Determinacy and reflection principles in second-order arithmetic

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MAIN THEOREM

Theorem (ACA_0)

1. Π^1_2-$\mathsf{Ref}(\mathsf{ACA}_0) \leftrightarrow \forall n.(\Sigma^0_1)_n$-$\mathsf{Det}^*$;
2. Π^1_3-$\mathsf{Ref}(\Pi^1_1$-$\mathsf{CA}_0) \leftrightarrow \forall n.(\Sigma^0_1)_n$-$\mathsf{Det}$;
3. Π^1_3-$\mathsf{Ref}(\Pi^1_2$-$\mathsf{CA}_0) \leftrightarrow \forall n.(\Sigma^0_2)_n$-$\mathsf{Det}$; *and*
4. Π^1_3-$\mathsf{Ref}(\mathsf{Z}_2) \leftrightarrow \forall n.(\Sigma^0_3)_n$-$\mathsf{Det}$.

Item 1 follows from (Nemoto et al., 2007). Item 2 from (Tanaka, 1991). Item 3 appears in (Kołodziejczyk and Michalewski, 2016) with a stronger base system. Item 4 follows from (Montalbán and Shore, 2014).

Determinacy Axioms

Gale-Steward games:

- Two players alternate building a sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$ (or $2^{\mathbb{N}}$).
- The payoff is some $A \subseteq \mathbb{N}^{\mathbb{N}}$ (or $2^{\mathbb{N}}$).
- Player I wins iff $\alpha \in A$.

In second-order arithmetic:

- Γ-Det states all games in Baire space with payoff in Γ are determined.
- Γ-Det* states all games in Cantor space with payoff in Γ are determined.

Difference Hierarchy

$\varphi(f)$ is a $(\Sigma^0_k)_x$ formula iff there is a Σ^0_k formula $\psi(y, f)$ (possibly with other parameters) such that:

- $\psi(x, f)$ always holds;
- if $z < y < x$ then $\psi(z, f)$ implies $\psi(y, f)$; and
- $\varphi(f)$ holds iff the least $y \leq z$ such that $x = y \vee \psi(y, f)$ holds is even.

Then $\forall n.(\Sigma^0_k)_n$-Det states that all games whose payoff is a finite difference of Σ^0_k sets are determined.

DETERMINACY AND THE DIFFERENCE HIERARCHY

Theorem (ACA_0)

- $(\Sigma^0_1)_2$-Det^* *is equivalent to* ACA_0 (Nemoto et al., 2007);
- $(\Sigma^0_1)_2$-Det *is equivalent to* Π^1_1-CA_0 (Tanaka, 1991); *and*
- $(\Sigma^0_2)_n$-Det *is equivalent to* $[\Sigma^1_1]^n$-ID (MedSalem and Tanaka, 2008).

Theorem (ACA_0)

- $\forall n.(\Sigma^0_1)_n$-Det^* *is equivalent to* ACA_0';
- $\forall n.(\Sigma^0_1)_n$-Det *is equivalent to* Π^1_1-CA_0';
- $\forall n.(\Sigma^0_2)_n$-Det *is equivalent to* $\forall n.[\Sigma^1_1]^n$-ID; *and*
- $\forall n.(\Sigma^0_2)_n$-Det^* *is equivalent to* $\forall n.(\Sigma^0_2)_n$-Det (Nemoto et al., 2007).

REFLECTION PRINCIPLES

Let

- T be a finitely axiomatizable theory;
- Pr_T be a standard provability predicate for T; and
- $\mathrm{Tr}_{\Pi^1_n}$ be a truth predicate for Π^1_n-sentences.

The reflection principle Π^1_n-$\mathsf{Ref}(T)$ is the sentence

$$\forall \varphi \in \Pi^1_n . \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Tr}_{\Pi^1_n}(\ulcorner \varphi \urcorner).$$

UNIFORM REFLECTION

The uniform reflection scheme Π^1_n-$\mathsf{RFN}(T)$ consists of the formulas

$$\forall x.\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(x)$$

for all Π^1_n-formulas.

Proposition

Let T be a finitely axiomatizable theory extending ACA_0. Π^1_n-$\mathsf{Ref}(T)$ and Π^1_n-$\mathsf{RFN}(T)$ are equivalent over ACA_0.

