The Alternation Hierarchy of the μ-calculus over Weakly Transitive Frames

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Slides available at
`leonardopacheco.github.io/slides-wollic2022.pdf`

BASIC DEFINITIONS

- The formulas of the μ-calculus are generated by the following grammar:

$$\varphi := P \mid \neg P \mid X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \mu X.\varphi \mid \nu X.\varphi.$$

- Given a μ-formula $\varphi(X)$ and a Kripke model M,

$$\|\mu X.\varphi\|^M \text{ is the least fixed-point of } \Gamma_\varphi;$$
$$\|\nu X.\varphi\|^M \text{ is the greatest fixed-point of } \Gamma_\varphi,$$

where $\Gamma_\varphi(X) = \|\varphi(X)\|^M$.

Example

- Let E be the "everyone knows" modality:

$$E\varphi := K_1\varphi \wedge \cdots \wedge K_n\varphi.$$

- Common knowledge can be defined as

$$\begin{aligned} C\varphi &: = \nu X.\varphi \wedge EX \\ &\quad (= \varphi \wedge E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge \cdots). \end{aligned}$$

Alternating fixed-points

- Fixed-point operators may be "entangled":

  $$W_n := \eta X_n \ldots \nu X_2 \mu X_1 \nu X_0. \bigvee_{0 \leq j \leq n} (P_j \vee P_\exists \vee \Diamond X_j) \vee (P_j \vee P_\forall \vee \Box X_j)$$

  W_n describes the winning region for player $\exists$ of a parity game using parities $0, \ldots, n$. The player $\exists$ wins an infinite play iff the greatest priority appearing infinitely often is even.
- A formula is alternation-free if it has no entangled fixed points.
  - $\mu X.(\nu Y.P \wedge \Box Y) \vee \Diamond X$ is alternation-free.
  - $\mu X \nu Y.(P \wedge \Box Y) \vee \Diamond X$ is not alternation-free.

THE COLLAPSE OF THE ALTERNATION HIERARCHY

Theorem

- *(Bradfield [3]) The alternation hierarchy is strict over all Kripke frames.*
- *(Alberucci and Facchini [1]) The alternation hierarchy collapses to the alternation-free fragment over transitive frames.*
- *(Alberucci and Facchini [1]) The alternation hierarchy collapses to modal logic over equivalence relations.*

Logic	Alternation Hierarchy
K	Strict
K4 S4	Alternation-free
S5	Modal Logic

Theorem (P. and Tanaka)

- *The alternation hierarchy collapses to the alternation-free fragment over weakly transitive frames.*
- *The alternation hierarchy collapses to modal logic over frames of* S4.3.2.

Logic	Alternation Hierarchy
K	Strict
wK4 K4/S4 S4.2 S4.3	Alternation-free
S4.3.2 S4.4 S5	Modal Logic

WEAKLY TRANSITIVE FRAMES

- The logic wK4 is obtained by adding to K the axiom scheme:
$$\Diamond\Diamond P \to P \vee \Diamond P.$$
- A frame $F = \langle W, R \rangle$ is weakly transitive iff
$$wRv \wedge vRu \text{ implies } wRu \vee w = u.$$
- wK4 is complete for weakly transitive frames.

COLLAPSE OVER WEAKLY TRANSITIVE FRAMES

Theorem (P., Tanaka)

The alternation hierarchy collapses to its alternation-free fragment over weakly transitive frames.

Lemma

Suppose X appears in the scope of some $\Box$ inside $\nu X.\varphi$. Then, over weakly transitive frames,

$$\nu X.\varphi(X) \equiv \varphi(\varphi(\varphi((\top)))).$$

Lemma

Over weakly transitive frames,

$$\Diamond \mu X.\varphi(X) \equiv \Diamond \varphi^2(\bot) \textit{ and } \Box \nu X.\varphi(X) \equiv \Box \varphi^2(\top).$$

Proof sketch.

- Let $\nu X.\varphi$ be a formula where X appears in the scope of some μY and only in the scope of $\Diamond$s.
- We may suppose $\mu Y.\psi$ is a subformula of some minimal formula of the form

$$\left(\bigwedge_{\theta\in\Gamma}\Diamond\theta\right)\wedge\Box\left(\bigvee_{\theta\in\Delta}\theta\right).$$

- ψ can only occur inside some $\theta\in\Gamma$ of the form

$$\theta_1\vee(\theta_2\vee(\cdots(\theta_k\vee\mu Y.\psi_2)\cdots).$$

- As we can commute $\Diamond$ and $\vee$, $\Diamond\theta$ is equivalent to

$$\Diamond(\theta_1\vee(\theta_2\vee(\cdots(\theta_k\vee\psi(\psi(\bot)))\cdots)).$$

$\Box$

Derivative topological semantics

- A derivative topological model is a triple $\mathcal{X} = \langle W, \tau, V \rangle$.
- Semantics for the topological μ-calculus are as in the modal μ-calculus, but we define

$$w \in \|\Diamond\varphi\|^{\mathcal{X}} \text{ iff } w \text{ is a limit point of } \|\varphi\|^{\mathcal{X}}.$$

- wK4 is complete for derivative topological semantics.

