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Mathematics for business Volume 2

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Undergraduate study in
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This guide was prepared for the University of London by:

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Preface

The second volume of this subject guide is a continuation of the mathematics studied in the first volume. You will need to make sure that you have finished working through Volume 1 before you start work on these chapters. You may also like to re-read the Introduction to Volume 1 again for general information, for example, relating to the examination. The Reading is the same and is reproduced below.

In Volume 1 we studied equations and functions, their graphs, and their applications in business and economics. Types of equation you should be familiar with from Volume 1 include linear equations, quadratic equations and simultaneous equations. We also saw how to represent and work with information stored in matrices and how to solve linear programming problems to maximise profit.

The material covered in Volume 2 includes differentiation and integration and the applications of these to economics. We will also be looking at exponential and logarithmic functions and their applications in the study of growth rates.

As in Volume 1, there are examples and learning activities for you to work through in each section. At the end of each chapter there are sample examination questions. Remember to always try the learning activities and examination questions yourself before you look at the solutions given in the back of the guide.

At the end of the course, there will be an examination consisting of 10 compulsory questions. These questions are designed to test your understanding of the material covered in the entire course. You will therefore need to revise the topics covered in both Volumes 1 and 2 before the examination.

At the end of this volume there are two examination papers (from 2004 and 2005) with detailed solutions. Use the given solutions to mark your own work, and to see the standard expected.

There are various books published on Mathematics for Business and Statistics which you might find useful. In particular it is recommended that you get a copy of the following book as it includes many examples and exercises similar to those in the subject guide which you can use for further practice if necessary.

> **Edward T. Dowling** *Mathematical Methods for Business and Economics* (McGraw-Hill, 1993). ISBN 0-07-017697-3.

Volumes 1 and 2 of the subject guide contain about half of the course each, so you are now already half way through CIS001. Congratulations on getting this far - keep up the hard work and good luck with the examination.

Chapter 1

Differentiation

Essential reading

See Chapter 9 of *Dowling* for further examples of the material covered in this chapter. In particular answer the supplementary problems 9.26 to 9.35 to practise your differentiation skills.

Differentiation is the process of finding the *derived function* $f'(x)$ from the function $f(x)$. This derived function is a measure of how the function $f(x)$ is changing with respect to x. Knowing how the function is changing tells us a lot of useful information about the function, such as where it has its turning points, maximum points and minimum points.

There are several *rules of differentiation* which can be followed to differentiate different types of function. In this chapter we start by differentiating simple functions and then introduce the rules of differentiation which can be used to differentiate more complicated functions.

There are many applications of differentiation and these are discussed in detail in the next chapter. This chapter concentrates solely on the mathematics of differentiation.

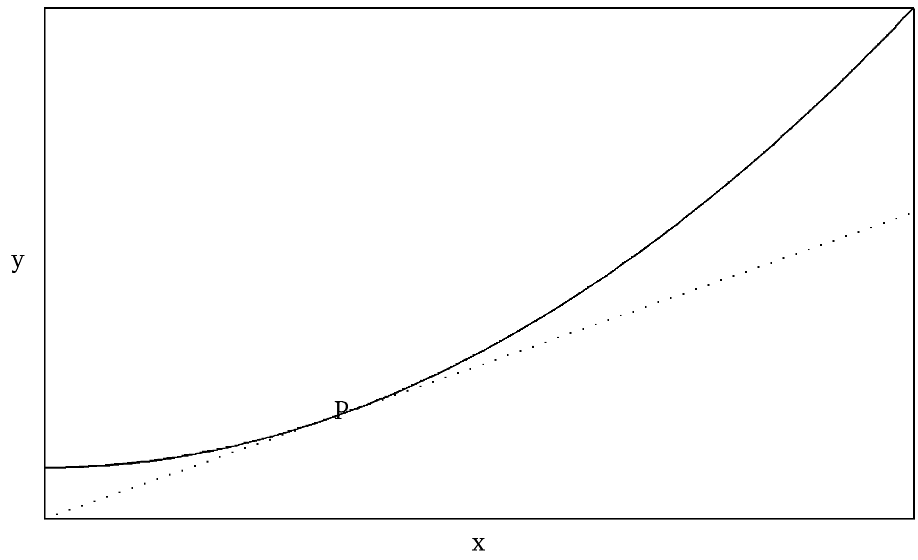
1.1 The gradient function $\frac{dy}{dx}$

In Chapter 2 of Volume 1 we saw that we could find the gradient (or measure of the steepness) of a straight line by drawing a triangle underneath the line and dividing the height of the triangle by the width.

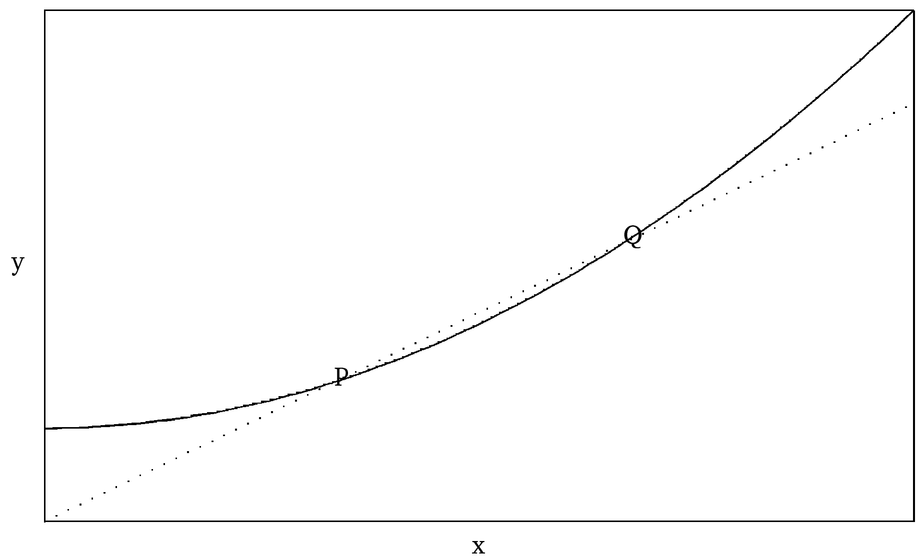
The gradient of a straight line is a constant value because the steepness of the line does not change.

The gradient of a curve however, is always changing.

We can find the gradient at a particular point P of a curve by drawing the *tangent* to the curve at point P and finding the gradient of the tangent as in the following diagram:



It is very hard to accurately draw a tangent to a curve. It is easier to draw a *secant* which is a straight line joining two points on a curve. In the following diagram the points P and Q are joined by a secant.



As the point Q is moved closer to the point P, the gradient of the secant joining P and Q gets closer to the gradient of the tangent at P.

The exact value of the gradient is found by *differentiation*. This is the *limit* of the gradient of PQ as Q moves towards P.

$\frac{dy}{dx}$ is called the *gradient function* and it measures the rate at which y changes with respect to x.

$\frac{dy}{dx}$ is also called the *derivative* of y with respect to x. Given a function $f(x)$ the derivative of the function may be written as $f'(x)$.

Thus if $y = f(x)$ then $\frac{dy}{dx} = f'(x)$.

Finding the derivative of a function either $\frac{dy}{dx}$ or $f'(x)$ is known as *differentiating* the function.

1.1.1 Example 1: differentiating $y = x^2$ using secants

Consider the curve $y = x^2$. We will find the gradient of the curve at the point P by finding the gradient of the secant PQ.

If we let P and Q be the points (p, p^2) and (q, q^2) respectively, then the gradient of the secant PQ is given by

$$gradient = \frac{q^2 - p^2}{q - p} = \frac{(q-p)(q+p)}{q-p} = p + q$$

As $Q \to P, q \to p$ and therefore $gradient \to p + p = 2p$.[1]

Since P could be any point (x, x^2) on the curve we have found that the gradient of the curve $y = x^2$ is given by $2x$. Thus $\frac{dy}{dx} = 2x$.

[1] The $\to$ symbol means 'gets closer and closer to' and is usually read as 'tends to'.

1.1.2 Example 2: differentiating $y = x^3$ using secants

Now consider the curve $y = x^3$ and let P and Q be the two points (p, p^3) and (q, q^3). The gradient of the secant PQ is given by

$$gradient = \frac{q^3 - p^3}{q - p} = \frac{(q-p)(q^2 + pq + p^2)}{q-p} = q^2 + pq + p^2$$

As $Q \to P, q \to p$ and therefore $gradient \to p^2 + p^2 + p^2 = 3p^2$

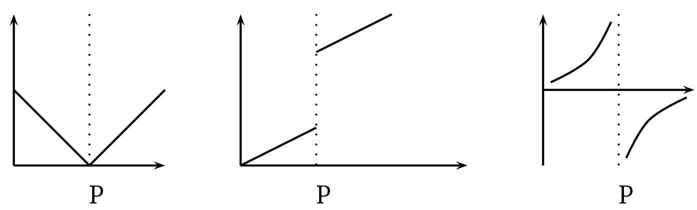
We have found that when $y = x^3$, $\frac{dy}{dx} = 3x^2$

Learning activity

Use the method of secants as above to show that the gradient of the curve $y = x^4$ is $\frac{dy}{dx} = 4x^3$

1.1.3 Differentiable functions

It is not always possible to find the derivative of a function at every point. For example, consider the three functions shown below. None of these functions can be differentiated at the point $x = p$.



A function is said to be *differentiable* at $x = p$ if $\frac{f(p)-f(q)}{q-p}$ gives a unique finite number as $q \to p$.

In these three examples, we cannot draw a unique tangent to the function at $x = p$ and therefore there is no unique gradient at this point.

In the second and third examples the functions are *discontinuous*.

A function is said to be *continuous* if it has no discontinuities. In general a function is continuous if you can draw it without taking the pen off the paper.

Learning activity

Decide whether or not the following functions are continuous and whether or not they can be differentiated at all points.

1. $f(x) = \frac{1}{x} \quad (x \neq 0)$
2. $f(x) = \sqrt{x} \quad (x \geq 0)$
3. $f(x) = \begin{cases} y = x & x < 4 \\ y = -x + 8 & x \geq 4 \end{cases}$
4. $f(x) = \begin{cases} y = -x + 2 & x \leq 2 \\ y = -x + 4 & x > 2 \end{cases}$

1.2 Differentiating x^n

In the preceeding examples and exercises we have found the gradient functions for $y = x^2, y = x^3$ and $y = x^4$. These results are summarised in the following table.

y	x^2	x^3	x^4
$\frac{dy}{dx}$	$2x$	$3x^2$	$4x^3$

Note that the *co-efficient* of the derivative $\frac{dy}{dx}$ is always the same as the *index* of the function y, and that the *index* of the derivative is always one less than the *index* of the function. This is not a coincidence and leads to the following rule:

$$\boxed{\text{The function } y = x^n \text{ has gradient function } \frac{dy}{dx} = nx^{n-1}}$$

For example if $y = x^8$ then $\frac{dy}{dx} = 8x^7$.

The rule also works for fractional indices. Thus if $y = \sqrt{x} = x^{\frac{1}{2}}$ then $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

Similarly for negative indices. If $y = \frac{1}{x^5} = x^{-5}$ then $\frac{dy}{dx} = -5x^{-6} = \frac{-5}{x^6}$.

Learning activity

Find the derivatives of the following functions:

1. x^6
2. x
3. $x^{\frac{5}{2}}$
4. $(\sqrt{x})^3$
5. x^{-3}
6. $\frac{1}{x^2}$

1.3 Differentiating kx^n

If a function is multiplied by a constant, then the constant is not changed by differentiating the function. The constant remains as a multiplier (or divisor) of the differentiated function.

$$\boxed{\text{The function } y = kx^n \text{ has gradient function } \frac{dy}{dx} = knx^{n-1}}$$

For example, if $y = 4x^2$ then $\frac{dy}{dx} = 4 * 2x = 8x$

If $y = \frac{x^3}{2} = \frac{1}{2} * x^3$ then $\frac{dy}{dx} = \frac{1}{2} * 3x^2 = \frac{3x^2}{2}$

If $y = \frac{-4}{x^2} = -4 * x^{-2}$ then $\frac{dy}{dx} = -4 * (-2x^{-3}) = 8x^{-3} = \frac{8}{x^3}$.

[2]

[2] Note that if $y = kx$ then we know that the graph of y is a straight line with gradient k. Differentiating kx gives k so the rules of differentiation give the gradient function as expected.

Learning activity

Find the derivatives of the following functions:

1. $5x^7$
2. $\frac{3x^2}{2}$
3. $-x^{-2}$
4. $\frac{-4}{x^3}$

1.4 Rules of differentiation

We have seen how to differentiate functions of the form $y = kx^n$ using the rule:

$$y = kx^n \quad \frac{dy}{dx} = knx^{n-1}$$

The following rules show how to differentiate more complicated functions. If $f(x)$ and $g(x)$ are functions of the form kx^n, then we will be able to use these rules to differentiate functions such as:

- $f(x) + g(x)$, for example $y = 3x^2 + 2x - 7$
- $f(g(x))$, for example $y = (3x + 8)^3$
- $f(x).g(x)$, for example $y = (x + 3)(7x - 4)$
- $\frac{f(x)}{g(x)}$, for example $y = \frac{3x^4-7}{x^3}$

1.4.1 The constant function rule

The graph of the function $y = k$ where k is a constant value, is a horizontal straight line with zero gradient. Thus the gradient function $\frac{dy}{dx} = 0$. [3]

For example,

$$\begin{array}{rr} y = 7 & \frac{dy}{dx} = 0 \\ f(x) = -6 & f'(x) = 0 \end{array}$$

[3] Recall that $\frac{dy}{dx}$ is a measure of how y changes with respect to x and since there is no x at all in the function $y = k$ we can see that y does not change at all in respect to x. Hence $\frac{dy}{dx} = 0$

1.4.2 The sum and difference rule

$$\boxed{\text{If } y = f(x) \pm g(x) \text{ then } \frac{dy}{dx} = f'(x) \pm g'(x)}$$

This means that we can differentiate a function with many terms added together by differentiating each term separately.

For example, if $y = 3x^2 + 2x - 7$ then we can differentiate the three terms $3x^2$, $2x$ and 7 separately and then combine the results as follows:

$$y = 3x^2 + 2x - 7 \quad \frac{dy}{dx} = 6x + 2$$

Sometimes we have to rearrange a function to make it easier to differentiate. In the following examples, we rearrange the functions so that each term is of the form kx^n.

Example 1: $f(x) = \frac{x^3-7x}{x^2}$

$$f(x) = \frac{x^3 - 7x}{x^2} = \frac{x^3}{x^2} - \frac{7x}{x^2} = x - 7x^{-1}$$
$$f'(x) = 1 + 7x^{-2} = 1 + \frac{7}{x^2} = \frac{x^2 + 7}{x^2}$$

Example 2: $y = \frac{7}{\sqrt{x}}$

$$y = \frac{7}{\sqrt{x}} = \frac{7}{x^{\frac{1}{2}}} = 7x^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = 7 * (-\frac{1}{2})x^{-\frac{3}{2}} = \frac{-7}{2x^{\frac{3}{2}}}$$

Learning activity

Differentiate the following functions. You may need to rearrange them first so that you can differentiate them term by term:

1. $4x^2 - 5x^3 + 8x - 9$
2. $(x+1)^2$
3. $(2x+4)(6-5x)$
4. $\frac{x-1}{x^2}$
5. $\frac{(7x^2+9x)}{x^4} + \frac{(4x^3-\sqrt{x})}{(\sqrt{x})^3}$

1.4.3 The chain rule

We can use the *chain rule* to differentiate a function of a function. If y is a function of u and u is a function of x then y is a function of a function of x and we can use the following rule:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

We use the chain rule in the following examples:

Example 3: $y = (3x+8)^3$

If we let $u = 3x + 8$, then we have $y = u^3$. We can differentiate y with respect to u and u with respect to x as follows:

$$y = u^3 \qquad u = 3x + 8$$
$$\frac{dy}{du} = 3u^2 \qquad \frac{du}{dx} = 3$$

Now by the chain rule we have:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3 \;.\; 3u^2 = 9u^2$$

And substituting back $u = 3x + 8$ gives the final result:

$$\frac{dy}{dx} = 9(3x+8)^2$$

Example 4: $y = \frac{1}{(x^2+2)^3}$

We have $y = u^{-3}$ where $u = x^2 + 2$.

Hence $\frac{dy}{du} = -3u^{-4}$ and $\frac{du}{dx} = 2x$.

So $\frac{dy}{dx} = -3u^{-4} \;.\; 2x = \frac{-6x}{u^4} = \frac{-6x}{(x^2+2)^4}$

Learning activity

Use the chain rule to differentiate the following functions:

1. $y = 3(x+4)^5$
2. $y = \surd(5x^2 + 7x)$
3. $y = \frac{2}{(x-3)^2}$
4. $y = \frac{1}{\surd(8x-x^2)}$

1.4.4 The product rule

If $y = uv$ where u and v are functions of x then we can use the *product rule* to differentiate y with respect to x:

$$\boxed{\text{If } y = uv \text{ then } \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}}$$

In the following examples we differentiate y using the product rule.

Example 5: $y = 4x^2(x+5)$

We have $y = uv$ where $u = 4x^2$ and $v = x + 5$. Differentiating u and v with respect to x gives us $\frac{du}{dx} = 8x$ and $\frac{dv}{dx} = 1$. Now using the product rule we have:

$$\frac{dy}{dx} = (x+5)(8x) + (4x^2)(1) = 8x(x+5) + 4x^2$$

Example 6: $y = (x^2+6)(3x+4)^7$

This time we have $y = uv$ where $u = x^2 + 6$ so $\frac{du}{dx} = 2x$ and $v = (3x+4)^7$. We use the chain rule to differentiate v and this gives us $\frac{dv}{dx} = 7(3x+4)^6(3) = 21(3x+4)^6$. Now using the product rule we have:

$$\frac{dy}{dx} = (3x+4)^7(2x) + (x^2+6)(21(3x+4)^6)$$

The answer above does not look very elegant and with a bit of work can be simplified to $y = (3x+4)^6(27x^2 + 8x + 126)$. Note however that unless a question specifically asks you to simplify your answers it is fine to leave differentiation results unsimplified.

Learning activity

Use the product rule to differentiate the following functions. Check your answers by expanding the brackets and differentiating the functions term by term.

1. $y = (x+3)(7x-4)$
2. $y = 6x^3(2x^2 + 5x - 4)$

1.4.5 The quotient rule

If $y = \frac{u}{v}$ where u and v are functions of x then we can use the quotient rule to differentiate y.

$$\boxed{\text{If } y = \frac{u}{v} \text{ then } \frac{dy}{dx} = \frac{v \,.\, \frac{du}{dx} - u \,.\, \frac{dv}{dx}}{v^2}}$$

We can differentiate the function y in the following examples using the quotient rule.

Example 7: $y = \frac{3x^4-7}{x^3}$

We have $y = \frac{u}{v}$ where $u = 3x^4 - 7$ and $v = x^3$. Differentiating u and v with respect to x gives us $\frac{du}{dx} = 12x^3$ and $\frac{dv}{dx} = 3x^2$. Now using the quotient rule we have:

$$\frac{dy}{dx} = \frac{(x^3)(12x^3) - (3x^4-7)(3x^2)}{(x^3)^2} = 3 - \frac{21}{x^4}$$

Example 8: $y = \frac{4x}{(x+1)^6}$

Now $u = 4x$ and $\frac{du}{dx} = 4$, $v = (x+1)^6$ and using the chain rule to differentiate v gives $\frac{dv}{dx} = 6(x+1)^5$. Using these values in the quotient rule gives us:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x+1)^6(4) - (4x)(6(x+1)^5)}{((x+1)^6)^2} \\
&= \frac{4(x+1)^6 - 24x(x+1)^5}{(x+1)^{12}} \\
&= \frac{4(x+1) - 24x}{(x+1)^7} \\
&= \frac{4-20x}{(x+1)^7}
\end{aligned}$$

Learning activity

Differentiate each of the following functions using the most appropriate method in each case:

1. $\frac{2}{x^2} - 3$
2. $(2x-3)x^2$
3. $\frac{1}{(2x-3)^2}$
4. $\frac{x^2}{2x-3}$
5. $2\sqrt{x} - 3$
6. $\sqrt{(x+5)}$
7. $\frac{x+5}{\sqrt{x}}$
8. $\frac{x-1}{(x+5)^2}$
9. $(\frac{1}{x^2})^5$
10. $(x+5)^9\sqrt{x}$

1.5 Second order derivatives

Sometimes we might want to differentiate a function twice to find the *second order derivative*. Finding the second order derivative gives us information about what the curve looks like. There will be more details on this in Chapter 2 Section 1.5.

If we have a function y, then we differentiate the function with respect to x to find the first order derivative $\frac{dy}{dx}$. We can then differentiate $\frac{dy}{dx}$ with respect to x to find the second order derivative, which is denoted by $\frac{d^2y}{dx^2}$.

If $f(x)$ notation is used, then the first order derivative is denoted by $f'(x)$ and the second order derivative is denoted by $f''(x)$.

Example 9: $f(x) = 3x^2 + 7x - 8$

We can differentiate the function $f(x)$ to find the first order derivative:

$$f'(x) = 6x + 7$$

Now we differentiate $f'(x)$ to find the second order derivative:

$$f''(x) = 6$$

Example 10: $y = (3x + 6)^3$

We can use the chain rule to differentiate this function to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = 3(3x + 6)^2(3) = 9(3x + 6)^2$$

Now we have to differentiate $\frac{dy}{dx}$ to find the second order derivate $\frac{d^2y}{dx^2}$. We use the chain rule again.

$$\frac{d^2y}{dx^2} = 18(3x + 6)(3) = 54(3x + 6)$$

Learning activity

Find the first and second order derivatives of the following functions:

1. $y = 6x^3 + 4x^2 - 7x + 11$
2. $f(x) = (x + 1)^3$
3. $f(x) = \frac{1}{x^2}$
4. $y = \sqrt{x}$

1.6 Learning outcomes

After studying this chapter and the relevant reading you should be able to:

- Explain how the gradient function $\frac{dy}{dx}$ measures the rate of change of y with respect to x.
- Recognize and properly use the notation $\frac{dy}{dx}$ and $f'(x)$ for the first order derivative of a function.
- Explain what is meant by differentiability and the notion that not all functions are differentiable at every point.
- Decide whether or not a function is continuous or discontinuous.
- Differentiate functions of the form $y = kx^n$.
- Differentiate functions by applying the chain rule, sum and difference rule, product rule and quotient rule as appropriate.
- Find the second order derivative $\frac{d^2y}{dx^2}$ or $f''(x)$ of a function.

1.7 Sample examination questions

Question 1

a) The function $f(x)$ is given by two different formulae; one formula when $x < 0$ and the other when $x \geq 0$:

$$\begin{array}{ll} f(x) = 2x + a & x < 0 \\ f(x) = x^2 + 3x - 2 & x \geq 0. \end{array}$$

Sketch the function for $-2 \leq x \leq 2$ when $a = 1$. Is the function continuous? For what values of a is the function continuous?

[5]

b) Differentiate the following functions. (You do not have to expand or simplify your answers.)

i) $3(5x - 2)^6$

ii) $\frac{(3x^3+4)}{(x-2)}$

iii) $(5x^2 - 2)(4x^2 - 3x - 1)$

[5]

Question 2

Given the functions $f(x) = 2 - 5x$, $g(x) = \frac{1}{\sqrt{x}}$ find the following functions and their derivatives:

a) $f \circ g(x)$

[2]

b) $\frac{f(x)}{g(x)}$

[2]

c) $g \circ f(x)$

[2]

d) $\frac{1}{f(x) \cdot g(x)}$

[2]

e) $f(\sqrt{(}x - 2)$

[2]

Chapter 2

Applications of differentiation

Essential reading

See Chapter 10 of *Dowling* for further examples of the material covered in this chapter. In particular attempt the supplementary problems 10.28 to 10.36.

In Chapter 1 we saw how to differentiate the function $f(x)$ to find the *derived function* $f'(x)$. We learnt how this derived function is the same as the *gradient function* $\frac{dy}{dx}$ which measures the rate at which y changes with respect to x. In this chapter we will discuss some of the applications of differentiation.

Given a function $f(x)$, we can use differentiation to find the *critical points* of the function. These are the points where the gradient function is equal to zero. At these points the curve changes in some way - such points may be *turning points* or *points of inflexion* where the *concavity* of the curve alters.

We can use second order derivatives to deduce whether turning points are *maximum points* or *minimum points*. If the function in question is a profit function, then we can use this information to determine when the profit is maximised.

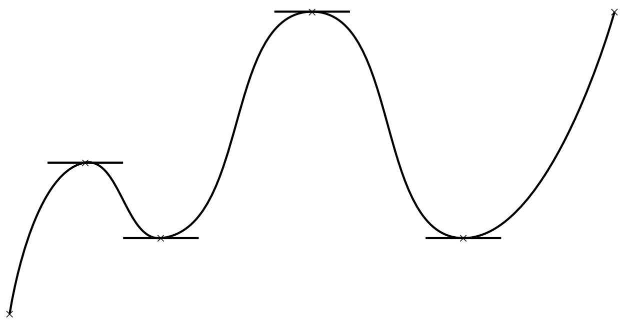
We can also find the marginal revenue and cost functions which tell us how revenue and cost vary as the quantity of goods produced and sold varies.

2.1 Critical points

A critical point of a function $f(x)$ is a point at which the gradient function $f'(x)$ is equal to zero. Such points are also known as stationary points. A stationary point may either be a turning point $\cap$ or $\cup$ or a point of inflexion $\wr$. In this section we will discuss how to use differentiation to find critical points and how to determine whether they are turning points; and if so whether they are maximum points or minimum points, or points of inflexion.

2.1.1 Turning points

A *turning point* of a curve, is a point where the slope of the curve changes direction. The gradient is positive on one side of the turning point and negative on the other side. A turning point is a type of stationary point.



This curve has four turning points. You can see that at these four points the tangent to the curve is horizontal.

At a stationary point of a function, the tangent to the curve is a horizontal line and therefore the *gradient function* is equal to zero at that point. We can therefore find the stationary points of a curve $y = f(x)$ by differentiating the function $f(x)$ to find the gradient function $\frac{dy}{dx} = f'(x)$ and then solving the equation $\frac{dy}{dx} = 0$ to find x.

Example

We will find the stationary points of the functions $f(x) = 3x^2 + 2x + 1$ and $g(x) = x^3 - 4x^2 + x - 9$

1. The function $f(x) = 3x^2 + 2x + 1$ is a quadratic and so will have one turning point. [1]

   The derived function is $f'(x) = 6x + 2$. Setting this equal to zero gives:

$$\begin{aligned} f'(x) = 6x + 2 &= 0 \\ 6x &= -2 \\ x &= -\frac{1}{3} \end{aligned}$$

   The function $f(x) = 3x^2 + 2x + 1$ has its turning point when $x = -\frac{1}{3}$. Evaluating $f(\frac{1}{3}) = 3(-\frac{1}{3})^2 + 2(-\frac{1}{3}) + 1 = \frac{2}{3}$. Hence the turning point of the function is the point $(-\frac{1}{3}, \frac{2}{3})$.

2. The function $g(x) = x^3 - 4x^2 + x - 9$ is not a quadratic but a cubic function. To find the stationary points we differentiate $g(x)$ to find

$$g'(x) = 3x^2 - 8x + 1$$

   Now $g'(x)$ *is* a quadratic and so we can set it equal to zero and then solve it to find two solutions for x using the quadratic formula (see Volume 1, Chapter 3).

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

   We have $a = 3$, $b = -8$ and $c = 1$, so

$$\begin{aligned} x &= \frac{8 \pm \sqrt{(-8^2 - 4(3)(1))}}{2(3)} \\ &= \frac{8 \pm \sqrt{(52)}}{6} \\ x &= 2.535 \quad or \quad x = 0.131 \quad (3d.p.) \end{aligned}$$

   Therefore the function $g(x)$ has two stationary points. One when $x = 2.535$ and another when $x = 0.131$. Evaluating $g(x)$ at these points tells us that the stationary points are $(2.535, -15.879)$ and $(0.131, -8.935)$.

[1] We saw in Chapter 3 of Volume 1 how to find the turning point or *vertex* of a quadratic using algebra - here we will use differentiation instead.

Learning activity

Find the co-ordinates of the three stationary points on the curve $y = 3x^4 + 4x^3 - 12x^2$.

2.1.2 Maximum and minimum points

We have seen how to use differentiation to find the stationary points of a function. The next step is to determine whether such points are maximum points - called *maxima*, or minimum points - called *minima*.

