

# Non-classical modal logics

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22 March 2024

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# NON-CLASSICAL MODAL LOGICS

- ▶ modal logic = propositional logic +  $\Box$  +  $\Diamond$ .
- ▶ Two main non-classical varieties:
  - ▶ constructive modal logic, and
  - ▶ intuitionistic modal logic.

# A BIT OF HISTORY

- ▶ Fitch (1948): intuitionistic first-order logic (with  $T$  and Barcan's formula)
- ▶ Prior (1957): MIPQ, an intuitionistic analogue of S5
- ▶ Ono (1977), Fischer Servi (1978): completeness of MIPQ
- ▶ Many people work on  $\Diamond$ -free intuitionistic modal logics
- ▶ Wijesekera (1990): constructive modal logic

For a better survey see Simpson (1994).

# CK — AXIOMATIZATION

Axioms:

- ▶ all intuitionistic tautologies;
- ▶  $K_{\Box} := \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ ;
- ▶  $K_{\Diamond} := \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$ .

Rules:

$$(\mathbf{Nec}) \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP}) \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}.$$

# IK — AXIOMATIZATION

Axioms:

- ▶ all intuitionistic tautologies;
- ▶  $K_{\Box} := \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ ;
- ▶  $K_{\Diamond} := \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$ ;
- ▶  $FS := (\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi)$ ;
- ▶  $DP := \Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi$ ; and
- ▶  $N := \neg\Diamond\perp$ .

Rules:

$$(\mathbf{Nec}) \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP}) \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}.$$

# KRIPKE MODELS FOR MODAL LOGIC

Tuples  $M = \langle W, R, V \rangle$  where:

- ▶  $W$  is the set of *possible worlds*;
- ▶  $R$  is a relation over  $W$ ; and
- ▶  $V : \text{Prop} \rightarrow \mathcal{P}(W)$  is a valuation function.

We define:

- ▶  $M, w \models \Box\varphi$  iff, for all  $v$ , if  $wRv$  then  $M, v \models \varphi$ ;
- ▶  $M, w \models \Diamond\varphi$  iff there is  $v$  such that  $wRv$  and  $M, v \models \varphi$ .

# KRIPKE MODELS FOR INTUITIONISTIC LOGIC

Bi-relational Kripke models  $M = \langle W, \preceq, V \rangle$  where:

- ▶  $W$  is the set of *possible worlds*;
- ▶  $\preceq$  is a reflexive and transitive relation over  $W$ ;
- ▶  $V : \text{Prop} \rightarrow \mathcal{P}(W)$  is a valuation function.

We require that:

- ▶  $w \preceq v$  and  $w \in V(P)$ , then  $v \in V(P)$ .

We define:

- ▶  $M, w \models \varphi \rightarrow \psi$  iff, for all  $v$ , if  $w \preceq v$  and  $M, v \models \varphi$ , then  $M, v \models \psi$ ;
- ▶  $M, w \models \neg\varphi$  iff, for all  $v$ , if  $w \preceq v$ , then  $M, v \not\models \varphi$ .

# CK — SEMANTICS

Bi-relational Kripke models  $M = \langle W, W^\perp, \preceq, R, V \rangle$  where:

- ▶  $W$  is the set of *possible worlds*;
- ▶  $W^\perp \subseteq W$  is the set of *fallible worlds*;
- ▶  $\preceq$  is a reflexive and transitive relation over  $W$ ;
- ▶  $\sqsubset$  is a relation over  $W$ ; and
- ▶  $V : \text{Prop} \rightarrow \mathcal{P}(W)$  is a valuation function.

We require that:

- ▶  $W^\perp \subseteq V(P)$ ;
- ▶  $w \preceq v$  and  $w \in V(P)$ , then  $v \in V(P)$ .



