

The mu-calculus collapses to modal logic over frames of IS5

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MOTIVATION

The μ -calculus = modal logic + fixed points.

Theorem (Alberucci, Facchini¹)

Over equivalence relations, every μ -formula is equivalent to a modal formula.

Two directions to generalize this theorem:

- ▶ Bigger classes of frames:
 - ▶ On frames of S4.3.2: collapse to modal logic.
 - ▶ On transitive frames: collapse to alternation-free fragment.
- ▶ Change the semantics:
 - ▶ Intuitionistic semantics.
 - ▶ Graded semantics.
 - ▶ Inflationary μ -calculus.

¹L. Alberucci, A. Facchini, *The Modal μ -Calculus Hierarchy over Restricted Classes of Transition Systems*.

COMPLETENESS FOR S5 AND IS5

Theorem

S5 is complete over equivalence relations $M = \langle W, R, V \rangle$.

IS5 is an intuitionistic variant of S5.

Theorem (Ono², Fischer Servi³)

IS5 is complete over birelational models $M = \langle W, \preceq, \equiv, V \rangle$, where \equiv is an equivalence relation.

We define IS5 and birelational semantics on the next slides.

²H. Ono, *On Some Intuitionistic Modal Logics*.

³G. Fischer Servi, *The Finite Model Property for MIPQ and Some Consequences*.

IS5

IS5 consists of following axioms:

- ▶ all intuitionistic tautologies;
- ▶ $K := \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) \wedge \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$;
- ▶ $T := \Box\varphi \rightarrow \varphi \wedge \varphi \rightarrow \Diamond\varphi$;
- ▶ $4 := \Box\varphi \rightarrow \Box\Box\varphi \wedge \Diamond\Diamond\varphi \rightarrow \Diamond\varphi$;
- ▶ $5 := \Diamond\varphi \rightarrow \Box\Diamond\varphi \wedge \Diamond\Box\varphi \rightarrow \Box\varphi$;
- ▶ $FS := (\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi)$;
- ▶ $DP := \Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi$;
- ▶ $N := \neg\Diamond\perp$;

and the following inference rules:

$$(\mathbf{Nec}) \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP}) \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}.$$

BI-RELATIONAL MODELS

A bi-relational Kripke model is a tuple $M = \langle W, \preceq, \equiv, V \rangle$ such that

- ▶ W is a set of worlds;
- ▶ $\preceq \subseteq W \times W$ is reflexive and transitive;
- ▶ $\equiv \subseteq W \times W$ is an equivalence relation;
- ▶ V is a valuation function.

Furthermore, we require:

- ▶ $w \preceq w'$ and $w \in V(P)$ imply $w' \in V(P)$;
- ▶ $w \preceq w'$ and $w \equiv v$ imply there is v' such that $v \preceq v'$ and $w' \equiv v'$;
- ▶ $w \equiv w' \preceq v'$ implies there is v such that $w \preceq v \equiv v'$.

These are models for IS5.

μ -FORMULAS

The μ -formulas are generated by the grammar:

$$\varphi := P \mid X \mid \perp \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \Box \varphi \mid \Diamond \varphi \mid \mu X. \varphi \mid \nu X. \varphi.$$

Use $\neg \varphi$ as a shorthand for $\varphi \rightarrow \perp$.

We require X to be positive in φ to define $\mu X. \varphi$ and $\nu X. \varphi$.

SEMANTICS — INTUITIONISTIC MODAL LOGIC

Let $M = \langle W, \preceq, \equiv, V \rangle$ be a birelational model.

- ▶ $M, w \models P$ iff $w \in V(P)$;
- ▶ $M, w \models \perp$ is false;
- ▶ $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$;
- ▶ $M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$;
- ▶ $M, w \models \varphi \rightarrow \psi$ iff, for all $v \succeq w$, $M, v \models \varphi$ implies $M, v \models \psi$;
- ▶ $M, w \models \Diamond \varphi$ iff, for all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$;
- ▶ $M, w \models \Box \varphi$ iff, for all v and u , if $w \preceq v \equiv u$ then $M, u \models \varphi$.

SEMANTICS — ... AND FIXED-POINTS

Let $M = \langle W, \preceq, \equiv, V \rangle$ be a birelational model.

Let φ be a μ -formula and X be positive in φ . Define:

$$\Gamma_{\varphi(X)}(A) \rightarrow \|\varphi(A)\|^M$$

Then

- ▶ $M, w \models \mu X. \varphi$ iff w is in the least fixed point of $\Gamma_{\varphi(X)}$;
- ▶ $M, w \models \nu X. \varphi$ iff w is in the greatest fixed point of $\Gamma_{\varphi(X)}$.

$\preceq; \equiv$ IS TRANSITIVE

Let $\preceq; \equiv$ be the composition of \preceq and \equiv .

