THIS PAPER IS NOT TO BE REMOVED FROM THE EXAMINATION HALLS

UNIVERSITY OF LONDON **CO3352 ZB**

BSc Examination

COMPUTING AND INFORMATION SYSTEMS and CREATIVE COMPUTING

Operations Research and Combinatorial Optimisation

Date and time: Wednesday 4 May: 14.30 – 16.45
Duration: 2 hours 15 minutes

There are FIVE questions on this paper. Candidates should answer **FOUR** questions. All questions carry equal marks, and full marks can be obtained for complete answers to a total of **FOUR** questions. The marks for each part of a question are indicated at the end of the part in [.] brackets.

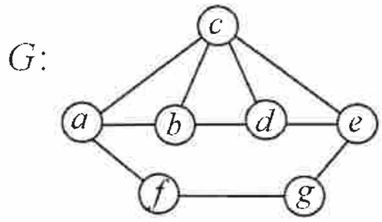
Only your first **FOUR** answers, in the order that they appear in your answer book, will be marked.

There are 100 marks available on this paper.

A handheld calculator may be used when answering questions on this paper but it must not be pre-programmed or able to display graphics text or algebraic equations. The make and type of machine must be stated clearly on the front cover of the answer book.

Question 1

A graph G is specified as shown in the following diagram



(a) Explain why G would be described as having **maximum degree 4**, a **Hamilton cycle** and **maximum path length 6**: [5]

(b) A **three-colouring** of this graph is an assignment of colours *red*, *blue* and *green* to the vertices such that no edge joins vertices of the same colour.

(i) Specify, either diagrammatically or by listing vertices, a three-colouring of the graph G. [4]

(ii) Suppose that it costs \$5 to colour a vertex *red*, \$10 to colour a vertex *blue* and \$15 to colour a vertex *green*. How might the Greedy Algorithm successfully find a minimum-cost three-colouring of G? Is this approach guaranteed to work? Justify your answer. [6]

(c) A subset X of vertices of a graph will be called **matchable** if there is a matching M for which every vertex in X belongs to an edge of M. It is known that maximum cardinality matchable sets can be found using the Greedy Algorithm.

(i) Explain briefly why the whole set of vertices of the graph G cannot be matchable. [3]

(ii) Suppose that each vertex v of the graph G in part (a) is given a weighting $w(v)$ as follows:

$$w(a) = 5; w(b) = 2; w(c) = 4; w(d) = 3; w(e) = 6; w(f) = 1; w(g) = 2.$$

Describe the steps by which the greedy algorithm would select a maximum-weight matchable set of vertices in G; give the total weight of the selected set and specify a matching which justifies that this set is matchable. [7]

Question 2

(a) Three matrices A, B and C are given as:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}.$$

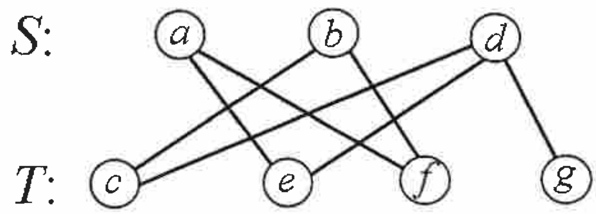
(i) Calculate

1. BA; 2. $\frac{1}{2}BB^T + C$; 3. $detC$. [5]

(ii) Give the row-echelon form of each of these three matrices and hence state their ranks. [5]

(iii) Each of the matrices A, B and C represents a matroid, with subsets of columns being independent if and only if the corresponding vectors are linearly independent. Write down a maximal independent set for each matroid. [3]

(b) A matroid is defined on the edge set E of a bipartite graph G, with vertex partition S and T by saying that a subset of E is independent if and only if no two edges in E share an end vertex in S. For the graph G shown below:



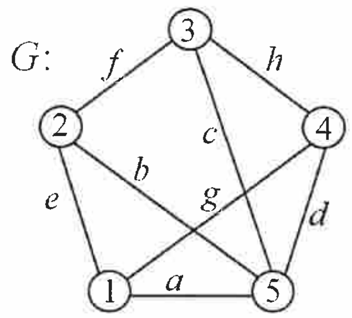
(i) explain why $\{ae, bc\}$ is an independent set but $\{ae, af\}$ is not an independent set; [4]

(ii) given that $X = \{ae, bc\}$ is independent and that $Y = \{af, bf, dc\}$ is also independent, what property of the pair X, Y does the Steinitz Exchange Lemma assert? Give an example of how this produces an independent set of size 3, starting with the set X; [4]

(iii) suppose that we try to create a new matroid in which independent sets are subsets of E in which no two edges share an end vertex in S or T. By finding suitable edge subsets in the above graph, show that this fails to define the independent sets of a matroid. [4]

Question 3

An undirected graph G with vertex set $V = \{1,2,3,4,5\}$ and edge set $E = \{a,b,c,d,e,f,g,h\}$ is specified by the following drawing:



The application of cycle and cocycle matroids to finding maximum-length paths in G will be investigated in this question.