# IK — SEMANTICS

Bi-relational Kripke models  $M = \langle W, \preceq, R, V \rangle$  where:

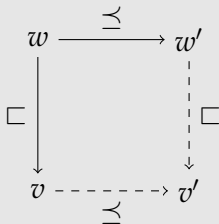
- ▶  $W$  is the set of *possible worlds*;
- ▶  $\preceq$  is a reflexive and transitive relation over  $W$ ;
- ▶  $\sqsubset$  is a relation over  $W$ ; and
- ▶  $V : \text{Prop} \rightarrow \mathcal{P}(W)$  is a valuation function.

We require that:

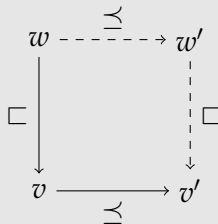
- ▶  $w \preceq v$  and  $w \in V(P)$ , then  $v \in V(P)$ ;
- ▶  $M$  is *forward confluent*:  $w \preceq w'$  and  $w \sqsubset v$  imply there is  $v'$  such that  $v \preceq v'$  and  $w' \sqsubset v'$ ;
- ▶  $M$  is *backward confluent*:  $w \sqsubset v \preceq v'$  implies then there is  $w'$  such that  $w \preceq w' \sqsubset v'$ .

# CONFLUENCE

Forward confluence



Backward confluence



# VALUATIONS

The valuation of  $\Box$ s are the same over CK and IK models:

- ▶  $M, w \models \Box\varphi$  iff, for all  $v, u$ , if  $w \preceq vRu$  then  $M, u \models \varphi$ .

Over CK models, define:

- ▶  $M, w \models \Box\varphi$  iff, for all  $v$  such that  $w \preceq v$ , there is  $u$  such that if  $vRu$  and  $M, u \models \varphi$ .

Over IK models, define:

- ▶  $M, w \models \Box\varphi$  iff there is  $v$  such that if  $wRv$  and  $M, v \models \varphi$ .

# SEPARATING CK AND IK

The following formulas are provable in IK but not in CK:

- ▶  $FS := (\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi);$
- ▶  $DP := \Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi;$  and
- ▶  $N := \neg\Diamond\perp.$

All of these involve  $\Diamond$ s.

## Question

*Do CK and IK prove the same  $\Diamond$ -free formulas?*

The answer is no!<sup>1</sup>

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<sup>1</sup>Das, Marin

# SEPARATION

- ▶  $\text{CK} \not\models \neg\neg\Box\perp \rightarrow \Box\perp$ :

$$w \preceq v \models \perp$$

- ▶  $w \models \neg\neg\Box\perp$  iff, for all  $w' \succeq w$ , there is  $w'' \succeq w'$  such that  $w'' \models \Box\perp$ .
- ▶ But  $\text{IK} \vdash \neg\neg\Box\perp \rightarrow \Box\perp$ .

# CS4 AND IS4

CS4 and IS4 are obtained by adding to CK and IK the axioms:

- ▶  $4_{\Box} := \Box P \rightarrow \Box \Box P;$
- ▶  $4_{\Diamond} := \Diamond \Diamond P \rightarrow \Diamond P;$
- ▶  $T_{\Box} := \Box P \rightarrow P;$
- ▶  $T_{\Diamond} := P \rightarrow \Diamond P.$

# AN EXAMPLE

- Consider the following model  $M$ :

$$x \preceq y \sqsubseteq z \preceq t \sqsubseteq w.$$

where  $P$  holds at  $\{x, y, z, t\}$ .

- The relation  $R$  is transitive, but the formula  $\Box P \rightarrow \Box\Box P$  fails at  $w$ .

# CS4 AND IS4 MODELS

A CS4 model is a CK model  $M = \langle W, W^\perp, \preceq, R, V \rangle$  where

- ▶  $R$  is transitive;
- ▶  $M$  is *backward confluent*:  $w \sqsubset v \preceq v'$  implies then there is  $w'$  such that  $w \preceq w' \sqsubset v'$ .

A CS4 model is a CK model  $M = \langle W, \preceq, R, V \rangle$  where

- ▶  $R$  is transitive.

CS4 and IS4 are complete w.r.t. CS4 and IS4 models.