Theorem (Baltag, Bezhanishvili, Fernández-Duque [2])

If a formula is satisfiable by some topological model, then it is satisfiable by a finite topological model.

Collapse for the topological μ-calculus

Theorem (P., Tanaka)

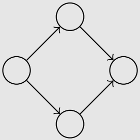
The alternation hierarchy collapses to its alternation-free fragment on topological semantics.

Proof sketch.

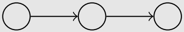
- Suppose φ is not equivalent to any alternation-free formula over topological models.
- Let ψ be an alternation-free formula.
- There is a (finite) topological model $\mathcal{X}$ which satisfies $\varphi \wedge \neg\psi$.
- $\mathcal{X}$ is equivalent to a weakly transitive model.
- Therefore $\varphi \wedge \neg\psi$ is satisfiable over weakly transitive models. $\square$

Between S4 and S5

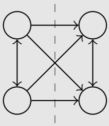
Logic	Frame Condition
S4.2	Convergent
S4.3	Weakly Connected
S4.3.2	Semi-Euclidean
S4.4	(no particular name)
S5	Equivalence Relation



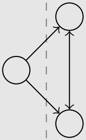
S4.2 $\wedge$ ¬S4.3



S4.3 $\wedge$ ¬S4.3.2



S4.3.2 $\wedge$ ¬S4.4



S4.4 $\wedge$ ¬S5

Game semantics for alternation-free formulas

We play a game to decide if $M, w \models \varphi$:

- Two players: Verifier and Refuter.
- Positions are of the form $\langle \psi, v \rangle$ with $\psi \in \mathsf{Sub}(\varphi)$ and $v \in W$.
- Initial position: $\langle \varphi, w \rangle$.

The rules are as follows:

- At $\langle \psi \vee \psi', v \rangle$, Verifier chooses $\langle \psi, v \rangle$ or $\langle \psi', v \rangle$.
- At $\langle \Box \psi, v \rangle$, Refuter chooses $\langle \psi, v' \rangle$ woth vRv'.
- At $\langle P, v \rangle$, Verifier wins iff $M, v \models P$.
- At $\langle \eta X.\psi, v \rangle$, move to $\langle \psi, v \rangle$.
- At $\langle X, v \rangle$, move to $\langle \eta X.\psi, v \rangle$.

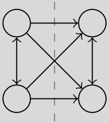
Verifier wins an infinite play iff some $\nu X.\psi$ appears infinitely often.

Theorem (P., Tanaka)

The alternation hierarchy collapses to modal logic over frames of S4.3.2.

Proof sketch.

We may suppose an S4.3.2 frame can be divided into two equivalence classes:



At any long enough game, we will have equivalent positions:

$$\langle \nu X.\varphi, w\rangle \to \cdots \to \langle \Box\psi, v\rangle \to \cdots \to \langle \Box\psi, v'\rangle \to \cdots \to \langle \Box\psi, v''\rangle \to \cdots$$

We can use this fact to show that $\varphi(\varphi(\top)) \equiv \varphi(\varphi(\varphi(\top)))$. $\Box$

IGNORANCE

Definition (Van der Hoek, Lomuscio)

The ignorance modality is defined by

$$I\varphi := \neg K\varphi \wedge \neg K\neg\varphi.$$

Read $I\varphi$ as "the agent is ignorant *whether* φ is true".

Theorem (Fine)

Define higher-order ignorance by:

$$I^1\varphi :\Leftrightarrow I\varphi; \textit{ and } I^{n+1}\varphi :\Leftrightarrow I(I^n\varphi).$$

If K satisfies S4 *then second-order ignorance is unobtainable. That is,*

$$\mathsf{S4} \models \neg I^2\varphi \textit{ for any } \varphi.$$

Degrees of ignorance

Fix a formula φ. Let

$$\alpha_\varphi(X) := \neg K(\varphi \wedge X) \wedge \neg K(\neg\varphi \wedge X).$$

The degrees of ignorance about φ are:

- $\alpha^1_\varphi := \alpha_\varphi(\top)$;
- $\alpha^{n+1}_\varphi := \alpha_\varphi(\alpha^n_\varphi)$;
- $\alpha^\infty_\varphi := \nu X.\alpha_\varphi$.

If K satisfies S4.2, then:

- $\alpha^1_\varphi \wedge \neg\alpha^2_\varphi \equiv$ the agent has a false belief but do not consider it possible to be wrong;
- $\alpha^2_\varphi \wedge \neg\alpha^3_\varphi \equiv$ the agent has a true belief but considers it possible to be wrong.

DEGREES OF IGNORANCE

Logic	Degrees
S4.2	ω
S4.3	ω
S4.3.2	2
S4.4	2
S5	1

Overview

Logic	Alternation Hierarchy
K	Strict
wK4	Alternation-free
K4	
S4	
S4.2	
S4.3	
S4.3.2	Modal Logic
S4.4	
S5	

OVERVIEW

Logic	Alternation Hierarchy
K	Strict
wK4	Alternation-free
K4	
S4	
S4.2	
S4.3	
S4.3.2	Modal Logic
S4.4	
S5	

Thank you!

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