Connecting reflection and β-models in second-order arithmetic

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REFLECTION PRINCIPLES

Let

▶ Pr_T be a standard provability predicate for a theory T;

▶ Tr be a truth predicate for Π_n^1-sentences.

$\Pi_n^1\text{-}\mathsf{Ref}(T)$ is the sentence

$$\forall \varphi \in \Pi_n^1 . \mathrm{Pr}_T(\varphi) \to \mathrm{Tr}(\varphi).$$

STRONG DEPENDENT CHOICES

Definition

Strong Σ^1_i-DC_0 is the schema containing

$$\forall Z \forall n \forall Y (\varphi(n, (Z)_{<n}, Y) \to \varphi(n, (Z)_{<n}, (Z)_n))$$

for all Σ^1_i formula φ.

β-MODELS

▶ Any set $\mathcal{M} \subseteq \mathbb{N}$ can be seen as a model whose sets are

$$(\mathcal{M})_n = \{i \in \mathbb{N} \mid \langle n, i \rangle \in \mathcal{M}\}.$$

▶ $\mathcal{M} \subseteq \mathbb{N}$ is a coded β-model iff, for all Π_1^1-sentence φ with parameters in $\mathcal{M}$,

$$\varphi \iff \mathcal{M} \models \varphi.$$

Theorem (ACA_0)

Strong Σ_1^1-DC_0 is equivalent to

  for all $X \subseteq \mathbb{N}$ there is a coded β-model $\mathcal{M}$ containing X.

β_k-MODELS

▶ $\mathcal{M} \subseteq \mathbb{N}$ is a coded β_k-model iff, for all Π_k^1-sentence φ with parameters in $\mathcal{M}$,

$$\varphi \iff \mathcal{M} \models \varphi.$$

Theorem (ACA_0)

Strong Σ_k^1-DC_0 is equivalent to

for all $X \subseteq \mathbb{N}$ there is a coded β_k-model $\mathcal{M}$ containing X.

SEQUENCES OF β_k-MODELS

$\psi_{i,e}(n)$ states that, for all $X \subseteq \mathbb{N}$, there are $Y_0, \ldots, Y_n$ such that:

$$Y_0 \subseteq_{\beta_i} Y_1 \subseteq_{\beta_i} \cdots \subseteq_{\beta_i} Y_n \subseteq_{\beta_e} \mathcal{N}$$
$$X \in Y_0 \in Y_1 \in \cdots \in Y_n$$

Note that $\psi_{i,e}(n)$ is a Π^1_{e+2}-formula.

Theorem (ACA_0)

If $e \leq i$, then $\forall n.\psi_{i,e}(n)$ is equivalent to Π^1_{e+2}-Ref(Strong Σ^1_i-DC_0).

SOME DETERMINACY RESULTS

For all standard $n \geq 2$,

► ACA_0 is equivalent to $(\Sigma^0_1)_n\text{-}\mathsf{Det}^*$;

► $\Pi^1_1\text{-}\mathsf{CA}_0$ is equivalent to $(\Sigma^0_1)_n\text{-}\mathsf{Det}$;

► $\Pi^1_2\text{-}\mathsf{CA}_0$ proves $(\Sigma^0_2)_n\text{-}\mathsf{Det}$; and

► Z_2 proves $(\Sigma^0_3)_n\text{-}\mathsf{Det}$.

CONSEQUENCES

Theorem

Over ACA_0,

- Π_2^1-$\mathsf{Ref}(\mathsf{ACA}_0) \leftrightarrow \forall n.(\Sigma_1^0)_n$-$\mathsf{Det}^*$;
- Π_3^1-$\mathsf{Ref}(\Pi_1^1$-$\mathsf{CA}_0) \leftrightarrow \forall n.(\Sigma_1^0)_n$-$\mathsf{Det}$;
- Π_3^1-$\mathsf{Ref}(\Pi_2^1$-$\mathsf{CA}_0) \leftrightarrow \forall n.(\Sigma_2^0)_n$-$\mathsf{Det}$; *and*
- Π_3^1-$\mathsf{Ref}(\mathsf{Z}_2) \leftrightarrow \forall n.(\Sigma_3^0)_n$-$\mathsf{Det}$.

OPEN PROBLEMS

Problem

Characterize Π_n^1-Ref(T) for other theories T.[1]

Problem

Study axioms stating the existence of transfinite sequences of models.

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[1]See also P., "Recent Results on Reflection Principles in Second-Order Arithmetic".

THANK YOU!

For more details see

▶ P., Yokoyama, "Determinacy and reflection principles in second-order arithmetic", arXiv:2209.04082.