Towards a characterization of the μ-calculus' collapse to modal logic

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Fixed-points in modal logic

Provability logic

Over GL, if X is in the scope of some $\Box$ in $\varphi(X)$, then there is ψ such that

$$\psi \leftrightarrow \varphi(\psi).$$

Epistemic logic

Common knowledge is

$$C\varphi := \varphi \wedge E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge \cdots$$

where R is the "everyone knows" modality. It can be thought as the greatest fixed-point of the operator

$$X \mapsto EX.$$

THE μ-CALCULUS

The μ-formulas are generated by the grammar:

$$\varphi := P \mid \neg P \mid X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \mu X.\varphi \mid \nu X.\varphi.$$

Let $M = \langle W, R, V\rangle$ be a Kripke model.
The semantics for μ and ν are as follows:

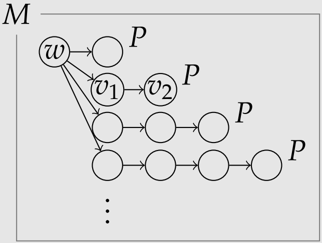
- $M, w \models \mu X.\varphi$ iff w is in the least fixed point of $\Gamma_{\varphi(X)}$;
- $M, w \models \nu X.\varphi$ iff w is in the greatest fixed point of $\Gamma_{\varphi(X)}$,

where

$$\Gamma_{\varphi(X)}(A) \to \|\varphi(A)\|^M$$

Game semantics — evaluation games

Verifier and Refuter discuss whether $\Box \mu X.P \vee \Diamond X$ holds at w.



V : $\Box \mu X.P \vee \Diamond X$ holds at w

R : $\mu X.P \vee \Diamond X$ fails at v_1

V : $P \vee \Diamond X$ holds at v_1

V : $\Diamond X$ holds at v_1

V : X holds at v_2

V : $P \vee \Diamond X$ holds at v_2

V : P holds at v_2

On an infinite run, if the variable with biggest scope which repeats infinitely often is ν, then Verifier wins.
Key point: on an infinite run, what matters is the *tail*.

Alternation depth

The valuation of νX and μY depend on each other:

$$\nu X.\, \underbrace{\mu Y.\, \overbrace{(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)}^{\text{scope of } \mu Y}}_{\text{scope of } \nu X}$$

Alternation depth of φ

Maximum number of codependent alternating μ and ν operators in φ.

Alternation hierarchy

Classifies μ-formulas with respect to their alternation depth.

GL has the fixed-point property

$$\mathsf{GL} := \mathsf{K} + \Box(\Box P \to P) \to \Box P$$

Theorem (de Jongh, Sambin)

Over GL*, if $\varphi(X)$ is a formula where X is in the scope of some $\Box$, then there is ψ such that*

$$\psi \leftrightarrow \varphi(\psi).$$

S5 DOES NOT HAVE THE FIXED-POINT PROPERTY

Theorem (Sacchetti)

Let L *be a logic with the fixed-point property. Then* L *can be invalidated in every frame containing a cycle, hence every finite frame for* L *is reverse well-founded.*

Therefore S5 does not have the fixed-point property. However:

Theorem (Alberucci, Facchini)

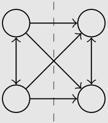
Over S5*, every μ-formula is equivalent to a formula without fixed-point operators.*

We say the μ-calculus collapses to modal logic over S5.

Theorem (P., Tanaka)

The alternation hierarchy collapses to modal logic over S4.3.2.

We may suppose an S4.3.2 frame can be divided into two equivalence classes:



At any long enough game, we will have equivalent positions:

$$\langle \nu X.\varphi, w\rangle \to \cdots \to \langle \Box\psi, v\rangle \to \cdots \to \langle \Box\psi, v'\rangle \to \cdots \to \langle \Box\psi, v''\rangle \to \cdots$$

We can use this fact to show that $\varphi(\varphi(\varphi((\top))) \equiv \varphi(\varphi(\varphi(\varphi(\top))))$.

Generalizing the proof

Definition

F is an n-pigeonhole frame iff for all sequence $w_0 R^* w_1 R^* \cdots R^* w_n$, there is $i < j \leq n$ such that $w_i R = w_j R$.

Definition

The μ-calculus n-uniformly collapses to modal logic over F iff, for all μ-formula φ with X positive,

$$\mu X.\varphi \equiv \varphi^n(\bot) \text{ and } \nu X.\varphi \equiv \varphi^n(\top).$$

Theorem

Fix $n \in \mathbb{N}$. Let F be a class of Kripke frames such that all frames F in F are n-pigeonhole frames. Then the μ-calculus $(n+1)$-uniformly collapses to modal logic over F.

AN OPEN PROBLEM

Proposition

Suppose that the μ-calculus $(n+1)$-uniformly collapses to modal logic over F. It does not follow that F is n-pigeonhole.

Question

Let F be a Kripke frame such that the μ-calculus n-uniformly collapses to modal logic over F. Is F is n-pigeonhole?

