THIS PAPER IS NOT TO BE REMOVED FROM THE EXAMINATION HALLS

UNIVERSITY OF LONDON **CO3352 ZA**

BSc Examination

COMPUTING AND INFORMATION SYSTEMS and CREATIVE COMPUTING

Operations Research and Combinatorial Optimisation

Date and time: Wednesday 4 May: 14.30 – 16.45
Duration: 2 hours 15 minutes

There are FIVE questions on this paper. Candidates should answer **FOUR** questions. All questions carry equal marks, and full marks can be obtained for complete answers to a total of **FOUR** questions. The marks for each part of a question are indicated at the end of the part in [.] brackets.

Only your first **FOUR** answers, in the order that they appear in your answer book, will be marked.

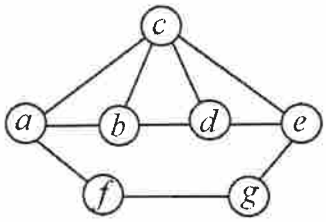
There are 100 marks available on this paper.

A handheld calculator may be used when answering questions on this paper but it must not be pre-programmed or able to display graphics text or algebraic equations. The make and type of machine must be stated clearly on the front cover of the answer book.

Question 1

A graph G is specified as shown in the following diagram

G:



(a) Explain why G would be described as having **maximum degree 4**, a **Hamilton cycle** and **maximum path length 6**: [5]

(b) A **three-colouring** of this graph is an assignment of colours *red*, *blue* and *green* to the vertices such that no edge joins vertices of the same colour.

(i) Specify, either diagrammatically or by listing vertices, a three-colouring of the graph G. [4]

(ii) Suppose that it costs \$5 to colour a vertex *red*, \$10 to colour a vertex *blue* and \$15 to colour a vertex *green*. How might the Greedy Algorithm successfully find a minimum-cost three-colouring of G? Is this approach guaranteed to work? Justify your answer. [6]

(c) A subset X of vertices of a graph will be called **matchable** if there is a matching M for which every vertex in X belongs to an edge of M. It is known that maximum cardinality matchable sets can be found using the Greedy Algorithm.

(i) Explain briefly why the whole set of vertices of the graph G cannot be matchable. [3]

(ii) Suppose that each vertex v of the graph G in part (a) is given a weighting $w(v)$ as follows:

$$w(a) = 5; w(b) = 2; w(c) = 4; w(d) = 3; w(e) = 6; w(f) = 1; w(g) = 2.$$

Describe the steps by which the greedy algorithm would select a maximum-weight matchable set of vertices in G; give the total weight of the selected set and specify a matching which justifies that this set is matchable. [7]

Question 2

A matroid M is specified on the ground set $A = \{a, b, c, d, e\}$. Three independent sets of M are given as

$$\{a, c\}, \{b, d\}, \{c, d, e\},$$

and three sets which are **not** independent are given as:

$$\{a, b\}, \{b, e\}, \{a, e\},$$

(a) Explain why
- (i) every subset of M of cardinality 1 is an independent set of M. [1]
- (ii) no independent set of M can have cardinality 4. [2]
- (iii) the subset $\{b, c, d\}$ can be deduced to be independent. [4]

(b) The following matrix X is given as a representation of M over the real numbers:

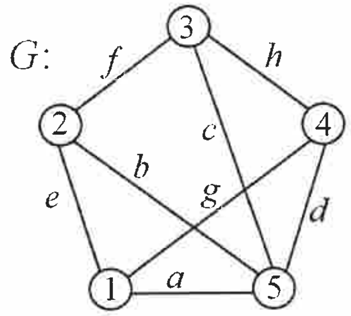
$$X = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 & 1 \end{pmatrix} \end{array}$$

- (i) Explain how the matrix X confirms that $\{a, c\}$ is independent and that $\{a, c, e\}$ is **not** independent. [4]
- (ii) Explain why the fact that bases have cardinality 3 means that there is a matrix representing M which has fewer rows than X. [2]
- (iii) Find a representation X' for M having three rows. [2]

(c) Using either of the matrices X or X' from part (b),
- (i) reduce the matrix to row echelon form; [6]
- (ii) for the resulting matrix, select a subset of the rows and reorder the columns (including columns headings *a, b, c, d, e*), to obtain a representation of M in standard form $[I_n|A]$ where A is a 3 x 2 matrix; [2]
- (iii) write down a matrix, with columns labelled from the set $\{a, b, c, d, e\}$, representing the dual matroid M^* of M. [2]

Question 3

An undirected graph G with vertex set $V = \{1,2,3,4,5\}$ and edge set $E = \{a,b,c,d,e,f,g,h\}$ is specified by the following drawing:



The application of cycle and cocycle matroids to finding maximum-length paths in G will be investigated in this question.

