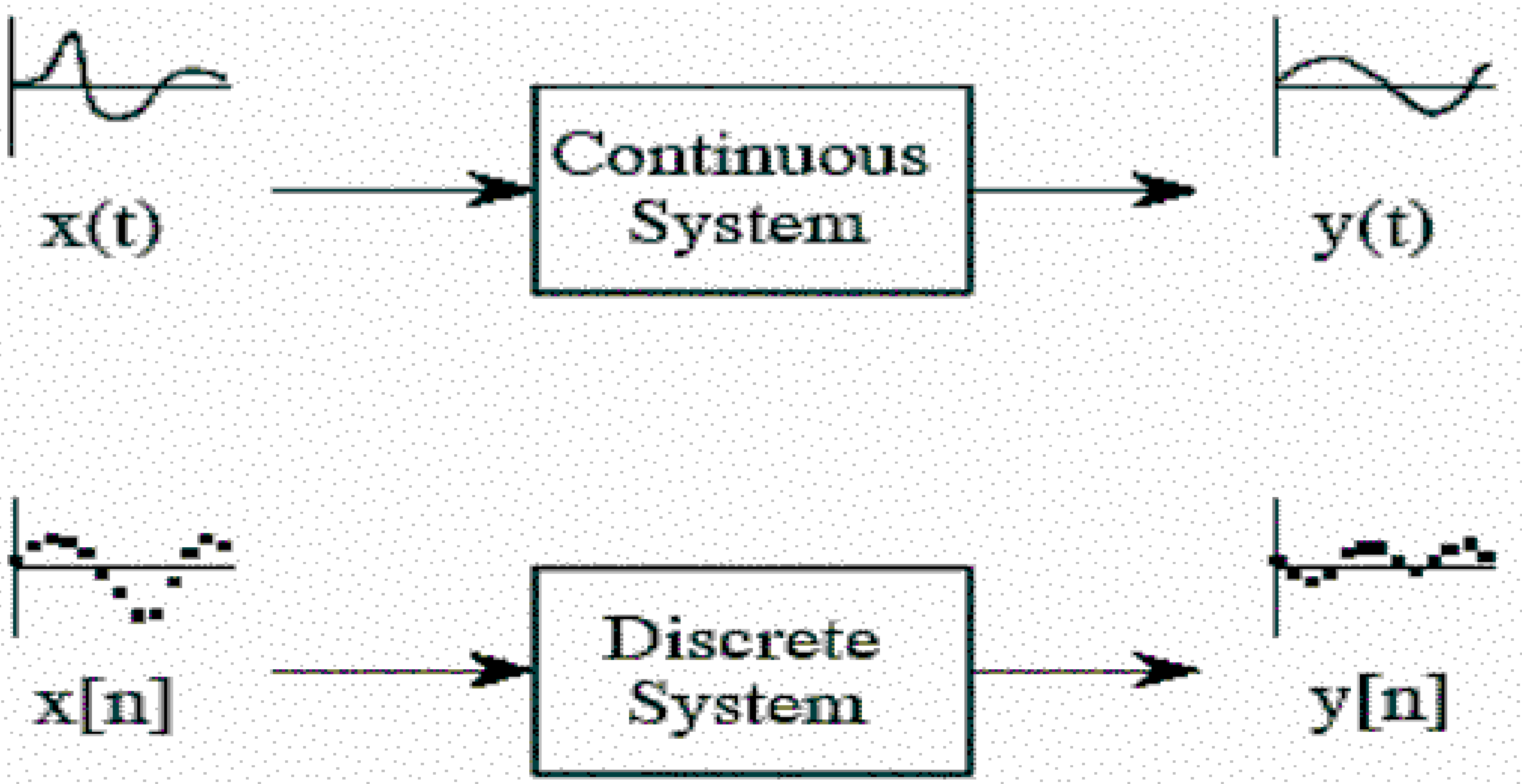
Linear Systems

The Engineer's Toolbox

A System

- **A system is any process that produces an *output signal* in response to an *input signal*.**



Reasons for understanding a *system*

- Remove noise
- Sharpen an out-of-focus image
- Remove echoes in an audio recording
- Characterize or Measure distortion or interfering effects
- Study or analyze physical processes of the system

Requirements for Linearity

- ***Temporal invariance* – Bell struck in the same way on different days**
- ***Spatial invariance* – Bell moved from Harare to Bulawayo**

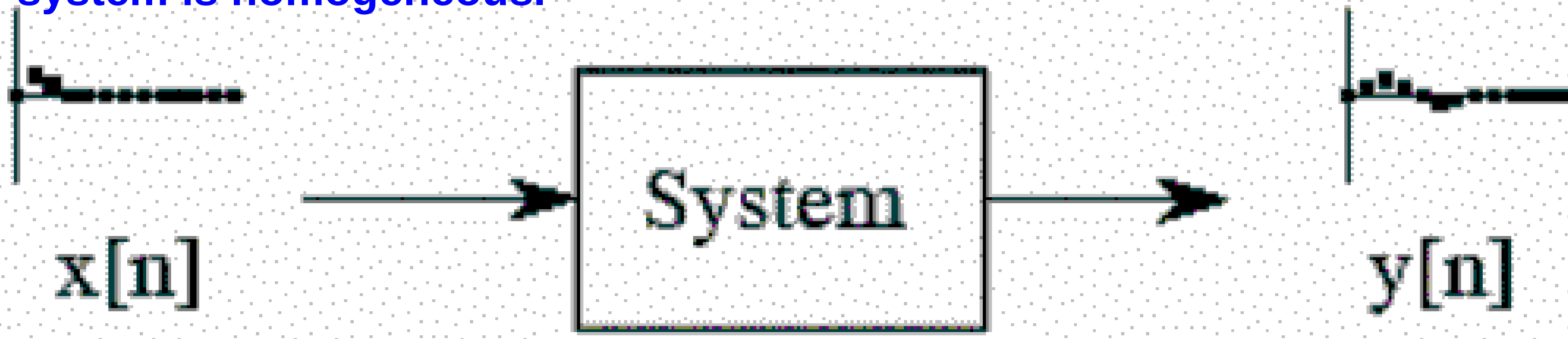
Mathematical properties

- **Homogeneity/Scaling**
- **Additivity**
- **Shift invariance**
- **Static Linearity**
- **Memoryless**
- **Sinusoidal Fidelity**
- **Zero-in/zero-out**
- **Causality**
- **Stability**
- ***Uniqueness***
- ***Invertibility***

