 11$$
$$11 = 11 \,.\, 1$$

We have found that $990 = 2 \,.\, 3^2 \,.\, 5 \,.\, 11$

Another method used for finding all the prime factors of a number is to draw a *factor tree*. For example, we start again with 990. An obvious factor of 990 is 10, so we divide 990 by 10 to find another factor is 99. Neither of these factors are prime, so now we need to find factors of them and so on until we come to a prime factor. We represent this factorisation by drawing a tree with root 990 and nodes for each factor. Circle any prime factors. Continue every branch until it is ended with a prime factor.



Write 990 as a product of all the circled prime numbers.
$990 = 2 \,.\, 5 \,.\, 3 \,.\, 3 \,.\, 11 = 2 \,.\, 3^2 \,.\, 5 \,.\, 11$

1.7.1 Highest common factor

The *highest common factor* (HCF) of two numbers, is the largest integer which divides exactly into both of the numbers. Note that the HCF of two numbers will never be greater than the smallest number.

One method of finding the HCF of two numbers is to write both numbers as a product of their prime powers. Then multiply together, those prime powers which appear in both lists. If a prime occurs in both lists but raised to a different power, then we can only include the prime raised to lower power. For example, we will find the HCF of the numbers 990 and 1188.

We need to write 1188 as a product of its prime powers.

$$\begin{aligned}1188 &= 2\ .\ 594\\ 594 &= 2\ .\ 297\\ 297 &= 3\ .\ 99\\ 99 &= 3\ .\ 33\\ 33 &= 3\ .\ 11\\ 11 &= 11\ .\ 1\end{aligned}$$

So now we can express the two numbers 990 and 1188 as the products of their prime powers:

$$\begin{aligned}990 &= 2\ .\ 3^2\ .\ 5\ .\ 11\\ 1188 &= 2^2\ .\ 3^3\ .\ 11\end{aligned}$$

To find the HCF, we multiply together the prime numbers which appear in both lists (we take the highest power possible for each prime). Thus $HCF(990, 1188) = 2\ .\ 3^2\ .\ 11 = 198$. So 198 is the biggest number which divides exactly into both 990 and 1188.

1.7.2 Lowest common multiple

The *lowest common multiple* (LCM) of two numbers, is the smallest number which both of the numbers divide into exactly. Note that the LCM of two numbers will never be less than the larger of the two numbers.

To find the LCM of two numbers which are written as a product of their prime powers, we multiply together **all** of the numbers which appear in the two lists but we do not include twice, any numbers which appear in both lists.

For example, the LCM of 990 and 1188 is given by $LCM(990, 1188) = 2^2\ .\ 3^3\ .\ 5\ .\ 11 = 5940$.

The following formula expresses the relationship between the HCF and the LCM of two numbers a and b.

$$HCF(a, b)\ .\ LCM(a, b) = a\ .\ b$$

Thus if the two numbers have highest common factor 1, i.e., there is no integer greater than 1 which divides into both of the numbers, then their lowest common multiple is equal to their product. Such numbers are called *co-prime numbers*.

Learning activity

1. Write each of the numbers 78, 240 and 420 as a product of their prime factors.
2. Find the HCF of 78 and 240.
3. Find the LCM of 240 and 420.

1.8 Polynomials

A *polynomial* is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0$$

where the values $a_n, a_{n-1}, \ldots, a_1, a_0$ are numbers called *co-efficients*. The number a_0 is called a *constant term*.

The following are all examples of polynomials:

$$\begin{gathered} x^3 + 4x^2 - 7x + 3 \\ 4x^2 - 5 \\ 10x^3 \\ x \end{gathered}$$

Simplifying polynomials

It is possible to simplify a polynomial by adding (or subtracting) *like terms*. That is, we can add together any terms in the polynomial which contain the same power of x. For example, the polynomial

$$3x^3 + 6x^2 + 7 - 2x^2 + 4x^3 + 4x + 10$$

can be simplified by adding together the like terms: $3x^3 + 4x^3 = 7x^3$, $6x^2 - 2x^2 = 4x^2$, $7 + 10 = 17$. We are thus left with the simplified polynomial:

$$\begin{aligned} &3x^3 + 6x^2 + 7 - 2x^2 + 4x^3 + 4x + 10 \\ &\qquad = 7x^3 + 4x^2 + 4x + 17 \end{aligned}$$

Two polynomials may be added together term by term in a similar way to form a single polynomial. For example:

$$\begin{aligned} &(x^2 + 4x - 6) + (5x^2 - 5x + 11) \\ &\qquad = 6x^2 - x + 5 \end{aligned}$$

We can also subtract one polynomial from another, again subtracting the like terms as in the following example:

$$\begin{aligned}(8x^2+3x+7)-(4x^2-9) \\ =(8x^2-4x^2)+(3x)+(7-9) \\ =4x^2+3x-2\end{aligned}$$

1.8.1 Multiplying polynomials

We can multiply a polynomial by a constant by multiplying all of the co-efficients in the polynomial by the constant. For example:

$$\begin{aligned}4\,.\,(5x^3-3x^2-2x+4) \\ =20x^3-12x^2-8x+16\end{aligned}$$

When we are multiplying a polynomial by a constant as in the preceding example, it is generally written as $4(5x^3-3x^2-2x+4)$ i.e., the . sign is usually omitted. A constant outside a bracket means that all of the terms inside the bracket are to be multiplied by that constant.

We can also multiply one polynomial by another. One of these polynomials might be a single term as in the following example. In this case we simply multiply each of the terms in the longer polynomial by the term in the first polynomial. We use the rules of indices to multiply together the terms.

$$\begin{aligned}(2x^2)(4x^2-5x-2) \\ =(2x^2\,.\,4x^2)+(2x^2\,.\,-5x)+(2x^2\,.\,-2) \\ =8x^4-10x^3-4x^2\end{aligned}$$

Notice how we must take account of the $+$ and $-$ signs when multiplying together polynomials, so that, as explained in section 1.1.4 two negative or two positive terms multiplied together result in a positive term. However a positive term multiplied by a negative term (and vice versa) will result in a negative term.

The distributive law

When multiplying a polynomial by a constant or a single term as in the preceding examples, we have made use of the *distributive law* which states that:

$$\boxed{\text{For any numbers } a, b \text{ and } c, \quad a(b+c)=(a\,.\,b)+(a\,.\,c)}$$

Expanding brackets

Finally we will consider multiplying together two polynomials, both with more than one term. For example, $(3x^2+5x)\,.\,(x^2-3x-1)$. Again the . sign is generally omitted so that this multiplication is usually written as $(3x^2+5x)(x^2-3x-1)$. What is meant here, is that every term in the second bracket is multiplied by every term in the first bracket. The easiest way to perform this multiplication, is to re-write it with the terms in the first bracket each written separately multiplying the second bracket as follows:

$$\begin{aligned} &(3x^2+5x)(x^2-3x-1) \\ =\ &3x^2(x^2-3x-1)+5x(x^2-3x-1) \end{aligned}$$

We can now multiply out each bracket using the Distributive Law.

$$\begin{array}{ccc} 3x^2(x^2-3x-1) & + & 5x(x^2-3x-1) \\ 3x^4-9x^3-3x^2 & + & 5x^3-15x^2-5x \end{array}$$

Finally, we can simplify the resulting polynomial by adding together the like terms as follows:

$$3x^4-4x^3-18x^2-5x$$

Multiplying polynomials together or multiplying a polynomial by a constant is also known as *expanding the brackets*.

Here are some further examples of *expanding brackets*:

$$\begin{aligned} (4x-6)(2x-2) \\ 4x(2x-2)-6(2x-2) \\ 8x^2-8x-12x+12 \\ 8x^2-20x+12 \end{aligned}$$

$$\begin{aligned} (3x-5)^2 \\ (3x-5)(3x-5) \\ 3x(3x-5)-5(3x-5) \\ 9x^2-15x-15x+25 \\ 9x^2-30x+25 \end{aligned}$$

Learning activity

Expand and simplify each of the following expressions:

1. $-3(x+4)$
2. $2b(4-2a)$
3. $4+3(2x+4)$
4. $x(2+3x)+2x(4-3x)$
5. $(x+3)(x+4)$
6. $(2y-4)(3y+7)$
7. $(2y+3)^2$
8. $(a+2b)^2$
9. $3x(x+2)^2$

1.8.2 Factorisation

In section 1.7 we saw that the *factors* of a number, n, are the numbers which divide exactly into n. Similarly, the factors of a polynomial are the polynomials which divide exactly into the polynomial.

The process of writing a polynomial as a product of its factors is called *factorisation*.

There are three different kinds of polynomial which can be factorised using three different methods. We will look at each of these methods in turn.

1.8.3 Finding common factors

If the terms of the polynomial can all be divided by a common factor (either a number or a single term polynomial), then this common factor can be placed outside brackets at the front of the expression, and divided out of all of the terms in the polynomial. Consider the following examples which can all be factorised by dividing out a common factor.

- $4x^2 + 6xy + 2x = 2x(2x + 3y + 1)$
- $3x^2y^2 - 4xy^2 + 5xy = xy(3xy - 4y + 5)$
- $4a + 6b + 2c - 8d = 2(2a + 3b + c - 4d)$
- $3x^4 - 12x^3 + 6x^2 = 3x^2(x^2 - 4x + 2)$

We can check that the factorisation is correct by expanding the brackets to make sure that the result is the original polynomial. For example, consider the last example above. Expanding the brackets we see that

$$3x^2(x^2 - 4x + 2) = 3x^4 - 12x^3 + 6x^2$$

and since this is the original polynomial, we are sure that the factorisation is correct.

Learning activity

Factorise the following polynomials by looking for common factors:

1. $2x + 4$
2. $2x^2 + x$
3. $30x + 12x^2$
4. $8x^3 - 6x^2$
5. $8p^2q - 4pq^2$

1.8.4 Factorising quadratics

A polynomial of the form $ax^2 + bx + c$ is called a *quadratic*. The co-efficients a, b and c can be any numbers, but we will only consider quadratics with integer co-efficients, and in particular $a = 1$. For example, $x^2 + 4x + 3$.

We cannot factorise a quadratic of this form by dividing out a common factor. However, there is a particular method used for factorising quadratics which results in finding the two polynomials which when multiplied together give the quadratic. Each of these polynomials is a *factor* of the quadratic. For example, the two factors of $x^2 + 4x + 3$ are $(x + 3)$ and $(x + 1)$. When we multiply together $(x + 3)$ and $(x + 1)$ the result is $x^2 + 4x + 3$.

$$\begin{aligned} & (x+3)(x+1) \\ = {} & x(x+1) + 3(x+1) \\ = {} & x^2 + 1x + 3x + 3 \\ = {} & x^2 + 4x + 3 \end{aligned}$$

Method for factorising quadratics

The method for factorising quadratics is summarised in the steps below. It may seem very complicated but this is only because it is easier to factorise a quadratic than it is to explain how to factorise a quadratic!

Step 1. If necessary, re-arrange the quadratic so that it is of the form $+x^2 \pm bx \pm c$.

Step 2. Look at the sign before the constant term c.

- (a) If the sign is $+c$, find two integers which multiply together to give c and add to give b. i.e., find two integers n_1 and n_2 such that $n_1 n_2 = c$ and $n_1 + n_2 = b$.
- (b) If the sign is $-c$, find two integers which multiply together to give c but whose *difference* is b. i.e., find two integers n_1 and n_2 such that $n_1 n_2 = c$ and $n_1 - n_2 = b$.

Step 3. The factors of the quadratic are:

- (a) $(x + n_1)(x + n_2)$ if the original quadratic is of the form $x^2 + bx + c$
- (b) $(x - n_1)(x - n_2)$ if the original quadratic is of the form $x^2 - bx + c$
- (c) $(x + n_1)(x - n_1)$ if the original quadratic is of the form $x^2 + bx - c$
- (d) $(x - n_1)(x + n_2)$ if the original quadratic is of the form $x^2 - bx - c$

The method is illustrated in the following examples, one of each of the four different possible cases.

Examples

1. $x^2 + 7x + 10$
   We are looking for two integers which multiply to give 10 and add to give 7. The numbers are therefore 5 and 2.
   This is case 3a) above so the factorisation is $(x + 5)(x + 2)$.
2. $x^2 - 6x + 9$
   We are looking for two integers which multiply to give 9 and add to give 6. The numbers are therefore 3 and 3.
   This is case 3b) above and so the factorisation is
   $(x - 3)(x - 3) = (x - 3)^2$.
3. $p^2 + 8p - 240$
   This time there is a minus sign before the constant term so we are looking for two integers which multiply to give 240, but whose difference is 8. The numbers are 20 and 12.
   This is case 3c) above and so the factorisation is $(p+20)(p-12)$.
4. $y^2 - 2y - 24$
   Again there is a minus sign before the constant and so we need to find two integers which multiply to give 24 and whose difference is 2. The numbers this time are 6 and 4.
   This is case 3d) above and so the factorisation is $(y - 6)(y + 4)$.

In the two following examples, we must first re-arrange the quadratics to make them into the required form as in Step 1.

5. $3x^2 + 3x - 36$
   There is a co-efficient in front of the x^2 term. However, this co-efficient 3 is a common factor in all of the terms and so we can divide it out of all the terms.
   $3(x^2 + x - 12)$
   We are now left with a quadratic of the correct form which can be factorised to give:
   $3(x + 4)(x - 3)$
6. $8x - x^2 - 15$
   The quadratic is not in the right order so first we re-order the terms:
   $-x^2 + 8x - 15$
   To get rid of the minus sign in front of the x^2 sign, we can take out the common factor -1 from each of the terms of the quadratic. This will have the effect of changing all of the signs in the quadratic. We must remember to put the -1 outside brackets to show that it is multiplying the whole quadratic:
   $-1(x^2 - 8x + 15)$
   Now we can factorise the quadratic
   $(x^2 - 8x + 15) = (x - 5)(x - 3)$. The factorisation is therefore:
   $-1(x - 5)(x - 3)$
   We can multiply the -1 back into one of the brackets to get the neater result:
   $(5 - x)(x - 3)$

Learning activity

Expand all of the brackets in the above examples to make sure that the factorisation is correct.

Difference of two squares

There is one special type of quadratic that can be factorised immediately without any work. If a quadratic is "the difference of two squares" it looks like this:

$$x^2 - y^2$$

where x and y here can be numbers, or polynomials. The factorisation is always:

$$(x - y)(x + y)$$

By expanding the brackets $(x - y)(x + y)$ we can see why this is the correct factorisation.

$$\begin{aligned} &(x - y)(x + y) \\ &= x(x + y) - y(x + y) \\ &= x^2 + xy - yx - y^2 \\ &= x^2 + xy - xy - y^2 \\ &= x^2 - y^2 \end{aligned}$$

Immediately below are some examples of the difference of two squares factorisation. In some cases, we need to do a little work first in order to make the quadratic look like two square terms.

- $x^2 - 1 = x^2 - 1^2 = (x - 1)(x + 1)$
- $4y^2 - 16x^2 = (2y)^2 - (4x)^2 = (2y - 4x)(2y + 4x)$
- $x^4 - 81 = (x^2)^2 - 9^2 = (x^2 - 9)(x^2 + 9) = (x^2 - 3^2)(x^2 + 9) = (x - 3)(x + 3)(x^2 + 9)$

The best way to learn how to do quadratic factorisation is lots of practice. You should therefore attempt to complete the following activity.

Learning activity

Factorise the following quadratics:

1. $x^2 + 3x + 2$
2. $x^2 + 3x - 10$
3. $x^2 - 2x + 1$
4. $9x^2 - 1$
5. $x^2 - 7x - 30$
6. $x^2 + 6x + 9$
7. $y^4 - 16$
8. $x^2 + x - 20$
9. $x^2 + 12x + 20$
10. $2x^2 + 12x + 16$
11. $6x - 7 + x^2$
12. $-x^2 + 9x - 20$

1.8.5 Algebraic fractions

We have so far considered adding, subtracting and multiplying polynomials. Of course, it is also possible to divide polynomials, and this gives rise to *algebraic fractions*. An algebraic fraction is simply a fraction which includes letters as well as numbers. For example:

$$\frac{x^2y^2}{xy} \quad or \quad \frac{p^3+4p^2-7p+1}{p^2+8}$$

The rules of indices apply to algebraic fractions, and can be used to simplify the fractions. For example, to simplify the following fraction, we work on the numbers, and individual letters separately.

$$\frac{16x^3y^2z}{8xy^3z^2} = \frac{16}{8} \, . \, \frac{x^3}{x} \, . \, \frac{y^2}{y^3} \, . \, \frac{z}{z^2} = 2 \, . \, x^2 \, . \, \frac{1}{y} \, . \, \frac{1}{z} = \frac{2x^2}{yz}$$

If we want to multiply together algebraic fractions, we multiply the denominators and multiply the numerators as when multiplying numeric fractions (see section 4). For example:

$$\frac{4x^2y}{3z^3} \cdot \frac{6xyz}{10x^3} = \frac{4x^2y \, . \, 6xyz}{3z^3 \, . \, 10x^3} = \frac{24x^3y^2z}{30x^3z^3} = \frac{4y^2}{5z^2}$$

Similarly, to divide algebraic fractions we turn the second fraction upside down and then multiply as with numeric fractions. For example:

$$\frac{4x^2y}{3z^3} \div \frac{6xyz}{10x^3} = \frac{4x^2y}{3z^3} \cdot \frac{10x^3}{6xyz} = \frac{40x^5y}{18xyz^4} = \frac{20x^4}{9z^4}$$

To add (or subtract) fractions we must have a common denominator. This is also true when adding or subtracting algebraic fractions. To find the lowest common multiple of two polynomials which can be used as the common denominator we find the lowest common multiple of any numbers in the polynomials and take the *highest* power of each different letter which appears in either of the two polynomials. This is just like finding the lowest common multiple of two numbers which are written as a product of their prime powers (see section 1.7.2). Once we have found the lowest common multiple of the denominators, we can write both of the fractions as equivalent fractions over this common denominator, and then add (or subtract) the numerators. For example:

$$\frac{2x^3}{5y^2z} + \frac{5x^2}{6z^3}$$

The lowest common multiple of $5y^2z$ and $6z^3$ is $30y^2z^3$ and this will be our common denominator. We multiply the first fraction on top and bottom by $6z^2$ and the second fraction on top and bottom by $5y^2$. This gives us two fractions equivalent to the original two fractions, but with a common denominator.

$$\frac{2x^3 \, . \, 6z^2}{5y^2z \, . \, 6z^2} + \frac{5x^2 \, . \, 5y^2}{6z^3 \, . \, 5y^2} = \frac{12x^3z^2}{30y^2z^3} + \frac{25x^2y^2}{30y^2z^3}$$

Now we can add the numerators and simplify to get the final result:

$$\frac{12x^3z^2 + 25x^2y^2}{30y^2z^3} = \frac{x^2(12xz^2 + 25y^2)}{30y^2z^2}$$

[7]

[7] Note that we cannot simplify this final expression any further. It is a common mistake to try and simplify expressions by adding the powers of unlike terms.

Learning activity

1. Simplify the following algebraic fractions:
   (a) $\frac{xy}{x^2}$
   (b) $\frac{x^2y^2}{xy^3}$
   (c) $\frac{xy}{x^2} \cdot \frac{x^2y^2}{xy^3}$
   (d) $\frac{xy}{x^2} \div \frac{x^2y^2}{xy^3}$
   (e) $\frac{xy}{x^2} + \frac{x^2y^2}{xy^3}$
   (f) $\frac{xy}{x^2} - \frac{x^2y^2}{xy^3}$
2. Show that $\frac{x^2-9}{x^2+6x+9} = \frac{x-3}{x+3}$

1.9 Accuracy

When we are using a calculator to do mathematics, we may get an answer on the calculator display which is something like 45.87836257. It is not usually necessary to give all of these digits when writing down the answer. We usually *round* the answer up or down. It is important to round numbers accurately because small rounding errors can accumulate to produce a larger error in a final answer.

1.9.1 Working to a specified number of decimal places

When we are rounding a number we usually work to a specific number of decimal places. If we are told to give an answer *correct to 2 decimal places*, this means that we include all the digits in front of the decimal point and 2 digits after the decimal point.

If we are working to 2 decimal places, we decide whether to round the number up or down by looking at the 3rd digit after the decimal point. If this digit is 5,6,7,8 or 9 then we round the number up by increasing the 2nd digit after the decimal point by 1 (if the 2nd digit is 9 then this digit becomes 0 and the 1st digit after the decimal point is increased by 1).

For example $7.90643 = 7.91(2d.p.)$ and $93.49812 = 93.50(2d.p.)$ [8]

[8] The $(2d.p.)$ shows that this number has been rounded to 2 decimal places.

If the 3rd digit after the decimal point is 0,1,2,3 or 4, then we round the number down by just chopping off all of the digits from this 3rd digit onwards.

For example $45.87426257 = 45.87(2d.p.)$.

We can round a number to any number of decimal places. We look at the digit immediately after the specified number of decimal places to decide whether to round up or down. We can also round a number to the nearest integer, nearest 10, nearest 100 etc. The rules for rounding up and down are the same. Look at the most significant digit past the rounding up point. If this digit is 5 or above round up, otherwise round down.

For example, we could write 45.87836257 as

50 (nearest 10)
46(nearest integer)
45.9 (1 d.p.)
45.88 (2 d.p.)
45.878 (3 d.p.)
45.8784 (4 d.p.)
45.87836 (5 d.p.)

If we are asked to give a final answer correct to 2 decimal places, then all intermediate calculations should be worked correct to 3 decimal places to avoid a large accumulation of rounding errors. For example, suppose we are asked to calculate $9.97425 \div 0.4517$ correct to 2 decimal places. The actual answer is $22.0815807 = 22.08(2d.p.)$.

If we round the numbers to 2 decimal places before doing the calculation, then we get $9.97 \div 0.45 = 22.15555556 = 22.16(2d.p.)$ which is 0.08 too high.

If we round the numbers to 3 decimal places before doing the calculation, then we get $9.974 \div 0.452 = 22.06637168 = 22.07(2d.p.)$ which is only 0.01 too low and so a much better approximation to the true answer.

This example shows that the more accuracy that is needed, the more decimal places should be used. It is not usually necessary however to give all the decimal places shown in a calculator display. If you are not told in the question how many decimal places to use, 2 is generally enough. If you do round an answer up or down be sure to include the $(2d.p.)$ to show that you have rounded the number.

Learning activity

1. Round the following numbers to the specified number of decimal places:

    (a) 79.367 (to 1 d.p.)

    (b) 864.8976 (to 2 d.p.)

    (c) 34.5634 to the nearest integer

    (d) 765.01 (to 1 d.p.).

2. Explain why a pencil which is 9.6cm long (correct to 1 d.p.) might not fit into a pencil tin which is 10cm long (correct to the nearest cm).

1.10 Learning outcomes

After studying this chapter, you should be able to:

- Represent positive and negative numbers on a number line.
- Add, subtract, multiply and divide directed numbers both with and without a calculator.
- Evaluate mathematical expressions applying the rules of BODMAS.
- Add, subtract, multiply and divide fractions.
- Multiply and divide roots and indices.
- Raise a root to a power and simplify the result.
- Express roots in their exponential form.
- Evaluate roots and indices using a calculator.
- Apply the rules of indices to evaluate or simplify expressions including indices and roots.
- Calculate percentage increase and decreases.
- Express an integer as the product of its prime powers.
- Find the Highest Common Factor and Lowest Common Multiple of two integers.
- Simplify polynomials by collecting like terms.
- Add, subtract and multiply polynomials.
- Expand brackets using the Distributive Law.
- Factorise polynomials by finding a common factor.
- Factorise quadratics of the form $x^2 + bx + c$ and $x^2 - y^2$.
- Reduce an algebraic fraction to its simplest form by cancelling through the highest common factor.
- Add, subtract, multiply and divide algebraic fractions.
- Round numbers up or down to a specified level of accuracy.

Chapter 2

Linear equations and graphs

Essential reading

See Chapter 2 of *Dowling* for many further examples of the material covered in this chapter. Attempt to answer all of the supplementary problems 2.35 to 2.55.

Additional reading

For further exercises on straight line graphs, see Sections 3.1, 3.2 and 3.3 of *Countdown to Mathematics volume 1*.

This chapter introduces the mathematics of linear equations and their graphs. After studying this chapter, you should be familiar with the equation of a straight line graph in the form $y = mx + c$. You should be able to draw a straight line graph given its equation and alternately, find the equation of a straight line given its graph. We will describe situations in business which can be modelled by straight line graphs, and apply the mathematics of linear equations to these business settings.

2.1 Equations

An *equation* is a mathematical statement which sets two algebraic expressions equal to one another. For example:

$$\begin{aligned} 3+8 &= 16-5 \\ a &= b+6 \\ y &= mx+c \\ y &= x^2+3x+7 \end{aligned}$$

If $a = b$ then so long as you perform the same mathematical operation on both a and b, the two sides of the equation will still be equal to one another. Thus:

$$\begin{aligned} a &= b \\ a+c &= b+c \\ a-c &= b-c \\ a\,.\,c &= b\,.\,c \\ \frac{a}{c} &= \frac{b}{c} \end{aligned}$$

Note that the letter c above could represent a number or a whole algebraic expression.

2.2 Re-arranging formulae

We often need to re-arrange equations, to make them easier to solve, or to plot as graphs. This is called *changing the subject*. We can change the subject of an equation by performing mathematical operations on both sides of the equation which make things *cancel* on one side and appear instead on the other side.

Example 1

We want to re-arrange the following equation to make c the subject:

$$y = mx + c$$

We can subtract mx from both sides:

$$y - mx = \cancel{m}\cancel{x} + c - \cancel{m}\cancel{x}$$

which leaves us with:

$$y - mx = c$$

and this is the same as:

$$c = y - mx$$

Example 2

Now suppose we start again with the original equation $y = mx + c$ but this time we want to make x the subject. This time we subtract c from both sides of the equation, and then divide both sides by m as follows:

$$\begin{aligned} y &= mx + c \\ y - c &= mx + \cancel{c} - \cancel{c} \\ y - c &= mx \\ \frac{y-c}{m} &= \frac{\cancel{m}x}{\cancel{m}} \\ \frac{y-c}{m} &= x \\ x &= \frac{y-c}{m} \end{aligned}$$

In both of the examples we have looked at so far, each variable (letter) has only appeared in the equation once. Sometimes however, the same variable appears in more than one place in an equation. Look at the following example where we are trying to make x the subject.

Example 3

$$10 + 2x - 3y = 4x - 4 - 7y$$

First we move all the x values to the right-hand side and all the other values (or *terms*) to the left-hand side.

$$10 + 4 - 3y + 7y = 4x - 2x$$

Next we add (or subtract depending on the signs) *like terms* to give

$$14 + 4y = 2x$$

Now dividing both sides by 2 we find the answer is $x = 7 + 2y$.

Example 4

In this final example, we want to make x, which is currently a denominator, the subject of the equation.

$$\frac{3}{x} - 4y = 7$$

First we add $4y$ to both sides to get the term including the x on its own on the left-hand side:

$$\frac{3}{x} = 7 + 4y$$

Now we multiply both sides by x. Remember that we must multiply all of the terms in the equation in order to preserve the equality. We use brackets on the right-hand side to show that both 7 and $4y$ are being multiplied by x. On the left-hand side, the x in the denominator will cancel with the multiplying x:

$$3 = x(7 + 4y)$$

Finally we divide both sides by the expression $(7 + 4y)$ so that this will cancel on the right-hand side and end up as the denominator on the left. Thus we are left with:

$$\frac{3}{7 + 4y} = x$$

Learning activity

$T = 2x + 3y$

1. Express x in terms of T and y.
2. Make y the subject.

2.2.1 Solving equations

If we are given an equation with one unknown value such as $5x + 4 = 14$ then we can *solve* the equation to find the solution. In this example the solution is $x = 2$. We can find this solution by trying different values of x in the right hand side until the equation is correct. When $x = 2$, $5x + 4 = 5 \,.\, 2 + 4 = 14$ which is the value on the right-hand-side of the equation, so that $x = 2$ must be the correct solution.

To solve more complicated equations, we can re-arrange the equation to make the unknown letter the subject. We can then work out the other side to find the solution. For example, suppose we want to solve the following equation:

$$\frac{2x}{3} + 1 = \frac{x}{4} + 6$$

If we re-arrange the equation to make x the subject, then the solution $x = 12$ is easy to find:

$$\begin{aligned} \frac{2x}{3} - \frac{x}{4} &= 6 - 1 \\ \frac{8x - 3x}{12} &= 5 \\ 8x - 3x &= 5\ .\ 12 \\ 5x &= 60 \\ x &= \frac{60}{5} \\ x &= 12 \end{aligned}$$

Learning activity

Solve the following equations i.e., find the value of x which makes the equations correct.

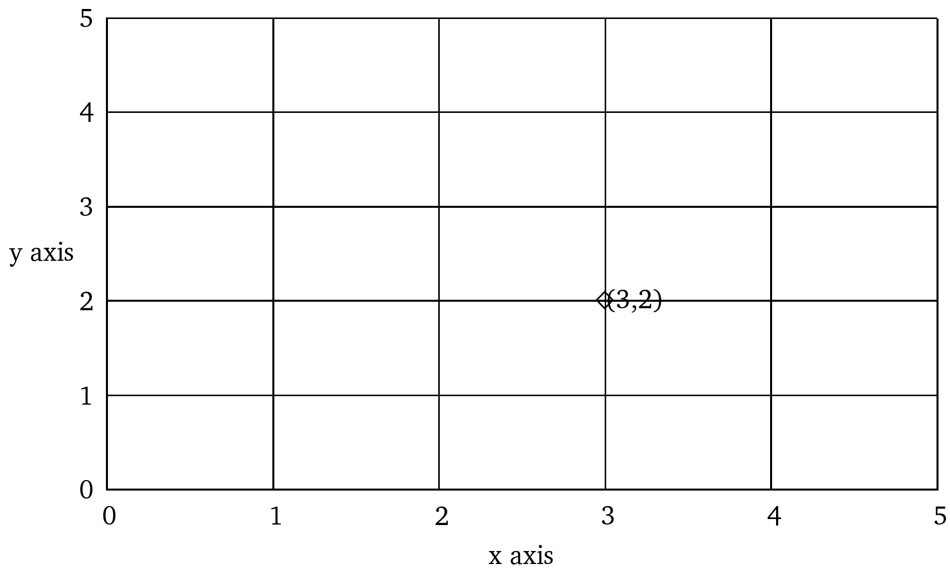
1. $2x = 3 - x$
2. $10 + 2x = 3x - 4$
3. $-5x = -x + 12$
4. $2(x - 1) = 3(x + 2)$
5. $5 - 4(3 - 2x) = 5(3x - 2)$
6. $\frac{x}{4} - 3 = \frac{x}{12} + 1$
7. $\frac{x}{3} - \frac{x}{5} = 2$
8. $\frac{1}{x} - 3 = -1$
9. $\frac{1}{1+x} = 1$
10. $\frac{2}{5} + \frac{3}{x+1} = \frac{9}{10}$

2.3 Graphs

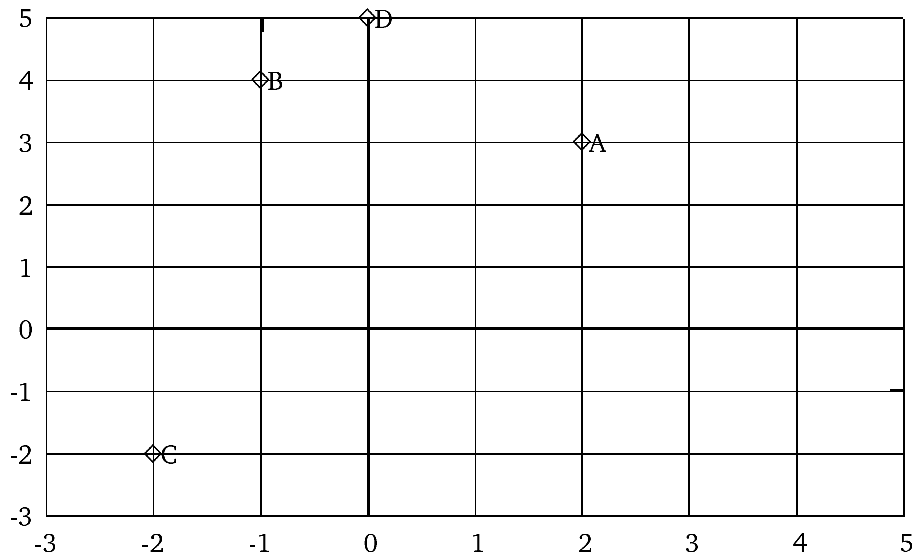
Graphs are used to represent data pictorially. Graphs are usually plotted on a grid of horizontal and vertical lines. The *axes* of the graph are a horizontal line called the *x-axis* and a perpendicular vertical line called the *y-axis*. The x-axis and the y-axis are usually labelled with the scale of the graph.

2.3.1 Plotting points

A *point* on the graph is represented by a pair of numbers (x, y), where x and y represent the distance the point is along the x-axis and y-axis respectively. For example, the graph below shows the point $(3, 2)$.



Given a graph with a point marked on it, we can find the *co-ordinates* of the point, i.e., the x and y values of the point, by measuring how far the point is along the x and y axes respectively. For example, the graph below shows points A, B, C and D.



The co-ordinates of these points are $A = (2, 3)$, $B = (-1, 4)$, $C = (-2, -2)$ and $D = (0, 5)$.

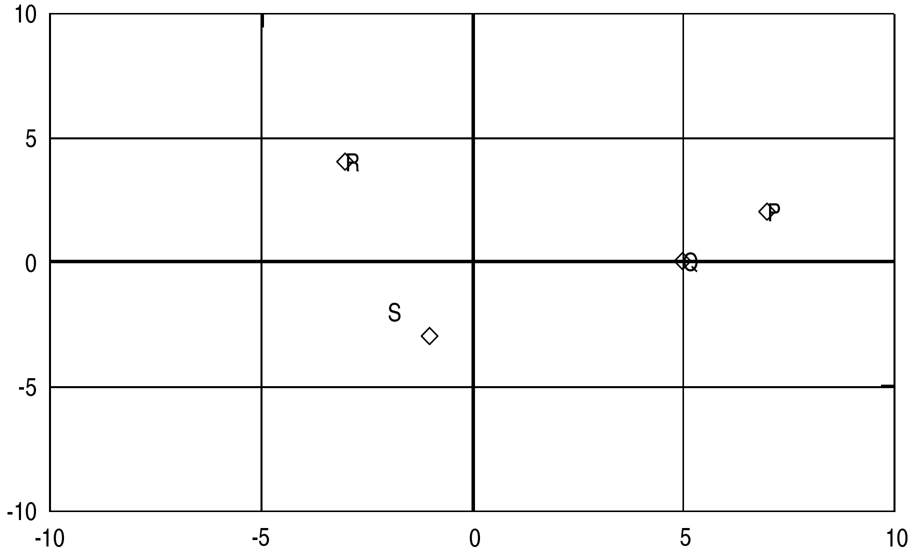
The x-axis crosses the y-axis at the point $(0, 0)$. The point $(0, 0)$ is sometimes called the *origin*. This method of plotting points on graphs is called the *Cartesian Co-ordinate System*.

1

Learning activity

On the graph below:

1. Find the co-ordinates of the points P, Q, R, S writing them in the form (x, y).
2. Plot the points $A = (4, 0), B = (-4, 5), C = (4, -5), D = (-6, -2)$.



