The μ-calculus' collapse on variations of S5

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μ-CALCULUS

μ-calculus = modal logic + fixed points

$$\varphi := P \mid X \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \mu X.\varphi \mid \nu X.\varphi$$

Example

$$\nu X.P \wedge \Diamond \mu Y.(X \vee \Diamond Y)$$

holds at M, w iff there is a path starting from w where P holds infinitely many times.

ALTERNATION HIERARCHY

Let Σ_n^μ be the set of formulas with n-many alternating least and greatest fixed-point operators and starting with μ.

Theorem (Bradfield)

For all n, there is a μ-formula in Σ_{n+1}^μ which is not equivalent to any formula in Σ_n^μ.

Theorem (Alberucci, Facchini)

Over equivalence relations, every μ-formula is equivalent to a modal formula.

SEMANTICS

Fix a Kripke model $M = \langle W, R, V \rangle$.
Given $\varphi(X)$ where X is positive, define

$$\Gamma_\varphi : A \mapsto \|\varphi(A)\|.$$

Then

- $\|\mu X.\varphi\|$ is the least fixed-point of Γ_φ.
- $\|\nu X.\varphi\|$ is the greatest fixed-point of Γ_φ.

THE EVALUATION GAME $\mathcal{G}(M, w \models \varphi)$

- Two players: Verifier and Refuter.
- Positions are of the form $\langle v, \psi \rangle$ with:
  - $v \in W$,
  - $\psi \in \mathrm{Sub}(\varphi)$.
- Game starts at $\langle w, \varphi \rangle$
- Some types of play:
  - at $\langle v, P \rangle$, V wins iff $v \in V(P)$.
  - at $\langle v, \psi_0 \vee \psi_1 \rangle$, V chooses one of $\langle v, \psi_0 \rangle$ and $\langle v, \psi_1 \rangle$.
  - at $\langle v, \Box\psi \rangle$, R moves to $\langle v', \psi \rangle$ with vRv'.
  - at $\langle v, \mu X.\psi \rangle$, move to $\langle v, \psi \rangle$.
  - at $\langle v, \psi \rangle$, move to $\langle v, \mu X.\psi \rangle$.
- V wins a play iff the outermost infinitely often regenerating operator is ν.

Theorem

V *wins* $\mathcal{G}(M, w \models \varphi)$ *iff* $M, w \models \varphi$.

INTUITIONISTIC SEMANTICS FOR S5

An intuitionistic Kripke model is a tuple $M = \langle W, \preceq, \equiv, V \rangle$ where

- W is a set of worlds;
- $\preceq$ is reflexive and transitive relation on W;
- $\equiv$ is an equivalence relation on W;
- V is a valuation function.

Furthermore, we require:

- $w \preceq w'$ and $w \in V(P)$ imply $w' \in V(P)$;
- $w \preceq w'$ and $w \equiv v$ imply there is v' such that $v \preceq v'$ and $w' \equiv v'$;
- $w \equiv w' \preceq v'$ implies there is v such that $w \preceq v \equiv v'$.

SEMANTICS

The modal semantics are defined as follows:

- $M, w \models \Diamond\varphi$ iff, for all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$;
- $M, w \models \Box\varphi$ iff, for all v and u, if $w \preceq v \equiv u$ then $M, u \models \varphi$.

Theorem (Ono, Fischer Servi)

IS5 *is complete over birelational models* $M = \langle W, \preceq, \equiv, V \rangle$, *where* $\equiv$ *is an equivalence relation.*

KEY LEMMA FOR S5

Lemma

Let $M = \langle W, R, V \rangle$ be an S5 *model, w' be accessible from w, φ be a μ-formula, and $\triangle \in \{\Box, \Diamond\}$. Then $w \in \|\triangle\varphi\|^M$ iff $w' \in \|\triangle\varphi\|^M$.*

Theorem (Alberucci, Facchini)

$\mu X.\varphi$ is equivalent to $\varphi(\varphi(\bot))$.

Intuitively, given a long enough game on an S5 frame:

$$\cdots \to \langle w, \Diamond\psi \rangle \to \cdots \to \langle w', \Diamond\psi \rangle \to \cdots$$

then V wins at $\langle w, \Diamond\psi \rangle$ iff they win at $\langle w', \Diamond\psi \rangle$.

KEY LEMMA FOR IS5

This lemma does not hold on intuitionistic semantics, but we can get a good enough version:

Lemma

Let $M = \langle W, \preceq, \equiv, V \rangle$ be a bi-relational model and $w \preceq ; \equiv w'$. Then

$$M, w \models \triangle\varphi \text{ implies } M, w' \models \triangle\varphi,$$

where $\triangle \in \{\Box, \Diamond\}$.

