IGL without sharps

Leonardo Pacheco
Institute of Science Tokyo
(j.w.w. Juan Pablo Aguilera)

7 July 2025

Available at: leonardopacheco.xyz/slides/lc2025.pdf

INTUITIONISTIC GÖDEL-LÖB LOGIC

- GL: $\Box(\Box P \to P) \to \Box P$.
- iGL: GL on an intuitionistic base, only boxes.
- IGL: GL on an intuitionistic base, boxes and diamonds. First developed by Das, van der Giessen and Marin.

INTUITIONISTIC GÖDEL-LÖB LOGIC

- Das, van der Giessen, Marin, "Intuitionistic Gödel-Löb logic, à la Simpson", 2024.
- Aguilera, Pacheco, "IGL without sharps", to appear.

$$
\begin{tikzcd}
\ell\mathsf{IGL} \vdash \varphi \arrow[rr] & & \mathcal{B}\mathsf{IGL} \models \varphi \arrow[d] \\
m\ell\mathsf{IGL} \vdash \varphi \arrow[u] & & \mathcal{P}\mathsf{IGL} \models \varphi \arrow[ll, "\Sigma^1_1\text{-Det}"'] \arrow[dl, "\Sigma^0_1\text{-Det}"] \\
& \mathsf{cm}\ell\mathsf{IGL} \vdash \varphi \arrow[ul] &
\end{tikzcd}
$$

$\omega m\ell$IGL

Let $\omega m\ell$IGL be the infinitary proof system with the ω-rule:

$$\frac{x : \Box^n \bot, \mathbf{R}, \Gamma \vdash \Delta \ (\forall n \in \omega)}{\mathbf{R}, \Gamma \vdash \Delta}.$$

Theorem

$\omega m\ell$IGL is complete w.r.t. IGL.

SOME INFERENCE RULES — I

$$\bot\mathrm{l}\;\frac{}{\mathbf{R}, \Gamma, x : \bot \vdash \Delta}$$

$$\wedge\mathrm{l}\;\frac{\mathbf{R}, \Gamma, x : A \wedge B, x : A, x : B \vdash \Delta}{\mathbf{R}, \Gamma, x : A \wedge B \vdash \Delta}$$

$$\Diamond\mathrm{r}\;\frac{\mathbf{R}, \Gamma \vdash \Delta, x : \Diamond A, \{y : A \mid xRy\}}{\mathbf{R}, \Gamma \vdash \Delta, x : \Diamond A}$$

$$\Box\mathrm{l}\;\frac{\mathbf{R}, \Gamma, x : \Box A, \{y : A \mid xRy\} \vdash \Delta}{\mathbf{R}, \Gamma, x : \Box A \vdash \Delta}$$

$$\Diamond\mathrm{l}\;\frac{\mathbf{R}, xRy, \Gamma, x : \Diamond A, y : A \vdash \Delta}{\mathbf{R}, \Gamma, x : \Diamond A \vdash \Delta}\;(y \text{ is fresh})$$

SOME INFERENCE RULES — II

Non-invertible rules:

$$\rightarrow\mathrm{r}\ \frac{\mathbf{R}, \Gamma, x : A \vdash x : B}{\mathbf{R}, \Gamma \vdash \Delta, x : A \rightarrow B}$$

$$\Box\mathrm{r}\ \frac{\mathbf{R}, xRy, \Gamma \vdash y : A}{\mathbf{R}, \Gamma \vdash \Delta, x : \Box A}\ (y \text{ is fresh})$$

cmℓIGL PROVES $\Box(\Box P \to P) \to \Box P$

$$
\to\mathrm{r}\,\dfrac{\Box\mathrm{r}\,\dfrac{\Box\mathrm{l}\,\dfrac{\to\mathrm{l}\,\dfrac{\text{id}\,\dfrac{}{xRy, x : \Box(\Box P \to P), y : \Box P \to P, y : P \vdash y : P} \qquad \Box\mathrm{l}\,\dfrac{\text{tr}\,\dfrac{xRy, yRz, xRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P \quad (*)}{xRy, yRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P}}{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P, y : \Box P}}{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P}}{xRy, x : \Box(\Box P \to P) \vdash y : P(*)}}{x : \Box(\Box P \to P) \vdash x : \Box P}}{\vdash x : \Box(\Box P \to P) \to \Box P}
$$

THE LOOP

$$xRy, yRz, xRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P$$

$$xRy, x : \Box(\Box P \to P) \vdash y : P$$

A PROOF SEARCH GAME FOR cmℓIGL

Given a sequent $\mathbf{R}, \Gamma \vdash \Delta$, we define an (open) game:

- Two players: Prover and Denier.
- Start on the sequent $\mathbf{R}, \Gamma \vdash \Delta$.
- When discussing a sequent S, Prover has two choices:
  - Pick an inference rule
    $$\frac{S_1 \quad \cdots \quad S_n}{S},$$
    and then Denier picks some S_i.
  - Draw a progressing loop from S to a previous sequent.
- Infinite plays are won by Denier.