2.3.2 Drawing straight line graphs from equations

Suppose an equation has 2 variables, x and y say. We can find pairs of x and y values that *satisfy* the equation i.e., when we substitute these values for x and y into the equation then the left-hand-side equals the right-hand-side. We can plot points (x, y) using these pairs of x and y values. This is called plotting the graph of the equation.

If there are no 'powers' in the equation i.e., no terms which look like x^2 or x^3 and no terms with variables multiplied together i.e. no terms like xy, then the equation is called a *linear equation*. The graph of a linear equation is always a straight line.

[1] When you are drawing a graph by hand it is usual to mark the scale on the axes i.e., on the horizontal and vertical lines through $(0, 0)$. This may mean that the scales are written through the middle of the graph rather than at the edges as in the computer generated graphs in the study guide. Do not confuse the line where the scales are written with the axes.

Example

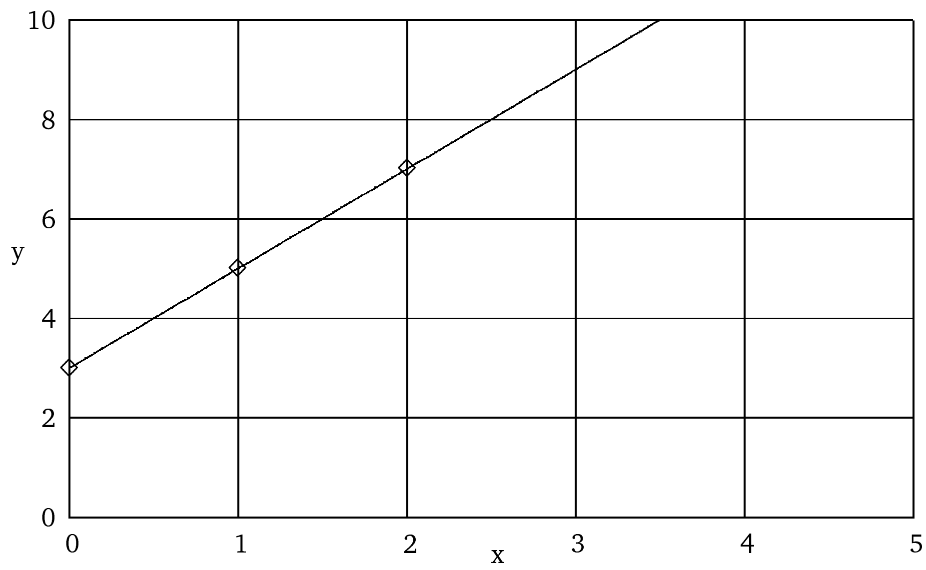
Consider the equation:

$$y = 2x + 3$$

We can find pairs of (x, y) values that satisfy the equation by giving x a value and then calculating the value of y. If we plot the (x, y) values of the equation on a graph then we should be able to join up the points to form a straight line. First we find three points by letting $x = 0, 1, 2$ in turn.

$$x = 0, \quad y = (2 \,.\, 0) + 3 = 3, \quad (0, 3)$$
$$x = 1, \quad y = (2 \,.\, 1) + 3 = 5, \quad (1, 5)$$
$$x = 2, \quad y = (2 \,.\, 2) + 3 = 7, \quad (2, 7)$$

Now when we plot the three points $(0, 3), (1, 5), (2, 7)$ on a graph, we see that they lie on a straight line. We can join up the three points and label the line $y = 2x + 3$.



[2]

[2] Note that when you are drawing a graph of a linear equation you should always find and plot three points on the line. You may think that you only need two points since two points are enough to draw a line. However, it is always best to plot three points, then if they are not in a straight line you know that you have made a mistake.

Learning activity

Draw the lines of the following equations by finding and plotting three points on each line. Use a graph with x and y axis both running from -10 to +10.

1. $y = x - 1$
2. $2y = x + 2$
3. $y = 1 - 3x$

2.3.3 Finding the equation of a straight line graph: $y = mx + c$

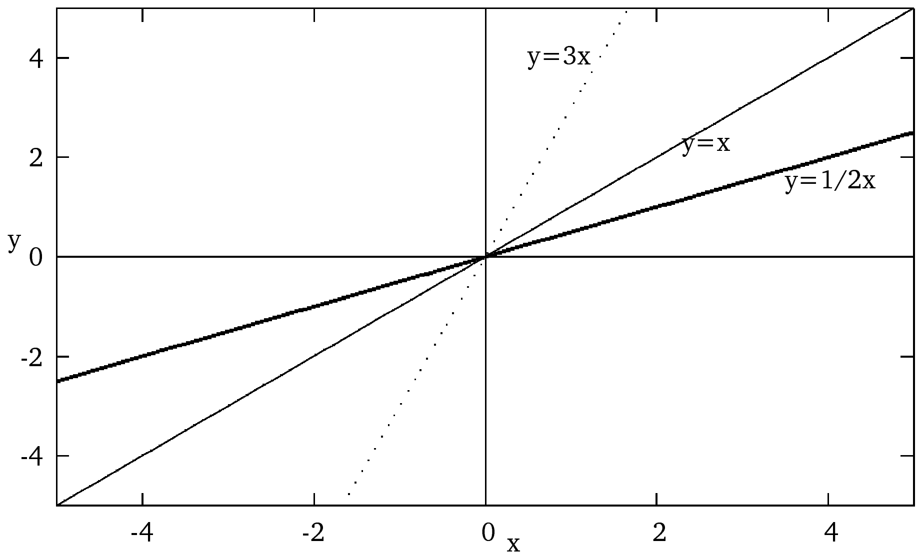
The equation of a linear graph can always be re-arranged so that it is of the form:

$$y = mx + c$$

The values m and c are *constants* i.e. numbers which do not change. When the equation of a straight line is written in the form above, the number m tells us how steep the line is and the number c tells us the position of the line.

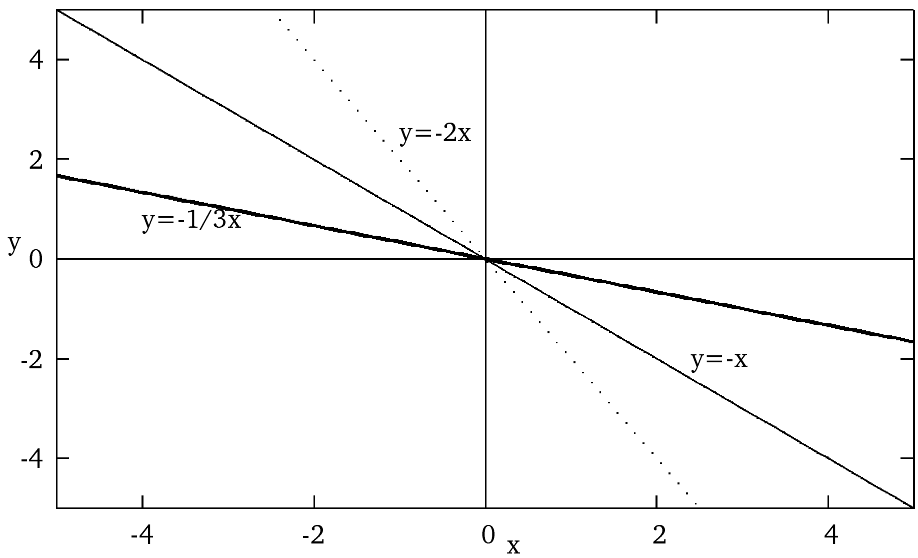
2.3.4 Gradient

When the equation of a straight line is written in the form $y = mx + c$ then the constant m represents the *gradient* of the line. The gradient is a measure of how steep the line is. Assume that the x and y axes have the same scale. Then if a line has a gradient of 1, the line will go up one square on the y-axis for every one square along on the x-axis. If the gradient is greater than 1, then the line will be steeper. If the gradient is between 0 and 1 then the line will be less steep. The graph below shows the lines $y = x$ which has gradient $m = 1$, $y = 3x$ which has gradient $m = 3$ and is much steeper, and $y = \frac{1}{2}x$ with gradient $m = \frac{1}{2}$ which is shallower.



All of the lines above have *positive gradient*. This means that they slope upwards from left to right.

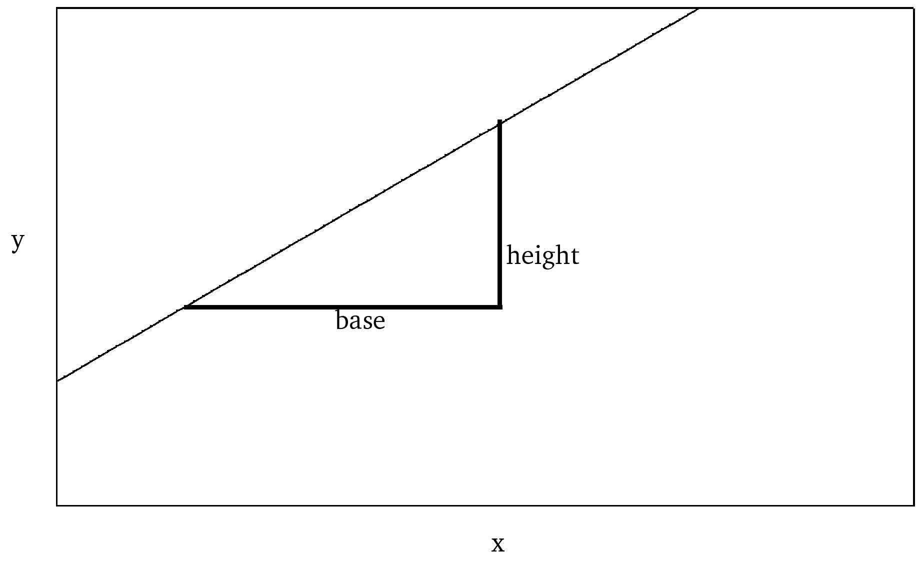
A line may also have *negative gradient* in which case the line will slope downwards from left to right as in the following graph.



The preceeding graph shows the lines $y = -x$ which has gradient $m = -1$, $y = -2x$ which has gradient $m = -2$ and is twice as steep, and $y = \frac{-1}{3}x$ which has gradient $m = \frac{-1}{3}$ and is very shallow.

Finding the gradient

Given a straight line drawn on a graph, we can find the gradient of the line by drawing a right-angled triangle underneath the line as in the picture below.

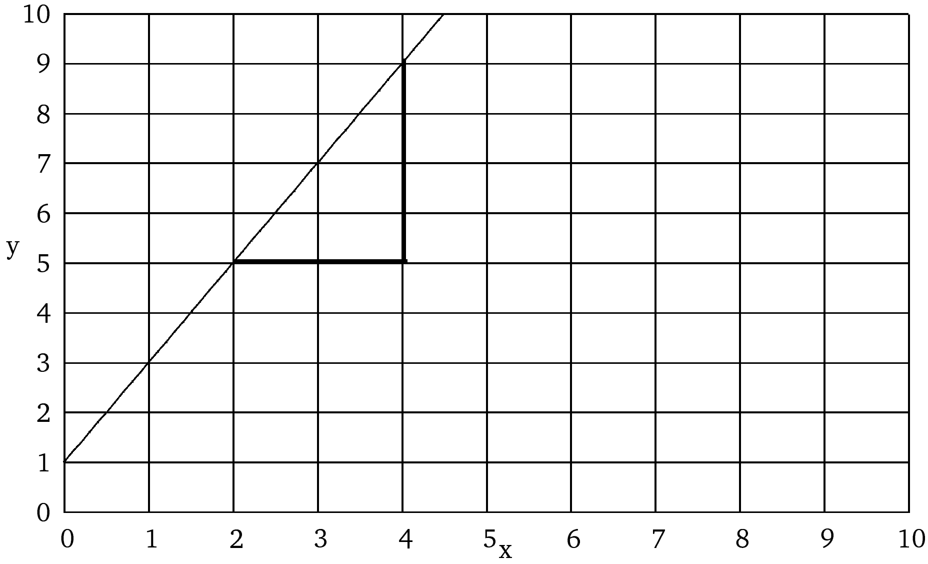


We can measure the height and base of the triangle using the scales on the y and x axes respectively. The gradient of the line can be found by dividing the height by the base. We make the gradient positive or negative depending on whether the line is sloping upwards or downwards from left to right. Note that it does not matter where you draw the triangle on the line, or how big the triangle is. Try and position the triangle so that the sides are easy to measure.

Example 1

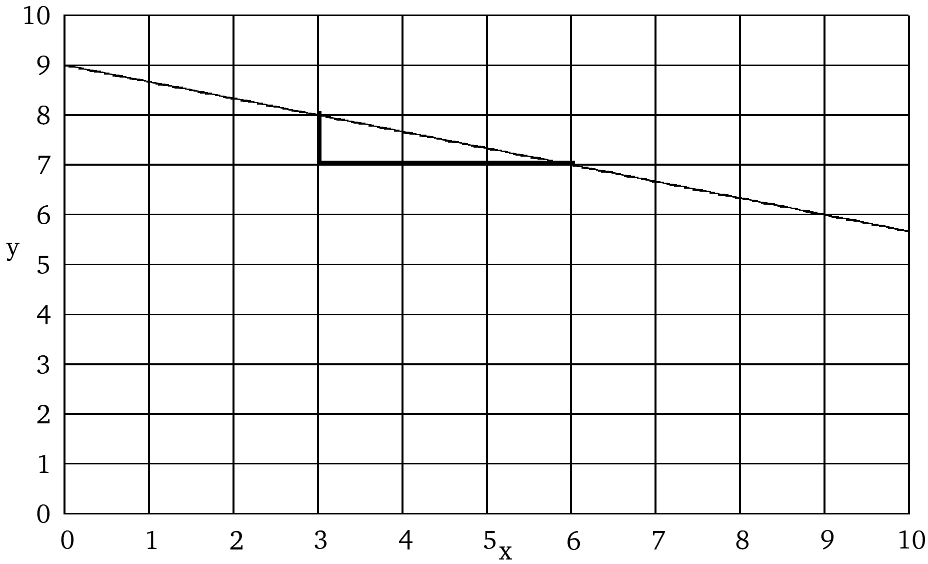
Consider the following straight line graph. We can use the triangle drawn below the line to find the gradient. The height of the triangle is 4 and the base of the triangle is 2. The line is sloping upwards so the gradient is positive and is given by

$$gradient = \frac{height}{base} = \frac{4}{2} = 2$$



Example 2

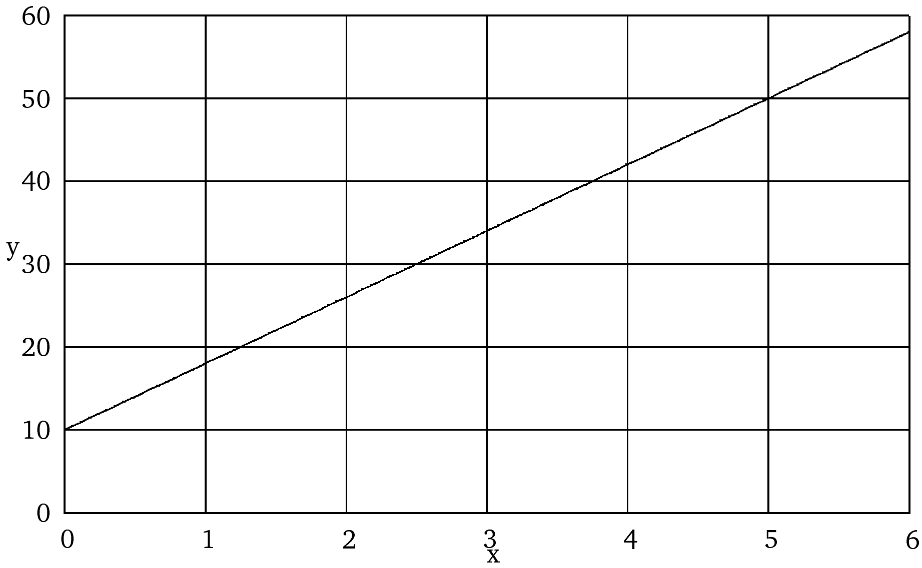
In the graph below, the line is sloping downwards. This time the gradient is negative and using the triangle below the line, we see that the gradient is $m = -\frac{1}{3}$.



Example 3

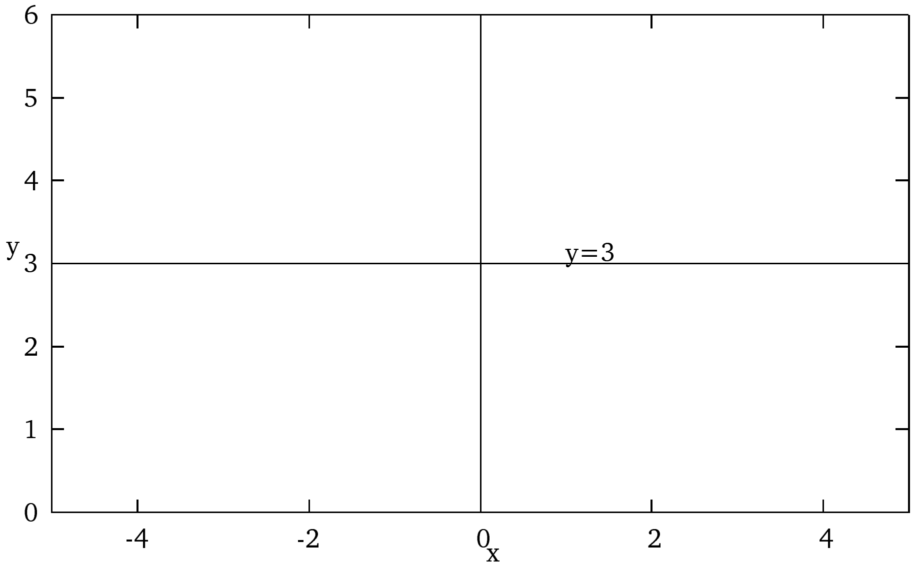
Be wary if the scales on the x and y axis are not the same. The measurements of height and base on the triangle used to find the gradient must be taken from the scales of the axes - not measured with a ruler. If the scales on the two axes are different, then lines which look steep can have a gradient less than 1 and lines which look shallow can have a high gradient.

For example, consider the line drawn on the graph below. The line looks shallow so we would expect a gradient between 0 and 1. However the gradient of this line is actually $m = 8$ because the scales on the x and y axes are different.



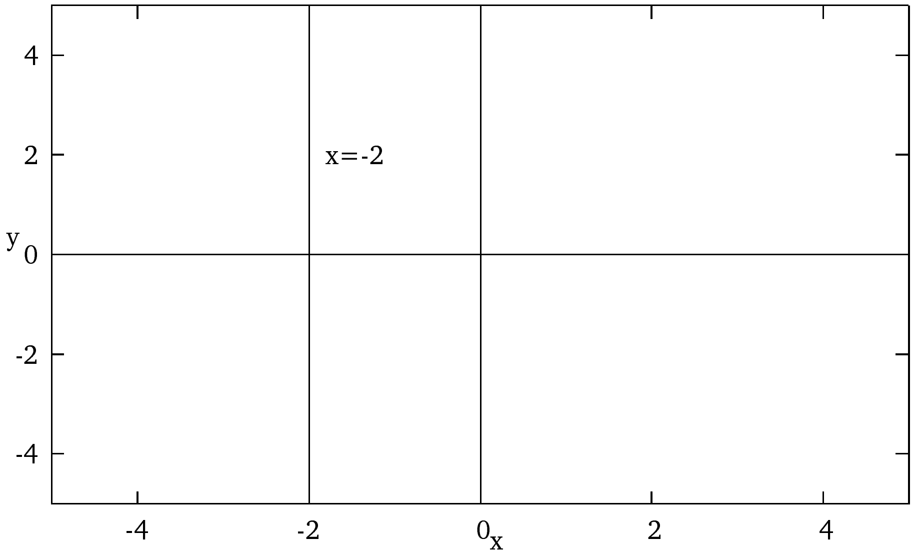
Lines with zero gradient

Sometimes a line has zero gradient, or $m = 0$. This means that it does not slope at all. In the equation $y = mx + c$ we have $m = 0$ and so we are left with an equation of the form $y = c$. Now the value of y does not depend on the value of x since y is always equal to c. Therefore a line with equation $y = c$ is a line parallel to the x-axis cutting the y-axis at c. For example, consider the line below which has equation $y = 3$.



Lines with infinite gradient

We have seen that a line may have zero gradient and that such lines are parallel to the x-axis. We may also have a line which is parallel to the y-axis. This happens when the equation of the line is of the form $x = a$. Now the value of x does not depend on the value of y but is always equal to the constant a. This gives us a vertical line which cuts the x-axis at a. For example, consider the line below which has equation $x = -2$.



Vertical lines like this have *infinite gradient*. If you tried to draw a triangle to measure the gradient then you would have a base measurement of 0 and dividing by 0 is impossible.

Learning activity

Decide which of these lines are parallel to each other. You may have to draw the lines to help you.

$$\begin{aligned} y &= x + 4 \\ y &= \frac{2}{3}x \\ y &= 2 - x \\ y + 3 &= 3 \\ 3y &= 2x + 6 \\ y &= x + 5 \end{aligned}$$

Can you determine a rule which tells you whether or not two straight lines are parallel just by looking at their equations?

2.3.5 Intercept

Remember that the equation of a straight line can always be written in the form $y = mx + c$ where the value m is the gradient of the line.

The value c is called the *intercept* and this tells us where the line crosses the y-axis (the vertical line $x = 0$). The point $(0, c)$ always lies on the line $y = mx + c$.

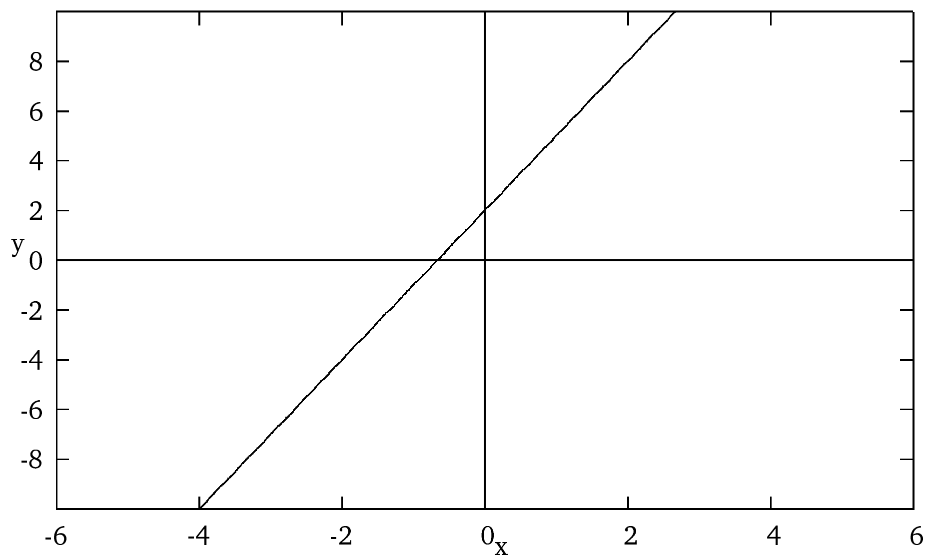
Given a diagram of a straight line, you can usually find the value of c simply by reading the value where the line cuts the line $x = 0$.

If the line goes through the origin $(0, 0)$ then the intercept is at $c = 0$ and the equation takes the form $y = mx$.

2.3.6 Finding the equation for a given line

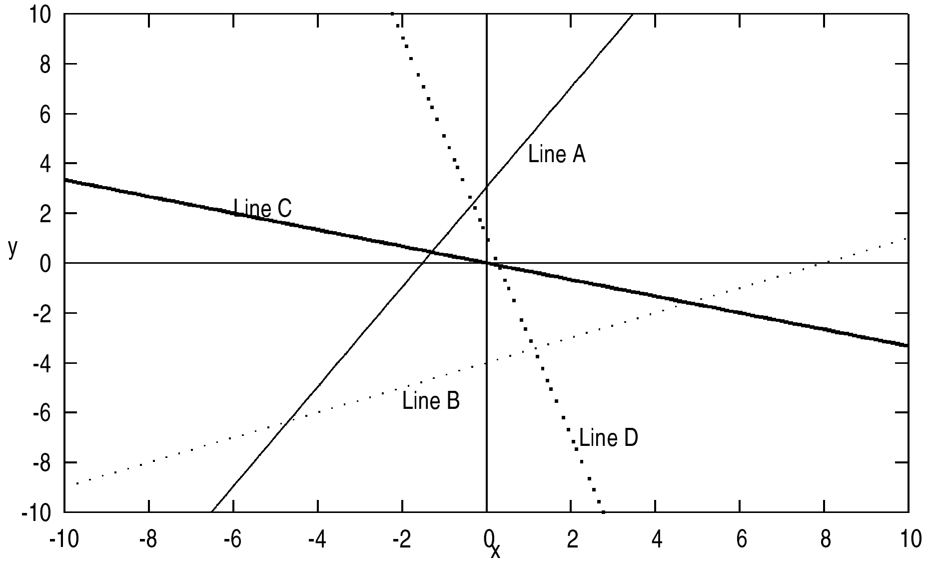
We are now in a position to find the equation, in the form $y = mx + c$ of a line drawn on a graph. We find the gradient, m, by drawing a triangle underneath the line to determine how steep it is, and the intercept, c, by finding the point where the line cuts through the y-axis.

For example, the graph below shows a straight line. By drawing a triangle underneath the line, we can find that the gradient, which is positive, is $m = 3$. The line cuts through the y-axis at 2 and therefore the equation of the line is $y = 3x + 2$.



Learning activity

Find the equations of the four lines drawn on the graph below.



Finding the equation given two points

We can also find the equation of a line if we are given the co-ordinates of any two points on the line. Suppose we are given two distinct points (x_1, y_1) and (x_2, y_2). Then we can find the gradient m of the line using the following formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

In effect we are using the same method to find the gradient as when we draw a triangle underneath a line. The height of the triangle is $y_2 - y_1$ and the base of the triangle is $x_2 - x_1$.

Now that we know the value of m, we can find c by substituting the values $x = x_1$, $y = y_1$ and m into the equation. We are left with only one unknown value c. Therefore we can re-arrange the equation to find the value of c.

Example

We can find the equation of the line which goes through the points $(1, 7)$ and $(4, -2)$ as follows:

The gradient of the line is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 7}{4 - 1} = \frac{-9}{3} = -3$$

Now we substitute $m = -3$, $x = 1$, $y = 7$ into the equation $y = mx + c$ to find c.

$$\begin{aligned} y &= mx + c \\ 7 &= (-3 \,.\, 1) + c \\ 7 &= -3 + c \\ c &= 10 \end{aligned}$$

Thus the equation of the straight line through the points $(1, 7)$ and $(4, -2)$ is:

$$y = -3x + 10$$

Learning activity

Find the equations of the lines which go through the following pairs of points.

1. $(1, -5)$ and $(-4, 5)$
2. $(2, 18)$ and $(-3, -7)$
3. $(4, 3)$ and $(-4, 1)$

Finding the equation of a line given one point and the gradient

If we are given one point (x_1, y_1) and the gradient of a line, then we can find the equation of the line in a similar method to that described above.

Example

Find the equation of the line with gradient $\frac{1}{2}$ which goes through the point $(-4, 3)$.

We substitute the values $x = -4$, $y = 3$ and $m = \frac{1}{2}$ into the equation $y = mx + c$ in order to find the value of c as follows:

$$\begin{aligned} y &= mx + c \\ 3 &= (\frac{1}{2} \,.\, -4) + c \\ 3 &= -2 + c \\ c &= 5 \end{aligned}$$

Thus the equation of the line with gradient $\frac{1}{2}$ through the point $(-4, 3)$ has equation

$$y = \frac{1}{2}x + 5$$

or, multiplying all the terms by 2 to get a ‘neater’ solution:

$$2y = x + 10$$

Learning activity

Find the equation of the line which has gradient -2 and goes through the point $(5, 6)$.

2.4 Applications to business

Linear equations and graphs can often be used to model business applications. The following section uses examples to show how to represent and interpret data using linear equations.

2.4.1 Production constraints

A business may have certain *production constraints*. For example, a firm may have 200 hours of skilled labour available, or a particular machine may have been hired for only 12 hours. The business will have to decide how to best use the available resources.

Example

A school has hired a badge making machine for eight hours. The machine can make two different types of badge, Badge X and Badge Y. It takes two minutes to produce a badge of type X, and five minutes to produce a badge of type Y.

We can model this information using a linear equation as follows:

$$time = (2 \,.\, Number\ \ of\ \ Badge\ \ X) + (5 \,.\, Number\ \ of\ \ Badge\ \ Y)$$

We know that the total time available is eight hours. We should convert this to $(8 \,.\, 60) = 480$ minutes since the other time data is in minutes. Now using x and y to represent the total number of each type of badge made, we have:

$$480 = 2x + 5y$$

We can re-arrange the equation above into the form $y = mx + c$ as follows:

$$y = -\frac{2}{5}x + 96$$

Now it is easy to see how the production of type Y badges will vary depending on how many type X badges the school produces.

If the school uses the machine to its full capacity and makes 200 type X badges, then how many type Y badges can be produced?

We can find out by substituting $x = 200$ into the equation above and solving to find y:

$$\begin{aligned} y &= -\frac{2}{5} \,.\, 200 + 96 \\ y &= 16 \end{aligned}$$

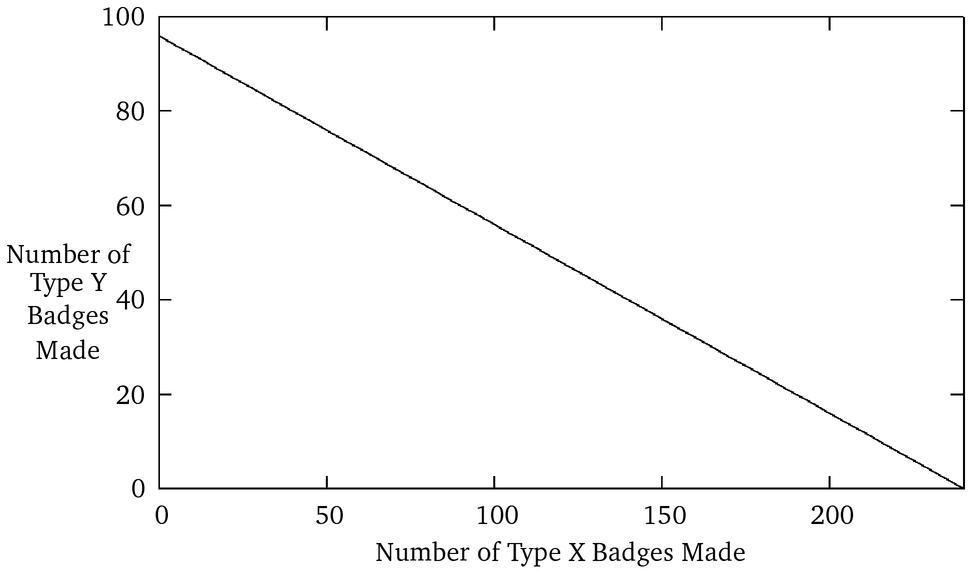
Thus if the school makes 200 badges of type X, it can only make 16 badges of type Y.

Similarly, we can find out how many badges of type X can be made if the school makes 50 badges of type Y:

$$
\begin{aligned}
50 &= -\frac{2}{5}x + 96 \\
-46 &= -\frac{2}{5}x \\
-46 \cdot \frac{5}{2} &= -x \\
x &= 115
\end{aligned}
$$

The school can make 50 type Y Badges and 115 type X Badges.

Modelling the production constraints using a linear graph shows the various possible outcomes depending on how the resources are distributed. This information can be illustrated using a graph as shown below.



Linear equations can also be used to solve the following problem.

Worked example

Essential oils cost \$14 per 10ml of rose and \$4 per 10ml of lavender. A chemist wants to make a blend of rose and lavender that costs \$5 per 10ml. What percentage of the blend should be lavender oil and what percentage should be rose oil?

Solution

We have $100\% = \%$ that is rose $+\%$ that is lavender. If we let x represent the percentage of the blend that is lavender, then (100-x)% is rose.

We can form the following linear equation to solve this problem:

$$5 = \frac{(100-x)}{100} \;.\; 14 + \frac{x}{100} \;.\; 4$$

Here the values 5, 14 and 4 represent the cost of the blend, rose oil and lavender oil respectively. $\frac{(100-x)}{100}$ is the fraction of the blend that is rose oil and $\frac{x}{100}$ is the fraction of the blend that is lavender oil.

We can solve the equation to find x which is the percentage of the blend that is lavender oil.

$$\begin{aligned} 5 &= \frac{(100-x)}{100} \;.\; 14 + \frac{x}{100} \;.\; 4 \\ 500 &= 1400 - 14x + 4x \\ 10x &= 900 \\ x &= 90 \end{aligned}$$

Therefore lavender oil should make up 90% of the blend and rose oil should make up the remaining 10%.

Learning activity

A baker has 2kg of flour. He can make a big cake using 175g of flour, or a smaller cake using 125g of flour.

a) Using x to represent the number of big cakes, and y to represent the number of small cakes model the baker's production constraint in terms of an equation.
b) How many big cakes can the baker produce if he makes five small cakes?

2.4.2 Linear depreciation

Although some goods such as jewels hold their value, many others become worth less and less over a period of time. For example, cars and computing equipment, which are expensive to buy new, lose their value as they get older.

If the value of an item has *linear depreciation* then the value will decrease by a fixed amount each year. This can be modelled by a straight line graph with negative gradient.

Example

The value y in \$ of a computer which is x years old can be modelled by the following linear equation:

$$y = -200x + 1400$$

a) How much was the computer originally worth?

b) How much will the computer be worth after four years?

c) How old will the computer be when it is worth less than \$500?

We can use the linear equation above to answer these questions as follows:

a) By putting $x = 0$ into the equation above, we can find that the initial value of the computer was \$1400.

b) Substituting $x = 4$ into the equation tells us the value of the computer after four years.

$$\begin{aligned} y &= -200 \,.\, 4 + 1400 \\ y &= 600 \end{aligned}$$

After 4 years, the computer is worth \$600.

c) Substituting $y = 500$ into the equation and then re-arranging to make x the subject will tell us how old the computer is when it is worth \$500.

$$\begin{aligned} 500 &= -200x + 1400 \\ x &= 4.5 \end{aligned}$$

When the computer is more than four and a half years old it will be worth less than \$500.

Learning activity

The value of a car has linear depreciation. The car which has an initial value of \$28,650 is worth \$15,000 after four years.

a) Using y to represent the value of the car, and x to represent the time in years, model the value of the car as a linear equation.

b) For how many years will the car be worth more than \$10,000?

2.4.3 Fixed costs and marginal costs

A business may have costs which are fixed, such as office rental, and costs which are variable, such as the price for raw materials. These variable costs may also be called *marginal costs*. If the marginal costs increase consistently i.e. with the number of units produced, then we can again use a linear equation to model the costs of the business.

Example

Suppose a mail order business charges \$3.25 per unit plus a flat fee of \$7.50 per order to cover handling charges and delivery. The charge of \$7.50 is a *fixed cost* and does not change depending on how many units are ordered. The charge of \$3.25 is called a *variable cost* or *marginal cost*. The total cost of an order will depend on how many units are ordered. This can be modelled by the following linear equation:

$$c = 3.25 \,.\, n + 7.50$$

where c represents the total cost of the order and n represents the number of units ordered.

If we wanted to work out the total cost if 15 units were ordered, then we can substitute $n = 15$ into the equation above.

$$c = 3.25 \,.\, 15 + 7.50 = 56.25$$

The 15 units will have a total cost of $56.25.

Alternatively, we may be given the price of two different orders and then be asked to find the fixed cost and price per unit. This is equivalent to being given two points on a straight line and finding the intercept c - which here represents the flat fee since it is the cost charged even if no units are ordered, and the gradient m - which here represents the price per unit since the gradient will tell us how much the price goes up for each extra unit ordered.

