Non-classical modal logics

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Non-classical modal logics

- modal logic = propositional logic + $\Box$ + $\Diamond$.
- Two main non-classical varieties:
  - constructive modal logic, and
  - intuitionistic modal logic.

A BIT OF HISTORY

- Fitch (1948): intuitionistic first-order modal logic (with T and Barcan's formula)
- Prior (1957): MIPQ, an intuitionistic analogue of S5
- Ono (1977), Fischer Servi (1978): completeness of MIPQ
- Many people work on $\Diamond$-free intuitionistic modal logics
- Wijesekera (1990): constructive modal logic

For a better survey see Simpson (1994).

CK — AXIOMATIZATION

Axioms:

- all intuitionistic tautologies;
- $K_\Box := \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
- $K_\Diamond := \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$.

Rules:

$$(\textbf{Nec})\ \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\textbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

IK — AXIOMATIZATION

Axioms:

- all intuitionistic tautologies;
- $K_\Box := \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
- $K_\Diamond := \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$;
- $FS := (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$;
- $DP := \Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$; and
- $N := \neg\Diamond\bot$.

Rules:

$$(\mathbf{Nec})\ \frac{\varphi}{\Box\varphi} \qquad \text{and} \qquad (\mathbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

KRIPKE MODELS FOR MODAL LOGIC

Tuples $M = \langle W, R, V\rangle$ where:

- W is the set of *possible worlds*;
- R is a relation over W; and
- V : Prop $\to \mathcal{P}(W)$ is a valuation function.

We define:

- $M, w \models \Box\varphi$ iff, for all v, if wRv then $M, v \models \varphi$;
- $M, w \models \Diamond\varphi$ iff there is v such that wRv and $M, v \models \varphi$.

KRIPKE MODELS FOR INTUITIONISTIC LOGIC

Relational Kripke models $M = \langle W, \preceq, V\rangle$ where:

- W is the set of *possible worlds*;
- $\preceq$ is a reflexive and transitive relation over W;
- $V : \text{Prop} \to \mathcal{P}(W)$ is a valuation function.

We require that:

- $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$.

We define:

- $M, w \models \varphi \to \psi$ iff, for all v, if $w \preceq v$ and $M, v \models \varphi$, then $M, v \models \psi$;
- $M, w \models \neg\varphi$ iff, for all v, if $w \preceq v$, then $M, v \not\models \varphi$.

CK — SEMANTICS

Bi-relational Kripke models $M = \langle W, W^{\perp}, \preceq, R, V \rangle$ where:

- W is the set of *possible worlds*;
- $W^{\perp} \subseteq W$ is the set of *fallible worlds*;
- $\preceq$ is a reflexive and transitive relation over W;
- R is a relation over W; and
- $V : \mathrm{Prop} \to \mathcal{P}(W)$ is a valuation function.

We require that:

- $W^{\perp} \subseteq V(P)$;
- $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$.

IK — SEMANTICS

Bi-relational Kripke models $M = \langle W, \preceq, R, V \rangle$ where:

- W is the set of *possible worlds*;
- $\preceq$ is a reflexive and transitive relation over W;
- R is a relation over W; and
- $V : \mathrm{Prop} \to \mathcal{P}(W)$ is a valuation function.

We require that:

- $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$;
- M is *forward confluent*: $w \preceq w'$ and $w \sqsubset v$ imply there is v' such that $v \preceq v'$ and $w' \sqsubset v'$;
- M is *backward confluent*: $w \sqsubset v \preceq v'$ implies then there is w' such that $w \preceq w' \sqsubset v'$.

