INTRODUCTION
000

CYCLIC PROOFS
00000

SEMANTICS
000

ωmℓIGL
000

CONCLUSION
00

IGL without sharps

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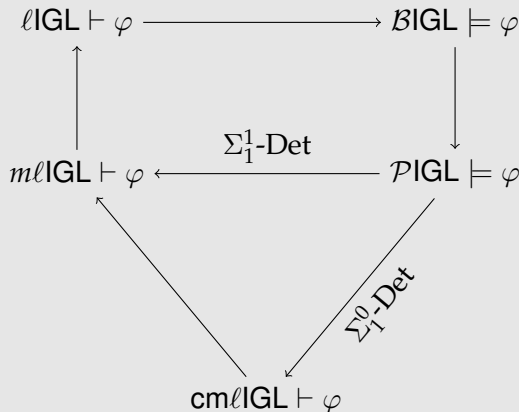(j.w.w. Juan Pablo Aguilera)

7 July 2025

Available at: leonardopacheco.xyz/slides/lc2025.pdf

INTUITIONISTIC GÖDEL-LÖB LOGIC

- ▶ GL: $\Box(\Box P \to P) \to \Box P$.
- ▶ iGL: GL on an intuitionistic base, only boxes.
- ▶ IGL: GL on an intuitionistic base, boxes and diamonds. First developed by Das, van der Giessen and Marin.

INTUITIONISTIC GÖDEL-LÖB LOGIC

- ▶ Das, van der Giessen, Marin, "Intuitionistic Gödel-Löb logic, à la Simpson", 2024.
- ▶ Aguilera, Pacheco, "IGL without sharps", to appear.

$$\ell\text{IGL} \vdash \varphi \longrightarrow \mathcal{B}\text{IGL} \models \varphi$$

$$m\ell\text{IGL} \vdash \varphi \xleftarrow{\quad \Sigma^1_1\text{-Det} \quad} \mathcal{P}\text{IGL} \models \varphi$$

$$\Sigma^0_1\text{-Det}$$

$$\text{cm}\ell\text{IGL} \vdash \varphi$$

$\omega m\ell$IGL

Let $\omega m\ell$IGL be the infinitary proof system with the ω-rule:

$$\frac{x : \Box^n \bot, \mathbf{R}, \Gamma \vdash \Delta \ (\forall n \in \omega)}{\mathbf{R}, \Gamma \vdash \Delta}.$$

Theorem

$\omega m\ell$IGL *is complete w.r.t.* IGL.

SOME INFERENCE RULES — I

$$\bot \text{l} \;\overline{\mathbf{R}, \Gamma, x : \bot \vdash \Delta}$$

$$\wedge \text{l} \;\frac{\mathbf{R}, \Gamma, x : A \wedge B, x : A, x : B \vdash \Delta}{\mathbf{R}, \Gamma, x : A \wedge B \vdash \Delta}$$

$$\Diamond \text{r} \;\frac{\mathbf{R}, \Gamma \vdash \Delta, x : \Diamond A, \{y : A \mid xRy\}}{\mathbf{R}, \Gamma \vdash \Delta, x : \Diamond A}$$

$$\Box \text{l} \;\frac{\mathbf{R}, \Gamma, x : \Box A, \{y : A \mid xRy\} \vdash \Delta}{\mathbf{R}, \Gamma, x : \Box A \vdash \Delta}$$

$$\Diamond \text{l} \;\frac{\mathbf{R}, xRy, \Gamma, x : \Diamond A, y : A \vdash \Delta}{\mathbf{R}, \Gamma, x : \Diamond A \vdash \Delta} \;(y \text{ is fresh})$$

SOME INFERENCE RULES — II

Non-invertible rules:

$$\to \text{r} \; \frac{\mathbf{R}, \Gamma, x : A \vdash x : B}{\mathbf{R}, \Gamma \vdash \Delta, x : A \to B}$$

$$\Box \text{r} \; \frac{\mathbf{R}, xRy, \Gamma \vdash y : A}{\mathbf{R}, \Gamma \vdash \Delta, x : \Box A} \; (y \text{ is fresh})$$

INTRODUCTION
000

CYCLIC PROOFS
00●00

SEMANTICS
000

ωmℓIGL
000

CONCLUSION
00

cmℓIGL PROVES $\Box(\Box P \to P) \to \Box P$

$$
\cfrac{
  \cfrac{
    \cfrac{
      \text{id} \cfrac{}{xRy, x : \Box(\Box P \to P), y : \Box P \to P, y : P \vdash y : P} \qquad
      \text{tr} \cfrac{
        \Box\text{l} \cfrac{
          \cfrac{xRy, yRz, xRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P \quad (*)}{xRy, yRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P}
        }{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P, y : \Box P}
      }{}
    }{\to\text{l} \; xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P}
  }{\Box\text{l} \cfrac{}{xRy, x : \Box(\Box P \to P) \vdash y : P \, (*)}}
}{\to\text{r} \cfrac{\Box\text{r} \; x : \Box(\Box P \to P) \vdash x : \Box P}{\vdash x : \Box(\Box P \to P) \to \Box P}}
$$

INTRODUCTION
000

CYCLIC PROOFS
00000

SEMANTICS
000

ωmℓIGL
000

CONCLUSION
00

THE LOOP

$$xRy, yRz, xRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P$$

$$xRy, x : \Box(\Box P \to P) \vdash y : P$$

A PROOF SEARCH GAME FOR cmℓIGL

Given a sequent $\mathbf{R}, \Gamma \vdash \Delta$, we define an (open) game:

- ► Two players: Prover and Denier.
- ► Start on the sequent $\mathbf{R}, \Gamma \vdash \Delta$.
- ► When discussing a sequent S, Prover has two choices:
  - ► Pick an inference rule

    $$\frac{S_1 \cdots S_n}{S} \, ,$$

    and then Denier picks some S_i.
  - ► Draw a progressing loop from S to a previous sequent.
- ► Infinite plays are won by Denier.

