THIS PAPER IS NOT TO BE REMOVED FROM THE EXAMINATION HALLS

UNIVERSITY OF LONDON **CO3352 ZA**

BSc Examination

COMPUTING AND INFORMATION SYSTEMS, CREATIVE COMPUTING AND COMBINED DEGREE SCHEME

Operations Research and Combinatorial Optimisation

Date and Time: Wednesday 6 May 2015: 14.30 16.45

Duration: 2 hours 15 minutes

There are FIVE questions in this paper. Candidates should answer **FOUR** questions. All questions carry equal marks, and full marks can be obtained for complete answers to a total of **FOUR** questions. The marks for each part of a question are indicated at the end of the part in [.] brackets.

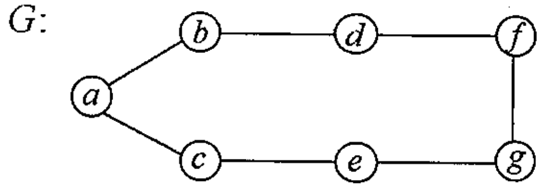
Only your first FOUR answers, in the order that they appear in your answer book, will be marked.

There are 100 marks available on this paper.

A hand held calculator may be used when answering questions on this paper but it must not be pre-programmed or able to display graphics text or algebraic equations. The make and type of machine must be stated clearly on the front cover of the answer book.

Question 1

A graph G is specified as shown in the following diagram



(a) Explain why G would be described as having **maximum degree 2**, a **Hamilton cycle** and **maximum path length 6**: [3]

(b) A **three-colouring** of this graph is an assignment of colours *red*, *blue* and *green* to the vertices such that no edge joins vertices of the same colour.

(i) Specify, either diagrammatically or by listing vertices, a three-colouring of the graph G. [2]

(ii) A simple algorithm can be specified to give any cycle either a two-colouring or a three-colouring without backtracking or look-ahead (i.e. referencing each edge and vertex at most once). Explain briefly what this algorithm might be. [3]

(iii) Suppose that it costs \$5 to colour a vertex *red* and \$10 to colour a vertex *blue* or *green*. Is it possible to give any cycle a *minimum cost* three-colouring without backtracking or look-ahead? Illustrate your answer using the graph G above. [4]

(iv) How would the Greedy Algorithm attempt to find a two-colouring of an even-length cycle (assuming the two colours have the same cost)? Explain why this approach will not work if cycle has length more than 4. [5]

(c) A subset X of vertices of a graph will be called **matchable** if there is a matching M for which every vertex in X belongs to an edge of M. It is known that maximum cardinality matchable sets can be found using the Greedy Algorithm.

(i) Explain briefly why the whole set of vertices of the graph G in part (a) cannot be matchable. [3]

(ii) Suppose that each vertex v of the graph G in part (a) is given a weighting $w(v)$ as follows:

$w(a)=5;\ w(b)=1;\ w(c)=4;\ w(d)=3;\ w(e)=6;\ w(f)=2;\ w(g)=3.$

Describe the steps by which the greedy algorithm would select a maximum-weight matchable set of vertices in G; give the total weight of the selected set and specify a matching which justifies that this set is matchable. [5]

Question 2

A matroid M is specified on the ground set $A = \{a, b, c, d, e\}$. Two independent sets of M are given as:

$$\{a, c, d\}, \ \{b, c, e\},$$

while neither $\{a, b\}$ nor $\{c, d, e\}$ is independent.