Confluence

Forward confluence

$$\begin{tikzcd}
w \arrow[r, "\preceq"] \arrow[d, "\sqsubset"'] & w' \arrow[d, dashed, "\sqsubset"] \\
v \arrow[r, dashed, "\preceq"'] & v'
\end{tikzcd}$$

Backward confluence

$$\begin{tikzcd}
w \arrow[r, dashed, "\preceq"] \arrow[d, "\sqsubset"'] & w' \arrow[d, dashed, "\sqsubset"] \\
v \arrow[r, "\preceq"'] & v'
\end{tikzcd}$$

VALUATIONS

The valuation of $\Box$s are the same over CK and IK models:

- $M, w \models \Box\varphi$ iff, for all v, u, if $w \preceq vRu$ then $M, u \models \varphi$.

Over CK models, define:

- $M, w \models \Diamond\varphi$ iff, for all v such that $w \preceq v$, there is u such that if vRu and $M, u \models \varphi$.

Over IK models, define:

- $M, w \models \Diamond\varphi$ iff there is v such that if wRv and $M, v \models \varphi$.

SEPARATING CK AND IK

The following formulas are provable in IK but not in CK:

- $FS := (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$;
- $DP := \Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$; and
- $N := \neg\Diamond\bot$.

All of these involve $\Diamond$s.

Question

Do CK and IK prove the same $\Diamond$-free formulas?

The answer is no![1]

[1] Das, Marin

SEPARATION

- CK $\nvdash \neg\neg\Box\bot \to \Box\bot$:

$$w \preceq v \models \bot$$

- $w \models \neg\neg\Box\bot$ iff, for all $w' \succeq w$, there is $w'' \succeq w'$ such that $w'' \models \Box\bot$.
- But IK $\vdash \neg\neg\Box\bot \to \Box\bot$.

CS4 AND IS4

CS4 and IS4 are obtained by adding to CK and IK the axioms:

- $4_\Box := \Box P \to \Box\Box P$;
- $4_\Diamond := \Diamond\Diamond P \to \Diamond P$;
- $T_\Box := \Box P \to P$;
- $T_\Diamond := P \to \Diamond P$.

AN EXAMPLE

- Consider the following model M:

$$x \preceq y \sqsubseteq z \preceq t \sqsubseteq w.$$

  where P holds at $\{x, y, z, t\}$.
- The relation R is transitive, but the formula $\Box P \to \Box\Box P$ fails at x.

CS4 AND IS4 MODELS

A CS4 model is a CK model $M = \langle W, W^{\perp}, \preceq, R, V \rangle$ where

- R is transitive and reflexive;
- M is *backward confluent*: $w \sqsubset v \preceq v'$ implies then there is w' such that $w \preceq w' \sqsubset v'$.

An IS4 model is a IK model $M = \langle W, \preceq, R, V \rangle$ where

- R is transitive and reflexive.

CS4 and IS4 are complete w.r.t. CS4 and IS4 models.

DECIDABILITY

The following was an open problem for $\sim$10 years:

Theorem (Balbiani, Dieguez, Fernández-Duque)

CS4 *is decidable.*

This was solved using canonical models and bisimilations.
The following was an open problem for $\sim$30 years:

Theorem (Girlando *et al.*)

IS4 *is decidable.*

This was solved using labeled proof systems.

GL

- GL is the logic obtaining by adding

$$L := \Box(\Box P \to P) \to \Box P$$

  to K.
- GL is complete with respect to models $M = \langle W, R, V \rangle$ where R is transitive and reverse well-founded.
- R is reverse well-founded iff there in no infinite sequence $w_0 R w_1 R w_2 R \cdots$

Rev. wf is not (intuitionistically) enough

In IK model below, no world satisfies $\Box(\Box P \to P) \to \Box P$.