The second derivative $\frac{d^2y}{dx^2} = f''(x)$ can give information about the nature of the stationary points.

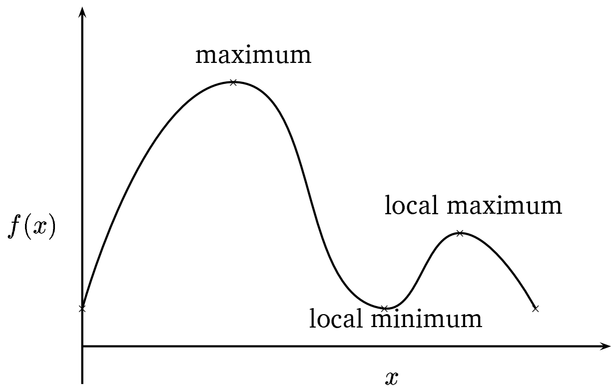
At a stationary point:

- if $f''(x) > 0$ then the stationary point is a local minimum
- if $f''(x) < 0$ then the stationary point is a local maximum

but

- if $f''(x) = 0$ then this does not give us any further information about the stationary point and we have to use other methods to determine its nature.

Note that we say a point is a *local minimum* or a *local maximum* because a turning point might be a maximum i.e., a $\cap$ shape; but the value of $f(x)$ at this turning point might not be the maximum value of the function. This is illustrated below.



Example

We will use second order derivatives to determine the nature of the stationary points of the functions $f(x)$ and $g(x)$ above.

1. We have found that the stationary point of the function $f(x) = 3x^2 + 2x + 1$ occurs when $x = -\frac{1}{3}$. By differentiating $f'(x) = 6x + 2$ we can find the second order derivative

$f''(x) = 6$. We have $f''(x) > 0$ and this tells us that the stationary point at $(-\frac{1}{3}, \frac{2}{3})$ is a minima. [2]

2. The function $g(x) = x^3 - 4x^2 + x - 9$ has two stationary points which occur when $x = 2.535$ and $x = 0.131$. Again we differentiate the derived function to obtain the second derivative:

$$\begin{aligned} g'(x) &= 3x^2 - 8x + 1 \\ g''(x) &= 6x - 8 \end{aligned}$$

When $x = 2.535$, $g''(x) = 6(2.535) - 8 = 7.21 > 0$. Therefore the turning point at $(2.535, -15.879)$ is a local minimum.

When $x = 0.131$, $g''(x) = 6(0.131) - 8 = -7.214 < 0$. Therefore the turning point at $(0.131, -8.935)$ is a local maximum.

[2] In this case, since $f(x)$ has only one turning point this is a global minima.

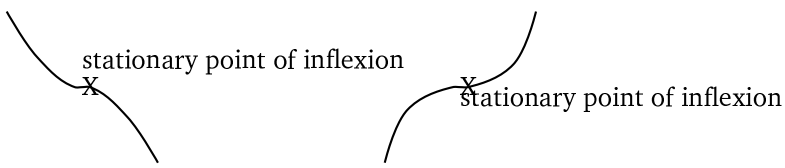
Learning activity

1. Find the turning points of the curve $y = 2x^3 - 6x^2 - 18x + 5$ and use the second derivative to determine whether they are maxima or minima.
2. Find the values of x for which the function $f(x) = x^3 - 6x^2 + 9x - 7$ is decreasing.

2.1.3 Points of inflexion

We said above that if $f''(x) = 0$ then this does not give us any further information about the nature of the stationary point at x. The stationary point might be a maxima or a minima or a *point of inflexion*.

At a point of inflexion, the graph changes the direction in which it is *curving*, but the function continues to increase or decrease as before.



If $f''(x) = 0$ then we have to evaluate the gradient function $f'(x)$ at points just before and just after x to determine whether the point x is a maxima or a minima or a point of inflexion.

- If the gradient is positive just before x and negative just after x then the stationary point at $(x, f(x))$ is a maximum.
- If the gradient is negative just before x but positive just after x, then the stationary point at $(x, f(x))$ is a minimum.
- If the gradient is positive both before x and after x, or negative both before x and after x, then the stationary point at $(x, f(x))$ is a point of inflexion.

Example

We will find, and determine the nature of, the stationary points of the function:

$$h(x) = \frac{1}{3}x^3 - 2x^2 + 4x$$

First of all we differentiate the function to find $h'(x) = x^2 - 4x + 4$. Now setting $h'(x) = 0$ and solving for x tells us where the stationary points occur.

$$\begin{aligned} h'(x) = x^2 - 4x + 4 &= 0 \\ (x-2)^2 &= 0 \\ x = 2 & \end{aligned}$$

There is a (repeated) stationary point at $x = 2$. Evaluating $h(2) = \frac{1}{3}(2)^3 - 2(2)^2 + 4(2) = \frac{8}{3}$ tells us that the stationary point is at $(2, \frac{8}{3})$.

Next we differentiate $h'(x)$ to find the second order derivative $h''(x) = 2x - 4$.

When $x = 2$, $h''(x) = 2(2) - 4 = 0$. This does not give us any information about the nature of the stationary point.

To find out more about the stationary point at $x = 2$, we will have to evaluate $h'(x)$ for values of x before and after $x = 2$.

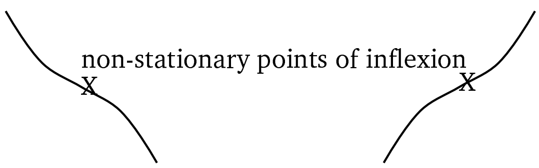
When $x = 1$, $h'(1) = (1-2)^2 = 1$ so the curve has positive gradient before the stationary point.

When $x = 3$, $h'(3) = (3-2)^2 = 1$ so the curve also has positive gradient after the stationary point.

Therefore the stationary point at $(2, \frac{8}{3})$ is a point of inflexion.

Non-stationary points of inflexion

A point of inflexion is not necessarily a stationary point. At a stationary point, the gradient function is equal to zero. This means that the tangent to a stationary point is horizontal. The tangent to a point of inflexion might not be horizontal as illustrated below.



To find the points of inflexion of a function $f(x)$, we find the values of x which make the second order derivative $f''(x)$ equal to zero.

For example, recall the function $g(x) = x^3 - 4x^2 + x - 9$ for which we have already found and classified the stationary points: $(2.535, -15.879)$ is a local minimum, and $(0.131, -8.935)$ is a local maximum.

For this function, the second order derivative is $g''(x) = 6x - 8$. When $x = \frac{4}{3}$, $g''(x) = 0$ and therefore there is a point of inflexion when $x = \frac{4}{3}$.

Evaluating $g(\frac{4}{3}) = (\frac{4}{3})^3 - 4(\frac{4}{3})^2 + \frac{4}{3} - 9 = -12\frac{11}{27}$ tells us that the point of inflexion is at $(\frac{4}{3}, -12\frac{11}{27})$.

Finding the points of inflexion of a curve as well as its stationary points makes it easier to draw an accurate sketch of the curve.

Note

Although at a point of inflexion $f''(x)$ is always equal to zero, it is not true that if $f''(x) = 0$ then $(x, f(x))$ must be a point of inflexion. Therefore if you are trying to determine the nature of a stationary point and find that $f''(x) = 0$ at that point then you cannot assume that it is a point of inflexion. You should consider the gradient function either side of the stationary point in order to determine its nature.

Learning activity

1. Find the co-ordinates of the stationary points on the following curve $y = x^4 - 4x^3$ and determine their nature. Show that the curve has a point of inflexion at $x = 2$.
2. Find the co-ordinates of the stationary points on the curve $y = (4 - 2x)^4$ and determine their nature.

2.1.4 Sketching functions

To draw an accurate sketch of a function $f(x)$ we need to find:

- The position and nature of any turning points — this tells us the shape of the curve.
- The position of any stationary or non-stationary points of inflexion — this tells us where the concavity of the curve changes.
- The value of $f(0)$ — this is the point where the function crosses the y-axis.
- If possible the *roots* of the function - these are the points where the function crosses the x-axis. [3]

Once we have all of this information it is possible to draw an accurate sketch by plotting the critical points and then joining them up using the information about maxima, minima and points of inflexion as appropriate.

For example, we have been working with the function $g(x) = x^3 - 4x^2 + x - 9$ throughout the previous section and have classified its turning points and points of inflexion as follows:

- $(2.535, -15.879)$ is a local minimum.

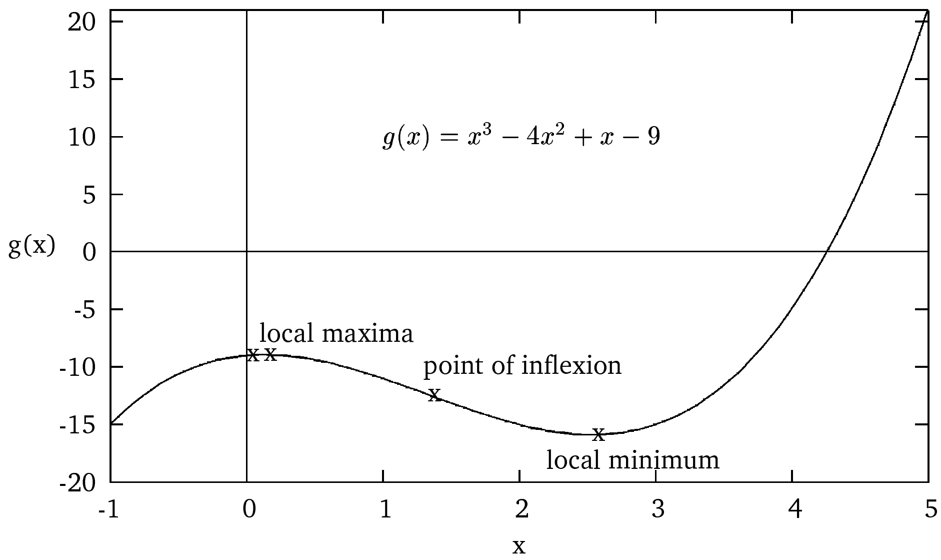
[3] If it is not possible to factorise the function then it may be difficult to find the roots. In this case a sketch of the graph may be used to estimate the roots. This might be useful, for example to determine the break-even points of a profit function.

- $(0.131, -8.935)$ is a local maximum.
- $(\frac{4}{3}, -12\frac{11}{27})$ is a point of inflexion.

We can evaluate $g(0) = -9$ which tells us that the function crosses the y-axis at $(0, -9)$.

We do not know the roots of the function, but since $g(4) = 4^3 - 4(4)^2 + 4 - 9 = -5$ and $g(5) = 5^3 - 4(5)^2 + 5 - 9 = 21$ we can tell that the curve crosses the x-axis between 4 and 5.

We have enough information to sketch the curve.



Learning activity

Find all of the critical values of the function $f(x) = 3x^5 - 5x^3 + 1$. For each critical point determine whether the point is a maximum or minimum or neither of these. Sketch the graph of $f(x)$ for values of x between -1 and +1.

2.2 Applications to economics

In this section we will apply the theory of differentiation to solve some practical problems. If functions are used to represent revenue, cost and profit, then differentiation techniques can be used to calculate *marginal revenue* and *marginal cost*, *average revenue, maximum and minimum profits* and *costs*. This knowledge enables us to *optimize* economic functions thus achieving the best possible output.

2.2.1 Calculating marginal revenue and costs

In economics, the *marginal revenue* is defined as the change in *total revenue* brought about by the sale of one additional unit.

If total revenue TR, is expressed as a function of q, where q is the quantity of units sold, then the derivative of TR with respect to q will tell us how the total revenue is changing as the quantity of units changes. This is the *marginal revenue*.

Thus marginal revenue is a function MR of q which is obtained by differentiating the function TR.

$$MR = \frac{dTR}{dq}$$

Similarly, *marginal cost* is defined to be the change in total cost brought about by the production of one additional unit.

If we express total cost as a function TC of q then marginal cost is also a function in q, denoted by MC, which can be obtained by differentiating TC with respect to q. This will tell us how the total cost changes with respect to the quantity of goods produced.

$$MC = \frac{dTC}{dq}$$

Worked example

A factory has total cost function and total revenue function given by $TC(q) = q^2 + 7q + 31$ and $TR(q) = -2q^2 + 65q$ where q represents the quantity of units produced and sold respectively.

Solution

The marginal cost is found by differentiating the total cost function:

$$\begin{aligned} TC(q) &= q^2 + 7q + 31 \\ \frac{dTC(q)}{dq} &= 2q + 7 \end{aligned}$$

Therefore the marginal cost is $MC(q) = 2q + 7$

The marginal revenue is found by differentiating the total revenue function:

$$\begin{aligned} TR(q) &= -2q^2 + 65q \\ \frac{dTR(q)}{dq} &= -4q + 65 \end{aligned}$$

Therefore the marginal revenue is $MR(q) = -4q + 65$.

Thus the cost of producing one extra unit is $MC(1) = 2(1) + 7 = 9$ and the revenue generated by producing one extra unit is $MR(1) = -4(1) + 65 = 61$.

2.2.2 Calculating average revenue and cost

The *average revenue* function can be found by dividing the total revenue function by q. If we use AR to denote the average revenue then:

$$AR(q) = \frac{TR(q)}{q}$$

Similarly, the *average cost* function can be found by dividing the total cost function by q:

$$AC(q) = \frac{TC(q)}{q}$$

Worked example

The total cost function and total revenue function for a production line are given by $TC = q^2 + 8q + 12$ and $TR = 48q - q^2$ respectively. Find the average cost function and the average revenue function and evaluate these functions assuming that a total of 10 units are produced and sold.

Solution

The average cost function is found by dividing TC by q as follows:

$$AC(q) = \frac{TC(q)}{q} = \frac{q^2}{q} + \frac{8q}{q} + \frac{12}{q} = q + 8 + \frac{12}{q}$$

The average revenue function is found by dividing TR by q:

$$AR(q) = \frac{TR(q)}{q} = \frac{48q}{q} - \frac{q^2}{q} = 48 - q$$

If 10 units are produced and sold then we have $q = 10$. The average cost and revenue for producing 10 units is given by:

$$\begin{aligned} AC(10) &= 10 + 8 + \frac{12}{10} = 19.2 \\ AR(10) &= 48 - 10 = 38 \end{aligned}$$

2.2.3 Finding maximum and minimum points of economic functions

We saw in section 2.1 how to find the critical points of a function. If the function in question is an economic function, representing profit or cost for example, then the same methods for finding the critical points can be used. For economic functions we are particularly interested in finding maximum and minimum points which will tell us where, for example, profits are maximised and costs are minimised.

The following worked example illustrates the use of differentiation of economic functions.

Worked example

Revenue for a company expressed in terms of output, q, is given as

$$R = 500q - 1000$$

Costs for this company are given by

$$C = q^3 - 60q^2 - 1000q + 3000$$

What value of q maximises profit?

Solution

Profit is given by revenue minus cost. Therefore we have the profit function:

$$\begin{aligned} \Pi(q) &= R(q) = C(q) = 500q - 1000 - (q^3 - 60q^2 - 1000q + 3000) \\ \Pi(q) &= -q^3 + 60q^2 + 1500q - 4000 \end{aligned}$$

By differentiating the function $\Pi(q)$ with respect to q, and setting the result equal to zero we can find the stationary points of the profit function.

$$\frac{d\Pi(q)}{dq} = -3q^2 + 120q + 1500$$

At turning points $\frac{d\Pi(q)}{dq} = 0$ therefore:

$$-3q^2 + 120q + 1500 = 0$$

dividing all terms by -3 gives:

$$q^2 - 40q - 500 = 0$$

10*50=500 and 10-50=-40 so we can immediately factorise this quadratic:

$$(q + 10)(q - 50) = 0$$

The solutions are at $q = -10$ and $q = 50$, and therefore the turning points of the profit function occur when $q = -10$ and when $q = 50$.

We cannot make a minus quantity of goods and therefore we can ignore the solution at $q = -10$.

We will use second order differentiation to verify that the critical point at $q = 50$ is a maximum.

$$\frac{d^2\Pi(q)}{dq^2} = -6q + 120$$

At $q = 50$, $\frac{d^2\Pi(q)}{dq^2} = -6(50) + 120 = -180$. This is less than zero so the turning point is a maximum.

Therefore the value of q which maximises profit is $q = 50$.

2.2.4 Sketching and interpreting graphs of economic functions

In section 2.1.4 we saw how to draw a graph of a function. The same rules apply when sketching an economic function. We can find and determine the nature of the critical points of the function and see where it crosses the x and y-axes.

Note that when sketching the graph of an economic function, quantity is usually on the x-axis. Since quantity must be greater than or equal to zero, we do not sketch the graph for negative values of q.

For example, suppose a factory has a profit function given by:

$$\Pi(q) = -3q^3 + 9q^2 + 27q - 15$$

where q represents the quantity of goods produced and sold.

We can differentiate $\Pi(q)$ and set the result equal to zero to find the critical points of the function as follows.

$$\begin{aligned} \frac{d\Pi(q)}{dq} = -9q^2 + 18q + 27 &= 0 \\ q^2 - 2q - 3 &= 0 \\ (q+1)(q-3) &= 0 \end{aligned}$$

The critical points occur when $q = -1$ and when $q = 3$. Since q represents quantity which cannot be negative, we disregard the critical point at $q = -1$.

When $q = 3$, $\frac{d^2\Pi(q)}{dq^2} = -18q + 18 = -18(3) + 18 < 0$ which shows that the turning point at $q = 3$ is a maximum.

Evaluating the profit function at $q = 3$ tells us that the maximum profit is 66.

$$\Pi(3) = -3(3)^3 + 9(3)^2 + 27(3) - 15 = 66$$

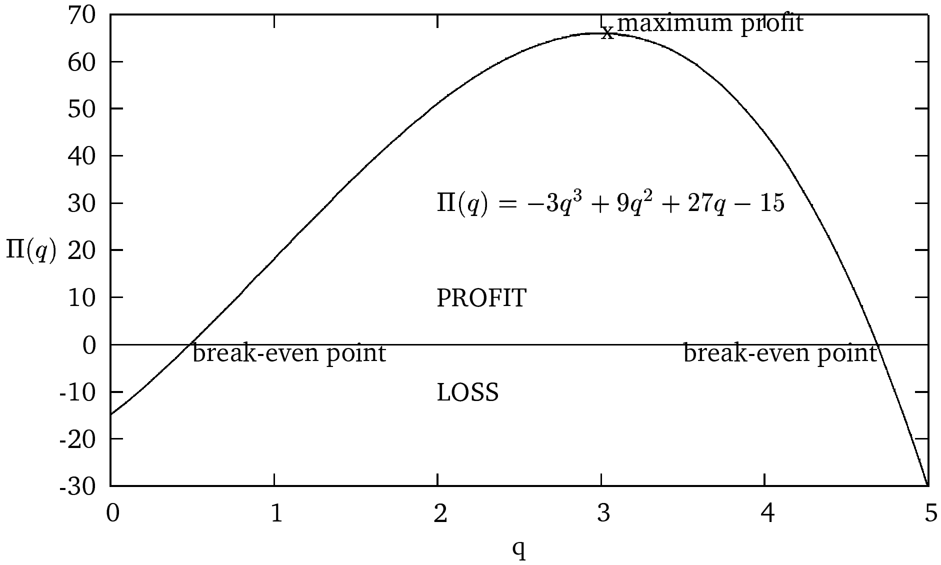
When $q = 0$, $\Pi(q) = -15$, therefore the function intercepts the y-axis at -15.

By evaluating the function at a few more values of q we can estimate where the function crosses the x-axis - this will tell us the break-even points of the function.

$$\begin{aligned} \Pi(1) &= -3(1)^3 + 9(1)^2 + 27(1) - 15 = 18 \\ \Pi(4) &= -3(4)^3 + 9(4)^2 + 27(4) - 15 = 45 \\ \Pi(5) &= -3(5)^3 + 9(5)^2 + 27(5) - 15 = -30 \end{aligned}$$

The function crosses the x-axis between 0 and 1 and again between 4 and 5.

Below is a sketch of the profit function.



2.3 Learning outcomes

After working through this chapter and the relevant readings you should be able to:

- Use differentiation to find the critical points of a function.
- Determine the nature of critical points using second order differentiation and by considering the gradient function on either side of the point.
- Find all of the points of inflexion of a function using second order differentiation.
- Sketch the graph of a function incorporating all of this information.
- Apply differentiation and function sketching techniques to economic functions in order to determine:
  - Marginal revenue and marginal cost;
  - Average revenue and average cost;
  - Maximum and minimum points of an economic function.

2.4 Sample examination questions

Question 1

a) Find the stationary points of the function

$$f(x) = x^3 + 3x^2 - 9x$$

and determine their nature.

[5]

b) Sketch the graph of $y = f(x)$, showing the turning points and the point of intersection with the y-axis.

[5]

Question 2

a) Write down the profit function Π, in terms of Q given the total revenue function

$$TR = Q^2 + 2Q$$

and the total cost function

$$TC = 2Q^3 + 4Q^2 - 10Q + 4$$

[1]

b) Find all the values of Q which give the relative extrema and points of inflexion for the function $\pi(Q)$.

[8]

c) What is the maximum profit?

[1]

Chapter 3

Exponential and logarithmic functions

Essential reading

See Chapter 11 of *Dowling* for many further examples of the material covered in this chapter.
For more practice answer the supplementary problems 11.40 to 11.51 and 11.62 to 11.68.

In this chapter we will explore the properties of an important family of functions known as the *exponential functions*.

Given a constant number a, we can define a function $y = a^x$ by raising constant a to the power x for varying values of x. This function $y = a^x$ is called the *exponential function of x (with base a)*. The exponent x is called the *logarithm of y to the base a*. We write this as $x = log_a y$.

Exponential functions are of considerable importance in computer science and in economic modelling. They are also used to study rates of growth and decay in the physical and biological sciences. In this and the following chapter, we will be considering the applications of exponential functions to economics. The applications to computing will be pursued in other units of this Diploma.

3.1 Exponential functions

An *exponential function* is a power function in which the variable is in the exponent. The *base* value remains constant.

$$f(x) = a^x \quad a > 0, \quad a \neq 0$$

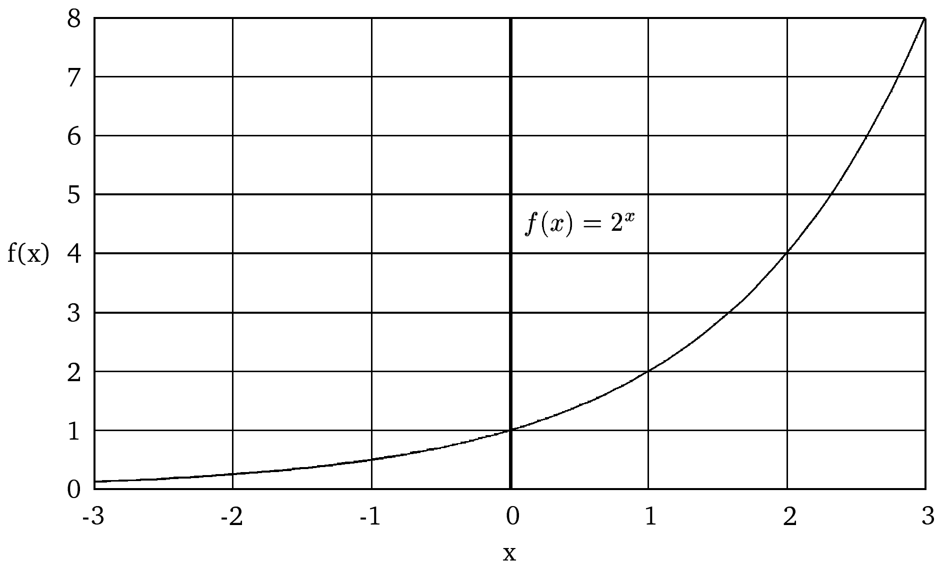
For example $f(x) = 2^x$ is an exponential function with base 2. We can evaluate the function $f(x)$ for all real values of x. [1]

[1] Use the $[x^y]$ button on your calculator to evaluate functions like this.

The table below shows the function $f(x) = 2^x$ evaluated for $-3 \leq x \leq 3$.

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

We can use the values in the table above to sketch the graph of $f(x) = 2^x$ between $-3 \leq x \leq 3$.



The function $f(x) = 2^x$ never has an output which is zero or negative. Therefore the curve never touches the x-axis. As $x \to -\infty, f(x) \to 0$.

For positive values of x, a small increase in x makes a big increase in $f(x)$. As $x \to +\infty, f(x) \to +\infty$.[2]

In general the following rules apply:

- For base $a > 1$ the graph of a^x is positive and increasing.
- For base $0 < a < 1$ the graph of a^x is positive and decreasing.

[2] Things which grow very quickly are said to have *exponential growth*. For example the number of rabbits in a colony - if you start off with two rabbits, you very soon have hundreds of rabbits - this is exponential growth!

Learning activity

Complete the table below and then sketch the graph of $f(x) = (\frac{1}{2})^x$ for values of x between -3 and 3.

x	-3	-2	-1	0	1	2	3
$(\frac{1}{2})^x$							

Compare the graphs of $f(x) = 2^x$ and $f(x) = (\frac{1}{2})^x$. What is the relationship between these two graphs?

3.2 Logarithmic functions

The *logarithmic function*

$$y = \log_a x \quad a > 0, \quad a \neq 0$$

is the *inverse* of the exponential function $y = a^x$.

Given the values of y and a, we can use the logarithmic function to find the value of x which satisfies the equation $y = a^x$.

For example, $\log_2 8 = 3$ (read 'log to the base two of eight equals three') because $2^3 = 8$, and $\log_{10} 100 = 2$ because $10^2 = 100$.

Learning activity

Evaluate the following logarithms. You may find it easier if you re-write the logarithm as an exponential function. For example, to evaluate $\log_2 16$ re-write as $2^x = 16$ and then find $x = 4$ since $2^4 = 16$. Hence $\log_2 16 = 4$.

1. $\log_5 125$
2. $\log_3 243$
3. $\log_3 \frac{1}{243}$
4. $\log_7 7$
5. $\log_{10} 100,000$
6. $\log_{\frac{1}{2}} \frac{1}{4}$
7. $\log_{18} 1$
8. $\log_{\frac{1}{4}} 16$
9. $\log_{0.1} 0.001$
10. $\log_{\frac{3}{4}} \frac{9}{16}$

3.3 Rules of indices and logarithms

The rules of indices which are listed below should be familiar from Volume 1 of the subject guide.

- **multiplication**: $a^x * a^y = a^{x+y}$
- **division**: $a^x \div a^y = a^{x-y}$
- **raising to a power**: $(a^x)^n = a^{xn}$
- **zero exponent**: $a^0 = 1$
- **negative exponent**: $a^{-x} = \frac{1}{a^x}$
- **fractional exponent**: $a^{\frac{x}{y}} =^y \sqrt{a^x}$

Following is the corresponding set of rules for logarithms.

- **multiplication**: $\log_a(x * y) = \log_a x + log_a y$
- **division**: $\log_a(x \div y) = \log_a x - \log_a y$
- **raising to a power**: $\log_a x^n = n \log_a x$
- **logarithm of unity**: $\log_a 1 = 0$
- **logarithm of the base**: $\log_a a = 1$

We will illustrate the rules of logarithms using examples with the base 2. The table below gives the values for $\log_2 x$ for $0 \leq x \leq 5$. Use the values in the table to check the answers in the following examples.

$\log_2 1 = 0$
$\log_2 2 = 1$
$\log_2 4 = 2$
$\log_2 8 = 3$
$\log_2 16 = 4$
$\log_2 32 = 5$

- $log_2 8 + \log_2 2 = \log_2(8 * 2) = \log_2 16 = 4$
- $\log_2 32 - \log_2 8 = \log_2(32 \div 8) = \log_2 4 = 2$
- $\log_2 4^2 = 2 * \log_2 4 = 2 * 2 = 4$
- $\log_2 1 = 0$
- $\log_2 2 = 1$

Learning activity

Evaluate the following expressions without using a calculator:

1. $\log_2 64 - \log_2 16$
2. $(\frac{81}{256})^{\frac{3}{4}}$
3. $\log_{10} 6 + \log_{10} 4 + \log_{10} 20 - \log_{10} 3 - \log_{10} 16$
4. $\log_3 9^5$

3.4 Solving exponential equations

An exponential equation is one in which the variable is an index. Such equations can often be solved by either:

- *taking logs*[3] on both sides of the equation and using the basic rules of logarithms; or
- using a substitution of the form $y = a^x$ to obtain an equation in y (usually a quadratic) which can then be solved.

[3] Just as we can square both sides of an equation, or divide each side by 2 etc, we can take logs on both sides of an equation. Thus if $a = b$ then $\log a = \log b$.

