Exploring the difference hierarchies on μ-calculus and arithmetic

from the point of view of Gale–Stewart games

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Two problems related to the difference hierarchies:

- Collapse of the alternation hierarchy over various semantics.
- Relation between determinacy and reflection in second-order arithmetic.

MODAL LOGIC

- Modal logic = propositional logic + modalities
- Modalities specify *how* some formula is true.
- Examples:
  - $\Box\varphi$:= φ is necessary
  - $\Diamond\varphi$:= φ is possible
  - $K_a\varphi$:= the agent a knows that φ
  - $[a]\varphi$:= after a run of the program a, φ holds
  - $O\varphi$:= it is obligatory that φ
- Different logics are interpreted over different classes of Kripke models (labeled directed graphs).

μ-CALCULUS

- μ-calculus = modal logic + fixed-point operators
- Let E be the "everyone knows" modality.
- φ is common knowledge iff

$$\varphi \wedge E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge \cdots$$

- In the μ-calculus, φ is common knowledge iff

$$\nu X.\varphi \wedge EX.$$

μ-FORMULAS

Definition

- $P, \neg P, X, \bot, \top$ are μ-formulas.
- If φ, ψ are μ-formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi$ are μ-formulas.
- If φ is a μ-formulas, then $\Box\varphi$, $\Diamond\varphi$, $\nu X.\varphi$, $\mu X.\varphi$ are μ-formulas.

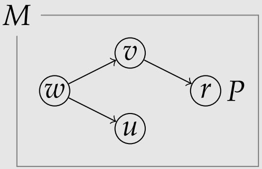
Examples:

- $\mu X.P \vee \Diamond X$,
- $\nu X.\mu Y.(P \wedge \Diamond X) \vee (\neg Q \wedge \Diamond Y)$.

SEMANTICS — MODALITIES

Kripke models := tuple $\langle\langle W, \rightarrow\rangle, V\rangle$ where

- $\langle W, \rightarrow\rangle$: directed graph,
- V: labeling.



$\Box\varphi$ holds at w iff φ holds at *all* accessible worlds.
$\Diamond\varphi$ holds at w iff φ holds at *some* accessible world.

- $\Diamond\Box P$ holds at w
- $\Box\Diamond P$ fails at w

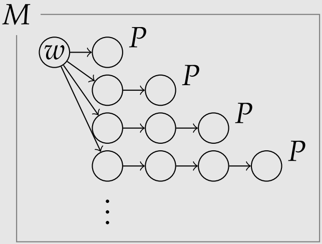
$\|\varphi\|$ is the set of worlds where φ holds.

SEMANTICS — FIXED-POINTS

Define $\Gamma_{\varphi(X)}(A) = \|\varphi(A)\| = \{w \mid \varphi(A) \text{ holds at } w\}$.

- $\mu X.\varphi(X)$ is the least fixed-point of $\Gamma_{\varphi(X)}$
- $\nu X.\varphi(X)$ is the greatest fixed-point of $\Gamma_{\varphi(X)}$

Let $\psi(X) := P \vee \Diamond X$:

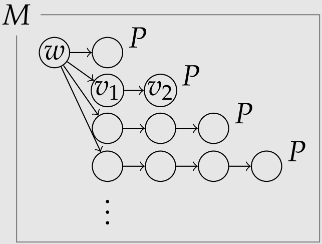


$$\psi(\bot) \equiv P$$
$$\psi(\psi(\bot)) \equiv P \vee \Diamond P$$
$$\psi(\psi(\psi(\bot))) \equiv P \vee \Diamond P \vee \Diamond\Diamond P$$
$$\psi(\psi(\psi(\psi(\bot)))) \equiv P \vee \Diamond P \vee \Diamond\Diamond P \vee \Diamond\Diamond\Diamond P$$
$$\vdots$$

- $\mu X.P \vee \Diamond X \approx P \vee \Diamond P \vee \Diamond\Diamond P \vee \cdots$
- $\Box \mu X.P \vee \Diamond X$ holds at w.
- $\Box \mu X.P \vee \Diamond X$ is not equivalent to any modal formula.