Lemma

If $M = \langle W, \preceq, \equiv, V \rangle$ is a birelational model, then $\preceq; \equiv$ is transitive.

Proof.

Suppose $w \preceq; \equiv v \preceq; \equiv u$. Then:

$$w \xrightarrow{\preceq} \cdot \xrightarrow{\equiv} v \xrightarrow{\preceq} \cdot \xrightarrow{\equiv} u$$



$\preceq; \equiv$ IS TRANSITIVE

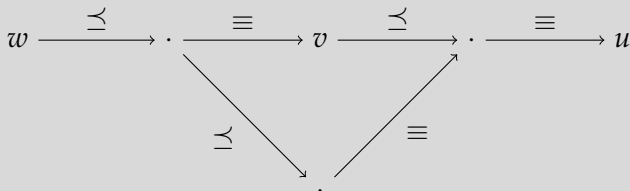
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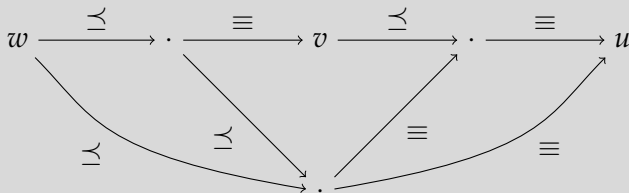
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If $M = \langle W, \preceq, \equiv, V \rangle$ is a birelational model, then $\preceq; \equiv$ is transitive.

Proof.

Suppose $w \preceq; \equiv v \preceq; \equiv u$. Then:



KEY LEMMA

Lemma (Alberucci, Facchini)

Let $M = \langle W, R, V \rangle$ be a transitive Kripke model, w' be a member of the strongly connected component of w , φ be a μ -formula, and $\Delta \in \{\Box, \Diamond\}$. Then $w \in \|\Delta\varphi\|^M$ iff $w' \in \|\Delta\varphi\|^M$.

This lemma does not generalize to intuitionistic semantics, but we can get a good enough version:

Lemma

Let $M = \langle W, \preceq, \equiv, V \rangle$ be a bi-relational model and $w \preceq; \equiv w'$. Then

$$M, w \models \Delta\varphi \text{ implies } M, w' \models \Delta\varphi,$$

where $\Delta \in \{\Box, \Diamond\}$.

KEY LEMMA — PROOF

- ▶ Suppose $w \preceq; \equiv w'$ and $M, w \models \Diamond\varphi$.
- ▶ For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
- ▶ Let $v' \succeq w'$, then:

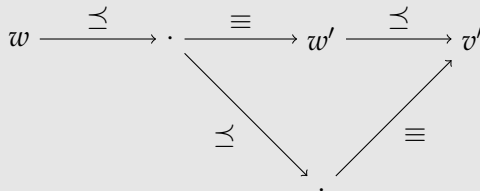
$$w \xrightarrow{\preceq} \cdot \xrightarrow{\equiv} w' \xrightarrow{\preceq} v'$$

- ▶ So $M, w' \models \Diamond\varphi$.



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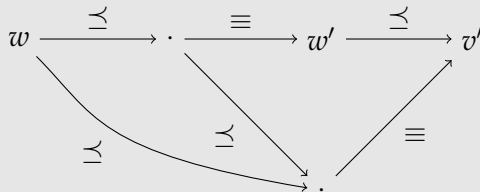


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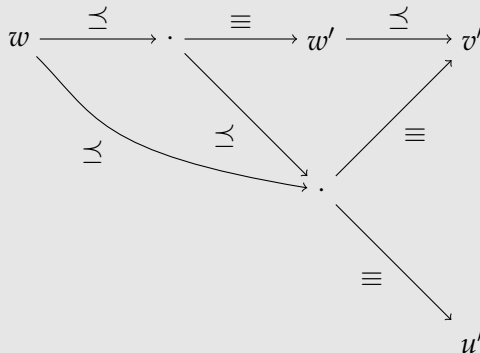


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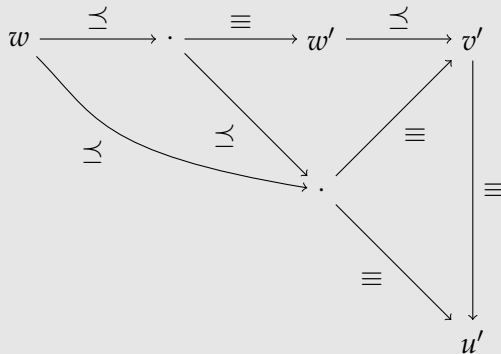


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- ▶ Suppose $w \preceq; \equiv w'$ and $M, w \models \Diamond\varphi$.
- ▶ For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
- ▶ Let $v' \succeq w'$, then:



- ▶ So $M, w' \models \Diamond\varphi$.