The following worked examples illustrate these two different methods of solving exponential equations.

Example 1: $7^x = 3$

To solve this equation, we will *take logs* on both sides. This will give us:

$$\log 7^x = \log 3$$

Note that there is no base value shown. It does not matter what the base value is so long as it is the same on both sides of the equation.

Using the rules of logarithms we can re-arrange the equation as follows:

$$\begin{aligned} x \log 7 &= \log 3 \\ x &= \frac{\log 3}{\log 7} \end{aligned}$$

Now we can use a calculator to evaluate $\log 3$ and $\log 7$ and compute x.[4]

$$x = \frac{\log 3}{\log 7} = \frac{0.4771}{0.8451} = 0.565(3d.p.)$$

[4] Your calculator should have a [log] button which allows you to calculate logs to the base 10. Try evaluating $\log_{10} 100$ using your calculator. The input sequence will either be [100][log][=] or [log][100][=] depending on the make of calculator. The answer should be 2.

Example 2: $\log_x 3 + log_x 27 = 2$

This time the variable is not in the index but we can still use the laws of indices and logarithms to solve this equation.

$$\begin{aligned}
\log_x 3 + log_x 27 &= 2\\
\log_x(3 * 27) &= 2\\
\log_x 81 &= 2\\
x^2 &= 81\\
x = \sqrt{}81 &= 9
\end{aligned}$$

Example 3: $2^{2x} - 5(2^{x+1}) + 9 = 0$

This equation can be re-written as

$$(2^x)^2 - 5(2 * 2^x) + 9 = 0$$

Now making the substitution $y = 2^x$ gives us

$$y^2 - 10y + 9 = 0$$

This quadratic can be factorised as

$$(y - 1)(y - 9) = 0$$

There are two solutions.

Either $y = 1$ so $2^x = 1$ and therefore $x = 0$.

Or $y = 9$ so $2^x = 9$. Taking logs on both sides gives $x \log 2 = \log 9$ and therefore the second solution is $x = \frac{\log 9}{\log 2} = \frac{0.9542}{0.3010} = 3.17(2d.p.)$

Example 4: $3^{2y} - 3^{y+1} - 3^y + 3 = 0$

This time we rewrite the equation in terms of (3^y) and then make the substitution $z = 3^y$.

$$\begin{aligned}
3^{2y} - 3^{y+1} - 3^y + 3 = 0\\
(3^y)^2 - 3(3^y) - 3^y + 3 = 0\\
z^2 - 3z - z + 3 = 0\\
z^2 - 4z + 3 = 0\\
(z - 1)(z - 3) = 0
\end{aligned}$$

Either $z = 1$, so $3^y = 1$ and therefore $y = 0$.

Or $z = 3$, so $3^y = 3$ and therefore $y = 1$.

Learning activity

Solve the following exponential equations using the methods illustrated above. Give your answers correct to three decimal places where appropriate.

1. $9^x = 16$
2. $2^{2x} - 16 = 0$
3. $5^{2x} - 5^{x+2} - 5^{x+1} + 125 = 0$ (use the substitution $z = 5^x$)
4. Rewrite $y = \log_4 x$ as a log with base 2. Hence solve the equation $\log_2 x - \log_4 x = 3$.

3.5 Natural exponential and logarithmic functions

The base which is most often used for logarithmic and exponential functions is the irrational number e which is defined mathematically as

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

Your calculator should have an $[e^x]$ button. Use this button to calculate $e^1 = e \approx 2.718281828...$[5]

[5] The reason why this number e is often used as the base value in exponential functions is because the differential of e^x is e^x.

The inverse of e^x is $log_e x$. This function is called the *natural logarithm* and is usually denoted by $\ln x$. Your calculator should also have a $[ln]$ button.

Learning activity

To make sure that you can use the $[\log]$ and $[\ln]$ and corresponding $[10^x]$ and $[e^x]$ buttons on your calculator work out the following giving your answers correct to three decimal places where appropriate:

1. $\log_{10} 95$
2. $\ln 95$
3. $\log_e 95$
4. 10^5
5. e^5
6. $10^{-0.5}$
7. $e^{-3.8}$

3.5.1 Solving equations involving e^x and $\ln x$

The methods which we used to solve exponential equations in section 3.4 can also be used to solve equations involving e^x or $\ln x$. The following worked examples illustrate this.

Example 5: $e^x = 9$

We take ln's on both sides of the equation and use the fact that $\ln e = 1$ to solve this equation.

$$\begin{aligned} e^x &= 9 \\ \ln e^x &= \ln 9 \\ x \ln e &= \ln 9 \\ x &= \ln 9 = 2.197(3d.p.) \end{aligned}$$

Example 6: $e^x = 8e$

We take ln's on both sides and then use the rule $\ln(x * y) = \ln x + \ln y$.

$$\begin{aligned} e^x &= 8e \\ \ln e^x &= \ln 8e \\ x \ln e &= \ln 8 + \ln e \\ x &= \ln 8 + 1 = 3.079(3d.p.) \end{aligned}$$

Example 7: $e^{2x} + e^x - 6 = 0$

We let $z = e^x$ and substitute z into the equation to get a quadratic.

$$\begin{aligned} e^{2x} + e^x - 6 &= 0 \\ z^2 + z - 6 &= 0 \\ (z-2)(z+3) &= 0 \end{aligned}$$

Either $z = 2$ so $e^x = 2$ and $x = \ln 2 = 0.693$ (3 d.p.) Or $z = -3$ so $e^x = -3$. This is not possible so this does not give us a solution.

The equation has only one solution for x, namely $x = 0.693$.

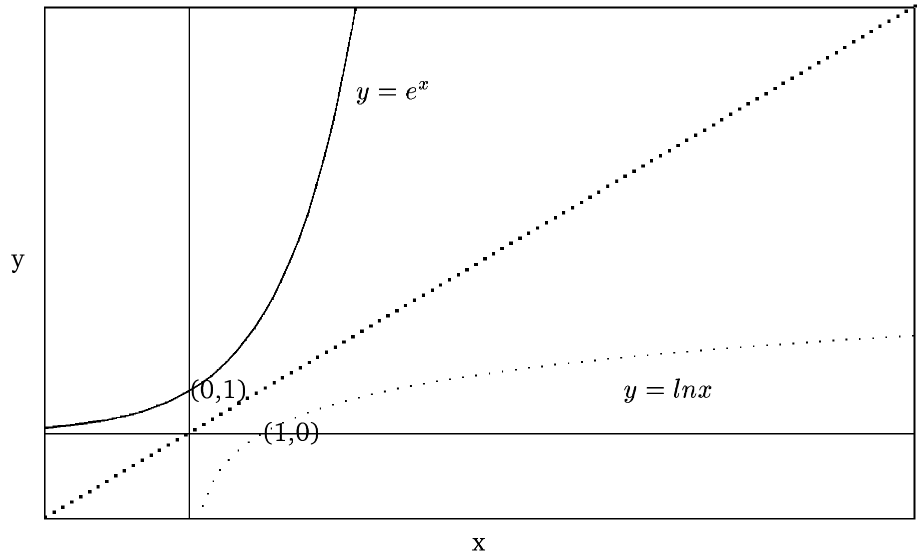
Learning activity

Solve the following exponential equations to find x correct to three decimal places.

1. $e^x = 4.75$
2. $e^{2x} = 15$
3. $5e^x = e$
4. $e^{2x} - 5e^x + 4 = 0$

3.5.2 Graphs of e^x and $\ln x$

The graph below shows the curves $y = e^x$ and $y = \ln x$



The graph of $y = e^x$ has the same shape as the other exponential graphs we have looked at. The graph of $y = \ln x$ is a reflection of $y = e^x$ about the line $y = x$. The graph of $y = \ln x$ is always increasing but at a much slower rate than the graph of $y = e^x$.

The graph of $y = e^x$ has no values of $y \leq 0$. The line $x = 0$ is a horizontal asymptote to the graph of $y = e^x$. Similarly the graph of $y = \ln x$ has no values of $x \leq 0$. The line $y = 0$ is a vertical asymptote to the graph of $y = \ln x$. (This is similar to the graphs of rational functions studied in Volume 1, Section 3.4.)

3.6 Differentiating natural exponential and logarithmic functions

3.6.1 Differentiating e^x

As mentioned above, the reason that e is often used as the base for exponential functions, is that the derivative of e^x is e^x.

$$\boxed{y = e^x,\ \frac{dy}{dx} = e^x}$$

We can differentiate more complicated functions involving e^x using the chain rule.

If $y = e^{f(x)}$ where $f(x)$ is some function in x then we let $u = f(x)$ so that $y = e^u$.

Now $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = f'(x)$.

By the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u f'(x) = e^{f(x)} f'(x)$

To differentiate functions involving $e^{f(x)}$ you can either use the chain rule, or you can directly use the rule which we have derived:

$$\boxed{y = e^{f(x)},\ \frac{dy}{dx} = e^{f(x)} f'(x)}$$

Example 8: $y = e^{(6x)}$

We have $y = e^u$ where $u = 6x$. Thus $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 6$.

By the chain rule $\frac{dy}{dx} = e^u * 6 = 6e^{6x}$

Example 9: $y = 8e^{(4x+1)}$

We have $f(x) = 4x + 1$ so $f'(x) = 4$.

Therefore $\frac{dy}{dx} = 8e^{(4x+1)} * 4 = 32e^{(4x+1)}$

Example 10: $y = e^{x^2} + 7e^x$

We can treat the two parts of this expression separately.

Let $a = e^{x^2}$ then $\frac{da}{dx} = e^{x^2} * 2x$

Let $b = 7e^x$ then $\frac{db}{dx} = 7e^x$

Therefore $\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} = 2xe^{x^2} + 7e^x$

Learning activity

Differentiate the following functions with respect to x.

1. $4e^{5x}$
2. xe^x (use the product rule)
3. $e^{(4x^2+2x+3)}$
4. $\frac{e^x}{5x}$ (use the quotient rule)

3.6.2 Differentiating $\ln x$

The derivative of $\ln x$ is $\frac{1}{x}$

$$\boxed{y = \ln x,\ \frac{dy}{dx} = \frac{1}{x}}$$

Again we can use the chain rule to differentiate more complicated functions involving $\ln x$.

If $y = \ln f(x)$, where $f(x)$ is some function in x then we let $u = f(x)$ so that $y = \ln u$.

Now $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = f'(x)$.

By the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} * f'(x) = \frac{f'(x)}{f(x)}$

You may either use the chain rule, or learn and use directly the rule derived above:

$$\boxed{y = \ln f(x),\ \frac{dy}{dx} = \frac{f'(x)}{f(x)}}$$

Example 11: $y = \ln 6x$

We let $u = 6x$, then $y = \ln u$ and $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 6$.

By the chain rule $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx} = \frac{1}{u} * 6 = \frac{6}{6x} = \frac{1}{x}$.

Example 12: $y = \ln 3x^2$

Here $f(x) = 3x^2$ so $f'(x) = 6x$.

Therefore $\frac{dy}{dx} = \frac{f'(x)}{f(x)} = \frac{6x}{3x^2} = \frac{2}{x}$.

Note that an alternative method would be to split up the function $f(x)$ using the rules of logarithms as follows:

$$\begin{aligned} y &= \ln 3x^2 \\ y &= \ln 3 + 2\ln x \\ \frac{dy}{dx} &= 0 + \frac{2}{x} = \frac{2}{x} \end{aligned}$$

[6]

[6] Note that $\ln 4$ is a constant and so its derivative is zero.

Example 13: $y = x^2 \ln(x^2 - 4)$

We use the product rule: $y = u * v$, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

Here $y = x^2 e^{2x}$ so $u = x^2$, $\frac{du}{dx} = 2x$ and $v = \ln(x^2 - 4)$, $\frac{dv}{dx} = \frac{2x}{x^2-4}$

By the product rule,

$\frac{dy}{dx} = x^2 * \frac{2x}{x^2-4} + \ln(x^2 - 4) * 2x = \frac{2x^3}{x^2-4} + 2x\ln(x^2 - 4)$.

Learning activity

Differentiate the following functions with respect to x:

1. $\ln x^2$
2. $3 \ln x$
3. $\ln 3x$
4. $\ln(4x^2 + 5)$
5. $x^3 \ln x^3$
6. $e^x \ln x$

3.7 Growth functions

One of the applications of exponential functions to economics is the study of *growth functions*. Many growth situations, for example the increase in sales after the launch of a new product, or the population of a country, may be modelled by exponential functions of the form $P = P_0 e^{kt}$ where P_0 and k are constants and t represents time.

Given suitable data, we can find the constants P_0 and k by forming and solving a system of equations. We can then make predictions about future growth.

The following worked examples demonstrate how to find and use exponential growth functions.

Worked example 1

Medical researchers studying the growth of a strain of bacteria observe that the number of bacteria, present after t hours, is given by the formula

$$N(t) = 40e^{1.5t}$$

a) State the number of bacteria present at the start of the experiment.

b) How many minutes will the bacteria take to double in number?

Solution

a) At the start of the experiment the time $t = 0$. $N(0) = 40e^{1.5*0} = 40e^0 = 40$. Therefore there are 40 bacteria present at the start of the experiment.

b) The bacteria will double in number when $e^{1.5t} = 2$. Taking ln's on both sides gives $\ln e^{1.5t} = \ln 2$ so $1.5t = \ln 2$ and therefore $t = \frac{\ln 2}{1.5} = 0.462$ hours. Multiply this answer by 60 to get the number of minutes: $0.462 \,.\, 60 = 27.73$ minutes.

Worked example 2

The value, \$V, of a particular car can be modelled by the equation $V = ke^{-pt}$ where t years is the age of the car. The car's original price was \$7499, and after one year it is valued at \$6000.

State the value of the constant k and calculate p giving your answer correct to two decimal places. Hence obtain the value of the car when it is three years old.

Solution The constant $k = 7499$ because when $t = 0, V = 7499 = ke^0 = k$.

When $t = 1, V = 6000$ so we have $6000 = 7499e^{-p}$ or $\frac{6000}{7499} = 0.800 = e^{-p}$. Taking $\ln$'s on both sides gives $-p = \ln 0.800 = 0.22$ (2 d.p.).

The equation is $V = 7499e^{-0.22t}$.

When $t = 3$, $V = 7499e^{-0.228(3)} = \$3875.87$ (2 d.p.)

Worked example 3

A company's profit has been growing consistently from \$5.2 million in 1995 to \$11.7 million in 2002. The profit p can be expressed as a natural exponentiation function of time t in years:

$$P = P_0 e^{rt}$$

for some constants P_0 and r and where the base year is 1995. (So $t = 0$ corresponds to the year 1995, $t = 1$ corresponds to 1996, etc.)

a) Determine P_0 and r.
b) Expressed as a percentage correct to two decimal places, what is the annual growth rate?
c) Estimate in millions of dollars, correct to two decimal places, the profit for 2003.

Solution

a) $P_0 = \$5.2$ million. To find r we use $t = 7$ which corresponds to the year 2002 when $P = 11.7$ million. $11.7 = 5.2e^{7r}$, $7r = \ln \frac{11.7}{5.2} = 0.81093$, $r = \frac{0.81093}{7} = 0.116$ (3 d.p.)
b) The annual growth rate is $e^{0.} = 1.122996$. This is 12.30%.
c) In 2003 $t = 8$ so the estimated profit in this year is $P = 5.2e^{0.116(8)} = \$13.15$ million dollars (2 d.p.)

Learning activity

1. A firm's profits have been growing exponentially over time from \$1.9 million in 1999 to \$4.3 million in 2003. Given that the profit, P, can be expressed as the following function of t, where t is the number of years since 1990:

$$P = P_0 e^{kt}$$

   (a) find P_0 and k

   (b) using these values for P_0 and k find the expected profits in 2005.

2. A country's population, P goes from 45 million in 1981 to 60.1 million in 1986. Express P as a natural exponential function of time and find the annual rate of growth.

3.8 Learning outcomes

After studying this chapter and the relevant readings you should be able to:

- Evaluate a^x where x is any real number.
- Simplify $\log_a x$ where x is a power of a.
- Use your calculator to evaluate e^x, $\ln x$, 10^x and $\log_{10} x$.
- Simplify algebraic expressions using the rules of indices and logarithms.
- *Take logs* to convert an exponential expression into a logarithmic expression.
- Solve exponential equations where the variable occurs as the exponent.
- Sketch graphs of exponential and logarithmic functions (in particular $y = e^x$ and $y = \ln x$) and understand the relationship between them.
- Differentiate expressions involving e^x and $\ln x$.
- Use the function $P = P_0 e^{kt}$ to solve problems involving growth rates.

3.9 Sample examination questions

Question 1

a) Simplify:

   i) $log_4 \frac{1}{64} - log_4 8$

   ii) $(\frac{1}{\sqrt{x}})^{-4}$

[3]

b) Differentiate with respect to x:

   i) $x^2 ln 2x$

   ii) $\frac{e^{x^2}}{x}$

[5]

c) Solve the equation $150 = 600e^{-5x}$

[2]

Question 2

Sales, S, of packets of a new washing powder are expected to grow according to the equation

$$S = 100000(1 - e^{-0.3t})$$

where t is given in weeks after its launch.

a) Find the number of packets of powder sold after:

  i) 2 weeks;

  ii) 5 weeks;

  iii) 10 weeks.

[2]

b) Find the time taken for the sales to reach 60000.

[2]

c) Comment on the general trend in sales.

[3]

d) Sketch the graph of S against t for $0 \le t \le 15$.

[3]

Chapter 4

Series and the mathematics of finance

Essential reading

See Chapter 11 of *Dowling* for many further examples of the material covered in this chapter. Answer supplementary problems 11.52 to 11.61 on interest compounding and discounting.

This chapter considers the mathematics of financial calculations. We will study the way that interest is calculated. We will introduce sigma notation and geometric series, and show how these may be used to calculate the future value of a savings plan.

4.1 Simple interest and compound interest

When money is invested, say in a bank or building society, interest is paid so that the amount you have invested keeps up with (or hopefully beats) inflation. Similarly, if you take out a loan then you will be expected to pay interest on the loan — usually at a higher rate than inflation.

Interest payments are usually a percentage of the amount invested or borrowed. There are two ways of calculating these payments - *simple interest* and *compound interest*.

Simple interest

With *simple interest*, the interest received on a sum invested is the same every year.

For example, if \$10,000 is invested for three years at a 10% rate of simple interest, then the interest paid would be \$1,000 each year. At the end of the three year period (assuming that the original investment and all interest payments have been left in the account) there would be a total of \$13,000 in the account.

Compound interest

Simple interest is in fact rarely used. *Compound interest*, which is calculated as a percentage of the total in the account i.e., including interest already received, is used instead.

If \$10,000 is invested for three years at a 10% rate of compound interest, then the interest paid in the first year would be \$1,000. Now there is \$11,000 in the account. So the interest paid in the second year would be 10% of \$11,000 which is \$1,100. In the third year there would be \$12,100 in the account so the interest paid at the end of this year would be \$1,210.

Hence at the end of the three years there would be a total of \$13,310 in the account. This is summarised in the following table.

Year	Amount in Account at Start of Year	Interest Earned	Amount in Account at End of Year
1	\$10,000	\$1,000	\$11,000
2	\$11,000	\$1,100	\$12,100
3	\$12,100	\$1,210	\$13,310

4.1.1 Calculating compound interest

If we are asked to calculate the total amount of compound interest paid after a number of years, then we could work out the interest paid each year and then add these individual payments up to get the total as in the example above.

This is not very efficient however. If the number of years in question is large, say 25 years, then this would be a very time-consuming method. We need to develop a quicker method for calculating compound interest.

Consider the example above again. We have \$10,000 invested for three years at an annual interest rate of 10%. We calculated:

$$final\ \ amount = (((10,000\ .\ 1.1)\ .\ 1.1)\ .\ 1.1)$$

Multiplying the initial investment of \$10,000 by 1.1 to add on 10% at the end of the first year, multiplying this amount by 1.1 and so on for three years.

The calculation above can be simplified to give:

$$final\ \ amount = 10,000\ .\ 1.1^3$$

In general, if a *principal investment* P is invested for t years at an interest rate of r% compounded annually, then the final amount is given by the formula:

$$\boxed{\text{final amount} = P(1+r)^t}$$

Worked example 1

\$1,500 is invested in a bank account which pays interest at a rate of 6% which is compounded annually. If no money is taken out of the bank account, how much will there be in the account at the end of: (i) 5 years, (ii) 8 years, (iii) 15 years?

Solution

Here the principal investment $P = 1500$, the interest rate $r = 6\%$ which means that we multiply the principal investment by 1.06 for each year.

Thus after 5 years, there will be $1500(1.06^5) = \$2007.34$ in the account.

After 8 years, there will be $1500(1.06^8) = \$2390.77$ in the account.

After 15 years, there will be $1500(1.06^{15}) = \$3594.84$ in the account.

Worked example 2

Peter invested a sum of money 10 years ago in an account which paid interest of 4% compounded annually. Peter now has \$555.09 in his account. How much money did Peter originally invest?

Tom invested \$650 into a similar account paying interest of 4% some years ago. The amount that Tom now has in his account is \$855.36. For how many years has Tom's money been invested in the account?

Solution

For Peter we have to solve the equation $555.09 = P \;.\; 1.04^{10}$ to find the principal P which Peter invested.

$$\begin{aligned} 555.09 &= 1.04^{10} \\ P &= \frac{555.09}{1.04^{10}} = 375 \end{aligned}$$

Therefore Peter originally invested \$375.

For Tom we have to solve the equation $855.36 = 650 \;.\; 1.04^t$ to find the number of years t for which Tom has had the account.

$$\begin{aligned} 650 \;.\; 1.04^t &= 855.36 \\ 1.04^t &= \frac{855.36}{650} = 1.3159 \\ t \ln 1.04 &= \ln 1.3159 \\ t &= \frac{\ln 1.3159}{\ln 1.04} = 7 \end{aligned}$$

Therefore Tom has had his money invested in the account for seven years.

Learning activity

1. I invest \$2000 in a bank account which pays interest at a rate of 3.5% compounded annually. How much money will be in the account after:
   (a) 2 years,
   (b) 7 years,
   (c) 20 years?

2. My friend also invests \$2000 for 20 years at an interest rate of 3.2% but her bank pays simple interest. How much more interest than my friend have I earned?
3. Six years ago Anna invested \$8000 in an account that pays interest which is compounded annually. There is now \$10720.77 in Anna's account. What was the rate of interest paid?

4.1.2 Multiple compounding

In all of the examples in the previous section, compound interest was calculated annually. In this section we will consider the case when the interest due is calculated more frequently — maybe every six months, or every month, or even every day.

Suppose \$10,000 is invested for one year at an interest rate of 10%.

If the interest is calculated at the end of the year, then the total amount in the account at the end of the year will be $10,000\ .\ 1.1 = \$11,000$.

If the interest is compounded every six months, then after one year there will be 2 interest calculations. Interest of $\frac{10}{2} = 5\%$ will be paid twice. We have:

$$final\ \ amount = 10,000\ .\ 1.05^2 = \$11,025$$

The amount of interest earned increases if the interest is compounded more frequently.

If the interest is compounded every month, then after one year there will be 12 interest calculations. Interest of $\frac{10}{12} = 0.83\%$ will be paid 12 times a year. Starting with the same principal of \$10,000 after one year we would have:

$$final\ \ amount = 10,000(1.0083^{12}) = \$11,047$$

In general, if interest of $r\%$ is awarded m times a year, then starting with a principal amount of P the final amount after t years is given by the following formula:

$$\boxed{\text{final amount} = P(1+\tfrac{r}{m})^{mt}}$$

Learning activity

If a principal of \$10,000 is invested at an interest rate of 10% show that the amount in the account after one year is \$11,051.56 if interest is compounded every day.

4.1.3 Continuous compounding

We have seen that the more frequently interest is compounded the higher the total interest. To push this idea to its limit, we will consider interest which is compounded continuously.

In the previous section we derived the formula:

$$final\ \ amount = P(1 + \frac{r}{m})^{mt}$$

where m is the number of times per year that interest is compounded.

If interest is compounded continuously then this formula becomes a *limit* as m tends to infinity.

$$final\ \ amount = P\ lim_{m\to\infty}(1 + \frac{r}{m})^{mt}$$

If we replace $\frac{m}{r}$ by n then we can change this formula to:

$$final\ \ amount = P\ \lim_{n\to\infty}(1 + \frac{1}{n})^{nrt}$$

Now recall that the definition of the irrational number e is:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

Therefore the formula for continual compounding is actually:[1]

$$\boxed{\text{final amount}= Pe^{rt}}$$

[1] You do not need to be able to derive this formula. It should look familiar as it is the same as the growth function formula considered in Chapter 3.

Suppose that our \$10,000 principal is in an account which pays interest of 10% and that the interest is compounded continually. After one year, we will have:

$$final\ \ amount = 10,000(e^{(0.1)(1)}) = \$11,051.71$$

Learning activity

1. \$400 is invested for 2 years in an account which pays interest of 5%. How much interest is earned if:
   (a) interest is compounded annually,
   (b) interest is compounded quarterly,
   (c) interest is compounded continuously?
2. \$750 was invested in an account 5 years ago. The amount in the account is now \$1012.40. Given that the interest was compounded continuously what was the interest rate applied?

4.1.4 Discounting

Sometimes we want to know how much money to invest in order to receive a particular sum in a certain number of years. Calculating the present value of a future sum is known as *discounting*.

Worked example 3

I want to invest some money for my son who is now 7 years old so that he has $10,000 when he is 18. The savings account I am investing in pays interest of 6% compounded annually. How much money do I need to put into the account now in order to achieve $10,000 in 11 years time?

There is another account which only pays 5.75% interest but this is compounded continuously. How much would I need to invest in this account to get the same return?

Solution

For the first account where interest is compounded annually we know that

$$final\ \ amount = P(1+r)^t$$

and final amount=$10,000, $r = 0.06$ and $t = 11$. Substituting these values into the equation and re-arranging we can find the value of the principal amount P needed.

$$\begin{aligned} 10,000 &= P(1.06)^{11} \\ P &= \frac{10,000}{1.06^{11}} \\ P &= 5,267.88 \end{aligned}$$

I would need to invest $5,267.88 into the account now in order to get a return of $10,000 in 11 years time.

For the second account where interest is compounded continuously we use the equation

$$final\ \ amount = Pe^{rt}$$

and we have final amount=$10,000, $= 0.0575$ and $t = 11$ as before. Substituting these values into the equation we can find the value of P.

$$\begin{aligned} 10,000 &= Pe^{(0.0575\ .\ 11)} \\ P &= \frac{10,000}{e^{(0.0575\ .\ 11)}} \\ P &= 5,312.62 \end{aligned}$$

To get a return of $10,000 from the second account I would need to invest $5,312.62.

Learning activity

Penny is saving up for a holiday. She wants to save some money so that in three years time she has $4,000. She is considering putting her money into one of three different accounts. The first account pays interest of 3.8% which is compounded annually. The second account pays interest of 3.8% which is compounded every six months. The third account pays interest of 3.75% which is compounded continuously.

Use discounting to find the principal sum that Penny would need to invest in each different account in order to achieve a return of \$4,000 in three years' time. Hence decide which of the accounts Penny should use.

4.2 Geometric series

A *geometric series* is a sum of terms where each term is some multiple of the previous term:

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n$$

In the geometric series above, the first term is a and all subsequent terms are r times the term before. The number r is called the *geometric ratio* of the geometric series.

Following are some examples of geometric series:

- $2 + 4 + 8 + 16 + 32$ $(a = 2, r = 2)$
- $3 + 6 + 12 + 24 + 48$ $(a = 3, r = 2)$
- $1 + 5 + 5^2 + 5^3 + \ldots + 5^8$ $(a = 1, r = 5)$
- $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ $(a = 2, r = \frac{1}{2})$
- $250 + 250(1.06) + 250(1.06)^2 + \ldots + 250(1.06)^{30}$ $(a = 250, r = 1.06)$

Learning activity

Decide which of the following are geometric series and for those which are, write down their geometric ratio and the next term.

1. $4 + 8 + 16 + 24 + \ldots$
2. $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \ldots$
3. $3 + 9 + 27 + 81 + 243 + 729 + \ldots$
4. $100 + 75 + 56.25 + 42.1875 + \ldots$
5. $0.1 + 0.01 + 0.001 + 0.0001 + \ldots$
6. $427 + 427(2) + 427(3) + 427(4) + 427(5) + 427(6) + \ldots$

It is easy to calculate the sum of a geometric series if just a few terms are involved. Simply add the terms together. However this becomes much harder if there are many terms in the series. We are going to derive a formula which can be used to calculate the sum of a geometric series. [2]

[2] You do not need to be able to derive this formula yourself - however it is often easier to remember a formula if you understand why it works.