Homogeneity/Scaling

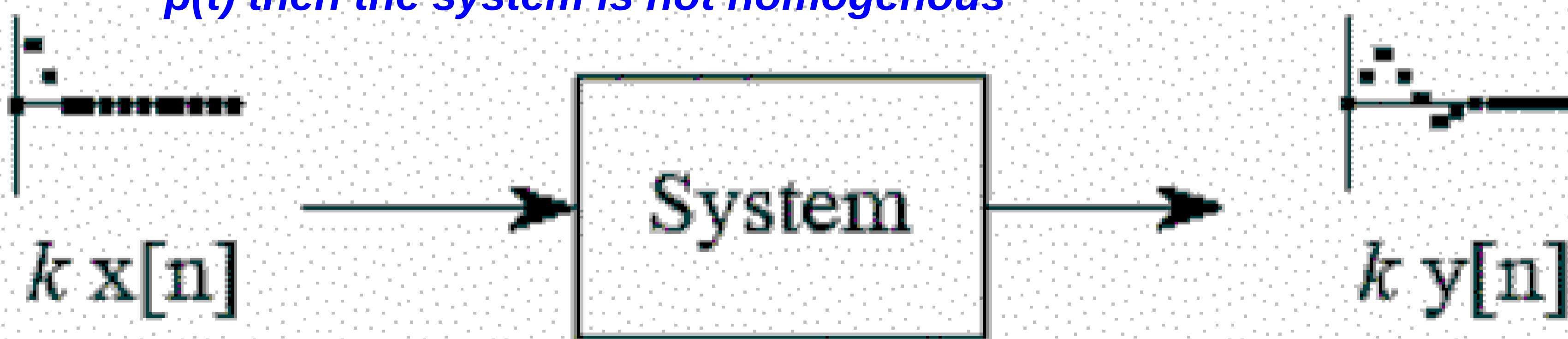
IF

If the input to the system is the voltage across the resistor, *v(t)*, and the output from the system is the current through the resistor, *i(t)* , the system is homogeneous.



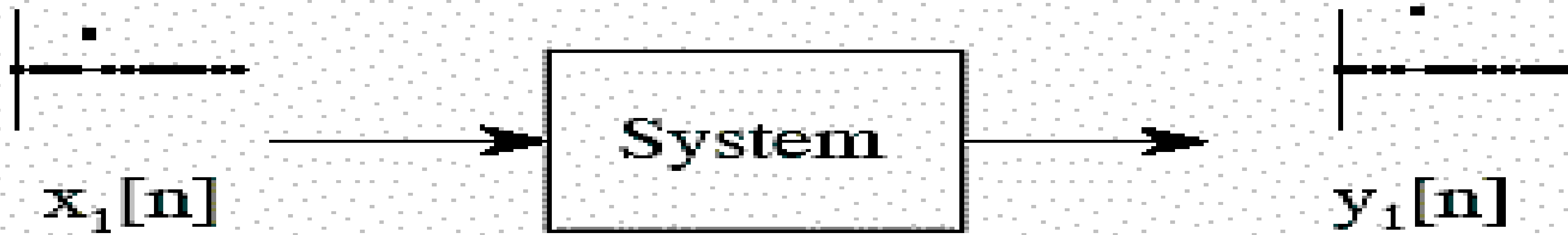
THEN

If the input signal is the voltage across the resistor, *v(t)*, but the output signal is the power being dissipated in the resistor, *p(t) then the system is not homogenous*



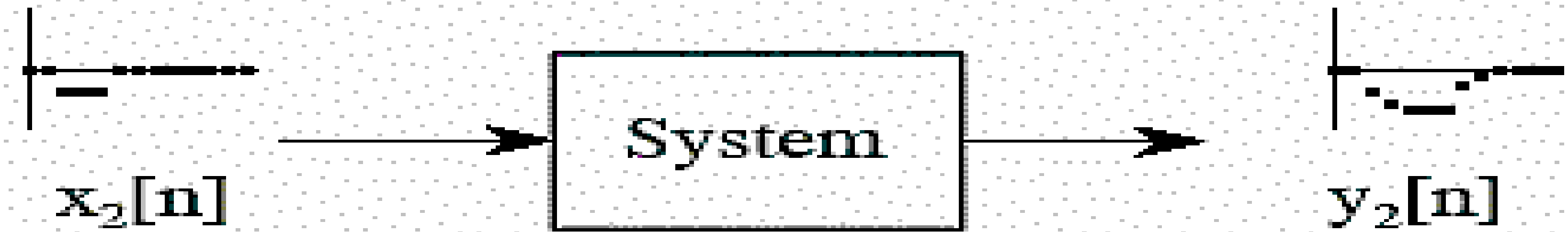
Additivity

IF



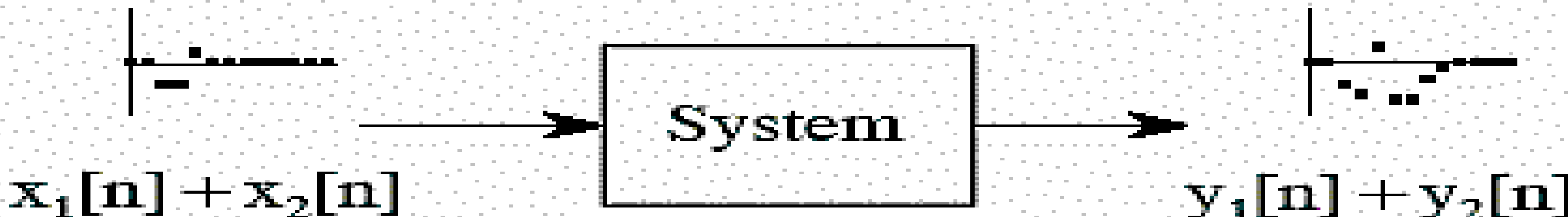
The important point is that *added* signals pass through the system without interacting.

AND IF



THEN

A good example of a *nonadditive* circuit is the mixer stage in a radio transmitter.