(a) Write a subset of E which is a **spanning tree** of G but which fails to be a **non-cut** (i.e., deleting the edges of E will cut the graph into two or more connected components.) [3]

(b) Write down a subset of E of size 4 which is a non-cut but which fails to be a spanning tree. [3]

(c) Write down a subset of E of which is simultaneously a spanning tree and a non-cut. [3]

(d) Explain briefly why a spanning tree of G which is also a non-cut and in which the degree of vertex 5 is at most 2 must be a path. Give an example of such a path. [5]

(e) Matrices B and B^* representing the cycle matroid and the cocycle matroids of G, respectively, are given below:

$$B = \begin{array}{cc} & \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{pmatrix} \end{array}, \quad B^* = \begin{array}{cc} & \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Let D be the matrix diag(a,b,c,d,e,f,g,h) whose only nonzero elements are the diagonal elements which are assigned the names of the edges of G.

(i) Construct the Binet-Cauchy product $\Phi = B \times D \times (B^*)^T$ [5]

(ii) Explain how $\det \Phi$ can be used to identify **eight** paths of length 4 in G and write down these paths [6]

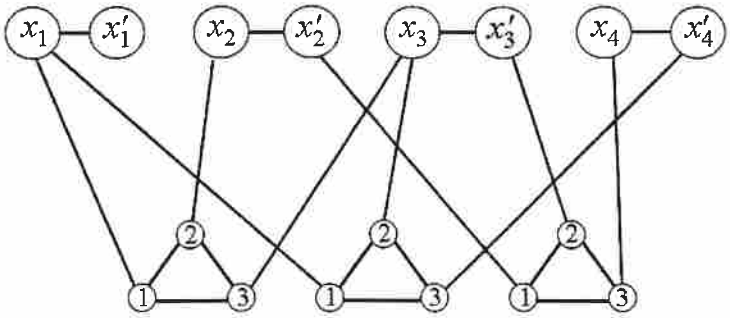
Question 4

An instance of 3-SAT is given as follows: there are four variables, x_1, x_2, x_3 and x_4, and three clauses:

$$(x_1, x_2, x_3), (x_1, x_3, x_4'), (x_2', x_3', x_4).$$

(a) An assignment $x_1 = x_3 = F, x_2 = x_4 = T$ is given for this instance of 3-SAT. Explain why this is **not** a satisfying truth assignment. A second assignment $x_1 = x_2 = F, x_3 = x_4 = T$ is given; explain why this assignment is satisfying.[6]

(b) The following graph makes a transformation of 3-SAT into vertex cover for the three clauses given above:



(x_1, x_2, x_3) (x_1, x_3, x_4') (x_2', x_3', x_4)

(i) Say what is meant by a vertex cover of a graph. Specify a lower bound for the size of a minimum cover for the above graph? [4]

(ii) Explain how the first truth assignment in part (a) maps to a non-minimum vertex cover of this graph. Explain how the second truth assignment maps to a minimum vertex cover, and give this cover. [4]

(c) Write down an integer linear programme which solves the given instance of 3-SAT. Show that the first truth assignment from part (a) is not a feasible solution to this programme but that the second assignment is feasible. [7]

(d) Explain why the Simplex Algorithm will not necessarily solve an instance of 3-SAT when it is applied to the integer linear programme of part (c). [4]

Question 5

(a) Three vectors in $\mathbb{R}^2$ are given as follows:

$$v_1 = (1,1), v_2 = (4,2), v_3 = (2,5).$$

(i) Sketch on the xy-axes the convex hull of these points. [4]

(ii) Suppose that f is a linear function of x and y and that $f(1,1) > f(4,2) > f(2,5)$ What can we deduce about the maximum and minimum values taken by f on the convex hull in part (i)? [3]

(b) A linear program with four basic variables x_1, x_2, x_3 and x_4 is given as:

$$\begin{array}{ll} \textbf{Maximize} & 3x_1 + x_2 + 5x_3 + x_4 \\ \textbf{subject to} & \begin{array}{rl} x_1 - x_2 - 3x_3 & \leq 1 \\ x_1 + 3x_3 - 2x_4 & \leq 2 \\ x_2 + 2x_3 + 2x_4 & \leq 2 \\ x_1, x_2, x_3, x_4 & \geq 0 \end{array} \end{array}$$

(i) What is meant by saying this linear programme is *feasible*? [3]

(ii) An optimal solution to this linear programme is given by

$$x_1 = \frac{5}{2}, \quad x_2 = \frac{3}{2}, \quad x_3 = 0, \quad x_4 = \frac{1}{4}.$$

Show that these values satisfy the constraints of the linear programme. [4]

(iii) Give one other set of values for x_1, x_2, x_3 and x_4, not all zero, which satisfies the constraints but which fails to optimise the linear programme. [4]

(iv) Give the dual of the given linear programme and give the value of an optimal solution of this dual. [7]

END OF PAPER