Consider the geometric series with n terms and sum S_n:

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

Multiplying every term by r gives us:

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n$$

Now if we subtract S_n from rS_n all of the terms will cancel except for ar^n and a. Thus we are left with:

$$rS_n - S_n = ar^n - a$$

Now rearranging to make S_n the subject of the formula:

$$S_n(r-1) = a(r^n - 1)$$

$$\boxed{S_n = \frac{a(r^n-1)}{r-1}}$$

For example, consider the geometric series $100 + 50 + 25 + 12.5 + \ldots$ which has $a = 100$ and $r = 0.5$.

The sum of the first five terms
$S_5 = 100 + 50 + 25 + 12.5 + 6.25 = 193.75$. Using the formula derived above we have:

$$S_5 = 100\frac{(0.5^5 - 1}{0.5 - 1} = 100\frac{-0.96875}{-0.5} = 193.75$$

The sum of the first 10 terms is given by:

$$S_{10} = 100\frac{0.5^{10} - 1}{0.5 - 1} = 100\frac{-0.999023}{-0.5} = 199.98\ (2d.p.)$$

Learning activity

Use the formula above to calculate the sum of the following geometric series to the number of terms stated.

1. $2 + 6 + 18 + 54 + \ldots$ (8 terms)
2. $2 + 10 + 50 + 250 + \ldots$ (12 terms)
3. $1 + 3 + 9 + 27 + \ldots$ (20 terms)
4. $8 + 4 + 2 + 1 + \frac{1}{2} + \ldots$ (10 terms)
5. $8 - 4 + 2 - 1 + \frac{1}{2} + \ldots$ (10 terms)

4.2.1 Geometric series and finance

Geometric series can be used to analyse savings and loans. In the case of saving we consider the situation where a regular sum of money is invested at the same time each year. For example, suppose I invest \$1,000 at the start of each year into an account which pays interest of 6%. If I make these payments each year for four years, then how much money do I have in the account at the end of four years?

The first payment of \$1000 is invested for four years and so its future value is $1000(1.06)^4 = \$1262.48$.

The second payment of \$1000 is invested for three years and its future value is $1000(1.06)^3 = \$1191.02$.

The third payment is invested for two years and its future value is $1000(1.06)^2 = \$1123.60$.

The fourth and final payment is only invested for one year and its value at the end of this year is $1000(1.06) = \$1060$.

Therefore at the end of the four years I would have a total of:

$$1262.48 + 1191.02 + 1123.60 + 1060.00 = \$4637.10$$

The sum that we have calculated is actually:

$$1000(1.06) + 1000(1.06)^2 + 1000(1.06)^3 + 1000(1.06)^4$$

Writing it like this, we can see that it is in fact a geometric series with $a = 1000(1.06) = 1060$ and $r = 1.06$. We can use the formula to find the sum in the account after four years:

$$S_4 = \frac{1060(1.06^4 - 1)}{1.06 - 1} = 4637.09$$

[3]

[3] The slight discrepancy in the answers here is due to rounding errors.

We can also calculate the amount that would be in the account if I continued to invest \$1000 per year at the same rate of interest for 20 years:

$$S_{20} = \frac{1060(1.06^{20} - 1)}{1.06 - 1} = \$38,992.73$$

And if I managed to save for 50 years I would have over \$300,000.

$$S_{50} = \frac{1060(1.06^{50} - 1)}{1.06 - 1} = \$307,756.06$$

Learning activity

1. The sum of \$4200 is invested annually at 5% interest per annum. What is the total sum of money in the account at the end of 50 years?
2. A regular saving of \$500 is made at the start of every year for 10 years. Determine the value of the savings at the end of the tenth year on the assumption that the rate of interest is:
   (a) 11% compounded annually
   (b) 10% compounded continuously. (Write down the first terms of the geometric series to find the value of a and r).
3. A sum of \$1000 is invested annually at 7.5% interest per annum.
   (a) What is the total sum of money at the end of n years?
   (b) What is the total sum of money at the end of 20 years?
4. The formula $S_n = \frac{a(r^n-1)}{r-1}$ does not apply for $r = 1$. What is the correct formula for S_n in the case when $r = 1$?

4.3 Sigma notation

Sigma notation is a useful mathematical shorthand for describing the sum of a sequence of numbers or terms. *Sigma* is the name of the Greek letter Σ.

$$\sum_{i=a}^{b} u_i$$

means the sum of all u_i evaluated for all i between a and b.

For example,

$$\sum_{i=1}^{6} 5^i = 5^1 + 5^2 + 5^3 + 5^4 + 5^5 + 5^6$$

$$\sum_{i=0}^{48} 2i + 1 = 1 + 3 + 5 + \ldots + 95 + 97$$

$$\sum_{i=5}^{8} \frac{i-1}{i} = \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \frac{7}{8}$$

To write a given series in Σ form, we have to identify the *general term* u_i and the starting and end points for the *counter* i.

For example, consider the series:

$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 20$$

This series can be rewritten as:

$$(1 \,.\, 2) + (2 \,.\, 2) + (3 \,.\, 2) + \ldots + (10 \,.\, 2)$$

Therefore the *general term* is $(i \,.\, 2)$ or $2i$ where i runs from $i = 1$ to $i = 10$. So the series can be written in Σ notation as:

$$\sum_{i=1}^{10} 2i$$

Learning activity

1. Write out the terms of the following series:

   (a)

   $$\sum_{i=1}^{5} \frac{1}{i}$$

   (b)

   $$\sum_{i=3}^{7} i^2$$

(c)

$$\sum_{i-1}^{5} \frac{1}{i(i+1)}$$

(d)

$$\sum_{i=1}^{6} e^i$$

(e)

$$\sum_{i=4}^{8} (-1)^i (2i+3)$$

(f)

$$\sum_{i=0}^{5} (i+1)^3$$

2. Rewrite using Σ notation:
   (a) $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{9}$
   (b) $2^2 + 4^2 + 6^2 + \ldots + 100^2$
   (c) $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{99}$
   (d) $1^3 - 2^3 + 3^3 - 4^3 + \ldots + 19^3$
   (e) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots + \frac{99}{100}$
3. Use Σ notation to write:
   (a) The sum of the first 60 odd numbers.
   (b) The sum of all the square numbers from 100 to 400 inclusive.
   (c) The sum of all the numbers between 1 and 100 inclusive that leave a remainder 1 when divided by 7.

4.4 Learning outcomes

After studying this chapter and the relevant reading you should be able to:

- Explain the difference between simple interest and compound interest.
- Calculate the future value of a principal under annual compounding, multiple compounding and continuous compounding.
- Use discounting to calculate the present value of a future sum.
- Recognise a geometric series and find the geometric ratio of such a series.
- Sum a geometric series using the formula $S_n = \frac{a(r^n-1)}{r-1}$
- Solve problems involving savings and loans which can be formulated using geometric series.
- Write out the terms of a series given in Σ notation.
- Identify the general term and number of terms of a given series and hence write the series in Σ notation.

4.5 Sample examination questions

Question 1

a) Write down the formula for the sum of a geometric progression of n terms which has the first term equal to a and geometric ratio r. Using the formula and showing all your working evaluate:

i)

$$\sum_{k=-2}^{8} 5\ .\ 2^k$$

ii)

$$\sum_{k=1}^{5}(3^k + 2^{k-1})$$

[5]

b) I invest P dollars in a bank for n years and receive interest at $r\%$ per annum, compounded annually. Express mathematically the value of my investment after n years. Calculate the value of the investment when $P = 2,000$, $n = 5$ and $r = 6$.

[2]

c) My friend invested $1,000 for 10 years at a different interest rate, again compounded annually. At the end of the 10 years her investment was worth $1,628.89. What was the rate of interest?

[3]

Question 2

a) Determine the value of

i)

$$\sum_{r=1}^{5} r(r-1)$$

ii)

$$\sum_{r=-2}^{2} 2^r$$

[3]

b) Rewrite the following sums using the *sigma* notation:

i) $1 + 4 + 9 + 16 + \dots + 100$;

ii) $25 + 22 + 19 + 16 + \dots + 1 + (-2) + (-5)$.

[3]

c) Mary invests A dollars at the beginning of *every* year in a bank which offers fixed rate interest of $r\%$ compounded annually. Use the *sigma* notation to express the total amount of Mary's investment after n years.

[4]

Chapter 5

Integration

Essential reading

See Chapter 12 of *Dowling* for many further examples of the material covered in this chapter. Answer the supplementary problems 12.42 to 12.46 and 12.49 to 12.55.

This chapter covers the basic methods and applications of *integration*. Integration is the reverse process of *differentiation*. We have seen in Chapters 1 and 2 how to find the differential $\frac{dy}{dx}$ of a given function $f(x)$. In this chapter, we will see how to find the function $f(x)$ given the differential $\frac{dy}{dx}$. This is written in mathematical notation as

$$\int \frac{dy}{dx}.dx = y$$

The sign $\int$ is called the *integral sign*, and the $.dx$ shows that we are integrating the function $\frac{dy}{dx}$ with respect to the variable x.

Integration has many applications. In this chapter we will use integration to find the area under a curve and the area between two curves. We can also use integration to find the *total* cost or revenue function given a *marginal* cost or revenue function.

5.1 Indefinite integrals

Consider the function $y = x^3$. We can differentiate the function to obtain $\frac{dy}{dx} = 3x^2$. To differentiate we have *multiplied* x by the power 3 and then *decreased* the power by 1.

Integration is the reverse process or *inverse* of differentiation. Hence to *integrate* the expression $3x^2$ with respect to x, we *increase* the power by 1 and then *divide* x by this power. This will give us $\frac{3x^3}{3} = x^3$. We are back to our original function y which is what we would expect since we have first differentiated and then integrated the same expression:

$$y = x^3, \frac{dy}{dx} = 3x^2, \int \frac{dy}{dx}.dx = \int 3x^2.dx = x^3 = y$$

Now consider the function $y = x^3 + 7$. If we differentiate this function we again obtain $\frac{dy}{dx} = 3x^2$. This means that if we integrate $\frac{dy}{dx}$ we will again obtain $y = x^3$ which is not quite right because now the constant value 7 is missing. When we integrate, we have no way of knowing (unless we are given some extra information) whether there should be a constant value in the answer. Therefore we always include a *constant of integration c*.

Thus $\int 3x^2.dx = x^3 + c$

This process is called *indefinite integration*.

Following are some further examples of indefinite integration. You really can think of integration as being *backwards differentiation*. Check that these examples are correct by differentiating the answers to see if you end up with the right result.

- $\int 5.dx = 5x + c$
- $\int 3x^4.dx = \frac{3}{5}x^5 + c$
- $\int x^{\frac{1}{2}}.dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x^{\frac{2}{3}} + c$
- $\int ax^n.dx = \frac{a}{n+1}x^{n+1} + c$ (for all $n \neq -1$)
- $\int \frac{1}{x^3}.dx = \int x^{-3}.dx = \frac{x^{-2}}{-2} = \frac{-1}{2x^2} + c$
- $\int \frac{-4}{x^2}.dx = -4\int x^{-2}.dx = \frac{-4}{-1}x^{-1} = \frac{4}{x} + c$

To integrate $\frac{1}{x} = x^{-1}$ we cannot simply increase the power by 1 since then we would have power zero. Again we use *backwards differentiation*. Recall that if $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$. Hence

- $\int \frac{1}{x}.dx = \ln x + c$[1]
- $\int \frac{5}{x}.dx = 5\ln kx$

[1]The constant of integration can be included in the $\ln$ as $\ln kx$ since $\ln kx = \ln x + \ln k = \ln x + c$ where k and c are arbitrary unknown constant values.

Recall that the differential of the function e^x is e^x. This means that integrating e^x also gives the result e^x. The differential of e^{ax} is ae^{ax} where a is some constant value. When differentiating we *multiply* by the constant a, so when integrating we will have to *divide* by the constant a.

- $\int e^x.dx = e^x + c$
- $\int 9e^x.dx = 9e^x + c$
- $\int e^{4x}.dx = \frac{1}{4}e^{4x} + c$
- $\int 2e^{8x}.dx = \frac{2}{8}e^{8x} = \frac{1}{4}e^{8x} + c$

We can integrate a function with more than one term added (or subtracted) together by integrating each term separately and combining the results at the end.

- $\int(3x^2 + 7x - 2).dx = \int 3x^2 + \int 7x - \int 2 = x^3 + \frac{7}{2}x^2 - 2x + c$
- $\int \frac{1}{x} - e^x.dx = \ln x - e^x + c$
- $\int \frac{x^3+7x^2+5}{x^2}.dx = \int x + 7 + 5x^{-2}.dx = \frac{x^2}{2} + 7x + \frac{5}{-1}x^{-1} = \frac{x^2}{2} + 7x - \frac{5}{x} + c$
- $\int(2 - x^2)^2.dx = \int 4 - 4x^2 + x^4.dx = 4x - \frac{4}{3}x^3 + \frac{1}{5}x^5 + c$

Learning activity

Integrate the following expressions checking your answers by using differentiation. Remember to include the constant of integration.

1. $\int 8.dx$

2. $\int \frac{x^2}{3}.dx$
3. $\int \frac{7}{x}.dx$
4. $\int (6x^2 + 6x + 1).dx$
5. $\int (x+1)^2.dx$
6. $\int 6(e^x + 1).dx$
7. $\int \frac{1}{3}e^{3x}.dx$
8. $\int \frac{3-2x}{x}.dx$
9. $\int \sqrt{x}^3.dx$
10. $\int x^2(1-\sqrt{x}).dx$

5.2 Definite integrals

The *definite integral from a to b of $f(x)$* is

$$\int_a^b f(x).dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x) = \int f(x)$ and the values a and b are called the *limits of integration*.

Definite integration gives a result which is a value rather than a function.

As an example, we will calculate the definite integral $\int_1^3 2x^3.dx$.

Here the *limits of integration* are 3 and 1 and the function is $f(x) = 2x^3$.

Integrating $f(x) = 2x^3$ gives us $F(x) = \frac{2}{4}x^4 = \frac{1}{2}x^4$

Thus $\int_1^3 2x^3.dx = \left[\frac{1}{2}x^4\right]_1^3$

We evaluate this expression by substituting each of the limits of integration into the place of x in turn:

$F(3) = \frac{1}{2}(3)^4 = \frac{81}{2}$ and $F(1) = \frac{1}{2}(1)^4 = \frac{1}{2}$

Finally we subtract $F(1)$ from $F(3)$ to obtain the result:

$$\int_1^3 2x^3.dx = \left[\frac{1}{2}x^4\right]_1^3 = \frac{81}{2} - \frac{1}{2} = 40$$

Note that we will always use box notation $[F(x)]_a^b$ to denote the limits of a definite integral. *Dowling* uses the alternative notation $|_a^b$ but the box notation is more generally used and is less ambiguous.

Learning activity

When we are calculating definite integrals we no longer need to include the *constant of integration c*. Why is that?

Following are two more worked examples of definite integration.

Example 1: $\int_5^7 (3x^2 - 4x + 1).dx$

$$\begin{aligned}
\int_5^7 (3x^2 - 4x + 1).dx &= [x^3 - 2x^2 + x]_5^7 \\
&= [(7)^3 - 2(7)^2 + (7)] - [(5)^3 - 2(5)^2 + (5)] \\
&= 252 - 80 \\
&= 172
\end{aligned}$$

Example 2: $\int_1^e \frac{1}{x}$

$$\begin{aligned}
\int_1^e \frac{1}{x} &= [\ln x]_1^e \\
&= [\ln e] - [\ln 1] \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

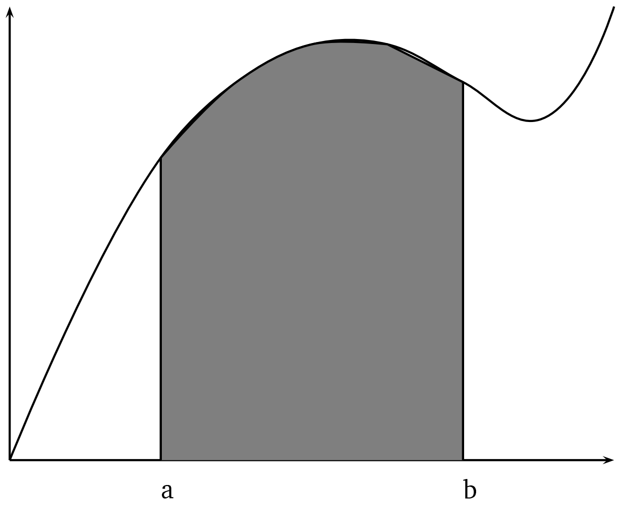
Learning activity

Evaluate the following definite integrals:

1. $\int_0^4 (x^2 + 5x).dx$
2. $\int_e^{e^2} \frac{1}{x}.dx$
3. $\int_1^2 \frac{x^4-1}{x^3}.dx$

5.3 Finding the area under a curve

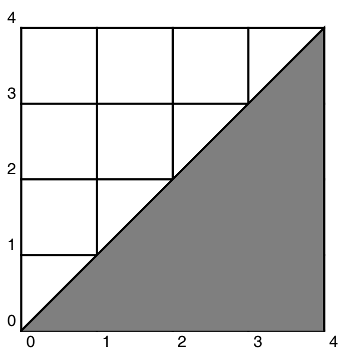
The definite integral $\int_a^b f(x).dx$ tells us the area under the curve $f(x)$ bounded by the lines $x = a$, $x = b$ and the x-axis.



To illustrate this, we will use integration to work out the area under the line $f(x) = x$ between $a = 0$ and $b = 4$. As you can see from the graph below, this area is a triangle and so we can work out directly that the area is equal to 8 (Area of a triangle is given by the formula $\frac{1}{2}$*base*height$= \frac{1}{2} * 4 * 4 = 8$).

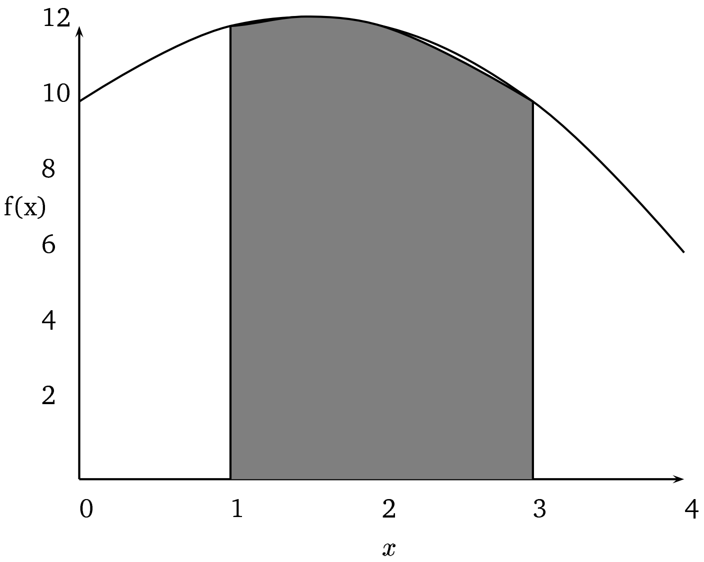
Using integration, we have:

$$\begin{aligned} Area &= \int_0^4 x.dx \\ &= \left[\frac{x^2}{2}\right]_0^4 \\ &= \left[\frac{4^2}{2}\right] - [0] \\ &= 8 \end{aligned}$$



Of course, we would not usually use integration to find out the area of a triangle. Integration is used when the curve $f(x)$ is more complicated.

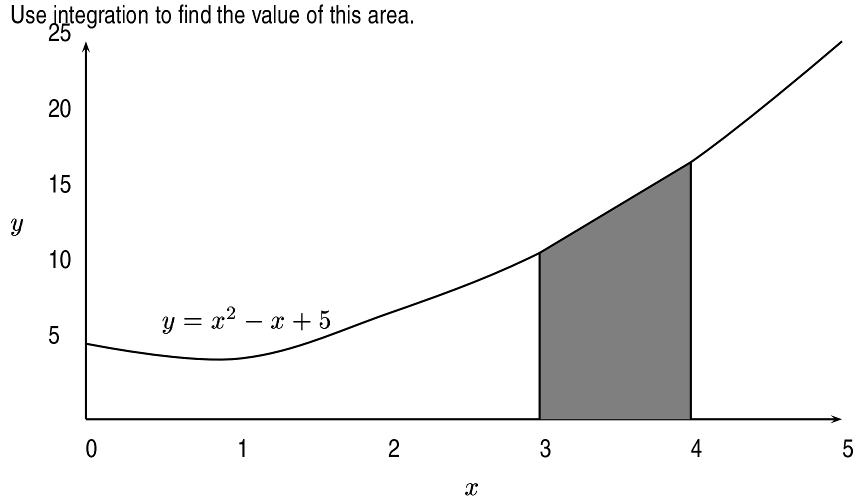
For example, we will find the area under the curve $f(x) = 10 + 3x - x^2$ between the lines $x = 1$ and $x = 3$. This area is illustrated on the graph below.



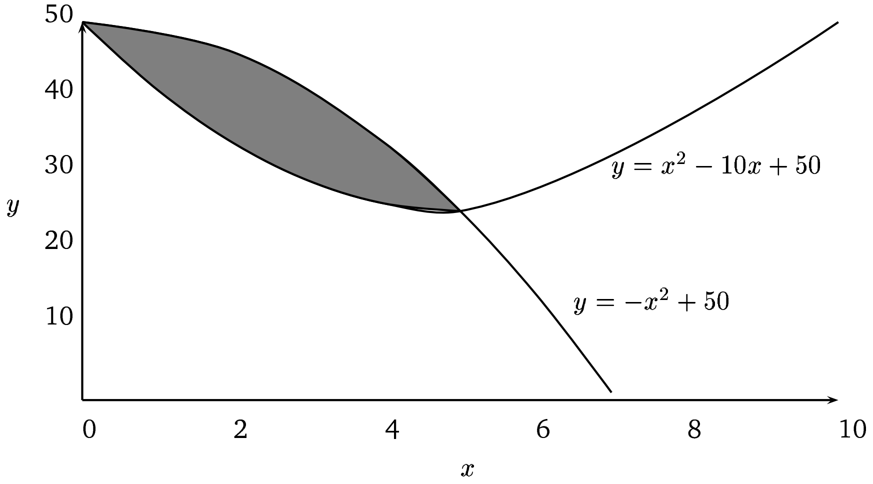
$$
\begin{aligned}
Area &= \int_1^3 (10 + 3x - x^2).dx \\
&= \left[10x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_1^3 \\
&= \left[10(3) + \frac{3(3)^2}{2} - \frac{(3)^3}{3}\right] - \left[10(1) + \frac{3(1)^2}{2} - \frac{(1)^3}{3}\right] \\
&= 34\frac{1}{2} - 11\frac{1}{6} \\
&= 23\frac{1}{3}
\end{aligned}
$$

Learning activity

1. Draw a graph to illustrate the area given by the following integrals. Use integration to find the value of each area.
   (a) $\int_1^5 (x^2 - 3x + 8).dx$
   (b) $\int_0^4 e^x.dx$
2. Write down the integral which represents the area illustrated on the graph below. Use integration to find the value of this area.



We can also use integration to find the area between two curves. Consider the graph below.



The shaded area is enclosed by the curves $y = -x^2 + 50$ and $y = x^2 - 10x + 50$. The two curves intersect when $x = 0$ and when $x = 5$.

To find the shaded area, we find the total area under the first curve between 0 and 5 and then subtract the area under the second curve between 0 and 5.

$$
\begin{aligned}
Area &= \int_0^5 (-x^2+50).dx - \int_0^5 (x^2-10x+50).dx \\
&= \left[\frac{-x^3}{3} + 50x\right]_0^5 - \left[\frac{x^3}{3} - 5x^2 + 50x\right]_0^5 \\
&= \left[\frac{-(5)^3}{3} + 50(5) - 0\right] - \left[\frac{(5)^3}{3} - 5(5)^2 + 50(5) - 0\right] \\
&= \left[-\frac{125}{3} + 250\right] - \left[\frac{125}{3} - 125 + 250\right] \\
&= 41\frac{2}{3}
\end{aligned}
$$

We can make this calculation easier by subtracting the second function from the first before integrating. In so doing we are using the following rule:

$$\boxed{\int_a^b f(x).dx \pm \int_a^b g(x).dx = \int_a^b f(x) \pm g(x).dx}$$

In our example we have:

$$
\begin{aligned}
\int_0^5 (-x^2+50).dx - \int_0^5 (x^2-10x+50).dx &= \int_0^5 ((-x^2+50)-(x^2-10x+50)).dx \\
&= \int_0^5 (-2x^2+10x).dx \\
&= \left[\frac{-2x^3}{3} + 5x^2\right]_0^5 \\
&= \frac{-2(5)^3}{3} + 5(5)^2 - 0 \\
&= 41\frac{2}{3}
\end{aligned}
$$

The following worked examination question provides another example.

Worked examination question

a) Find the following indefinite integrals:

  i) $\int (6x^2 - \frac{2}{\sqrt{x}}).dx$

  ii) $\int 2e^{-0.2t}.dt$

b) Sketch on the same axes for $-3 \leq x \leq 4$, the graphs of the functions $f(x) = x^2 - 4$ and $g(x) = x + 2$. Find the area enclosed by these two functions.

Solution

a) i)

$$\begin{aligned}
\int (6x^2 - \frac{2}{\sqrt{x}}).dx &= \int 6x^2 - 2x^{-\frac{1}{2}}.dx \\
&= \frac{6x^3}{3} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
&= 2x^3 - 4\sqrt{x} + c
\end{aligned}$$

ii)

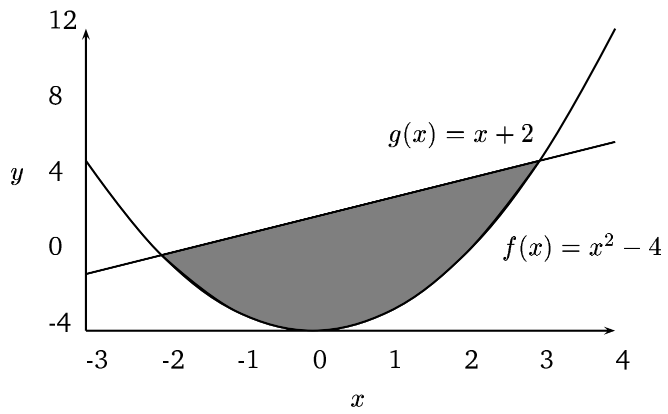
$$\begin{aligned}
\int 2x^{-0.2t}.dt &= \frac{2e^{-0.2t}}{-0.2} \\
&= -10e^{-0.2t} + c
\end{aligned}$$

b)

x	-3	-2	-1	0	1	2	3	4
$f(x)$	5	0	-3	-4	-3	0	5	12
$g(x)$	-1	0	1	2	3	4	5	6

From the table above we can see that $f(x) = g(x)$ when $x = -2$ and when $x = 3$. Therefore the two lines cross at $(-2, 0)$ and $(3, 5)$. We could also find the points where the two lines cross by equating $f(x)$ and $g(x)$ and solving the resulting equation:

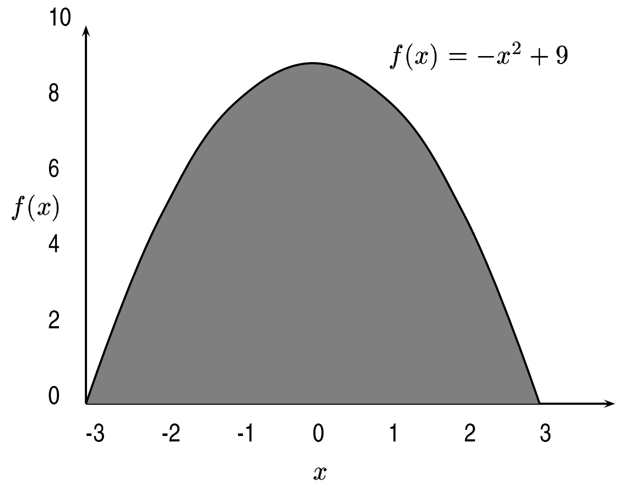
$$\begin{aligned}
x^2 + 4 &= x + 2 \\
x^2 - x + 6 &= 0 \\
(x - 3)(x + 2) &= 0 \\
x = 3 \ \ &or \ \ x = -2
\end{aligned}$$

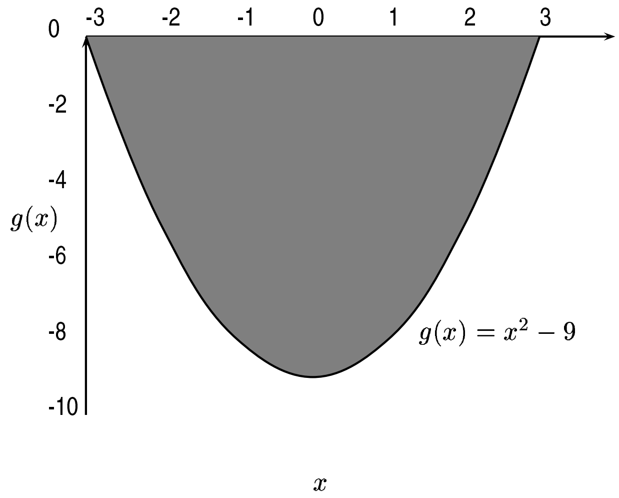


$$\begin{aligned}
Area &= \int_{-2}^{3} (x+2).dx - \int_{-2}^{3} (x^2-4).dx \\
&= \int_{-2}^{3} (x+2-(x^2-4)).dx \\
&= \int_{-2}^{3} -x^2+x+6.dx \\
&= \left[\frac{-x^3}{3}+\frac{x^2}{2}+6x\right]_{-2}^{3} \\
&= \left[\frac{-(3)^3}{3}+\frac{(3)^2}{2}+6(3)\right] - \left[\frac{-(-2)^3}{3}+\frac{(-2)^2}{2}+6(-2)\right] \\
&= 13\frac{1}{2}-(-7\frac{1}{3}) \\
&= 20\frac{5}{6}
\end{aligned}$$

Learning activity

1. The graphs below show the curves $f(x) = -x^2 + 9$ and $g(x) = x^2 - 9$. Both of these curves cut the x-axis at $x = -3$ and $x = 3$. You can see that the size of the area enclosed by the curve and the x-axis is the same for both curves.