Generalised Additivity and Homogeneity Property

- **The additive and homogeneity properties can be generalized to the superposition of many inputs:**

If

$$x[n] = \sum_k a_k x_k[n]$$

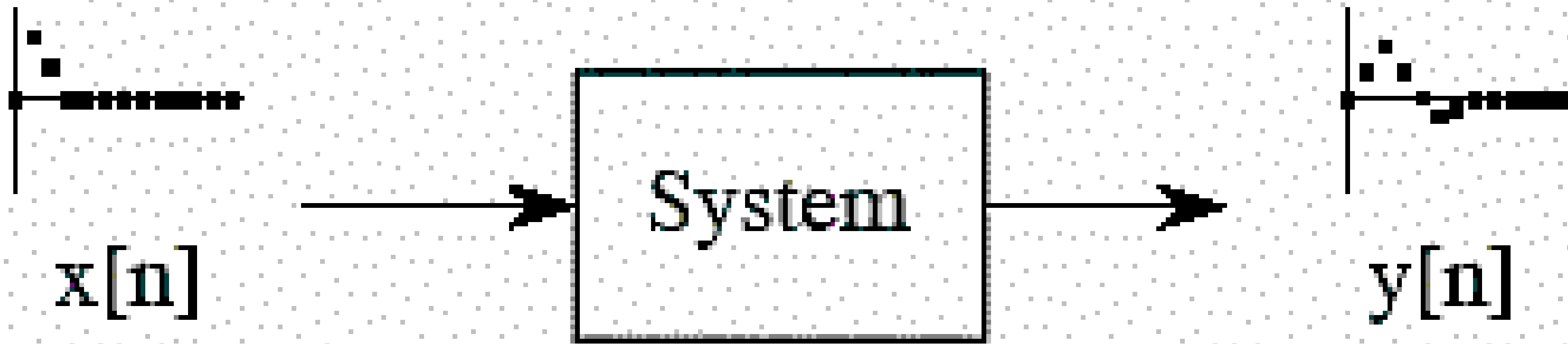
Then

$$y[n] = \sum_k a_k y_k[n]$$

Shift invariance

IF

Means that a shift in the input signal will result in nothing more than an identical shift in the output signal.

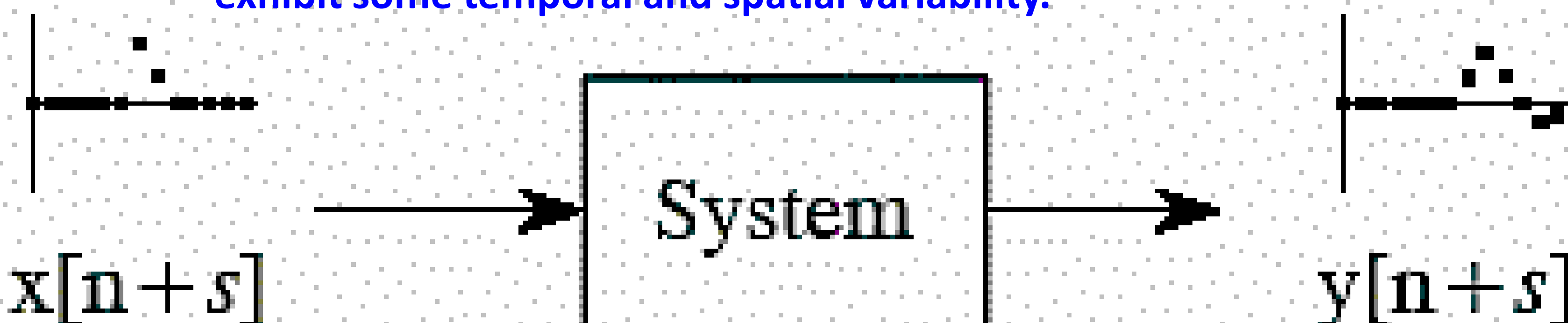


If the index *n* is associated with time, shift- invariance corresponds to time-invariance

Shift invariance is important because it means the characteristics of the system do not change with time (or whatever the independent variable happens to be) nor is it dependent upon its position in space.

THEN

In the strictest sense, almost all supposedly linear physical systems exhibit some temporal and spatial variability.



Homogeneity, additivity, and Shift invariance

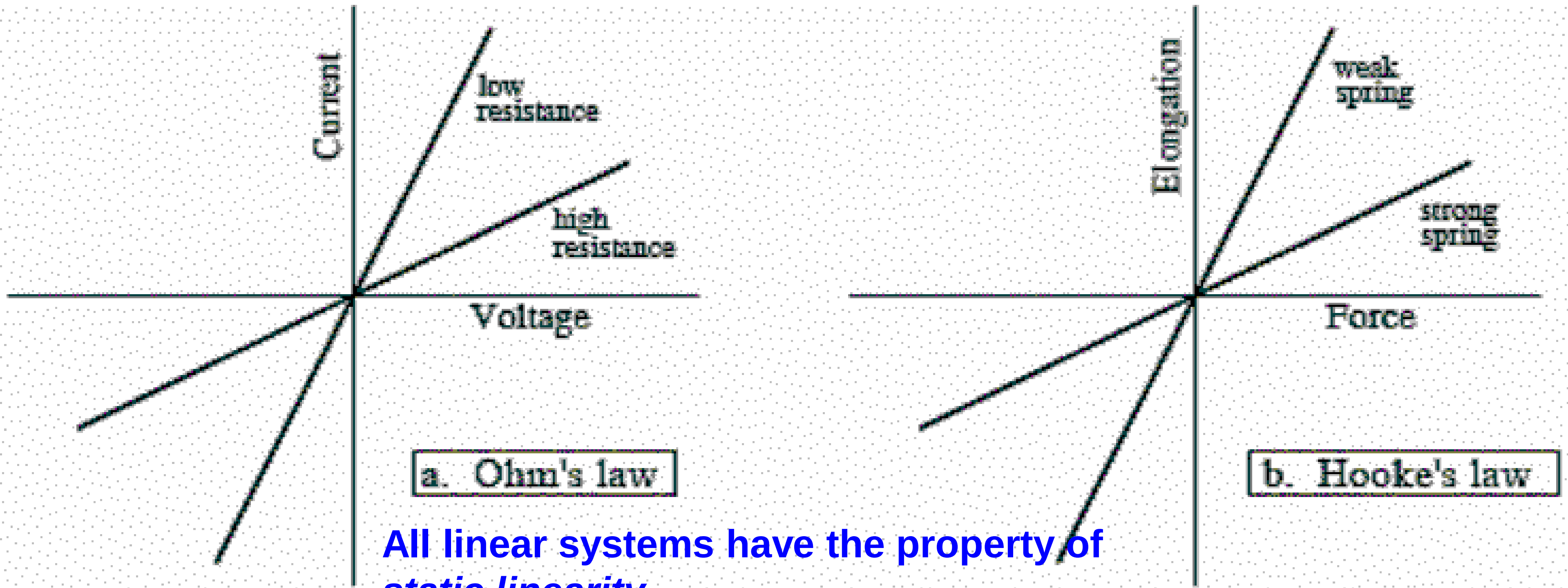
- Examples of physical systems that are linear (or nearly so) include:
  - Acoustic musical instruments,
  - Audio amplifiers,
  - Ultrasonic transducers,
  - Electronic filters,
  - The acoustic properties of a room,
  - Pendulums,
  - Springs and suspension systems, etc.
- Note that we put in the caveat *or nearly so*, because all of them are associated with small non-linearities that can for the most part be ignored when describing their behaviours
- linear system need not necessarily be a *physical* thing – it could, for example, be *conceptual*, such as in the case of a mathematical process, transform or algorithm.