Example

A business charges $59.75 for an order of eight boxes of stationery and $40.25 for an order of five boxes of the same stationery. These prices include a flat fee for handling and delivery. We can find the price per box of stationery and the flat fee by writing the information we have as a linear equation as follows:

$$cost = price \;\; per \;\; unit \,.\, number \;\; of \;\; units + flat \;\; fee$$

Substituting the two pairs $(number \;\; of \;\; units = 8, \;\; cost = 59.75)$ and $(number \;\; of \;\; units = 5, \;\; cost = 40.25)$ into this equation we have:

$$\begin{aligned} 59.75 &= 8P + f \\ 40.25 &= 5P + f \end{aligned}$$

where P represents the price per box and f represents the flat fee.

Now subtracting the second equation from the first we have

$$19.50 = 3P$$

This tells us that three boxes of stationery cost $19.50 and therefore each box costs $\$19.50/3 = \$6.50$.

We can now find the fixed cost by substituting $P = 6.5$ into the equation above.

$$\begin{aligned} 40.25 &= (5 \,.\, 6.5) + f \\ 40.25 - (5 \,.\, 6.5) &= f \\ f &= 7.75 \end{aligned}$$

Thus the price per unit is \$6.50 and the fixed cost is \$7.75.

Learning activity

A firm has a fixed running cost of \$8000 and a variable cost of \$230 for each item produced. Find the total cost of producing:

a) 15 units of output
b) 40 units of output.

2.4.4 Profit, revenue and costs

In the previous section, we were able to find the *total costs* incurred by a business to produce a number of items. We have the linear equation:

$$C = Mx + F$$

where C is the total cost, M is the marginal cost per unit produced, x is the number of units produced, and F is the fixed cost.

The business wants to make a profit

$$R = Px$$

where R is the total revenue, P is the price per unit, and x is the number of units sold.

The *profit* of the business is the total revenue minus the total cost. This too can be modelled by a linear equation:

$$\pi = R - C$$

where π is used to represent profit.

In the following example, we make use of all of these linear equations, to find the costs, revenue and profit.

Example

A firm has fixed costs of \$1,850 per month for rent of premises and salaries. It also has marginal costs of \$25 per unit for each unit produced. The firm produces 234 units in one month. The units are sold for \$48 each.

The total cost C can be represented by the following equation:

$$C = 25x + 1850$$

where x represents the total number of units produced.

The total revenue R can be represented by the equation:

$$R = 48x$$

where x represents the total number of units sold.

The profit π can be represented as the difference between the total revenue and the total cost:

$$\pi = R - C$$
$$\pi = 48x - (25x + 1850)$$
$$\pi = 23x - 1850$$

In the month in question, 234 units were produced and sold. Thus the profit for that month can be found by substituting $x = 234$ into the above equation.

$$\pi = (23\ .\ 234) - 1850$$
$$\pi = \$3,532$$

Learning activity

A firm makes toy cars which are sold to a wholesaler for $60 each. The firm has a variable cost of $18 per car plus a fixed cost of $2000. How much profit does the firm make if it sells:

a) 100 cars
b) 250 cars
c) 40 cars?

How many cars does the firm have to manufacture and sell in order to start making a profit?

2.4.5 Budget lines and isocost lines

In section 2.4.1 we considered the constraints put upon a business by limited resources. The business must decide how to best use the resources available. The business must also decide how to best use its available budget.

For example there may be a budget of $n dollars available to spend on two products x and y. Product x costs P_x dollars and product y costs P_y dollars. We can draw a budget line showing all of the different combinations of products x and y that could be purchased with the budget of n dollars.

The equation for this budget line is the following linear equation:

$$n = xP_x + yP_y$$

Example

A teacher has $75 to spend on books and pens. Each book costs $7.50 and each pen costs $2.

The budget line for the teacher is given by the equation:

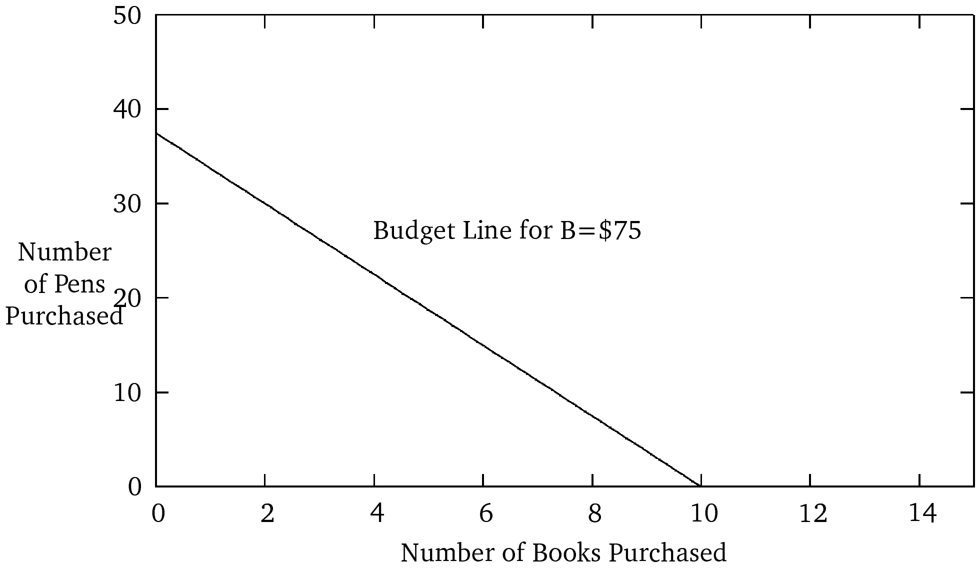
$$75 = 7.5x + 2y$$

where x represents the number of books purchased and y represents the number of pens purchased.

In order to make the budget line easier to draw, we can arrange it into the form $y = mx + c$ as follows:

$$\begin{aligned} 2y &= -7.5x + 75 \\ y &= -3.75x + 37.5 \end{aligned}$$

The graph below shows the budget line of the teacher.

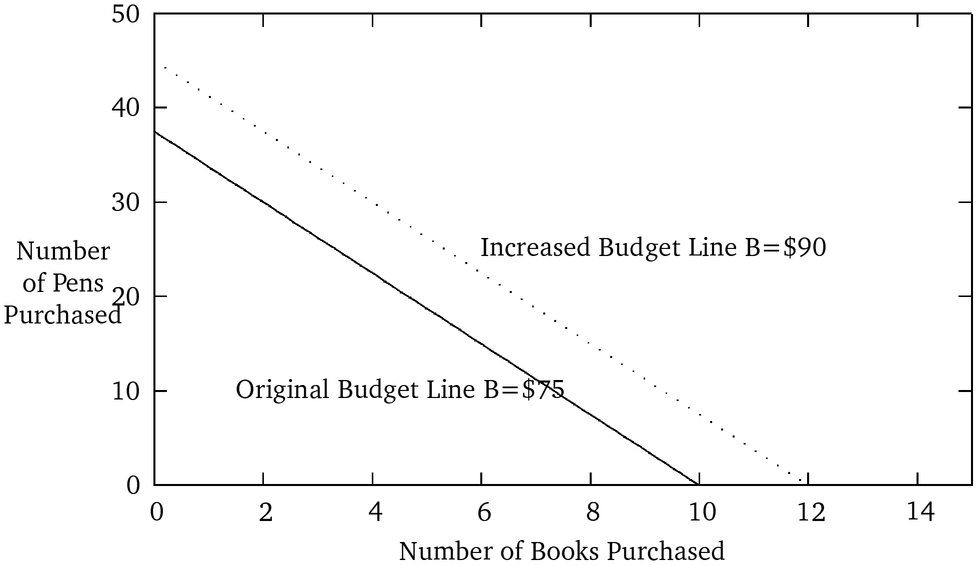


It can also be useful to show on the graph what difference a change in the budget could make.

For example, suppose the teacher's budget is increased by 20%. Now the budget is $75 + (0.2 . $75) = $90. The budget line for this graph is given by the equation:

$$\begin{aligned} 90 &= 7.5x + 2y \\ y &= -3.75x + 45 \end{aligned}$$

The graph below shows the original budget line and the new budget line representing the increased budget.



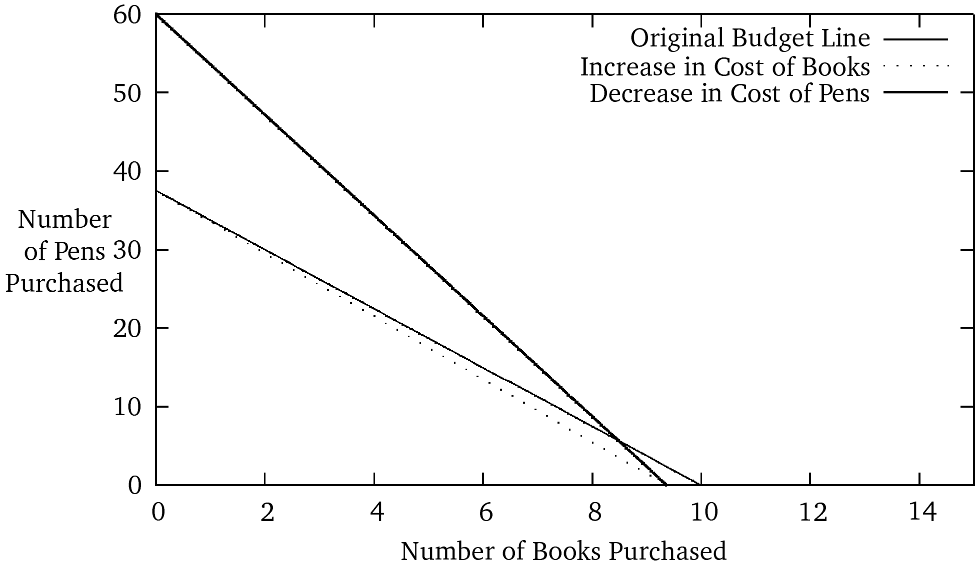
There may also be a change in the cost of the goods to be purchased. For example, if the teacher's budget remains at $75, but the cost of books increases to $8.00 per book then we will have a new budget line given by the equation:

$$\begin{aligned} 75 &= 8x + 2y \\ y &= -4x + 37.5 \end{aligned}$$

The cost of the pens may also change. Suppose now that the teacher still has a budget of $75, that books cost $8.00 each, but the pens are on special offer and their price has been reduced to $1.25 per pen. The equation of the budget line is now given by:

$$\begin{aligned} 75 &= 8x + 1.25y \\ y &= -6.4x + 60 \end{aligned}$$

The graph below shows the original budget line (budget: $75, price of books: $7.50, price of pens: $2), the budget line representing the increased purchase price of books (budget: $75, price of books: $8, price of pens: $2) and the final budget line representing the increased price of books together with the decreased price of pens (budget: $75, price of books: $8, price of pens: $1.25).



A budget line which is used to illustrate the different combinations of *two* goods that can be purchased with a given budget, as in the preceding example, is also called an *isocost line*.

Learning activity

A car can have a diesel engine or a petrol engine.

a) Using x to represent diesel and y to represent petrol, and given that $P_x = 60$ and $P_y = 110$ draw the isocost line for a budget of 1500.

b) Assuming that the price of fuel remains unchanged, draw a new isocost line for a budget decreased by 25%.

c) Draw an isocost line depicting the original budget of 1500 and a 20% increase in the price of petrol.

2.5 Learning outcomes

After studying this chapter and the relevant reading, you should be able to:

- plot and identify points in a Cartesian co-ordinate system
- draw the graph of a given linear equation and find its gradient and intercept
- find the equation of a straight line in the form $y = mx + c$ given:
  - a point on the line and its gradient
  - two points on the line.

You should also be able to apply the above mathematics to solve problems relating to business and economics. In particular you should be able to explain the following concepts:

- production constraints
- linear depreciation
- fixed and marginal costs
- profit
- budget lines and isocost lines.

2.6 Sample examination questions

Question 1

a) A straight line passes through the point with co-ordinates $(2, -6)$ and cuts the x-axis at $x = 3$. Find the equation of this line. The point $(-1, b)$ lies on this line. Find b. [5]

b) A consumption function has equation

$$C = 0.5Y + 12$$

where Y represents income. Sketch the graph of this function. Find the value of C when $Y = 15$ and the value of Y when $C = 20$. [5]

Question 2

a) Giving your answer in the form of $ax + cy = d$, find the equation of the straight line passing through the point $(5, 3)$ and cutting the y-axis at $y = -12$. Where does this line cut the x-axis? Show that $(6, 6)$ also lies on this line. Does the point $(2, -15)$ lie on the line? [5]

b) A company buys peanuts at 60 cents per kilogram and buys cashew nuts at 100 cents per kilogram. It wishes to mix the nuts so that a blend costs 75 cents per kilogram. What percentage of peanuts should be in the blend? [5]

Chapter 3

Functions

Essential reading

See Chapter 3 of *Dowling* for many further examples of the material covered in this chapter. In particular try the supplementary problems 3.33 to 3.51 to test your understanding of functions and their applications.

Functions are often used in business and economics. For example, functions may be used to model the costs or the profits of a business. By working with these business functions, we can deduce when profit will be maximised, how many items should be manufactured and what price they should be sold for.

In this chapter we introduce the idea of functions and show how they may be manipulated. We will look at the graphs of functions; in particular we will see how to sketch the graph of a quadratic function by finding its vertex and roots.

Finally we will apply the mathematics of functions and their graphs to problems relating to business and economics.

3.1 Introduction and definitions

A *function* is a rule which is used to map one number to another. We can think of a function as a type of machine into which you feed the input number. The result of the function with the given input is then the output of the machine.

$$x \longrightarrow \boxed{\mathrm{f}} \longrightarrow f(x)$$

A function is usually denoted by a letter, often f. For example

$$f(x) = 6x - 3$$

is a function which takes input x and outputs $6x - 3$. So if we input 4 into the function f, the output is equal to $6 \,.\, 4 - 3 = 21$. This is written as follows:

$$f(4) = 6 \,.\, 4 - 3 = 21$$

When we input a value x into the function and work out the output of the function, we say that we are *evaluating the function at* x.

For example, if we evaluate the function $f(x) = 3x^2 + 4x + 2$ at $x = 7$, the result is $f(7) = 3(7)^2 + 4(7) + 2 = 177$. So 177 is the value of f at 7.

Every number x which is input into a function must have an unambiguous output. Every time the same number x is input into the function the result must be the same. This means that, for example, $f(x) = \sqrt{x}$ is **not** a function because if we input, say 9 into the function, the result could be either $+3$ or -3 so the output is not unambiguous.

This is not the same as saying that every input x must lead to a different output. For example $f(x) = x^2$ is a function because any value of x input will result in an unambiguous output. However $f(2) = f(-2) = 4$ so the values 2 and -2 both map to the same output.

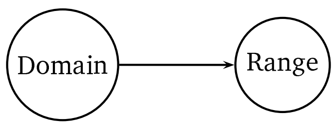
Learning activity

1. $f(x) = 4x^2 + 2x - 9$ find:
   (a) $f(5)$
   (b) $f(-2)$
   (c) $f(0)$.
2. Explain why $f(x) =^4 \sqrt{x}$ is not a function.

3.1.1 Domain and range

The set of numbers which can be input into the function is called the *domain* of the function. The set of numbers which is output by the function is called the *range* of the function.

$$x \longrightarrow \boxed{\text{f}} \longrightarrow f(x)$$



For $f(x) = 6x - 3$ the domain is the set of all real numbers - we can input any number into the function. The range is also the set of all real numbers because depending on the number input, we can make the function output any number.

However the domain and range of a function are not always the same as illustrated in the following examples.

Example 1

Consider the function $g(x) = x^2$. Any real number can be input into the function; however, the output will always be positive (or zero if zero is input) because the square of a number is always positive. (Recall that multiplying two negative numbers gives a positive result.) Therefore the domain of g is the set of all real numbers, but the range of g is the set of all positive real numbers.

Example 2

We can limit the domain of a function and this will have an effect on the range of the function. For example, if the function $h(x) = 3x - 5$ has the domain of *all integers greater than zero*, what is its range?

The output of the function h will be an integer, but not necessarily a positive integer since $h(1) = 3 \,.\, 1 - 5 = -2$. There cannot be an output less than -2 and so the range of this function is *all integers greater than* -3.

Example 3

Sometimes, a number, or a set of numbers, must be excluded from the domain of a function because including them would result in a division by zero. For example consider the following function:

$$f(x) = \frac{2x - 7}{x + 1} \quad (x \neq -1)$$

We have to specify that the input $x \neq -1$ since this would result in attempting to divide by zero.

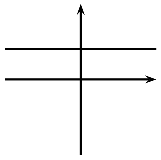
Learning activity

1. Find the range of the function $f(x) = x^2 - 10$ given that:
   (a) The domain is the set of all real numbers.
   (b) The domain is the set of all real numbers ≥ 0.
   (c) The domain is the set of integers $\{-3, -2, -1, 0, 1, 2, 3\}$.
2. Specify a suitable domain for the function $f(x) = \frac{x^2+4}{x-3}$

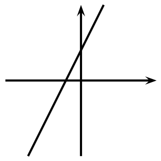
3.1.2 Types of function

There are several different types of function. These include:

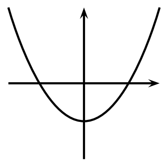
- The **constant** function, for example $f(x) = 4$. Regardless of the input to this function, the output is always equal to 4.



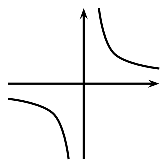
- The **linear** function, for example $f(x) = 7x + 5$. Linear functions can be represented by straight line graphs.



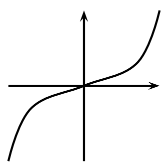
- The **quadratic** function, for example $f(x) = x^2 + 3x - 4$. We will discuss quadratic functions in detail in section 3.3.



- The **rational function**, for example $f(x) = \frac{3x-2}{4x+8}$. When defining a rational function, it is necessary to exclude from the domain any value of x which makes the denominator equal to zero.



- The **power** function, for example $f(x) = x^5$ or $g(x) = x^{-2}$ $(x \neq 0)$.



3.2 Sketching linear functions

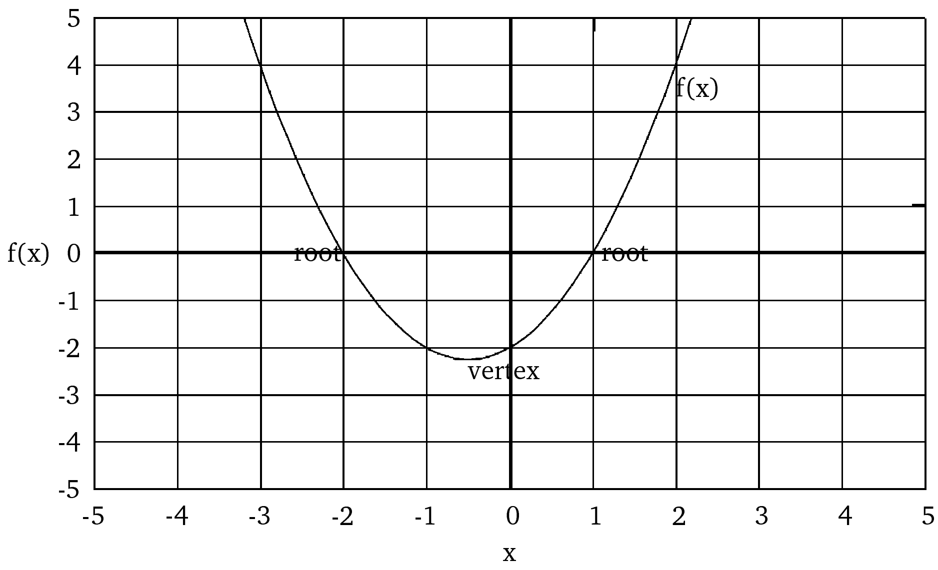
It is often useful to sketch the graph of a function plotting the input values on the x-axis and the output values on the y-axis. This makes it easier to see what happens to the output as the input varies. We can also see from the graph what input is needed to produce a particular output. For example, if the function is representing profit then we can find the *break-even* point.

The graph of a linear function, f, is always a straight line, so plotting three points $(x_1, f(x_1)), (x_2, f(x_2))$ and $(x_3, f(x_3))$ and joining them with a straight line gives the graph of the function. We sketched several graphs of linear functions in section 2.4.

The graph of a constant function $f(x) = a$ is also a straight line. The line is parallel to the x-axis and passes through the point $(0, a)$ on the y-axis.

3.3 Sketching quadratic functions

The graph of a quadratic function is a $\cup$ or $\cap$ shape called a *parabola*. The shape will be $\cup$ if the leading term of the quadratic is x^2, and $\cap$ if the leading term is $-x^2$. The points where the parabola cuts through the x-axis are called the *roots* of the quadratic. The parabola is symmetrical about its *turning point* which is also called the *vertex*.



We can sketch the graph of a quadratic function f by:

- deciding whether the parabola is a $\cup$ or a $\cap$ shape;
- finding the roots - these will tell us where the graph cuts the x-axis;
- finding the value of $f(0)$ - this will tell us where the graph cuts the y-axis;
- finding the vertex - this will tell us where the graph has its turning point.

Finding the roots of a quadratic function

In section 1.8.4 we learned how to factorise quadratics. That work will be important in this section.

The roots of a quadratic are the points where the graph cuts the x-axis. This means that the roots are the inputs to the function which evaluate to zero. When the quadratic has been written as a product of its linear factors (i.e., after it has been factorised), it is easy to find the roots - they will be the values that make the linear factors equal to zero. This is because when two (or more) numbers multiply together to give zero, then at least one of the numbers must itself be equal to zero.

If we have $f(x) = (x + a)(x + b) = 0$ then either $(x + a) = 0$ in which case $x = -a$ or $(x + b) = 0$ in which case $x = -b$. Hence $f(-a) = 0$ and $f(-b) = 0$. The graph of the function $f(x)$ will cut the x-axis at $-a$ and $-b$ and these are the roots of the function.

For example, consider the following quadratic function:

$$f(x) = x^2 + 6x - 7$$

This quadratic can be factorised into the linear factors $(x + 7)$ and $(x - 1)$ so we can re-write the function as:

$$f(x) = (x + 7)(x - 1)$$

If $f(x) = 0$ then either $(x + 7)$ or $(x - 1)$ must equal zero.

If $x + 7 = 0$ then we must have $x = -7$. Alternatively if $x - 1 = 0$ then we must have $x = 1$. These are the two roots of the quadratic: $x = -7$ and $x = 1$. We can check this by evaluating the original function $f(x) = x^2 + 6x - 7$ at these two values - the result should be equal to zero.

$$\begin{aligned} f(-7) &= (-7)^2 + 6(-7) - 7 = 49 - 42 - 7 = 0 \\ f(1) &= (1)^2 + 6(1) - 7 = 1 + 6 - 7 = 0 \end{aligned}$$

So now we know that the graph of the function $f(x) = x^2 + 6x - 7$ cuts the x-axis at the points $(-7, 0)$ and $(1, 0)$.

By putting $x = 0$ into the function we can find $f(0)$ and this will tell us where the graph cuts though the y-axis.

$$f(0) = 0^2 + 6(0) - 7 = -7$$

The graph of the function cuts through the $y - axis$ at the point $(0, -7)$.

Finding the vertex of a quadratic function

The graph of a quadratic function is symmetrical. Its two halves are reflections of each other about the line which is parallel to the y-axis and cuts through the x-axis midway between the two roots. The vertex of the quadratic will also lie on this line. Therefore, to find the x co-ordinate of the vertex we need to find the midpoint of the two roots. This is done by adding together the roots and then dividing the result by 2.

The function $f(x) = x^2 + 6x - 7$ has roots -7 and $+1$. The midpoint of these roots is $(-7 + 1) \div 2 = -6 \div 2 = -3$. Thus the x co-ordinate of the vertex is -3. To find the y co-ordinate we evaluate the function at the point $x = -3$:

$$f(-3) = (-3)^2 + 6(-3) - 7 = 9 - 18 - 7 = -16$$

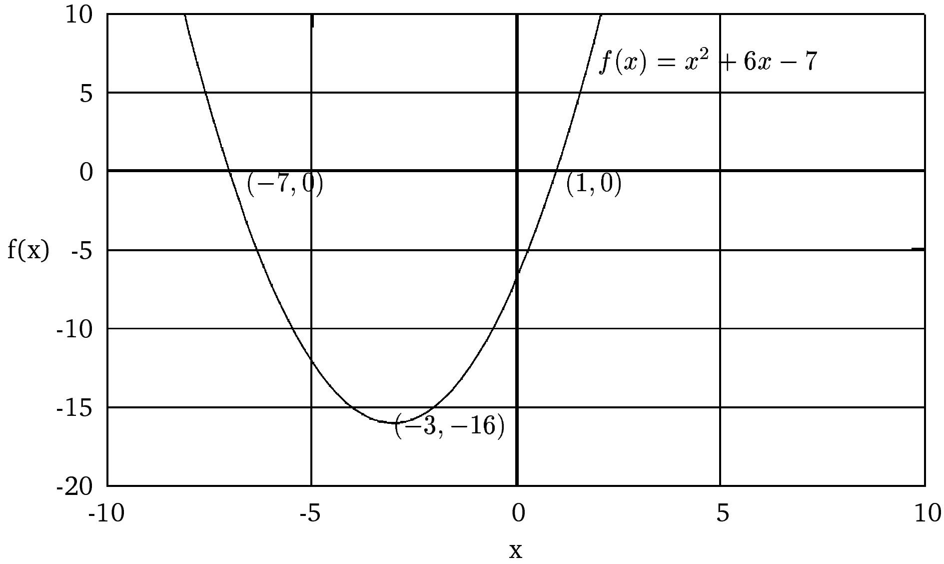
Therefore the turning point of the function is at the point $(-3, -16)$.

Sketching the quadratic function

We now know that our function $f(x) = x^2 + 6x - 7$ cuts the $x - axis$ at -7 and $+1$, cuts the y-axis at -7 and turns at the point $(-3, -16)$.

We also know that the shape of the graph will be $\cup$ because the leading term of the function is positive.

We can now sketch a graph of the function accurately showing the vertex and the points where the graph cuts the axes.



Following is another worked example.

Example $f(x) = -x^2 + x + 6$

The function $f(x) = -x^2 + x + 6$ has a negative leading term, and so this time, the graph will be a $\cap$ shape. We can find the roots of the function by factorising the quadratic as follows:

$$\begin{aligned} -x^2 + x + 6 \\ -(x^2 - x - 6) \\ -(x-3)(x+2) \\ (3-x)(x+2) \end{aligned}$$

Thus $f(x) = (3-x)(x+2)$. If $f(x) = 0$ then either $(3-x) = 0$ in which case $x = 3$, or $(x+2) = 0$ in which case $x = -2$.

Thus the roots of the quadratic are at $x = 3$ and $x = -2$ and these are the points where the graph cuts the x-axis.

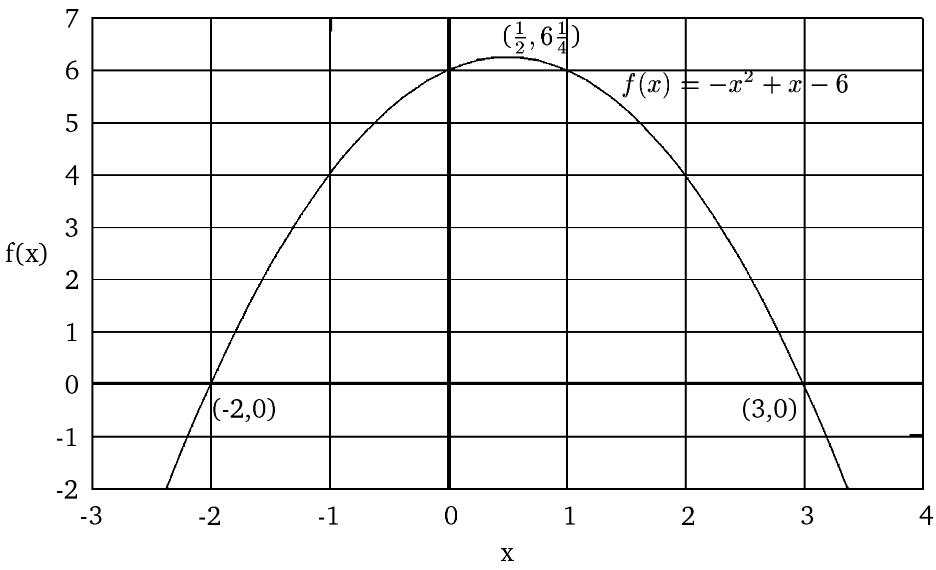
The mid-point of the roots is at $3 + -2 \div 2 = 1 \div 2 = \frac{1}{2}$ and this is the x co-ordinate of the vertex. The y co-ordinate is given by:

$$f(\frac{1}{2}) = -(\frac{1}{2})^2 + (\frac{1}{2}) + 6 = -\frac{1}{4} + \frac{1}{2} + 6 = 6\frac{1}{4}$$

Therefore the vertex is at the point $(\frac{1}{2}, 6\frac{1}{4})$.

Evaluating $f(0) = -(0)^2 + (0) + 6 = 6$ tells us that the graph cuts the y-axis at 6.

We now have enough information to sketch the graph showing the turning point and the points where the graph cuts the axes.



Learning activity

Sketch graphs of the following quadratic functions clearly marking the vertices and any points where the graph cuts the axes:

1. $f_1(x) = x^2 + 4x + 3$
2. $f_2(x) = -x^2 + 4x$
3. $f_3(x) = x^2 - 6x + 9$

The function f_3 has repeated roots. This means that the graph of f_3 only cuts the x-axis at one point. The line of symmetry of this graph cuts though this point. What does that tell us about the vertex?

3.3.1 Finding roots using the quadratic formula

All of the quadratic functions that we have looked at so far have had integer roots and this means that it is easy to find the roots using the quadratic factorisation method described in 1.8.4. However, not all quadratics have integer roots. If a quadratic cannot be factorised using the factorisation method then we can instead use the following *quadratic formula*.

The roots of the quadratic $ax^2 + bx + c$ are given by:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

You will need to learn this formula for the examination.

Use of the quadratic formula is illustrated in the following example.

Example: $f(x) = 3x^2 + 2x - 7$

We have $a = 3, b = 2$ and $c = -7$. Substituting these values into the quadratic formula gives:

$$\begin{aligned} x &= \frac{-(2) \pm \sqrt{((2)^2 - 4(3)(-7)}}{2(3)} \\ x &= \frac{-2 \pm \sqrt{88}}{6} \\ x &= \frac{-2 \pm 9.381}{6} \end{aligned}$$

$$x = \frac{-2 + 9.381}{6} \quad \text{or} \quad x = \frac{-2 - 9.381}{6}$$

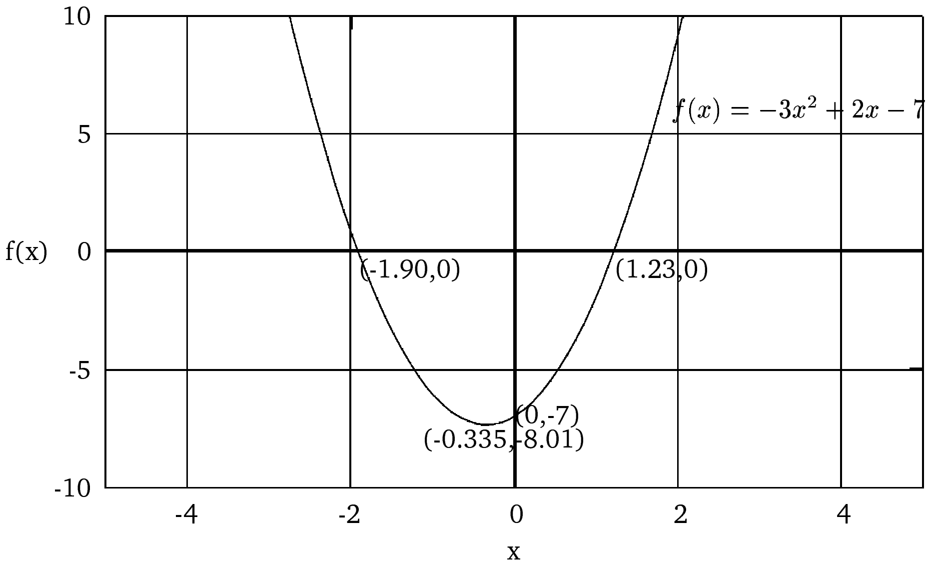
$$x = 1.23(2d.p.) \quad \text{or} \quad x = -1.90(2d.p.)$$

We have found the roots of the function are $x = 1.23$ and $x = -1.90$. We can find the x co-ordinate of the vertex by finding the mid-point of the roots as before. $(1.23 + -1.90) \div 2 = -0.335$. Now we find the y co-ordinate of the vertex by evaluating $f(-0.335)$.

$$f(-0.335) = 3(-0.335)^2 + 2(-0.335) - 7 = -8.01(2d.p.)$$

Therefore the vertex is at the point $(-0.335, -8.01)$.

The graph of the function is a $\cup$ shape, and $f(0) = -7$ so the graph cuts the y-axis at $(0, -7)$. We now have enough information to sketch the function.



Note that when using the quadratic formula it is helpful to list the values of a, b and c before you substitute them into the formula. Remember to include any minus signs.

For example, if $f(x) = -3x^2 - 6x + 6$ then we have $a = -3$, $b = -6$ and $c = 6$. The minus signs must also be included when substituting the values of a, b and c into the quadratic formula:

$$\begin{array}{rcl} x & = & \dfrac{-(-6) \pm \sqrt{((-6)^2 - 4(-3)(6)}}{2(-3)} \\ x & = & \dfrac{6 \pm \sqrt{108}}{-6} \\ x & = & \dfrac{6 \pm 10.392}{-6} \\ x = \dfrac{6 + 10.392}{-6} & \text{or} & x = \dfrac{6 - 10.392}{-6} \\ x = -2.73(2d.p.) & \text{or} & x = 0.73(2d.p.) \end{array}$$

Finding the vertex without finding the roots

Sometimes we might want to find the vertex of a quadratic function but are not interested in the value of the roots. For example, if a negative quadratic function represents profit then we can find the maximum profit by finding the vertex of the function. Recall that the x co-ordinate of the vertex of the quadratic is the mid-point of the two roots. Rewriting the quadratic formula as:

$$x = \frac{-b}{2a} \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}$$

we can see that the mid-point of the two roots must be

$$x = \frac{-b}{2a}$$

since the actual roots are either $+\frac{\sqrt{(b^2-4ac)}}{2a}$ and $-\frac{\sqrt{(b^2-4ac)}}{2a}$ away from this value.

Therefore we can use the formula $x = \frac{-b}{2a}$ to find the x co-ordinate of the vertex without finding the roots. We can then find the y co-ordinate of the vertex by evaluating the function at the x co-ordinate.

For example, we will find the vertex of the quadratic function $f(x) = -5x^2 + 15x + 6$.

We have $a = -5$, $b = 15$ and $c = 6$. Therefore the x co-ordinate of the vertex is given by

$$x = \frac{-b}{2a} = \frac{-15}{2(-5)} = 1.5$$

Evaluating the function at $x = 1.5$ gives us the y co-ordinate of the vertex:

$$f(1.5) = -5(1.5)^2 + 15(1.5) + 6 = -11.25 + 22.5 + 6 = 17.25$$

Therefore the vertex of the function $f(x) = -5x^2 + 15x + 6$ is at the point $(1.5, 17.25)$.