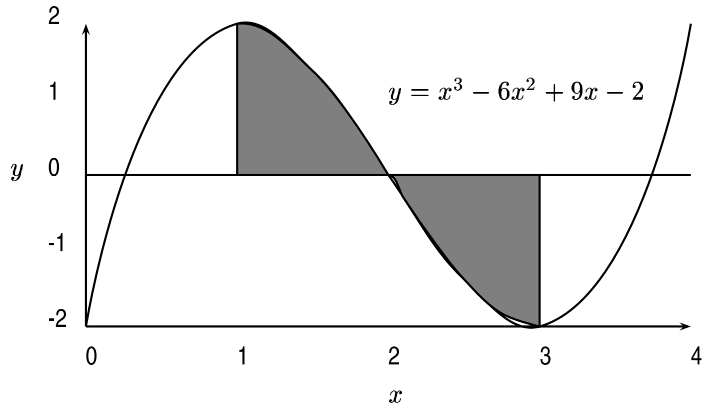


   (a) By integrating $\int_{-3}^{3} f(x).dx$ show that the area enclosed by $f(x)$ and the x -axis is equal to 36 $units^2$.

   (b) Now evaluate $\int_{-3}^{3} g(x).dx$. Since the enclosed area on the second graph is the same size as the enclosed area on the first graph, we would expect the answer to be 36. You should find however that the answer is -36.

   We have a negative answer because the area enclosed by $g(x)$ is *below* the x-axis. However an area cannot have a negative value. Thus although the integral $\int_{-3}^{3} g(x).dx = -36$ the area that this integral represents has a positive value 36 $unit^2$.

2. The graph below shows the curve $y = x^3 - 6x^2 + 9x - 2$ which cuts the x-axis at $(2, 0)$. Use integration to find the shaded area. (Hint: Calculate separately the areas below and above the x-axis).



3. On the same axes sketch the graphs of $f(x) = x^2 + 1$ and $g(x) = 4x + 3$ for values of x between 0 and 2. Draw the lines $x = 1$ and $x = 2$ on your graph. Shade the area enclosed by the four lines and use integration to find the size of this shaded area.

5.4 Applications in economics

5.4.1 Finding the total cost and revenue function

In section 2.2 we looked at applications of differentiation to economics. Starting with a basic economic function we can differentiate it to find the corresponding marginal function. Integration allows us to reverse this process and obtain the original function from a marginal function.

For example, a total revenue function can be found by integrating a marginal revenue function. A total cost function can be found by integrating a marginal cost function. This is illustrated in the following worked examples.

Worked example 1

A firm's marginal cost function is $MC = q^2 + 2q + 5$. Find the total cost function if the fixed costs are 80.

Solution

We know that $MC = \frac{d(TC)}{dq}$ i.e. the marginal cost function is the differential of the total cost function.

Therefore $TC = \int MC.dq$

In this example,

$$TC = \int (q^2 + 2q + 5).dq$$

$$= \frac{q^3}{3} + q^2 + 5q + c$$

We can find the value of c because we have the extra information that the fixed costs are 80. This means that when $q = 0, TC = 80$. Therefore $c = 80$.

Hence the total cost function is $TC = \frac{q^3}{3} + q^2 + 5q + 80$.

Worked example 2

A firm's marginal revenue function is $MR = 12 - 4q$. Find the total revenue function.

Solution

We know that $MR = \frac{d(TR)}{dq}$. i.e. the marginal revenue function is the differential of the total revenue function.

Therefore $TR = \int MR.dq$

$$\begin{aligned} TR &= \int (12 - 4q).dq \\ &= 12q - 2q^2 + c \end{aligned}$$

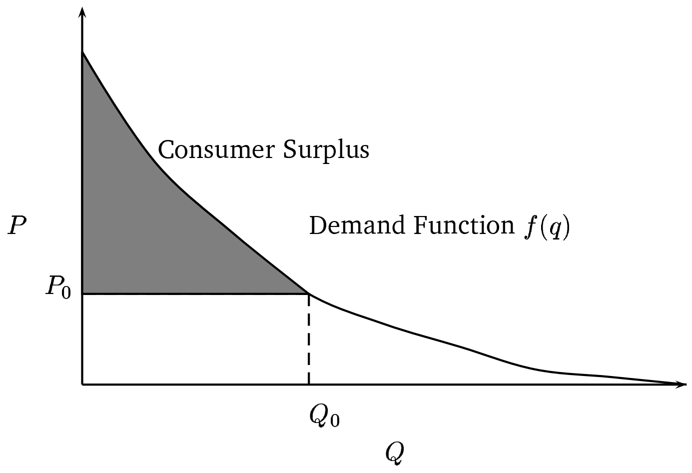
When the firm produces no goods the revenue will be equal to zero. Hence when $q = 0$ we have $TR = 0$. Substituting this information into the function above we find that $c = 0$.

Hence the total revenue function is $TR = 12q - 2q^2$.

5.4.2 Calculating consumer and producer surplus

We saw in the previous volume (section 4.4) how a demand function $f(q)$ may be used to model demand for a product plotting quantity produced against price. Typically as the quantity of products becoming available increases the price decreases. We have *equilibrium* at the point (Q_0, P_0) when supply equals demand and P_0 is called the *equilibrium price*.

If all of the products produced are sold at the equilibrium price, then consumers who would have paid more than this price have benefitted. The area shaded on the graph below is called the *consumer's surplus*. This area represents the amount that consumers have saved because they have not had to pay more than the equilibrium price.



We can use the integration techniques of this chapter to calculate the size of the consumer surplus.

The shaded area is equal to the area under the graph $f(q)$ between 0 and Q_0 minus the area of the rectangle with corners $(0,0),(Q_0,0),(Q_0,P_0)$ and $(0,P_0)$. This area is given by:

$$\boxed{\text{Consumer Surplus} = (\textstyle\int_0^{Q_0} f(q).dq) - (P_0 * Q_0)}$$

Worked example 3

Given that $Q_0 = 5$ brings market equilibrium to the demand function $P = q^2 - 25q + 400$, find the consumer surplus.

Solution

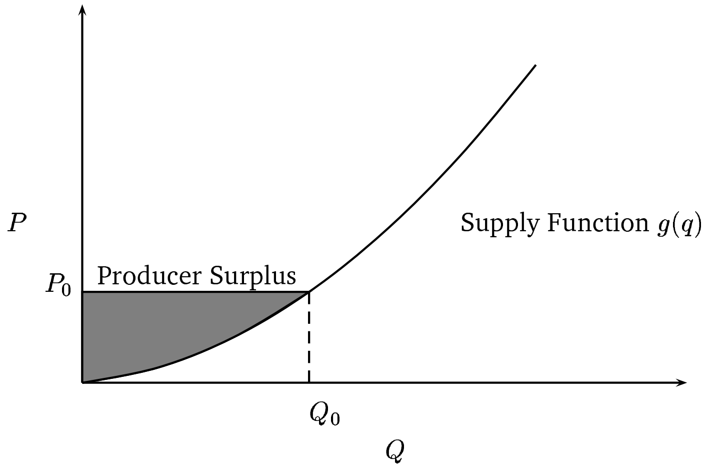
We substitute $Q_0 = 5$ into the demand function to find the equilibrium price P_0 as follows:

$$P_0 = 5^2 - 25(5) + 400 = 300$$

Now we can apply the formula given above to calculate the consumer surplus.

$$\begin{aligned}
consumersurplus &= (\int_0^{Q_0} f(q).dq) - (P_0 * Q_0) \\
&= \int_0^5 (q^2 - 25q + 400).dq - (300 * 5) \\
&= \left[\frac{q^3}{3} - \frac{25q^2}{2} + 400q\right]_0^5 - (1500) \\
&= \left[\frac{(5)^3}{3} - \frac{25(5)^2}{2} + 400(5)\right] - [0] - (1500) \\
&= 1729.167 - 1500 \\
&= 229.167(3d.p.)
\end{aligned}$$

We can also use integration to find the *producer surplus*. This area, shown on the graph below, represents the amount saved by producers, who would have supplied products for less than the equilibrium price. The function $g(q)$ is the *supply function* which models the prices at which producers will supply different quantities of a good.



The shaded area can be found by subtracting the area under the curve between $q = 0$ and $q = Q_0$ from the area of the rectangle with corners $(0, 0)$, $(Q_0, 0)$, (Q_0, P_0) and $(0, P_0)$. This area is given by:

$$\boxed{\text{Producer Surplus} = (P_0 * Q_0) - (\int_0^{Q_0} g(q).dq)}$$

Worked example 4

Given the supply function $P = q^2 + 20q + 100$ and assuming that at market equilibrium $Q_0 = 5$ and $P_0 = 225$, calculate the total producer surplus.

Solution

We substitute the given values $P_0 = 225, Q_0 = 5, g(q) = q^2 + 20q + 100$ into the formula above as follows:

$$\begin{aligned}
ProducerSurplus &= (P_0 * Q_0) - (\int_0^{Q_0} g(q).dq) \\
&= (225 * 5) - \int_0^5 (q^2 + 20q + 100).dq \\
&= 1125 - \left[\frac{q^3}{3} + 10q^2 + 100q\right]_0^5 \\
&= 1125 - \left[\frac{(5)^3}{3} + 10(5)^2 + 100(5)\right] - [0] \\
&= 1125 - 791.333 \\
&= 333.333(3d.p.)
\end{aligned}$$

5.5 Learning outcomes

After studying this chapter and the relevant readings you should be able to:

- Recognize and use correctly the notation for definite and indefinite integration.
- Write down the integrals of simple power, logarithmic and exponential functions.
- Evaluate definite integrals in simple cases.
- Evaluate the area underneath a curve, or in between two curves.
- Use integration to find the total cost or revenue function given the marginal cost or revenue function.
- Calculate the consumer and producer surplus by finding the area underneath the demand or supply function as appropriate.

5.6 Sample examination questions

Question 1

a) Evaluate:
   i) $\int_0^2 (x^2 + x - 2)dx$
   ii) $\int_1^e (\frac{1}{x})dx$

[3]

b) A firm's marginal revenue function is

$$MR = 28 - 6x - 3x^2$$

Find the total revenue function TR.

[3]

c) Find the area between the curves

$$f(x) = x^2 + 1, \quad g(x) = 4x + 3$$

from $x = 1$ to $x = 2$.

[4]

Question 2

a) Find the following indefinite integrals:

i) $\int (\frac{3}{x^2}) dx$

ii) $\int (5 + 3\surd(x + 1)) dx$

[3]

b) On the same graph, sketch the two functions:

$$\begin{aligned} y_1 &= x + 2 \\ y_2 &= x^2 - 6x + 15 \end{aligned}$$

for $0 \leq x \leq 5$.

[2]

On the graph shade the region between the two functions from $x = 1$ to $x = 4$. Using the methods of integration, evaluate the area of this region.

[5]

Appendix A

Specimen examination papers

A.1 2004

UNIVERSITY OF LONDON

Goldsmiths College

Diploma Examination 2004

CIS001 Mathematics for Business

Duration: 3 hours

There are TEN questions on this examination paper.

Full marks will be awarded for complete answers to all TEN questions.

Electronic calculators may be used. The make and model should be specified on the script and the calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

Graph paper should be provided for this examination.

1. (a) Use your calculator to evaluate the following expressions correct to 2 decimal places where $x = 2.481$ and $y = 0.0842$.
   i. $x^{\frac{1}{4}} - y^{-\frac{1}{3}}$
   ii. $4.842 - \frac{xy}{2}$

   [6]

   (b) Simplify the expressions:
   i. $(a+b)^2 - (a-b)^2$
   ii. $\frac{1}{(a+1)} - \frac{1}{(a-1)}$

   [4]

2. (a) A straight line passes through the points $(1, -1)$ and $(-1, 4)$. Find the equation of the line in the form $ax + by + c = 0$. Where does the line cut the axes?

   [5]

   (b) Another line has gradient $m = 2$ and passes through the point $(1, 4)$. Find the equation of this line. Find the point where this line intersects the line found in (a).

   [5]

3. A company buys peanuts at 70 cents per kilogram and buys cashews at 90 cents per kilogram. The company wishes to sell a blend of these nuts at 120 cents per kilogram. Of this price, 20% is profit, 70% is the cost of the nuts and the remaining 10% is packaging and distribution costs.

   (a) What proportion of peanuts should be in the blend?

   [5]

   (b) Market research shows that, for the same price of 120 cents per kilogram, the company can sell a blend of peanuts and cashews in equal proportions without affecting sales. If the packaging and distribution costs are still 10% of this price, what percentage of the price is now profit?

   [5]

4. A factory's cost is $C(n) = 400n + 1000$ where n is the number of units produced. Its revenue is $R(n) = 5n^2 + 200n - 1500$.

   (a) Determine the profit P in terms of n. Determine the values of n for which the profit P is positive.

   [5]

   (b) For financial reasons, the costs C for the year 2005 need to be restricted to between 100,000 and 120,000.
   i. How many units will the factory be able to produce? Express your answer as $a \le n \le b$ for suitable integers a and b.
   ii. What can you say about the profits for 2005?

   [5]

5. A parabola is given by $f(x) = -3x^2 + 8x + 3$. Determine its vertex and where it cuts the axes.

   [5]

   Sketch the graph of the parabola for $-1 \le x \le 4$

   [3]

   For what value of x is the gradient of the parabola equal to 2? Indicate on your sketch the gradient at this value of x.

   [2]

6. A company makes two products A and B. Each item of product A requires 6 hours to manufacture, 4 hours to package and makes a profit of \$30. Each item of product B requires 2 hours to manufacture, 3 hours to package and makes a profit of \$20. In one week a company has 80 hours available for manufacturing and 80 hours available for packaging.

   The company wishes to find the number of items of each product that it should produce in one week in order to maximize its profit.

   (a) Formulate the problem as a linear programming problem. [3]

   (b) Draw a sketch graph to show a feasible region and determine the co-ordinates of the corners of the region. [5]

   (c) Evaluate the objective function at the corners of the feasible region and hence solve the linear programming problem. [2]

7. Consider the matrices

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} B = \begin{pmatrix} -2 & 1 \\ 3 & 1 \\ 0 & -2 \end{pmatrix} C = \begin{pmatrix} 1 & 5 & 0 \\ -4 & 1 & -1 \end{pmatrix}$$

   (a) Determine which of the following expressions are valid (conformable):

      i. $A + B$
      ii. AC
      iii. $A + 3BC$
      iv. $CB - 2A$

   [2]

   (b) Compute the valid expressions in part (a). [4]

   (c) A company has four stores S_1, S_2, S_3 and S_4 and three products P_1, P_2 and P_3. The number of items of each product in each store is given by the following table.

	P_1	P_2	P_3
S_1	100	120	150
S_2	200	150	100
S_3	300	160	250
S_4	200	100	120

   The value of the products P_1, P_2 and P_3 are \$20, \$15 and \$30 respectively. Express these facts in terms of a matrix and a vector and illustrate matrix multiplication to compute the value of the stock in each of the stores. [4]

8. Consider the two functions

$$f(x) = -x^2 + 2x + 2 \quad g(x) = -3x + 1$$

   (a) Express $f(g(x))$ as a polynomial in x.

[2]

   (b) Differentiate the following functions. (You do not have to expand or simplify your answers.)

      i. $f(x)$
      ii. $g(x)$
      iii. $(f(x) + g(x))^7$
      iv. $\frac{(f(x))^2}{g(x)}$
      v. $e^{f(x)}$
      vi. $ln(g(x))$

[8]

9. (a) Rewrite the following expressions using the sigma notation:

      i. $1 \,.\, 2 + 2 \,.\, 3 + 3 \,.\, 4 + \ldots + 19 \,.\, 20$
      ii. $54 + 49 + 44 + 39 + \ldots + 14$

[2]

   (b) Richard invests B dollars at the beginning of the year in a bank which offers a fixed interest of $r\%$ per annum compounded annually. Write an expression for the value of his investment after m years. If $r = 5$, after how many years will an initial investment of \$1000 first exceed \$1600?

[4]

   (c) Judy invests A dollars at the beginning of every year in a bank which offers a fixed rate of interest of $r\%$ per annum compounded annually. Express T, the total amount of Judy's investment after n years, in terms of A, r and n using the sigma notations for summation.

[2]

   (d) Showing all your working and without using a calculator, determine:

      i. $log_4 8$
      ii. $log_5(\frac{1}{25})$

[2]

10. (a) Evaluate

      i. $\int_1^2 (-x^5 + 3x^2 - \frac{1}{x^2}).dx$
      ii. $\int_{-1}^1 e^{3x}.dx$
      iii. $\int_1^2 (\frac{1}{x}).dx$

[6]

   (b) A company's marginal cost function is

$$MC = 3Q^2 + 5Q + 7$$

   Find the total cost function if the fixed costs are 80.

[4]

A.2 2005

UNIVERSITY OF LONDON

Goldsmiths College

Diploma Examination 2005

CIS001 Mathematics for Business

Duration: 3 hours

There are TEN questions on this examination paper.

Full marks will be awarded for complete answers to all TEN questions.

Electronic calculators may be used. The make and model should be specified on the script and the calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

Graph paper should be provided for this examination.

1. (a) Find the equation of the line in the form $ax + by + c = 0$ which passes through the points $(1, 2)$ and $(3, -1)$. A second line has slope 2 and passes through the point $(2, 1)$. Find the equation of the line in the form $ax + by + c = 0$. Determine where the second line intercepts the axes and the co-ordinates of the point where the two lines meet.

[7]

(b) The value of a house has linear appreciation. It was valued at \$120,000 in 2000 and \$140,000 four years later. Write down an expression for the value of the house in terms of the time in years. In what year will the value of the house exceed \$158,000?

[3]

2. (a) A quadratic revenue function has a graph which has a maximum value of 507 at $Q = 16$ and which is zero at $Q = 3$. Determine the second value of Q for which the revenue function is zero. Hence find the equation for the revenue in terms of Q.

[5]

(b) A profit function has the equation $P = -5Q^2 + 30Q - 15$. Find the break-even points correct to 2 decimal places. Determine the maximum profit. Find the values of Q where the profit is $P = 10$.

[5]

3. (a) The matrices A, B and C are given by:

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} B = \begin{pmatrix} 3 & -2 \\ 0 & 1 \\ 4 & -2 \end{pmatrix} C = \begin{pmatrix} 4 & -1 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

Indicating your reasons, which of the following expressions can be evaluated (i.e. which are conformable with respect to dimensions)?

i. $B^t - AC$
ii. BCA
iii. CBC
iv. $B + B^t$

Evaluate those expressions which are conformable.

[6]

(b) A company has four retail stores and sells three items A, B and C. The inventory, representing the stock of items in the stores, is given by the following table:

Store	A	B	C
1	900	500	300
2	700	800	500
3	600	400	300
4	800	900	700

Express the inventory as a matrix M. Item A sells for \$120, item B sells for \$100 and item C sells for \$150. Write down the selling price of the items as an appropriate vector V and evaluate the product MV. What does the product MV represent?

[4]

4. A company produces two products A and B. In one week, it can sell at most 70 items of A and at most 40 items of B. The cost to produce each item of A is \$10 and to produce each item of B is \$20. There is a total budget which limits costs to at most \$1400 a week. The profit for each item of A is \$4 and for each item of B is \$5.

   The company wishes to know how many items of A and B need to be produced each week in order to maximize profits. Formulate this as a linear programming problem.

   [4]

   Draw a sketch graph to show the feasible region and determine the co-ordinates of each of the corners of the feasible region. Solve this linear programming problem.

   [6]

5. Differentiate the following functions with respect to x. (You do not need to simplify your answer.)

   (a) $(3x^3 - 2x + 2)^{\frac{1}{2}}$

   [2]

   (b) $(4x - 1)(2 + x^{\frac{1}{2}})$

   [2]

   (c) $\frac{(2x-1)}{(3+x)}$

   [2]

   (d) $ln(3x^2 - 5)$

   [2]

   (e) $\frac{e^{-2x}}{(1+x)}$

   [2]

6. (a) Simplify the following expressions:

   i. $\frac{(a+4)^2-(a-1)^2}{2a+3}$

   ii. $\frac{1}{(3x^2)^{-\frac{3}{2}}}$

   iii. $log_b(b\sqrt{b})$

   [3]

   (b) Solve the following equations, correct to 2 decimal places:

   i. $e^{x^2} = 3$

   ii. $2^{x-1}3^x = 25$

   [4]

   (c) Using substitution methods, or otherwise, find values of s and t which satisfy

$$s^2 - t = 3$$
$$3s + 2t = 8$$

   [3]

7. Show that $x = 1$ is a solution to the equation

$$2x^3 - x - 1 = 0$$

[1]

Show that there are no other solutions to this equation.

[2]

Find the stationary points of the curve

$$y = 2x^3 - x - 1$$

and determine their nature. Sketch the curve for $-1 \leq x \leq 2$ showing stationary points and intercepts on both axes.

[7]

8. (a) Kofi invests \$1,000 in a bank every year for 5 years receiving a fixed interest of 5% per annum. What is the value of his investment after the end of five years?

[3]

(b) Sasha invests one sum of \$1,000 at a different bank at a different fixed interest rate per annum. After 5 years the investment of \$1,000 is worth \$1,246.18. What is the rate of interest?

[3]

(c) Julie invests one sum of \$5,000 at yet another bank, recieving a fixed interest rate of 4% per annum. After how many complete years will her investment be worth more than \$10,000?

[4]

9. (a) Express the following sums using the sigma ($\sum$) notation:

i. $\frac{2}{1^2} + \frac{3}{2^2} + \frac{4}{3^2} + \ldots + \frac{18}{17^2}$

ii. $1^3 + 4^3 + 7^3 + \ldots + 28^3$

[3]

(b) Write down the formula for the sum of a geometric progression of n terms which has the first term equal to a and geometric ratio r.

[1]

Using the formula, evaluate the following expressions, correct to two decimal places:

i. $\sum_{k=-1}^{7} (1.4)^k$

ii. $\sum_{t=0}^{9} 2^{t-1}$

iii. $\sum_{s=2}^{7} \frac{2}{3^s}$

[6]

10. (a) Evaluate the following expressions, correct to two decimal places:

i. $\int_0^1 (2 + \sqrt{(x+2)}).dx$

ii. $\int_1^2 e^{2x}.dx$

iii. $\int_2^4 \frac{3}{(x-1)}.dx$

[6]

(b) Marginal costs are given by $M = 5Q^{-1} + 16 + 20Q$ and total costs are 224 when $Q = 4$. Find the expressions for the total costs in terms of Q.

[4]

Appendix B

Solutions to the specimen examination papers

B.1 2004 solutions

1. (a) i. $2.481^{\frac{1}{4}} - 0.0842^{-\frac{1}{3}} = 1.255 - 2.282 = -1.027$
   $= -1.03(2\text{ d.p.})$
   ii. $4.842 - \frac{(2.481)(0.0842)}{2} = 4.74\ (2\text{ d.p.})$

   (b) i. $(a+b)^2 - (a-b)^2 = a^2 + 2ab + b^2 - (a^2 - 2ab + b^2) = 4ab$
   ii. $\frac{1}{(a+1)} - \frac{1}{(a-1)} = \frac{a-1-(a+1)}{(a+1)(a-1)} = \frac{-2}{a^2-1} = \frac{2}{1-a^2}$

2. (a) $m = \frac{4-(-1)}{-1-1} = \frac{5}{-2}$
   $y = mx + c$ Substituting in the given values:
   $4 = \frac{5}{-2}(-1) + c$, $4 = \frac{5}{2} + c$ so $c = \frac{3}{2}$.
   $y = -\frac{5}{2}x + \frac{3}{2}$ or $2y = -5x + 3$. Hence $5x + 2y - 3 = 0$.
   The line cuts the y-axis at c i.e. at $(0, \frac{3}{2})$.
   It cuts the x-axis when $y = 0$ so $5x + 2(0) - 3 = 0$ or $5x = 3$ so $x = \frac{3}{5}$. This is the point $(\frac{3}{5}, 0)$.

   (b) Substituting the given values into $y = mx + c$ we have $4 = (2)(1) + c$ so $c = 2$. The equation of the line is $y = 2x + 2$.
   The two lines intersect when $2x + 2 = \frac{-5}{2}x + \frac{3}{2}$.
   Re-arranging this gives $\frac{9}{2}x = -\frac{1}{2}$ so $x = \frac{-1}{9}$ and $y = 2(\frac{-1}{9}) + 2 = \frac{16}{9}$. The two lines intersect at $(\frac{-1}{9}, \frac{16}{9})$.
   [Check this answer in line 1: $5(\frac{-1}{9}) + 2(\frac{16}{9}) - 3 = 0$]

3. (a) 70% of $120 = 0.7(120) = 84$. The nuts in the blend will cost 84c.
   Let x be the % of peanuts in the blend, then $(100 - x)$ is the % of cashews.

$$\begin{aligned} 84 &= 70(\frac{x}{100}) + 90\frac{(100-x)}{100} \\ 84 &= 0.7x + 90 - 0.9x \\ 0.2x &= 6 \\ x &= 30 \end{aligned}$$

   The blend should be 30% peanuts and 70% cashews.

   (b) Cost of blend $= 70(0.5) + 90(0.5) = 80$ cents. Packaging and distribution costs = 10% of $120 = 12$ cents. So profit $= 120 - 80 - 12 = 28$ cents in every 120 cents. This is $\frac{100}{120}(28) = 23\frac{1}{3}\%$ profit.

4. (a) Profit=Revenue-Cost so
   $P(n) = R(n) - C(n) = 5n^2 + 200n - 1500 - (400n + 1000)$
   Therefore $P(n) = 5n^2 - 200n - 2500$.
   We factorise $P(n)$ to find the values of n for which P is positive:

$P(n) = 5(n^2 - 40n - 500) = 5(n+10)(n-50)$. A sketch of $P(n)$ would be a $\cup$ shape (because $P(n)$ is a quadratic with positive square term) which cuts the axis at $n = -10$ and $n = 50$. Therefore the profit is positive when $n > 50$. (You may find it helpful to draw a quick sketch of the function.)

(b) i. $100,000 \leq 400n + 1000 \leq 120,000$. Taking each side of the inequalitiy seperately we have

$$\begin{aligned} 100,000 &\leq 400n + 1000 \\ \frac{99,000}{400} &\leq n \\ 247.5 &\leq n \end{aligned}$$

$$\begin{aligned} 400n + 1000 &\leq 120,000 \\ n &\leq \frac{119,000}{400} \\ n &\leq 297.5 \end{aligned}$$

Putting these results together taking care with the rounding we have: $248 \leq n \leq 297$.

ii. Profit for 2005 will be between
$P(248) = 5(248)^2 - 200(248) - 2500 = 255,420$ and
$P(297) = 5(297)^2 - 200(297) - 2500 = 379,145$.