# DECIDABILITY

The following was an open problem for  $\tilde{10}$  years:

Theorem (Balbiani, Dieguez, Fernández-Duque)

*CS4 is decidable.*

This was solved using canonical models and bisimulations.  
The following was an open problem for  $\tilde{30}$  years:

Theorem (Girlando *et al.*)

*IS4 is decidable.*

This was solved using labeled proof systems.

# GL

- ▶ GL is the logic obtaining by adding

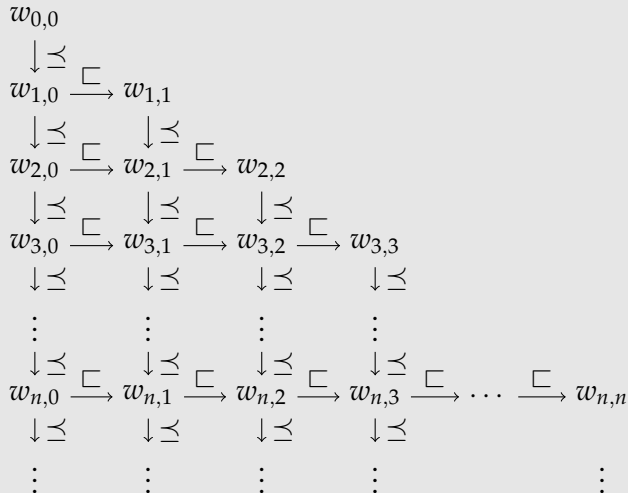
$$L := \Box(\Box P \rightarrow P) \rightarrow \Box P$$

to  $K$ .

- ▶ GL is complete with respect to models  $M = \langle W, R, V \rangle$  where  $R$  is transitive and reverse well-founded.
- ▶  $R$  is reverse well-founded iff there is no infinite sequence  $w_0 R w_1 R w_2 R \dots$

# REV. WF IS NOT (INTUITIONISTICALLY) ENOUGH

In IK model below, no world satisfies  $\Box(\Box P \rightarrow P) \rightarrow \Box P$ .



( $P$  holds nowhere)

# IGL MODELS

An IGL model is an IK model  $M = \langle W, \preceq, R, V \rangle$  where

- ▶  $R$  is transitive;
- ▶ the composition  $\preceq; R$  is reverse well-founded.

# A PROOF SYSTEM FOR IGL

Das, van der Giessen and Marin proved that an infinitary proof system based on the following is complete over IGL frames:

$$\Box_I \frac{R, xRy, \Gamma, y : A \Rightarrow \Delta}{R, xRy, \Gamma, x : \Box A \Rightarrow \Delta}$$

$$\Box_r \frac{R, xRy, \Gamma \Rightarrow \Delta, y : A}{R, \Gamma \Rightarrow \Delta, x : \Box A} (y \text{ fresh})$$

$$\text{tr} \frac{R, xRy, yRz, xRz, \Gamma \Rightarrow \Delta}{R, xRy, yRz, \Gamma \Rightarrow \Delta}$$

# CGL AND IGL MODELS

A CGL model is a CK model  $M = \langle W, W^\perp, \preceq, R, V \rangle$  where

- ▶  $R$  is transitive;
- ▶  $R$  is forward confluent;
- ▶ the composition  $\preceq$ ;  $R$  is reverse well-founded.

An IGL model is an IK model  $M = \langle W, \preceq, R, V \rangle$  where

- ▶  $R$  is transitive;
- ▶ the composition  $\preceq$ ;  $R$  is reverse well-founded.

# AN AXIOMATIZATION FOR CGL AND IGL?

To obtain CGL and IGL, add to CK and IK the axioms:

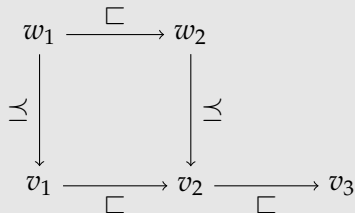
- ▶  $4_{\Box} := \Box\varphi \rightarrow \Box\Box\varphi$ ;
- ▶  $4_{\Diamond} := \Diamond\Diamond\varphi \rightarrow \Diamond\varphi$ ; and
- ▶  $L_{\Box} := \Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$ .