Homogeneity, additivity, and Shift invariance

- Homogeneity, additivity, and shift invariance provide the mathematical basis for defining linear systems.
- Unfortunately, these properties alone don't provide most scientists and engineers with an intuitive feeling of what linear systems are about.
- The properties of static linearity and sinusoidal fidelity are often of help here.
- These are not especially important from a mathematical standpoint, but relate to how humans think about and understand linear systems.

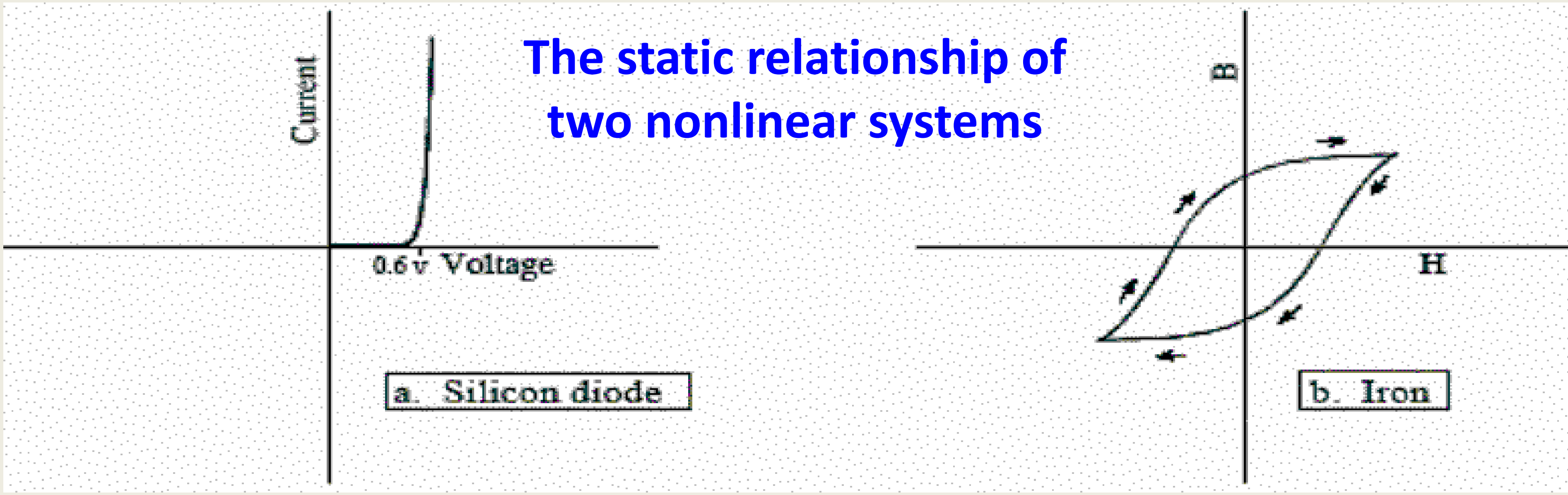
Static Linearity

- **Static linearity defines how a linear system reacts when the signals aren't changing, i.e., when they are *DC* or *static*.**
- **The static response of a linear system is very simple: *the output is the input multiplied by a constant*.**
- **Fig. Below for two common linear systems: Ohm's law for resistors, and Hooke's law for springs.**



All linear systems have the property of *static linearity*.

Static Linearity



- Although all linear systems have the property of *static linearity*.
- The opposite is usually true, but not always.
- There are systems that show static linearity, but are not linear with respect to changing signals.
- However, a very common class of systems can be completely understood with static linearity alone.
- In these systems it doesn't matter if the input signal is static or changing. These are called memoryless systems, because the output depends only on the present state of the input, and not on its history.

Memoryless

- A system is memoryless if output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value.
- For example, the instantaneous current in a resistor depends only on the instantaneous voltage across it, and not on how the signals came to be the value they are.
- If a system has static linearity, and is memoryless, then the system must be linear.
- With continuous-time systems, a resistor is memoryless:

$$y(t) = Rx(t)$$

- Example of a continuous-time system with memory is a capacitor:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

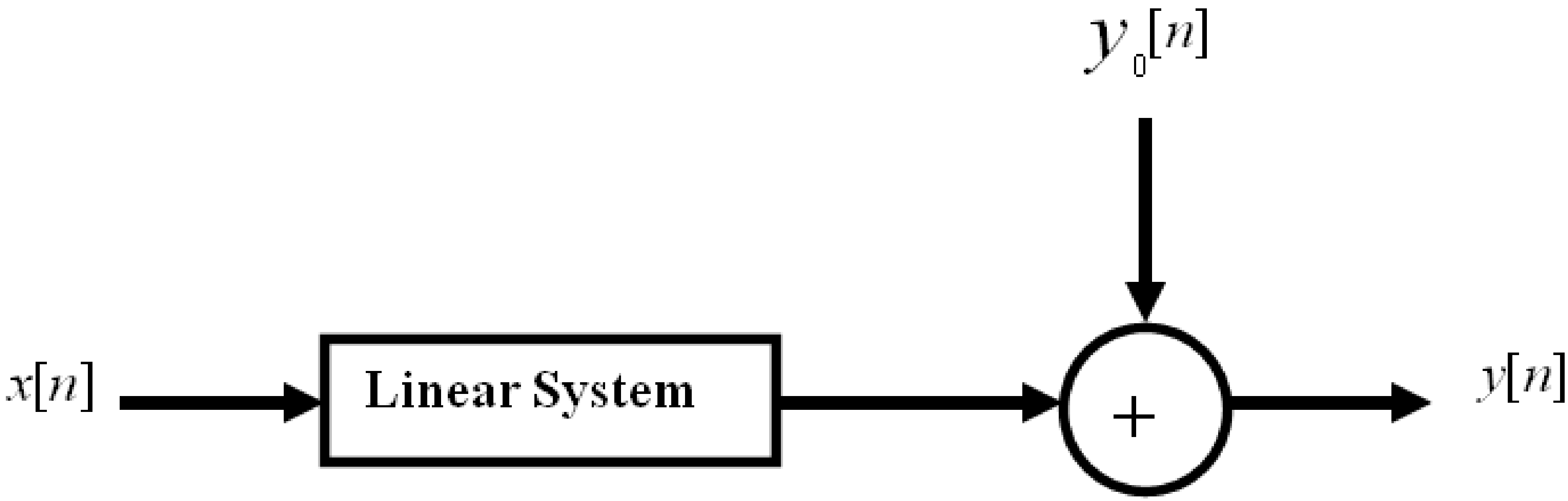
- All analogue electronic systems are band limited and so they effectively store energy in the same way a capacitor does, releasing it over a time longer than it took to acquire it.
- Only digital systems can be perfectly memoryless.

