

Epistemic possibility in intuitionistic epistemic logic

Leonardo Pacheco
TU Wien

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MODAL LOGIC

Modal logic is

propositional logic + \Box + \Diamond

In epistemic logic:

- ▶ $K\varphi$:= it is known that φ ;
- ▶ $\hat{K}\varphi$:= it is epistemically possible(?) that φ .

CLASSICAL EPISTEMIC LOGIC

Axioms:

- ▶ $K := K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi);$
- ▶ $T := K\varphi \rightarrow \varphi;$
- ▶ $4 := K\varphi \rightarrow KK\varphi;$
- ▶ $5 := \neg K\varphi \rightarrow K\neg K\varphi.$

Rules:

- ▶ $MP := \varphi \rightarrow \psi, \varphi \vdash \psi;$
- ▶ $Nec := \varphi \vdash \Box\varphi.$

Models:

- ▶ Kripke models $\langle W, R, V \rangle$ where R is an equivalence relation.

INTUITIONISTIC EPISTEMIC LOGIC

Artemov and Protopopescu defined a logic IEL such that:

Intuitionistic truth implies intuitionistic knowledge.

IEL consists of

- ▶ intuitionistic tautologies;
- ▶ $K := K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$;
- ▶ $coT := \varphi \rightarrow K\varphi$;
- ▶ $T' := K\varphi \rightarrow \neg\neg\varphi$;

closed under *modus ponens*.

BHK INTERPRETATION

- ▶ a proof of $\varphi \wedge \psi$ consists in a proof of φ and a proof of ψ ;
- ▶ a proof of $\varphi \vee \psi$ consists in giving either a proof of φ or a proof of ψ ;
- ▶ a proof of $\varphi \rightarrow \psi$ consists in a construction which given a proof of φ returns a proof of ψ ;
- ▶ $\neg\varphi$ is an abbreviation for $\varphi \rightarrow \perp$.

Artemov and Protopopescu proposed:

- ▶ a proof of $K\varphi$ is conclusive evidence of verification that φ has a proof.

SEMANTICS

An IEL model is a tuple $M = \langle W, \preceq, R, V \rangle$ where:

- ▶ \preceq is a preorder on W ;
- ▶ V is monotone w.r.t. \preceq ;
- ▶ wRv implies $w \preceq v$;
- ▶ $w \preceq v$ implies $vR \subseteq wR$;
- ▶ $wR \neq \emptyset$ for all w .

Define:

- ▶ $w \models K\varphi$ iff, for all v , wRv implies $v \models \varphi$.

Proposition

If $w \models \varphi$ and $w \preceq v$, then $v \models \varphi$.

PROPERTIES — NECESSITATION

Proposition

IEL is closed under necessitation:

$$\text{IEL} \vdash \varphi \text{ implies } \text{IEL} \vdash \Box\varphi.$$

PROPERTIES — INTROSPECTION

Proposition

IEL *proves positive introspection*:

$$K\varphi \rightarrow KK\varphi.$$

Proposition

IEL *proves negative introspection*:

$$\neg K\varphi \rightarrow K\neg K\varphi.$$

POSSIBILITY — SEMANTICS

$w \models \hat{K}\varphi$ holds iff

for all $v \succeq w$, there is u such that vRu and $u \models \varphi$.

Proposition

If $w \models \hat{K}\varphi$ and $w \preceq v$, then $v \models \hat{K}\varphi$.

POSSIBILITY — DOUBLE NEGATION

Proposition

$\text{IEL} \vdash \hat{K}P \rightarrow \neg\neg P.$

Proof.

We have $\neg\neg\varphi$ iff

for all $v \succeq w$, there is u such that $v \preceq u$ and $u \models \varphi$.

From $R \subseteq \preceq$, we have $\hat{K}P \rightarrow \neg\neg P.$



POSSIBILITY — DOUBLE NEGATION (REVERSAL)

Proposition

$\text{IEL} \vdash \neg\neg P \rightarrow \hat{K}P.$

Proof.

By contradiction:

- ▶ If $\hat{K}P$ fails at w , there is v such that $w \preceq v$ and, for all v' , vRv' implies $v' \not\models P$.
- ▶ If $\neg\neg P$ holds at w , there is u such that $v \preceq u$ and $u \models P$.
- ▶ uR is not empty; fix $u' \in uR$.
- ▶ Since $R \subseteq \preceq$, $u' \models P$.
- ▶ As $v \preceq u$, $uR \subseteq vR$.
- ▶ Therefore $v \preceq u' \not\models P$.



POSSIBILITY — BHK INTERPRETATION

Proposition

$$\text{IEL} \vdash \hat{K}P \leftrightarrow \neg\neg P.$$

Epistemic possibility is impossibility of proof of negation.

THANK YOU!

For more details on intuitionistic epistemic logic, see

- ▶ Artemov, Protopopescu, “Intuitionistic Epistemic Logic”.

For more details on intuitionistic modal logics, see

- ▶ van der Giessen, “Uniform Interpolation and Admissible Rules: Proof-theoretic investigations into (intuitionistic) modal logics ”.