Lemma

- *If Prover wins this game,* cmℓIGL *proves* $\mathbf{R}, \Gamma \vdash \Delta \in$ cmℓIGL.
- *If Denier wins this game, we can build a countermodel for* $\mathbf{R}, \Gamma \vdash \Delta$.

PREDICATE IGL MODELS

Tuple $M = \langle W, \preceq, \{D_w\}_{w \in W}, \{Pr_w\}_{w \in W}, \{R_w\}_{w \in W} \rangle$ such that:

1. W is a non-empty set of possible worlds;
2. $\preceq$ is a partial order on W;
3. D_w is the domain of $w \in W$;
4. Pr_w is the valuation over D_w;
5. R_w is a transitive relation in D_w;
6. all relations are monotone in $\preceq$.

We also require there is no infinite path moving through the R_w infinitely often.

THE GLOBAL CONDITION IS NECESSARY

$$\begin{array}{l}
\begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & & \vdots \\ \rotatebox{90}{$\preceq$} & & & & & \end{array} \\
\boxed{d_0 \longrightarrow d_1 \longrightarrow d_2 \longrightarrow d_3 \longrightarrow \cdots \longrightarrow d_n}
\end{array}$$

$$\begin{array}{l}
\begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ \rotatebox{90}{$\preceq$} & & & \end{array} \\
\boxed{d_0 \longrightarrow d_1 \longrightarrow d_2 \longrightarrow d_3} \\
\rotatebox{90}{$\preceq$} \\
\boxed{d_0 \longrightarrow d_1 \longrightarrow d_2} \\
\rotatebox{90}{$\preceq$} \\
\boxed{d_0 \longrightarrow d_1} \\
\rotatebox{90}{$\preceq$} \\
\boxed{d_0}
\end{array}$$

$V(P) = \emptyset$ implies $\models \Box(\Box P \to P)$ and $\models \neg\Box P$.

IGL DOES NOT PROVE $\Diamond P \to \Diamond(P \wedge \neg\Diamond P)$

w_2 $\boxed{x \longrightarrow y \longrightarrow z}$

$\curlyvee|$

w_1 $\boxed{x \longrightarrow y}$

P holds at y and z.

Definition

$\omega m\ell$IGL is obtained by adding to the basic rules the ω-rule:

$$\frac{x : \Box^n \bot, \mathbf{R}, \Gamma \vdash \Delta \ (\forall n \in \omega)}{\mathbf{R}, \Gamma \vdash \Delta}.$$

- No loops or infinite paths.
- If we restrict the right-hand side, we can define $\omega\ell$IGL.

A similar ω-rule for classical GL was studied by Yoshihito Tanaka.

COMPLETENESS

Lemma

$m\ell\mathsf{IGL} \vdash \varphi$ implies $\omega m\ell\mathsf{IGL} \vdash \varphi$.

Proof.

- Let T be an $m\ell\mathsf{IGL}$-proof of φ with $\vdash x : \varphi$ at its root.
- Let $n \in \omega$.
- Append $x : \Box^n \bot$ to the left-hand side of all sequents of T.
- The infinite paths of the new tree can be trimmed with applications of $\Box$l and $\bot$l.
- The trimmed proof T_n is an $\omega m\ell\mathsf{IGL}$-proof of $x : \Box^n \bot \vdash x : \varphi$.
- From all the T_n, an application of the ω-rule gives us an $\omega m\ell\mathsf{IGL}$-proof of φ. $\Box$

SOUNDNESS

Lemma

$\omega m\ell\mathsf{IGL} \vdash \varphi$ implies $\mathcal{P}\mathsf{IGL} \vdash \varphi$.

Proof.

- Suppose $\mathcal{P}\mathsf{IGL} \not\models \varphi$, then $\mathsf{cm}\ell\mathsf{IGL} \nvdash \varphi$.
- In the countermodel built in the completeness proof of $\mathsf{cm}\ell\mathsf{IGL}$, we can show that each R_w only has paths of length less than:

$$f(\varphi) := 2^{2^{|\mathrm{Sub}(\varphi)|}} \times 2^{|\mathrm{Sub}(\varphi)|} \times |\mathrm{Sub}(\varphi)| + 1.$$

- So $\Box^{f(\varphi)}\bot \to \varphi$ is not $\mathcal{P}\mathsf{IGL}$-valid.
- (This is a **extremely** rough bound.) $\square$

RESULTS

Theorem

For all modal formula φ, the following are equivalent:

- $\mathcal{P}\mathsf{IGL} \models \varphi$;
- $\ell\mathsf{IGL} \vdash \varphi$;
- $\mathsf{cm}\ell\mathsf{IGL} \vdash \varphi$; *and*
- $\omega m\ell\mathsf{IGL} \vdash \varphi$.

OPEN QUESTIONS

Question

Is $\mathsf{IK4} + \Box(\Box\varphi \to \varphi) \to \Box\varphi$ *complete w.r.t.* IGL*?*

Question

Is IGL *decidable?*

Question

Does IGL *have some form of interpolation?*

Question

Can the theory of IGL *be done in a constructive setting?*