

# IGL without sharps

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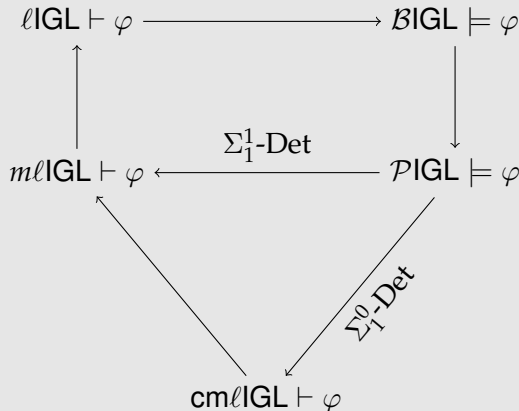
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# INTUITIONISTIC GÖDEL-LÖB LOGIC

- ▶ GL:  $\Box(\Box P \rightarrow P) \rightarrow \Box P$ .
- ▶ iGL: GL on an intuitionistic base, only boxes.
- ▶ IGL: GL on an intuitionistic base, boxes and diamonds.  
First developed by Das, van der Giessen and Marin.

# INTUITIONISTIC GÖDEL-LÖB LOGIC

- Das, van der Giessen, Marin, “Intuitionistic Gödel-Löb logic, à la Simpson”, 2024.
- Aguilera, Pacheco, “IGL without sharps”, to appear.



$\omega m\ell$ IGL

Let  $\omega m\ell$ IGL be the infinitary proof system with the  $\omega$ -rule:

$$\frac{x : \Box^n \perp, \mathbf{R}, \Gamma \vdash \Delta \ (\forall n \in \omega)}{\mathbf{R}, \Gamma \vdash \Delta}.$$

**Theorem**

$\omega m\ell$ IGL is complete w.r.t. IGL.

## SOME INFERENCE RULES — I

$$\perp\text{I} \frac{}{\mathbf{R}, \Gamma, x : \perp \vdash \Delta}$$

$$\wedge\text{I} \frac{\mathbf{R}, \Gamma, x : A \wedge B, x : A, x : B \vdash \Delta}{\mathbf{R}, \Gamma, x : A \wedge B \vdash \Delta}$$

$$\diamond\text{r} \frac{\mathbf{R}, \Gamma \vdash \Delta, x : \diamond A, \{y : A \mid xRy\}}{\mathbf{R}, \Gamma \vdash \Delta, x : \diamond A}$$

$$\Box\text{I} \frac{\mathbf{R}, \Gamma, x : \Box A, \{y : A \mid xRy\} \vdash \Delta}{\mathbf{R}, \Gamma, x : \Box A \vdash \Delta}$$

$$\diamond\text{I} \frac{\mathbf{R}, xRy, \Gamma, x : \diamond A, y : A \vdash \Delta}{\mathbf{R}, \Gamma, x : \diamond A \vdash \Delta} \text{ (} y \text{ is fresh)}$$

# SOME INFERENCE RULES — II

Non-invertible rules:

$$\rightarrow_r \frac{\mathbf{R}, \Gamma, x : A \vdash x : B}{\mathbf{R}, \Gamma \vdash \Delta, x : A \rightarrow B}$$

$$\Box_r \frac{\mathbf{R}, xRy, \Gamma \vdash y : A}{\mathbf{R}, \Gamma \vdash \Delta, x : \Box A} \text{ (} y \text{ is fresh)}$$

cmℓIGL PROVES  $\Box(\Box P \rightarrow P) \rightarrow \Box P$

$$\begin{array}{c}
 \text{id} \frac{}{xRy, x : \Box(\Box P \rightarrow P), y : \Box P \rightarrow P, y : P \vdash y : P} \\
 \rightarrow l \frac{}{\frac{}{xRy, x : \Box(\Box P \rightarrow P), y : \Box P \rightarrow P, y : P \vdash y : P} \quad \frac{\text{tr} \frac{xRy, yRz, xRz, x : \Box(\Box P \rightarrow P), y : \Box P \rightarrow P \vdash z : P \quad (*)}{xRy, yRz, x : \Box(\Box P \rightarrow P), y : \Box P \rightarrow P \vdash z : P}}{\Box l \frac{}{xRy, x : \Box(\Box P \rightarrow P), y : \Box P \rightarrow P \vdash y : P, y : \Box P}} \\
 \Box l \frac{xRy, x : \Box(\Box P \rightarrow P), y : \Box P \rightarrow P \vdash y : P}{\Box r \frac{xRy, x : \Box(\Box P \rightarrow P) \vdash y : P(*)}{x : \Box(\Box P \rightarrow P) \vdash x : \Box P}} \\
 \rightarrow r \frac{}{\vdash x : \Box(\Box P \rightarrow P) \rightarrow \Box P}
 \end{array}$$

## THE LOOP

$$xRy, yRz, xRz, x : \Box(\Box P \rightarrow P), y : \Box P \rightarrow P \vdash z : P$$

$$xRy, x : \Box(\Box P \rightarrow P) \vdash y : P$$



# A PROOF SEARCH GAME FOR $\text{cm}\ell\text{IGL}$

Given a sequent  $\mathbf{R}, \Gamma \vdash \Delta$ , we define an (open) game:

- ▶ Two players: Prover and Denier.
- ▶ Start on the sequent  $\mathbf{R}, \Gamma \vdash \Delta$ .
- ▶ When discussing a sequent  $S$ , Prover has two choices:
  - ▶ Pick an inference rule

$$\frac{S_1 \cdots S_n}{S},$$

and then Denier picks some  $S_i$ .

