Connecting reflection and β-models in second-order arithmetic

Leonardo Pacheco
TU Wien
(jww Keita Yokoyama)

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Strong dependent choices

Definition

Strong Σ^1_i-DC_0 is the schema containing

$$\forall Z \forall n \forall Y (\varphi(n, (Z)_{<n}, Y) \rightarrow \varphi(n, (Z)_{<n}, (Z)_n))$$

for all Σ^1_i formula φ.

REFLECTION PRINCIPLES

Let

- Pr_T be a standard provability predicate for a theory T;
- Tr be a truth predicate for Π^1_n-sentences.

Π^1_n-$\mathsf{Ref}(T)$ is the sentence

$$\forall \varphi \in \Pi^1_n . \mathrm{Pr}_T(\varphi) \to \mathrm{Tr}(\varphi).$$

β-MODELS

- Any set $\mathcal{M} \subseteq \mathbb{N}$ can be seen as a model whose sets are

$$(\mathcal{M})_n = \{i \in \mathbb{N} \mid \langle n, i \rangle \in \mathcal{M}\}.$$

- $\mathcal{M} \subseteq \mathbb{N}$ is a coded β-model iff, for all Π^1_1-sentence φ with parameters in $\mathcal{M}$,

$$\varphi \iff \mathcal{M} \models \varphi.$$

Theorem (ACA_0)

Strong Σ^1_1-DC_0 is equivalent to

for all $X \subseteq \mathbb{N}$ there is a coded β-model $\mathcal{M}$ containing X.

β_k-models

- $\mathcal{M} \subseteq \mathbb{N}$ is a coded β_k-model iff, for all Π^1_k-sentence φ with parameters in $\mathcal{M}$,

$$\varphi \iff \mathcal{M} \models \varphi.$$

Theorem (ACA_0)

Strong Σ^1_k-DC_0 is equivalent to

$$\textit{for all } X \subseteq \mathbb{N} \textit{ there is a coded } \beta_k\textit{-model } \mathcal{M} \textit{ containing } X.$$

Sequences of β_k-models

$\psi_{i,e}(n)$ states that, for all $X \subseteq \mathbb{N}$, there are $Y_0, \ldots, Y_n$ such that:

$$\begin{array}{l} Y_0 \subseteq_{\beta_i} Y_1 \subseteq_{\beta_i} \cdots \subseteq_{\beta_i} Y_n \subseteq_{\beta_e} \mathcal{N} \\ X \in Y_0 \in Y_1 \in \cdots \in Y_n \end{array}$$

Note that $\psi_{i,e}(n)$ is a Π^1_{e+2}-formula.

Theorem (ACA_0)

If $e \leq i$, then $\forall n.\psi_{i,e}(n)$ is equivalent to Π^1_{e+2}-Ref(Strong Σ^1_i-DC_0).

SOME DETERMINACY RESULTS

For all standard $n \geq 2$,

- ACA_0 is equivalent to $(\Sigma^0_1)_n$-Det^*;
- Π^1_1-CA_0 is equivalent to $(\Sigma^0_1)_n$-Det;
- Π^1_2-CA_0 proves $(\Sigma^0_2)_n$-Det; and
- Z_2 proves $(\Sigma^0_3)_n$-Det.

Consequences

Theorem

Over ACA_0*,*

- Π^1_2-$\mathsf{Ref}(\mathsf{ACA}_0) \leftrightarrow \forall n.(\Sigma^0_1)_n$-$\mathsf{Det}^*$*;*
- Π^1_3-$\mathsf{Ref}(\Pi^1_1$-$\mathsf{CA}_0) \leftrightarrow \forall n.(\Sigma^0_1)_n$-$\mathsf{Det}$*;*
- Π^1_3-$\mathsf{Ref}(\Pi^1_2$-$\mathsf{CA}_0) \leftrightarrow \forall n.(\Sigma^0_2)_n$-$\mathsf{Det}$*; and*
- Π^1_3-$\mathsf{Ref}(\mathsf{Z}_2) \leftrightarrow \forall n.(\Sigma^0_3)_n$-$\mathsf{Det}$.

Open problems

Problem

Characterize Π^1_n-$\mathsf{Ref}(T)$ *for other theories* T.[1]

Problem

Study axioms stating the existence of transfinite sequences of models.

[1] See also P., "Recent Results on Reflection Principles in Second-Order Arithmetic".

Thank you!

For more details see

- P., Yokoyama, "Determinacy and reflection principles in second-order arithmetic", arXiv:2209.04082.