$$
\begin{tikzcd}
w_{0,0} \arrow[d, "\preceq"] \\
w_{1,0} \arrow[r, "\sqsubset"] \arrow[d, "\preceq"] & w_{1,1} \arrow[d, "\preceq"] \\
w_{2,0} \arrow[r, "\sqsubset"] \arrow[d, "\preceq"] & w_{2,1} \arrow[r, "\sqsubset"] \arrow[d, "\preceq"] & w_{2,2} \arrow[d, "\preceq"] \\
w_{3,0} \arrow[r, "\sqsubset"] \arrow[d, "\preceq"] & w_{3,1} \arrow[r, "\sqsubset"] \arrow[d, "\preceq"] & w_{3,2} \arrow[r, "\sqsubset"] \arrow[d, "\preceq"] & w_{3,3} \arrow[d, "\preceq"] \\
\vdots \arrow[d, "\preceq"] & \vdots \arrow[d, "\preceq"] & \vdots \arrow[d, "\preceq"] & \vdots \arrow[d, "\preceq"] \\
w_{n,0} \arrow[r, "\sqsubset"] \arrow[d, "\preceq"] & w_{n,1} \arrow[r, "\sqsubset"] \arrow[d, "\preceq"] & w_{n,2} \arrow[r, "\sqsubset"] \arrow[d, "\preceq"] & w_{n,3} \arrow[r, "\sqsubset"] \arrow[d, "\preceq"] & \cdots \arrow[r, "\sqsubset"] & w_{n,n} \\
\vdots & \vdots & \vdots & \vdots & & \vdots
\end{tikzcd}
$$

(P holds nowhere)

IGL models

An IGL model is an IK model $M = \langle W, \preceq, R, V \rangle$ where

- R is transitive;
- the composition $\preceq; R$ is reverse well-founded.

A proof system for IGL

Das, van der Giessen and Marin proved that an infinitary proof system based on the following is complete over IGL frames:

$$\Box_{\mathbf{l}} \frac{R, xRy, \Gamma, y : A \Rightarrow \Delta}{R, xRy, \Gamma, x : \Box A \Rightarrow \Delta}$$

$$\Box_{\mathbf{r}} \frac{R, xRy, \Gamma \Rightarrow \Delta, y : A}{R, \Gamma \Rightarrow \Delta, x : \Box A} (y \text{ fresh})$$

$$\mathbf{tr} \frac{R, xRy, yRz, xRz, \Gamma \Rightarrow \Delta}{R, xRy, yRz, \Gamma \Rightarrow \Delta}$$

CGL AND IGL MODELS

A CGL model is a CK model $M = \langle W, W^{\perp}, \preceq, R, V\rangle$ where

- R is transitive;
- R is forward confluent;
- the composition $\preceq; R$ is reverse well-founded.

An IGL model is an IK model $M = \langle W, \preceq, R, V\rangle$ where

- R is transitive;
- the composition $\preceq; R$ is reverse well-founded.

An axiomatization for CGL and IGL?

To obtain CGL and IGL, add to CK and IK the axioms:

- $4_\Box := \Box\varphi \to \Box\Box\varphi$;
- $4_\Diamond := \Diamond\Diamond\varphi \to \Diamond\varphi$; and
- $L_\Box := \Box(\Box\varphi \to \varphi) \to \Box\varphi$.

Question

Are CGL *and* IGL *complete over* CGL *and* IGL *models?*

The dual of Löb's Theorem

$L_\Diamond := \Diamond P \to \Diamond(P \wedge \Box\neg P)$ of fails at w_1:

$$\begin{tikzcd} w_1 \arrow[r, "\sqsubset"] \arrow[d, "\preceq"'] & w_2 \arrow[d, "\preceq"] & \\ v_1 \arrow[r, "\sqsubset"'] & v_2 \arrow[r, "\sqsubset"'] & v_3 \end{tikzcd}$$

(P holds everywhere.)