GAME SEMANTICS — INTUITIONISTIC MODAL LOGIC

We define the evaluation game $\mathcal{G}(M, w \models \varphi)$:

- ▶ Two players: Verifier and Refuter.
- ▶ At $\langle v, P \rangle$, V wins iff $v \in V(P)$.
- ▶ At $\langle v, \varphi \vee \psi \rangle$, V chooses to move to $\langle v, \varphi \rangle$ or $\langle v, \psi \rangle$.
- ▶ ...
- ▶ At $\langle v, \Diamond \varphi \rangle$, R chooses $v \succeq w$ and V chooses $u \equiv v$. The players move to $\langle u, \varphi \rangle$.
- ▶ At $\langle v, \Box \varphi \rangle$, R chooses $v \succeq w$ and $u \equiv v$. The players move to $\langle u, \varphi \rangle$.

DEFINING THE FIXED-POINTS

Lemma

Let $M = \langle W, \preceq, \equiv, V \rangle$ be a birelational model and φ be a formula where X is positive. Then

$$\|\varphi(\varphi(\top))\| = \|\varphi(\varphi(\varphi(\top)))\| \text{ and } \|\varphi(\varphi(\perp))\| = \|\varphi(\varphi(\varphi(\perp)))\|.$$

- ▶ $\|\varphi(\varphi(\varphi(\top)))\| \subseteq \|\varphi(\varphi(\top))\|$ as X is positive in φ .
- ▶ We show $\|\varphi(\varphi(\top))\| \subseteq \|\varphi(\varphi(\varphi(\top)))\|$.
- ▶ Suppose $M, w \models \varphi(\varphi(\top))$ and $M, w \not\models \varphi(\varphi(\varphi(\top)))$.
- ▶ \mathbf{V} has a winning strategy σ for $\mathcal{G}(M, w \models \varphi(\varphi(\top)))$ and \mathbf{R} has a winning strategy τ for $\mathcal{G}(M, w \models \varphi(\varphi(\varphi(\top))))$.

DEFINING THE FIXED-POINTS — CONT.

\mathbf{V} and \mathbf{R} play $\mathcal{G}(M, w \models \varphi(\varphi(\mathsf{T})))$ and $\mathcal{G}(M, w \models \varphi(\varphi(\varphi(\mathsf{T}))))$ simultaneously, using analogous strategies σ' and τ' .

For example:

$$\mathcal{G}(M, w \models \varphi(\varphi(\mathsf{T}))) : \rightarrow^* \langle v, (\psi \vee \theta)(\mathsf{T}) \rangle \rightarrow \langle v, \psi(\mathsf{T}) \rangle$$

$$\mathcal{G}(M, w \models \varphi(\varphi(\varphi(\mathsf{T})))) : \rightarrow^* \langle v, (\psi \vee \theta)(\varphi(\mathsf{T})) \rangle \rightarrow \langle v, \psi(\varphi(\mathsf{T})) \rangle$$

Eventually, the players will reach positions as follows:

$$\mathcal{G}(M, w \models \varphi(\varphi(\mathsf{T}))) : \rightarrow^* \langle w', \Delta\psi(\varphi(\mathsf{T})) \rangle \rightarrow^* \langle w'', \Delta\psi(\mathsf{T}) \rangle$$

$$\mathcal{G}(M, w \models \varphi(\varphi(\varphi(\mathsf{T})))) : \rightarrow^* \langle w', \Delta\psi(\varphi(\varphi(\mathsf{T}))) \rangle \rightarrow^* \langle w'', \Delta\psi(\varphi(\mathsf{T})) \rangle$$

By the lemma we proved above, $M, w'' \models \Delta\psi(\varphi(\mathsf{T}))$; since $w' \preceq; \equiv w''$ and $M, w' \models \Delta\psi(\varphi(\mathsf{T}))$. □

THE COLLAPSE

Theorem

Over birelational models of IS5, every μ -formula is equivalent to a modal formula.

Proof.

Let φ be a μ -formula and ψ be an equivalent modal formula.
Then

$$\mu X.\varphi \equiv \mu X.\psi \equiv \psi(\psi(\perp)),$$

and

$$\nu X.\varphi \equiv \nu X.\psi \equiv \psi(\psi(\top)).$$



PRELIMINARIES

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LEMMATA

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COLLAPSE OVER IS5 FRAMES

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THANKS!

A QUESTION — INFLATIONARY μ -CALCULUS

The μ -calculus allows only positive fixed-point operators.
 What happens on equivalence relations if we allow
 non-positive fixed-points operators?

	positive fp	non-positive fp
GL	$\mu\text{-calc} \equiv \text{ML}$	$\mu\text{-calc} \equiv \text{ML}$
S5	$\mu\text{-calc} \equiv \text{ML}$???