(a) Write a subset of E which is a **spanning tree** of G but which fails to be a **non-cut** (i.e., deleting the edges of E will cut the graph into two or more connected components.) [3]

(b) Write down a subset of E of size 4 which is a non-cut but which fails to be a spanning tree. [3]

(c) Write down a subset of E of which is simultaneously a spanning tree and a non-cut. [3]

(d) Explain briefly why a spanning tree of G which is also a non-cut and in which the degree of vertex 5 is at most 2 must be a path. Give an example of such a path. [5]

(e) Matrices B and B^* representing the cycle matroid and the cocycle matroids of G, respectively, are given below:

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{pmatrix} \end{array}, \quad B^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Let D be the matrix diag(a,b,c,d,e,f,g,h) whose only nonzero elements are the diagonal elements which are assigned the names of the edges of G.

(i) Construct the Binet-Cauchy product $\Phi = B \times D \times (B^*)^T$ [5]

(ii) Explain how $\det \Phi$ can be used to identify **eight** paths of length 4 in G and write down these paths [6]

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Question 4

The knapsack problem in combinatorial optimization is the following:

> Given a set of items, each having a size and a value, and a limit L, to choose a subset of the items with total size at most L and having maximum total value.

For example, suppose $L = 5$ and that A, B and C have the sizes and values shown on the right. Then the subset $\{A, C\}$ has total size $2 + 3 = 5 \leq L$ and total value $6 + 7 = 13$; and the subset $\{B\}$ is also an optimal solution, having total size $4 \leq L$ and the same total value 13 as $\{A, C\}$.

	A	B	C
size	2	4	3
value	6	13	7

(a) Find an optimal solution to the knapsack problem specified below, given the limit $L = 12$. [5]

	A	B	C	D	E	F
size	4	6	1	5	2	7
value	7	5	3	3	4	6

(b) Suppose $x_1, x_2, \ldots, x_6$ are six integer variables taking values in the set $\{0, 1\}$. Represent the optimisation goal of the knapsack problem of part (a) as an objective function in the six variables. [4]

(c) Using the same six variables as part (b), use an inequality to represent the size limit on choice of items. [3]

(d) Explain briefly how your solution to part (a) constitutes an optimal integer solution to the integer linear programme specified by your answers to parts (b) and (c) [4]

(e) Deciding whether a given instance of the knapsack problem has a solution whose total value exceeds some required target is NP-Complete.

  (i) Explain briefly why an integer linear programming representation of the knapsack problem does not provide a polynomial-time algorithm for finding optimal solutions to the problem. [3]

  (ii) Explain what is meant by saying that the variable values $x_1 = x_3 = x_5 = 1, x_2 = x_4 = 0$ and $x_6 = 5/7$ solve the **linear relaxation** of the integer linear programme from parts (b)-(d). Give the value of the objective function for these variable values and explain why this does not constitute a valid solution to the given instance of the knapsack problem. [6]

Question 5

(a) Three vectors in $\mathbb{R}^2$ are given as follows:

$$v_1 = (1,3), v_2 = (4,7), v_3 = (6,1).$$

(i) Sketch on the xy-axes the convex hull of these points. [4]

(ii) Suppose that the convex hull in part (i) is defined by three inequalities. If these inequalities are the constraints of a linear programme then what can we say about the optimal value of this linear programme? [3]

(b) A linear program with four basic variables x_1, x_2, x_3 and x_4 is given as:

$$\begin{array}{lrcl} \textbf{minimise} & 2x_1 + x_2 + 3x_3 + 2x_4 & & \\ \textbf{subject to} & x_1 - 3x_3 & \geq & 1 \\ & x_1 + 3x_3 - x_4 & \geq & 2 \\ & x_1 + x_2 + x_3 + x_4 & \geq & 2 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

(i) State the Duality Theorem of linear programming. [3]

(ii) An optimal solution to this linear programme is given by

$$x_1 = \frac{3}{2}, \quad x_2 = \frac{1}{3}, \quad x_3 = \frac{1}{6}, \quad x_4 = 0\,.$$

Show that these values satisfy the constraints of the linear programme. [4]

(iii) Give one other set of values for x_1, x_2, x_3 and x_4 which satisfies the constraints but which fails to optimise the linear programme. [4]

(iv) Give the dual of the given linear programme and give the value of an optimal solution of this dual. [7]

END OF PAPER