(a) Explain why

(i) every subset of M of cardinality 1 is an independent set of M. [1]

(ii) no independent set of M can have cardinality 4. [2]

(iii) the subset $\{b, c, d\}$ can be deduced to be independent. [4]

(b) The following matrix X is given as a representation of M over the real numbers:

$$X = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \end{array}.$$

(i) Explain how the matrix X confirms that $\{a, c\}$ is independent and that $\{c, d, e\}$ is **not** independent. [4]

(ii) Explain why the fact that bases have cardinality 3 means that there is a matrix representing M which has fewer rows than X. [2]

(iii) Find a representation X' for M having three rows. [4]

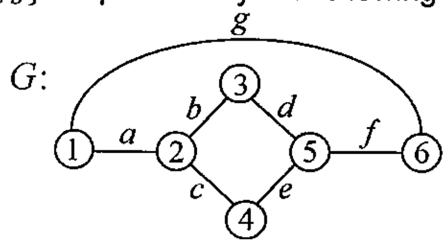
(c) Given that the matrix X in part (b) is a representation of M,

(i) draw a graph G on four vertices whose cycle matroid is isomorphic to M. [4]

(ii) draw a graph on five vertices, with no vertex having zero degree, whose cycle matroid is also isomorphic to M, and give the incidence matrix of this graph. [4]

Question 3

An undirected graph G with vertex set $V = \{1,2,3,4,5,6\}$ and edge set $E = \{a,b,c,d,e,f,g\}$ is specified by the following drawing:



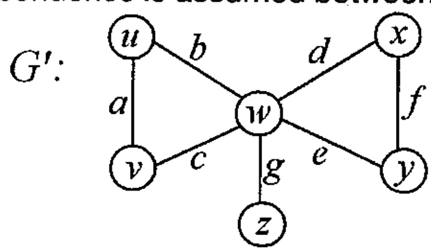
The use of matroid intersection to find maximum-length paths in G will be investigated in this question.

(a) Write a subset of E which constitutes a **maximum-length** cycle, and a subset which constitutes a **maximal** path which is not maximum-length. [4]

(b) Why does your answer to part (a) suggest that a single matroid cannot be used to maximise path length in G? [2]

(c) Write down two maximum independent sets in the cycle matroid of G, of which just one is a maximum-length path. [4]

(d) The graph G' shown below is defined over the same edge set E as graph G (but no correspondence is assumed between the vertex sets of G and G'):



(i) Write down a maximum independent set in the cycle matroid of G' [2]

(ii) Explain why a set of edges which is simultaneously independent in the cycle matroids of both G and G' will constitute a collection of paths in G. [2]

(e) Let D be the matrix diag(a,b,c,d,e,f,g) whose only nonzero elements are the diagonal elements which are assigned the names of the edges of G.

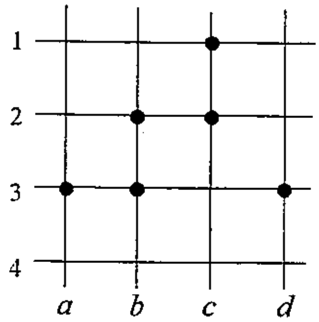
(i) Write down the incidence matrix B of G and the incidence matrix B' of G'. [3]

(ii) Construct the Binet-Cauchy product $\Phi = B \times D \times (B')^T$. [3]

(iii) By taking an appropriate 4×4 submatrix X of Φ and using an appropriate example, explain how $\det X$ enumerates all maximum-length paths in G. [5]

Question 4

The diagram below shows the grid locations of a network of six radio transmitters



these locations being referenced as $a3, b2, b3, c1, c2$ and $d3$. The grid squares measure $1km \times 1km$.

We will consider two transmitters to be 'adjacent' if they are within a range of $2km$ of each other (e.g. $a3$ is $\sqrt{2} < 2km$ from $b2$ but $\sqrt{5} > 2km$ from $c2$).

(a) Model the network adjacencies as an undirected graph G whose vertices correspond to the labelled locations. [4]

(b) It is required to test the radio signal between every pair of transmitters that are within range of each other. We want to find a minimum set of transmitters to carry out this test.

(i) Explain how this corresponds to a **vertex cover** of the graph G in part (a). [2]

(ii) Give an example of a minimum vertex cover for G and one which is minimal but not minimum. [4]

(c) The problem of finding a minimum-size vertex cover for the graph G in part (a) is to be solved as an integer linear programme in which the six vertices $c1, b2, c2, a3, b3$ and $d3$ of G are used as variables.

