IGL without sharps

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Intuitionistic Gödel-Löb Logic

- GL: $\Box(\Box P \to P) \to \Box P$
- iGL: GL on an intuitionistic base, only boxes
  See [3] for more on iGL.
- IGL: GL on an intuitionistic base, boxes and diamonds
  First developed by Das, van der Giessen and Marin [2]

IGL

Das, van der Giessen, and Marin proved:

$$
\begin{tikzcd}
\ell\mathsf{IGL} \vdash \varphi \arrow[r] & \mathcal{B}\mathsf{IGL} \models \varphi \arrow[d] \\
m\ell\mathsf{IGL} \vdash \varphi \arrow[u] & \mathcal{P}\mathsf{IGL} \models \varphi \arrow[l, "\Sigma^1_1\text{-Det}"']
\end{tikzcd}
$$

IGL

We prove completeness with less determinacy:

$$\begin{tikzcd}
\ell\mathsf{IGL} \vdash \varphi \arrow[rr] & & \mathcal{B}\mathsf{IGL} \models \varphi \arrow[d] \\
m\ell\mathsf{IGL} \vdash \varphi \arrow[u] & & \mathcal{P}\mathsf{IGL} \models \varphi \arrow[ll, "\Sigma^1_1\text{-Det}"'] \arrow[dl, "\Sigma^0_1\text{-Det}"] \\
& \mathsf{cm}\ell\mathsf{IGL} \vdash \varphi \arrow[ul] &
\end{tikzcd}$$

Some rules of cmℓIGL — I

$$\text{id}\ \frac{}{\mathbf{R}, \Gamma, x : P \vdash \Delta, x : P}$$

$$\text{tr}\ \frac{\mathbf{R}, xRy, yRz, xRz, \Gamma \vdash \Delta}{\mathbf{R}, xRy, yRz, \Gamma \vdash \Delta}$$

$$\wedge\mathrm{l}\ \frac{\mathbf{R}, \Gamma, x : A \wedge B, x : A, x : B \vdash \Delta}{\mathbf{R}, \Gamma, x : A \wedge B \vdash \Delta}$$

$$\to\mathrm{l}\ \frac{\mathbf{R}, \Gamma, x : A \to B \vdash \Delta, x : A \qquad \mathbf{R}, \Gamma, x : A \to B, x : B \vdash \Delta}{\mathbf{R}, \Gamma, x : A \to B \vdash \Delta}$$

Some rules of cmℓIGL — II

$$\Diamond\mathrm{r}\ \frac{\mathbf{R}, \Gamma \vdash \Delta, x : \Diamond A, \{y : A \mid xRy\}}{\mathbf{R}, \Gamma \vdash \Delta, x : \Diamond A}$$

$$\Box\mathrm{l}\ \frac{\mathbf{R}, \Gamma, x : \Box A, \{y : A \mid xRy\} \vdash \Delta}{\mathbf{R}, \Gamma, x : \Box A \vdash \Delta}$$

$$\Diamond\mathrm{l}\ \frac{\mathbf{R}, xRy, \Gamma, x : \Diamond A, y : A \vdash \Delta}{\mathbf{R}, \Gamma, x : \Diamond A \vdash \Delta}\ (y \text{ is fresh})$$

SOME RULES OF cmℓIGL — III

Non-invertible rules:

$$\to\mathrm{r}\ \frac{\mathbf{R}, \Gamma, x : A \vdash x : B}{\mathbf{R}, \Gamma \vdash \Delta, x : A \to B}$$

$$\Box\mathrm{r}\ \frac{\mathbf{R}, xRy, \Gamma \vdash y : A}{\mathbf{R}, \Gamma \vdash \Delta, x : \Box A}\ (y \text{ is fresh})$$

Loops

Loop v_S from $\mathbf{R}, \Gamma \vdash \Delta$ to $\mathbf{R}', \Gamma' \vdash \Delta'$:

- if $x : \varphi \in \Gamma'$ then $v_S(x) : \varphi \in \Gamma$;
- if $x : \varphi \in \Delta'$ then $v_S(x) : \varphi \in \Delta$;
- if $xR'y$ then $v_S(x) R v_S(y)$;
- for each x in $\mathbf{R}'$, $v(x) = x$ or $xRv(x)$
- if there is $x \in \mathrm{Var}(R')$ such that $xRv_S(x) \in \mathbf{R}$.

