

Non-classical modal logics

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NON-CLASSICAL MODAL LOGICS

- ▶ modal logic = propositional logic + \Box + \Diamond .
- ▶ Two main non-classical varieties:
 - ▶ constructive modal logic, and
 - ▶ intuitionistic modal logic.

A BIT OF HISTORY

- ▶ Fitch (1948): intuitionistic first-order modal logic (with T and Barcan's formula)
- ▶ Prior (1957): MIPQ, an intuitionistic analogue of S5
- ▶ Ono (1977), Fischer Servi (1978): completeness of MIPQ
- ▶ Many people work on \Diamond -free intuitionistic modal logics
- ▶ Wijesekera (1990): constructive modal logic

For a better survey see Simpson (1994).

CK — AXIOMATIZATION

Axioms:

- ▶ all intuitionistic tautologies;
- ▶ $K_{\Box} := \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$;
- ▶ $K_{\Diamond} := \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$.

Rules:

$$(\mathbf{Nec}) \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP}) \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}.$$

IK — AXIOMATIZATION

Axioms:

- ▶ all intuitionistic tautologies;
- ▶ $K_{\Box} := \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$;
- ▶ $K_{\Diamond} := \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$;
- ▶ $FS := (\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi)$;
- ▶ $DP := \Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi$; and
- ▶ $N := \neg\Diamond\perp$.

Rules:

$$(\mathbf{Nec}) \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP}) \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}.$$

KRIPKE MODELS FOR MODAL LOGIC

Tuples $M = \langle W, R, V \rangle$ where:

- ▶ W is the set of *possible worlds*;
- ▶ R is a relation over W ; and
- ▶ $V : \text{Prop} \rightarrow \mathcal{P}(W)$ is a valuation function.

We define:

- ▶ $M, w \models \Box\varphi$ iff, for all v , if wRv then $M, v \models \varphi$;
- ▶ $M, w \models \Diamond\varphi$ iff there is v such that wRv and $M, v \models \varphi$.

KRIPKE MODELS FOR INTUITIONISTIC LOGIC

Relational Kripke models $M = \langle W, \preceq, V \rangle$ where:

- ▶ W is the set of *possible worlds*;
- ▶ \preceq is a reflexive and transitive relation over W ;
- ▶ $V : \text{Prop} \rightarrow \mathcal{P}(W)$ is a valuation function.

We require that:

- ▶ $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$.

We define:

- ▶ $M, w \models \varphi \rightarrow \psi$ iff, for all v , if $w \preceq v$ and $M, v \models \varphi$, then $M, v \models \psi$;
- ▶ $M, w \models \neg\varphi$ iff, for all v , if $w \preceq v$, then $M, v \not\models \varphi$.

CK — SEMANTICS

Bi-relational Kripke models $M = \langle W, W^\perp, \preceq, R, V \rangle$ where:

- ▶ W is the set of *possible worlds*;
- ▶ $W^\perp \subseteq W$ is the set of *fallible worlds*;
- ▶ \preceq is a reflexive and transitive relation over W ;
- ▶ R is a relation over W ; and
- ▶ $V : \text{Prop} \rightarrow \mathcal{P}(W)$ is a valuation function.

We require that:

- ▶ $W^\perp \subseteq V(P)$;
- ▶ $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$.

IK — SEMANTICS

Bi-relational Kripke models $M = \langle W, \preceq, R, V \rangle$ where:

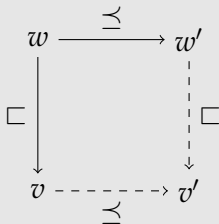
- ▶ W is the set of *possible worlds*;
- ▶ \preceq is a reflexive and transitive relation over W ;
- ▶ R is a relation over W ; and
- ▶ $V : \text{Prop} \rightarrow \mathcal{P}(W)$ is a valuation function.

We require that:

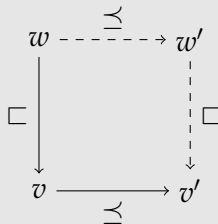
- ▶ $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$;
- ▶ M is *forward confluent*: $w \preceq w'$ and $w \sqsubset v$ imply there is v' such that $v \preceq v'$ and $w' \sqsubset v'$;
- ▶ M is *backward confluent*: $w \sqsubset v \preceq v'$ implies then there is w' such that $w \preceq w' \sqsubset v'$.

CONFLUENCE

Forward confluence



Backward confluence



VALUATIONS

The valuation of \Box s are the same over CK and IK models:

- ▶ $M, w \models \Box\varphi$ iff, for all v, u , if $w \preceq vRu$ then $M, u \models \varphi$.

Over CK models, define:

- ▶ $M, w \models \Diamond\varphi$ iff, for all v such that $w \preceq v$, there is u such that if vRu and $M, u \models \varphi$.

Over IK models, define:

- ▶ $M, w \models \Diamond\varphi$ iff there is v such that if wRv and $M, v \models \varphi$.