STRONG DEPENDENT CHOICES

Strong Σ^1_k-DC_0 is ACA_0 plus the following scheme:

$$\exists Z \forall n \forall Y.(\eta(n,(Z)^n,Y) \rightarrow \eta(n,(Z)^n,(Z)_n)),$$

where $\eta(n,X,Y)$ is a Σ^1_k-formula in which Z does not occur, $(Z)^n = \{(i,m) | (i,m) \in Z \wedge m < n\}$, and $(Z)_n = \{i | (i,n) \in Z\}$.

Theorem

For $0 \leq k < \omega$,

- *(Simpson, Lemma VII.6.6)* Strong Σ^1_k-DC_0 *implies* Π^1_k-CA_0 *and* Σ^1_k-DC_0.
- *(Simpson, Theorem VII.6.20)* Strong Σ^1_{k+3}-DC_0 *is conservative over* Π^1_{k+3}-CA_0 *over* Π^1_4 *sentences.*

β_k-models and Strong Dependent Choices

$\mathcal{M} \subseteq \mathbb{N}$ is a coded β_k-model iff, for all Π^1_k-sentence φ with parameters in $\mathcal{M}$,

$$\varphi \iff \mathcal{M} \models \varphi.$$

The elements of $\mathcal{M}$ are $(\mathcal{M})_n = \{i \in \mathbb{N} \mid \langle n, i \rangle \in \mathcal{M}\}$.

Theorem (Simpson, Theorem VII.6.9)

- Π^1_1-$\mathsf{CA}_0 \equiv$ *Strong* Σ^1_1-DC $\equiv \forall X \exists \mathcal{M} \ni X. \mathcal{M}$ *is a coded* β*-model;*
- Π^1_2-$\mathsf{CA}_0 \equiv$ *Strong* Σ^1_2-DC $\equiv \forall X \exists \mathcal{M} \ni X. \mathcal{M}$ *is a coded* β_2*-model;*
- *Strong* Σ^1_k-$\mathsf{DC} \equiv \forall X \exists \mathcal{M} \ni X. \mathcal{M}$ *is a coded* β_k*-model, if* $k \geq 3$*;*
- *If* $V = L$*, then* Π^1_k-$\mathsf{CA}_0 \equiv$ *Strong* Σ^1_k-DC.

SEQUENCES OF β_k-MODELS

$\psi_{i,e}(n)$ (roughly) states that there are finite sequences of β_i-models of any length where the last model is a β_e-submodel of $\mathcal{N}$:

$$\begin{array}{l} X \in Y_0 \in \cdots \in Y_n \\ \quad Y_0 \subseteq_{\beta_i} \cdots \subseteq_{\beta_i} Y_n \subseteq_{\beta_e} \mathcal{N}. \end{array}$$

Each $\psi_{i,e}(n)$ is a Π^1_{e+2}-formula.

Theorem (ACA_0)

If $e \leq i$, then $\forall n.\psi_{i,e}(n)$ is equivalent to Π^1_{e+2}-Ref(Strong Σ^1_i-DC_0).

Π^1_{e+2}-Ref(*Strong* Σ^1_i-DC$_0$) PROVES $\forall n.\psi_{i,e}(n)$

- $\mathrm{Pr}_{\mathit{Strong}\ \Sigma^1_i\text{-}\mathsf{DC}_0}(\ulcorner \psi_e(i,0) \urcorner)$ and $\mathrm{Pr}_{\mathit{Strong}\ \Sigma^1_i\text{-}\mathsf{DC}_0}(\ulcorner \forall n.\psi_e(i,n) \to \psi_e(i,n+1) \urcorner)$ hold.
- By Σ^0_1-induction, $\mathrm{Pr}_{\mathit{Strong}\ \Sigma^1_i\text{-}\mathsf{DC}_0}(\ulcorner \psi_e(i,n) \urcorner)$ for each $n \in \mathbb{N}$.
- By reflection, $\psi_e(i,n)$ holds for any $n \in \mathbb{N}$.
- Thus $\forall n.\psi_e(i,n)$ holds.

$\forall n.\psi_{i,e}(n)$ PROVES Π^1_{e+2}-Ref(*Strong* Σ^1_i-DC_0)

Suppose that $\theta(X_0) \in \Sigma^1_{e+1}$ holds but $\mathrm{Pr}_{\mathit{Strong}\ \Sigma^1_i\text{-}\mathsf{DC}_0}(\ulcorner \forall X.\theta(X) \urcorner)$.
Consider a first-order theory T which includes the following axioms:

- M is a discrete ordered semiring;
- $\mathcal{M}_j \models \mathsf{ACA}_0$ for all $j \in \mathbb{N}$;
- $\mathcal{M}_j \subseteq_{\beta_i} \mathcal{M}_{j+1}$ for all $j \in \mathbb{N}$;
- $\mathcal{M}_j$ is "β_e-submodel enough" of the ground model; and
- $X_0 \in M_0$.