GAME SEMANTICS — EVALUATION GAMES

Verifier and Refuter discuss whether $\Box \mu X.P \vee \Diamond X$ holds at w.



V : $\Box \mu X.P \vee \Diamond X$ holds at w

R : $\mu X.P \vee \Diamond X$ fails at v_1

V : $P \vee \Diamond X$ holds at v_1

V : $\Diamond X$ holds at v_1

V : X holds at v_2

V : $P \vee \Diamond X$ holds at v_2

V : P holds at v_2

- μ repeats infinitely often $\implies$ Refuter wins
- ν repeats infinitely often $\implies$ Verifier wins

Kripke semantics and game semantics are equivalent.

ALTERNATION DEPTH

The valuation of νX and μY depend on each other:

$$\nu X. \underbrace{\mu Y. \overbrace{(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)}^{\text{scope of } \mu Y}}_{\text{scope of } \nu X}$$

Alternation depth of φ

Maximum number of codependent alternating μ and ν operators in φ.

Alternation hierarchy

Classifies μ-formulas with respect to their alternation depth.

ALTERNATION-HIERARCHY — EXISTING RESULTS

The AH ...	condition on graph	logic
is strict	none	K
($\approx \mathcal{B}(G_\delta)$)	reflexive	T
	symmetric	B
collapses to	weakly transitive	wK4
alternation-free frag.	transitive	K4
($\approx \mathcal{B}(G)$)	reflexive and transitive	S4
	trans, refl., convergent	S4.2
	trans., refl., weakly connected	S4.3
collapses to	trans., refl., semi-euclidean	S4.3.2
modal logic	(no name)	S4.4
	equivalence relations	S5

Results in red are new.

COLLAPSE TO MODAL LOGIC

Theorem (P., Tanaka)

Fix $n \in \omega$. Let M be a Kripke model such that, for all $\{w_0, \dots, w_n\} \subseteq W$ and $i < n$, there is a $\to$-path from w_i to w_{i+1}. Then

- $\mu X.\varphi$ *is equivalent to* $\overbrace{\varphi(\cdots(\varphi}^{n \text{ many times}}(\bot)))$, *and*
- $\nu X.\varphi$ *is equivalent to* $\overbrace{\varphi(\cdots(\varphi}^{n \text{ many times}}(\top)))$.

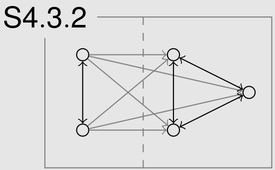
Why? Write $\mu X, \varphi(X)$ as $\mu X.\alpha(\Diamond\beta(X))$.
In an evaluation game:

$$\cdots \to \Diamond\beta(X) \text{ holds at } w_i \to \cdots \to \Diamond\beta(X) \text{ holds at } w_j \to \cdots$$

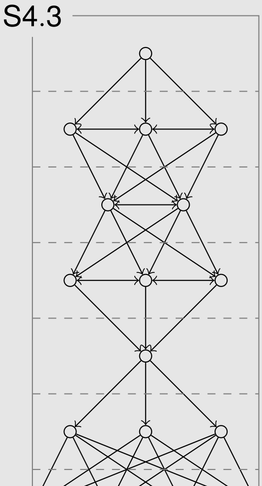
COLLAPSE TO MODAL LOGIC — EXAMPLES

The models below are reflexive and transitive.

Collapse to modal logic



Collapse to alternation-free fragment



VARIATIONS

Theorem (P., Tanaka)

The alternation hierarchy collapses to modal logic over equivalence relations on

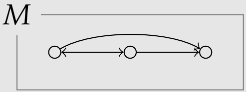
- *Non-normal semantics: worlds where everything is possible and nothing is necessary.*
- *Graded semantics: $\Diamond^{>n}\varphi$ holds iff there are more than n accessible worlds where φ holds.*
- *Intuitionistic semantics: no excluded middle.*

The alternation hierarchy is strict over equivalence relations on

- *Multimodal semantics.*

A model is weakly transitive iff

$$wRuRv \implies wRv \text{ or } w = v.$$



On weakly transitive models,

- $\Diamond\mu X.\varphi$ is equivalent to $\Diamond\varphi(\varphi(\bot))$,
- $\Box\nu X.\varphi$ is equivalent to $\Box\varphi(\varphi(\top))$.