5. To find the turning point we differentiate and set $f'(x) = 0$

$$\begin{aligned} f(x) &= -3x^2 + 8x + 3 \\ f'(x) &= -6x + 8 \\ -6x + 8 &= 0 \\ x &= \frac{8}{6} = \frac{4}{3} \\ f(\frac{4}{3}) &= -3(\frac{4}{3})^2 + 8(\frac{4}{3}) + 3 = \frac{25}{3} \end{aligned}$$

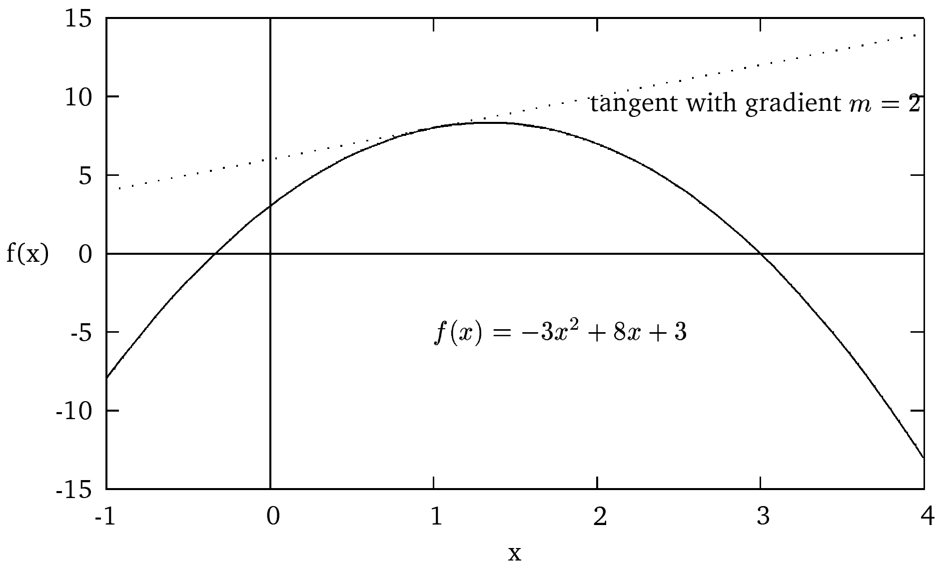
Since $f(x)$ is a quadratic it only has one turning point which is at the vertex. Hence the vertex of the curve is at $(\frac{4}{3}, \frac{25}{3})$.

The curve cuts the y-axis when $x = 0$. $f(0) = 3$ so the curve cuts the y-axis at $(0, 3)$.

Use the quadratic formula with $a = -3, b = 8$ and $c = 3$ to find the roots and hence the poitns where the line cuts the x-axis.

$$x = \frac{-8 \pm \sqrt{(8^2 - 4(-3)(3))}}{2(-3)} = \frac{-8 \pm 10}{-6} = \frac{-1}{3} \;\; or \;\; 3$$

The curve cuts the x-axis at the points $(-\frac{1}{3}, 0)$ and $(3, 0)$.



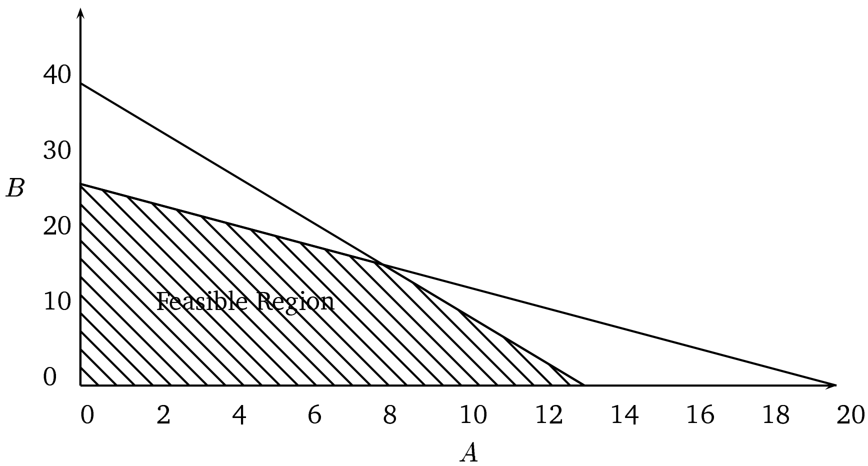
The gradient of the parabola is equal to 2 when $f'(x) = -6x + 8 = 2$. Re-arranging gives $x = 1$. A tangent with gradient 2 can be drawn on the graph at the point $(1, 8)$.

6. (a) Objective equation : $\Pi = 30A + 20B$ (maximise)

Production constraints : $\begin{cases} 6A + 2B \leq 80 & (make) \\ 4A + 3B \leq 80 & (pack) \end{cases}$

Non-negativity constraints : $\begin{cases} A \geq 0 \\ B \geq 0 \end{cases}$

(b)



(c) Corners of the feasible region are at $(0, 0)$, $(13\frac{1}{3}, 0)$, $(0, 26\frac{2}{3})$ and $(8, 16)$. The maximum profit will occur at one corner of the feasible region by the extreme point theorem so we evaluate the profit function at these points:

$$\begin{array}{ll} \Pi & (0,0) = 0 \\ \Pi & (13,0) = 30(13) + 20(0) = 390 \\ \Pi & (0,26) = 30(0) + 20(26) = 520 \\ \Pi & (8,16) = 30(8) + 20(16) = 560 \end{array}$$

To maximise profit the company should manufacture 8 units of product A and 16 units of product B.

7. (a) i. $A + B$ is $(2 * 2) + (3 * 2)$ which is non-conformable.

ii. AC is $(2 * 2)(2 * 3) = (2 * 3)$ which is conformable.

iii. $A + 3BC$ is $(2*2) + (3*2)(2*3) = (2*2) + (3*3)$ which is non-conformable.

iv. $CB - 2A$ is
$(2*3)(3*2) - (2*2) = (2*2) + (2*2) = (2*2)$ which is conformable.

(b)

$$AC = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 5 & 0 \\ -4 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 1 \\ -14 & -7 & -3 \end{pmatrix}$$

$$\begin{aligned} CB - 2A &= \begin{pmatrix} 1 & 5 & 0 \\ -4 & 1 & -1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3 & 1 \\ 0 & -2 \end{pmatrix} - 2\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 13 & 6 \\ 11 & -1 \end{pmatrix} - \begin{pmatrix} 2 & -2 \\ -4 & 6 \end{pmatrix} = \begin{pmatrix} 11 & 8 \\ 15 & -7 \end{pmatrix} \end{aligned}$$

(c)

$$M = \begin{pmatrix} 100 & 120 & 150 \\ 200 & 150 & 100 \\ 300 & 160 & 250 \\ 200 & 100 & 120 \end{pmatrix} \quad V = \begin{pmatrix} 20 \\ 15 \\ 30 \end{pmatrix}$$

Where matrix M represents the number of each product ($\leftrightarrow$) held at each store ($\updownarrow$), and V represents the individual value of the three different products.
Multiplying the matrix and the vector together gives:

$$MV = \begin{pmatrix} 8300 \\ 9250 \\ 15900 \\ 9100 \end{pmatrix} \begin{matrix} total\ value\ of\ products\ at\ Store\ 1 \\ total\ value\ of\ products\ at\ Store\ 2 \\ total\ value\ of\ products\ at\ Store\ 3 \\ total\ value\ of\ products\ at\ Store\ 4 \end{matrix}$$

8. (a)

$$\begin{aligned} f(g(x)) &= -(-3x+1)^2 + 2(-3x+1) + 2 \\ &= -(9x^2 - 6x + 1) - 6x + 2 + 2 \\ &= -9x^2 + 3 \end{aligned}$$

(b) i. $y = f(x) = -x^2 + 2x + 2$, $\frac{dy}{dx} = -2x + 2$

ii. $y = g(x) = -3x + 1$, $\frac{dy}{dx} = -3$

iii. $y = (f(x) + g(x))^7 = (-x^2 + 2x + 2 - 3x + 1)^7 = (-x^2 - x + 3)^7$
$\frac{dy}{dx} = 7(-x^2 - x + 3)^6(-2x - 1)$ (chain rule)

iv. $y = \frac{(f(x))^2}{g(x)} = \frac{(-x^2+2x+2)^2}{-3x+1}$
$\frac{dy}{dx} = \frac{-(-x^2+2x+2)^2(-3)+(-3x+1)(2)(-x^2+2x+2)(-2x+2)}{(-3x+1)^2}$
(chain rule and quotient rule)

v. $y = e^{f(x)} = e^{-x^2+2x+2}$, $\frac{dy}{dx} = (-2x+2)e^{-x^2+2x+2}$

vi. $y = \ln(g(x)) = \ln(-3x+1)$, $\frac{dy}{dx} = \frac{-3}{-3x+1} = \frac{3}{3x-1}$

9. (a) i. $\sum_{n=1}^{19} n(n+1)$
   ii. $\sum_{n=0}^{8} 54 - 5n$ or $\sum_{n=11}^{3} 5n - 1$
   (b) $value = Br^m$ (where $r\%$ is expressed as $(1 + \frac{r}{100})$ when using this expression)
   Putting $r = 5\%$ and $B = 1000$ and $1600 < value$ to find m:

$$\begin{aligned} 1600 &< 1000(1.05)^m \\ 1.6 &< 1.05^m \\ \ln 1.6 &< m \ln 1.05 \\ \frac{\ln 1.6}{\ln 1.05} &< m \\ 9.6 &< m \end{aligned}$$

   During the ninth year the investment will become worth more than \$1600.
   (c)

$$\begin{aligned} Total\ \ value &= Ar^n + Ar^{n-1} + Ar^{n-2} + \ldots + Ar^2 + Ar \\ &= \sum_{k=1}^{n} Ar^k \end{aligned}$$

   (d) i. $x = log_4 8$ means $4^x = 8$; re-write this as $(2^2)^x = 2^3$ so $2x = 3$ thus $x = \frac{3}{2}$.
   ii. $x = log_5(\frac{1}{25})$ means $5^x = \frac{1}{25}$; $5^{-2} = \frac{1}{25}$ so $x = -2$.

10. (a) i.

$$\int_1^2 (-x^5 + 3x^2 - \frac{1}{x^2}).dx = \left[\frac{-x^6}{6} + x^3 + x^{-1}\right]_1^2$$

$$= \left[\frac{-(2)^6}{6} + (2)^3 + \frac{1}{(2)}\right] - \left[\frac{-(1)^6}{6} + (1)^3 + \frac{1}{(1)}\right]$$

$$= \left[-2\frac{1}{6}\right] - \left[1\frac{5}{6}\right] = -4$$

   ii. $\int_{-1}^{1} e^{3x}.dx = \left[\frac{e^{3x}}{3}\right]_{-1}^{1} = \frac{e^3}{3} - \frac{e^{-3}}{3} = 6.68$ (2 d.p.)
   iii. $\int_1^2 (\frac{1}{x}).dx = [\ln x]_1^2 = \ln 2 - \ln 1 = \ln 2 = 0.69$ (2 d.p.)
   (b)

$$TC = \int MC.dQ = \int (3Q^2 + 5Q + 7).dQ = Q^3 + \frac{5Q^2}{2} + 7Q + C$$

   When $Q = 0$, $TC = 80$ so $C = 80$
   Therefore the total cost function is
   $TC = Q^3 + \frac{5Q^2}{2} + 7Q + 80$

B.2 2005 solutions

1. (a) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-2}{3-1} = \frac{-3}{2}$
   $y = mx + c$ substituting in values for y, m, x gives
   $2 = (\frac{-3}{2})(1) + c$ so $c = \frac{7}{2}$.
   Equation line 1: $y = \frac{-3}{2}x + \frac{7}{2}$ or $3x + 2y - 7 = 0$.
   Line 2 has gradient $m = 2$ and passes through $(2, 1)$
   substituting gives $1 = 2(2) + c$ so $c = -3$.
   Equation line 2: $y = 2x - 3$ or $2x - y - 3 = 0$
   Line 2 intercepts the y-axis at c i.e. at $(0, -3)$.
   Putting $y = 0$ into the equation of line 2 gives:
   $2x - 3 = 0, \ \ x = \frac{3}{2}$, so the line intercepts the x-axis at $(\frac{3}{2}, 0)$.
   Solve simultaneous equations to find the point of intersection of the two lines:

$$\begin{aligned} & 3x + 2y - 7 = 0 && \text{(B.1)} \\ (multiply\ \ by\ \ 2) \quad & 2x - y - 3 = 0 && \text{(B.2)} \\ (add\ \ B.1\ \ and\, B.3) \quad & 4x - 2y - 6 = 0 && \text{(B.3)} \end{aligned}$$

$$\begin{aligned} & 7x - 13 = 0 \\ & x = \frac{13}{7} \\ (substitute\ \ into\ \ B.1) \quad & 3(\frac{13}{7}) + 2y - 7 = 0 \\ & y = \frac{5}{7} \\ (check\ \ solution\ \ in\ \ B.2) \quad & 2(\frac{13}{7} - \frac{5}{7} - 3 = 0 \end{aligned}$$

   The two lines intersect at the point $(\frac{13}{7}, \frac{5}{7})$.

   (b) In four years the house value has increased by \$20,000; this is \$5,000 per year so we have $m = 5000$, $c = 120,000$ since this is the initial value of the house.
   Substituting these values for m and c into the straight line equation $y = mx + c$ using $y = V$ and $x = (t - 2000)$ to represent the value and year respectively gives the value of the house $V = 5000(t - 2000) + 120,000$ where t is the year. This can be re-arranged as $V = 5000t - 9880000$.

$$\begin{aligned} 158000 &< 5000t - 9880000 \\ 2007.6 &< t \end{aligned}$$

   The value of the house first exceeds \$158,000 in the year 2007.

2. (a) Since a quadratic function is symmetrical, the second value of Q where the revenue function is zero is the same distance away from 16 as 3 i.e. $13 + 16 = 29$. (You might find it helpful to draw a quick sketch of the function.)
   Hence $R(Q) = C(Q - 3)(Q - 29)$ is the factorisation of the function. We can find the constant C by substituting in the given values $Q = 16$, $R(16) = 507$:

$$R(Q) = C(Q - 3)(Q - 29)$$

$$
\begin{aligned}
507 &= C(16-3)(16-29) \\
507 &= C(13)(-13) \\
\frac{507}{-169} &= C \\
C &= -3
\end{aligned}
$$

The quadratic revenue function is $R(Q) = 3(Q-3)(Q-29)$.

(b) $P = -5Q^2 + 30Q - 15$. We find the break-even points by factorising using the quadratic formula with $a = -5, b = 30$ and $c = -15$. (Note that the quadratic does not factorise using other methods - this is hinted by the question asking for two decimal places.)

$$
\begin{aligned}
Q &= \frac{-30 \pm \sqrt{30^2 - 4(-5)(-15)}}{2(-5)} \\
&= \frac{-30 \pm \sqrt{600}}{-10} \\
&= 5.45 \;\; or \;\; 0.55 \; (2 \; d.p.)
\end{aligned}
$$

Since the function is symmetrical, the maximum profit will occur at the midpoint of the two break-even points. This is at $Q = \frac{5.45+0.55}{2} = 3$.
The maximum profit is $P(3) = -5(3)^2 + 30(3) - 15 = 30$.
Substitute $P = 10$ and factorise to find the values of Q when profit is 10.

$$
\begin{aligned}
10 &= -5Q^2 + 30Q - 15 \\
-2 &= Q^2 - 6Q + 3 \\
0 &= Q^2 - 6Q + 5 \\
0 &= (Q-5)(Q-1)
\end{aligned}
$$

The profit is 10 when $Q = 5$ and when $Q = 1$.

3. (a) i. $B^t - AC$ is
$(3*2)^t - (2*2)(2*3) = (2*3) - (2*3) = (2*3)$ which can be evaluated:

$$
\begin{aligned}
B^t - AC &= \begin{pmatrix} 3 & 0 & 4 \\ -2 & 1 & -2 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 4 & -1 & 0 \\ -2 & 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 0 & 4 \\ -2 & 1 & -2 \end{pmatrix} - \begin{pmatrix} 6 & -2 & -2 \\ -14 & 5 & 6 \end{pmatrix} \\
&= \begin{pmatrix} -3 & 2 & 6 \\ 12 & -4 & -8 \end{pmatrix}
\end{aligned}
$$

ii. BCA is $(3*2)(2*3)(2*2) = (3*3)(2*2)$ which is non-conformable.

iii. CBC is $(2*3)(3*2)(2*3) = (2*2)(2*3) = (2*3)$ which can be evaluated:

$$
\begin{aligned}
CB &= \begin{pmatrix} 4 & -1 & 0 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 0 & 1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 12 & 9 \\ 2 & 1 \end{pmatrix} \\
CBC &= \begin{pmatrix} 12 & 9 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 & 0 \\ -2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 66 & -21 & -18 \\ 6 & -1 & 2 \end{pmatrix}
\end{aligned}
$$

iv. $B + B^t$ is $(3 * 2) + (2 * 3)$ which is non-conformable.

(b)

$$M = \begin{pmatrix} 900 & 500 & 300 \\ 700 & 800 & 500 \\ 600 & 400 & 300 \\ 800 & 900 & 700 \end{pmatrix} \quad V = \begin{pmatrix} 120 \\ 100 \\ 150 \end{pmatrix}$$

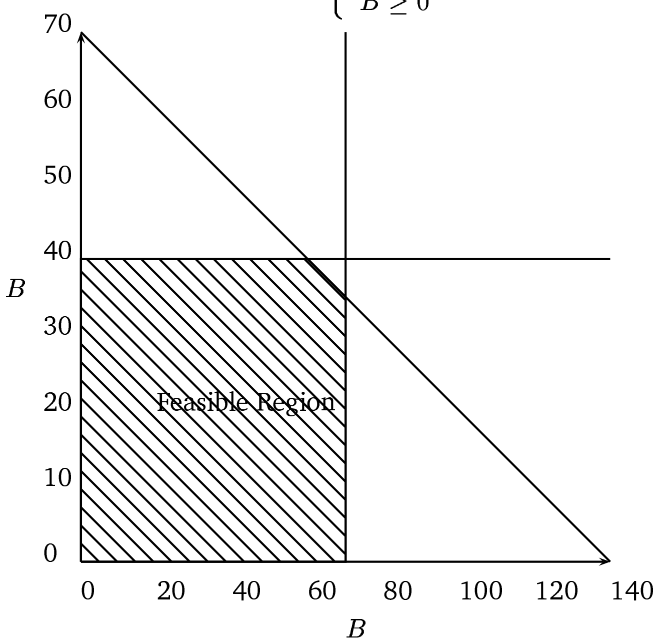
$$MV = \begin{pmatrix} 203,000 \\ 239,000 \\ 157,000 \\ 291,000 \end{pmatrix}$$

The values in the vector MV represent the total amount the items at each store are worth. The items in stock at Store A retail for a total of \$203,000; the items at Store B retail for a total of \$239,000 and so on.

4. Objective function: $\Pi = 4A + 5B$ (maximise)

Production constraints: $\begin{cases} 10A + 20B \leq 1400 \\ A \leq 70 \\ B \leq 40 \end{cases}$

Non-negativity constraints : $\begin{cases} A \geq 0 \\ B \geq 0 \end{cases}$



The corners of the feasible region are at $(0, 40)$, $(0, 0)$, $(70, 0)$, $(60, 40)$ and $(70, 35)$. Evaluating the objective function at these points will show us the maximum profit:

$$\begin{aligned} \Pi(0, 40) &= 4(0) + 5(40) = 200 \\ \Pi(0, 0) &= 4(0) + 5(0) = 0 \\ \Pi(70, 0) &= 4(70) + 5(0) = 280 \\ \Pi(60, 40) &= 4(60) + 5(40) = 440 \\ \Pi(70, 35) &= 4(70) + 5(35) = 455 \end{aligned}$$

The company should produce 70 items of product A and 35 items of product B each week in order to maximise their profit. This maximum profit is \$455.

5. (a)

$$\begin{aligned} y &= (3x^3 - 2x + 2)^{\frac{1}{2}} \ \ (chain \ \ rule) \\ \frac{dy}{dx} &= \frac{1}{2}(3x^3 - 2x + 2)^{-\frac{1}{2}}(9x^2 - 2) \end{aligned}$$

(b)

$$\begin{aligned} y &= (4x-1)(2+x^{\frac{1}{2}}) \ \ (product \ \ rule) \\ \frac{dy}{dx} &= (4x-1)(\frac{x^{-\frac{1}{2}}}{2}) + (4)(2+x^{\frac{1}{2}}) \end{aligned}$$

(c)

$$\begin{aligned} y &= \frac{(2x-1)}{(3+x)} \ \ (quotient \ \ rule) \\ \frac{dy}{dx} &= \frac{2(3+x) - 1(2x-1)}{(3+x)^2} \end{aligned}$$

(d)

$$\begin{aligned} y &= ln(3x^2 - 5) \ \ (ln \ \ rule) \\ \frac{dy}{dx} &= \frac{6x}{3x^2 - 5} \end{aligned}$$

(e)

$$\begin{aligned} y &= \frac{e^{-2x}}{(1+x)} \ \ (quotient \ \ rule) \\ \frac{dy}{dx} &= \frac{-2e^{-2x}(1+x) - 1(e^{-2x})}{(1+x)^2} = \frac{e^{-2x}(-3-2x)}{(1+x)^2} \end{aligned}$$

6. (a) i.

$$\begin{aligned} & \frac{(a+4)^2 - (a-1)^2}{2a+3} \\ &= \frac{a^2 + 8a + 16 - (a^2 - 2a + 1)}{2a+3} \\ &= \frac{10a + 15}{2a+3} = \frac{5(2a+15)}{2a+15} = 5 \ \ (if \ \ a \neq \frac{-3}{2}) \end{aligned}$$

ii.

$$\frac{1}{(3x^2)^{-\frac{3}{2}}} = (3x^2)^{\frac{3}{2}} = 3^{\frac{3}{2}}x^3 = 3\sqrt{3}x^3$$

iii.

$$log_b(b\sqrt{b}) = \log_b(b^{\frac{3}{2}}) = \frac{3}{2}$$

(b) i.

$$\begin{aligned} e^{x^2} &= 3 \\ x^2 &= \ln 3 \\ x &= \sqrt{\ln 3} = \pm 1.05 \ \ (2d.p.) \end{aligned}$$

ii.

$$\begin{aligned} 2^{x-1}3^x &= 25 \\ 2^{x-1}3^{x-1}3 &= 25 \\ 6^{x-1} &= \frac{25}{3} \\ (x-1)\ln 6 &= \ln(\frac{25}{3}) \\ x &= 2.18 \ \ (2d.p.) \end{aligned}$$

(c)

$$s^2 - t = 3 \quad \text{(B.4)}$$
$$3s + 2t = 8 \quad \text{(B.5)}$$

From B.4 $2t = 2s^2 - 6$. Substituting into B.5 gives:

$$3s + 2s^2 - 6 = 8$$
$$2s^2 + 3s - 14 = 0$$

Solve for s using the quadratic formula with $a = 2$, $b = 3$ and $c = -14$:

$$\begin{aligned} s &= \frac{-3 \pm \sqrt{(3)^2 - 4(2)(-14)}}{2(2)} \\ &= \frac{-3 \pm \sqrt{121}}{4} \end{aligned}$$

$s = \frac{-3+11}{4} = 2$ or $s = \frac{-3-11}{4} = -3.5$
Substitute $s = 2$ into B.4 to find $t = 1$
Substitute $s = -3.5$ into B.4 to find $t = 9.25$
Check the solutions $(s = 2, t = 1)$ and $(s = -3.5, t = 9.25)$ by substituting these values into B.5.

7. Substituting $x = 1$ into $2x^3 - x - 1 = 0$ gives $2(1)^3 - (1) - 1 = 0$. Therefore $x = 1$ is a solution.

   Since $x = 1$ is a solution $(x - 1)$ must be a factor of the equation. We can divide $2x^3 - x - 1$ by $(x - 1)$ to find the other factor is $(2x^2 + 2x + 1)$ i.e. $(x - 1)(2x^2 + 2x + 1) = 2x^3 - x - 1 = 0$.

   If there were any other solutions to the equation then we would be able to factorise $(2x^2 + 2x + 1)$. However this is impossible since the discriminant $b^2 - 4ac = 2^2 - 4(2)(1) = -4$. In the quadratic formula we have to take the square root of the discriminant, but that is not possible here since the discriminant is negative. Therefore $(2x^2 + 2x + 1)$ cannot be factorised and hence $2x^3 - x - 1 = 0$ has no solutions other than $x = 1$.

   Stationary points occur when $\frac{dy}{dx} = 0$.

$$\begin{aligned} y &= 2x^3 - x - 1 \\ \frac{dy}{dx} &= 6x^2 - 1 \\ 6x^2 - 1 &= 0 \\ x = \pm\sqrt{\frac{1}{6}} &= \pm 0.408 \;\; (3d.p.) \end{aligned}$$

   When $x = 0.408$, $y = 2(0.408)^3 - (0.408) - 1 = -1.272$ (3 d.p.)
   When $x = -0.408$, $y = 2(-0.408)^3 - (-0.408) - 1 = -0.728$ (3 d.p.)

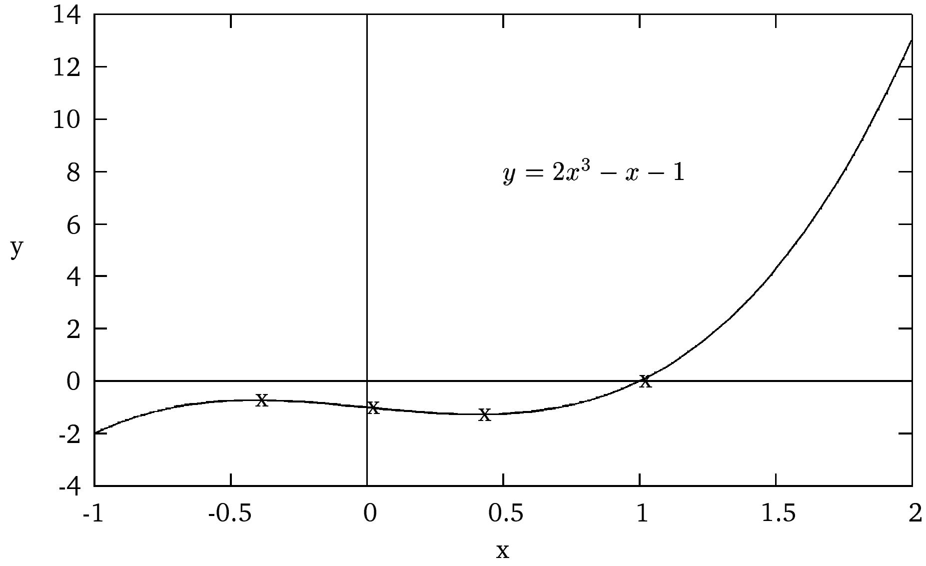
   The stationary points are at $(0.4.8, -1.272)$ and $(-0.498, -0.728)$.

   Use double differentiation to determine the nature of the stationary points:

   $\frac{d^2y}{dx^2} = 12x$, when $x = 0.408$, $\frac{d^2y}{dx^2} > 0$ so this point is a minimum.

When $x = -0.408$, $\frac{d^2y}{dx^2} < 0$ so this point is a maximum.

x	-1	-0.408	0	0.408	1	2
y	-2	-0.728	-1	-1.272	0	13



8. (a) Kofi recieves $1000(1.05)^5 + 1000(1.05)^4 + \ldots + 1000(1.05)$.
This is a G.P. with $a = 1050$ and $r = 1.05$. Hence
$S_5 = \frac{1050(1.05^5-1)}{1.05-1} = \$5801.91$

(b) Here $1000r^5 = 1246.18$ so $r =^5 \sqrt{1.24618} = 1.045$ (3 d.p.)
Therefore the rate of interest Sasha recieves is 4.5%.

(c)

$$\begin{aligned} 5000(1.04)^t &> 10000 \\ 1.04^t &> 2 \\ t \ln 1.04 &> \ln 2 \\ t &> \frac{ln2}{ln1.04} \\ t &> 17.67 \end{aligned}$$

Julie's investment will be worth more than \$10000 after 18 years.