FAILURE TO PROVE THE COMPLETENESS

Proofs of completeness using finitary canonical models seem to need some diamond version of

$$L_\Box := \Box(\Box\varphi \to \varphi) \to \Box\varphi.$$

Reiterating:

Question

Are CGL *and* IGL *complete over* CGL *and* IGL *models?*

Question

If the answer to the above is negative:

- *Are there a complete axiomatization for* CGL *and* IGL *models?*
- *What class of models are characterized by the systems* CGL *and* IGL*?*

THE μ-CALCULUS

μ-calculus = modal logic + fixed-point operators
If X is positive, then:

- $\|\mu X.\varphi\|^M :=$ least fixed-point of $A \mapsto \|\varphi(A)\|^M$
- $\|\nu X.\varphi\|^M :=$ greatest fixed-point of $A \mapsto \|\varphi(A)\|^M$

Alternation Depth

The valuation of νX and μY depend on each other:

$$\nu X.\, \underbrace{\mu Y.\, \overbrace{(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)}^{\text{scope of } \mu Y}}_{\text{scope of } \nu X}$$

Alternation depth of φ

Maximum number of codependent alternating μ and ν operators in φ.

Alternation hierarchy

Classifies μ-formulas with respect to their alternation depth.

Some results on the unimodal μ-calculus

Theorem (Bradfield)

The μ-calculus alternation hierarchy is strict over all frames.

Theorem (Alberucci–Facchini)

The μ-calculus alternation hierarchy collapses to the alternation-free fragment over transitive frames.

Theorem (Alberucci–Facchini)

The μ-calculus alternation hierarchy collapses to modal logic over equivalence relations.

For a survey, see my PhD thesis.

VARIATIONS OF S5

CS5 and IS5 are obtained by adding to CK and IK the axioms:

- $4_\Box := \Box P \to \Box\Box P$;
- $4_\Diamond := \Diamond\Diamond P \to \Diamond P$;
- $5_\Box := \Diamond P \to \Box\Diamond P$;
- $5_\Diamond := \Diamond\Box P \to \Box P$;
- $T_\Box := \Box P \to P$;
- $T_\Diamond := P \to \Diamond P$.

CS5 and IS5 models are CS4 and IS4 models where the the modal relation is an equivalence relation.

Collapse over CS5/IS5 models

Lemma

Let $M = \langle W, W^{\perp}, \preceq, \equiv, V\rangle$ be a CS5 *model and $w \preceq ; \equiv w'$. Then*

$$M, w \models \triangle\varphi \text{ implies } M, w' \models \triangle\varphi,$$

where $\triangle \in \{\Box, \Diamond\}$.

At any long enough evaluation game, we will have positions:

$$\langle \nu X.\varphi, w\rangle \to \cdots \to \langle \Box\psi, v\rangle \to \cdots \to \langle \Box\psi, v'\rangle \to \cdots$$

We can use this fact to show that $\varphi(\varphi(\top)) \equiv \varphi(\varphi(\varphi(\top)))$.

COLLAPSE OVER CS4/IS4 MODELS

Question

Does the μ-calculus collapse to its alternation free-fragment over CS4 *and* IS4 *models?*

All the proofs I know of the collapse over S4 fail on non-classical settings.

REFERENCES

[1] Das, van der Giessen, Marin, "Intuitionistic Gödel–Löb Logic, à La Simpson: Labelled Systems and Birelational Semantics", 2024.

[2] Balbiani, Dieguez, Fernández-Duque, "Some Constructive Variants of S4 with the Finite Model Property", 2021.

[3] Alberucci, Facchini, "The modal μ-calculus hierarchy over restricted classes of transition systems", 2009.

[4] Bradfield, "Simplifying the modal mu-calculus alternation hierarchy", 1998.

[5] Pacheco, "Exploring the difference hierarchies on μ-calculus and arithmetic—from the point of view of Gale–Stewart games", PhD Thesis, 2023.

[6] Pacheco, "Game Semantics for the Constructive μ-Calculus", arXiv:2308.16697, 2023.

[7] Simpson, "The proof theory and semantics of intuitionistic modal logic", 1994.

[8] Girlando, Kuznets, Marin, Morales, Straßburger, "Intuitionistic S4 is decidable", 2023.