Learning activity

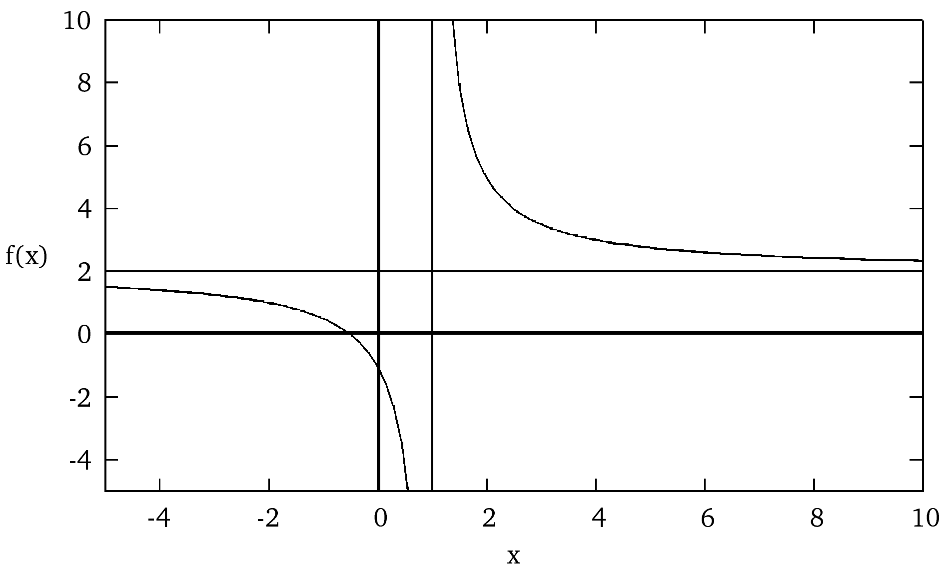
1. We have found that the roots of the function $f(x) = -3x^2 - 6x + 6$ are $x = -2.73$ and $x = 0.73$. Find the turning point of the function and sketch a graph showing the turning point and all the points where the graph cuts the axes.
2. Use the quadratic formula to find the roots of the following functions giving your answers correct to two decimal places where appropriate:
   (a) $g_1(x) = 2x^2 - 3x - 5$
   (b) $g_2(x) = 2x^2 - 6x + 1$
   (c) $g_3(x) = 3x^2 - x - 5$
3. Find the vertices of the following functions without finding the roots. Give your answers correct to two decimal places where appropriate:
   (a) $f_1(x) = 4x^2 + 8x - 12$
   (b) $f_2(x) = -3x^2 - 5x + 7$
   (c) $f_3(x) = -10x^2 + 200x - 32$

3.4 Sketching rational functions

We have seen that when defining a rational function, we must exclude from the domain any values of x that would make the denominator equal to zero. For example, consider the rational function

$$f(x) = \frac{2x+1}{x-1} \quad (x \neq 1)$$

The graph of $f(x)$ is shown below.



Since the function cannot be evaluated at $x = 1$, the graph of the function will never cross or touch the line $x = 1$. This line is called the *vertical asymptote*. As x gets larger, $f(x)$ gets closer and closer to, but never reaches, 2. Therefore the graph never crosses or touches the line $f(x) = 2$. This line is called the *horizontal asymptote*.

To sketch the graph of a rational function, first draw the asymptotes and then work out what happens to the graph near where the asymptotes cross by evaluating the function at some suitable values of x.

Worked example: $f(x) = \frac{5-x}{x+2}$ $(x \neq -2)$

The vertical asymptote is the line $x = -2$ since $f(-2)$ cannot be evaluated.

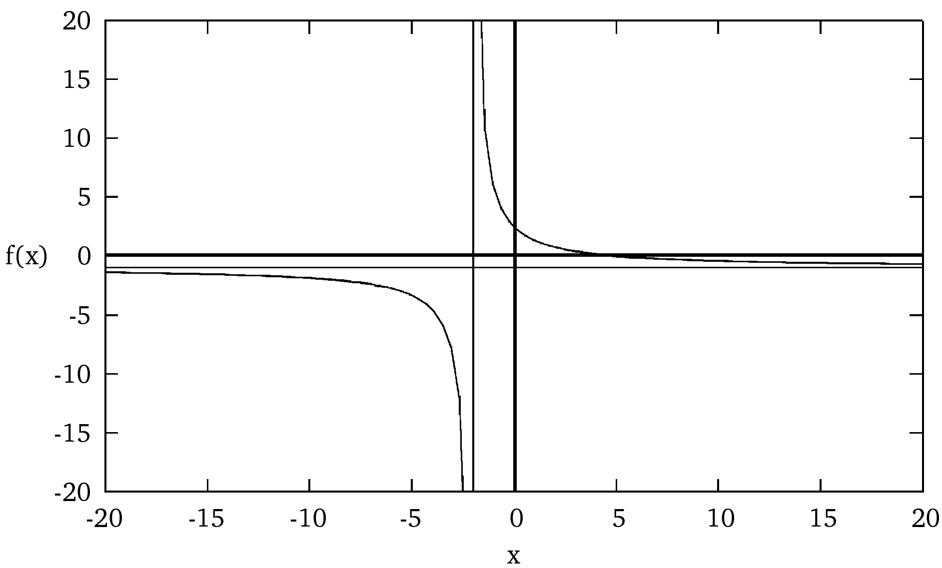
The horizontal asymptote is the line $f(x) = -1$ because $f(x)$ gets closer and closer to -1 when evaluated at larger and larger values of x.

To get more information about what the graph looks like, we evaluate the function at several values:

x	-1	0	1	2	5	10
$f(x)$	6	$\frac{5}{2}$	$\frac{4}{3}$	$\frac{3}{4}$	0	$-\frac{5}{12}$

x	-3	-4	-5	-6	-9	-14
$f(x)$	-8	$-\frac{9}{2}$	$-\frac{10}{3}$	$-\frac{11}{4}$	-2	$-\frac{19}{12}$

We now have enough information to sketch the function.



Learning activity

The rational function f is given by $f(x) = \frac{2x+1}{x}$, $(x \neq 0)$. On a graph with both axes labelled from -10 to $+10$ mark the horizontal and vertical asymptotes of the function f. Hence sketch a graph of the function.

3.5 Combining functions

Two or more functions can be combined by addition, subtraction, multiplication or division. To combine functions we can think of the functions as polynomials and add, subtract, multiply or divide as in section 1.8.

For example, suppose we have functions $f(x) = 2x - 3$ and $g(x) = x^2 + x - 6$. Then we have:

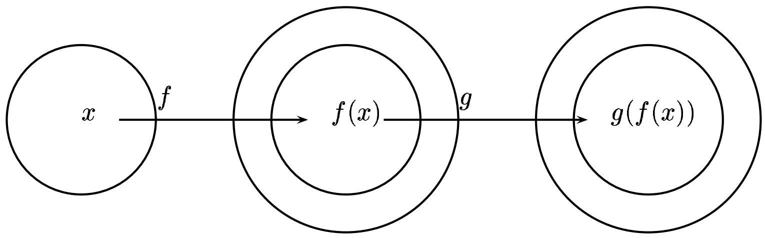
$$\begin{aligned}
f(x) + g(x) &= 2x - 3 + x^2 + x - 6 = x^2 + 3x - 9 \\
f(x) - g(x) &= 2x - 3 - (x^2 + x - 6) = -x^2 + x + 3 \\
f(x).g(x) &= (2x - 3)(x^2 + x - 6) = 2x^3 - x^2 - 15x + 18 \\
\tfrac{f(x)}{g(x)} &= \frac{2x - 3}{x^2 + x - 6} \ (x \neq 2, x \neq -3)
\end{aligned}$$

Learning activity

1. Given the functions $f(x) = 4x + 5$ and $g(x) = x^2$ find:
   (a) $f(x) + g(x)$
   (b) $g(x) + f(x)$
   (c) $f(x) - g(x)$
   (d) $g(x) - f(x)$
   (e) $f(x).g(x)$
   (f) $g(x).f(x)$
   (g) $f(x) \div g(x)$
   (h) $g(x) \div f(x)$
2. Hence decide which of the operations $+$, $-$, . and $\div$ are *commutative*. (Recall that an operation is commutative if the order of operation does not matter.)

3.5.1 Composition of functions

If f and g are two functions, and the range of $f(x)$ is the same as (or is contained within) the domain of $g(x)$ then a value x can be evaluated first by f and then the result $f(x)$ can be evaluated by the function g to give $g(f(x))$. This is called *composition of functions*.



Domain of f

Range of f
inside
Domain of g

Range of gf
inside
Range of g

For example, if $f(x) = x^2 + 7$ and $g(x) = 4x - 5$ then we can work out $g(f(x))$ or $f(g(x))$ for any given value of x. Suppose $x = 5$ then we have:

$$x = 5, \quad f(x) = f(5) = 5^2 + 7 = 32, \quad g(f(x)) = g(32) = 4(32) - 5 = 123$$
$$x = 5, \quad g(x) = g(5) = 4(5) - 5 = 15, \quad f(g(x)) = f(15) = 15^2 + 7 = 232$$

We have calculated $f(g(x)) = 232$ and $g(f(x)) = 123$.

Note that $f(g(x) \neq g(f(x))$ which means that composition of functions is not a commutative operation.

The $\circ$ operation

Suppose we want to evaluate $f(g(x))$. We could evaluate $g(x)$ first and then find $f(g(x))$ as above. Alternately, we can find one new function $f \circ g(x)$ which has the same output.

We find the function $f \circ g(x)$ by replacing every instance of x in $f(x)$ by the function $g(x)$. Using the same functions as before $f(x) = x^2 + 7$ and $g(x) = 4x - 5$ this means that we replace the x^2 in $f(x)$ with $(4x - 5)^2/$. This gives us:

$$f \circ g(x) = (4x - 5)^2 + 7 = 16x^2 - 40x + 32$$

Now we can calculate $f(g(5))$ by evaluating $f \circ g(5)$:

$$f \circ g(5) = 16(5)^2 - 40(5) + 32 = 232$$

Similarly we can find the function $g \circ f(x)$ by replacing $4x$ in $g(x)$ by $4(x^2 + 7)$. This gives us:

$$g \circ f(x) = 4(x^2 + 7) - 5 = 4x^2 + 23$$

Now we can find $g(f(5))$ by evaluating the function $g \circ f(x)$ at $x = 5$. This gives us:

$$g \circ f(5) = 4(5)^2 = 23 = 123$$

Learning activity

1. Given the two functions $f(x) = x^2 + 4$ and $g(x) = 6x - 2$ find $f \circ g(x)$ and $g \circ f(x)$. Hence evaluate:
   (a) f(g(2))
   (b) f(g(-1))
   (c) g(f(0))
   (d) g(f(-3))
2. Find correct to one decimal place, the values of x which make $f \circ g(x) = g \circ f(x)$

3.6 Applications in business and economics

Functions are often used in business and economics. For example, a function may be used to model the profit of a company depending on units manufactured. Another function may be used to model the costs of the company. Functions representing fixed costs and variable costs may be combined to produce another function representing total costs and so on.

3.6.1 Profit, cost and revenue functions

Instead of using f and g to represent business functions, we usually use C to represent *cost*, R to represent *revenue* and π to represent *profit*.

Thus if x is the number of units produced then $C(x)$ gives the cost of producing the units, $R(x)$ gives the revenue generated by selling the units, and $\pi(x)$ gives the profit made by producing and selling the units.

The profit is the amount left after all of the costs have been paid out of the revenue. Therefore:

$$\pi(x) = R(x) - C(x)$$

Finding maximum profit

Modelling business information using functions means that we can apply the mathematics of functions to solve business problems. For example, suppose the profit generated by a particular unit produced by a factory can be modelled by a quadratic function $\pi(x)$ where x represents the number of units produced. Since profit is unlikely to go on increasing infinitely, the quadratic is not a $\cup$ shape but a $\cap$ shape. By finding the vertex of the quadratic, we can find the maximum profit, and the quantity of units that should be produced in order to attain this maximum profit.

3.6.2 Break-even analysis

When planning to manufacture a new product, break-even analysis is very important. Such analysis tells us the quantity Q of products that must be manufactured and sold in order to make a profit rather than a loss. The point where neither profit nor loss is made is called the *break-even point*. In terms of functions this occurs when

- $\pi(Q) = 0$ - the profit function evaluates to zero; or
- $R(Q) = C(Q)$ - the total revenue function is equal to the total cost function.

Given a revenue function $R(Q)$ and a cost function $C(Q)$, we can find the profit function $\pi(Q)$ by setting $\pi(Q) = R(Q) - C(Q)$. We can then find the break-even point either by:

- finding the values of Q which makes $\pi(Q) = 0$ - if $\pi(Q)$ is a quadratic function this is equivalent to finding the roots of the function; or
- finding the value of Q which makes $R(Q) = C(Q)$ by equating the functions and solving the resulting equation.

The following worked examples demonstrate how the mathematics of functions can be applied in a business environment.

Worked example 1

a) A quadratic revenue function has a graph which has a maximum value of 576 at $Q = 12$ and which is zero of $Q = 0$.

   i) Sketch the graph of this quadratic revenue function showing the second point where the revenue is 0.

   ii) Hence find the equation for revenue in terms of Q.

b) A profit function has equation:

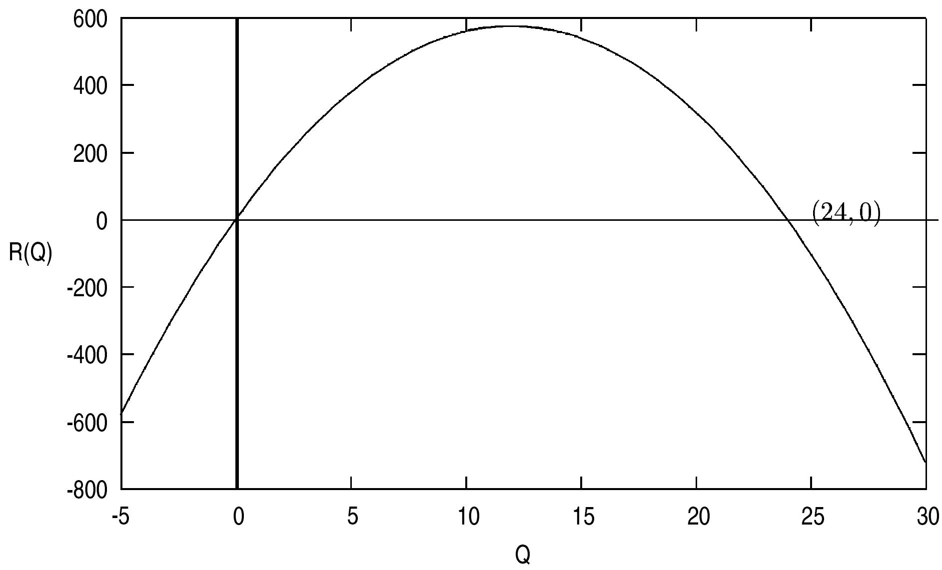
$$\pi = -3Q^2 + 42Q - 50$$

   i) Find the break-even points correct to two decimal places.

   ii) Find the maximum profit.

Solution

a) i) We have to sketch a quadratic which has Q on the x-axis and $R(Q)$ on the y-axis. The function is a quadratic and we are told it has a maximum value and so it must be a $\cap$ shape. The graph goes through the points $(0, 0)$ and $(12, 576)$. The graph of a quadratic is always symmetrical, so by sketching the part of the curve between the two points that we already know, we can sketch the rest using symmetry. We can calculate that the second root will be at $(24, 0)$ since the turning point of the curve must be at the mid-point of the two roots. We now have enough information to draw the graph of the function:



ii) The roots of the revenue function are at 0 and 24, and this tells us that $(Q-0)$ and $(Q-24)$ are linear factors of the revenue function. Thus

$$R(Q) = Q(Q-24) = Q^2 - 24Q$$

b) i) Here we have another quadratic function. The break-even points of a profit function are when the profit is zero. This occurs at the roots of the function. The question asks for the answers correct to two decimal places which is a clue that the quadratic does not have integer roots. Therefore we have to use the quadratic formula (see section 3.3.1) to find the roots of the function $\pi = -3Q^2 + 42Q - 50$. We have $a = -3, b = 42, c = -50$. Substituting these values into the formula gives:

$$Q = \frac{-42 \pm \sqrt{(42^2 - 4(-3)(-50))}}{2(-3)} = \frac{-42 \pm \sqrt{1164}}{-6}$$

Either $Q = \frac{-42+34.117}{-6} = 1.32$ or $Q = \frac{-42-34.117}{-6} = 12.69$.
Therefore the break-even points of the profit function are at $Q = 1.32$ and $Q = 12.69$ correct to two decimal places.

ii) The maximum profit will occur at the turning point of the function which will be at the mid-point of the two roots. Therefore we can find the maximum profit by evaluating the function π at the midpoint of 1.32 and 12.69. The mid-point is $(1.32 + 12.69) \div 2 = 7.00$. Therefore the maximum profit is:

$$\pi(7) = -3(7)^2 + 42(7) - 50 = 97$$

Worked example 2

a) A company has fixed costs of \$2000 and a marginal cost of \$18 for each item produced.

i) Express the cost, $C(q)$, as a function of q, the number of items produced.

ii) The revenue function is $R(q) = 250 - q^2$. Express profit as a function of q.

b) Sketch the graph of the function $f(x) = -x^2 + 7x - 10$, showing where it cuts the axes and the co-ordinates of its turning point. Given that a company's profit function is

$$\pi(q) = -q^2 + 7q - 10$$

state:

i) its maximum profit

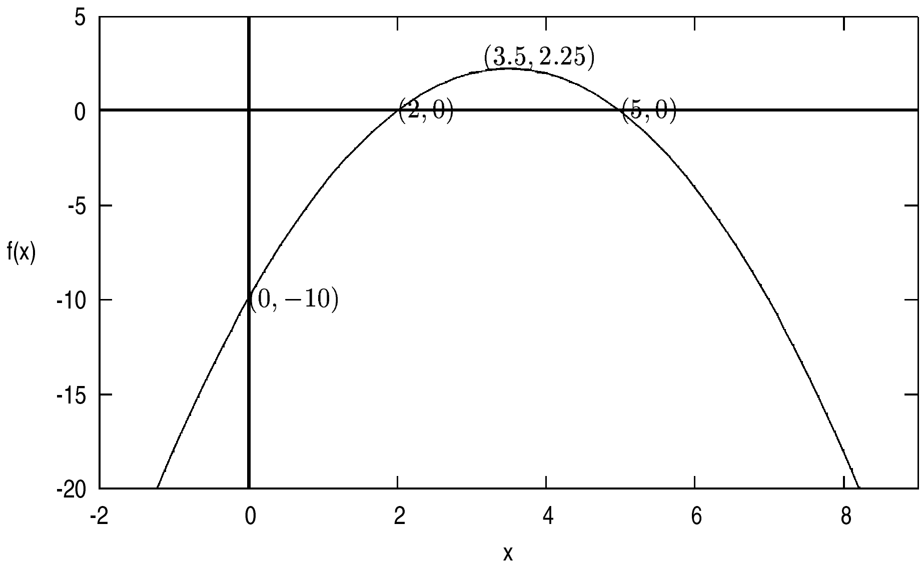
ii) its break-even points

Solution

a) i) The total cost is given by adding the fixed and variable costs. Therefore the cost function is $C(q) = 2000 + 18q$.

ii) $\pi(q) = R(q) - C(q) = 250q - q^2 - (2000 + 18q) = -q^2 + 232q - 2000$. Therefore the profit function is $\pi(q) = -q^2 + 232q - 2000$.

b) We have to sketch the graph of the quadratic $-x^2 + 7x - 10$. The first step is to factorise the quadratic using the quadratic factorisation method of section 1.8.4. This gives us $f(x) = (5 - x)(x - 2)$ and this means that the roots of the quadratic are at $x = 5$ and $x = 2$. These are the points where the graph cuts the x-axis. By evaluating $f(0) = -10$ we see that the graph cuts the y-axis at the point $(0, -10)$. The graph will be a $\cap$ shape since the function is a negative quadratic. Next we need to find the turning point of the graph. The x co-ordinate of the turning point lies in the middle of the roots. Thus $x = (5 + 2) \div 2 = 3.5$ is the x co-ordinate of the turning point. We can find the y co-ordinate by evaluating the function at $x = 3.5$. We have$f(3.5) = -(3.5)^2 + 7(3.5) - 10 = 2.25$. Therefore the turning point is at $(3.5, 2.25)$. We now have enough information to sketch the graph.



The profit function of the company is the same as the quadratic function $f(x)$. Therefore the maximum profit is the same as the maximum value of $f(x)$ and the break-even points are the roots of $f(x)$:

i) maximum profit = 2.25

ii) break-even points are at $q = 2$ and $q = 5$.

Worked example 3

a) If $fx) = x^2 + 3$ and $g(x) = 2x - 1$, express $f(g(x))$ as a polynomial in x. If $x = 2$ what are the values of $f(g(x))$ and $g(f(x))$?

b) A factory's cost $C(q)$ is a function of q, the number of units produced, and given by $C(q) = 1000q + 2000$. Its profit, $P(q)$ is given by $P(q) = 400q - 500$. Express the profit P in terms of the cost C.

c) Sketch the graph of the parabola $f(x) = -2x^2 + 7x - 3$, showing clearly where it cuts the axes and where it attains its maximum value. A company has revenue function $R(x) = 1 + 10x - 2x^2$ and cost function $C(x) = 3x + 4$. Find the maximum profit.

Solution

a) $f(g(x))$ as a polynomial in x is
$f \circ g(x) = (2x - 1)^2 + 3 = 4x^2 - 4x + 1 + 3 = 4x^2 - 4x + 4.$
$f(g(2) = 4(2)^2 - 4(2) + 4 = 16 - 8 + 4 = 12$
$g(f(2)) = g(2^2 + 3) = g(7) = 2(7) - 1 = 13$

b) The idea here is to write the profit function using the cost function in the place of the variable q. We have $400q$ in the profit function, so we need to find out what $400q$ is in terms of $C(q)$. We have $C(q) = 1000q + 2000$. Multiplying all the terms by 0.4 gives:

$$0.4C(q) = 400q + 800$$
$$400q = 0.4C(q) - 800$$

Now replacing the $400q$ in the profit function by $0.4C(q) - 800$ gives us the profit in terms of the cost:

$$P(q) = 0.4C(q) - 800 - 500 = 0.4C(q) - 1300$$

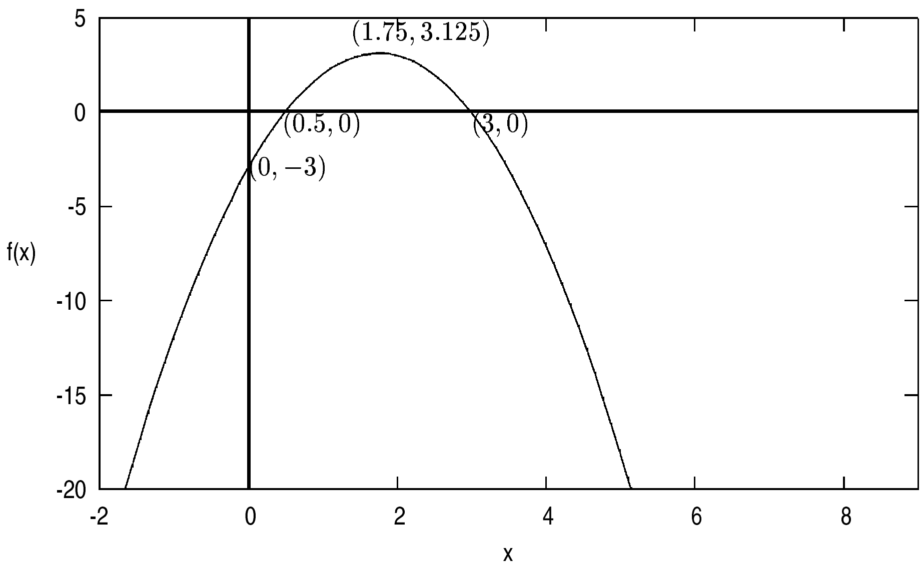
c) We have negative quadratic function $f(x) = -2x^2 + 7x - 3$ so the graph will be a $\cap$ shape. We can use the quadratic formula to find the roots of the equation - this will tell us where the graph cuts the x-axis:

$$x = \frac{-7 \pm \sqrt{(7^2 - 4(-2)(-3))}}{2(-2)} = \frac{-7 \pm \sqrt{25}}{-4} = \frac{-7 \pm 5}{-4}$$

We have $x = \frac{-7+5}{-4} = 0.5$ or $x = \frac{-7-5}{-4} = 3$. Therefore the graph cuts the x-axis at 0.5 and 3.

We can find the point where the graph cuts the y-axis by evaluating $f(0) = -3$. Therefore the graph cuts the y-axis at the point $(0, -3)$.

Finally we need to find the turning point of the graph. The x co-ordinate will be at the mid-point of the two roots: $x = (0.5 + 3) \div 2 = 1.75$. The y co-ordinate will be at $f(1.75) = -2(1.75)^2 + 7(1.75) - 3 = 3.125$. Therefore the turning point is at $(1.75, 3.125)$. We now have enough information to sketch the graph.



$\pi(x) = R(x) - C(x)$
$\pi(x) = 1 + 10x - 2x^2 - (3x + 4)$
$\pi(x) = -2x^2 + 7x - 3$
This profit function is the same as the function $f(x)$. Therefore the maximum profit is 3.125.

3.7 Learning outcomes

After studying this chapter and the relevant questions from *Dowling* you will be able to:

- Define the meaning of the term *function* and make correct use of the terms *domain* and *range* as applied to functions.
- Define and describe a function giving an appropriate domain and range.
- Combine two functions using $+, -, ., \div$ and $\circ$ operations.
- Sketch the graph of a constant, linear, quadratic or simple rational function.
- Apply the algebra of functions to problems in business and economics in order to:
  - identify and sketch simple revenue and cost functions
  - solve problems involving quadratic revenue, profit and cost functions
  - find the break-even point by equating revenue to cost and solving the resulting equations graphically or algebraically
  - identify and sketch the profit function and find the break-even point by equating it to zero by solving the resulting equation either graphically or algebraically.

3.8 Sample examination questions

Question 1

a) Given $f(x) = x^3$ and $g(x) = (x-2)^2$ find:

   i) $f(g(x))$

   ii) $g(f(x))$

[2]

b) Factorise the quadratic expression $x^2 - 8x + 12$. Sketch the graph of the function $f(x) = x^2 - 8x + 12$, showing where it cuts the axes.

[3]

c) Given the revenue function $R(x) = 10x - 2x^2$, and the cost function $C(x) = 4 + x$, express the profit π, as a function of x. Find the maximum profit.

[5]

Question 2

a) A company has fixed costs of $500 and a marginal cost of $24 for each item produced.

   i) Express the cost, $C(q)$, as a function of q, the number of items produced.

   ii) The revenue function is $R(q) = 1000q - q^2$. Express profit as a function of q.

[3]

b) Sketch the graph of the function $f(x) = -x^2 + 16x - 48$, showing where it cuts the axes and the co-ordinates of its turning point.
Given that a company's profit function is

$$\pi(q) = -q^2 + 16q - 48$$

state (i) its maximum profit, and (ii) its break-even points.

[7]

Chapter 4

Simultaneous equations

Essential reading

See Chapter 4 of *Dowling* for many further examples of the material covered in this chapter. In particular, use the supplementary problems 4.31 to 4.40 to test your understanding of systems of equations and their applications.

Some practical problems require several equations and variables to describe or model them in mathematical terms. A solution must satisfy all of the equations at the same time and hence such equations are known as *simultaneous equations*.

Simultaneous equations can be solved using graphs or using algebra. We will be looking at both methods of solution in this chapter. We will concentrate on solving two equations with two variables. This is known as a $2 * 2$ *system of equations*.

Systems of simultaneous equations occur naturally in business and economics. For example, one equation may represent supply and another demand. Solving these equations simultaneously will tell us when supply and demand are balanced, bringing equilibrium to the market. As well as supply and demand analysis, we will also study income determination, and IS-LM analysis which links national income and interest rates with the money market.

4.1 Systems of equations

We know how to solve an equation with one variable using algebraic methods such as changing the subject or quadratic factorisation.

However, if we have an equation with two unknown variables such as:

$$2y = 4x + 6$$

then we cannot find a unique solution. We can only say what y will equal if we give x a particular value. For example if $x = 4$ then $y = 11$ and if $x = 9$ then $y = 21$. There are *infinitely many solutions* because x can take infinitely many values and whatever value we give x there will be a corresponding solution for y.

In order to find a unique solution for two variables, we need two equations.

For example, consider the following equations:

$$8 = x + 2y$$
$$11 = x + 3y$$

Here we have two equations each with two variables x and y. This is called a $2 * 2$ system of equations. We want to find a solution which *satisfies* both of the equations i.e., we want to find a value for x and a value for y so that both of the equations are correct.

The second equation has the same number of x's but one more y than the first equation. Therefore the difference between the left-hand-side of the first and second equation must equal one y. $11 - 8 = 3$ so we must have $y = 3$. Now we know $y = 3$ we can substitute this into the first equation to make one equation in one variable:

$$8 = x + 2(3)$$

Now we can solve this equation to find $x = 2$.

We have used an algebraic method known as *elimination* to find the solution, $x = 2$ and $y = 3$, for this system of equations. We will look at the elimination method in more detail later on.

4.1.1 Unique, multiple, infinite and impossible solutions

A set of n equations with m variables is called an $n * m$ *system of equations*.

Unique solutions

The solution $x = 2, y = 3$ to the $2 * 2$ system of equations in the example above is *unique*. No other values of x and y can make both equations correct at the same time.

In order to have a unique solution, a system of equations must have at least as many equations as variables. Therefore we can usually find a unique solution to an $n * m$ system of equations - that is a system of n equations with m variables - if $n \geq m$. Note however that there are some exceptions to this - in particular if the equations are non-distinct (see section 4.2.1).

Infinite solutions

We saw earlier that if we have one equation with two variables then we cannot find a unique solution. In our example, $2y = 4x + 6$, there are infinitely many solutions.

It is true in general that if we have more variables than equations i.e., $n < m$ in an $n * m$ system, then we will not be able to find a unique solution. For example, below is a $2 * 3$ system of equations which has only two equations but three variables:

$$\begin{aligned} 10 &= x + y + z \\ -4 &= x - y - z \end{aligned}$$

We can find a solution to this system of equations:
$x = 3, y = 5, z = 2$ is a solution.

However, so is $x = 3, y = 3, z = 4$ or $x = 3, y = 1, z = 6$.

The value of x must be 3, but we can pick any values for y and z so long as $y + z = 7$. (See section 4.3.1 to understand why this is so.)

This means that there are infinitely many solutions because we can assign whatever value we like to y and then make $z = 7 - y$. For example, $x = 3$, $y = 100$, $z = -93$.

Multiple solutions

If one of the equations in a system is a quadratic then there may be two correct solutions. For example consider the following $2 * 2$ system:

$$\begin{aligned} y &= x^2 \\ y &= 5x - 6 \end{aligned}$$

We can solve this system by *equating* the two equations. This means that we set the two right-hand-sides equal to each other to make a quadratic equation in one variable x which we can solve to find two solutions for x. Then by substituting these values for x into one of the original equations, we can find the corresponding values of y.

$$\begin{aligned} x^2 &= 5x - 6 \\ x^2 - 5x + 6 &= 0 \\ (x-2)(x-3) &= 0 \end{aligned}$$

Either $x = 2$ or $x = 3$. When $x = 2$, $y = 2^2$ and when $x = 3$, $y = 3^2$ therefore this system has two possible solutions. The first is $x = 2, y = 4$ and the second is $x = 3, y = 9$.

No solutions

Consider the following $2 * 2$ system:

$$\begin{aligned} 4 &= y - x \\ 6 &= y - x \end{aligned}$$

There are no values which can be assigned to x and y which will make both of these equations correct. Try giving x a value and then making $y - x$ equal both 4 and 6 at the same time - it's impossible. Therefore this system of equations has no solution.

4.2 Using graphs to solve simultaneous equations

It is much easier to understand why some simultaneous equations have no solution and why others have multiple solutions when we consider the equations graphically.

As we have seen in chapters 2 and 3, we can draw the graph of an equation. If the equation is linear it will take the form $y = mx + c$

and will be a straight line graph. If the equation is quadratic then it will include a square term and the graph will be a parabola.

We can solve a system of simultaneous equations by drawing the line of each equation on the same graph. The point(s) where the lines cross are the solutions to the equations - read the x value off the x axis and the y value off the y axis.

4.2.1 Systems of linear equations

If we draw the lines of two linear equations (straight lines) on the same graph then they might cross:

- never - if the two lines are parallel - no solution;
- once - if the two lines are different and not parallel - unique solution;
- infinitely - if the two lines are actually the same line - infinite solutions.

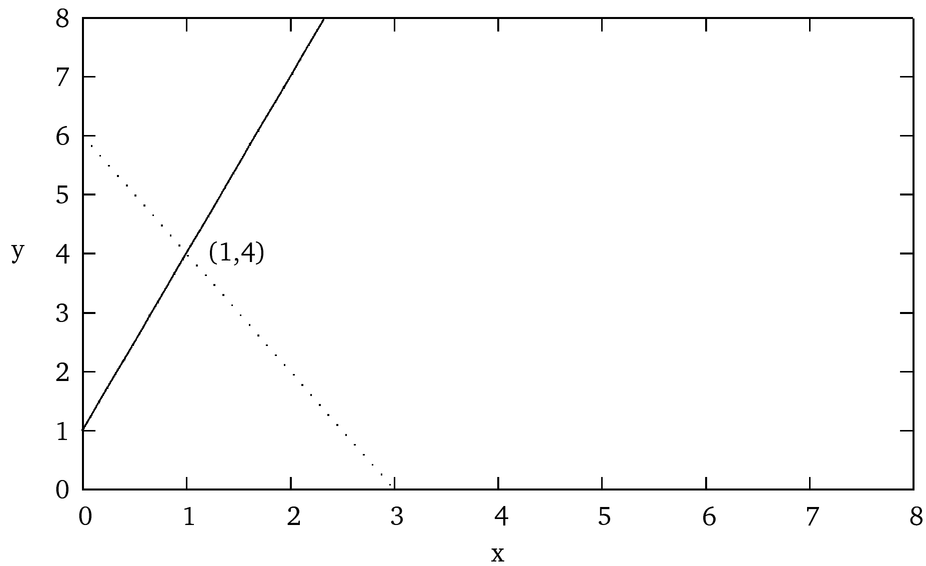
Independent equations : unique solution

Two linear equations are said to be *independent* if their lines are distinct and not parallel. Two independent lines will cross each other at exactly one point and at this point both equations are solved simultaneously.