The answer is yes for $n = 1$ and $n = 2$.

COMMON KNOWLEDGE

- Common knowledge is

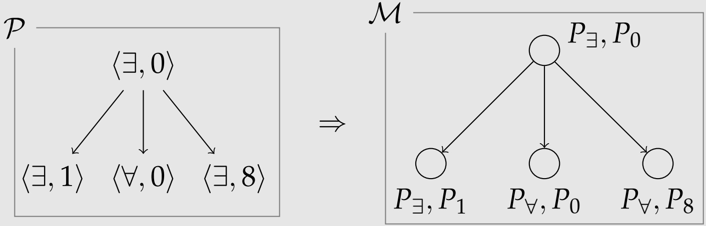
$$\begin{aligned} C\varphi :=& \varphi \wedge E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge \cdots \\ \equiv& \mu X.\varphi \wedge EX. \end{aligned}$$

  where R is the “everyone knows” modality.
- If there are two or more agents, common knowledge is not equivalent to a modal formula.

PARITY GAMES

- Two players $\exists$ and $\forall$ move a token in a graph.
- Each vertex is labeled with a natural number and an owner.
- $\exists$ wins a run $\rho = v_0, v_1, v_2, \dots$ iff the greatest label which appears infinitely often in ρ is even.
- Key point: on an infinite run, what matters is the *tail*.
- Evaluation games for the μ-calculus are parity games.

Parity games as Kripke models



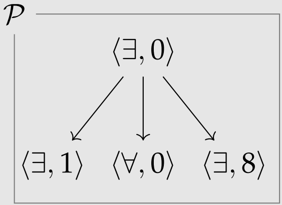
W_n describes the winning region for $\exists$ in parity games where n is the maximum parity:

$$W_n := \eta X_n \ldots \nu X_0. \bigvee_{0 \le j \le n} [(P_j \wedge P_\exists \wedge \Diamond X_j) \vee (P_j \wedge P_\forall \wedge \Box X_j)].$$

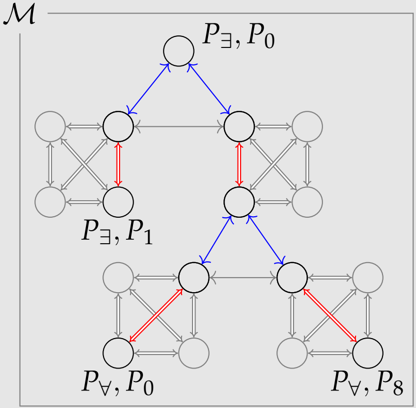
Theorem (Bradfield)

Let $n \in \omega$, then W_n is not equivalent to any formula with less alternation.

PARITY GAMES AS $\mathsf{S5}_2$ FRAMES



$\Rightarrow$



Bimodal winning region formulas

$$W'_n := \eta X_n \ldots \nu X_0. \bigvee_{0 \le j \le n} [(P_j \wedge P_\exists \wedge \blacklozenge X_j) \vee (P_j \wedge P_\forall \wedge \blacksquare X_j)].$$

Where

- $\blacklozenge \varphi := \mu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \wedge \Diamond_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \wedge \Diamond_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \wedge ((Y \wedge \neg \mathrm{st}) \vee (\varphi \wedge \mathrm{st}))))$; and
- $\blacksquare \varphi := \nu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \rightarrow \Box_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \rightarrow \Box_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \rightarrow ((Y \wedge \neg \mathrm{st}) \wedge (\varphi \wedge \mathrm{st}))))$,

Generalizing the non-collapse around $\mathsf{S5}_2$

Let L_0, L_1 and L_2 have disjoint sets of modal operators.

Theorem

If $\circ \to \circ$ *is a subframe of some* L_0*-frame and*

- $\circ \leftarrow \circ \to \circ$ *is a subframe of some* L_1*-frame; or*
- $\circ \to \circ \to \circ$ *is a subframe of some* L_1*-frame.*

Then the μ*-calculus' alternation hierarchy is strict over frames of* $\mathsf{L}_0 \otimes \mathsf{L}_1$.

Theorem

If $\circ \to \circ$ *is a subframe of some frames of* L_0, L_1, *and* L_2.
Then the μ*-calculus' alternation hierarchy is strict over frames of* $\mathsf{L}_0 \otimes \mathsf{L}_1 \otimes \mathsf{L}_2$.

An open problem

When does the μ-calculus' alternation hierarchy collapse over an *interesting* multimodal logic?

Example

The fixed-point theorem holds over GLP.

Example

The μ-calculus collapses to modal logic over CS5.

Non-example

The μ-calculus collapses over epistemic logic (knowledge + belief) for one agent.

OVERVIEW

- The μ-calculus $(n+1)$-uniformly collapses to modal logic over n-pigeonhole frames.
- Are n-uniformly collapsing frames also n-pigeonhole?
- The μ-calculus' alternation hierarchy is strict over most multimodal settings.
- Which restriction do we need to add between the modalities for the μ-calculus to collapse?

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