

Collapsing Constructive and Intuitionistic Modal Logics

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18 March 2025

INTRODUCTION

Theorem (Das, Marin)

CK and IK do not prove the same \Diamond -free formulas:

- ▶ $CK \not\vdash \neg\neg\Box\perp \rightarrow \Box\perp$, and
- ▶ $IK \vdash \neg\neg\Box\perp \rightarrow \Box\perp$

Theorem (P.)

CKB and IKB prove the same formulas.

THE LOGIC CK

CK is the least set of formulas containing:

- ▶ intuitionistic tautologies;
- ▶ $K_{\Box} := \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$;
- ▶ $K_{\Diamond} := \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$;

and closed under

$$(\mathbf{Nec}) \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP}) \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}.$$

THE LOGICS CKB, IK, AND IKB

Let

- ▶ $FS := (\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi);$
- ▶ $DP := \Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi;$
- ▶ $N := \neg\Diamond\perp;$
- ▶ $B_{\Box} := P \rightarrow \Box\Diamond P;$ and
- ▶ $B_{\Diamond} := \Diamond\Box P \rightarrow P.$

Then:

- ▶ $CKB := CK + \{B_{\Box}, B_{\Diamond}\};$
- ▶ $IK := CK + \{FS, DP, N\};$ and
- ▶ $IKB := IK + \{B_{\Box}, B_{\Diamond}\} = CKB + \{FS, DP, N\}.$

CK-MODELS

A CK-model is a tuple $M = \langle W, W^\perp, \preceq, R, V \rangle$ where:

- ▶ W is the set of *possible worlds*;
- ▶ $W^\perp \subseteq W$ is the set of *fallible worlds*;
- ▶ the *intuitionistic relation* \preceq is a reflexive and transitive relation over W ;
- ▶ the *modal relation* R is a relation over W ;
- ▶ $V : \text{Prop} \rightarrow \mathcal{P}(W)$ is a *valuation function*.

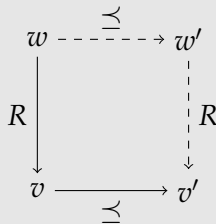
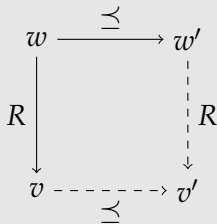
We require:

- ▶ if $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$;
- ▶ for all $P \in \text{Prop}$, $W^\perp \subseteq V(P)$;
- ▶ if $w \in W^\perp$ and either $w \preceq v$ or wRv , then $v \in W^\perp$.

IK-MODELS

An IK-model is a CK-model where:

- ▶ $W^\perp = \emptyset$;
- ▶ R is forward and backward confluent:



An IKB-model is an IK-model where R is symmetric.

VALUATION

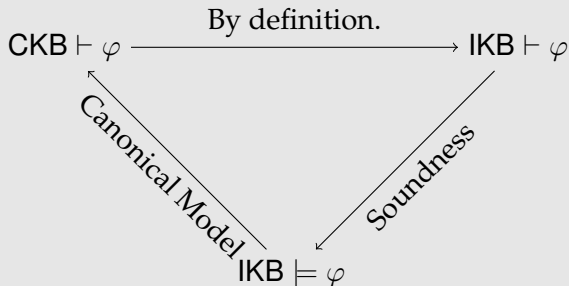
- ▶ $M, w \models P$ iff $w \in V(P)$;
- ▶ $M, w \models \perp$ iff $w \in W^\perp$;
- ▶ $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$;
- ▶ $M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$;
- ▶ $M, w \models \varphi \rightarrow \psi$ iff, for all $v \in W$, if $w \preceq v$ and $M, v \models \varphi$, then $M, v \models \psi$;
- ▶ $M, w \models \Box\varphi$ iff, for all $v, u \in W$, if $w \preceq v$ and vRu , then $M, u \models \varphi$; and
- ▶ $M, w \models \Diamond\varphi$ iff, for all $v \in W$, if $w \preceq v$ then, there is u such that vRu and $M, u \models \varphi$.

MAIN THEOREM

Theorem

For all modal formula φ , the following are equivalent:

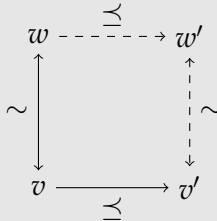
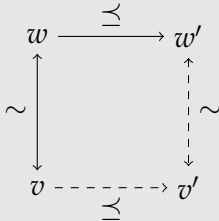
1. $\text{CKB} \vdash \varphi$;
2. $\text{IKB} \vdash \varphi$; and
3. $\text{IKB} \models \varphi$.