For example consider the system of linear equations below:

$$y = 3x + 1$$
$$y = -2x + 6$$

Lines representing each of these equations are drawn on the graph below:

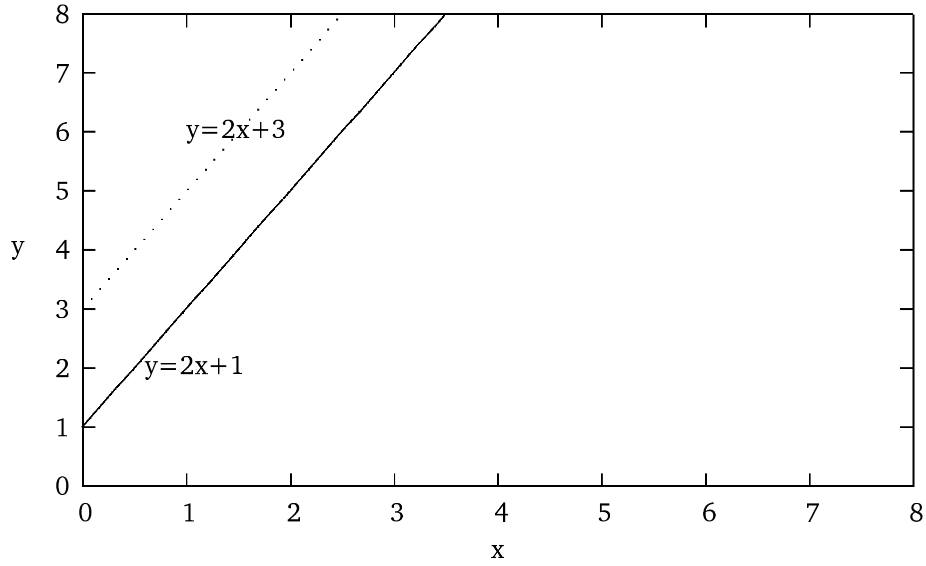


The point where the two lines cross is $(1, 4)$ and the solution is $x = 1, y = 4$.

Parallel lines : no solution

If two straight lines have the same gradient then they will be parallel to each other and they will never cross.

The graph below shows the two lines $y = 2x + 1$ and $y = 2x + 3$.



These two lines both have gradient 2 and are parallel. They will never cross each other. This explains why the system of equations below has no solution.

$$y = 2x + 1$$
$$y = 2x + 3$$

Non-independent lines : infinitely many solutions

Consider the two linear equations:

$$4 = x + 3y$$
$$12 = 3x + 9y$$

If you plot these two lines on the same graph, you will end up with just one line. This is because the terms in the second equation are all 3 times the terms in the first equation. The second equation is a *multiple* of the first.

Such equations are said to be *non-independent* and any solution of the first equation will also be a solution for the second.

Since in effect we have just one equation but two variables we cannot find a unique solution to a system with non-independent equations.

In our example the first equation has infinitely many solutions (since we can pick any value for x and then compute $y = (4 - x)/3$), and therefore the system of equations also has infinitely many solutions.

Learning activity

State whether the following systems of equations have no solution, a unique solution or infinitely many solutions. For those systems with a unique solution find this solution by sketching a graph of the two lines and finding their meeting point.

1.

$$\begin{aligned} y &= 2x + 3 \\ y &= 3x - 1 \end{aligned}$$

2.

$$\begin{aligned} y + 2x &= 3 \\ -2y - 8x &= -6 \end{aligned}$$

3.

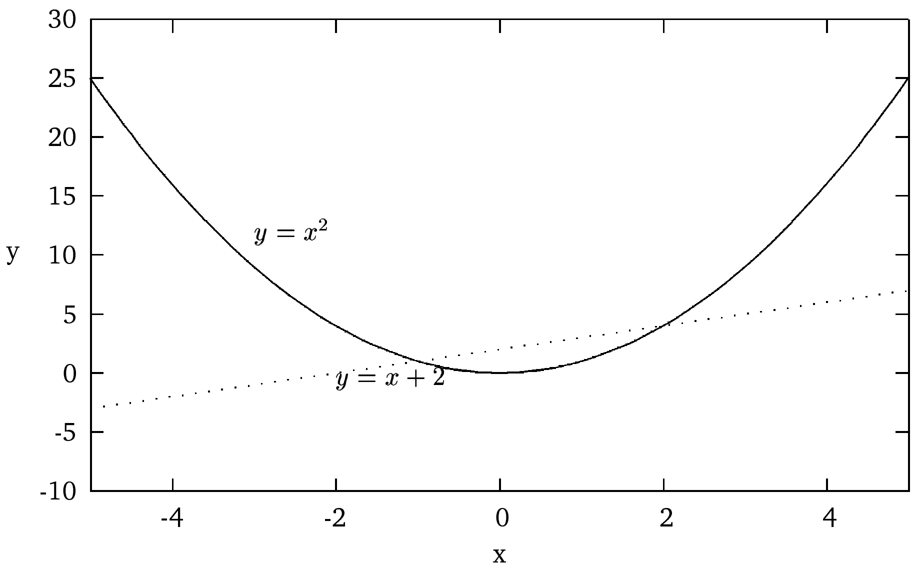
$$\begin{aligned} 2y - 4x - 2 &= 0 \\ y - 2x + 1 &= 0 \end{aligned}$$

4.

$$\begin{aligned} y &= -5x + 8 \\ y &= 5x - 2 \end{aligned}$$

4.2.2 Systems involving quadratic equations

If we draw the graph of a quadratic equation and a straight line then the straight line might cut through the quadratic twice as in the graph below.



The lines in the graph above represent $y = x^2$ and $y = x + 2$. The lines meet at two points $(2, 4)$ and $(-1, 1)$ and therefore the system of simultaneous equations:

$$\begin{aligned} y &= x^2 \\ y &= x + 2 \end{aligned}$$

has two solutions. Namely $x = 2, y = 4$ and $x = -1, y = 1$.

Learning activity

As shown in the graph above, a straight line may cut a quadratic at two points. It is also possible for a straight line to cut a quadratic just once or not at all. Draw graphs which show:

1. A quadratic and a straight line which meet at exactly one point.
2. A quadratic and a straight line which do not meet at all.

4.3 Using algebra to solve simultaneous equations

Although it can sometimes be helpful to draw a graph when solving simultaneous equations, it is not always necessary. We can use algebraic methods of solution instead. There are two different techniques which we can use - *elimination* or *substitution*. When using either technique the aim is the same. If we start off with two equations in two unknowns, we try to get rid of one of the variables to make one equation in one unknown.

4.3.1 Solution by elimination

Suppose we want to solve a $2 * 2$ system of linear equations. Each equation has 2 variables x and y say. The method of elimination combines the two equations, by adding or subtracting them term by term, to make one equation in one variable. This equation can then be solved to find the value of the one of the variables. The value of the second variable can then be found by substituting the value of the known variable into one of the original equations.

Example 1

Consider the following system of equations:

$$3y + 2x = -5 \tag{4.1}$$

$$y + 2x = 1 \tag{4.2}$$

Both of the equations include the term $+2x$. If we subtract the second equation from the first term by term then the $+2x$ terms are eliminated because they cancel each other out. We are left with:

$$2y = -6$$

This equation can be solved for y and gives us $y = -3$.

By substituting $y = -3$ back into equation 4.1 we can find the value of x.

$$\begin{aligned} 3(-3) + 2x &= -5 \\ 2x &= -5 + 9 \\ x &= 4/2 = 2 \end{aligned}$$

We have found the solution $x = 2, y = -3$. We can check this answer by making sure that it also satisfies equation 4.2.

$$(-3) + 2(2) = -3 + 4 = 1$$

Our solution $x = 2, y = -3$ satisfies both of the equations 4.1 and 4.2 and therefore we can be sure that it is the correct solution.

Example 2

In the previous example, it was easy to eliminate the x term because exactly the same term $+2x$ appeared in both of the equations. If the equations do not both include the same term then we must manipulate one or both of the equations to make elimination possible. For example, consider the following system of equations:

$$2y + 3x = 5 \qquad (4.3)$$

$$3y + 6x = 6 \qquad (4.4)$$

This time we do not have a pair of matching terms. However, if we multiply all of the terms in equation 4.3 by 2 then we get a new equation:

$$4y + 6x = 10 \qquad (4.5)$$

Equations 4.3 and 4.5 are non-independent and will have the same solutions. We can solve the system of equations 4.4 and 4.5 by subtracting one from the other to eliminate the $+6x$ terms.

$$\begin{aligned} 3y + 6x &= 6 \\ -(\underline{4y + 6x = 10}) \\ -y &= -4 \end{aligned}$$

We have found that $y = 4$ and by substituting this value for y into equation 4.3 we can find the value of x.

$$\begin{aligned} 2(4) + 3x &= 5 \\ 3x &= 5 - 8 \\ x &= -3/3 = -1 \end{aligned}$$

Our solution is $x = -1, y = 4$. We can check this answer by substituting these values for x and y into equation 4.4:

$$3y + 6x = 3(4) + 6(-1) = 12 - 6 = 6$$

Both equations 4.3 and 4.4 are satisfied by the solution $x = -1, y = 4$.

Example 3

In both of the previous examples, we have subtracted one of the equations from the other in order to eliminate one of the variables. In the following example, we will add the equations together because the term $3x$ that we want to eliminate is positive in the first equation and negative in the second equation.

$$4y + 3x = 16 \tag{4.6}$$
$$5y - 3x = 47 \tag{4.7}$$

Adding equations 4.6 and 4.7 together term by term gives:

$$9y = 63$$

We can solve this equation to find $y = 7$. Substituting $y = 7$ into equation 4.6 will tell us the value of x:

$$\begin{aligned} 4(7) + 3x &= 16 \\ 3x &= 16 - 28 \\ x &= -12/3 = -4 \end{aligned}$$

We check the solution $x = -4, y = 7$ by substituting these values in equation 4.7:

$$5(7) - 3(-4) = 35 + 12 = 47$$

Since both equations are satisfied we are sure that the solution $x = -4, y = 7$ is correct.

Example 4

What if there are no terms in the equations which can cancel with each other? Sometimes we have to multiply both of the equations by a constant so that each will contain a cancelling term. This is the case in the next example.

$$2y + 5x = 34 \tag{4.8}$$
$$3y + 2x = 7 \tag{4.9}$$

Neither the x nor the y terms will cancel if we add or subtract the equations. To remedy this, we will multiply equation 4.8 by 3 and equation 4.9 by 2. This will make the term $6y$ appear in both equations.

$$6y + 15x = 102 \tag{4.10}$$
$$6y + 4x = 14 \tag{4.11}$$

Note that equation 4.10 has the same solutions as equation 4.8 and equation 4.11 has the same solutions as equation 4.9. Therefore we can solve the original system of equations by solving this second system of equations.

This can be done by subtracting equation 4.11 from equation 4.10 to obtain:

$$\begin{aligned} 11x &= 88 \\ x &= 88/11 = 8 \end{aligned}$$

Now substituting $x = 8$ into equation 4.8 gives us:

$$\begin{aligned} 2y + 5(8) &= 34 \\ 2y &= 34 - 40 \\ y &= -6/2 = -3 \end{aligned}$$

Finally we check our solution by substituting the values $x = 8$, $y = -3$ into equation 4.9:

$$3(-3) + 2(8) = -9 + 16 = 7$$

Both of the equations 4.8 and 4.9 are satisfied and so we can be sure that the solution $x = 8, y = -3$ is correct.

Learning activity

Solve the following simultaneous equations using the method of elimination.

1.
$$\begin{aligned} x + 2y &= 9 \\ 3x + y &= 7 \end{aligned}$$

2.
$$\begin{aligned} 7x + 3y &= 27 \\ 2x - y &= 4 \end{aligned}$$

3.
$$\begin{aligned} 2x + 3y &= 27 \\ 3x + 2y &= 28 \end{aligned}$$

4.
$$\begin{aligned} 2x + 5y &= 16 \\ 5x + 3y &= 21 \end{aligned}$$

4.3.2 Solution by substitution

An alternative algebraic method is to rearrange one of the equations so that one of the variables is expressed in terms of the other. Substituting this into the second equation results in one equation in one variable. The following examples show how the method works.

Example 5

$$
\begin{aligned}
2x + 3y &= 14 \qquad (4.12)\\
3x - y &= 10 \qquad (4.13)
\end{aligned}
$$

We can rearrange equation 4.13 to make y the subject:

$$y = 3x - 10$$

Now substituting $y = 3x - 10$ into equation 4.12 gives:

$$
\begin{aligned}
2x + 3(3x - 10) &= 14\\
2x + 9x - 30 &= 14\\
11x &= 44\\
x &= 4
\end{aligned}
$$

We have found that $x = 4$, substituting this into equation 4.12 tells us the value of y:

$$
\begin{aligned}
2(4) + 3y &= 14\\
3y &= 14 - 8\\
y &= 6/3 = 2
\end{aligned}
$$

We can check the solution by substituting $x = 4, y = 2$ into equation 4.13:

$$3(4) - (2) = 10$$

Example 6

The following example involves a quadratic equation and a linear equation.

$$
\begin{aligned}
y &= 11x - 2 \qquad (4.14)\\
y &= 5x^2 \qquad (4.15)
\end{aligned}
$$

This time there is no need to rearrange either of the equations. We simply substitute $y = 5x^2$ into equation 4.14 to obtain a quadratic equation in one variable:

$$
\begin{aligned}
5x^2 &= 11x - 2\\
5x^2 - 11x + 2 &= 0
\end{aligned}
$$

Now we can use the quadratic formula (3.3.1) with $a = 5$, $b = -11$ and $c = 2$ to find the values of x which satisfy this quadratic:

$$x \quad = \quad \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$
\begin{aligned}
x &= \frac{11 \pm \sqrt{((-11)^2 - 4(5)(2))}}{2(5)} \\
x &= \frac{11 \pm 9}{10} \\
x &= 2 \quad or \quad x = 0.2
\end{aligned}
$$

We now substitute the solutions $x = 2$ and $x = 0.2$ into equation 4.14 in turn to find the corresponding solutions for y:

When $x = 2$ we have $y = 11(2) - 2 = 20$.

When $x = 0.2$ we have $y = 11(0.2) - 2 = 0.2$.

We can check both of these solutions by substituting the values of x and y into equation 4.15:

$$
\begin{aligned}
5x^2 = 5(0.2)^2 = 0.2 = y \\
5x^2 = 5(2)^2 = 20 = y
\end{aligned}
$$

Equation 4.15 is satisfied and therefore both pairs of solutions $x = 0.2, y = 0.2$ and $x = 4, y = 20$ are correct.

Example 7

$$
\begin{aligned}
x + y &= 7 && (4.16) \\
x^2 + 2y^2 &= 54 && (4.17)
\end{aligned}
$$

From equation 4.16 we have $x = y - 7$. Substituting this into equation 4.17 and rearranging gives:

$$
\begin{aligned}
(y-7)^2 + 2y^2 &= 54 \\
y^2 - 14y + 49 + 2y^2 &= 54 \\
3y^2 - 14y - 5 &= 0
\end{aligned}
$$

Using the quadratic formula with $a = 3$, $b = -14$ and $c = -5$ gives us the solutions $y = -\frac{1}{3}$ and $y = 5$. We substitute these solutions for y into equation 4.16 to find the corresponding solutions for x:

When $y = -\frac{1}{3}$, $x = -7\frac{1}{3}$ and when $y = 5$, $x = 2$.

We can check these solutions by substituting them into equation 4.17:

$$
\begin{aligned}
(2)^2 + 2(5)^2 = 4 + 2(25) = 54 \\
(-7\frac{1}{3})^2 + 2(-\frac{1}{3})^2 = 53\frac{7}{9} + 2(\frac{1}{9}) = 54
\end{aligned}
$$

Sometimes we are given information that we can use to form simultaneous equations as in the following example.

Example 8

In the St James' School Hall, some rows seat 25 people and some seat 15 people. There are 26 rows altogether. When the hall is full it seats 550 people. How many rows seat 25 people and how many seat 15?

We can solve this problem by letting x represent the number of rows with 25 seats and y represent the number of rows with 15 seats and writing the information given as two equations:

$$x + y = 26 \tag{4.18}$$
$$25x + 15y = 550 \tag{4.19}$$

Equation 4.18 represents the information that there are 26 rows of seats altogether. Equation 4.19 uses the information that there are 550 seats altogether.

We can solve this system of equations using substitution. We replace y in equation 4.19 by $(26 - x)$.

$$\begin{aligned} 25x + 15(26 - x) &= 550 \\ 25x + 390 - 15x - 550 &= 0 \\ 10x - 160 &= 0 \\ x &= 16 \end{aligned}$$

We have found that there are 16 rows with 25 seats and this means that there must be 10 rows with 15 seats. We can check this answer by substituting these values for x and y into equation 4.19.

$$25(16) + 15(10) = 400 + 150 = 550$$

Learning activity

1. Solve the following systems of equations using the method of substitution. Give your answers correct to two decimal places where appropriate.

   (a)

   $$\begin{aligned} x + 2y &= 3 \\ x^2 + 2y^2 &= 3 \end{aligned}$$

   (b)

   $$\begin{aligned} x + y &= 5 \\ xy &= 6 \end{aligned}$$

   (c)

   $$\begin{aligned} x + y &= 2 \\ 3x^2 - y^2 &= 1 \end{aligned}$$

2. At a concert tickets cost either \$4 or \$6. A total of 700 tickets are sold at a cost of \$3360. How many \$4 tickets were sold?

3. The difference between two numbers p and q is 21. The same two numbers add to 95. Write down two equations expressing the relationship between p and q. Hence find the values of p and q.
4. The Taylors and the Smiths have booked the same holiday. The Taylor family have to pay \$1880 for two adults and three children. The Smith family will pay \$2110 for three adults and two children. Write down the equation for each family and hence find the cost for each adult and each child.

4.4 Applications in business and economics

4.4.1 Supply and demand analysis

Balancing supply and demand is very important in business - if supply is greater than demand then there will be a surplus of goods and the price will fall. On the other hand if demand is greater than supply then prices will rise but ultimately customers will be unsatisfied. With supply and demand modelled by equations, we can find the point of *equilibrium* by solving the equations simultaneously. At this point, supply and demand are balanced.

Let P represent *price* and Q represent *quantity*. Then equations representing *supply* and *demand* can be written using the two variables P and Q. By solving these equations simultaneously, we can find the *Equilibrium Point* (P_E, Q_E) where supply and demand are equal.

The solution for P will be the *Equilibrium Price* P_E and the solution for Q will be the *Equilibrium Quantity* Q_E.

Worked example

A firm has the following supply and demand equations where Q is the quantity and P is the price of goods produced:

$$\begin{array}{lll} \text{Supply equation} & Q & = -40 + 6P \\ \text{Demand equation} & Q & = 240 - 8P \end{array}$$

We can find the point of equilibrium by solving the supply and demand equations simultaneously. This can be done graphically or algebraically as described in the previous sections. To demonstrate both methods, we will find the point of equilibrium using the elimination method and then draw the lines on a graph to show that they cross at this point.

Subtracting the supply equation from the demand equation leaves us with:

$$\begin{aligned} 0 &= 280 - 14P \\ 14P &= 280 \\ P &= 280/14 = 20 \end{aligned}$$

This means that the equilibrium price $P_E = 20$.

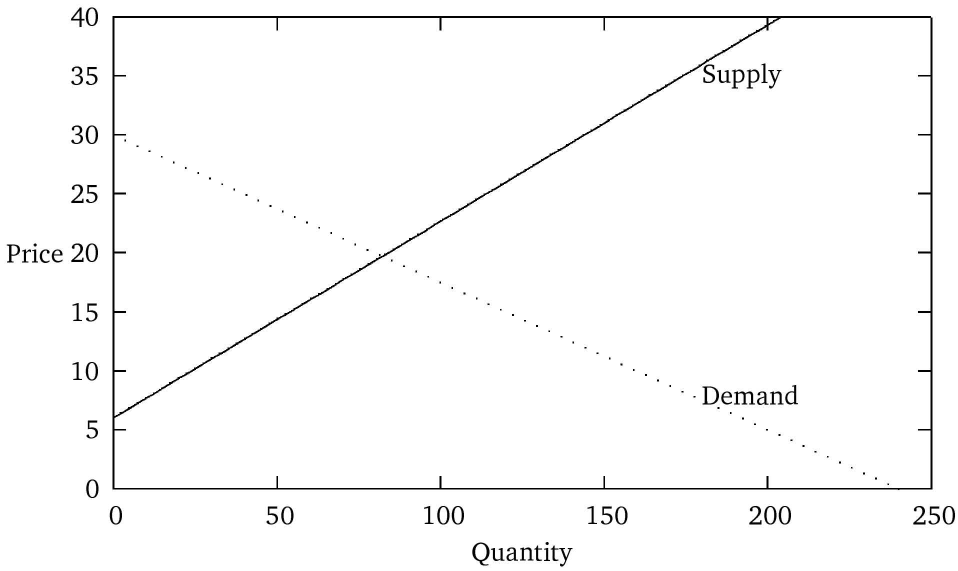
Substituting $P_E = 20$ into the supply equation will tell us the equilibrium quantity Q_E.

$$Q_E = -40 + 6(P_E) = -40 + 6(20) = 80$$

Thus the point of equilibrium is $(20, 80)$.

By drawing the two lines on a graph, we will show that the supply and demand lines cross at the point $(20, 80)$. Since *Quantity* is the controlling variable and *Price* is the dependent variable we should put *Quantity* on the x-axis and *Price* on the y-axis. This means that we need to re-arrange the supply and demand equations to make *Price* the subject:

$$\begin{array}{lll} \text{Supply equation} & 6P & = Q + 40 \\ \text{Demand equation} & 8P & = -Q + 240 \end{array}$$



Looking at the graph, we can see that there is a *surplus* of goods when $Q > 80$ which means that the price falls below the equilibrium price; and a *shortage* when $Q < 80$ which means that the price rises until it reaches the equilibrium price.

4.4.2 Income determination

The economy is in equilibrium if *income* is equal to *expenditure*. In a simple two-sector model, expenditure can be thought of as consumption plus investment. Thus the economy is in equilibrium if

$$income = consumption + investment$$

We usually use the letters:

- I to represent investment;

- C to represent consumption;
- Y to represent income.

When the equation below is satisfied there is income and expenditure balance. This equation is known as the *equilibrium equation*:

$$Y = C + I$$

The *Consumption Function* C is a function of *income* Y. In a simple model, consumption increases as income increases and the relationship between consumption and income can be expressed as:

$$C = C_0 + bY$$

where C_0 and b are constant values.

The amount of investment I can be fixed. This means that we are left with two variables Y and C. The two equations:

$$\begin{aligned} Y &= C + I \\ C &= C_0 + bY \end{aligned}$$

form a $2 * 2$ system which can be solved simultaneously to find the values of Y and C. This is known as *income determination*.

Worked example

For a *closed economy* in which there is no government intervention the consumption function is given by $C = 10 + 0.6Y$ and planned investment is $I = 12$.

Find the equilibrium levels of income and consumption.

Solution We know that $Y = C + I$ and since in this example $I = 12$ we have the following system of equations:

$$\begin{aligned} Y &= C + 12 \\ C &= 10 + 0.6Y \end{aligned}$$

We can solve this system of equations to find the equilibrium income Y by replacing C in the second equation by $Y - 12$ (from the first equation) . We can then solve the resulting equation for Y as follows:

$$\begin{aligned} Y - 12 &= 10 + 0.6Y \\ 0.4Y &= 22 \\ Y &= 22/0.4 = 55 \end{aligned}$$

Now we can find C by substituting $Y = 55$ back into the equilibrium equation:

$$\begin{aligned} Y &= C + 12 \\ 55 &= C + 12 \\ C &= 43 \end{aligned}$$

Thus the equilibrium levels in this model are $income = 55$ and $consumption = 43$.

Government expenditure

It is not very realistic to assume that an economy is closed with no government intervention. To make the model more realistic we can include *government expenditure* G, in the equilibrium equation.

$$Y = C + I + G$$

Given a consumption function and planned values for I and G we can find the equilibrium income by solving simultaneous equations as before.

Worked example

Find the equilibrium level of income given that $C = 135 + 0.8Y$, $I = 75$ and $G = 30$.

Substituting the given values of G and I into equilibrium equation $Y = C + I + G$ gives:

$$\begin{aligned} Y &= C + 75 + 30 \\ Y &= C + 105 \\ C &= Y - 105 \end{aligned}$$

Now by replacing C by $Y - 105$ in the consumption equation $C = 135 + 0.8Y$ we can find the equilibrium value of Y.

$$\begin{aligned} Y - 105 &= 135 + 0.8y \\ 0.2Y &= 240 \\ Y &= 240/0.2 = 1200 \end{aligned}$$

The equilibrium income in this model is $Y = 1200$.

4.4.3 IS-LM analysis

Up until now we have assumed that investment I is a constant value. However it is often more realistic to take I as a function of the rate of interest i. As the rate of interest rises the rate of investment falls. There is a linear relationship between I and i. If we are given the investment function and the consumption function of an economy, we can combine these functions to express the relationship between national income Y and interest rate i.

For example, suppose $C = 100 + 0.8Y$ and $I = -20i + 1000$.

Substituting these values into the equilibrium equation $Y = C + I$ gives:

$$\begin{aligned} Y &= (100 + 0.8Y) + (-20i + 1000) \\ 0.2Y &= 1100 - 20i \end{aligned}$$

This equation relating national income Y to interest rate i is called the **IS schedule**.

We may wish to find the values of Y and i. However we currently have one equation in two variables and so there is no unique solution. The correct solution will occur when the *money market* is in equilibrium. This means that the supply of money M_s is equal to the demand for money M_d.

The demand for money comes from three sources:

- The *transactions demand* — used for the daily exchange of goods and services.
- The *precautionary demand* — used to fund any emergencies requiring unforeseen expenditure.
- The *speculative demand* — used as a reserve fund for investment which falls as interest rates rise.

We use M_t - a linear function in Y, to represent the transactions and precautionary demand, and M_w - a linear function in i to represent the speculative demand. The demand for money $M_d = M_t + M_w$.

Thus the money market is in equilibrium if:

$$M_s = M_t + M_w$$

This equation is called the **LM schedule**.

For example, suppose $M_s = 2375$, $M_t = 0.1Y$ and $M_w = -25i + 2000$. Then the money market is in equilibrium if:

$$2375 = 0.1Y - 25i + 2000$$

Now we have two equations in two variables, namely the IS schedule and the LM schedule. Putting them together, we can solve the equations simultaneously to find the values of i and Y which satisfy the equilibrium equation when the money market is also in equilibrium.

$$\begin{array}{lll} \text{IS schedule} & 0.2Y & = 1100 - 20i \\ \text{LM schedule} & 2375 & = 0.1Y - 25i + 2000 \end{array}$$

Re-arranging the IS schedule gives:

$$Y = 5500 - 100i$$

Now substituting this value for Y in the LM schedule we can find the value of i as follows:

$$\begin{aligned} 2375 &= 0.1(5500 - 100i) - 25i + 2000 \\ 2375 &= 550 - 10i - 25i + 2000 \\ 35i &= 175 \\ i &= 5 \end{aligned}$$

Substituting $i = 5$ into the IS schedule we can now find the value of Y:

$$\begin{aligned} 0.2Y &= 1100 - 20(5) \\ Y &= 1000/0.2 \\ Y &= 5000 \end{aligned}$$

4.5 Learning outcomes

After working through this chapter and the relevant reading you should be able to:

- State the dimension of a system of equations and be familiar with the terms and concepts of *no solution*, *unique solution*, *multiple solutions* and *infinite solutions* in relation to that system.
- Use graphical methods to solve a $2 * 2$ system of equations.
- Use elimination and substitution to solve a $2 * 2$ system of equations algebraically.
- Identify and sketch simple supply and demand functions.
- Find the equilibrium price and quantity by graphical and algebraic methods.
- Calculate equilibrium income and consumption using the equilibrium equation.
- Analyse IS and LM schedules.

4.6 Sample examination questions

Question 1

a) Solve the equations

$$3x + 5y = 24$$
$$x - 3y = -20$$

[4]

b) A firm has the following supply and demand equations where Q is the quantity and P the price of goods produced.

$$\begin{aligned} \text{Supply equation } Q &= -36 + 4P \\ \text{Demand equation } Q &= 174 - 6P \end{aligned}$$

Find:

i) the value of P which brings equilibrium to the market
ii) the values of P which bring a surplus to the market
iii) the values of P which bring a shortage to the market.

[6]

Question 2

a) Solve the system of equations

$$2x - y = 11$$
$$x + 3y = -5$$

The lines $2x - y = 11$ and $x - 3y = -5$ divide the plane into 4 regions. Which of the two points $(0, 0), (2, -1)$ and $(-2, -2)$ are in the same region?

[6]

b) A company's profits are $P(x) = 20x - 3x^2$ and its costs are $C(x) = 6x + 8$. Sketch these functions for x between 0 and 5 on one graph. Use the graph to give the range of values of x for which the company makes a profit.

[4]

Question 3

a) Solve the system of equations

$$\begin{aligned} 3x - 2y &= 8 \\ -x - 4y &= 2 \end{aligned}$$

[3]

b) Find the break-even point for the monopolistic firm with revenue and cost functions:

$$\begin{aligned} R(x) &= -2x^2 + 14x \\ C(x) &= 2x + 10 \end{aligned}$$

[4]

c) Find the level of income Y that brings equilibrium to the economy, given:

$$\begin{aligned} Y &= C + I + G \\ C &= 300 + 0.75Y \\ I &= 25 + 0.15Y \\ G &= 175 \end{aligned}$$

[3]

Chapter 5

Matrices

Essential reading

See Chapter 5 of *Dowling* for further examples of the matrix operations. Answer questions 5.40 to 5.46 for further practice on this topic. Note that you do not need to perform row operations or Guassian elimination.

In this chapter we will introduce the concept of a *matrix* which is a convenient mathematical way of representing numerical data in a table. We will show how to manipulate matrices by addition, subtraction, multiplication and transposition.

Since tables of data occur naturally in many applications it is easy to see why being able to manipulate such information is important. We will also show how to represent systems of simultaneous equations in matrix form and how to form the related *augmented matrix* which is used to store systems of equations on a computer.

5.1 Introduction and definitions

A *matrix* (plural *matrices*) is a way of storing information in columns and rows. Each position in the matrix contains a particular piece of information. If a matrix has m rows and n columns then it is called a $(m * n)$ matrix. For example, the matrix M shown below is a $(3 * 2)$ matrix because it has 3 rows and 2 columns:

$$M = \begin{pmatrix} 2 & 4 \\ 1 & -1 \\ 7 & 0 \end{pmatrix}$$

A *vector* is a matrix which has only one row or one column.

A *row vector* is a $(1 * n)$ matrix such as:

$$V = \begin{pmatrix} 2 & 4 & -1 \end{pmatrix}$$

A *column vector* is a $(m * 1)$ matrix such as:

$$C = \begin{pmatrix} 2 \\ -1 \\ 8 \end{pmatrix}$$

A *square* matrix is one in which the number of rows equals the number of columns. For example S is a $(3 * 3)$ square matrix:

$$S = \begin{pmatrix} 2 & 4 & -2 \\ 1 & -1 & 0 \\ 7 & 0 & 3 \end{pmatrix}$$

Using subscripts enables us to refer to the position of a particular piece of data in the matrix. For example $m_{(2,1)}$ refers to the data in *row* 2 and *column* 1 of matrix M and $m_{(1,3)}$ refers to the data in *row* 1 and *column* 3. In the matrix M below we have $m_{(2,1)} = 1$ and $m_{(1,2)} = 4$.

$$M = \begin{pmatrix} 2 & 4 \\ 1 & -1 \\ 7 & 0 \end{pmatrix}$$

The identity matrix

The *identity* matrix is a special $(n * n)$ square matrix with 1's in all of the diagonal positions $m_{(1,1)}, m_{(2,2)}, \ldots, m_{(n,n)}$ and 0's in every other position. The identity matrix of dimensions $(n * n)$ is sometimes referred to as I_n. Shown below are the identity matrices of dimension $(1 * 1)$, $(2 * 2)$, $(3 * 3)$ and $(4 * 4)$.

$$I_1 = \begin{pmatrix} 1 \end{pmatrix}, I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The transpose of a matrix

Given a $(m * n)$ matrix M which has m rows and n columns, we can find the *transpose of* M written M^T by writing the rows of M as the columns of M^T. The dimensions of M^T will be $(n * m)$ since it will have n rows and m columns.

For example

$$M = \begin{pmatrix} 2 & 4 \\ 1 & -1 \\ 7 & 0 \end{pmatrix} M^T = \begin{pmatrix} 2 & 1 & 7 \\ 4 & -1 & 0 \end{pmatrix}$$

Learning activity

$$M = \begin{pmatrix} 3 & 6 & -1 \\ 4 & 0 & 7 \\ -4 & 2 & 5 \end{pmatrix}$$

1. What are the values of $m_{2,2}, m_{3,1}, m_{1,3}, m_{1,2}$ and $m_{2,3}$?
2. Write down the transpose of M.

5.2 Basic matrix operations

5.2.1 Addition and subtraction

We can only add or subtract matrices if they have exactly the same dimensions. If the dimensions of two matrices are the same, then we can add them by adding the terms in the same position. The resulting matrix will also have the same dimensions. For example the two matrices A and B below are both $2 * 2$ matrices and so we can add them together.

$$A = \begin{pmatrix} 3 & 2 \\ 7 & -1 \end{pmatrix}, B = \begin{pmatrix} -5 & 3 \\ 0 & 4 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 3+(-5) & 2+3 \\ 7+0 & -1+4 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 7 & 3 \end{pmatrix}$$

Similarly we can find the $2 * 2$ matrix $A - B$ by subtracting the terms in the same positions.

$$A - B = \begin{pmatrix} 3-(-5) & 2-3 \\ 7-0 & -1-4 \end{pmatrix} = \begin{pmatrix} 8 & -1 \\ 7 & -5 \end{pmatrix}$$

5.2.2 Scalar multiplication

We can multiply a matrix by a constant and this is known as *scalar multiplication*. Here it does not matter what the dimensions of the matrix are. We simply multiply every term in the matrix by the constant so the resulting matrix will have the same dimensions as the original matrix. For example:

$$3A = \begin{pmatrix} 3\,.\,3 & 3\,.\,2 \\ 3\,.\,7 & 3\,.\,(-1) \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 21 & -3 \end{pmatrix}$$

5.2.3 Matrix multiplication

We can only multiply two matrices together if the number of columns in the first matrix is equal to the number of rows in the second matrix. The resulting matrix has the same number of rows as the first matrix and the same number of columns as the second matrix.

$$\underset{(3*2)}{\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}} \underset{(2*3)}{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}} = \underset{(3*3)}{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}$$

We will start with an example of multiplying a $(2 * 2)$ matrix and a $(2 * 1)$ column vector. This will result in a $(2 * 1)$ column vector.

$$\begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} (1\,.\,3) + (2\,.\,4) \\ (5\,.\,3) + (7\,.\,4) \end{pmatrix} = \begin{pmatrix} 11 \\ 43 \end{pmatrix}$$

First we multiply *along* the top row of the matrix and *down* the vector, multiplying the first number in the row by the first number in the column and the second number in the row by the second number in the column and adding the results. The answer goes in the top row of the resulting vector. We repeat this process for the bottom row of the matrix.

When the second matrix has more than one column we treat each column in turn. Multiply along row i of the first matrix and down column j of the second matrix to get the answer for position $m_{(i,j)}$ in the resulting matrix.

For example using the same matrices A and B as before:

$$A = \begin{pmatrix} 3 & 2 \\ 7 & -1 \end{pmatrix}, B = \begin{pmatrix} -5 & 3 \\ 0 & 4 \end{pmatrix}$$

If we multiply A and B together then to find the answer in position $m_{(1,1)}$ we multiply the first row of A by the first column of B:

$$\begin{pmatrix} 3 & 2 \end{pmatrix}\begin{pmatrix} -5 \\ 0 \end{pmatrix} = (3\,.\,-5) + (2\,.\,0) = -15$$

To find the answer in position $m_{(1,2)}$ we multiply the first row of A by the second column of B:

$$\begin{pmatrix} 3 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = (3\,.\,3) + (2\,.\,4) = 17$$

The entire calculation looks like this:

$$AB = \begin{pmatrix} 3 & 2 \\ 7 & -1 \end{pmatrix}\begin{pmatrix} -5 & 3 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} (3\,.\,-5) + (2\,.\,0) & (3\,.\,3) + (2\,.\,4) \\ (7\,.\,-5) + (-1\,.\,0) & (7\,.\,3) + (-1\,.\,4) \end{pmatrix} = \begin{pmatrix} -15 & 17 \\ -35 & 17 \end{pmatrix}$$

Note that matrix multiplication is not commutative. This means that $AB \neq BA$.