KEY LEMMA FOR IS5 — PROOF

- Suppose $w \preceq ; \equiv w'$ and $M, w \models \Diamond\varphi$.
- For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
- Let $v' \succeq w'$, then:

$$w \xrightarrow{\preceq} \cdot \xrightarrow{\equiv} w' \xrightarrow{\preceq} v'$$

$\square$

KEY LEMMA FOR IS5 — PROOF

- Suppose $w \preceq ; \equiv w'$ and $M, w \models \Diamond\varphi$.
- For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
- Let $v' \succeq w'$, then:

$$\begin{tikzcd} w \arrow[r, "\preceq"] & \cdot \arrow[r, "\equiv"] \arrow[dr, "\preceq"'] & w' \arrow[r, "\preceq"] & v' \\ & & \cdot \arrow[ur, "\equiv"'] & \end{tikzcd}$$

$\square$

Key lemma for IS5 — Proof

- Suppose $w \preceq;\equiv w'$ and $M, w \models \Diamond\varphi$.
- For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
- Let $v' \succeq w'$, then:

$$
\begin{tikzcd}
w \arrow[r, "\preceq"] \arrow[drr, bend right, "\preceq"'] & \cdot \arrow[r, "\equiv"] \arrow[dr, "\preceq"] & w' \arrow[r, "\preceq"] & v' \\
 & & \cdot \arrow[ur, "\equiv"'] &
\end{tikzcd}
$$

$\square$

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- For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
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& & \cdot \arrow[ur, "\equiv"'] \arrow[dr, "\equiv"'] & \\
& & & u'
\end{tikzcd}
$$

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KEY LEMMA FOR IS5 — PROOF

- Suppose $w \preceq ; \equiv w'$ and $M, w \models \Diamond\varphi$.
- For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
- Let $v' \succeq w'$, then:

$$
\begin{tikzcd}
w \arrow[r, "\preceq"] \arrow[drr, bend right, "\preceq"'] & \cdot \arrow[r, "\equiv"] \arrow[dr, "\preceq"] & w' \arrow[r, "\preceq"] & v' \arrow[dd, "\equiv"] \\
 & & \cdot \arrow[ur, "\equiv"'] \arrow[dr, "\equiv"'] & \\
 & & & u'
\end{tikzcd}
$$

- So $M, w' \models \Diamond\varphi$. $\square$

THE COLLAPSE

- Suppose $M, w \models \varphi(\varphi(\top))$ and $M, w \not\models \varphi(\varphi(\varphi(\top)))$.
- Play games for both $\varphi(\varphi(\top))$ and $\varphi(\varphi(\varphi(\top)))$ simultaneously.
- Write φ as $\theta(\triangle\psi(X))$.
- Eventually, the players will reach positions as follows:

$$
\begin{aligned}
&\mathcal{G}(M,w \models \varphi(\varphi(\top))) : \\
&\qquad \cdots \to \langle w', \triangle\psi(\varphi(\top))\rangle \to \cdots \to \langle w'', \triangle\psi(\top)\rangle \\
&\mathcal{G}(M,w \models \varphi(\varphi(\varphi(\top)))) : \\
&\qquad \cdots \to \langle w', \triangle\psi(\varphi(\varphi(\top)))\rangle \to \cdots \to \langle w'', \triangle\psi(\varphi(\top))\rangle
\end{aligned}
$$

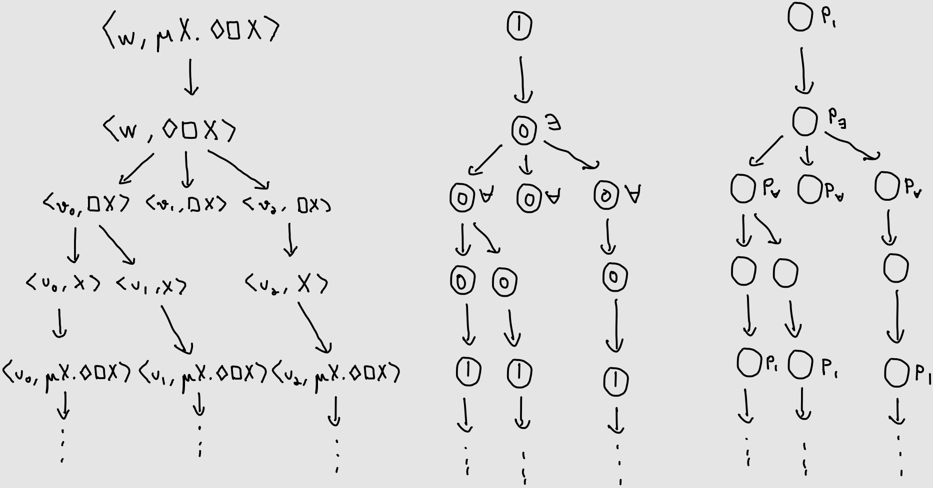
- By the key lemma, $M, w'' \models \triangle\psi(\varphi(\top))$; since $w' \preceq;\equiv w''$ and $M, w' \models \triangle\psi(\varphi(\top))$.
- Therefore $\|\varphi(\varphi(\top))\| = \|\varphi(\varphi(\varphi(\top)))\|$.