Lemma

- ► *If Prover wins this game,* cmℓIGL *proves* $\mathbf{R}, \Gamma \vdash \Delta \in$ cmℓIGL.
- ► *If Denier wins this game, we can build a countermodel for* $\mathbf{R}, \Gamma \vdash \Delta$.
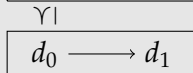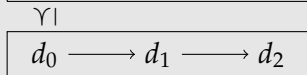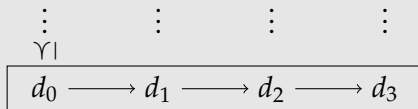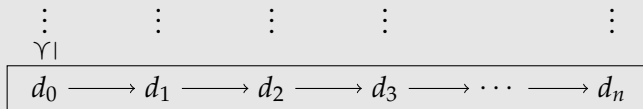
PREDICATE IGL MODELS

Tuple $M = \langle W, \preceq, \{D_w\}_{w \in W}, \{Pr_w\}_{w \in W}, \{R_w\}_{w \in W} \rangle$ such that:

1. W is a non-empty set of possible worlds;
2. $\preceq$ is a partial order on W;
3. D_w is the domain of $w \in W$;
4. Pr_w is the valuation over D_w;
5. R_w is a transitive relation in D_w;
6. all relations are monotone in $\preceq$.

We also require there is no infinite path moving through the R_w infinitely often.

THE GLOBAL CONDITION IS NECESSARY

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad\quad\quad \vdots$$

$\curlyvee|$

$$\boxed{d_0 \longrightarrow d_1 \longrightarrow d_2 \longrightarrow d_3 \longrightarrow \cdots \longrightarrow d_n}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$\curlyvee|$

$$\boxed{d_0 \longrightarrow d_1 \longrightarrow d_2 \longrightarrow d_3}$$

$\curlyvee|$

$$\boxed{d_0 \longrightarrow d_1 \longrightarrow d_2}$$

$\curlyvee|$

$$\boxed{d_0 \longrightarrow d_1}$$

$\curlyvee|$

$$\boxed{d_0}$$

$V(P) = \emptyset$ implies $\models \Box(\Box P \to P)$ and $\models \neg \Box P$.

INTRODUCTION
000

CYCLIC PROOFS
00000

SEMANTICS
00●

ωmℓIGL
000

CONCLUSION
00

IGL DOES NOT PROVE $\Diamond P \to \Diamond(P \land \neg\Diamond P)$

P holds at y and z.

DEFINITION

$\omega m\ell$IGL is obtained by adding to the basic rules the ω-rule:

$$\frac{x : \Box^n \bot, \mathbf{R}, \Gamma \vdash \Delta \ (\forall n \in \omega)}{\mathbf{R}, \Gamma \vdash \Delta}.$$

- ▶ No loops or infinite paths.
- ▶ If we restrict the right-hand side, we can define $\omega\ell$IGL.

A similar ω-rule for classical GL was studied by Yoshihito Tanaka.

COMPLETENESS

Lemma

$m\ell$IGL $\vdash \varphi$ *implies* $\omega m\ell$IGL $\vdash \varphi$.

Proof.

- ▶ Let T be an $m\ell$IGL-proof of φ with $\vdash x : \varphi$ at its root.
- ▶ Let $n \in \omega$.
- ▶ Append $x : \Box^n \bot$ to the left-hand side of all sequents of T.
- ▶ The infinite paths of the new tree can be trimmed with applications of $\Box$l and $\bot$l.
- ▶ The trimmed proof T_n is an $\omega m\ell$IGL-proof of $x : \Box^n \bot \vdash x : \varphi$.
- ▶ From all the T_n, an application of the ω-rule gives us an $\omega m\ell$IGL-proof of φ. $\qquad \Box$

SOUNDNESS

Lemma

$\omega m\ell$IGL $\vdash \varphi$ implies $\mathcal{P}$IGL $\vdash \varphi$.

Proof.

- ▶ Suppose $\mathcal{P}$IGL $\not\models \varphi$, then cmℓIGL $\not\vdash \varphi$.
- ▶ In the countermodel built in the completeness proof of cmℓIGL, we can show that each R_w only has paths of length less than:

$$f(\varphi) := 2^{2^{|\mathrm{Sub}(\varphi)|}} \times 2^{|\mathrm{Sub}(\varphi)|} \times |\mathrm{Sub}(\varphi)| + 1.$$

- ▶ So $\Box^{f(\varphi)}\bot \to \varphi$ is not $\mathcal{P}$IGL-valid.
- ▶ (This is a **extremely** rough bound.) □

RESULTS

Theorem

For all modal formula φ, the following are equivalent:

- ► $\mathcal{P}$IGL $\models \varphi$;
- ► ℓIGL $\vdash \varphi$;
- ► cmℓIGL $\vdash \varphi$; *and*
- ► $\omega m\ell$IGL $\vdash \varphi$.

INTRODUCTION
000

CYCLIC PROOFS
00000

SEMANTICS
000

$\omega m\ell$IGL
000

CONCLUSION
○●

OPEN QUESTIONS

Question

Is IK4 $+ \Box(\Box\varphi \to \varphi) \to \Box\varphi$ *complete w.r.t.* IGL?

Question

Is IGL *decidable?*

Question

Does IGL *have some form of interpolation?*

Question

Can the theory of IGL *be done in a constructive setting?*