## Question

*Are CGL and IGL complete over CGL and IGL models?*

## THE DUAL OF LÖB'S THEOREM

$L_{\Diamond} := \Diamond P \rightarrow \Diamond(P \wedge \Box \neg P)$  of fails at  $w_1$ :



( $P$  holds everywhere.)



# FAILURE TO PROVE THE COMPLETENESS

Proofs of completeness using finitary canonical models seem to need some diamond version of

$$L_{\Box} := \Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi.$$

Reiterating:

## Question

*Are CGL and IGL complete over CGL and IGL models?*

## Question

*If the answer to the above is negative:*

- ▶ *Are there a complete axiomatization for CGL and IGL models?*
- ▶ *What class of models are characterized by the systems CGL and IGL?*

# THE $\mu$ -CALCULUS

$\mu$ -calculus = modal logic + fixed-point operators

If  $X$  is positive, then:

- ▶  $\|\mu X.\varphi\|^M := \text{least fixed-point of } A \mapsto \|\varphi(A)\|^M$
- ▶  $\|\mu X.\varphi\|^M := \text{greatest fixed-point of } A \mapsto \|\varphi(A)\|^M$

# ALTERNATION DEPTH

The valuation of  $\nu X$  and  $\mu Y$  depend on each other:

$$\nu X. \underbrace{\mu Y. \overbrace{(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)}^{\text{scope of } \mu Y}}_{\text{scope of } \nu X}$$

Alternation depth of  $\varphi$

Maximum number of codependent alternating  $\mu$  and  $\nu$  operators in  $\varphi$ .

Alternation hierarchy

Classifies  $\mu$ -formulas with respect to their alternation depth.

# SOME RESULTS ON THE UNIMODAL $\mu$ -CALCULUS

## Theorem (Bradfield)

*The  $\mu$ -calculus alternation hierarchy is strict over all frames.*

## Theorem (Alberucci–Facchini)

*The  $\mu$ -calculus alternation hierarchy collapses to the alternation-free fragment over transitive frames.*

## Theorem (Alberucci–Facchini)

*The  $\mu$ -calculus alternation hierarchy collapses to modal logic over equivalence relations.*

For a survey, see my PhD thesis.

# VARIATIONS OF S5

CS5 and IS5 are obtained by adding to CK and IK the axioms:

- ▶  $4_{\Box} := \Box P \rightarrow \Box \Box P;$
- ▶  $4_{\Diamond} := \Diamond \Diamond P \rightarrow \Diamond P;$
- ▶  $5_{\Box} := \Diamond P \rightarrow \Box \Diamond P;$
- ▶  $5_{\Diamond} := \Diamond \Box P \rightarrow \Box P;$
- ▶  $T_{\Box} := \Box P \rightarrow P;$
- ▶  $T_{\Diamond} := P \rightarrow \Diamond P.$

CS5 and IS5 models are CS4 and IS4 models where the modal relation is an equivalence relation.

# COLLAPSE OVER CS5/IS5 MODELS

## Lemma

Let  $M = \langle W, W^\perp, \preceq, \equiv, V \rangle$  be a **CS5** model and  $w \preceq; \equiv w'$ . Then

$$M, w \models \Delta\varphi \text{ implies } M, w' \models \Delta\varphi,$$

where  $\Delta \in \{\Box, \Diamond\}$ .

At any long enough evaluation game, we will have positions:

$$\langle \nu X.\varphi, w \rangle \rightarrow \cdots \rightarrow \langle \Box\psi, v \rangle \rightarrow \cdots \rightarrow \langle \Box\psi, v' \rangle \rightarrow \cdots$$

We can use this fact to show that  $\varphi(\varphi(\top)) \equiv \varphi(\varphi(\varphi(\top)))$ .

# COLLAPSE OVER CS4/IS4 MODELS

## Question

*Does the  $\mu$ -calculus collapse to its alternation free-fragment over CS4 and IS4 models?*

All the proofs I know of the collapse over S4 fail on non-constructive settings.

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