Sinusoidal Fidelity/Frequency Preservation

- **If the input to a linear system is a sinusoidal wave, the output will also be a sinusoidal wave, and containing the same frequencies as the input.**
- **Sinusoids are the only waveforms that have this property.**
- **For instance, there is no reason to expect that a square wave entering a linear system will produce a square wave on the output.**
- **Although a sinusoid on the input guarantees a sinusoid on the output, the two may be different in amplitude and phase.**
- **If a system always produces a sinusoidal output in response to a sinusoidal input, is the system guaranteed to be linear?**
- **The answer is no, but the exceptions are rare and usually obvious.**
- ***Phase locked loops* are a good example.**

Zero-in/zero-out

Consider a system: $y[n] = 2x[n] + 3$



- The system is not linear since if the system violets the zero-in/zero-out property of linear systems.
- The system above is called an incremental linear system
- The added signal is the zero input response of the overall system.

Additional Constraints

- **We have shown that Homogeneity, Additivity, and Shift-Invariance are important because they provide the mathematical basis for defining linear systems.**
- **The additional constraint of causality and stability define a more restricted class of linear time-invariant systems of practical importance.**

Causality

- **A system is causal if the output at any given time depends only on values of the input at the present time and in the past.**
- **The system is non-anticipative, as the system output does not anticipate future values of the input.**
- **At first glance such a property might appear self-evident – how could any system predict the future?**
- **In fact, the possibility is not as improbable as one might think, and is regularly encountered in off-line DSP systems.**
- **The motion of a vehicle is causal since it does not anticipate future actions of the driver.**
- ***Examples of causal systems:***

$$y(t) = x(t-1)$$

$$y(t) = \frac{1}{c}\int_{-\infty}^{t} x(\tau)d\tau$$

$$y[n] = x[n] - x[n-1]$$

- ***Examples of non-causal systems:***

$$y(t) = x(t+1)$$

$$y[n] = x[n+1] - x[n]$$

$$y[n] = x[n] - x[n+1]$$

Stability

- **A system is stable if a bounded input (gain limited) sequence produces a bounded output sequence.**

Input $x[n]$ is bounded if there exist a fixed positive finite value B_x Such that:

$$|x[n]| \leq B_x < \infty \quad \text{for all } n$$

Stability requires that for every bounded input there exist a fixed positive value B_y Such that:

$$|y[n]| \leq B_y < \infty \quad \text{for all } n$$

Examples of stable systems:

$$y[n] = x[n - n_d] \quad -\infty \leq n < \infty$$

$$y[n] = \left(x[n]\right)^2$$

$$y[n] = x[Mn] \quad -\infty < n < \infty$$

Example of an unstable system:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Both physical linear systems and DSP algorithms often incorporate feedback, and if the design procedures are not followed with care, instability can result.

Uniqueness.

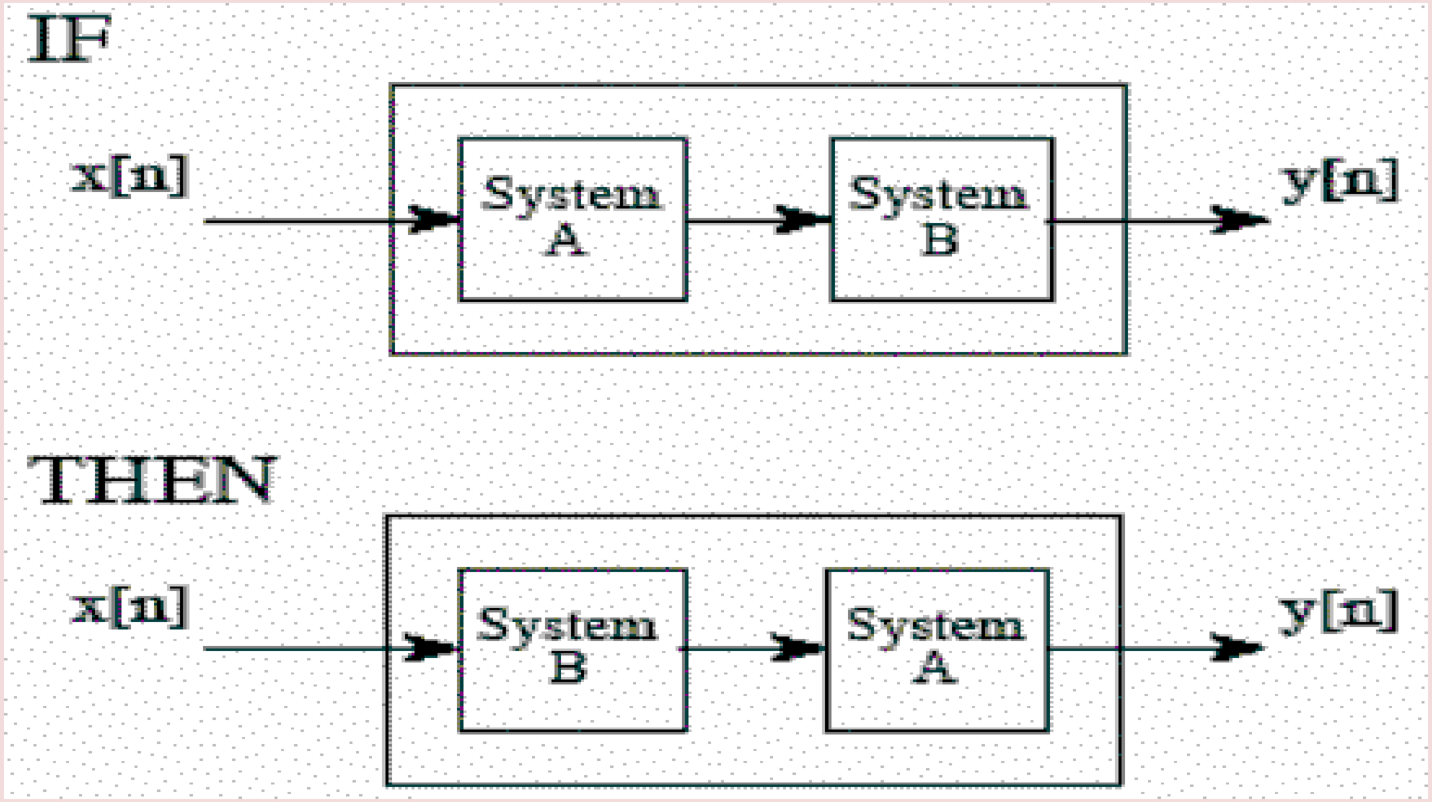
- A linear system is said to be unique if it always produces a unique output in response to a unique input.
- As a consequence, the same output cannot be generated by two different inputs, and two different outputs cannot be generated in response to the same input.