9. (a) i. $\sum_{n=1}^{17} \frac{n+1}{n^2}$

ii. $\sum_{n=0}^{9}(3n+1)^3$

(b) $S_n = \frac{a(r^n-1)}{r-1}$

i. $\sum_{k=-1}^{7}(1.4)^k = 1.4^{-1} + 1.4^0 + 1.4^1 + \ldots + 1.4^7$
$a = \frac{1}{1.04}$, $r = 1.4$ and $n = 9$
$S_9 = \frac{\frac{1}{1.04}(1.04^9-1)}{1.4-1} = 35.11$ (2 d.p.)

ii. $\sum_{t=0}^{9} 2^{t-1} = 2^{-1} + 2^0 + 2^1 + \ldots + 2^8$
$a = \frac{1}{2}$, $r = 2$ and $n = 10$
$S_{10} = \frac{\frac{1}{2}(2^{10}-1)}{2-1} = 511\frac{1}{2}$

iii. $\sum_{s=2}^{7} \frac{2}{3^s} = 2\sum_{s=2}^{7} \frac{1}{3^s} = 2\left(\frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^7}\right)$
Inside the brackets there is a G.P. with $a = \frac{1}{9}$, $r = \frac{1}{3}$ and $n = 6$.
The total sum is therefore $2\left(\frac{\frac{1}{9}((\frac{1}{3})^6-1)}{\frac{1}{3}-1}\right) = \frac{728}{2187}$

10. (a) i.

$$\begin{aligned}\int_0^1 (2+\sqrt{(x+2)}).dx &= \left[2x+\frac{2}{3}(x+2)^{\frac{3}{2}}\right]_0^1 \\ &= \left[2(1)+\frac{2}{3}((1)+2)^{\frac{3}{2}}\right]-\left[2(0)+\frac{2}{3}((0)+2)^{\frac{3}{2}}\right] \\ &= 5.464-1.886 \\ &= 3.58 \;\; (2d.p.)\end{aligned}$$

ii.

$$\begin{aligned}\int_1^2 e^{2x}.dx &= \left[\frac{1}{2}e^{2x}\right]_1^2 \\ &= \frac{e^4}{2}-\frac{2^2}{2}=23.60 \;\; (2d.p.)\end{aligned}$$

iii.

$$\begin{aligned}\int_2^4 \frac{3}{(x-1)}.dx &= [3\ln(x-1)]_2^4 \\ &= 3(\ln 3-\ln 1)=3.30 \;\; (2d.p.)\end{aligned}$$

(b) $T=\int M.dQ$
$T=\int(\frac{5}{Q}+16+20Q).dQ=5\ln Q+16Q+10Q^2+C$
Substitute $T=224$, $Q=4$ into T to find C:
$224=5\ln 4+16(4)+10(4)^2+C$
$C=-6.93$ (2 d.p.)
Therefore the total costs are $T=5\ln Q+16Q+10Q^2-6.93$

Appendix C

Solutions to the subject guide activities

C.1 Chapter 1 activity solutions

Page 3

Let P and Q be the points (p, p^4) and (q, q^4) respectively. Then the gradient of the secant PQ is given by

$$gradient = \frac{q^4 - p^4}{q - p} = \frac{(q^2 - p^2)(q^2 + p^2)}{q - p} = (q + p)(q^2 + p^2)$$

As $Q \to P, q \to p$ and therefore
$gradient \to (p + p)(p^2 + p^2) = 2p(2p^2) = 4p^3$

Therefore if $y = x^4$, $\frac{dy}{dx} = 4x^3$.

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1. $f(x) = \frac{1}{x}$ is discontinuous - it cannot be differentiated at the point $x = 0$.
2. $f(x) = \sqrt{x}$ is continuous for $x \geq 0$ and can be differentiated for all values of $x \geq 0$.
3. This function is continuous since the two parts meet up. However it cannot be diffentiated at the point when $x = 4$ since there is no unique tangent at this point.
4. This function is discontinuous since there is a break at the point when $x = 2$. The function cannot be differentiated at this point.

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1. $\frac{dy}{dx} = 6x^5$
2. $\frac{dy}{dx} = 1$
3. $\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}}$
4. $y = x^{\frac{3}{2}}$, $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}$
5. $\frac{dy}{dx} = -3x^{-4}$
6. $y = x^{-2}$, $\frac{dy}{dx} = -2x^{-3}$

Page 5

1. $\frac{dy}{dx} = 35x^6$
2. $\frac{dy}{dx} = 3x$
3. $\frac{dy}{dx} = 2x^{-3}$

4. $\frac{dy}{dx} = \frac{12}{x^4}$

Page 7

1. $\frac{dy}{dx} = 8x - 15x^2 + 8$
2. $y = x^2 + 2x + 1, \ \frac{dy}{dx} = 2x + 2$
3. $y = -10x^2 - 8x + 24 \ \frac{dy}{dx} = -20x - 8$
4. $y = x^{-1} - x^{-2} \ \frac{dy}{dx} = -x^{-2} + 2x^{-3} = \frac{-x+2}{x^3}$
5. $y = 7x^{-2} + 9x^{-3} + 4x^{\frac{3}{2}} - x^{-1} \ \frac{dy}{dx} = -14x^{-3} - 27x^{-4} + 6x^{\frac{1}{2}} + x^{-2}$

Page 8

1. $y = 3u^5, u = x + 4, \frac{dy}{du} = 15u^4, \frac{du}{dx} = 1, \frac{dy}{dx} = 15(x+4)^4$
2. $y = u^{\frac{1}{2}}, u = 5x^2 + 7x, \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}, \frac{du}{dx} = 10x + 7, \frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} * (10x + 7) = \frac{(10x+7)}{2\sqrt{(5x^2+7x)}}$
3. $y = 2u^{-2}, u = x - 3, \frac{dy}{du} = -4u^{-3}, \frac{du}{dx} = 1, \frac{dy}{dx} = -4u^{-3} = \frac{-4}{(x-3)^3}$
4. $y = u^{-\frac{1}{2}}, u = 8x - x^2, \frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}, \frac{du}{dx} = 8 - 2x, \frac{dy}{dx} = \frac{-(8-2x)}{2\sqrt{(8x-x^2)^3}} = \frac{x-4}{\sqrt{(8x-x^2)^3}}$

Page 8

1. $u = x + 3, \frac{du}{dx} = 1, v = 7x - 4, \frac{dv}{dx} = 7, \frac{dy}{dx} = (7x-4)(1) + (x+3)(7) = 14x + 17$
2. $u = 6x^3, \frac{du}{dx} = 18x^2, v = 2x^2 + 5x - 4, \frac{dv}{dx} = 4x + 5, \frac{dy}{dx} = (2x^2 + 5x - 4)(18x^2) + (6x^3)(4x + 5) = 60x^4 + 120x^3 - 72x^2$

Page 9

1. (Rearrange to $y = 2x^{-2} - 3$) $\frac{dy}{dx} = -4x^{-3} = \frac{-4}{x^3}$
2. (Product rule) $\frac{dy}{dx} = (2x - 3)2x + 2x^2$
3. (Rearrange to $(2x - 3)^{-2}$ and then use the chain rule) $\frac{dy}{dx} = -2(2x - 3)^{-3}(2) = \frac{-4}{(2x-3)^3}$
4. (Quotient rule) $\frac{dy}{dx} = \frac{(2x-3)(2x) - 2x^2}{(2x-3)^2}$
5. (Rearrange to $y = 2x^{\frac{1}{2}} - 3$) $\frac{dy}{dx} = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$
6. (Chain rule) $\frac{dy}{dx} = \frac{1}{2}(x + 5)^{-\frac{1}{2}} = \frac{1}{2\sqrt{(x+5)}}$
7. (Quotient rule) $\frac{dy}{dx} = \frac{\sqrt{x} - (x+5)\frac{1}{2}x^{-\frac{1}{2}}}{x}$
8. (Quotient rule) $\frac{dy}{dx} = \frac{(x+5)^2 - 2(x-1)(x+5)}{(x+5)^4}$
9. (Rearrange to $y = \frac{1}{x^{10}} = x^{-10}$) $\frac{dy}{dx} = -10x^{-11} = \frac{-10}{x^{11}}$
10. (Product rule) $\frac{dy}{dx} = (x + 5)^9 \frac{1}{2}x^{-\frac{1}{2}} + \sqrt{x}(9(x + 5)^8 = \frac{(x+5)^9}{2\sqrt{x}} + 9\sqrt{x}(x + 5)^8$

Page 10

1. $\frac{dy}{dx} = 19x^2 + 8x - 7, \ \frac{d^2y}{dx^2} = 36x + 8$
2. $f'(x) = 3(x + 1)^2, \ f''(x) = 6(x + 1)$
3. $f'(x) = -2x^{-3}, \ f''(x) = 6x^{-4}$
4. $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}, \ \frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}}$

C.2 Chapter 2 activity solutions

Page 15

$\frac{dy}{dx} = 12x^3 + 12x^2 - 24x = 12x(x-1)(x+2)$
When $\frac{dy}{dx} = 0, x = 0$ or 1 or -2
When $x = 0, y = 0$, when $x = 1, y = -5$, when $x = -2, y = -32$
Therefore the stationary points are $(0,0), (1,-5)$ and $(-2,-32)$.

Page 16

1. $\frac{dy}{dx} = 6x^2 - 12x - 18 = 6(x+1)(x-3)$
   $\frac{dy}{dx} = 0$ when $x = -1$ or $x = 3$ so the stationary points are $(-1, 15)$ and $(3, -49)$.
   $\frac{d^2y}{dx^2} = 12x - 12$
   When $x = -1$, $\frac{d^2y}{dx^2} < 0$ so the point $(-1, 15)$ is a maximum.
   When $x = 3$, $\frac{d^2y}{dx^2} > 0$ so the point $(3, -49)$ is a minimum.
2. $f'(x) = 3x^2 - 12x + 9 = 3(x-3)(x-1)$
   $f'(x) = 0$ when $x = 3$ and $x = 1$. Therefore there are turning points when $x = 1$ and when $x = 3$.
   $f''(x) = 6x - 12$
   When $x = 3, f''(x) > 0$ so this turning point is a minimum.
   When $x = 1, f''(x) <)$ so this turning point is a maximum.
   Therefore the graph is decreasing in value in between the two turning points i.e. when $1 < x < 3$. (A rough sketch of the function will help you to see this result.)

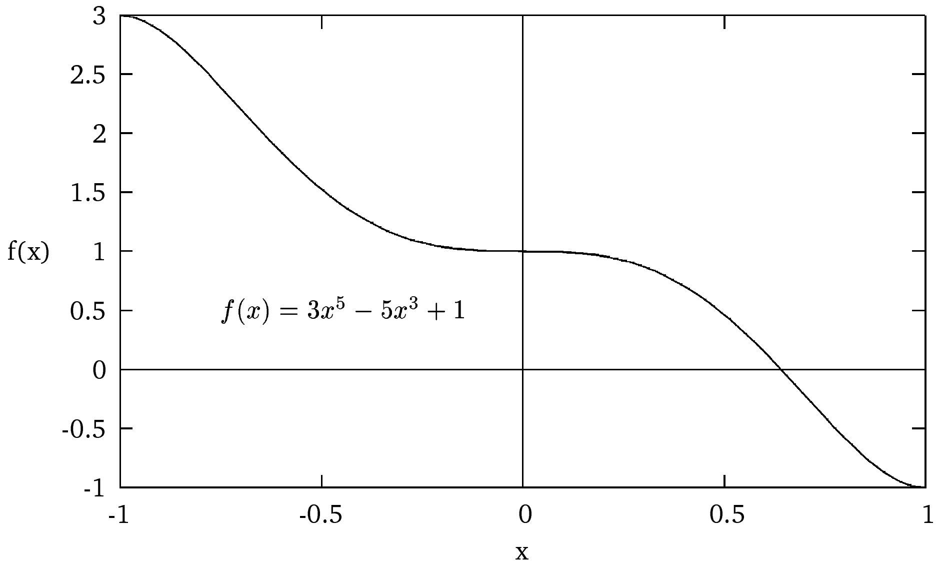
Page 18

1. $\frac{dy}{dx} = 4x^3 - 12x^2 = 4x^2(x-3)$. There are stationary points when $x = 0, y = 0$ and $x = 3, y = -27$.
   $\frac{d^2y}{dx^2} = 12x^2 - 24x$
   When $x = 3$, $\frac{d^2y}{dx^2} > 0$ so the point $(3, -27)$ is a minimum.
   When $x = 0$, $\frac{d^2y}{dx^2} = 0$ so we look at the gradient function either side of $x = 0$. When $x = -1$, $\frac{dy}{dx} < 0$ so the function is decreasing, when $x = 1$, $\frac{dy}{dx} < 0$ so the function is still decreasing. Therefore the point $(0,0)$ is a point of inflexion.
   $\frac{d^2y}{dx^2} = 12x(x-2) = 0$ when $x = 0$ and when $x = 2$. Therefore there is a second point of inflexion when $x = 2$.
2. Using the chain rule $h'(x) = -8(4-2x)^3$. Therefore there is a stationary point when $x = 2$. $h(2) = 0$ so the stationary point is at $(2,0)$. Using the chain rule again, $h''(x) = 48(4-2x)^2$. But $h''(2) = 0$ so this test is inconclusive. We look at the gradient function either side of $x = 2$. $h'(1) < 0$ so the curve is decreasing before the stationary point. $h'(3) > 0$ so the curve is increasing after the stationary point. Therefore the stationary point at $(2,0)$ is a minimum.

Page 19

$f'(x) = 15x^4 - 15x^2 = 15x^2(x-1)(x+1)$
Stationary points occur when $x = 0, 1, -1$
Evaluating the function at these values gives us the co-ordinates of the critical points $(0,1), (1,-1)$ and $(-1,3)$.
$f''(x) = 60x^3 - 30x$

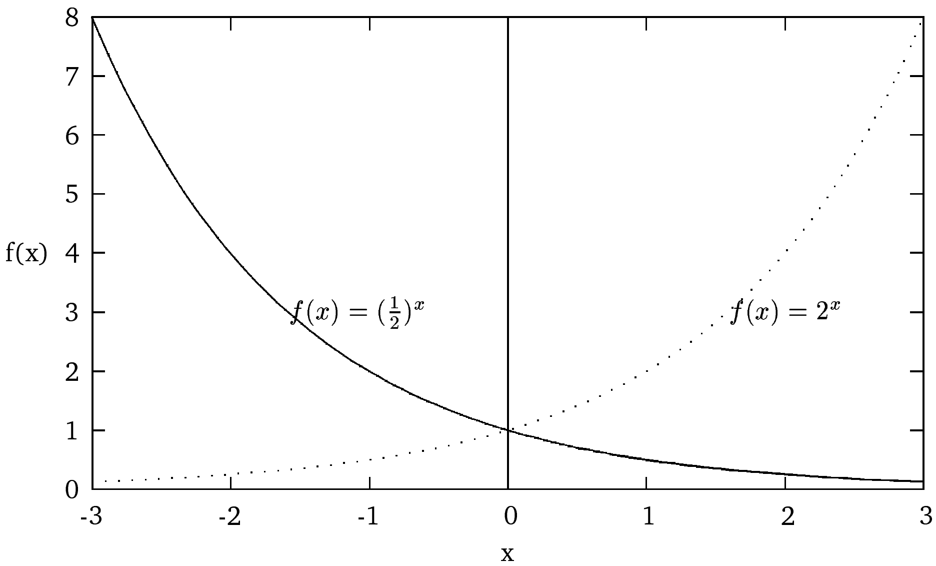
When $x = 0$, $f''(x) = 0$ so no information.
When $x = 1$, $f''(x) > 0$ so $(1, -1)$ is a local minimum.
When $x = -1$, $f''(x) < 0$ so $(-1, 3)$ is a local maximum.
Since the critical point $(0, 1)$ is in between a maximum and a minimum and there are no other turning points, the point $(0, 1)$ must be a point of inflexion.



C.3 Chapter 3 activity solutions

Page 28

x	-3	-2	-1	0	1	2	3
$\frac{1}{2}^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



The graphs of $f(x) = 2^x$ and $f(x) = (\frac{1}{2})^x$ are *mirror images* of each other reflected in the line $x = 0$.

Page 29

1. $5^x = 125, x = 3$
2. $3^x = 243, x = 5$
3. $3^x = \frac{1}{243}, x = -5$
4. $7^x = 7, x = 1$
5. $10^x = 100000, x = 5$
6. $(\frac{1}{2})^x = \frac{1}{4}, x = 2$
7. $18^x = 1, x = 0$
8. $\frac{1}{4}^x = 16, x = -2$
9. $0.1^x = 0.001, x = 3$
10. $(\frac{3}{4})^x = \frac{9}{16}, x = 2$

Page 30

1. $\log_2 64 - \log_2 16 = \log_2(\frac{64}{16}) = \log_2 4 = 2$
2. $(\frac{81}{256})^{\frac{3}{4}} = (\frac{\sqrt[4]{81}}{\sqrt[4]{256}})^3 = \frac{3^3}{4^3} = \frac{27}{64}$
3. $\log_{10} 6 + \log_{10} 4 + \log_{10} 20 - \log_{10} 3 - \log_{10} 16 = \log_{10}(\frac{6*4*20}{3*16}) = \log_{10} 10 = 1$
4. $\log_3 9^5 = 5 * \log_3 9 = 5 * 2 = 10$

Page 32

1. $x = \frac{\log 16}{\log 9} = 1.262$
2. $2^{2x} = 16$, $2^{2x} = 2^4$ so $2x = 4$ therefore $x = 2$
   or $2x \log 2 = \log 16$, $2x = \frac{\log 16}{\log 2} = 4$ therefore $x = 2$
3. $(5^x)^2 - 5^2(5^x) - 5(5^x) + 125 = 0$, let $z = 5^x$ then
   $z^2 - 25z - 5z + 125 = 0$, $z^2 - 30z + 125 = 0$, $(z-5)(z-25) = 0$.
   Either $z = 5$ so $5^x = 5$ so $x = 1$, or $z = 25$ so $5^x = 25$ so $x = 2$.
4. $y = \log_4 x$ so $4^y = x$ or $(2^2)^y = x$ so $2^{2y} = x$. Hence $\log_2 x = 2y$
   or $y = \frac{\log_2 x}{2}$.
   $\log_2 x - \log_4 x = 3$
   $\log_2 x - \frac{\log_2 x}{2} = 3$
   $\frac{\log_2 x}{2} = 3$
   $\log_2 x = 6$
   $x = 2^6 = 64$

Page 32

1. 1.978
2. 4.554
3. 4.554 ($\log_e$ is the same as ln)
4. 100000
5. 148.413
6. 0.316
7. 0.022

Page 33

1. $x = \ln 4.75 = 1.558$
2. $2x = \ln 15, x = \frac{\ln 15}{2} = 1.354$
3. $e^x = \frac{e}{5}$, $\ln e^x = \ln(\frac{e}{5}) = \ln e - \ln 5 = 1 - \ln 5 = -0.609$
4. Let $z = e^x$ then $z^2 - 5z + 4 = 0$ so $(z-4)(z-1) = 0$. Either
   $z = 4$ so $e^x = 4, x = \ln 4 = 1.386$ or $z = 1$ so $e^x = 1, x = \ln 1 = 0$

Page 35

1. $y = 4e^{5x}$, $\frac{dy}{dx} = 4(5e^{5x}) = 20e^{5x}$
2. $y = xe^x$, $\frac{dy}{dx} = 1 * e^x + x * e^x = e^x(1+x)$
3. $y = e^{(4x^2+2x+3)}$, $\frac{dy}{dx} == e^{(4x^2+2x+3)}(8x+2)$
4. $y = \frac{e^x}{5x}$, $\frac{dy}{dx} = \frac{5xe^x - e^x 5}{(5x)^2} = \frac{e^x(x-1)}{5x^2}$

Page 37

1. $y = \ln x^2 = 2\ln x$, $\frac{dy}{dx} = \frac{2}{x}$
2. $y = 3\ln x$, $\frac{dy}{dx} = \frac{3}{x}$
3. $y = \ln 3x$, $\frac{dy}{dx} = \frac{3}{3x} = \frac{1}{x}$
4. $y = \ln(4x^2+5)$, $\frac{dy}{dx} = \frac{8x}{4x^2+5}$
5. $y = x^3 \ln x^3$, $\frac{dy}{dx} = x^3 * \frac{3x^2}{x^3} + 3x^2 * \ln x^3 = 3x^2(1 + \ln x^3)$
6. $y = e^x \ln x$, $\frac{dy}{dx} = e^x \ln x + \frac{e^x}{x}$

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1. (a) $P_0 = 1.9$ million
   Since $P = 4.3$ million when $t = 4$ we have

$$\begin{aligned} 4.3 &= 1.9e^{4k} \\ 4k &= \ln \frac{4.3}{1.9} \\ k &= 0.20419 (5 d.p.) \end{aligned}$$

   (b) In 2005, $t = 6$ so $P = 1.9e^{6(0.20419)} = \$6.47$ million (2 d.p.)

2. Let the growth function be $P = P_0 e^{kt}$ where $P_0 = 45$ million.
   Then $60.1 = 45e^{5k}$ so $k = 0.058$ (3 d.p.)

   The growth function is $P = 45e^{0.058t}$ where t is the time in years since 1981 and P is the population in millions.

   The annual growth rate is $e^{0.058} = 1.0597$ or 5.6% (1 d.p.).

C.4 Chapter 4 activity solutions

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1. We have final amount= $2000(1.035^t)$ where t is the number of years.
   (a) $t = 2$, amount= $2000(1.035^2) = \$2142.45$
   (b) $t = 7$, amount= $2000(1.035^7) = \$2544.56$
   (c) $t = 20$, amount= $2000(1.035^{20}) = \$3979.58$
2. I will have earned $3979.58 - 2000 = \$1979.58$ in interest over 20 years.

   My friend will earn simple interest of $2000 \, . \, 0.035 = \$70$ per year for 20 years. This will be a total of \$1400 which is \$579.58 less than I have earned with compound interest.
3. We have to solve the equation $8000 * r^6 = 10720.77$ to find the rate of interest r.

$$r^6 = \frac{10720.77}{8000} = 1.34$$
$$r = 1.34^{\frac{1}{6}} = 1.05$$

   Therefore the rate of interest paid to Anna was 5%.

Page 44
We use the formula $final \;\; amount = P(1 + \frac{r}{m})^{mt}$ where $P = 10,000$, $r = 10\% = 0.1$, $m = 365$ and $t = 1$. This gives us $final \;\; amount = 10,000(1 + \frac{0.1}{365})^{365} = \$11,051.56$

Page 45

1. We have $P = 400, t = 2, r = 0.05$
   (a) final amount $= 400(1 + 0.05)^2 = 441$, interest= \$41
   (b) final amount $= 400(1 + \frac{0.05}{4})^8 = 441.79$, interest= \$41.79
   (c) final amount $= 400e^{0.05 \, . \, 2} = 442.07$, interest= \$42.07
2.

$$1012.40 = 750e^{5r}$$
$$\frac{1012.40}{750} = e^{5r}$$
$$\ln(\frac{1012}{750}) = 5r$$
$$r = 0.06 = 6\%$$

Page 46

- For the first account: $P = \frac{4000}{1.038^3} = 3576.58$
- For the second account: $P = \frac{4000}{(1+0.038/2)^6} = 3572.85$
- For the third account: $P = \frac{4000}{e^{(0.0375*3)}} = 3574.39$

Therefore Penny should invest in the second account.

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1. This is not a geometric series.

2. $r = 1$ next term $= 1$
3. $r = 3$ next term $= 2187$
4. $r = 0.75$ next term $= 31.640625$
5. $r = 0.1$ next term $= 0.00001$
6. This is not a geometric series.

Page 48

1. $n = 8, r = 3, a = 2$ $S_8 = \frac{2(3^8-1)}{3-1} = 6560$
2. $n = 12, r = 5, a = 2$ $S_{12} = \frac{2(5^{12}-1)}{5-1} = 122070312$
3. $n = 20, r = 3, a = 1$ $S_{20} = \frac{1(3^{20}-1)}{3-1} = 1743392201$
4. $n = 10, r = 0.5, a = 8$ $S_{10} = \frac{8(0.5^{10}-1)}{0.5-1} = 15.984375$
5. $n = 10, r = -0.5, a = 8$ $S_{10} = \frac{8(-0.5^{10}-1)}{-0.5-1} = 5.328125$

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1. We have $r = 1.05$, $a = 4200 * 1.05 = 4410$ and $n = 50$.
   $S_{50} = \frac{4410(1.05^{50}-1)}{1.05-1} = \$923,224.66$
2. (a) We have $r = 1.11$, $a = 500 * 1.11 = 555$ and $n = 10$.
   $S_{10} = \frac{555(1.11^{10}-1)}{1.11-1} = \$9,280.71$
   (b) The sum is
   $500e^{0.1*10} + 500e^{0.1*9} + 500e^{0.1*8} + \ldots + 500e^{0.1*1}$. This is a geometric series with $a = 500e^{0.1}$ and $r = e^{0.1}$. Therefore
   $S_{10} = \frac{500e^{0.1}((e^{0.1})^{10}-1)}{e^{0.1}-1} = \$9,028.14$
3. We have $a = 1000 * 1.075 = 1075$ and $r = 1.075$.
   (a) At the end of n years we would have a total sum of
   $S_n = \frac{1075(1.075^n-1)}{0.075}$.
   (b) At the end of 20 years we would have
   $S_{20} = \frac{1075(1.075^{20}-1)}{0.075} = \$46,552.53$
4. If $r = 1$ the geometric series is simply $a + a + a + a + \ldots + a + a$.
   Therefore when $r = 1$, $S_n = a * n$.

Page 50

1. (a) $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$
   (b) $3^2 + 4^2 + 5^2 + 6^2 + 7^2$
   (c) $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}$
   (d) $e + e^2 + e^3 + e^4 + e^5 + e^6$
   (e) $11 - 13 + 15 - 17 + 19$
   (f) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3$
2. (a) $\sum_{i=1}^{9} \sqrt{i}$
   (b) $\sum_{i=1}^{50} (2i)^2$
   (c) $\sum_{i=1}^{97} \frac{1}{i+2}$
   (d) $\sum_{i=1}^{19} (-1)^{i+1} i^3$
   (e) $\sum_{i=1}^{99} \frac{i}{i+1}$
3. (a) $\sum_{i=1}^{60} 2i - 1$
   (b) $\sum_{i=10}^{20} i^2$
   (c) $\sum_{i=0}^{14} 7i + 1$

C.5 Chapter 5 activity solutions

Page 54

1. $\int 8.dx = 8x + c$
2. $\int \frac{x^2}{3}.dx = \frac{x^3}{9} + c$
3. $\int \frac{7}{x}.dx = 7\ln x + c$ or $7\ln kx$
4. $\int 6x^2 + 6x + 1.dx = 2x^3 + 3x^2 + x + c$
5. $\int (x+1)^2.dx = \int x^2 + 2x + 1.dx = \frac{x^3}{3} + x^2 + x + c$
6. $\int 6(e^x + 1).dx = 6(e^x + x) + c$
7. $\int \frac{1}{3}e^{3x}.dx = \frac{1}{9}e^{3x} + c$
8. $\int \frac{3-2x}{x}.dx = \int \frac{3}{x} - 2.dx = 3\ln x - 2x + c$
9. $\int \sqrt{x}^3.dx = \int x^{\frac{3}{2}}.dx = \frac{2}{5}x^{\frac{5}{2}} + c$
10. $\int x^2(1 - \sqrt{x}).dx = \int x^2 - x^{\frac{3}{2}}.dx = \frac{1}{3}x^3 - \frac{2}{5}x^{\frac{52}{+}}c$

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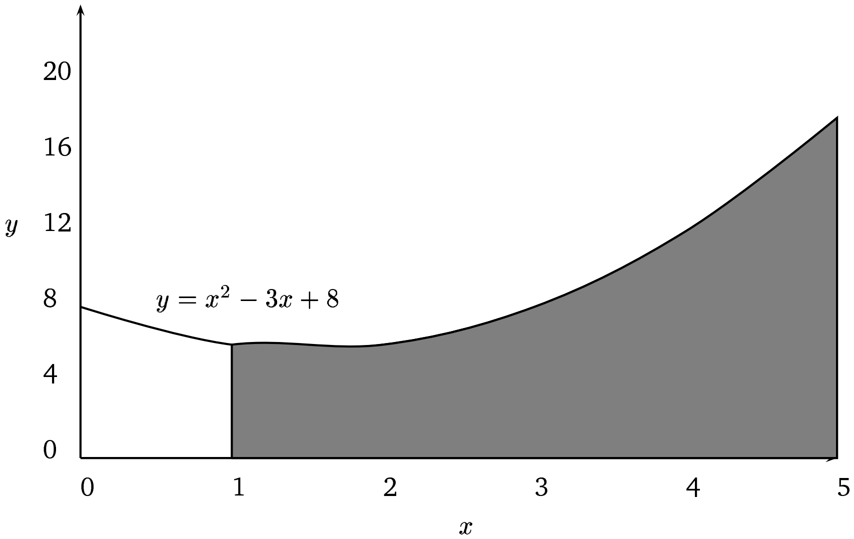
We could include the constant of integration but it is not necessary to do so because when we subtract $F(a)$ from $F(b)$ the constants would cancel each other out. For example $\int_1^2 5 = [5x+c]_1^2 = [5(2)+c] - [5(1)+c] = 10 + c - 5 - c = 10 - 5 = 5$.