$\forall n.\psi_{i,e}(n)$ implies T is consistent, and so has a model $\mathcal{M}$.
$\mathcal{M}$ satisfies both $\theta(X_0)$ and $\neg\theta(X_0)$.
This is a contradiction.

CONSEQUENCES - I

By taking $e = i = 1$:

Corollary (ACA_0)

$\forall n.\psi_{1,1}(n)$ is equivalent to Π^1_3-Ref(Strong Σ^1_1-DC_0).

Corollary (ACA_0)

$\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_1\text{-}\mathsf{CA}_0) \leftrightarrow \forall n.(\Sigma^0_1)_n\text{-}\mathsf{Det}$

Consequences - II

We can also modify the proof above to get:

Theorem (ACA_0)

ACA_0' *is equivalent to* $\Pi^1_2\text{-}\mathsf{Ref}(\mathsf{ACA}_0)$.

Corollary (ACA_0)

$\Pi^1_2\text{-}\mathsf{Ref}(\mathsf{ACA}_0) \leftrightarrow \forall n.(\Sigma^0_1)_n\text{-}\mathsf{Det}^*$.

PROVING $\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_2\text{-}\mathsf{CA}_0) \leftrightarrow \forall n.(\Sigma^0_2)_n\text{-}\mathsf{Det}$

$$
\begin{tikzcd}
\text{Determinacy} \arrow[r, leftrightarrow] & \text{Multiple Inductions} \arrow[d, red] \\
\text{Reflection} \arrow[u] \arrow[r, leftrightarrow, red] & \text{Seq. of } \beta_2\text{-models}
\end{tikzcd}
$$

Σ^1_1 Inductive Definition

Let $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be a Σ^1_1 operator. Define

$$\Gamma'(X) = \Gamma(X) \cup X.$$

The set inductively defined by Γ is the least fixed point of Γ'.

MULTIPLE INDUCTIVE DEFINITIONS

$[\Sigma^1_1]^k$-ID is the states the existence of the sets inductively definable by combinations of k many Σ^1_1 operators.

Theorem (MedSalem and Tanaka, Theorem 3.4)

The following assertions are equivalent over ATR_0*:*

- $\forall n.(\Sigma^0_2)_n$-Det*; and*
- $\forall n.[\Sigma^1_1]^n$-ID.

The set defined by $\Gamma_0, \ldots, \Gamma_{k-1}$ is the least simultaneous fixed point of all the $\Gamma_0, \ldots, \Gamma_{k-1}$.

Building Sequences of β_2-models

We want to build:

$$\begin{aligned} X \in M_0 \in M_1 & \\ M_0 \subseteq_{\beta_2} M_1 & \subseteq_\beta \mathcal{N}. \end{aligned}$$

We define using inductive operators $\Gamma_0, \Gamma_1, \Gamma_2$:

Models	M_0, M_1
Recipes	M_0^r, M_1^r
Copy	M_0^c

Γ_0: CLOSURE UNDER Π^1_1-COMPREHENSION

$$e \in \mathbb{N} \text{ and } (M_i)_j \neq \emptyset \text{ implies } \langle \mathtt{comp}, e, j \rangle \in M_0^r$$

$$\begin{tikzcd}
 & \boxed{M_0} & \boxed{M_1} & \\
\langle \mathtt{comp}, e, j \rangle \arrow[r] & \boxed{M_0^r} & \boxed{M_1^r} & \langle \mathtt{comp}, e, j \rangle \arrow[l] \\
 & \boxed{M_0^c} & &
\end{tikzcd}$$