Learning activity

Using the matrices A and B from above calculate BA to see for yourself that $BA \neq AB$.

Following is another example, this time we will multiply a $(3 * 2)$ matrix R by a $(2 * 3)$ matrix Q. The result will be a $(3 * 3)$ matrix RQ.

$$R = \begin{pmatrix} 2 & 7 \\ 4 & -1 \\ 0 & 3 \end{pmatrix}, Q = \begin{pmatrix} -8 & 5 & 6 \\ 1 & -4 & -2 \end{pmatrix}$$

$$RQ = \begin{pmatrix} (2\,.\,-8)+(7\,.\,1) & (2\,.\,5)+(7\,.\,-4) & (2\,.\,6)+(7\,.\,-2) \\ (4\,.\,-8)+(-1\,.\,1) & (4\,.\,5)+(-1\,.\,-4) & (4\,.\,6)+(-1\,.\,-2) \\ (0\,.\,-8)+(3\,.\,1) & (0\,.\,5)+(3\,.\,-4) & (0\,.\,6)+(3\,.\,-2) \end{pmatrix}$$

$$= \begin{pmatrix} -9 & -18 & -2 \\ -33 & 24 & 26 \\ 3 & -12 & -6 \end{pmatrix}$$

Note that this time there is no question of calculating QR since the number of rows in Q does not equal the number of columns in R. We say that the matrix QR is *not defined*.

The identity matrix and multiplication

Multiplying a matrix by the identity matrix has no effect on the matrix. It is the same as multiplying a number by 1. $MI = M$ and $IM = M$ for any matrix M.

For example, multiply the $2 * 3$ matrix Q by the $3 * 3$ identity matrix I_3 to check that $QI = Q$. Multiply the $2 * 2$ identity matrix I_2 by Q to check that $IQ = Q$.

$$QI = \begin{pmatrix} -8 & 5 & 6 \\ 1 & -4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Q$$

$$IQ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -8 & 5 & 6 \\ 1 & -4 & -2 \end{pmatrix} = Q$$

Learning activity

Given the matrices

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 6 & 3 \\ 7 & -2 \\ 1 & 5 \end{pmatrix}, C = \begin{pmatrix} 5 \\ 2 \end{pmatrix} D = \begin{pmatrix} 5 & 0 & -2 \\ 1 & 7 & 2 \end{pmatrix}$$

a) State the dimensions of each of these matrices.

b) Find:

   i) $2B$

   ii) $-3A$

   iii) A^T

   iv) D^T

c) Say whether the following matrices are defined:

   i) $A + B$

   ii) AC

   iii) CA

   iv) $C + AC$

   v) BD

vi) $BD - A$

vii) DA

viii) $C^T A$

ix) $B + D^T$

d) Compute the matrices which are defined in part c) above.

5.3 Representing simultaneous equations as matrices

Consider the $2 * 2$ system of simultaneous equations:

$$\begin{aligned} 2x + 3y &= 19 \\ 4x - y &= 3 \end{aligned}$$

The left-hand sides of these equations can be represented by a matrix of *co-efficients* multiplied by a column vector of *variables*:

$$\begin{array}{cc} \text{co-efficients} & \text{variables} \\ \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} & \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

If we multiply the co-efficient matrix and the variable vector together, we end up with a $(2 * 1)$ column vector. Each position in this vector holds one of the left-hand sides of the original simultaneous equations:

$$\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (2 \,.\, x) + (3 \,.\, y) \\ (4 \,.\, x) + (-1 \,.\, y) \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 4x - y \end{pmatrix}$$

By forming a $(2 * 1)$ column vector of the right-hand sides of the equations, the *constants*, we can represent the entire system of equations in matrix form:

$$\begin{array}{ccccc} co-efficients & variables & & constants \\ \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} & \begin{pmatrix} x \\ y \end{pmatrix} & = & \begin{pmatrix} 19 \\ 3 \end{pmatrix} \end{array}$$

This is called a *matrix equation*.

5.3.1 The augmented matrix

Writing three matrices to represent co-efficients, variables and constants is time and space consuming. We can represent the same information more efficiently using an *augmented matrix*. This is the form in which systems of equations are stored in a computer.

The augmented matrix for the system of simultaneous equations above is:

$$\left(\begin{array}{cc|c} 2 & 3 & 19 \\ 4 & -1 & 3 \end{array}\right)$$

A vertical line separates the co-efficient matrix from the constant vector. The variable vector is omitted since the *names* of variables are not important. We can tell from the augmented matrix how many variables there are in the system by counting the number of columns in the co-efficient matrix. When we expand the augmented matrix back into a system of equations we can call the variables $x_1, x_2, x_3 \ldots$.

For example, the augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 12 \\ 3 & -1 & 4 & -9 \\ 1 & 5 & -2 & 25 \end{array}\right)$$

represents the following system of simultaneous equations:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 12 \\ 3x_1 - x_2 + 4x_3 &= -9 \\ x_1 + 5x_2 - 2x_3 &= 25 \end{aligned}$$

There are methods which can be used to solve simultaneous equations using matrices. However these methods are beyond the scope of this course. You only have to be able to represent a system of simultaneous equations using an augmented matrix.

Learning activity

1. Represent the following system of equations as an augmented matrix

$$\begin{aligned} 4x - 2y + 3z &= 1 \\ x + 3y - z &= 2 \\ 6x + 4y - 2z &= -4 \end{aligned}$$

2. Write the system of simultaneous equations that are represented by the following augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & 3 \\ 4 & -2 & 1 & -11 \\ 3 & 3 & -2 & -4 \end{array}\right)$$

5.4 Applications in business and economics

Matrices can be used to store or represent tables of information. Matrix equations can then be used to answer questions about the information. This is demonstrated in the following examples.

Worked example 1

The daily sales for a fast-food chain with three shops are as follows:

Units Sold	Shop A	Shop B	Shop C
Burgers	700	600	500
Chips	900	650	600
Drinks	550	800	550

For each unit sold the profit is \$0.30 for burgers, \$0.35 for chips and \$0.40 drinks. Use matrix multiplication to find the profit for each shop.

We can represent this information using matrices as follows:

$$\begin{pmatrix} 700 & 900 & 550 \\ 600 & 650 & 800 \\ 500 & 600 & 550 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.35 \\ 0.4 \end{pmatrix} = \begin{pmatrix} Profit\ A \\ Profit\ B \\ Profit\ C \end{pmatrix}$$

Applying matrix multiplication we can calculate the profit for each shop.

$Profit\ A = (700\ .\ 0.30) + (900\ .\ 0.35) + (550\ .\ 0.40) = \$745$
$Profit\ B = (600\ .\ 0.30) + (650\ .\ 0.35) + (800\ .\ 0.40) = \$727.5$
$Profit\ C = (500\ .\ 0.30) + (600\ .\ 0.35) + (550\ .\ 0.40) = \$580$

Notice how we have had to *transpose* the original table of information when writing the matrix equation. Why is that?

Worked example 2

A team of four salespersons record the following sales over a period of one month:

Person	Televisions	Fridges	Freezers	Cookers
A	15	20	17	23
B	19	26	25	31
C	12	30	31	20

The commission each salesperson receives is \$15 for each television, \$10 for each fridge, \$12 for each freezer and \$14 for each cooker.

We can express these figures in matrix form as follows:

$$\begin{pmatrix} 15 & 20 & 17 & 23 \\ 19 & 26 & 25 & 31 \\ 12 & 30 & 31 & 20 \end{pmatrix} \begin{pmatrix} 15 \\ 10 \\ 12 \\ 14 \end{pmatrix} = \begin{pmatrix} Commission\ A \\ Commission\ B \\ Commission\ C \end{pmatrix}$$

Using matrix multiplication we can calculate the commission earned by each of the salespersons.

$Commission\ A = (15\ .\ 15) + (20\ .\ 10) + (17\ .\ 12) + (23\ .\ 14) = \$951$
$Commission\ B = (19\ .\ 15) + (26\ .\ 10) + (25\ .\ 12) + (31\ .\ 14) = \$1279$
$Commission\ C = (12\ .\ 15) + (30\ .\ 10) + (31\ .\ 12) + (20\ .\ 14) = \$1132$

Worked example 3

The market share for two products x and y changes from week to week according to the following matrix equation where t is the time in weeks.

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

Given that $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$ we can find the sequence of matrices:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}, \begin{pmatrix} x_4 \\ y_4 \end{pmatrix}, \begin{pmatrix} x_5 \\ y_5 \end{pmatrix}, \ldots$$

In this example

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.38 \\ 0.62 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{pmatrix} \begin{pmatrix} 0.38 \\ 0.62 \end{pmatrix} = \begin{pmatrix} 0.362 \\ 0.638 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{pmatrix} \begin{pmatrix} 0.362 \\ 0.638 \end{pmatrix} = \begin{pmatrix} 0.3638 \\ 0.6362 \end{pmatrix}$$

$$\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{pmatrix} \begin{pmatrix} 0.3638 \\ 0.6362 \end{pmatrix} = \begin{pmatrix} 0.36362 \\ 0.63638 \end{pmatrix}$$

Thus the market share after five weeks is 0.36362 for products x and 0.63638 for product y.

5.5 Learning outcomes

After studying this chapter and the relevant reading you should be able to:

- Use the correct notation and terminology of matrices.
- Find the transpose of a matrix.
- Add and subtract matrices of the same dimension.
- Multiply a matrix by a scalar (single value).
- Multiply matrices together or say that the resulting matrix is undefined.
- Represent a system of linear equations in matrix notation.
- Form the augmented matrix for a system of equations.
- Represent and manipulate tabular information using matrices.

5.6 Sample examination questions

Question 1

a) Given the matrices

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

determine which of the following expressions represent valid matrices:

i) $A + B$

ii) AB

iii) $A + BC$

iv) $CB - A$

Compute the valid matrices.

[4]

b) A company has three stores S_1, S_2, S_3 and sells four products P_1, P_2, P_3, P_4. The number of items of each product at each store is given by the following table:

	P_1	P_2	P_3	P_4
S_1	100	200	300	200
S_2	120	150	160	100
S_3	150	100	250	120

The profit for each product is $10, $12, $20, $10 respectively for P_1, P_2, P_3, P_4. Express these facts in terms of a matrix and a vector and illustrate matrix multiplication to compute the profit made by each store.

[6]

Question 2

a) Given the matrices

$$A = \begin{pmatrix} 3 & 0 \\ -2 & 1 \\ 5 & 2 \end{pmatrix}, B = \begin{pmatrix} 6 & -4 \\ 3 & 4 \end{pmatrix}, C = \begin{pmatrix} 2 & 4 & 2 \\ -7 & 3 & 1 \end{pmatrix}$$

find the following matrices:

i) AC

ii) $CA + B$

[4]

b) The market share for two products x and y changes from week to week according to the following matrix equation:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

where t is the time in weeks.

Given that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.24 \\ 0.76 \end{pmatrix}$ find:

i) $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

ii) $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$

iii) the market share for x after 5 weeks.

[6]

Chapter 6

Linear programming

Essential reading

See Chapter 7 of Dowling for further examples of the material covered in this chapter. Answer supplementary problems 7.24 to 7.43 to test your understanding of inequalities and linear programming techniques.

We looked at linear equations and their graphs in Chapter 2. However, not all relationships between two variables can be represented by equations. Sometimes it is more appropriate to represent the relationship using *inequalities* where one of the symbols $>, \geq, <$ or $\leq$ is used instead of an equals sign.

A common problem that arises in industry is the determination of a production schedule that will maximize the profit a firm can make on selling a range of articles. The number of articles of each type the firm can produce will be limited by the availability of the resources needed for their production; for example, raw materials, machine time, labour, investment capital, etc.

These constraints on production lead us to use a mathematical model that involves inequalities rather than equations. The variables represent the numbers of each type of article the firm might produce. If the constraints and the profit can be expressed as linear functions of these variables then the problem of maximizing the profit can be solved by a technique known as *linear programming*.

We will first look at inequalities in more detail and then introduce a graphical approach for solving a problem involving several linear inequalities in two variables. The region on a graph in which all constraints are satisfied is called the *feasible region*. Profit is always maximised at a corner of the feasible region; by sketching the feasible region given by a set of production constraints we can optimise production and maximise profit.

6.1 Inequalities

An *equation* must include an *equals* sign $=$. In an inequality, the equals sign is replaced by one of the signs $<, \leq, >$ or $\geq$ where:

- $x < y$ means x is *strictly less than* y
- $x \leq y$ means x is *less than or equal to* y
- $x > y$ means x is *strictly greater than* y
- $x \geq y$ means x is *greater than or equal to* y.

Inequalities can be re-arranged in the same way as equations. This might be necessary for example in order to change the subject of the inequality. The same rules apply as when manipulating equations. We can add (or subtract) the same amount to both sides of the inequality. This means that we can move a term from one side of the inequality to the other remembering that the sign will change when the term appears on the other side. We can also multiply (or divide) all of the terms in the inequality by the same value. Note however that if we multiply (or divide) the terms of an inequality by a negative number then the inequality sign changes direction. This is because if $a \geq b$ then $-a \leq -b$.

For example, we will make y the subject of the following inequality:

$$\begin{aligned} 2y + 3x &> 4x + 3y \\ 2y - 3y &> 4x - 3x \\ -y &> x \\ y &< -x \end{aligned}$$

Notice how the inequality sign has changed from $>$ to $<$ in the last line. This is because in order to make y positive, we have multiplied both sides of the inequality by -1.

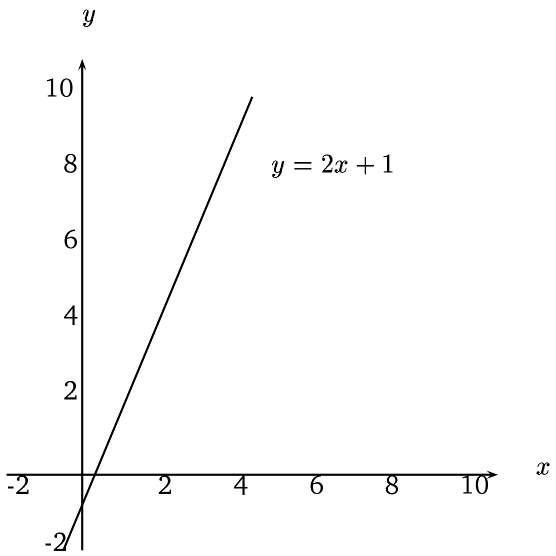
6.1.1 Sketching inequalities on a graph

In Chapter 2 we discovered that a linear equation of the form $y = mx + c$ can be represented by a straight line graph. We can give a similar graphical interpretation to a linear inequality of the form:

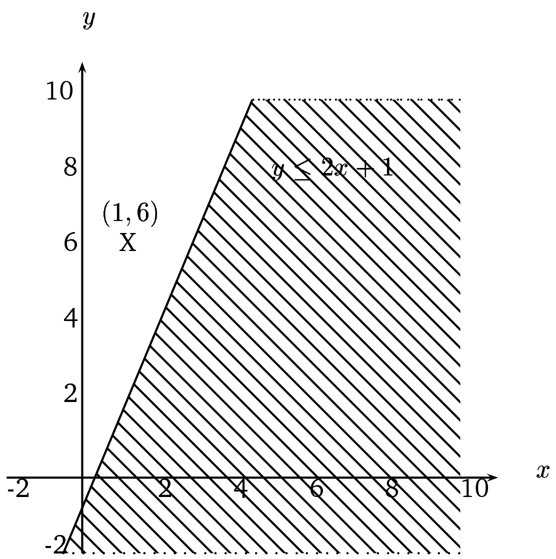
$$\begin{aligned} y &< mx + c \\ y &\leq mx + c \\ y &> mx + c \\ y &\geq mx + c \end{aligned}$$

To draw a graph representing one of these inequalities, we first sketch the line $y = mx + c$ and then decide which side of the line the points which satisfy the inequality lie. Such points will all be on the same side of the line.

For example, to illustrate the inequality $y \leq 2x - 1$ we first draw the line $y = 2x - 1$.

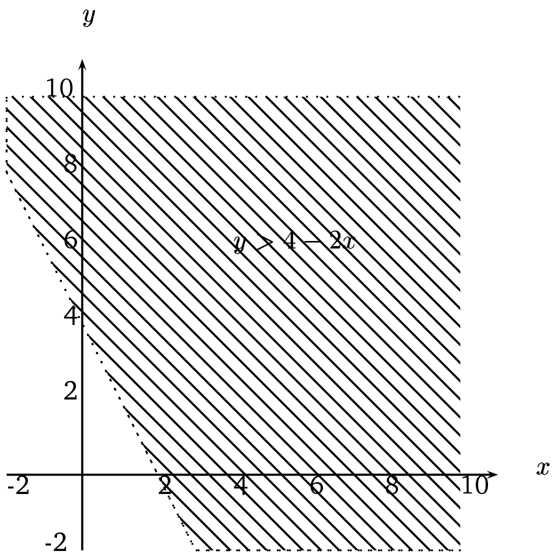


Next we choose any point which is not on the line and see if its x and y co-ordinates satisfy the inequality. We will take the point $(1, 6)$. This point is not in the region where the inequality is satisfied since $6 \not\leq 2(1) - 1$. The point $(1, 6)$ is above the line and does not satisfy the inequality. Therefore the region which *is* satisfied by the inequality is the region under the line. We show this by shading part of the region as below.



Note that when we are illustrating inequalities, we draw an unbroken line to represent $\leq$ or $\geq$ as in the example above. A point which lies on the line is also included in the region satisfied by the inequality.

However, if the inequality is $<$ or $>$ then we use a broken line. This time a point which lies on the line is not included in the region satisfied by the inequality. For example, the graph below shows the region satisfied by the inequality $y > 4 - 2x$.



Learning activity

Sketch graphs showing the region defined by each of the following inequalities:

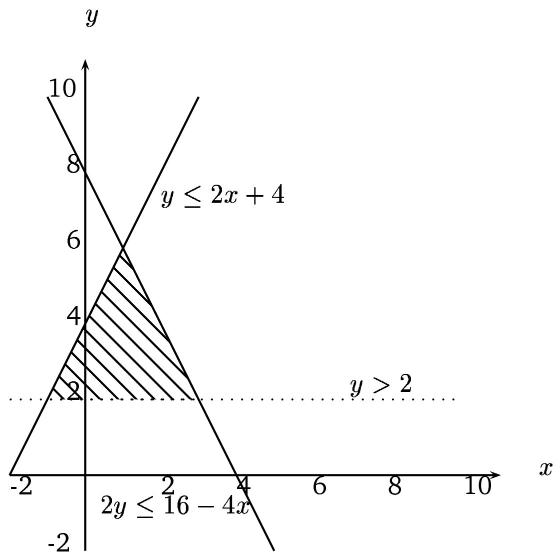
1. $x + 2y \leq 12$
2. $y \leq 3 + x$
3. $2x > 8 - 4y$

6.1.2 Sketching regions identified by inequalities on a graph

Just as we can solve simultaneous *equations* (see Chapter 4), we can also solve simultaneous *inequalities*. Assuming that we have a system of linear inequalities with two unknowns x and y, this means that we find the set of points (x, y) which simultaneously satisfy all of the inequalities. We can do this by drawing lines representing each of the inequalities, shading the correct side of each line and finding the region which is on the correct side of all of the lines.

For example, the following graph shows the region which satisfies the system of inequalities:

$$\begin{aligned} y &> 2 \\ y &\leq 2x + 4 \\ 2y &\leq 16 - 4x \end{aligned}$$



Learning activity

Sketch lines representing the following inequalities on the same graph:

$$x > -2$$
$$y \ge x - 2$$
$$y + 2x \le 4$$

Hence shade the region which simultaneously satisfies all of these inequalities.

6.2 Graphical solutions of linear programming problems

6.2.1 Production constraints and the feasible region

In Chapter 2, we saw how to use graphs of linear functions to model production constraints. For example, suppose that a factory can manufacture two different products. The first takes 1 hour per unit and the second 2 hours per unit. There is a maximum time constraint of 50 hours. We can draw the linear graph of $x_1 + 2x_2 = 50$ to see what combination of products x_1 and x_2 could be manufactured in the time available.

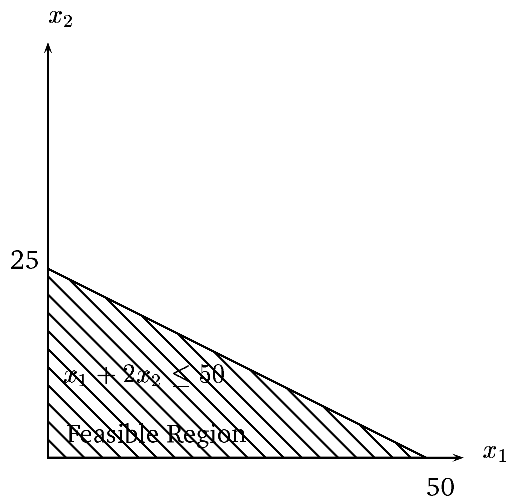
The factory might not use all of the 50 available manufacturing hours. Thus the *time constraint* can be expressed as a linear inequality instead of an equation:

$$x_1 + 2x_2 \le 50$$

The factory cannot make less than zero of either product and therefore we also have the two *non-negativity constraints*:

$$x_1 \ge 0$$
$$x_2 \ge 0$$

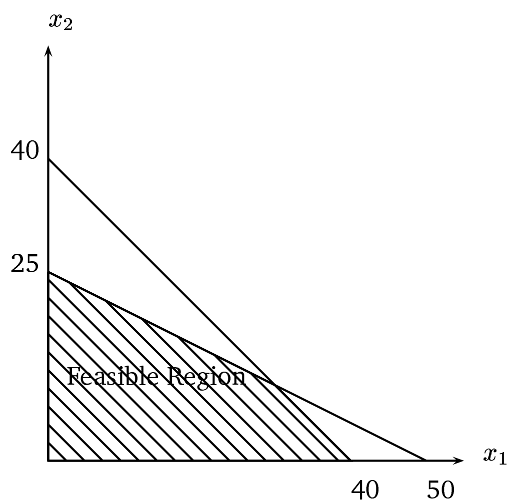
Below is a graph which shows the region which is satisfied by these three inequalities. This is called the *feasible region*. Any point (x_1, x_2) which lies inside the feasible region is a possible combination of products x_1 and x_2 which could be manufactured within the time constraint.



Very often there is more than one production constraint. Continuing with the previous example, suppose the raw materials for either of the two products x_1 and x_2 cost \$10, and that there is a maximum budget of \$400 for buying raw materials. Now we have a *cost constraint* which can be expressed as:

$$10x_1 + 10x_2 \leq 400$$

By drawing this inequality onto the previous graph, we can see how this cost constraint affects the feasible region.



We can see that the size of the feasible region has decreased with the addition of another constraint.

6.2.2 Isoprofit lines and profit maximisation

The idea of drawing a feasible region is to see how many of each type of product should be manufactured. Generally we would want to choose to manufacture the combination of products which will generate the most profit. We can see which product combination will make the most profit by drawing parallel *isoprofit lines* on the graph as demonstrated in the example below.

Again continuing with our previous example, suppose the profit made by each product x_1 is \$15 and the profit made by each product x_2 is \$20. Then using π to represent profit we have:

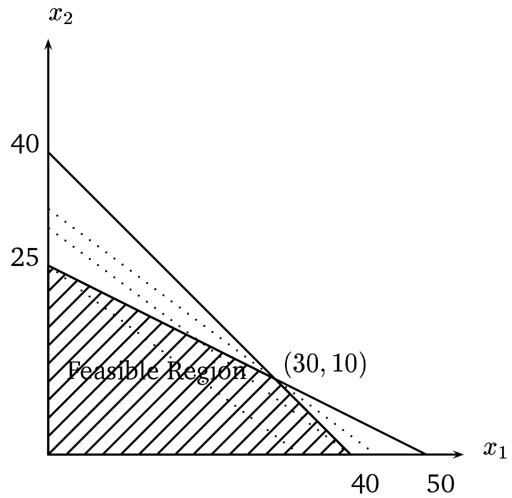
$$\pi = 15x_1 + 20x_2$$

This function is called the *objective function* - the object being to maximise the profit π.

If we re-arrange the objective function into the form $x_2 = mx_1 + c$, we will find the *gradient* m of the isoprofit lines.

$$x_2 = -\frac{3}{4}x_1 + \frac{\pi}{20}$$

The gradient of the isoprofit lines is equal to $-\frac{3}{4}$. We draw parallel lines of gradient $-\frac{3}{4}$ through points in the feasible region.



The isoprofit line which allows for the highest profit (it is the highest line) goes through the point $(30, 10)$ at one corner of the feasible region. This point is the point where profit is maximised.

Therefore in this example, the factory should manufacture 30 units of product x_1 and 10 units of product x_2. This will generate a maximum profit of

$$\pi = 15(30) + 20(10) = \$650$$

6.2.3 The extreme-point theorem

It is no coincidence that in the preceding example the maximum profit point was found to be at a corner of the feasible region. The *extreme-point theorem* states that if an optimal feasible value of the objective function exists, it will always be found at one of the extreme points of the boundary.

This means that the point where profit is maximised will always be at an intersection of the constraint lines. Or more simply, at a corner of the feasible region.

Therefore it is possible to find the maximum profit by drawing the feasible region and calculating the profit at each corner point.

The feasible region in our example has four corner points, namely $(0, 25), (30, 10), (40, 0)$ and $(0, 0)$. The profit made at each of these points is \$500, \$650, \$600 and \$0 respectively.

6.2.4 Worked example

Following is another worked example demonstrating use of the objective function, production constraints, non-negativity constraints, isoprofit lines and the extreme-point theorem.

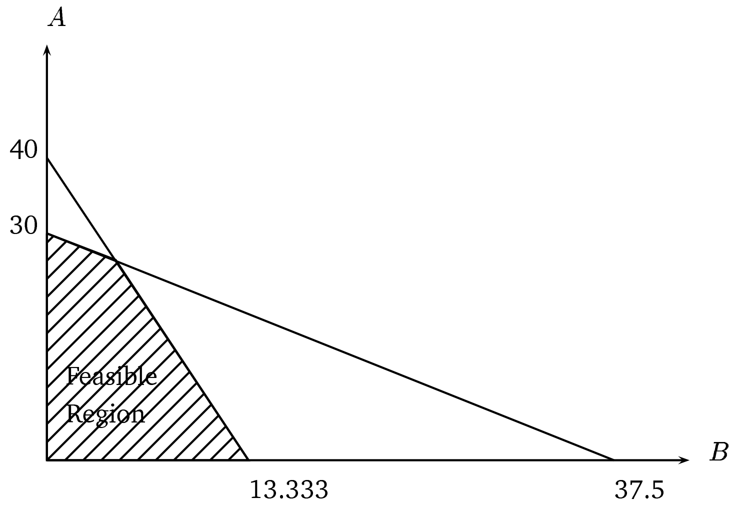
A factory produces two products, type A and type B. Each batch of type A requires three hours of machine P time and five hours of machine Q time, whereas each batch of type B requires nine hours of machine P time and four hours of machine Q time. The factory has 120 hours of machine P time and 150 hours of machine Q time available each week.

The factory makes a profit of \$3 on a batch of product A and \$6 on a batch of product B.

a) Formulate the problem of finding how many batches of products A and B the factory should produce each week in order to maximise profits as a linear programming problem in two variables.
b) Draw a sketch graph to show the feasible region for this problem.
c) Determine the number of batches of products A and B that the factory should produce in order to maximise the weekly profit and find this profit.

Solution

a) We have
   - *objective function* $\pi = 3A + 6B$
   - *production constraints* $3A + 9B \leq 120$ and $5A + 4B \leq 150$
   - *non-negativity constraints* $A \geq 0$ and $B \geq 0$.
b) If we put A on the y-axis and B on the x-axis, then we can find the feasible region by drawing the lines $A = -3B + 40$, $A = -\frac{4}{5}B + 30$, $A = 0$ and $B = 0$. This is shown on the graph below. Note that we have chosen to put A on the y-axis because in this particular example it is easier to re-arrange the production constraints to make A the subject rather than B. It does not matter which product goes on which axis, but you must re-arrange all the equations in the same way.

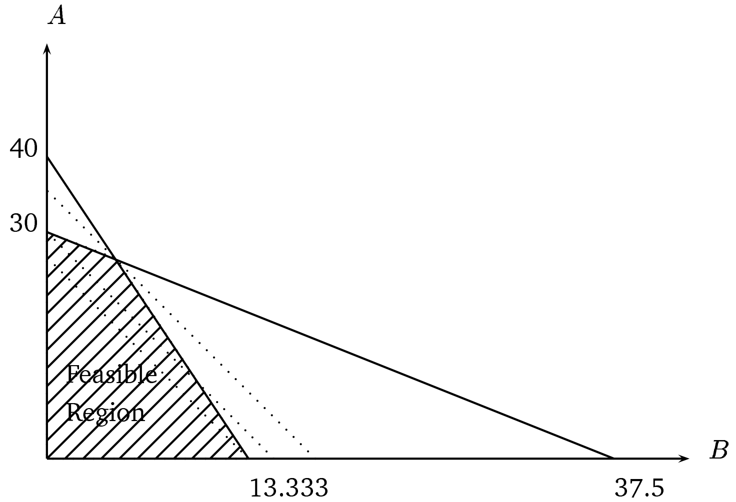


c) We can find the point where the maximum profit is made by either:

- finding the gradient of the isoprofit line and drawing parallel isoprofit lines on the graph, or
- finding the corners of the feasible region and calculating the profit at each of these points.

We will use both methods here for demonstration purposes but be aware that you need only use one method.

First, the objective function is $\pi = 3A + 6B$. Re-arranging to make A the subject gives us $A = -2B + \frac{\pi}{3}$. Therefore the isoprofit lines have gradient -2. Drawing isoprofit lines through each of the corner points shows us that the profit is maximised at the point where the two production constraint lines intersect.



This is at the point where $B = 4.5$ (1 $d.p$) and $A = 26.5$ (1 $d.p.$). Since the factory cannot manufacture half of a product we take the closest point inside the feasible region which represents a whole number of both products. That point is $B = 4, A = 26$. The profit at this point is given by:

$$\pi = 3(26) + 6(4) = \$102$$

Alternatively we could arrive at the same answer by calculating the profit at each of the corner points of the feasible region.

These corner points are $(0,0), (0,30), (13\frac{1}{3}, 0)$ and $(4.5, 26.5)$. Again we cannot calculate profit except at a point which represents whole numbers of products so the third corner is taken to be the point $(13, 0)$ and the fourth corner is taken to be the point $(4, 26)$. We calculate the profit at each of these points using the objective function $\pi = 6B + 3A$ as follows:

Point $(0, 0)$ profit = \$0
Point $(0, 30)$ profit = 6(0)+3(30)=\$90
Point $(13, 0)$ profit = 6(13)+3(0)=\$78
Point $(4, 26)$ profit = 6(4)+3(26)=\$102

By the extreme-point theorem the maximum profit must occur at one of the corners of the feasible region and therefore we can say that the maximum profit in this case is \$102 which occurs when 26 batches of A and 4 batches of B are produced.

6.3 Learning outcomes

After studying this chapter and the relevant readings you should be able to:

- Make correct use of the symbols $\leq, <, \geq, >$ and manipulate linear inequalities.
- Identify the region of a graph defined by a linear inequality.
- Sketch the feasible region defined by simultaneous linear inequalities.
- Identify and graph the objective function as a series of isoprofit lines.
- Solve linear programming problems graphically making use of the extreme-point theorem.

6.4 Sample examination questions

Question 1

A company makes two products x_1 and x_2. Each item of the first product requires three hours to manufacture and two hours to package. Each item of the second product requires one hour to manufacture and one and a half hours to package. The plant has 40 hours available for manufacturing and 40 hours available for packaging. Each item of the first product makes a profit of \$3 and each item of the second product makes a profit of \$2.

a) Model the problem of finding the number of items of each product that the company should make in order to maximise its profit as a *linear programming problem* in two unknowns.

[3]

b) Draw a sketch graph to show the feasible region for this problem.

[3]

c) Find the co-ordinates of the four corner points of the feasible region. Hence determine the number of each product that the company should make to maximise its profit and find this maximum profit.

[4]

Question 2

A manufacturer produces two types of machines in a factory and the hours of operator time, the cost and floor space involved in each are given in the following table:

	Floor space in m^2	Costs	Hours of operator time	Profit
Machine A	3	4	3	20
Machine B	2	8	4	25

There is a maximum floor space available of $150m^2$, a maximum budget of $320 and the maximum number of operator hours available is 180.

a) Formulate the problem of finding how many of each type of machine the firm should produce per week in order to maximise profits as a linear programming problem in two variables.

[3]

b) Draw a sketch graph to show the feasible region for this problem.

[4]

c) Determine the number of each type of machine the firm should produce per week in order to maximise the profit and find this profit.

[3]

Appendix A

Sample test papers

The examination for CIS001 will cover the whole of the syllabus. Since we have not covered the whole syllabus in Volume 1 of the study guide it would be inappropriate to include a sample examination paper here. Instead there follow two tests similar to those set as assignments in previous years.

Each of the following tests has a duration of one hour. There is a maximum of 50 marks for each test.

A.1 Test 1

Question 1

Use your calculator to evaluate the following expressions correct to two decimal places where $x = \sqrt{2}$.

a) $\frac{x^3}{2} - \frac{2}{x^2}$

[3]

b) $\frac{1}{(3x)^3}$

[3]

Question 2

Simplify the following expressions:

a) $(2p - q)q + (2p - q)^2$

[3]

b) $\frac{4u^2 - 6uv^2}{2uv}$ where $u, v \neq 0$

[3]

Question 3

a) Solve the following equation:

$$\frac{3}{x} + \frac{x}{x+1} = 1$$

[6]

b) Given $f(x) = x^3 + 4x^2 + x - 6$ find $f(2)$ and $f(-2)$. Hence solve the equation

$$x^3 + 4x^2 + x - 6 = 0$$

[6]

Question 4

A company's profits are increasing at a constant rate and were \$5.5 million in 2000 and \$6.7 million in 2003.

a) If this trend continues find the expected profit in 2010.

[4]

b) Find an expression for the profit (in millions of dollars) in terms of the years since 2000.

[4]

Question 5

a) Solve the equation

$$x^2 - 14x + 40 = 0$$

Hence sketch the graph of $y = x^2 - 14x + 40$ for $0 \le x \le 12$, showing clearly any intercepts with the axes.

[8]

b) A company's revenue function is given by:

$$R(x) = 48x - 3x^2$$

and its profit function is given by

$$P(x) = -3x^2 + 42x - 120$$

Find:

i) the break-even points
ii) the maximum profit
iii) the total cost function.

[10]

A.2 Test 2

Question 1

Given the matrices

$$L = \begin{pmatrix} 3 & 0 \\ 4 & 1 \\ -2 & 2 \end{pmatrix}, M = \begin{pmatrix} 2 & -3 & 1 \\ 5 & 7 & 1 \end{pmatrix}, N = \begin{pmatrix} 1 & 2 \\ -1 & 6 \end{pmatrix}$$

which of the following matrices may be evaluated:

$$MN; ML; LM - N; NML?$$

Calculate the valid matrices.