Theorem (P., Tanaka)

Every μ-formula is equivalent to an alternation-free formula, over weakly transitive models.

TOPOLOGICAL SEMANTICS

Instead of directed graphs $\langle W, \rightarrow \rangle$, we consider topological spaces $\langle W, \tau \rangle$.
Over $\mathcal{X} = \langle \langle W, \tau \rangle, V \rangle$,

$$\Diamond\varphi \text{ holds at } w \text{ iff } w \text{ is a limit point of } \|\varphi\|.$$

Theorem (P., Tanaka)

Every μ-formula is equivalent to an alternation-free formula over topological semantics.

Proof sketch.

- Suppose $\mathcal{X}, w \models \varphi \wedge \neg\psi$.
- Then $\mathcal{X}_{\text{fin}}, w \models \varphi \wedge \neg\psi$, with $\mathcal{X}_{\text{fin}}$ finite.
- Then $M, w \models \varphi \wedge \neg\psi$, with M weakly transitive. $\square$

DIFFERENCE HIERARCHY

Difference hierarchy of open sets:

- First level: open sets.
- Second level: $A \setminus B$, with A, B open.
- Third level: $A \setminus (B \setminus C)$, with A, B, C open.
- $\ldots$

Booleans combinations of open sets are obtained by $\cap$, $\cup$, and $\neg$ from open sets.
The difference hierarchy classifies all boolean combinations.

GALE–STEWART GAMES

Fix a payoff $A \subseteq \mathbb{N}^{\mathbb{N}}$.
I and II alternate picking natural numbers.

$$\begin{array}{lcccccccccccc} \mathsf{I} & x_0 & & x_2 & & x_4 & & \cdots & & x_{2n} & & \cdots & \\ \mathsf{II} & & x_1 & & x_3 & & x_5 & & \cdots & & x_{2n+1} & & \cdots \end{array}$$

I wins iff $\alpha = x_0 x_1 x_2 \cdots \in A$.
A game is determined iff one of the players has a winning strategy.

Determinacy axioms

Γ-Det states that every game with payoff in Γ is determined.

μ-calculus and determinacy

Consider graphs over the natural numbers.

Theorem (Bradfield, Quickert, Duparc)

The μ-calculus defines the winning regions of Gale–Stewart games whose payoffs are boolean combinations of G_δ sets.

Theorem (P., Li, Tanaka)

The alternation-free μ-calculus defines the winning regions of Gale–Stewart games whose payoffs are boolean combinations of open sets.

REVERSE MATHEMATICS

Which set existence axioms are needed to prove the theorems of mathematics?

- RCA_0 proves
  - the intermediate value theorem;
  - existence of an algebraic closure of a countable field;
  - determinacy of finite games.
- ACA_0 is equivalent to
  - the Ascoli lemma;
  - every countable commutative ring has a maximal ideal;
  - König's lemma.
- Π^1_1-CA_0 is equivalent to
  - the Cantor/Bendixson theorem;
  - every countable Abelian group is the direct sum of a divisible group and a reduced group;
  - determinacy for difference of two open sets.

REFLECTION PRINCIPLES

- Π^1_n-formula: $\forall X_n \exists X_{n-1} \dots \mathsf{Q} X_1.\varphi(X_1, \dots, X_n)$, where φ has no set quantifier.
- Π^1_n-$\mathsf{Ref}(T)$ states that

$$\text{if } \varphi \text{ is a } \Pi^1_n\text{-formula provable in } T\text{, then } \varphi \text{ is true.}$$

Theorem (Kołodziejczyk, Michalewski)

Over Π^1_2-CA_0,

$$\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_2\text{-}\mathsf{CA}_0) \textit{ is equivalent to } \forall n.(\Sigma^0_2)_n\text{-}\mathsf{Det}.$$

We improved this theorem and proved variations.

