Game semantics for the constructive μ-calculus

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AXIOMATIZATION

The axioms of CK are:

- all intuitionistic tautologies;
- $K := \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \wedge \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$;

CS5 is closed under necessitation and *modus ponens*:

$$\frac{\varphi}{\Box\varphi} \quad \text{and} \quad \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

Semantics for constructive modal logic

Constructive Kripke model are tuples $\langle W, W^{\perp}, \preceq, R, V \rangle$ with:

- W = set of possible worlds;
- $W^{\perp}$ = set of fallible worlds;
- $\preceq$ = intuitionistic relation;
- R = modal relation;
- V = valuation;

We require that $wR;\preceq v$ implies $w \preceq; Rv$.
Define

- $M, w \models \Box\varphi$ iff $\forall v \succeq w \forall u.vRu$ implies $M, u \models \varphi$;
- $M, w \models \Diamond\varphi$ iff $\forall v \succeq w \exists u.vRu$ and $M, u \models \varphi$.

Theorem

CK *is complete over constructive Kripke frames.*

GAME SEMANTICS — I

Given a formula φ, a Kripke model M and a world w, we define an evaluation game for $M, w \models \varphi$

- Two players: I and II;
- Two roles: Verifier and Refuter, I starts as Verifier;
- Examples of moves;
  - At $\langle \psi \wedge \theta, w \rangle$, Refuter moves to one of $\langle \psi, w \rangle$ and $\langle \theta, w \rangle$.
  - At $\langle \Diamond \psi, w \rangle$, Refuter picks $v \succeq w$ and Verifier picks u with vRu, the players move to $\langle \psi, u \rangle$.
  - At $\langle \neg \psi, w \rangle$, the players switch roles and move to $\langle \psi, w \rangle$.
  - At $\langle P, w \rangle$, Verifier wins iff $w \in V(P)$.
- $M, w \models \varphi$ iff Verifier wins the evaluation game.

AN ASIDE — OTHER CONSTRUCTIVE VARIATIONS

We get IK by adding to CK the axioms:

- $FS := (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$;
- $DP := \Diamond(\varphi \lor \psi) \to \Diamond\varphi \lor \Diamond\psi$; and
- $N := \neg\Diamond\bot$.

We get GK by adding to IK the axiom:

- $GD := (\varphi \to \psi) \lor (\psi \to \varphi)$.

These systems are complete over certain classes of constructive Kripke frames.

BASIC DEFINITIONS

The constructive μ-formulas are defined by the following grammar:

$$\varphi := P \mid X \mid \bot \mid \top \mid \neg\varphi \mid \varphi\wedge\varphi \mid \varphi\vee\varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \mu X.\varphi \mid \nu X.\varphi,$$

$\mu X.\varphi$ and $\nu X.\varphi$ are defined iff X is positive* in φ.
The semantics for μ and ν are as follows:

- $M, w \models \mu X.\varphi$ iff w is in the least fixed point of $\Gamma_{\varphi(X)}$;
- $M, w \models \nu X.\varphi$ iff w is in the greatest fixed point of $\Gamma_{\varphi(X)}$,

where

$$\Gamma_{\varphi(X)}(A) := \|\varphi(A)\|^M.$$

EVALUATING $\mu X.P \vee \Diamond X$

- Let M be a Kripke model.
- $\|\mu X.P \vee \Diamond X\|^M$ is the least fixed-point of $A \mapsto P \vee \Diamond A$.
- We can approximate this fixed-point by

$$\emptyset \mapsto \|P\|^M \mapsto \|P \vee \Diamond P\|^M \mapsto \|P \vee \Diamond P \vee \Diamond\Diamond P\|^M \mapsto \cdots$$

- We have

$$\mu X.P \vee \Diamond X \equiv P \vee \Diamond P \vee \Diamond\Diamond P \vee \cdots$$

GAME SEMANTICS — II

Consider the evaluation game for $\mu X.P \vee \Diamond X$:

- From $\langle \mu X.P \vee \Diamond X, v\rangle$, the players move to $\langle P \vee \Diamond X, v\rangle$.
- From $\langle X, v\rangle$, the players move to $\langle \mu X.P \vee \Diamond X, v\rangle$.
- Verifier loses if the operator μX is not regenerated infinitely often.
- That is, Verifier loses all infinite runs.

EVALUATING $\nu X \mu Y.\varphi(X, Y) := \nu X \mu Y.(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)$

- Let M be a constructive Kripke model.
- $\|\nu X \mu Y.\varphi(X, Y)\|^M$ is the least fixed-point of $A \mapsto \mu Y.\varphi(A, Y)$.
- To approximate this fixed-point, we do as follows:
  - $X_0 := W$;
  - Y_0 is the least-fixed point of $\Gamma_{\varphi(X_0, Y)}$;
  - $X_1 := \|\varphi(X_0, Y_0)\|^M$;
  - Y_1 is the least-fixed point of $\Gamma_{\varphi(X_1, Y)}$;
  - $\dots$
  - $X_{\alpha+1} := \|\varphi(X_\alpha, Y_\alpha)\|^M$;
  - $Y_{\alpha+1}$ is the least-fixed point of $\Gamma_{\varphi(X_{\alpha+1}, Y)}$;
  - $\dots$
- $\|\nu X \mu Y.\varphi(X, Y)\|$ is the least X_α such that $X_\alpha = X_{\alpha+1}$.