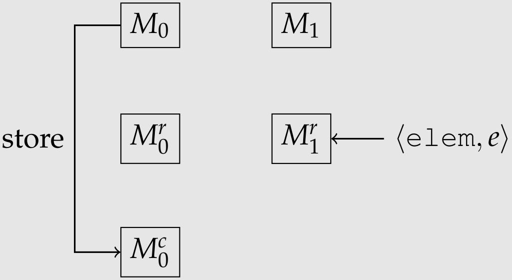
Γ_1: $M_0 \subseteq_{\beta_2} M_1$

$$M_1 \models \forall Z.\theta((M_1)_e, Z) \text{ and } M_0 \not\models \exists Y \forall Z \theta(Y, Z) \text{ implies } \langle \mathtt{subm}, e\rangle \in M_0^r$$

$$\begin{tikzcd}
 & \boxed{M_0} & \boxed{M_1} \\
\langle \mathtt{subm}, e\rangle \arrow[r] & \boxed{M_0^r} & \boxed{M_1^r} \\
 & \boxed{M_0^c} &
\end{tikzcd}$$

Γ_2: $M_0 \in M_1$

$\langle \texttt{elem}, e \rangle$ with e being the least such $\exists i \in (M_0)_e$ and $\neg\exists i \in (M_0^c)_e$



Γ_0 again

$\langle \texttt{comp}, e, \bar{j} \rangle$, $\langle \texttt{subm}, e \rangle$ and $\langle \texttt{elem}, e \rangle$ are recipes

$$\begin{tikzcd}
\boxed{M_0} & & \boxed{M_1} \\
\boxed{M_0^r} \arrow[u] & \text{make the recipes} & \boxed{M_1^r} \arrow[u] \\
\boxed{M_0^c} & &
\end{tikzcd}$$

Overview

- Γ_0: M_0, M_1 are coded β-models;
- Γ_1: $M_0 \subseteq_{\beta_2} M_1$; and
- Γ_2: $M_0 \in M_1$.

$$\begin{tikzcd} \boxed{M_0} \arrow[loop left, "\Gamma_0"] \arrow[r, bend left, "\Gamma_2"] & \boxed{M_1} \arrow[l, bend left, "\Gamma_1"] \arrow[loop right, "\Gamma_0"] \end{tikzcd}$$

PROVING Π^1_3-Ref$(\Pi^1_2$-CA$_0) \leftrightarrow \forall n.(\Sigma^0_2)_n$-Det

$$
\begin{array}{ccc}
\forall n.(\Sigma^0_2)_n\text{-Det} & \longleftrightarrow & \forall n.[\Sigma^1_1]^n\text{-ID} \\
\uparrow & & \downarrow \\
\Pi^1_3\text{-Ref}(\Pi^1_2\text{-CA}_0) & \longleftrightarrow & \forall n.\psi_{2,1}(n)
\end{array}
$$

RESULTS FROM MONTALBÁN AND SHORE

Theorem (Montalbán and Shore, Theorem 1.1)

For every $m \geq 1$, Π^1_{m+2}-CA_0 proves $(\Pi^0_3)_m$-Det.

Theorem (Montalbán and Shore, Theorem 1.10.5)

Let $m \geq 1$ and $X \subseteq \mathbb{N}$, then $(\Pi^0_3)_m$-Det proves the existence of a β-model $\mathcal{M}$ of Δ^1_m-CA_0 with $X \in \mathcal{M}$.

PROVING $\Pi^1_3\text{-}\mathsf{Ref}(\mathsf{Z}_2) \leftrightarrow \forall n.(\Sigma^0_3)_n\text{-}\mathsf{Det}$

Formalizing Montalbán and Shore's results we get:

Corollary

ACA_0 *proves* $\forall m.\mathrm{Pr}_{\Pi^1_{m+2}\text{-}\mathsf{CA}_0}(\ulcorner(\Pi^0_3)_m\text{-}\mathsf{Det}\urcorner)$.

Corollary

ACA_0 *proves* $\forall m.\mathrm{Pr}_{(\Pi^0_3)_m\text{-}\mathsf{Det}}(\ulcorner\beta(\Delta^1_{m+2}\text{-}\mathsf{CA}_0)\urcorner)$. *Here,* $\beta(\Delta^1_{m+2}\text{-}\mathsf{CA}_0)$ *is the sentence which states that for every* X *there is a* β*-model* $\mathcal{M}$ *of* $\Delta^1_m\text{-}\mathsf{CA}_0$ *with* $X \in \mathcal{M}$.

Using these corollaries, we can prove:

Theorem

Over ACA_0, $\forall n.(\Sigma^0_3)_n\text{-}\mathsf{Det}$ *is equivalent to* $\Pi^1_3\text{-}\mathsf{Ref}(\mathsf{Z}_2)$.

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