Page 56

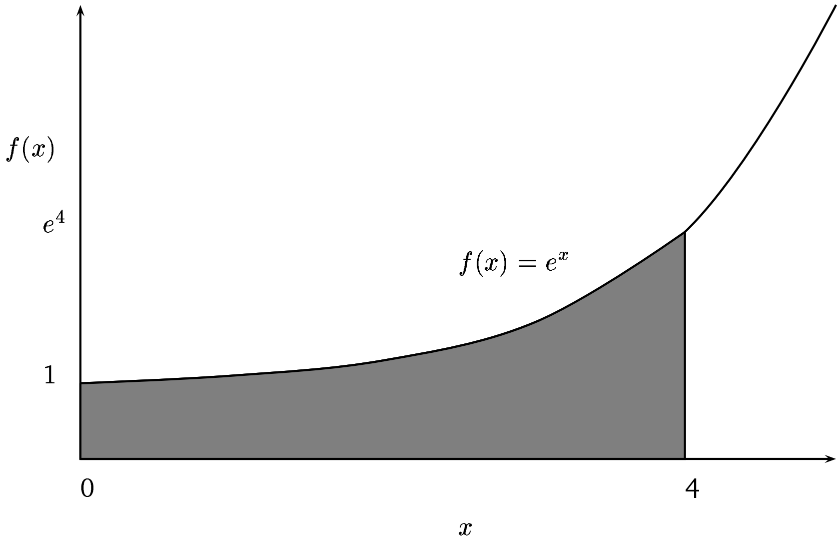
1. $\int_0^4 (x^2 + 5x).dx = \left[\frac{1}{3}x^3 + \frac{5}{2}x^2\right]_0^4 = \left[\frac{1}{3}(4)^3 + \frac{5}{2}(4)^2\right] - [0] = 61\frac{1}{3}$
2. $\int_e^{e^2} \frac{1}{x}.dx = [\ln x]_e^{e^2} = [\ln e^2] - [\ln e] = 2 - 1 = 1$
3. $\int_1^2 \frac{x^4-1}{x^3}.dx = \int_1^2 (x - x^{-3}).dx = \left[\frac{x^2}{2} + \frac{1}{2}x^{-2}\right]_1^2 = \left[\frac{x^2}{2} + \frac{1}{2x^2}\right]_1^2 = \left[\frac{2^2}{2} + \frac{1}{2^3}\right] - \left[\frac{1}{2} + \frac{1}{2}\right] = 2\frac{1}{8} - 1 = 1\frac{1}{8}$

Page 58

1.



$$
\begin{aligned}
\int_1^5 (x^2 - 3x + 8).dx &= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 8x\right]_1^5 \\
&= \left[\frac{(5)^3}{3} - \frac{3(5)^2}{2} + 8(5)\right] - \left[\frac{(1)^3}{3} - \frac{3(1)^2}{2} + 8(1)\right] \\
&= 44\frac{1}{6} - 6\frac{5}{6} \\
&= 37\frac{1}{3}
\end{aligned}
$$



$$
\begin{aligned}
\int_0^4 e^x.dx &= [e^x]_0^4 \\
&= [e^4] - [e^0] \\
&= 54.598 - 1 \\
&= 53.598(3d.p.)
\end{aligned}
$$

2.

$$
\begin{aligned}
\int_3^4 (x^2 - x + 5).dx &= \left[\frac{x^3}{3} - \frac{x^2}{2} + 5x\right]_3^4 \\
&= \left[\frac{(4)^3}{3} - \frac{(4)^2}{2} + 5(4)\right] - \left[\frac{(3)^3}{3} - \frac{(3)^2}{2} + 5(3)\right] \\
&= 33\frac{1}{3} - 19\frac{1}{2} \\
&= 13\frac{5}{6}
\end{aligned}
$$

Page 62

1. (a) $\int_{-3}^{3}(-x^2+9).dx = \left[\frac{-x^3}{3} + 9x\right]_{-3}^{3}$
   $= \left[\frac{-(3)^3}{3} + 9(3)\right] - \left[\frac{-(-3)^3}{3} + 9(-3)\right] = 18 - (-18) = 36$

   (b) $\int_{-3}^{3}(x^2-9).dx = \left[\frac{x^3}{3} - 9x\right] =$
   $\left[\frac{(3)^3}{3} - 9(3)\right] - \left[\frac{(-3)^3}{3} - 9(-3)\right] = -18 - 18 = -36$

2. We will calculate the area above the x-axis and then the area below the x-axis. Call these Area 1 and Area 2 respectively.

$$\begin{aligned}
Area\ \ 1 &= \int_1^2 (x^3 - 6x^2 + 9x - 2).dx \\
&= \left[\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2} - 2x\right]_1^2 \\
&= \left[\frac{(2)^4}{4} - 2(2)^3 + \frac{9(2)^2}{2} - 2(2)\right] - \left[(1)^4 4 - 2(1)^3 + \frac{9(1)^2}{2} - 2(1)\right] \\
&= 2 - \frac{3}{4} \\
&= 1\frac{1}{4}
\end{aligned}$$

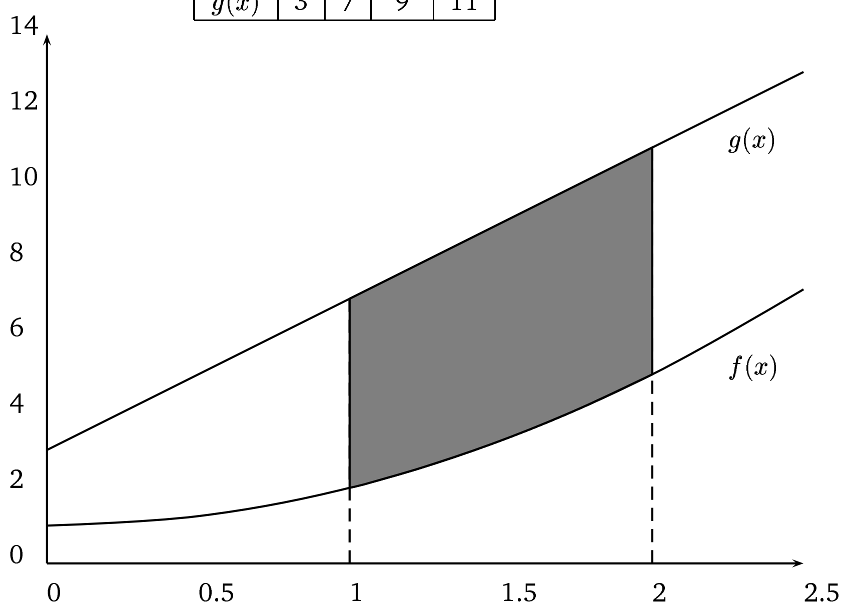
$$\begin{aligned}
Area\ \ 2 &= \int_2^3 (x^3 - 6x^2 + 9x - 2).dx \\
&= \left[\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2} - 2x\right]_2^3 \\
&= \left[\frac{(3)^4}{4} - 2(3)^3 + \frac{9(3)^2}{2} - 2(3)\right] - [2] \\
&= \frac{3}{4} - 2 \\
&= -1\frac{1}{4}
\end{aligned}$$

But area is positive so Area 2$= +1\frac{1}{4}$

Therefore the total shaded area is equal to $1\frac{1}{4} + 1\frac{1}{4} = 2\frac{1}{2}$.

3. Drawing a table as below helps to sketch the graphs of $f(x)$ and $g(x)$.

x	0	1	$1\frac{1}{2}$	2
$f(x)$	1	2	$3\frac{1}{4}$	5
$g(x)$	3	7	9	11



$$\begin{aligned}
Area &= \int_1^2 (4x+3).dx - \int_1^2 (x^2+1).dx \\
&= \int_1^2 (4x+3-(x^2+1)).dx \\
&= \int_1^2 (4x-x^2+2).dx \\
&= \left[2x^2 - \frac{x^3}{3} + 2x\right]_1^2 \\
&= \left[2(2)^2 - \frac{(2)^3}{3} + 2(2)\right] - \left[2(1)^2 - \frac{(1)^3}{3} + 2(1)\right] \\
&= 9\frac{1}{3} - 3\frac{2}{3} \\
&= 5\frac{2}{3}
\end{aligned}$$

Appendix D

Solutions to the sample examination questions

D.1 Chapter 1 examination solutions

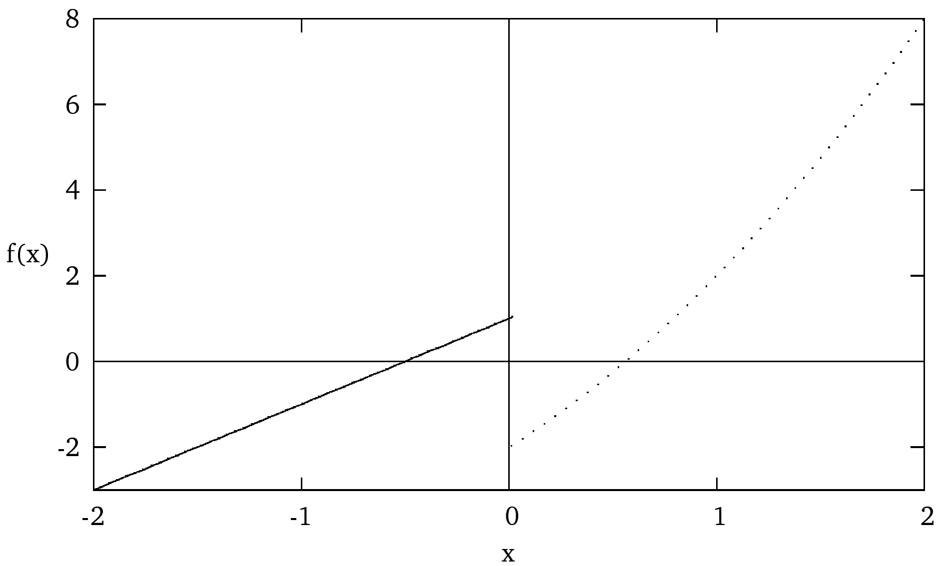
Question 1

a) When $a = 1$ we have

$$f(x) = 2x + 1 \qquad x < 0$$
$$f(x) = x^2 + 3x - 2 \qquad x \geq 0$$

Computing values from $-2 \leq x \leq 2$ gives us

x	-2	-1	0	1	2
$f(x)$	-3	-1	-2	2	8



The function is not continuous when $a = 1$ since there is a break in the line when $x = 0$.

The function would be continuous if the two parts joined up when $x = 0$. This occurs when $2(0) + a = (0)^2 + 3(0) - 2$. The only solution is when $a = -2$. Therefore the function $f(x)$ is a continuous function only when $a = -2$.

b) i) We use the chain rule: $y = 3(5x - 2)^6 = 3u^6$ where $u = 5x - 2$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 18u^5 \cdot 5 = 90(5x - 2)^5$$

ii) We use the quotient rule: $y = \frac{(3x^3+4)}{(x-2)} = \frac{u}{v}$ where $u = 3x^3 + 4$, $u' = 9x^2$, $v = x - 2$, $v' = 1$.

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(x-2)(9x^2) - (3x^3+4)(1)}{(x-2)^2}$$

iii) We use the product rule: $y = (5x^2 - 2)(4x^2 - 3x - 1) = uv$ where $u = 5x^2 - 2$, $u' = 10x$, $v = 4x^2 - 3x - 1$, $v' = 8x - 3$.

$$\frac{dy}{dx} = uv' + vu' = (5x^2 - 2)(8x - 3) + (4x^2 - 3x - 1)(10x)$$

Question 2

a) $y = f \circ g(x) = 2 - 5(\frac{1}{\sqrt{x}}) = 2 - 5x^{-\frac{1}{2}}$, $\frac{dy}{dx} = \frac{5}{2}x^{-\frac{3}{2}} = \frac{5}{2\sqrt{(x^3)}}$

b) $y = \frac{f(x)}{g(x)} = \frac{2-5x}{\frac{1}{\sqrt{x}}} = 2\sqrt{x} - 5x\sqrt{x} = 2x^{\frac{1}{2}} - 5x^{\frac{3}{2}}$,
$\frac{dy}{dx} = x^{-\frac{1}{2}} - \frac{15}{2}x^{\frac{1}{2}} = \frac{1}{\sqrt{x}} - \frac{15\sqrt{x}}{2}$

c) $y = g \circ f = \frac{1}{\sqrt{(2-5x)}} = (2 - 5x)^{-\frac{1}{2}}$,
$\frac{dy}{dx} = -\frac{1}{2}(2 - 5x)^{-\frac{3}{2}}(-5) = \frac{5}{2(\sqrt{(2-5x)^3})}$ (using the chain rule)

d) $y = \frac{1}{f(x).g(x)} = \frac{1}{(2-5x)(\frac{1}{\sqrt{x}})} = \frac{\sqrt{x}}{(2-5x)}$,
$\frac{dy}{dx} = \frac{(\frac{1}{2}x^{-\frac{1}{2}})(2-5x)-(x^{\frac{1}{2}})(-5)}{(2-5x)^2} = \frac{\frac{1}{\sqrt{x}}+\frac{5}{2}\sqrt{x}+5\sqrt{x}}{(2-5x)^2}$ (using the quotient rule with $u = \sqrt{x}$ and $v = (2 - 5x)$)

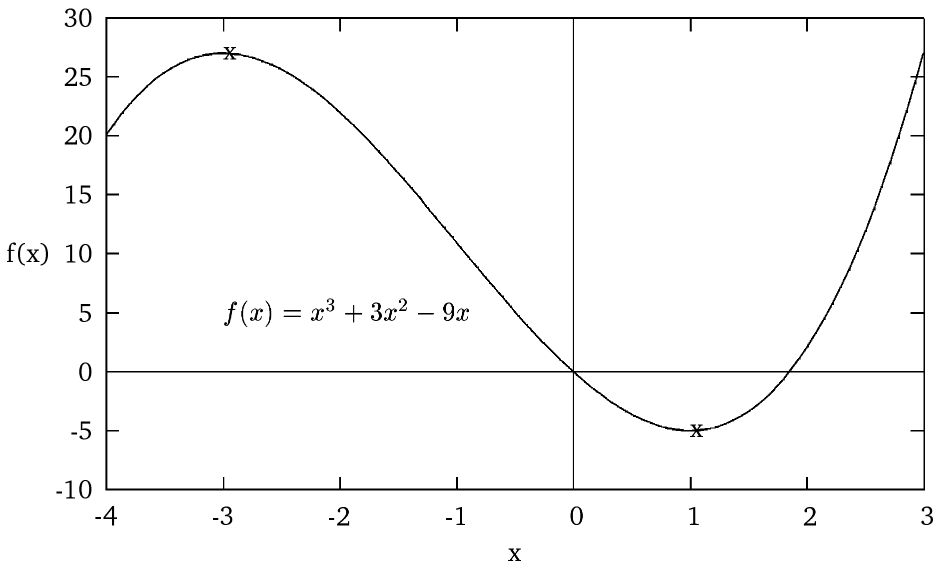
e) $y = f(\sqrt{(x-2)}) = 2 - 5(\sqrt{(x-2)}) = 2 - 5(x-2)^{\frac{1}{2}}$,
$\frac{dy}{dx} = -5(\frac{1}{2}(x-2)^{-\frac{1}{2}}(1)) = \frac{-5}{2\sqrt{(x-2)}}$ (using the chain rule)

D.2 Chapter 2 examination solutions

Question 1

a) $f(x) = x^3 + 3x^2 - 9x$
$f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1)$
Stationary points occur when $x = -3$ and $x = 1$.
When $x = -3$, $f(x) = -27 + 27 + 27 = 27$, when $x = 1$, $f(x) = 1 + 3 - 9 = -5$
Therefore the stationary points are $(-3, 27)$ and $(1, -5)$
$f''(x) = 6x + 6$
When $x = -3$, $f''(x) < 0$ so $(-3, 27)$ is a local maximum.
When $x = 1$, $f''(x) > 0$ so $(1, -5)$ is a local minimum.

b) $f(0) = 0$ so the curve cuts through the origin.
$f''(x) = 0$ when $x = -1$ so there is a point of inflexion at $(-1, 11)$.



Question 2

a) $\Pi = TR - TC = Q^2 + 2Q - (2Q^3 + 4Q^2 - 10Q + 4)$
$\Pi(Q) = -2Q^3 - 3Q^2 + 12Q - 4$

b) $\Pi'(Q) = -6Q^2 - 6Q + 12 = -6(Q + 2)(Q - 1)$
Therefore there are stationary points when $Q = -2$ and $Q = 1$.
$\Pi''(Q) = -12Q - 6$
When $Q = -2, \Pi''(Q) > 0$ so $(-2, \Pi(-2))$ is a minimum point.
When $Q = 1, \Pi''(Q) < 0$ so $(1, \Pi(1))$ is a maximum point.
$\Pi''(Q) = 0$ when $Q = -\frac{1}{2}$ so there is a non-stationary point of inflexion at $(-\frac{1}{2}, \Pi(-\frac{1}{2}))$.

c) The maximum profit occurs when $Q = 1$. At this point the profit is given by $\Pi(1) = -2(1)^3 - 3(1)^2 + 12(1) - 4 = 3$. Therefore the maximum profit is 3.

D.3 Chapter 3 examination solutions

Question 1

a) i) Let $4^x = 64^{-1} = (4^3)^{-1} = 4^{-3}$ so $x = -3$. Therefore $\log_4 \frac{1}{64} = -3$. Let $4^x = 8$, then $(2^2)^x = 2^3$ giving $2x = 3$ so $x = 1.5$. Therefore $\log_4 8 = 1.5$. Altogether $\log_4 \frac{1}{64} - \log_4 8 = -3 - 1.5 = -4.5$

ii) $(\frac{1}{\sqrt{x}})^{-4} = (x^{-\frac{1}{2}})^{-4} = x^{-\frac{1}{2}*-4} = x^2$

b) i) $y = x^2 \ln 2x$ using the product rule gives $\frac{dy}{dx} = 2x \ln 2x + x^2(\frac{1}{x}) = 2x \ln 2x + x$

ii) $y = \frac{e^{x^2}}{x}$ using the quotient rule with $u = e^{x^2}$, $\frac{du}{dx} = 2xe^{x^2}$ and $v = x$, $\frac{dv}{dx} = 1$, gives

$$\frac{dy}{dx} = \frac{x(2xe^{x^2}) - e^{x^2}(1)}{x^2} = \frac{e^{x^2}(2x^2 - 1)}{x^2}$$

c)

$$\begin{aligned} 150 &= 600e^{-5x} \\ \frac{1}{4} &= \frac{1}{e^{5x}} \\ 4 &= e^{5x} \\ \ln 4 &= 5x \\ x = \frac{\ln 4}{5} &= 0.277(3d.p.) \end{aligned}$$

Question 2

a) i) $t = 2, S = 100000(1 - e^{-0.6}) = 45119$ (nearest integer)
ii) $t = 5, S = 100000(1 - e^{-1.5}) = 77687$ (nearest integer)
iii) $t = 10, S = 100000(1 - e^{-3}) = 95021$ (nearest integer)

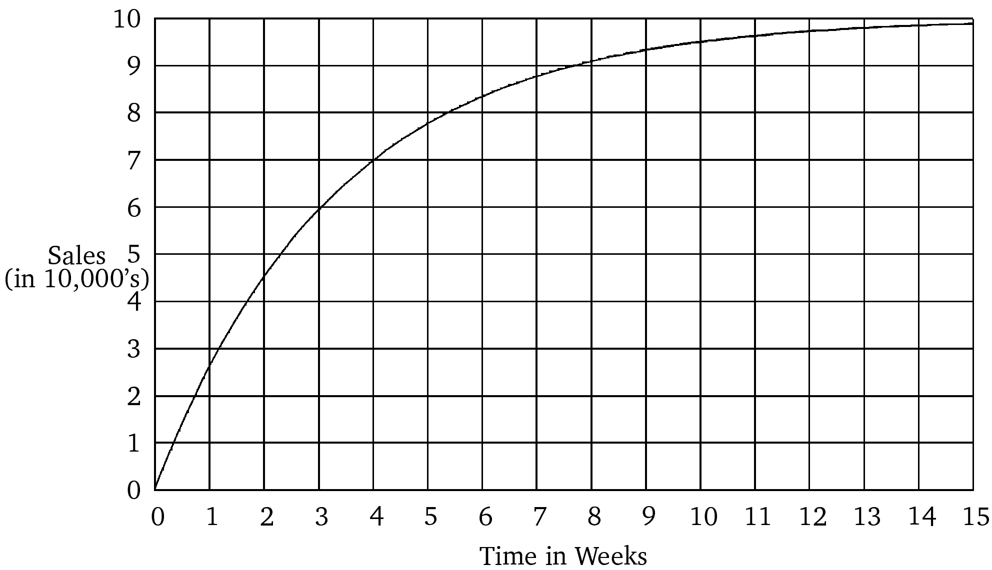
b)

$$\begin{aligned} 100000(1 - e^{-0.3t}) &= 60000 \\ 1 - e^{-0.3t} &= 0.6 \\ e^{-0.3t} &= 0.4 \\ -0.3t &= \ln 0.4 \\ t = \frac{\ln 0.4}{-0.3} &= 3.05 \end{aligned}$$

So sales will reach 60,000 between 3 and 4 weeks after launch.

c) The sales increase rapidly for the first few months but then level off at around 95,000 with only a slight increase in sales per week as the number of weeks increase. The number of sales approaches but never reaches 100,000.

d)



D.4 Chapter 4 examination solutions

Question 1

a)

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

i) $a = 5(2^{-2}) = 1.25$, $r = 2$, $n = 11$,
$S_{11} = \frac{1.25(2^{11}-1)}{2-1} = 2558.75$

ii)

$$\sum_{k=1}^{5}(3^k + 2^{k-1}) = \sum_{k=1}^{5} 3^k + \sum_{k=0}^{4} 2^k$$

$$\sum_{k=1}^{5} 3^k = \frac{3(3^5-1)}{3-1} = 363$$

$$\sum_{k=0}^{4} 2^k = \frac{1(2^5-1)}{2-1} = 31$$

Therefore

$$\sum_{k=1}^{5}(3^k + 2^{k-1}) = 363 + 31 = 394$$

b) $V = P(1+r)^n$, $V = 2000(1+0.06)^5 = \$2676.45$

c)

$$\begin{aligned}1628.89 &= 1000(1+r)^{10}\\ 1.62889 &= (1+r)^{10}\\ 1.62889^{\frac{1}{10}} &= 1+r\\ 1+r &= 1.05\\ r &= 5\%\end{aligned}$$

Question 2

a) i)

$$\sum_{r=1}^{5} r(r-1) = 1(0)+2(1)+3(2)+4(3)+5(4) = 0+2+6+12+20 = 40$$

ii)

$$\sum_{r=-2}^{2} 2^r = 2^{-2} + 2^{-1} + 2^0 + 2^1 + 2^2 = \frac{1}{4} + \frac{1}{2} + 1 + 2 + 4 = 7\frac{3}{4}$$

or
this is a geometric series with $a = 0.25$, $r = 2$ and $n = 5$ so
$S_5 = \frac{0.25(2^5-1)}{2-1} = 7.75$

b) i) $1+4+9+16+\ldots+100 = 1^2+2^2+3^2+\ldots+10^2 = \sum_{i=1}^{10} i^2$

ii) $25 + 22 + 19 + 16 + \ldots + 1 + (-2) + (-5) =$
$(1+3(8)) + (1+3(7)) + (1+3(6)) + \ldots + (1+3(0)) + (1+$
$3(-1)) + (1+3(-2)) = \sum_{i=-2}^{8}(1+3i)$

c) After n years Mary has
$A(1+r)^n + A(1+r)^{n-1} + \ldots + A(1+r)^2 + A(1+r)^1 = \sum_{i=1}^{n} A(1+r)^i$

D.5 Chapter 5 examination solutions

Question 1

a) i) $\int_0^2 (x^2 + x - 2)dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x\right]_0^2$
$= \left[\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 - 2(2)\right] - [0]$
$= \frac{8}{3} + 2 - 4$
$= \frac{2}{3}$

ii) $\int_1^e \frac{1}{x}dx = [lnx]_1^e$
$= \ln e - \ln 1 = 1 - 0 = 1$

b) $TR = \int MR.dx$
$TR = \int (28 - 6x - 3x^2).dx = 28x - \frac{6}{2}x^2 - \frac{3}{3}x^3 + c$

The constant of integration $c = 0$ because there is no revenue when there are no sales. Therefore $TR = 28x - 3x^2 - x^3$

c) Area under $f(x) = \int_1^2 (x^2 + 1)dx = \left[\frac{x^3}{3} + x\right]_1^2$
$= \left[\frac{2^3}{3} + 2\right] - \left[\frac{1^3}{3} + 1\right] = 4\frac{2}{3} - 1\frac{1}{3} = 3\frac{1}{3}$

Area under $g(x) = \int_1^2 (4x + 3)dx = [2x^2 + 3x]_1^2$
$= [2(2)^2 + 3(2)] - [2(1)^2 + 3(1)] = 14 - 5 = 9$

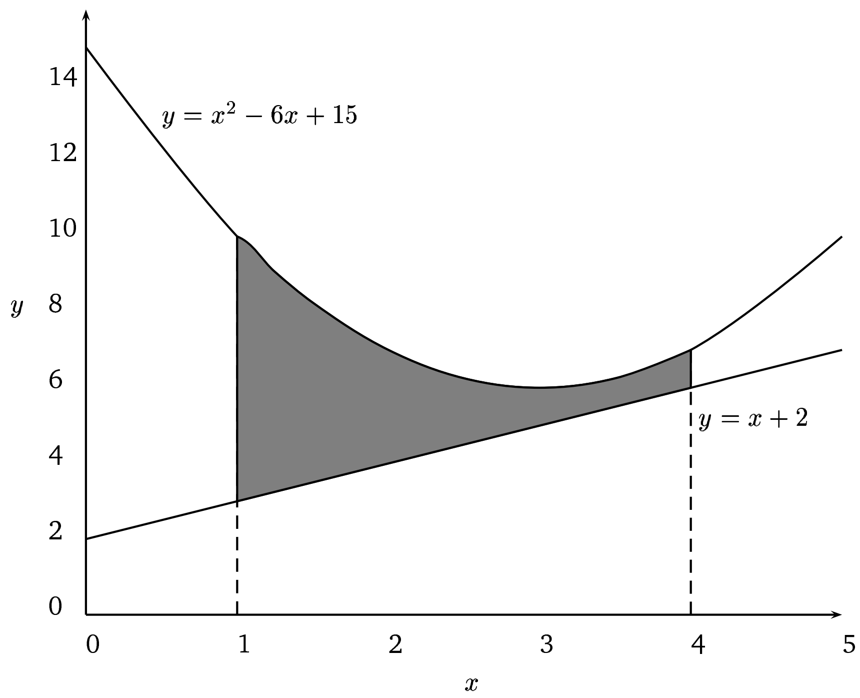
Therefore the area between the $f(x)$ and $g(x)$ is given by $9 - 3\frac{1}{2} = 5\frac{2}{3}$.

Question 2

a) i) $\int \frac{3}{x^2}dx = \int 3x^{-2}dx = -3x^{-1} + c = \frac{-3}{x} + c$

ii) $\int (5 + 3\sqrt{(x+1)})dx = 5x + 3\int (x+1)^{\frac{1}{2}}dx =$
$5x + 3(x+1)^{\frac{3}{2}} * (\frac{2}{3}) + c = 5x + 2\sqrt{(x+1)^3} + c$

b)



The shaded area is given by
$\int_1^4 (x^2 - 6x + 15).dx - \int_1^4 (x + 2).dx$

These can be combined to give $\int_1^4 (x^2 - 7x + 13).dx$
$= \left[\frac{1}{3}x^3 - \frac{7}{2}x^2 + 13x\right]_1^4$
$= \left[\frac{1}{3}(4)^3 - \frac{7}{2}(4)^2 + 13(4)\right] - \left[\frac{1}{3}(1)^3 - \frac{7}{2}(1)^2 + 13(1)\right]$
$= 17\frac{1}{3} - 9\frac{5}{6} = 7\frac{1}{2}$

Notes

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Notes

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