- ▶ Draw a progressing loop from  $S$  to a previous sequent.
- ▶ Infinite plays are won by Denier.

## Lemma

- ▶ *If Prover wins this game,  $\text{cm}\ell\text{IGL}$  proves  $\mathbf{R}, \Gamma \vdash \Delta \in \text{cm}\ell\text{IGL}$ .*
- ▶ *If Denier wins this game, we can build a countermodel for  $\mathbf{R}, \Gamma \vdash \Delta$ .*

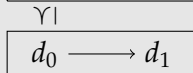
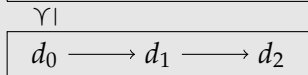
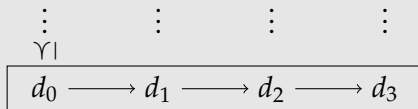
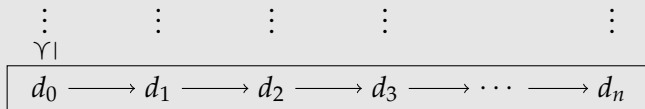
# PREDICATE IGL MODELS

Tuple  $M = \langle W, \preceq, \{D_w\}_{w \in W}, \{Pr_w\}_{w \in W}, \{R_w\}_{w \in W} \rangle$  such that:

1.  $W$  is a non-empty set of possible worlds;
2.  $\preceq$  is a partial order on  $W$ ;
3.  $D_w$  is the domain of  $w \in W$ ;
4.  $Pr_w$  is the valuation over  $D_w$ ;
5.  $R_w$  is a transitive relation in  $D_w$ ;
6. all relations are monotone in  $\preceq$ .

We also require there is no infinite path moving through the  $R_w$  infinitely often.

## THE GLOBAL CONDITION IS NECESSARY



$V(P) = \emptyset$  implies  $\models \Box(\Box P \rightarrow P)$  and  $\models \neg \Box P$ .

# IGL DOES NOT PROVE $\Diamond P \rightarrow \Diamond(P \wedge \neg \Diamond P)$

$$\begin{array}{c} w_2 \\ \boxed{x \longrightarrow y \longrightarrow z} \end{array}$$

 $\Upsilon \mid$ 

$$\begin{array}{c} w_1 \\ \boxed{x \longrightarrow y} \end{array}$$

$P$  holds at  $y$  and  $z$ .

# DEFINITION

$\omega m\ell$ IGL is obtained by adding to the basic rules the  $\omega$ -rule:

$$\frac{x : \Box^n \perp, \mathbf{R}, \Gamma \vdash \Delta \ (\forall n \in \omega)}{\mathbf{R}, \Gamma \vdash \Delta}.$$

- ▶ No loops or infinite paths.
- ▶ If we restrict the right-hand side, we can define  $\omega\ell$ IGL.

A similar  $\omega$ -rule for classical GL was studied by Yoshihito Tanaka.

# COMPLETENESS

## Lemma

$m\ell$ IGL  $\vdash \varphi$  implies  $\omega m\ell$ IGL  $\vdash \varphi$ .

## Proof.

- ▶ Let  $T$  be an  $m\ell$ IGL-proof of  $\varphi$  with  $\vdash x : \varphi$  at its root.
- ▶ Let  $n \in \omega$ .
- ▶ Append  $x : \Box^n \perp$  to the left-hand side of all sequents of  $T$ .
- ▶ The infinite paths of the new tree can be trimmed with applications of  $\Box l$  and  $\perp l$ .
- ▶ The trimmed proof  $T_n$  is an  $\omega m\ell$ IGL-proof of  $x : \Box^n \perp \vdash x : \varphi$ .
- ▶ From all the  $T_n$ , an application of the  $\omega$ -rule gives us an  $\omega m\ell$ IGL-proof of  $\varphi$ . □

# SOUNDNESS

## Lemma

$\omega$ mℓIGL  $\vdash \varphi$  implies  $\mathcal{P}$ IGL  $\vdash \varphi$ .

## Proof.

- ▶ Suppose  $\mathcal{P}$ IGL  $\not\models \varphi$ , then  $\text{cm}\ell$ IGL  $\not\models \varphi$ .
- ▶ In the countermodel built in the completeness proof of  $\text{cm}\ell$ IGL, we can show that each  $R_w$  only has paths of length less than:

$$f(\varphi) := 2^{2^{|\text{Sub}(\varphi)|}} \times 2^{|\text{Sub}(\varphi)|} \times |\text{Sub}(\varphi)| + 1.$$

- ▶ So  $\Box^{f(\varphi)} \perp \rightarrow \varphi$  is not  $\mathcal{P}$ IGL-valid.
- ▶ (This is a **extremely** rough bound.)



# RESULTS

## Theorem

*For all modal formula  $\varphi$ , the following are equivalent:*

- ▶  $\mathcal{P}\text{IGL} \models \varphi$ ;
- ▶  $\ell\text{IGL} \vdash \varphi$ ;
- ▶  $\text{cm}\ell\text{IGL} \vdash \varphi$ ; and
- ▶  $\omega\text{ml}\text{IGL} \vdash \varphi$ .



# OPEN QUESTIONS

## Question

*Is  $\text{IK4} + \Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$  complete w.r.t. IGL?*

## Question

*Is IGL decidable?*

## Question

*Does IGL have some form of interpolation?*

## Question

*Can the theory of IGL be done in a constructive setting?*