SEPARATING CK AND IK

The following formulas are provable in IK but not in CK:

- ▶ $FS := (\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi);$
- ▶ $DP := \Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi;$ and
- ▶ $N := \neg\Diamond\perp.$

All of these involve \Diamond s.

Question

Do CK and IK prove the same \Diamond -free formulas?

The answer is no!¹

¹Das, Marin

SEPARATION

- ▶ $\text{CK} \not\models \neg\neg\Box\perp \rightarrow \Box\perp$:

$$w \preceq v \models \perp$$

- ▶ $w \models \neg\neg\Box\perp$ iff, for all $w' \succeq w$, there is $w'' \succeq w'$ such that $w'' \models \Box\perp$.
- ▶ But $\text{IK} \vdash \neg\neg\Box\perp \rightarrow \Box\perp$.

CS4 AND IS4

CS4 and IS4 are obtained by adding to CK and IK the axioms:

- ▶ $4_{\Box} := \Box P \rightarrow \Box \Box P;$
- ▶ $4_{\Diamond} := \Diamond \Diamond P \rightarrow \Diamond P;$
- ▶ $T_{\Box} := \Box P \rightarrow P;$
- ▶ $T_{\Diamond} := P \rightarrow \Diamond P.$

AN EXAMPLE

- Consider the following model M :

$$x \preceq y \sqsubseteq z \preceq t \sqsubseteq w.$$

where P holds at $\{x, y, z, t\}$.

- The relation R is transitive, but the formula $\Box P \rightarrow \Box\Box P$ fails at x .

CS4 AND IS4 MODELS

A CS4 model is a CK model $M = \langle W, W^\perp, \preceq, R, V \rangle$ where

- ▶ R is transitive and reflexive;
- ▶ M is *backward confluent*: $w \sqsubset v \preceq v'$ implies then there is w' such that $w \preceq w' \sqsubset v'$.

An IS4 model is a IK model $M = \langle W, \preceq, R, V \rangle$ where

- ▶ R is transitive and reflexive.

CS4 and IS4 are complete w.r.t. CS4 and IS4 models.

DECIDABILITY

The following was an open problem for ~ 10 years:

Theorem (Balbiani, Dieguez, Fernández-Duque)

CS4 is decidable.

This was solved using canonical models and bisimulations.
The following was an open problem for ~ 30 years:

Theorem (Girlando *et al.*)

IS4 is decidable.

This was solved using labeled proof systems.

GL

- ▶ GL is the logic obtaining by adding

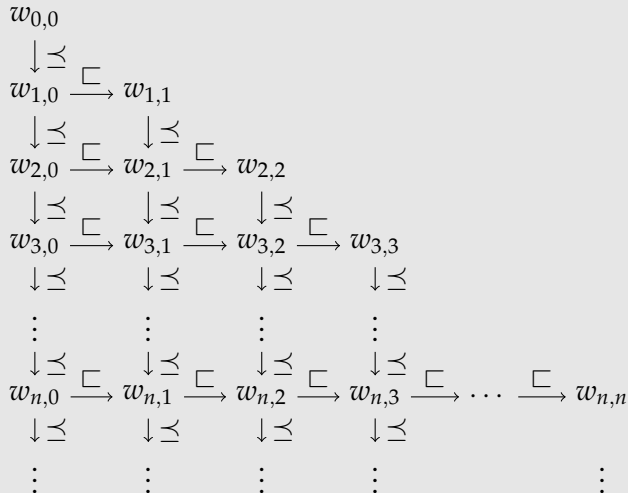
$$L := \Box(\Box P \rightarrow P) \rightarrow \Box P$$

to K .

- ▶ GL is complete with respect to models $M = \langle W, R, V \rangle$ where R is transitive and reverse well-founded.
- ▶ R is reverse well-founded iff there is no infinite sequence $w_0 R w_1 R w_2 R \dots$

REV. WF IS NOT (INTUITIONISTICALLY) ENOUGH

In IK model below, no world satisfies $\Box(\Box P \rightarrow P) \rightarrow \Box P$.



(P holds nowhere)

IGL MODELS

An IGL model is an IK model $M = \langle W, \preceq, R, V \rangle$ where

- ▶ R is transitive;
- ▶ the composition $\preceq; R$ is reverse well-founded.

A PROOF SYSTEM FOR IGL

Das, van der Giessen and Marin proved that an infinitary proof system based on the following is complete over IGL frames:

$$\Box_I \frac{R, xRy, \Gamma, y : A \Rightarrow \Delta}{R, xRy, \Gamma, x : \Box A \Rightarrow \Delta}$$

$$\Box_r \frac{R, xRy, \Gamma \Rightarrow \Delta, y : A}{R, \Gamma \Rightarrow \Delta, x : \Box A} (y \text{ fresh})$$

$$\text{tr} \frac{R, xRy, yRz, xRz, \Gamma \Rightarrow \Delta}{R, xRy, yRz, \Gamma \Rightarrow \Delta}$$

CGL AND IGL MODELS

A CGL model is a CK model $M = \langle W, W^\perp, \preceq, R, V \rangle$ where

- ▶ R is transitive;
- ▶ R is forward confluent;
- ▶ the composition $\preceq; R$ is reverse well-founded.