IGL PROVES $\Box(\Box P \to P) \to \Box P$

$$
\to\mathrm{r}\,\dfrac{\Box\mathrm{r}\,\dfrac{\Box\mathrm{l}\,\dfrac{\to\mathrm{l}\,\dfrac{\mathsf{id}\,\dfrac{}{xRy, x : \Box(\Box P \to P), y : \Box P \to P, y : P \vdash y : P} \qquad \Box\mathrm{l}\,\dfrac{\mathsf{tr}\,\dfrac{xRy, yRz, xRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P \quad (=: S)}{xRy, yRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P}}{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P, y : \Box P}}{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P}}{xRy, x : \Box(\Box P \to P) \vdash y : P (=: S')}}{x : \Box(\Box P \to P) \vdash x : \Box P}}{\vdash x : \Box(\Box P \to P) \to \Box P}
$$

$$l(S) = S' \text{ and } v_S(x) = x, v_S(y) = z.$$

Predicate Kripke Frames

Tuple $M = \langle W, \preceq, \{D_w\}_{w \in W}, \{Pr_w\}_{w \in W}, \{R_w\}_{w \in W} \rangle$ such that:

1. W is a non-empty set of possible worlds;
2. the intuitionistic relation $\preceq$ is a partial order on W;
3. $\{D_w\}_{w \in W}$ is a family of domains $D_w \subseteq \mathrm{Var}$;
4. $\{Pr_w\}_{w \in W}$ is a family of mappings $Pr_w : \mathrm{Prop} \to \mathcal{P}(D_w)$;
5. $\{R_w\}_{w \in W}$ is a family of modal relations $R_w \subseteq D_w \times D_w$;
6. all relations are monotone in $\preceq$, i.e., if $w \preceq w'$, then we have $D_w \subseteq D_{w'}$, $Pr_w \subseteq Pr_{w'}$, and $R_w \subseteq R_{w'}$.

Predicate IGL Frames

Define

- $(w, d) \preceq_M (w', d')$ if $w \preceq w'$ and $d = d'$;
- $(w, d) R_M (w', d')$ if $w = w'$ and $d R_w d'$;

where $d \in D_w$ and $d' \in D_{w'}$.

M is a $\mathcal{P}$IGL-model if the following conditions hold:

1. $M = \langle W, \preceq, \{D_w\}_{w \in W}, \{Pr_w\}_{w \in W}, \{R_w\}_{w \in W} \rangle$ is a Kripke structure;
2. for all $w \in W$, R_w is transitive; and
3. the composition $\preceq_M \circ R_M$ is reverse well-founded.

VALUATION

If $M = \langle W, \preceq, \{D_w\}_{w \in W}, \{Pr_w\}_{w \in W}, \{R_w\}_{w \in W} \rangle$, then

- $M, w \models x : P$ iff $x \in Pr_w(P)$.
- $M, w \not\models x : \bot$.
- $M, w \models x : A \wedge B$ iff $M, w \models x : A$ and $M, w \models x : B$.
- $M, w \models x : A \vee B$ iff $M, w \models x : A$ or $M, w \models x : B$.
- $M, w \models x : A \rightarrow B$ iff for all $w' \succeq w$, if $M, w' \models x : A$ then $M, w' \models x : B$.
- $M, w \models x : \Box A$ iff, for all $w' \succeq w$ and for all $y \in D_{w'}$, if $xR_{w'}y$ then $M, w' \models y : A$.
- $M, w \models x : \Diamond A$ iff there is $y \in D_w$ such that xR_wy and $M, w \models y : A$.

IGL does not prove $\Diamond P \to \Diamond(P \wedge \neg\Diamond P)$

$$\begin{array}{l} w_2\ \boxed{x \longrightarrow y \longrightarrow z} \\ \curlyvee| \\ w_1\ \boxed{x \longrightarrow y} \end{array}$$

P holds at y and z.

IGL does not prove $\Diamond P \to \Diamond(P \wedge \neg\Diamond P)$

$$
\to\mathrm{r}\ \dfrac{\Diamond\mathrm{l}\ \dfrac{\Diamond\mathrm{r}\ \dfrac{\wedge\mathrm{r}\ \dfrac{\mathsf{id}\ \dfrac{}{xRy, x : \Diamond P, y : P \vdash x : \Diamond(P \wedge \neg\Diamond P), y : P} \qquad \to\mathrm{r}\ \dfrac{\Diamond\mathrm{l}\ \dfrac{\mathsf{tr}\ \dfrac{xRy, yRz, xRz, x : \Diamond P, y : \Diamond P, y : P, z : P \vdash y : \bot\ (*)}{xRy, yRz, x : \Diamond P, y : \Diamond P, y : P, z : P \vdash y : \bot}}{xRy, x : \Diamond P, y : \Diamond P, y : P \vdash y : \bot}}{xRy, x : \Diamond P, y : P \vdash x : \Diamond(P \wedge \neg\Diamond P), y : \neg\Diamond P\ (*)}}{xRy, x : \Diamond P, y : P \vdash x : \Diamond(P \wedge \neg\Diamond P), y : P \wedge \neg\Diamond P}}{xRy, x : \Diamond P, y : P \vdash x : \Diamond(P \wedge \neg\Diamond P)}}{x : \Diamond P \vdash x : \Diamond(P \wedge \neg\Diamond P)}}{\vdash x : \Diamond P \to \Diamond(P \wedge \neg\Diamond P)}
$$