Invertibility.

If a linear system produces $y[n]$ in response to $x[n]$, then the inverse of the linear system will produce $x[n]$ in response to $y[n]$, then the system is invertible. Once again, this may seem like a self-evident, but entire books and journals are devoted to it. It forms the basis of a technique called *deconvolution*, also termed *inverse filtering*.

Special Properties of Linearity

- **Linearity is commutative, a property involving the combination of two or more systems.**

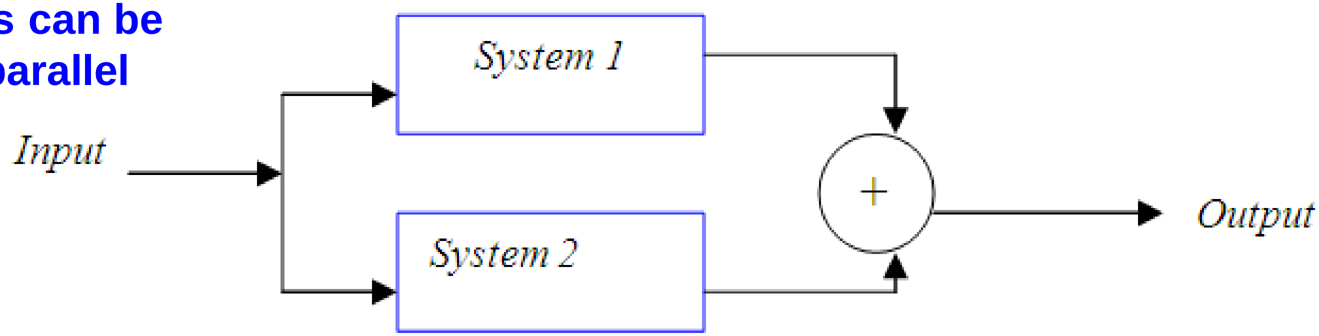


The commutative property states that the order of the systems in the cascade can be rearranged without affecting the characteristics of the overall combination.

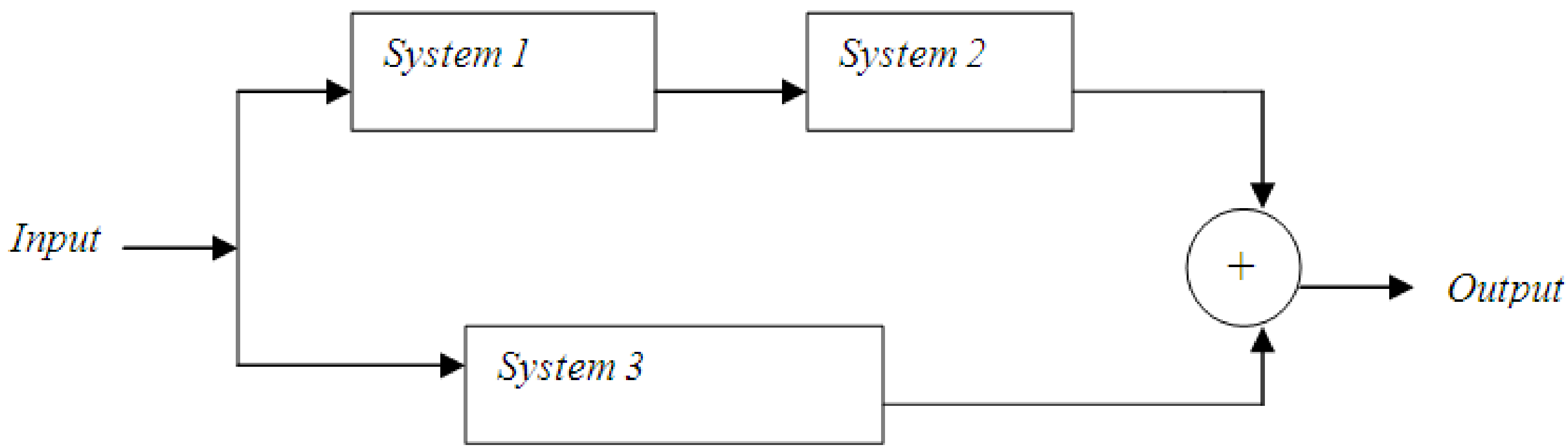
Keep in mind that actual electronics has *nonlinear* effects that may make the order important, for instance: interference, DC offsets, internal noise, slew rate distortion, etc.