Game Semantics — III

- Let $\eta X.\psi_X$ be the infinitely often regenerated formula with biggest scope.
- The player on role of Verifier at $\langle \eta X.\psi_X, v\rangle$ wins iff η is ν.
- The complete set of rules is:

Verifier

Position	Admissible moves
$\langle v, \psi_1 \vee \psi_2\rangle$	$\{\langle v, \psi_1\rangle, \langle v, \psi_2\rangle\}$
$\langle v, \psi_0 ? \psi_1\rangle$	$\{\langle v, \psi_0\rangle$ and exchange roles, $\langle v, \psi_1\rangle\}$
$\langle \langle v\rangle, \psi\rangle$	$\{\langle u, \psi\rangle \mid v \sqsubseteq u\}$
$\langle v, P\rangle$ and $v \not\in V(P)$	$\emptyset$
$\langle v, \mu X.\psi_X\rangle$	$\{\langle v, \psi_X\rangle\}$
$\langle v, X\rangle$	$\{\langle v, \psi_X\rangle\}$

Refuter

Position	Admissible moves
$\langle v, \psi_1 \wedge \psi_2\rangle$	$\{\langle v, \psi_1\rangle, \langle v, \psi_2\rangle\}$
$\langle v, \neg\psi\rangle$	$\{\langle u, \psi\rangle \mid v \preceq v\}$ and exchange roles
$\langle v, \psi_1 \to \psi_2\rangle$	$\{\langle u, \psi_0 ? \psi_1\rangle \mid v \preceq v\}$
$\langle v, \Diamond\psi\rangle$	$\{\langle \langle u\rangle, \psi\rangle \mid v \preceq u\}$
$\langle v, \Box\psi\rangle$	$\{\langle [u], \psi\rangle \mid v \preceq u\}$
$\langle [v], \psi\rangle$	$\{\langle u, \psi\rangle \mid v \sqsubseteq u\}$
$\langle v, P\rangle$ and $v \in V(P)$	$\emptyset$
$\langle v, \nu X.\psi_X\rangle$	$\{\langle v, \psi_X\rangle\}$
$\langle v, X\rangle$	$\{\langle v, \psi_X\rangle\}$
$\langle v, \psi\rangle, v \in W^{\perp}$ and $\psi \in \mathrm{Sub}(\varphi)$	$\emptyset$

POSITIVENESS — A TECHNICAL POINT

Proposition

Suppose that X is positive in $\varphi(X)$, then $\Gamma_\varphi(X)$ is monotone.

- X is positive and negative in P;
- X is positive in X ;
- if $Y \neq X$, X is positive and negative in Y ;
- if X is positive (negative) in φ, then X is negative (positive) in $\neg\varphi$;
- if X is positive (negative) in φ and ψ, then X is positive (negative) in $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\triangle\varphi$;
- if X is negative (positive) in φ and positive (negative) in ψ, then X is negative (positive) in $\varphi \to \psi$;
- X is positive and negative in $\eta X.\varphi$.

CS5

CS5 is obtained by adding to CK the axioms:

- $T := \Box\varphi \to \varphi \wedge \varphi \to \Diamond\varphi$;
- $4 := \Box\varphi \to \Box\Box\varphi \wedge \Diamond\Diamond\varphi \to \Diamond\varphi$; and
- $5 := \Diamond\varphi \to \Box\Diamond\varphi \wedge \Diamond\Box\varphi \to \Box\varphi$.

A CS5 model is a constructive Kripke model $\langle W, W^{\perp}, \preceq, R, V\rangle$ where R is an equivalence relation.

Theorem (Essentially Ono and Fischer-Servi)

CS5 *is complete over* CS5 *models.*

COLLAPSE

Lemma

Let $M = \langle W, W^{\perp}, \preceq, \equiv, V \rangle$ *be a* CS5 *model and* $w \preceq;\equiv w'$. *Then*

$$M, w \models \triangle\varphi \textit{ implies } M, w' \models \triangle\varphi,$$

where $\triangle \in \{\Box, \Diamond\}$.

At any long enough game, we will have positions:

$$\langle \nu X.\varphi, w \rangle \to \cdots \to \langle \Box\psi, v \rangle \to \cdots \to \langle \Box\psi, v' \rangle \to \cdots$$

We can use this fact to show that $\varphi(\varphi(\top)) \equiv \varphi(\varphi(\varphi(\top)))$.

μCS5 is complete

μCS5 is obtained by adding to CS5 the axioms:

- $\nu X.\varphi \to \varphi(\nu X.\varphi)$;
- $\varphi(\mu X.\varphi) \to \mu X.\varphi$;

and the rules:

$$\frac{\psi \to \varphi(\psi)}{\psi \to \nu X.\varphi} \quad \text{and} \quad \frac{\varphi(\psi) \to \psi}{\mu X.\varphi \to \psi}.$$

Theorem

μCS5 *is complete over* CS5 *frames.*

Using the collapse to modal logic, we can lift the completeness of CS5 to the completeness of μCS5.

THANK YOU!

- Game semantics for the constructive μ-calculus.
- The μ-calculus collapses to modal logic over $\mathsf{CS5}$ frames.
- $\mu\mathsf{CS5}$ is complete over $\mathsf{CS5}$ frames.
- Next step: show that $\mu\mathsf{CK}$ is complete over CK frames.

For detailed proofs and references, see the preprint: L. Pacheco, “Game semantics for the constructive μ-calculus”, arXiv:2308.16697.