An IGL model is an IK model $M = \langle W, \preceq, R, V \rangle$ where

- ▶ R is transitive;
- ▶ the composition $\preceq; R$ is reverse well-founded.

AN AXIOMATIZATION FOR CGL AND IGL?

To obtain CGL and IGL, add to CK and IK the axioms:

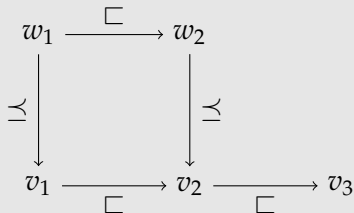
- ▶ $4_{\Box} := \Box\varphi \rightarrow \Box\Box\varphi$;
- ▶ $4_{\Diamond} := \Diamond\Diamond\varphi \rightarrow \Diamond\varphi$; and
- ▶ $L_{\Box} := \Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$.

Question

Are CGL and IGL complete over CGL and IGL models?

THE DUAL OF LÖB'S THEOREM

$L_{\Diamond} := \Diamond P \rightarrow \Diamond(P \wedge \Box \neg P)$ of fails at w_1 :



(P holds everywhere.)

FAILURE TO PROVE THE COMPLETENESS

Proofs of completeness using finitary canonical models seem to need some diamond version of

$$L_{\Box} := \Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi.$$

Reiterating:

Question

Are CGL and IGL complete over CGL and IGL models?

Question

If the answer to the above is negative:

- ▶ *Are there a complete axiomatization for CGL and IGL models?*
- ▶ *What class of models are characterized by the systems CGL and IGL?*

THE μ -CALCULUS

μ -calculus = modal logic + fixed-point operators

If X is positive, then:

- ▶ $\|\mu X.\varphi\|^M :=$ least fixed-point of $A \mapsto \|\varphi(A)\|^M$
- ▶ $\|\nu X.\varphi\|^M :=$ greatest fixed-point of $A \mapsto \|\varphi(A)\|^M$

ALTERNATION DEPTH

The valuation of νX and μY depend on each other:

$$\nu X. \underbrace{\mu Y. \overbrace{(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)}^{\text{scope of } \mu Y}}_{\text{scope of } \nu X}$$

Alternation depth of φ

Maximum number of codependent alternating μ and ν operators in φ .

Alternation hierarchy

Classifies μ -formulas with respect to their alternation depth.

SOME RESULTS ON THE UNIMODAL μ -CALCULUS

Theorem (Bradfield)

The μ -calculus alternation hierarchy is strict over all frames.

Theorem (Alberucci–Facchini)

The μ -calculus alternation hierarchy collapses to the alternation-free fragment over transitive frames.

Theorem (Alberucci–Facchini)

The μ -calculus alternation hierarchy collapses to modal logic over equivalence relations.

For a survey, see my PhD thesis.

VARIATIONS OF S5

CS5 and IS5 are obtained by adding to CK and IK the axioms:

- ▶ $4_{\Box} := \Box P \rightarrow \Box \Box P;$
- ▶ $4_{\Diamond} := \Diamond \Diamond P \rightarrow \Diamond P;$
- ▶ $5_{\Box} := \Diamond P \rightarrow \Box \Diamond P;$
- ▶ $5_{\Diamond} := \Diamond \Box P \rightarrow \Box P;$
- ▶ $T_{\Box} := \Box P \rightarrow P;$
- ▶ $T_{\Diamond} := P \rightarrow \Diamond P.$

CS5 and IS5 models are CS4 and IS4 models where the modal relation is an equivalence relation.

COLLAPSE OVER CS5/IS5 MODELS

Lemma

Let $M = \langle W, W^\perp, \preceq, \equiv, V \rangle$ be a CS5 model and $w \preceq; \equiv w'$. Then

$$M, w \models \Delta\varphi \text{ implies } M, w' \models \Delta\varphi,$$

where $\Delta \in \{\Box, \Diamond\}$.

At any long enough evaluation game, we will have positions:

$$\langle \nu X.\varphi, w \rangle \rightarrow \cdots \rightarrow \langle \Box\psi, v \rangle \rightarrow \cdots \rightarrow \langle \Box\psi, v' \rangle \rightarrow \cdots$$

We can use this fact to show that $\varphi(\varphi(\top)) \equiv \varphi(\varphi(\varphi(\top)))$.

COLLAPSE OVER CS4/IS4 MODELS

Question

Does the μ -calculus collapse to its alternation free-fragment over CS4 and IS4 models?

All the proofs I know of the collapse over S4 fail on non-classical settings.

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