PARITY GAME: $\mathcal{P} = \langle V_\exists, V_\forall, v_0, E, \Omega \rangle$

- Two players: $\exists$ and $\forall$
- Positions on the graph $\langle V_\exists \cup V_\forall, E \rangle$
- Game starts at v_0.
- $\exists$ moves at nodes of $V_\exists$.
- $\forall$ moves at nodes of $V_\forall$.
- $\Omega : V_\exists \cup V_\forall \to n$ assigns parities to nodes.
- $\exists$ wins $\rho = v_0, v_1, v_2, \ldots$ iff the greatest $\Omega(v_i)$ which appears infinitely often in ρ is even.

EVALUATION GAMES AS KRIPKE MODELS

Evaluation game $\rightarrow$ parity game $\rightarrow$ Kripke model



WINNING REGION FORMULAS

Bradfield described W_n, which defines the winning region for $\exists$ in parity games with $\Omega(v) \leq n$:

$$W_n := \eta X_n \ldots \nu X_0. \bigvee_{0 \leq j \leq n} [(P_j \wedge P_\exists \wedge \Diamond X_j) \vee (P_j \wedge P_\forall \wedge \Box X_j)].$$

Theorem (Bradfield)

Let $n \in \omega$, then W_n is not equivalent to any formula in $\Sigma^\mu_n \cup \Pi^\mu_n$. Therefore the alternation hierarchy is strict (over K*).*

Proof sketch.

- Let n be even. Then $W_n \in \Pi^{\mu}_{n+1}$.
- Suppose that W_n is equivalent to some formula in Π^{μ}_{n}. Let $\varphi \in \Sigma^{\mu}_{n}$ be equivalent to $\neg W_n$.
- Define $f_{\varphi}(M, w) = (\mathcal{G}^{\mathrm{K}}(M, w \models \varphi), \langle w, \varphi \rangle)$.
- Let (M, w) be a fixed-point of $f_{\varphi \wedge \varphi}$. Then

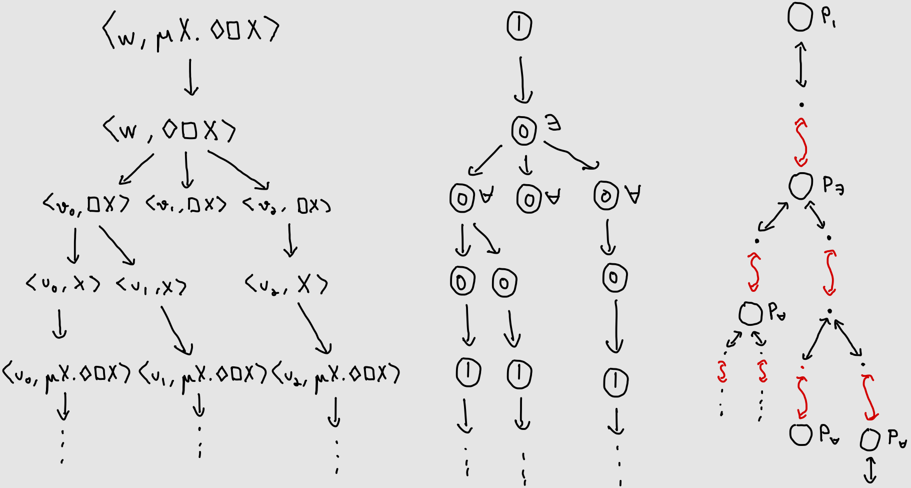
$$\begin{aligned} M, w \models \neg W_n &\iff M, w \models \varphi \wedge \varphi \\ &\iff f_{\varphi \wedge \varphi}(M, w) \models W_n \\ &\iff M, w \models W_n. \end{aligned}$$

- This is a contradiction.

$\square$

EVALUATION GAMES AS MULTIMODAL KRIPKE MODELS

Evaluation game $\rightarrow$ parity game $\rightarrow$ Kripke model



Multimodal winning region formulas

$$W'_n := \eta X_n \ldots \nu X_0. \bigvee_{0 \leq j \leq n} [(P_j \wedge P_\exists \wedge \blacklozenge X_j) \vee (P_j \wedge P_\forall \wedge \blacksquare X_j)].$$

Where

- $\blacklozenge\varphi := \mu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \wedge \Diamond_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \wedge \Diamond_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \wedge ((Y \wedge \neg\mathrm{st}) \vee (\varphi \wedge \mathrm{st}))))$; and
- $\blacksquare\varphi := \nu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \to \Box_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \to \Box_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \to ((Y \wedge \neg\mathrm{st}) \wedge (\varphi \wedge \mathrm{st}))))$,

Theorem

Let $n \in \omega$, then W'_n is not equivalent to any formula in $\Sigma^\mu_n \cup \Pi^\mu_n$. Therefore the alternation hierarchy is strict (over bimodal S5).

REFERENCES

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