[10]

Question 2

Given $f(x) = 6x^2 - 5x$ and $g(x) = -3x + 4$:

a) $f(x) + g(x)$

b) $f(x) - g(x)$

c) $f(x).g(x)$

d) $f(g(x))$

e) $g(f(x))$

[10]

Question 3

a) Sketch the region which satisfies the inequalities

$$\begin{aligned} x - 3y &\leq 9 \\ 2x + y &\geq 4 \\ x &\geq 0 \\ y &\leq 0 \end{aligned}$$

indicating the region clearly by shading. Show any intercepts with the axes.

[8]

b) Find the point of intersection of the lines $x - 3y = 9$ and $2x + y = 4$.

[2]

Question 4

A company sells three products called p_1, p_2 and p_3 and has two warehouses w_1 and w_2. In warehouse 1 it has 30000, 25000 and 10000 items of p_1, p_2 and p_3 respectively. In warehouse 2 it has 15000, 35000 and 20000 items of p_1, p_2 and p_3.

a) Write down a matrix M corresponding to the company stock which is held in the two warehouses, with rows indexed by warehouse and columns by products.

[4]

b) The value of products p_1, p_2 and p_3 is 5, 4 and 2 dollars respectively. Writing these as a column vector, X, determine $Y = MX$ and say what this product Y represents.

[6]

Question 5

A manufacturer produces radios and cassette recorders. Each radio requires 1 hour to make and the materials cost \$32. Each cassette recorder takes 40 minutes to make and the materials cost \$40. The company can devote at most 34 hours per week to producing radios and cassette recorders and the weekly cost for materials must not exceed \$1200. The company can sell each radio they produce at a profit of \$12 and each cassette recorder at a profit of \$14.

a) Model the problem of finding the number of radio and cassette recorders that the company should make each week in order to maximise their profits as a linear programming problem in two unknowns.

[3]

b) Draw a sketch graph to show the feasible region for this problem.

[3]

c) Find the co-ordinates of the four corner points of the feasible region. Hence determine the number of radios and cassette recorders the company should make each week to maximise their profit and find this maximum profit.

[4]

Appendix B

Solutions to the sample test papers

B.1 Test 1 solutions

Question 1

a) $\frac{(\sqrt{2})^3}{2} - \frac{2}{(\sqrt{2})^2} = \frac{2\sqrt{2}}{2} - \frac{2}{2} = \sqrt{2} - 1 = 0.41(2d.p.)$

b) $\frac{1}{(3\sqrt{2})^3} = \frac{1}{27\ .\ 2\ .\ \sqrt{2}} = 0.01(2d.p.)$

Question 2

a) $(2p-q)q + (2p-q)^2 = (2p-q)(q+(2p-q)) = 2p(2p-q)$

b) $\frac{4u^2-6uv^2}{2uv} = \frac{2u(2u-3v^2)}{2uv} = \frac{2u-3v^2}{v}$

Question 3

a)

$$\begin{aligned} \frac{3}{x} + \frac{x}{x+1} &= 1 \\ 3(x+1) + x^2 &= x(x+1) \\ 3x + 3 + x^2 &= x^2 + x \\ 2x &= -3 \\ x &= -\frac{3}{2} \end{aligned}$$

b) $f(2) = (2)^3 + 4(2)^2 + (2) - 6 = 8 + 16 + 2 - 6 = 20$
$f(-2) = (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0$
Since $f(-2) = 0$ we know that $(x+2)$ is a linear factor of $x^3 + 4x^2 + x - 6$. Dividing $x^3 + 4x^2 + x - 6$ by $(x+2)$ gives:

$$\begin{aligned} x^3 + 4x^2 + x - 6 &= (x+2)(x^2+2x-3) \\ &= (x+2)(x-1)(x+3) \end{aligned}$$

Therefore the equation has three solutions, namely:
$x = -2, x = 1, x = -3$.

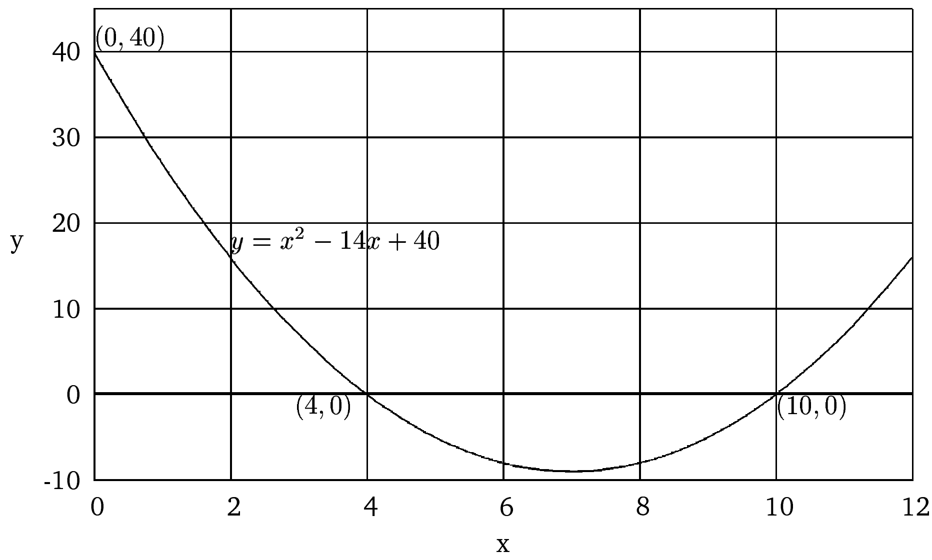
Question 4

a) The yearly increase is given by $\frac{6.7-5.5}{3} = 0.4$ million dollars. After 10 years the increase will be $0.4\ .\ 10 = 4$ million dollars. Therefore the total profit in 2010 is expected to be $5.5 + 4 = 9$ million dollars.

b) $profit = (Year - 2000)\ .\ 0.4 + 5.5$

Question 5

a) $x^2 - 14x + 40 = (x - 4)(x - 10) = 0$. Therefore the roots of the quadratic are at $x = 4$ and $x = 10$. These are the points where the curve cuts the x-axis. When $x = 0$ we have $y = 40$ so the curve cuts the y-axis at 40. This is a positive quadratic and so the curve will be a $\cup$ shape.



b) i) Break-even occurs when $P(x) = 0$.
$P(x) = -3x^2 + 42x - 120 = -3(x - 4)(x - 10)$. Therefore break-even occurs when $x = 4$ and when $x = 10$.

ii) The maximum profit occurs at the turning point which will be at the midpoint of the two roots i.e., when $x = (4 + 10)/2 = 7$. When $x = 7$ profit is given by $P(7) = -3(7 - 4)(7 - 10) = 27$. Therefore the maximum profit is 27.

iii) $P(x) = R(x) - C(x)$. So
$C(x) = R(x) - P(x) = 48x - 3x^2 - (-3x^2 + 42x - 120) = 48x - 42x + 120 = 6x + 120$

B.2 Test 2 solutions

Question 1

$MN = (2 * 3)(2 * 2)$ not valid

$ML = (2 * 3)(3 * 2)$ is valid

$$ML = \begin{pmatrix} 2\,.\,3 + -3\,.\,4 + 1\,.\,-2 & 2\,.\,0 + -3\,.\,1 + 1\,.\,2 \\ 5\,.\,3 + 7\,.\,4 + 1\,.\,-2 & 5\,.\,0 + 7\,.\,1 + 1\,.\,2 \end{pmatrix} = \begin{pmatrix} -8 & -1 \\ 41 & 9 \end{pmatrix}$$

$LM - N = (3 * 2)(2 * 3) - (2 * 2)$ not valid because LM is a $3 * 3$ matrix so we cannot subtract the $(2 * 2)$ matrix N from LM.

$NML = (2 * 2)(2 * 3)(3 * 2)$ is valid

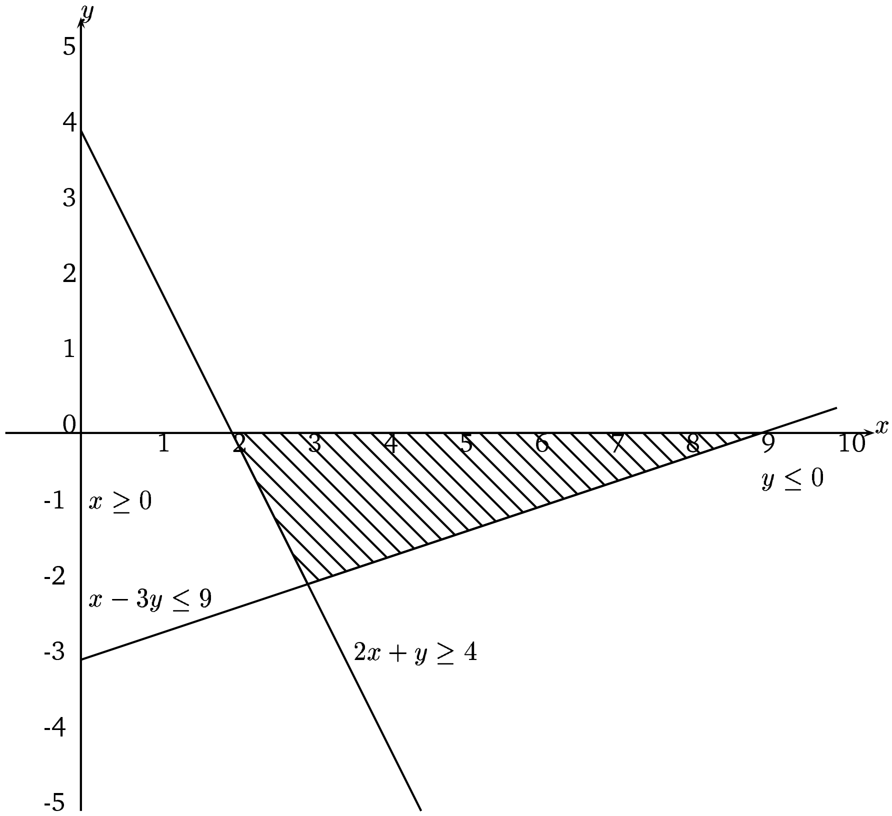
$$NML = \begin{pmatrix} 1 & 2 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} 2\,.\,3 + -3\,.\,4 + 1\,.\,-2 & 2\,.\,0 + -3\,.\,1 + 1\,.\,2 \\ 5\,.\,3 + 7\,.\,4 + 1\,.\,-2 & 5\,.\,0 + 7\,.\,1 + 1\,.\,2 \end{pmatrix} = \begin{pmatrix} 74 & 17 \\ 254 & 55 \end{pmatrix}$$

Question 2

a) $f(x) + g(x) = 6x^2 - 5x - 3x + 4 = 6x^2 - 8x + 4$
b) $f(x) - g(x) = 6x^2 - 5x - (-3x + 4) = 6x^2 - 2x - 4$
c) $f(x).g(x) = (6x^2 - 5x)(-3x + 4) = -18x^3 + 39x^2 - 20x$
d) $f(g(x)) = 6(-3x + 4)^2 - 5(-3x + 4) = 54x^2 - 129x + 76$
e) $g(f(x)) = -3(6x^2 - 5x) + 4 = -18x^2 + 15x + 4$

Question 3

a)



b) From the graph we can see that the lines $x - 3y = 9$ and $2x + y = 4$ intersect at the point $x = 3, y = -2$.

Question 4

a)

$$M = \begin{array}{c c} & \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{array}{c} w_1 \\ w_2 \end{array} & \begin{pmatrix} 30000 & 25000 & 10000 \\ 15000 & 35000 & 20000 \end{pmatrix} \end{array}$$

b) $Y = MX$ where $X = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$

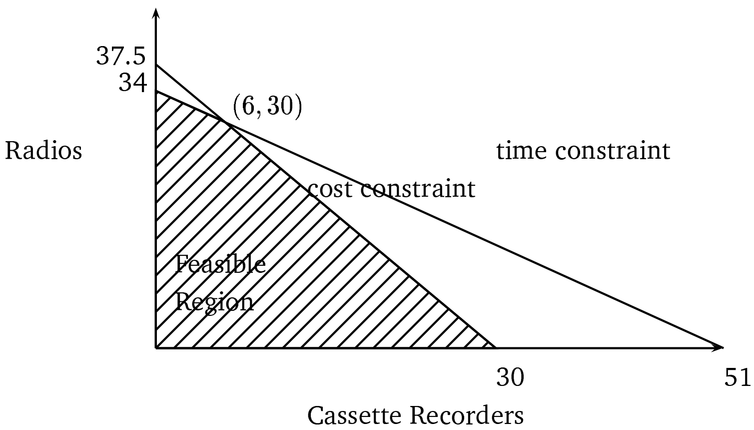
$$Y = \begin{pmatrix} 30000 \,.\, 5 + 25000 \,.\, 4 + 10000 \,.\, 2 \\ 15000 \,.\, 5 + 35000 \,.\, 4 + 20000 \,.\, 2 \end{pmatrix} = \begin{pmatrix} 270000 \\ 255000 \end{pmatrix}$$

The values in matrix Y represent the value in dollars of the goods in the two warehouses.

Question 5

a) Letting r represent the number of radios and c the number of cassette recorders produced, we have the following equations:
objective function: $\pi = 12r + 14c$
production constraints $\begin{cases} 1r + \frac{2}{3}c \leq 34 \\ 32r + 40c \leq 1200 \end{cases}$
non-negativity constraints: $\begin{cases} r \geq 0 \\ c \geq 0 \end{cases}$

b) The feasible region for this problem is shown below:



c) The four corners of the feasible region are at: $(0, 0)$ profit $= \$0$
$(30, 0)$ profit $= 14 \,.\, 30 = \$420$
$(0, 34)$ profit $= 12 \,.\, 34 = \$408$
$(6, 30)$ profit $12 \,.\, 30 + 14 \,.\, 6 = \$444$

The maximum profit is \$444 which occurs when the company makes 6 cassette recorders and 30 radios each week.

Appendix C

Solutions to subject guide activities

C.1 Chapter 1 activity solutions

Page 3

1. 2
2. -7
3. -12
4. 4
5. -3
6. 32
7. -12
8. 48
9. 1

Page 4

1. -8
2. 12
3. -5
4. 56
5. 2
6. 8
7. -21
8. -5

Page 5

1. 5
2. 25
3. 7
4. 8
5. $(17 + 13) \div (2 * 3) = 5$
6. $10 * (4 * 5 - 20) = 0$

Page 6

$\frac{2}{3} = \frac{24}{36} = \frac{40}{60} = \frac{18}{27}$

Page 8

1. $\frac{5}{6}$
2. $1\frac{4}{15}$

3. $2\frac{1}{14}$
4. $\frac{2}{15}$
5. $\frac{3}{4}$
6. $\frac{4}{15}$
7. $\frac{21}{40}$
8. $3\frac{1}{2}$
9. 2
10. $1\frac{1}{15}$
11. 4
12. $1\frac{5}{8}$

Page 10

1. 7^{10}
2. 7^{6}
3. 7^{3}
4. $7^{1} = 7$
5. 7^{-5}

Page 11

1. (a) $4^{\frac{1}{2}} = \sqrt{4} = \pm 2$
   (b) $81^{\frac{1}{4}} =^{4} \sqrt{81} = \pm 3$
   (c) $16^{\frac{3}{4}} = (^{4}\sqrt{16})^{3} = (\pm 2)^{3} = \pm 8$
   (d) $5^{0} = 1$
   (e) $3^{-3} = \frac{1}{3^{3}} = \frac{1}{27}$
   (f) $25^{-\frac{1}{2}} = \frac{1}{\sqrt{25}} = \pm\frac{1}{5}$
   (g) $8^{-\frac{2}{3}} = \frac{1}{(^{3}\sqrt{8})^{2}} = \frac{1}{2^{2}} = \frac{1}{4}$
   (h) $\left(\frac{1}{2}\right)^{-1} = \frac{1}{1/2} = 2$
   (i) $\left(\frac{1}{4}\right)^{\frac{1}{2}} = \sqrt{\left(\frac{1}{4}\right)} = \pm\frac{1}{2}$
   (j) $\left(\frac{1}{4}\right)^{-\frac{1}{2}} = \frac{1}{\pm 1/2} = \pm 2$
2. (a) 8
   (b) 0
   (c) $\frac{1}{100}$
   (d) 1
   (e) 81

Page 13

1. $10\sqrt{21}$
2. $2\sqrt{3}$
3. $13\sqrt{2}$
4. $9\sqrt{5} - 4\sqrt{(4*5)} = 9\sqrt{5} - 8\sqrt{5} = \sqrt{5}$
5. $\sqrt{50} = \sqrt{(25*2)} = 5\sqrt{2}$
6. $\frac{\sqrt{80}}{2} = \frac{\sqrt{(16*5)}}{2} = \frac{4\sqrt{5}}{2} = 2\sqrt{5}$
7. $\sqrt{\left(\frac{1}{8}\right)} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

8. $\frac{\sqrt{300}}{\sqrt{50}} = \sqrt{60} = \sqrt{4}\sqrt{15} = 2\sqrt{15}$

Page 14

1. $\frac{84}{420} = 20\%$
2. $1\% = 420 \div 100 = 4.2, 15\% = 15 * 4.2 = 63$ people
3. All of the people is 100%. $20\% + 15\% = 35\%$ of the people have already been accounted for. Therefore $100 - 35 = 65\%$ of the people preferred drink B.

Page 14

1. \$280
2. 92.8m
3. 8190
4. \$2616
5. \$1762.50
6. \$138
7. \$220
8. \$9775
9. 799

Page 15

1. \$42000
2. \$450

Page 15

2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97

Page 18

1. $78 = 2 * 3 * 13$, $240 = 2^4 * 3 * 5$, $420 = 2^2 * 3 * 5 * 7$
2. $HCF(78, 240) = 2 * 3 = 6$
3. $LCM(240, 420) = 2^4 * 3 * 5 * 7 = 1680$

Page 20

1. $-3x - 12$
2. $8b - 4ab$
3. $4 + 6x + 12 = 6x + 16$
4. $2x + 3x^2 + 8x - 6x^2 = -3x^2 + 10x$
5. $x(x + 4) + 3(x + 4) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12$
6. $2y(3y + 7) - 4(3y + 7) = 6y^2 + 14y - 12y - 28 = 6y^2 + 2y - 28$
7. $(2y + 3)(2y + 3) = 2y(2y + 3) + 3(2y + 3) = 4y^2 + 6y + 6y + 9 = 4y^2 + 12y + 9$
8. $(a + 2b)(a + 2b) = a(a + 2b) + 2b(a + 2b) = a^2 + 2ab + 2ab + 4b^2 = a^2 + 4ab + 4b^2$
9. $3x(x + 2)(x + 2) = 3x[x(x + 2) + 2(x + 2)] = 3x[x^2 + 2x + 2x + 4] = 3x[x^2 + 4x + 4] = 3x^3 + 12x^2 + 12x$

Page 21

1. $2(x + 2)$

2. $x(2x+1)$
3. $6x(5+2x)$
4. $2x^2(4x-3)$
5. $4pq(2p-q)$

Page 24

1. $(x+2)(x+1)$
2. $(x+5)(x-2)$
3. $(x-1)^2$
4. $(3x-1)(3x+1)$
5. $(x-10)(x+3)$
6. $(x+3)^2$
7. $(y^2-4)(y^2+4) = (y-2)(y+2)(y^2+4)$
8. $(x+5)(x-4)$
9. $(x+10)(x+2)$
10. $2(x+4)(x+2)$
11. $(x+7)(x-1)$
12. $-(x-5)(x-4) = (5-x)(x-4)$

Page 26

1. (a) $\frac{y}{x}$
   (b) $\frac{x}{y}$
   (c) $\frac{xy}{xy} = 1$
   (d) $\frac{y^2}{x^2}$
   (e) $\frac{y^2+x^2}{xy}$
   (f) $\frac{y^2-x^2}{xy} = \frac{(y-x)(y+x)}{xy}$
2. $\frac{x^2-9}{x^2+6x+9} = \frac{(x-3)(x+3)}{(x+3)(x+3)} = \frac{x-3}{x+3}$

Page 27

1. (a) 79.4 (1 d.p.)
   (b) 864.90 (2 d.p.)
   (c) 35 (nearest integer)
   (d) 765.0 (1 d.p.)
2. It is possible that the pencil is 9.64 cm long and the tin is 9.5 cm long. Thus the pencil might not fit into the tin.

C.2 Chapter 2 activity solutions

Page 31

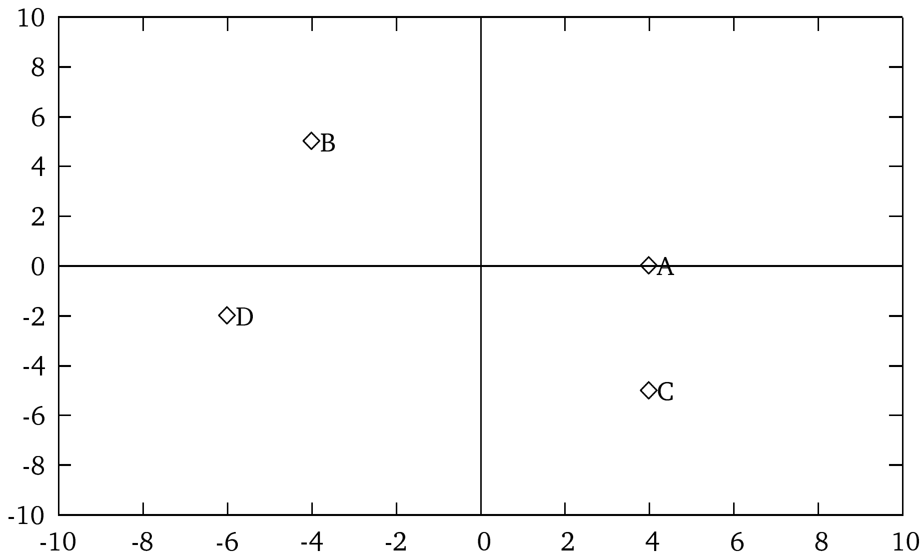
1. $x = \frac{T-3y}{2}$.
2. $y = \frac{T-2x}{3}$.

Page 32

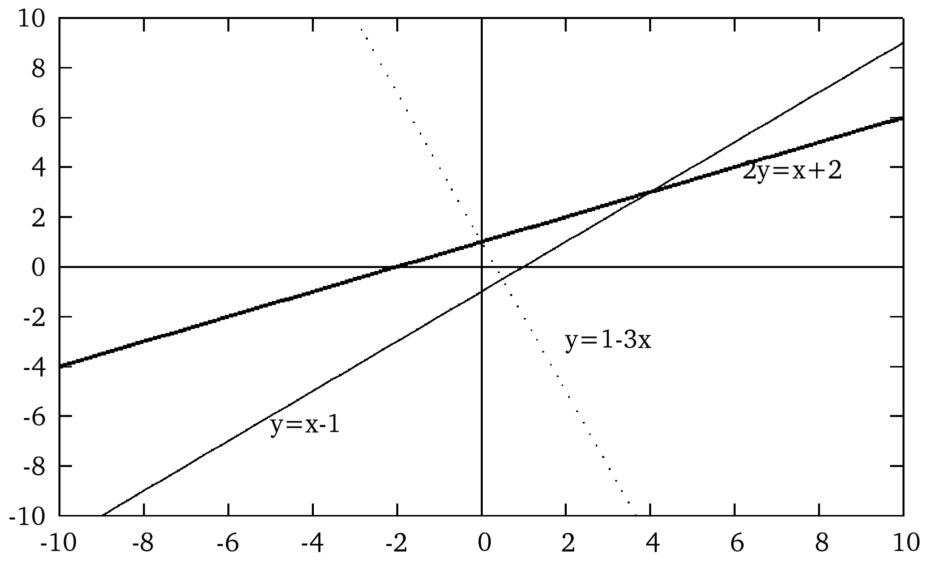
1. $x = 1$
2. $x = 14$
3. $x = -3$
4. $x = -8$
5. $x = \frac{3}{7}$
6. $x = 24$
7. $x = 15$
8. $x = \frac{1}{2}$
9. $x = 0$
10. $x = 5$

Page 34

1. $P = (7, 2), Q = (5, 0), R = (-3, 4), S = (-1, -3)$
2.



Page 35



Page 40

The line $y = x + 4$ is parallel to $y = x + 5$; both have gradient $m = 1$.

The line $\frac{2}{3}x$ is parallel to $3y = 2x + 6$; both have gradient $m = \frac{2}{3}$.

The line $y = 2 - x$ is parallel to $y + x = 3$; both have gradient $m = -1$.

The rule for finding parallel lines is 'Lines which are parallel have the same gradient'.

Page 42

A $y = 2x + 3$

B $y = \frac{1}{2}x - 4$

C $y = \frac{-1}{3}x$

D $y = -4x + 1$

Page 43

1. $y = -2x - 3$
2. $y = 5x + 8$
3. $y = \frac{1}{4}x + 2$

Page 44

$$y = -2x + 16 \quad or \quad y + 2x = 16$$

Page 46

a) $y = -1.4x + 16$

b) $5 = -1.4x + 16$ therefore $x = (5 - 16)/ - 1.4 = 7.857$.
The baker can't make 0.857 of a big cake so can only make 7 big cakes along with 5 small cakes.

Page 47

a) The car loses $28650 − $15,000 = $13650 in 4 years. This represents an annual depreciation of $13650 ÷ 4 = $3412.50. Thus the linear depreciation of the car can be modelled by the following linear equation $y = -3412.50x + 28650$.

b) Substituting $y = 10000$ into the equation gives $10000 = -3412.50x + 28650$ and rearranging we find $x = 5.5$ (1 $d.p.$). Thus the car is worth more than $10,000 for $5\frac{1}{2}$ years.

Page 49

a) $cost = 230 * 15 + 8000 = \$11,450$

b) $cost = 230 * 40 + 8000 = \$17,200$

Page 50

The cost is given by $C = 18x + 2000$
The revenue is given by $R = 60x$
The profit is given by $\pi = 60x - (18x + 2000) = 42x - 2000$

a) $x = 100$, $\pi = (42 * 100) - 2000 = 2200$; profit is $2,200

b) $x = 250$, $\pi = (42 * 250) - 2000 = 8500$; profit is $8,500

c) $x = 40$, $\pi = (42 * 40) - 2000 = -320$; profit is $-320 i.e. the firm makes a loss of $320 if it only sells 40 cars.

In order to break-even we must have $\pi = 0$. So $0 = 42x - 2000$. Re-arranging we find $x = 47.6$. Thus the firm must manufacture and sell 48 cars in order to break-even.

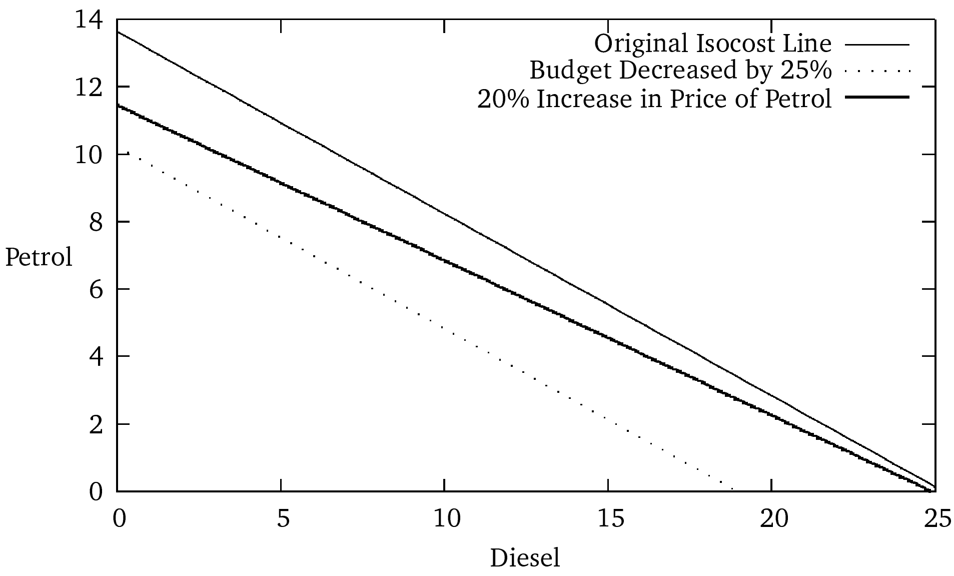
Page 53

a) Original Isocost Line: $1500 = 60x + 110y$ or $y = 0.54x + 13.64$

b) Budget decreases by 25% giving a new budget of 1125. New Isocost Line: $1125 = 60x + 110y$ or $y = -0.54x + 10.23$

c) Price of petrol increases by 20% to 131. New Isocost Line: $1500 = 60x + 131y$ or $y = -0.46x + 11.45$

These three isocost lines are shown on the graph below.



C.3 Chapter 3 activity solutions

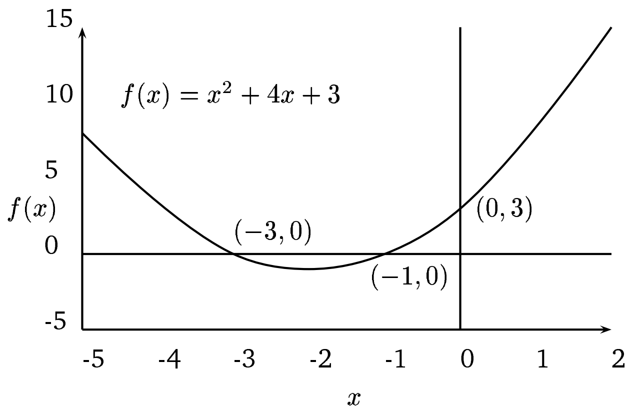
Page 56

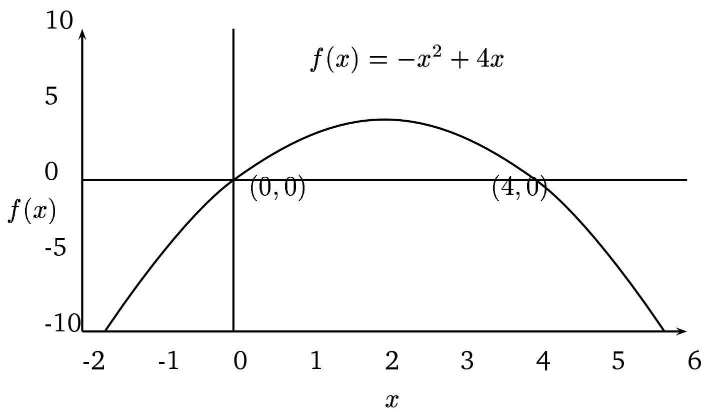
1. $f(5) = 4(5)^2 + 2(5) - 9 = 101$
   $f(-2) = 4(-2)^2 + 2(-2) - 9 = 3$
   $f(0) = 4(0)^2 + 2(0) - 9 = -9$
2. $f(x) =^4 \sqrt{x}$ is not a function because it is possible to find a single input which could result in different outputs. For example, $f(16) = 2$ or -2.

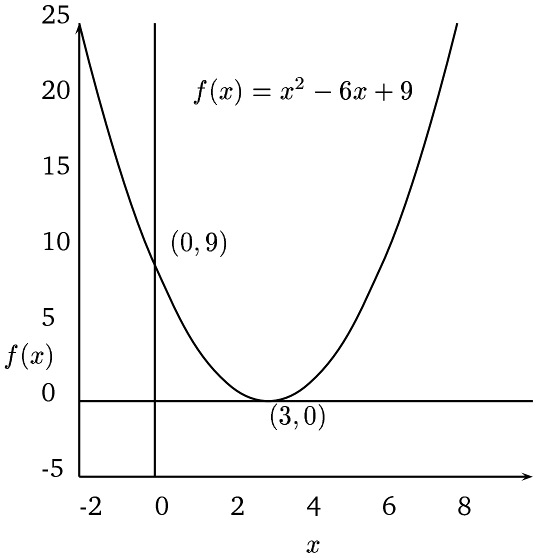
Page 57

1. (a) The range is the set of all real numbers ≥ -10.
   (b) The range is the set of all real numbers ≥ -10.
   (c) The range is the set of integers $\{-1, -6, -9, -10\}$.
2. We must ensure that the denominator is non-zero so we must specify that the domain excludes the value $x = 3$. Thus a suitable domain is the set of all real numbers $(x \neq 3)$.

Page 62

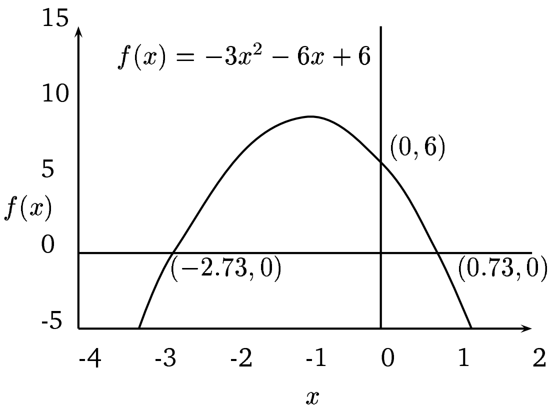






If a quadratic function has a repeated root as in $f_3(x)$ then in effect the graph of the function cuts the x-axis twice in the same place. This means that the vertex of the graph must also be at this root.

Page 65

1. 

2. (a) $x = 2.5$ or $x = -1$

   (b) $x = 2.82(2d.p.)$ or $x = 0.18(2d.p.)$

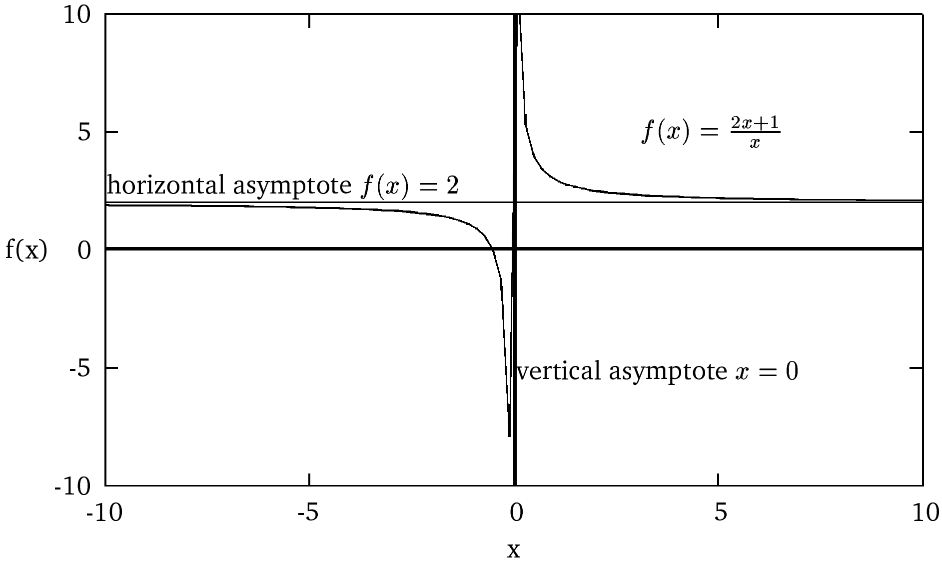
   (c) $x = 1.47(2d.p.)$ or $x = -1.14(2d.p.)$

3. (a) $(-1, -16)$

   (b) $(0.83, 9.08)$

   (c) $(10, 968)$

Page 66



Page 67

1. (a) $f(x) + g(x) = x^2 + 4x + 5$
   (b) $g(x) + f(x) = x^2 + 4x + 5$
   (c) $f(x) - g(x) = 4x + 5 - x^2$
   (d) $g(x) - f(x) = x^2 - 4x - 5$
   (e) $f(x).g(x) = 4x^3 + 5x^2$
   (f) $g(x).f(x) = 4x^3 + 5x^2$
   (g) $f(x) \div g(x) = \frac{4x+5}{x^2} \quad (x \neq 0)$
   (h) $g(x) \div f(x) = \frac{x^2}{4x+5} \quad (x \neq -\frac{4}{5})$
2. $f(x) + g(x) = g(x) + f(x)$ so $+$ is commutative.
   $f(x) - g(x) \neq g(x) - f(x)$ so $-$ is not commutative.
   $f(x).g(x) = g(x).f(x)$ so . is commutative.
   $f(x) \div g(x) \neq g(x) \div f(x)$ so $\div$ is not commutative.