$$
\begin{tikzcd}
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m\ell\mathsf{IGL} \vdash \varphi \arrow[u] & & \mathcal{P}\mathsf{IGL} \models \varphi \arrow[ll, "\Sigma^1_1\text{-Det}"'] \arrow[dl, "\Sigma^0_1\text{-Det}"] \\
& \mathsf{cm}\ell\mathsf{IGL} \vdash \varphi \arrow[ul] &
\end{tikzcd}
$$

$\mathsf{cm}\ell\mathsf{IGL} \vdash \varphi$ IMPLIES $\mathcal{P}\mathsf{IGL} \models \varphi$

Lemma

$\mathsf{cm}\ell\mathsf{IGL} \vdash \varphi$ *implies* $m\ell\mathsf{IGL} \vdash \varphi$.

Proof.

Unfold the cyclic proof of φ into a non-wellfounded proof. $\square$

Lemma

$m\ell\mathsf{IGL} \vdash \varphi$ *implies* $\mathcal{P}\mathsf{IGL} \models \varphi$.

Proof.

Follows from Das, van der Giessen and Marin's paper. $\square$

A Proof Search Game

Given a sequent $\mathbf{R}, \Gamma \vdash \Delta$, we define a game:

- Two players: Prover and Denier.
- Start on the sequent $\mathbf{R}, \Gamma \vdash \Delta$.
- When discussing a sequent S, Prover has two choices:
  - Pick an inference rule
    $$\frac{S_1 \quad \cdots \quad S_n}{S},$$
    and then Denier picks some S_i.
  - Draw a progressing loop from S to a previous sequent.
- Infinite plays are won by Denier.

Lemma

If Prover wins this game, then $\mathbf{R}, \Gamma \vdash \Delta \in \mathsf{cm\ell IGL}$.

Building a Tree

Suppose $\mathbf{R}, \Gamma \vdash \Delta$ is not provable and let τ be a winning strategy for Denier.

We build a tree T as follows:

- The root is $\mathbf{R}, \Gamma \vdash \Delta$.
- If S is non-saturated, let

$$\frac{S_1 \ \cdots \ S_n}{S},$$

  be an inference saturating S. Add above S the sequent S_i picked by τ.
- If S is saturated, add above S all the premises of the applicable non-invertible rules.

Lemma

If S_0 is a non-saturated sequent in T, there is no path starting in S_0 where all sequents are non-saturated.

THE COUNTERMODEL

Let $M = \langle W, \preceq, \{D_S\}_{S \in S}, \{Pr_S\}_{S \in S}, \{R_S\}_{S \in W} \rangle$ be defined as follows:

- W is the set of saturated sequents of T.
- $\preceq$ is the tree ordering of T.
- D_w is the set of variables occurring in w.
- $Pr_S(P)$ is the set of variables labeling P in Γ_S.
- xR_Sy iff it occurs in $\mathbf{R}$.

Lemma

M is an $\mathcal{P}$IGL-model.

Truth Lemma

Lemma

For all sequents $w \in W$,

- *if $x : \varphi \in \Gamma_w$ then $M, w \models x : \varphi$; and*
- *if $x : \varphi \in \Delta_w$ then $M, w \not\models x : \varphi$.*

Lemma

Suppose $\mathbf{R}, \Gamma \vdash \Delta$ is not provable, then the constructed model does not validate $\mathbf{R}, \Gamma \vdash \Delta$.

RESULTS

Theorem

Σ^0_1-Det *proves the Kripke completeness of* IGL.

Theorem

IGL *is recursively enumerable.*

OPEN QUESTIONS

Question

Is IGL *recursive?*

Question

Does IGL *have the finite model property?*

Question

Does IGL *have a finite Hilbert-style axiomatization?*

REFERENCES

[1] Aguilera, Pacheco, "IGL without sharps", preprint soon™.

[2] Das, van der Giessen, Marin, "Intuitionistic Gödel-Löb logic, à la Simpson: labelled systems and birelational semantics", 2024.

[3] Van der Giessen, "Uniform Interpolation and Admissible Rules: Proof-theoretic investigations into (intuitionistic) modal logics", 2022.