RESULTS ON DETERMINACY

- Π^1_2-Ref(ACA_0) is equivalent to $\forall n.(\Sigma^0_1)_n$-Det*.
- ATR_0 is equivalent to Δ^0_1-Det and Σ^0_1-Det.
- Π^1_1-CA_0 is equivalent to $(\Sigma^0_1)_2$-Det.
- Π^1_3-Ref(Π^1_1-CA_0) is equivalent to $\forall n.(\Sigma^0_1)_n$-Det.
- Σ^1_1-ID is equivalent to Σ^0_2-Det.
- Π^1_2-CA_0 proves $(\Sigma^0_2)_2$-Det.
- Π^1_3-Ref(Π^1_2-CA_0) is equivalent to $\forall n.(\Sigma^0_2)_n$-Det.

Results in red are new.

REFLECTION AND SEQUENCES OF MODELS

Let $\psi_e(i,n)$ state the existence of $Y_0, \ldots, Y_n$ such that:

$$\begin{aligned} X \in Y_0 \; \in \; \cdots \; \in \; Y_n, \\ Y_0 \subseteq_{\beta_i} \cdots \subseteq_{\beta_i} Y_n \subseteq_{\beta_e} \mathcal{N}. \end{aligned}$$

Theorem (P., Yokoyama)

Over ACA_0*, if* $e \leq i$ *then* $\forall n.\psi_e(i,n)$ *is equivalent to* Π^1_{e+2}-Ref(Strong Σ^1_i-DC_0).

OUR RESULTS — KEY IDEAS

Π^1_3-$\mathsf{Ref}(\Pi^1_1$-$\mathsf{CA}_0)$ is equivalent to $\forall n.(\Sigma^0_1)_n$-Det

- Both are equivalent to $\forall n.\psi_1(1, n)$.

Π^1_3-$\mathsf{Ref}(\Pi^1_2$-$\mathsf{CA}_0)$ is equivalent to $\forall n.(\Sigma^0_2)_n$-Det

- Both are equivalent to $\forall n.\psi_1(2, n)$, needs to use multiple inductive definitions.

Π^1_2-$\mathsf{Ref}(\mathsf{ACA}_0)$ is equivalent to $\forall n.(\Sigma^0_1)_n$-Det^*

- Modified version of the $\forall n.\psi_e(i, n)$.

References I

- Luca Alberucci and Alessandro Facchini. "The Modal μ-Calculus Hierarchy over Restricted Classes of Transition Systems". In: *The Journal of Symbolic Logic* 74.4 (2009), pp. 1367–1400.
- Julian C. Bradfield. "The Modal Mu-Calculus Alternation Hierarchy Is Strict". In: *Theoretical Computer Science* 195.2 (1998), pp. 133–153.
- Leszek Aleksander Kołodziejczyk and Henryk Michalewski. "How Unprovable Is Rabin's Decidability Theorem?". In: *2016 31st Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. 2016, pp. 1–10.

REFERENCES II

- Leonardo Pacheco, Wenjuan Li, and Kazuyuki Tanaka. "On One-Variable Fragments of Modal μ-Calculus". In: *Computability Theory and Foundations of Mathematics*. Wuhan, China: WORLD SCIENTIFIC, 2022, pp. 17–45.
- Leonardo Pacheco and Kazuyuki Tanaka. "The Alternation Hierarchy of the μ-Calculus over Weakly Transitive Frames". In: *Logic, Language, Information, and Computation*. Vol. 13468. 2022, pp. 207–220.
- Leonardo Pacheco and Keita Yokoyama. *Determinacy and Reflection Principles in Second-Order Arithmetic*. 2022. arXiv: `2209.04082 [math]`.