Page 69

1. $f \circ g = (6x - 2)^2 + 4 = 36x^2 - 24x + 8$
   $g \circ f = 6(x^2 + 4) - 2 = 6x^2 + 22$
   (a) $f(g(2)) = 36(2)^2 - 24(2) + 8 = 104$
   (b) $f(g(-1)) = 36(-1)^2 - 24(-1) + 8 = 68$
   (c) $g(f(0)) = 6(0)^2 + 22 = 22$
   (d) $g(f(-3)) = 6(-3)^2 + 22 = 76$
2. $36x^2 - 24x + 8 = 6x^2 + 22, 30x^2 - 24x - 14 = 0$. Using the quadratic formula with $a = 30, b = -24, c = -14$ gives $x = 1.2$ and $x = -0.4$ (1 d.p.)

C.4 Chapter 4 activity solutions

Page 82

1. Unique solution: $x = 4, y = 11$

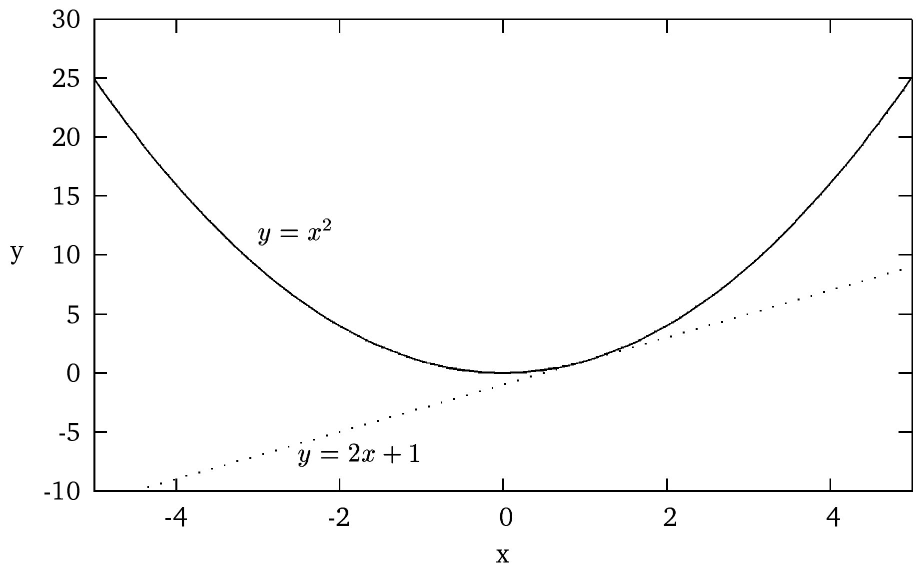
2. Infinitely many solutions satisfying: $y = 3 - 2x$

3. No solutions: both lines have gradient 2 and so they are parallel and do not cross.

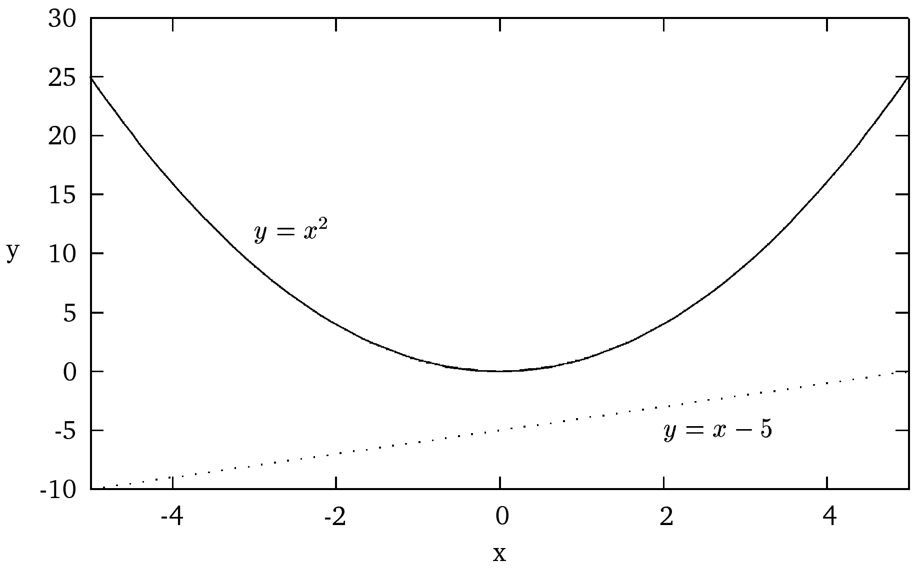
4. Unique solution: $x = 1, y = 3$

Page 83

1. The straight line will cut the parabola just once if the line is *tangent* to the curve. This means that the line just touches the curve at one point. The graph below shows the quadratic $y = x^2$ and tangent line $y = 2x - 1$.



2. The straight line does not cut the parabola if the turning point is above the straight line and the gradient of the line is less than the gradient of the parabola near its turning point (or the turning point is below the straight line in the case of a ∩ shape parabola). For example, the graph below shows the quadratic $y = x^2$ which has its turning point at $(0, 0)$ and straight line $y = x - 5$.



Page 86

1. $x = 1, y = 4$
2. $x = 3, y = 2$
3. $x = 6, y = 5$
4. $x = 3, y = 2$

Page 89

1. (a) $x = 1, y = 1$
   (b) $x = 2, y = 3$ or $x = 3, y = 2$
   (c) $x = -2.87, y = 4.87$ or $x = 0.87, y = 1.13$
2. $x + y = 700, 4x + 6y = 3360$, $x = 420, y = 280$
3. $p - q = 21, p + q = 95$, $p = 58, q = 37$
4. $2a + 3c = 1880, 3a + 2c = 2110$, $a = 514, c = 284$

C.5 Chapter 5 activity solutions

Page 98

1. $m_{2,2} = 0, m_{3,1} = -4, m_{1,3} = -1, m_{1,2} = 6, m_{2,3} = 7$
2. $M^T = \begin{pmatrix} 3 & 4 & -4 \\ 6 & 0 & 2 \\ -1 & 7 & 5 \end{pmatrix}$

Page 100 $BA = \begin{pmatrix} 6 & -13 \\ 28 & -4 \end{pmatrix}$

Page 101

a) $A(2*2), B(3*2), C(2*1), D(2*3)$

b) i) $2B = \begin{pmatrix} 12 & 6 \\ 14 & -4 \\ 2 & 10 \end{pmatrix}$

ii) $-3A = \begin{pmatrix} -6 & -9 \\ 3 & 0 \end{pmatrix}$

iii) $A^T = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$

iv) $D^T = \begin{pmatrix} 5 & 1 \\ 0 & 7 \\ -2 & 2 \end{pmatrix}$

c) i) $A + B$ is not defined

ii) AC is defined

iii) CA is not defined

iv) $C + AC$ is defined

v) BD is defined

vi) $BD - A$ is not defined

vii) DA is not defined

viii) C^TA is defined

ix) $B + D^T$ is defined

d) $AC = \begin{pmatrix} 16 \\ -5 \end{pmatrix}$

$C + AC = \begin{pmatrix} 21 \\ -3 \end{pmatrix}$

$BD = \begin{pmatrix} 33 & 21 & -6 \\ 33 & -14 & -18 \\ 10 & 35 & 8 \end{pmatrix}$

$C^TA = (-1 \;\; 6)$

$B + D^T = \begin{pmatrix} 11 & 4 \\ 7 & 5 \\ -1 & 7 \end{pmatrix}$

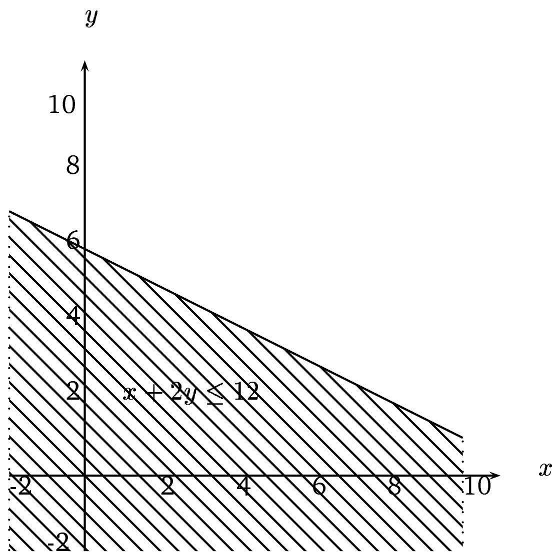
Page 103

1. $\left(\begin{array}{ccc|c} 4 & -2 & 3 & 1 \\ 1 & 3 & -1 & 2 \\ 6 & 4 & -2 & -4 \end{array}\right)$
2. $x_1 + 5x_2 - 3x_3 = 3$
   $4x_1 - 2x_2 + x_3 = -11$
   $3x_1 + 3x_2 - 2x_3 = -4$

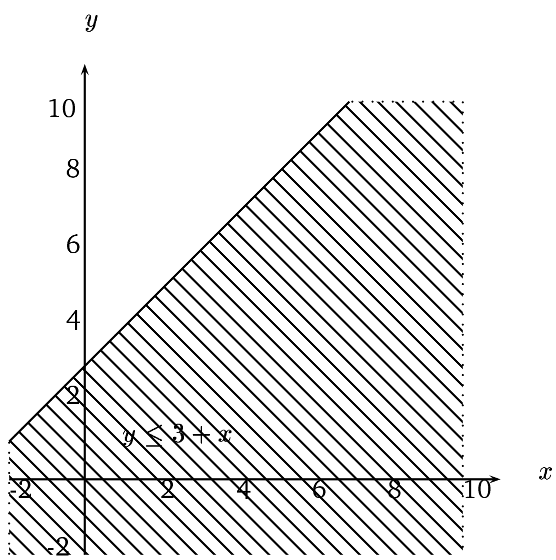
C.6 Chapter 6 activity solutions

Page 112

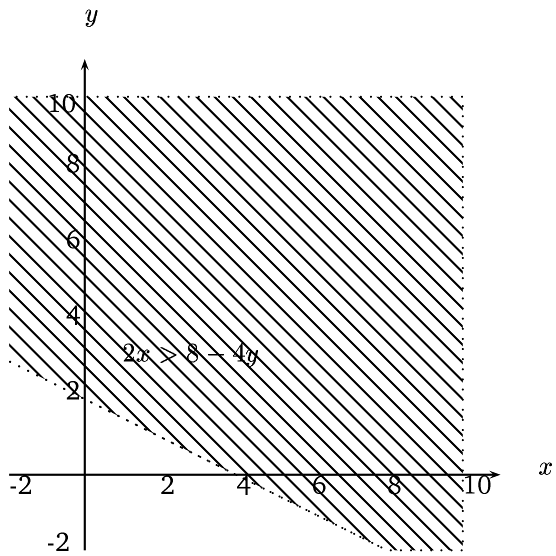
1.



2.

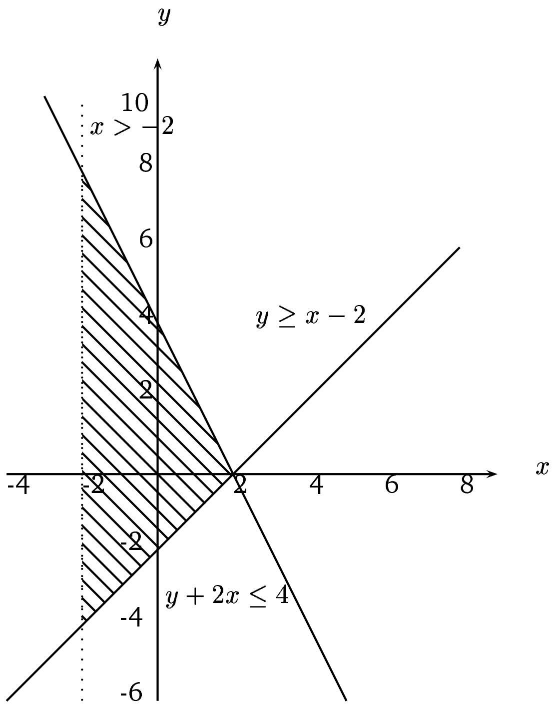


3.



y
10
8
6
4
2
-2
-2
2
4
6
8
10
x
$2x > 8 - 4y$

Page 113



y
10
$x > -2$
8
6
4
2
-2
-4
-6
-4
-2
2
4
6
8
x
$y \geq x - 2$
$y + 2x \leq 4$

Appendix D

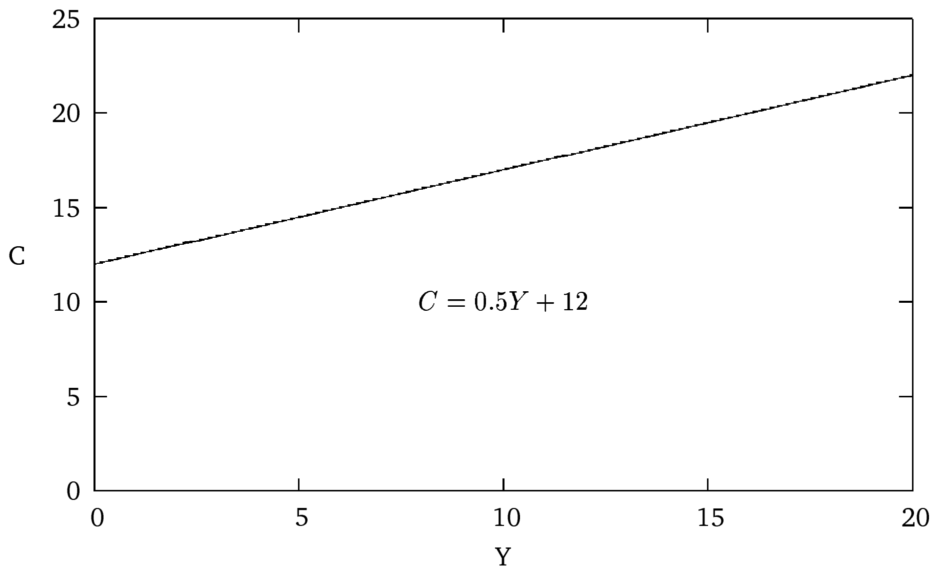
Solutions to sample examination questions

D.1 Chapter 2

Question 1

a) The points $(2, -6)$ and $(3, 0)$ lie on the line. The gradient is given by $m = \frac{(-6-0)}{2-3} = 6$. Substituting $m = 6, x = 3, y = 0$ into the equation $y = mx + c$ gives $0 = (6)(3) + c$. Therefore $c = -18$. The equation of the line is $y = 6x - 18$. When $x = -1$, $y = b$. Substituting $x = -1$ into the equation will give the value of b: $b = 6(-1) - 18 = -24$.

b) The line $C = 0.5Y + 12$ has gradient $+0.5$ and cuts the y-axis at 12:



When $Y = 15, C = 0.5(15) + 12 = 19.5$
When $C = 20, Y = \frac{(C-12}{0.5} = \frac{20-12}{0.5} = 16$
Note that you could use your graph to find these values.

Question 2

a) The points $(5, 3)$ and $(0, -12)$ lie on the line. The gradient is given by $m = \frac{-12-3}{0-5} = 3$ and the y intercept is -12. Therefore the equation of the line is $y = 3x - 12$ or $3x - y = 12$. The line cuts the x-axis when $y = 0$. Substituting $y = 0$ into the equation gives $3x = 12$ therefore $x = 4$. So the line cuts the x-axis at

$(4, 0)$. The point $(6, 6)$ lies on the line because substituting $x = 6, y = 6$ into the equation gives $3(6) - 6 = 18 - 6 = 12$ so the equation is satisfied. Substituting $x = 2, y = -15$ into the equation gives $3(2) - (-15) = 21 \neq 12$ and therefore the point $(2, -15)$ does not lie on the line.

b) Let x be the percentage of peanuts in the blend and $(100 - x)$ be the percentage of cashews. Then

$$75 = \frac{60x}{100} + \frac{100(100 - x)}{100} = 0.6x + 100 - x$$

Re-arranging to find x gives:

$$0.4x = 25, x = \frac{25}{0.4} = 62.5$$

Therefore the blend of nuts should include 62.5% peanuts.

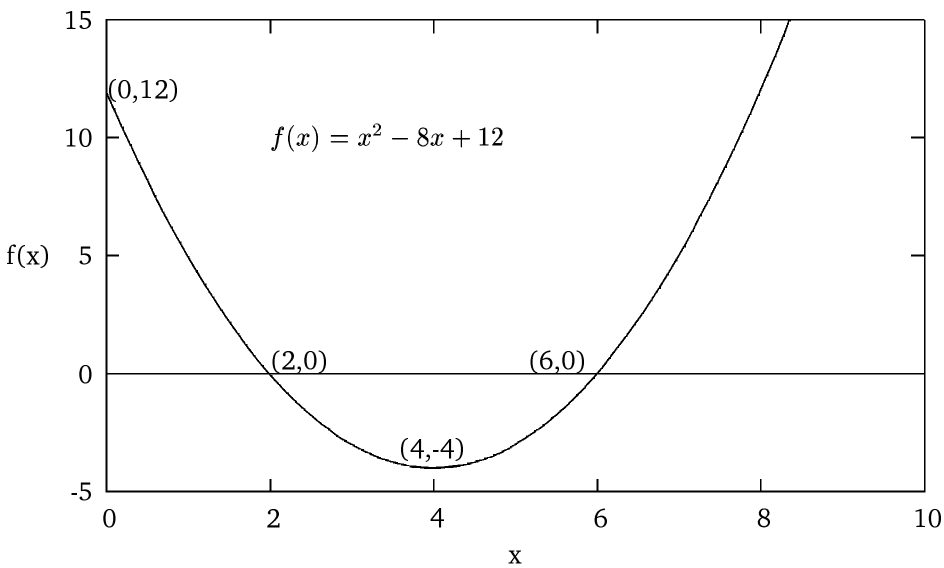
D.2 Chapter 3

Question 1

a) i) $f[g(x)] = ((x - 2)^2)^3 = (x - 2)^6$

ii) $g[f(x)] = (x^3 - 2)^2$

b) $x^2 - 8x + 12 = (x - 6)(x - 2)$. The graph cuts the x-axis at $x = 6$ and $x = 2$. The graph cuts the y-axis at $y = 12$. The x co-ordinate of the turning point is at $(2 + 6) \div 2 = 4$ and the y co-ordinate is at $4^2 - 8(4) + 12 = -4$. Therefore the turning point is at $(4, -4)$.



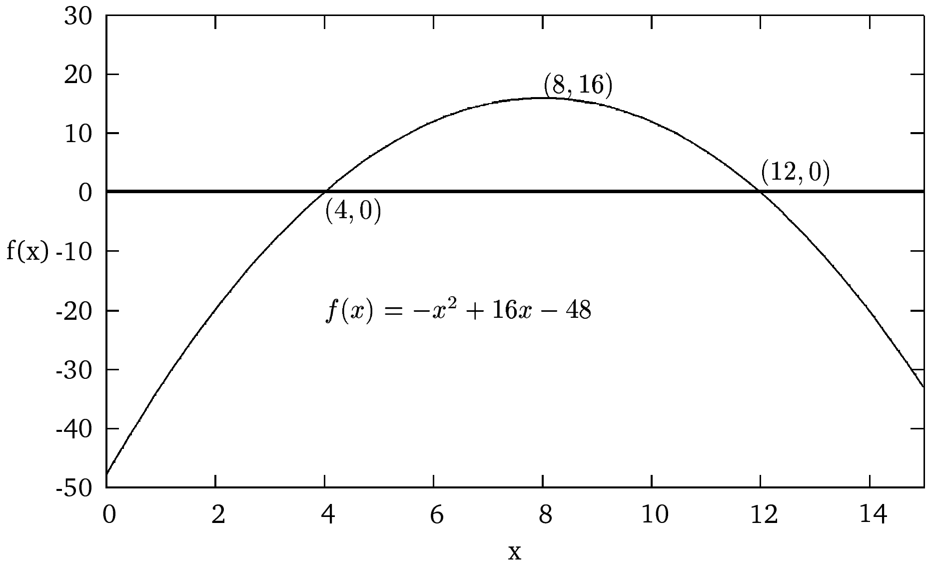
c) $R(x) = 10x - 2x^2$ and $C(x) = 4 + x$. The profit $\pi(x) = R(x) - C(x) = 1 - x - 2x^2 - (4 + x) = -2x^2 + 9x - 4$. The profit expressed as a function of x is $\pi(x) = -2x^2 + 9x - 4$. The maximum profit will occur at the vertex. The x co-ordinate of the vertex is given by $x = \frac{-b}{2a} = \frac{-9}{2(-2)} = 2.25$. The y co-ordinate is given by $f(2.25) = -2(2.25)^2 + 9(2.25) - 4 = 6.125$. Therefore the maximum profit is 6.125.

Question 2

a) i) $C(q) = 500 + 24q$

ii) $\pi(q) = R(q) - C(q) = 1000q - q^2 - (500 + 24q) = -q^2 + 976q - 500$

b) $f(x) = -x^2 + 16x - 48 = (4 - x)(x - 12)$. The graph cuts the x-axis at $x = 4$ and $x = 12$, and the y-axis at $f(0) = 48$. The x co-ordinate of the turning point is at $(4 + 12) \div 2 = 8$ and the y co-ordinate is given by $f(8) = 16$. Therefore the turning point is at $(8, 16)$.



The profit function $\pi(q) = -q^2 + 16q - 48$ is the same function as $f(x)$ above. Therefore (i) the maximum profit is 16 and (ii) the break-even points are at $q = 4$ and $q = 12$.

D.3 Chapter 4

Question 1

a) Multiplying the second equation by 3 gives $3x - 9y = -60$. Subtracting this equation term by term from the first equation gives $14y = 84$ and therefore $y = 6$. Substituting $y = 6$ into the second equation gives $x - 3(6) = 20$ and therefore $x = -2$. We can check the solution $x = -2, y = 6$ by substituting these values into the first equation: $3(-2) + 5(6) = -6 + 30 + 24$.

b) Subtracting the demand equation from the supply equation term by term gives $0 = -210 + 10P$ and therefore supply and demand are equated when $P = 21$. Thus

i) the value of P which brings equilibrium to the market is $P = 21$

ii) there is a surplus when the price is too high - therefore there is a surplus when $P > 21$

iii) there is a shortage when the price is too low - therefore there is a shortage when $P < 21$.

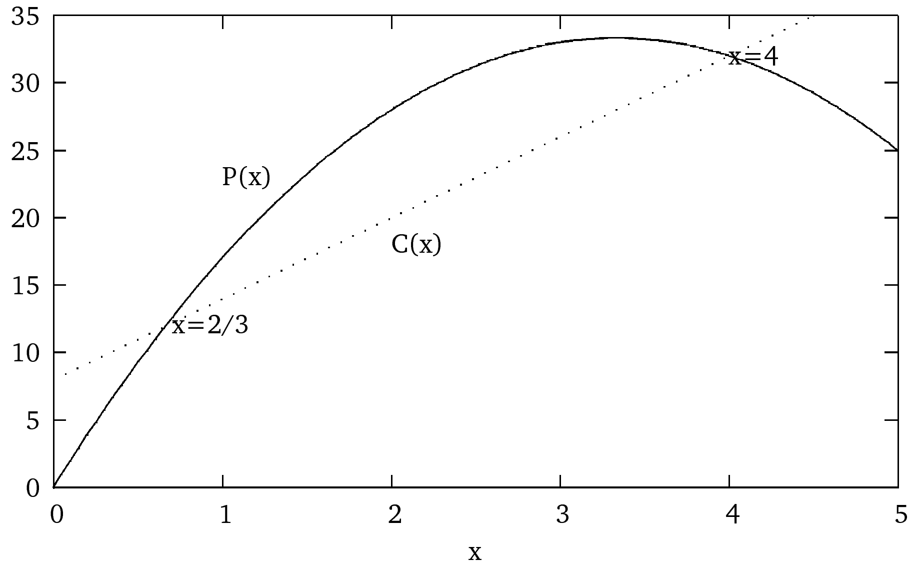
Question 2

a) Multiplying the second equation by 2 gives $2x + 6y = -10$. Subtracting this equation term by term from the first equation gives $-7y = 21$ and therefore $y = -3$. Substituting $y = -3$ into the second equation gives $x + 3(-3) = -5$ and therefore $x = 4$. We can check the solution $x = 4, y = -3$ by substituting these values into the first equation: $2(4) - (-3) = 11$.

b) We can draw a table of values evaluating the functions $P(x)$ and $C(x)$ at $x = 0, 1, 2, 3, 4, 5$.

x	0	1	2	3	4	5
$P(x)$	0	17	28	33	32	25
$C(x)$	8	14	20	26	32	38

Now we can plot these points on the graph below:



The company makes a profit when $P(x)$ is greater than $C(x)$. From the graph we can see that the two lines cross when $x = \frac{2}{3}$ and when $x = 4$. Therefore the company makes a profit when $\frac{2}{3} < x < 4$.

Question 3

a) Multiplying the second equation by 3 gives $-3x - 12y = 6$. Subtracting this equation term by term from the first equation gives $-14y = 14$ and therefore $y = -1$. Substituting $y = -1$ into the second equation gives $-x - 4(-1) = 2$ and therefore $x = 2$. We can check the solution $x = 2, y = -1$ by substituting these values into the first equation: $3(2) - 2(-1) = 8$.

b) The break-even points are when the profit function $\pi(x)$ is zero. This is at the roots of the function. We find the profit function and then factorise it to find the roots and hence the break-even points.

$$\begin{aligned} \pi(x) &= R(x) - C(x) \\ &= -2x^2 + 14x - (2x + 10) \\ &= -2x^2 + 12x - 10 \\ &= -2(x^2 - 6x + 5) \\ &= -2(x - 5)(x - 1) \end{aligned}$$

Therefore the break-even points are at $x = 5$ and $x = 1$.

c) $Y = C + I + G$. Substituting the given values $I = 25 + 0.15Y$ and $G = 175$ into this equation gives $Y = C + 25 + 0.15Y + 175$ or re-arranging $C = 0.85Y - 200$. Now we can solve the two equations in Y and C simulataneously to find the value of Y.

$$C = 300 + 0.75Y$$
$$C = 0.85Y - 200$$

Subtracting the second equation from the first gives $0 = 500 - 0.1Y$ and therefore $Y = 5000$ is the level of income which brings equilibrium to the market.

D.4 Chapter 5

Question 1

a) i) $A + B$ is not valid

ii) $AB = \begin{pmatrix} 6 & -1 & 4 \\ 3 & 1 & 2 \end{pmatrix}$

iii) $BC = \begin{pmatrix} 1 & 8 \\ -1 & 0 \end{pmatrix}$, $A + BC = \begin{pmatrix} 3 & 9 \\ 0 & -1 \end{pmatrix}$

iv) $CB - A$ is not a valid matrix because although it is possible to compute CB, this matrix would have dimensions $(3 * 3)$ and so we cannot subtract the $(2 * 2)$ matrix A. There is no need to compute CB.

b)

$$\begin{array}{ccc} Number\ of\ Products & Profit\ per\ Product & Profit\ per\ Store \end{array}$$

$$\begin{pmatrix} 100 & 200 & 300 & 200 \\ 120 & 150 & 160 & 100 \\ 150 & 100 & 250 & 120 \end{pmatrix} \begin{pmatrix} 10 \\ 12 \\ 20 \\ 10 \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

Profit
$S_1 = (100 * 10) + (200 * 12) + (300 * 20) + (200 * 10) = \$11,400$
Profit
$S_2 = (120 * 10) + (150 * 12) + (160 * 20) + (100 * 10) = \$7,200$
Profit
$S_3 = (150 * 10) + (100 * 12) + (250 * 20) + (120 * 10) = \$19,700$

Question 2

a) i) $AC = \begin{pmatrix} 6 & 12 & 6 \\ -11 & -5 & -3 \\ -4 & 26 & 12 \end{pmatrix}$

ii) $CA = \begin{pmatrix} 8 & 4 \\ -22 & 5 \end{pmatrix}$, $CA + B = \begin{pmatrix} 14 & 0 \\ -19 & 9 \end{pmatrix}$

b) i)

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix} \begin{pmatrix} 0.24 \\ 0.76 \end{pmatrix} = \begin{pmatrix} 0.524 \\ 0.476 \end{pmatrix}$$

ii)

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix} \begin{pmatrix} 0.524 \\ 0.476 \end{pmatrix} = \begin{pmatrix} 0.5524 \\ 0.4476 \end{pmatrix}$$

iii)

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix} \begin{pmatrix} 0.5524 \\ 0.4476 \end{pmatrix} = \begin{pmatrix} 0.55524 \\ 0.44476 \end{pmatrix}$$

$$x_5 = (0.6 * 0.55524) + (0.5 * 0.44476) = 0.555524$$

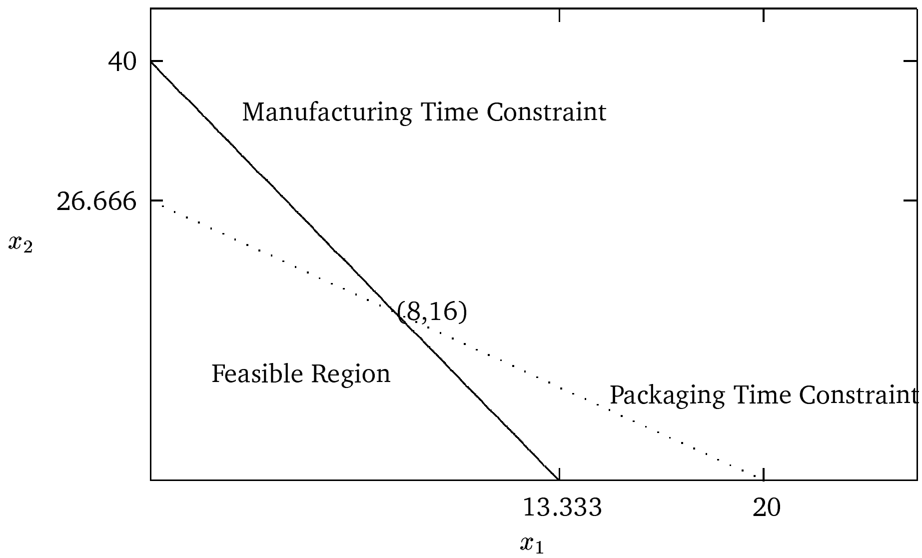
The market share for x after five weeks is 0.555524

D.5 Chapter 6

Question 1

a) objective function: $\pi = 3x_1 + 2x_2$
production constraints $\begin{cases} 3x_1 + x_2 \leq 40 \\ 2x_1 + 1.5x_2 \leq 40 \end{cases}$
non-negativity constraints: $\begin{cases} x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$

b) Putting x_2 on the y-axis and x_1 on the x-axis means that we have to rearrange the production constraints to make x_2 the subject. This gives us the lines $x_2 = -3x_1 + 40$ and $x_2 = -\frac{4}{3}x_1 + 40$. Sketching these lines gives us the feasible region.



c) The four corners of the feasible region are at $(0,0)$, $(0, 26\frac{2}{3})$, $(13\frac{1}{3}, 0)$ and $(8, 16)$. We can calculate the profit made at each of these points using the objective function. We round down to whole numbers where necessary.

point $(0,0)$ profit= \$0
point $(0,26)$ profit= $3(0) + 2(26) = \$52$
point $(13,0)$ profit= $3(13) + 2(0) = \$39$
point $(8,16)$ profit= $3(8) + 2(16) = \$56$

By the extreme-point theorem we know that the maximum profit will occur at one of the corner points of the feasible region, and therefore the maximum profit is \$56 and this occurs when the company makes 8 x_1 products and 16 x_2 products.

Question 2

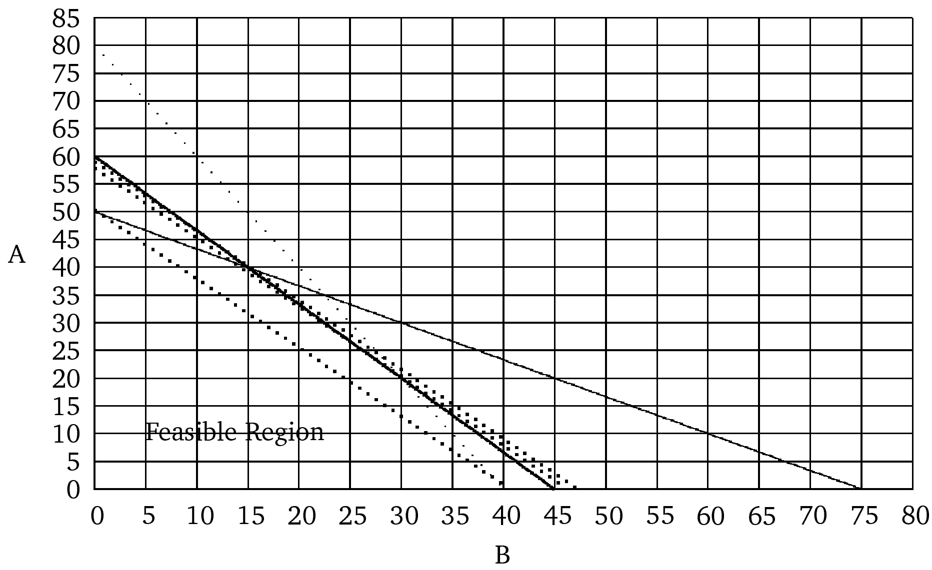
a) Let A be the number of type A machines produced, and B be the number of type B machines produced. Then we have:

objective function: $\pi = 20A + 25B$

production constraints $\begin{cases} 3A + 2B \leq 150 \\ 4A + 8B \leq 320 \\ 3A + 4B \leq 180 \end{cases}$

non-negativity constraints: $\begin{cases} A \geq 0 \\ B \geq 0 \end{cases}$

b) We will put A on the y-axis so first we rearrange each of the production constraints to make A the subject. This gives $A = -\frac{2}{3}B + 50$, $A = -2B + 80$ and $A = -\frac{4}{3}B + 60$. Plotting these lines gives us the feasible region shown on the graph below.



c) Re-arranging the objective function to make A the subject gives us the equation $A = -\frac{5}{4}B + \frac{\pi}{20}$. Therefore the isoprofit lines have gradient $-\frac{5}{4}$. Drawing isoprofit lines on the graph as shown above, we can see that the highest isoprofit line is the one through the corner point $(15, 40)$. Therefore the maximum profit will occur when the firm produces 40 machines of Type A and 15 machines of Type B. The profit in this case will be $\pi = 20(40) + 25(15) = \$1175$.

Notes

Notes

Notes

Comment form

We welcome any comments you may have on the materials which are sent to you as part of your study pack. Such feedback from students helps us in our effort to improve the materials produced for the International Programmes.
If you have any comments about this guide, either general or specific (including corrections, non-availability of Essential readings, etc.), please take the time to complete and return this form.

Title of this subject guide: ....

....

Name ....

Address ....

....

Email ....

Student number ....

For which qualification are you studying? ....

Comments

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

Please continue on additional sheets if necessary.

Date: ....

Please send your completed form (or a photocopy of it) to:
Publishing Manager, Publications Office, University of London International Programmes, Stewart House, 32 Russell Square, London WC1B 5DN, UK.