Special Properties of Linearity

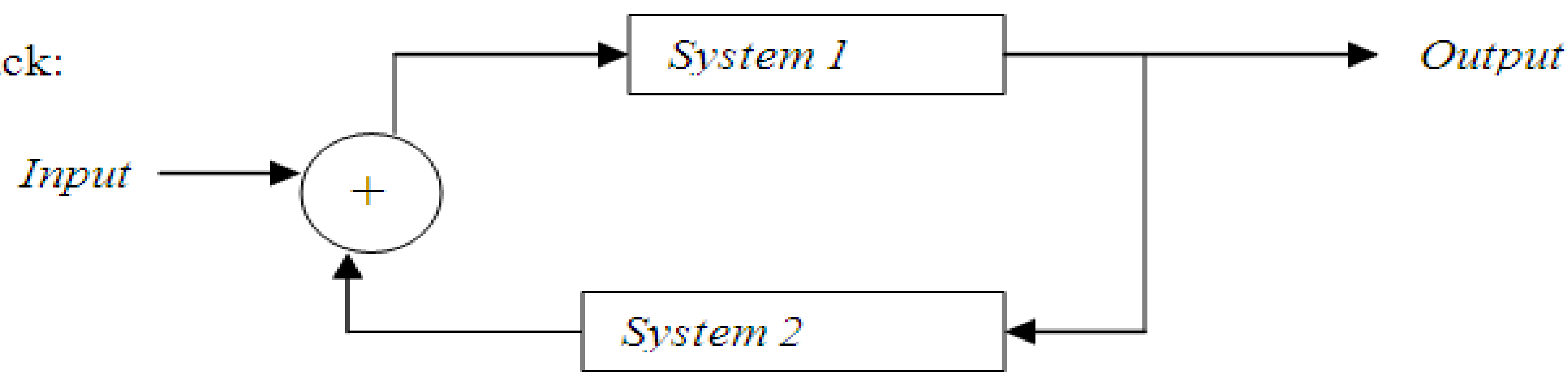
Linear systems can be connected in parallel



Series/Parallel:

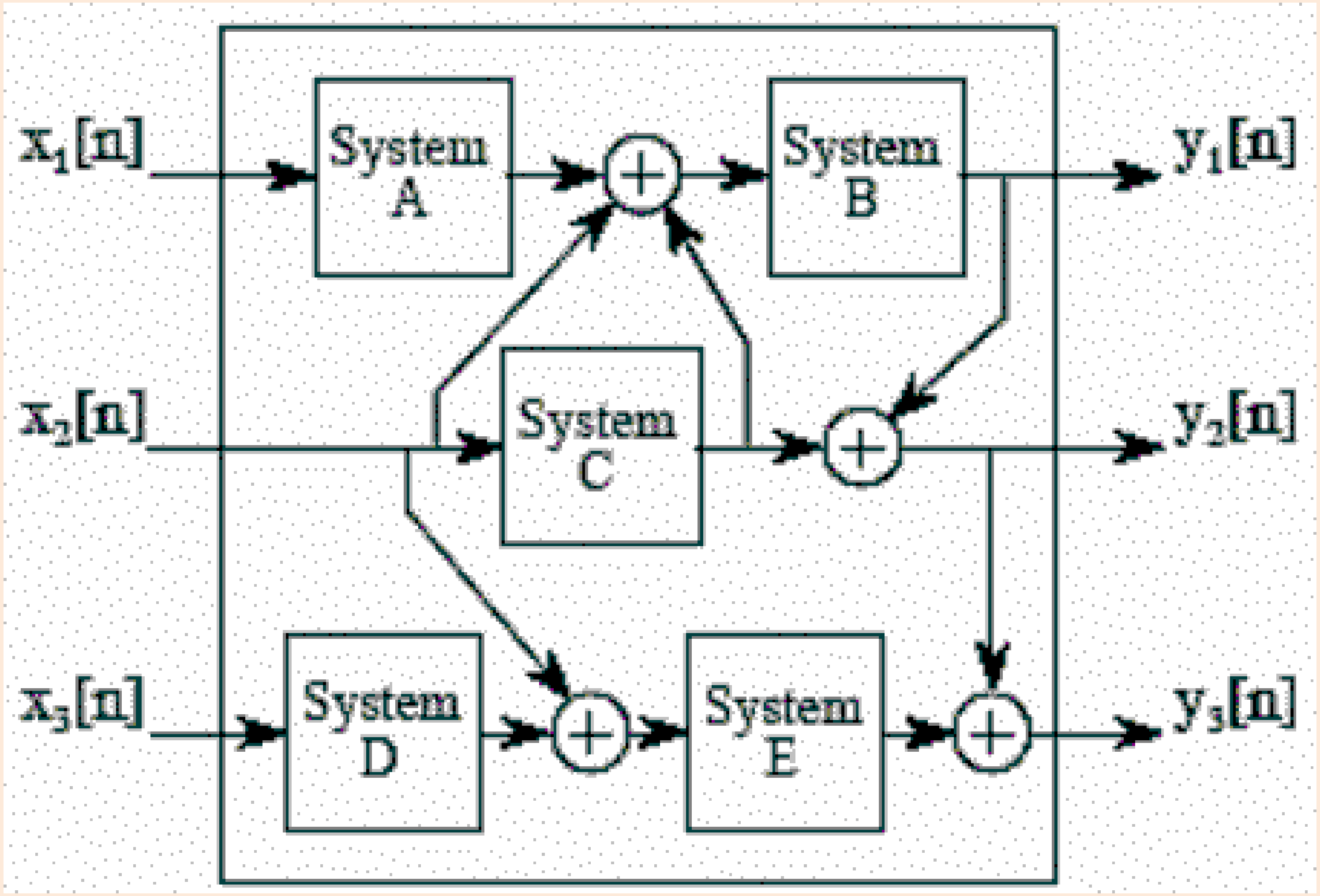


Feed Back:



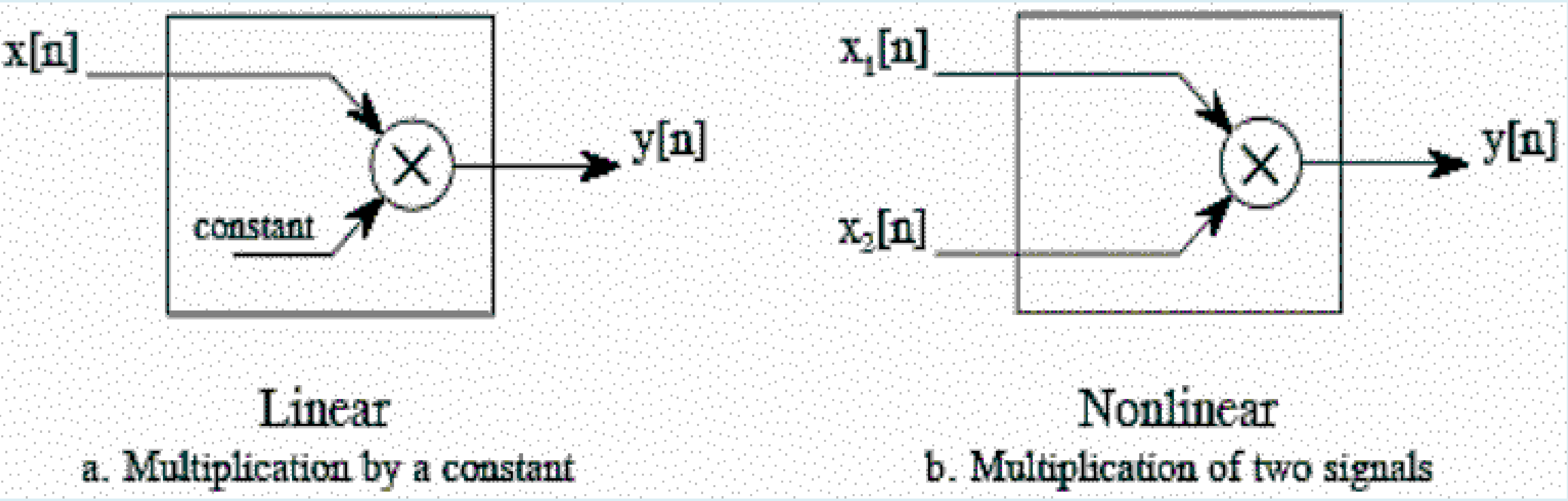
Special Properties of Linearity

- **A system with multiple inputs and/or outputs will be linear if it is composed of *linear subsystems* and *additions of signals*.**



Special Properties of Linearity

- **The use of multiplication in linear systems is frequently misunderstood.**
- **This is because multiplication can be either linear or nonlinear, depending on what the signal is multiplied by.**



Special Properties of Linearity

- Another commonly misunderstood situation relates to parasitic signals added in electronics, such as DC offsets and thermal noise.
- Is the addition of these extraneous signals linear or nonlinear? The answer depends on where the contaminating signals are viewed as originating.
- If they are viewed as coming from *within* the system, the process is nonlinear.
- This is because a sinusoidal input does not produce a pure sinusoidal output.
- Conversely, the extraneous signal can be viewed as *externally* entering the system on a separate input of a multiple input system.
- This makes the process linear, since only a signal addition is required within the system.

Examples of *Linear* Systems

- Wave propagation such as sound and electromagnetic waves
- Electrical circuits composed of resistors, capacitors, and inductors
- Electronic circuits, such as amplifiers and filters
- Mechanical motion from the interaction of masses, and springs.
- Systems described by differential equations such as resistor-capacitor-inductor networks
- Multiplication by a constant, that is, amplification or attenuation of the signal
- Signal changes, such as echoes, resonances, and image blurring
- The unity system where the output is always equal to the input
- The null system where the output is always equal to the zero, regardless of the input
- Differentiation and integration, and the analogous operations of *first difference* and *running sum* for discrete signals

Examples of *Linear* Systems

- **Small perturbations in an otherwise nonlinear system, for instance, a small signal being amplified by a properly biased transistor**
- **Convolution, a mathematical operation where each value in the output is expressed as the sum of values in the input multiplied by a set of weighing coefficients.**
- **Recursion, a technique similar to convolution, except previously calculated values in the output are used in addition to values from the input**

Alternatives to Linearity

- **There is only *one* major strategy for analyzing systems that are nonlinear.**
- **That strategy is to make the nonlinear system *resemble* a linear system.**
- **There are three common ways of doing this:**
    - **First, ignore the nonlinearity. If the nonlinearity is small enough, the system can be approximated as linear. Errors resulting from the original assumption are tolerated as noise or simply ignored.**
    - **Second, keep the signals very small. Many nonlinear systems appear linear if the signals have a very small amplitude.**
    - **Third, apply a linearizing transform. For example, consider two signals being multiplied to make a third: $a[n] = b[n] \times c[n]$. Taking the logarithm of the signals changes the nonlinear process of multiplication into the linear process of addition: $\log(a[n]) = \log(b[n]) + \log(c[n])$. The fancy name for this approach is *homomorphic* signal processing.**

Examples of Desirable nonlinear systems

- Don't make the mistake of thinking that nonlinear systems are unusual or in some way undesirable.
- It is very common in science and engineering to make use of systems whose responses change over time, adapt to the input or generate a non-unique output signal.
- Take for example, the amplifier of an ultrasound scanner used for foetal imaging.
- The image is generated using the strength of the echoes reflected from the various tissue interfaces.
- However, as the ultrasound beam propagates deeper into the body, it is attenuated through scatter and absorption.
- To correct for this factor, the amplifier increases its gain over time, applying the greatest gain to the most distant signals.
- This kind of device is called a *swept gain amplifier*, and is also widely used in radar systems.
- Because it violates the principle of temporal invariance, the system is nonlinear.

Examples of *Nonlinear* Systems

- **Systems that do not have static linearity**, for instance, the voltage and power in a resistor: $P=V^2R$, the radiant energy emission of a hot object depending on its temperature: $R = kT^4$, the intensity of light transmitted through a thickness of translucent material: $I = e^{-\alpha T}$, etc.
- **Systems that do not have sinusoidal fidelity**, such as electronics circuits for: peak detection, squaring, sine wave to square wave conversion, frequency doubling, etc.
- **Common electronic distortion**, such as clipping, crossover distortion and slewing
- **Multiplication of one signal by another signal**, such as in amplitude modulation and automatic gain controls
- **Hysteresis phenomena**, such as magnetic flux density versus magnetic intensity in iron, or mechanical stress versus strain in vulcanized rubber
- **Saturation**, such as electronic amplifiers and transformers driven too hard
- **Systems with a threshold**, for example, digital logic gates, or seismic vibrations that are strong enough to pulverize the intervening rock

Complexities of Linear Systems

- **Just because a system is linear does not necessarily mean that its behaviour is simple.**
- **Many physicists, for example, have spent their lives trying to characterise the way musical instruments perform.**
- **Conversely, some nonlinear systems are pretty simple, as described above.**
- **However, the critical difference between linear and nonlinear systems, regardless of their complexity, is predictability.**