(i) Explain what the constraint $a3+b2 \geq 1$ represents in respect of a vertex cover of G. [2]

(ii) Give a complete set of constraints in the variables $a3, b2, b3, c1, c2$ and $d3$ such that an assignment of nonnegative integer values to these variables will represent a vertex cover if and only if these values satisfy all the constraints. [4]

(iii) By representing the constraints in part (ii) as a matrix A and by writing down a suitable linear objective function in terms of $x = (a3, b2, b3, c1, c2, d3)$, specify the minimum cover problem for G as an integer linear programme. [4]

(iv) Explain the purpose of taking the **linear programming relaxation** of your integer linear programme. Show that this relaxation will **not** yield a valid solution to the minimum vertex cover problem for G. [5]

Question 5

A new company is planning to launch with some or all of four products W, X, Y and Z. The expected income, per million units, during the first trading year is projected to be:

$$W : \$3m,\ X : \$7m,\ Y : \$5m,\ Z : \$2m.$$

Manufacturing costs per million units are given as:

$$W : \$1.2m,\ X : \$2.1m,\ Y : \$1.0m,\ Z : \$1.1m.$$

Selling costs (storage, shipping commission etc) per million units are given as:

$$W : \$0.2m,\ X : \$0.7m,\ Y : \$0.5m,\ Z : \$0.1m.$$

There is a maximum manufacturing budget of \$10m and for selling of \$3m.

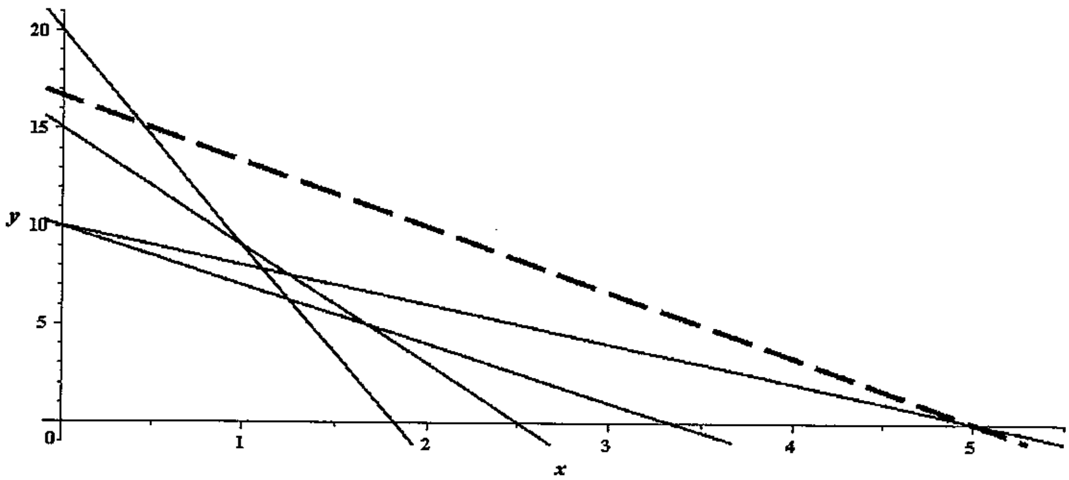
Linear programming is to be used to maximise income from the four products, subject to the given constraints.

(a) Write down the appropriate linear programme in the four variables W, X, Y and Z. [6]

(b) Supposing the company decides to manufacture and sell an equal number, K millions, of each product. Explain why the value $K = 1.8$ yields a feasible vector for the linear programme, while the value $K = 1.9$ does not yield a feasible vector. [4]

(c) Write down the dual programme in the two new variables x and y. [5]

(d) The constraints of the dual programme are plotted as shown in the graph below, together with a dotted straight line which represents one possible value of the objective function.



Explain briefly how the dotted line will identify a vertex of the constraint polyhedron for the dual programme which optimises this programme. Give the value of this programme. [7]

(e) Explain how linear programming duality now establishes an optimal value for the maximum income for the company and show that this value may be achieved by manufacturing only products Y and Z. [3]

END OF PAPER