SYMMETRY IMPLIES CONFLUENCES COINCIDE

Lemma

Let M be a **CK**-model where the modal relation \sim is symmetric. Then \sim is forward confluent iff \sim is backward confluent.

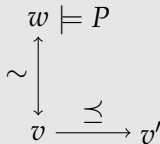


SYMMETRY IMPLIES CONFLUENCE IS NECESSARY

Lemma

There is a CK-model $M = \langle W, W^\perp, \preceq, \sim, V \rangle$ and $w \in W$ such that:

- ▶ *\sim is a symmetric relation;*
- ▶ *$B_\square := P \rightarrow \square\Diamond P$ does not hold at w .*



EXISTING RESULTS

Theorem (Arisaka, Das, Straßburger)

$\text{CKB} \vdash DP$ and $\text{CKB} \vdash N$.

Theorem (De Groot, Shillito, Clouston)

Let $M = \langle W, W^\perp, \preceq, R, V \rangle$ be a **CK-model**. Then:

- ▶ Suppose that, for all $w, v \in W$, wRv , and $v \in W^\perp$ implies $w \in W^\perp$. Then $M \models N$.
- ▶ Suppose that R is forward and backward confluent. Then $M \models DP$ and $M \models FS$.

A CANONICAL MODEL FOR CKB

A (consistent) **CKB**-theory Γ is a set of formulas such that:

- ▶ Γ contains all the axioms of **CKB** and is closed under **MP**;
- ▶ if $\varphi \vee \psi \in \Gamma$, then $\varphi \in \Gamma$ or $\psi \in \Gamma$;
- ▶ $\perp \notin \Gamma$.

Definition

The **CKB**-canonical model is $M_c := \langle W_c, W_c^\perp, \preceq_c, \sim_c, V_c \rangle$ where:

- ▶ $W_c := \{\Gamma \mid \Gamma \text{ is a CKB-theory}\}$;
- ▶ $W_c^\perp = \emptyset$;
- ▶ $\Gamma \preceq_c \Delta$ iff $\Gamma \subseteq \Delta$;
- ▶ $\Gamma \sim_c \Delta$ iff $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Delta$ and $\Delta \subseteq \{\varphi \mid \Diamond\varphi \in \Gamma\}$;
- ▶ $\Gamma \in V_c(P)$ iff $P \in \Gamma$.

TRUTH LEMMA

Lemma

The CKB-canonical model M_c is an IKB-model.

The following lemma uses standard techniques:

Lemma

Let M_c be the CKB-canonical model.

For all formula φ and for all CKB-theory Γ ,

$$M_c, \Gamma \models \varphi \text{ iff } \varphi \in \Gamma.$$

Above, we use Zorn's Lemma to prove:

- ▶ $\Box\varphi \notin \Gamma$ implies $\Gamma \not\models \Box\varphi$; and
- ▶ $\Diamond\varphi \in \Gamma$ implies $\Gamma \models \Diamond\varphi$.

CONCLUSION

While, in general, constructive and intuitionistic versions of the same logic do not coincide, we have:

Theorem (P.)

CKB and IKB prove the same formulas.

This result extends to logics proving the axiom *B*. For example:

Corollary

$CS5 = IS5$.

AN OPEN PROBLEM





Characterize necessary and sufficient conditions for CK-frames to validate the axioms in the modal cube:

- ▶ $B_{\Box} := P \rightarrow \Box\Diamond P, B_{\Diamond} := \Diamond\Box P \rightarrow P;$
- ▶ $4_{\Box} := \Box P \rightarrow \Box\Box P, 4_{\Diamond} := \Diamond\Diamond P \rightarrow \Diamond P;$
- ▶ $5_{\Box} := \Diamond P \rightarrow \Box\Diamond P, 5_{\Diamond} := \Diamond\Box P \rightarrow \Box P;$
- ▶ $T_{\Box} := \Box P \rightarrow P, T_{\Diamond} := P \rightarrow \Diamond P;$ and
- ▶ $D := \Box P \rightarrow \Diamond P.$

Characterize necessary and sufficient conditions for CK-frames to validate the axioms:

- ▶ $L_{mix} := \Box(\Diamond\neg P \vee P) \rightarrow \Box P;$
- ▶ $L_{\Box} := \Box(\Box P \rightarrow P) \rightarrow \Box P;$
- ▶ $L_{\Diamond} := \Diamond P \rightarrow \Diamond(P \wedge \neg\Diamond P).$

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