The mu-calculus collapses to modal logic over frames of IS5

Leonardo Pacheco
Tohoku University

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MOTIVATION

The μ-calculus = modal logic + fixed points.

Theorem (Alberucci, Facchini[1])

Over equivalence relations, every μ-formula is equivalent to a modal formula.

Two directions to generalize this theorem:

- Bigger classes of frames:
  - On frames of $\mathsf{S4.3.2}$: collapse to modal logic.
  - On transitive frames: collapse to alternation-free fragment.
- Change the semantics:
  - Intuitionistic semantics.
  - Graded semantics.
  - Inflationary μ-calculus.

[1] L. Alberucci, A. Facchini, *The Modal μ-Calculus Hierarchy over Restricted Classes of Transition Systems*.

COMPLETENESS FOR S5 AND IS5

Theorem

S5 *is complete over equivalence relations* $M = \langle W, R, V \rangle$.

IS5 is an intuitionistic variant of S5.

Theorem (Ono[2], Fischer Servi[3])

IS5 *is complete over birelational models* $M = \langle W, \preceq, \equiv, V \rangle$, *where* $\equiv$ *is an equivalence relation.*

We define IS5 and birelational semantics on the next slides.

[2] H. Ono, *On Some Intuitionistic Modal Logics*.

[3] G. Fischer Servi, *The Finite Model Property for* MIPQ *and Some Consequences*.

IS5

IS5 consists of following axioms:

- all intuitionistic tautologies;
- $K := \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \wedge \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$;
- $T := \Box\varphi \to \varphi \wedge \varphi \to \Diamond\varphi$;
- $4 := \Box\varphi \to \Box\Box\varphi \wedge \Diamond\Diamond\varphi \to \Diamond\varphi$;
- $5 := \Diamond\varphi \to \Box\Diamond\varphi \wedge \Diamond\Box\varphi \to \Box\varphi$;
- $FS := (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$;
- $DP := \Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$;
- $N := \neg\Diamond\bot$;

and the following inference rules:

$$(\mathbf{Nec})\ \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

BI-RELATIONAL MODELS

A bi-relational Kripke model is a tuple $M = \langle W, \preceq, \equiv, V \rangle$ such that

- W is a set of worlds;
- $\preceq \subseteq W \times W$ is reflexive and transitive;
- $\equiv \subseteq W \times W$ is an equivalence relation;
- V is a valuation function.

Furthermore, we require:

- $w \preceq w'$ and $w \in V(P)$ imply $w' \in V(P)$;
- $w \preceq w'$ and $w \equiv v$ imply there is v' such that $v \preceq v'$ and $w' \equiv v'$;
- $w \equiv w' \preceq v'$ implies there is v such that $w \preceq v \equiv v'$.

These are models for IS5.

μ-FORMULAS

The μ-formulas are generated by the grammar:

$$\varphi := P \mid X \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \mu X.\varphi \mid \nu X.\varphi.$$

Use $\neg\varphi$ as a shorthand for $\varphi \to \bot$.
We require X to be positive in φ to define $\mu X.\varphi$ and $\nu X.\varphi$.

SEMANTICS — INTUITIONISTIC MODAL LOGIC

Let $M = \langle W, \preceq, \equiv, V \rangle$ be a birelational model.

- $M, w \models P$ iff $w \in V(P)$;
- $M, w \models \bot$ is false;
- $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$;
- $M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$;
- $M, w \models \varphi \to \psi$ iff, for all $v \succeq w$, $M, v \models \varphi$ implies $M, v \models \psi$;
- $M, w \models \Diamond\varphi$ iff, for all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$;
- $M, w \models \Box\varphi$ iff, for all v and u, if $w \preceq v \equiv u$ then $M, u \models \varphi$.

Semantics — … and Fixed-Points

Let $M = \langle W, \preceq, \equiv, V\rangle$ be a birelational model.
Let φ be a μ-formula and X be positive in φ. Define:

$$\Gamma_{\varphi(X)}(A) \to \|\varphi(A)\|^M$$

Then

- $M, w \models \mu X.\varphi$ iff w is in the least fixed point of $\Gamma_{\varphi(X)}$;
- $M, w \models \nu X.\varphi$ iff w is in the greatest fixed point of $\Gamma_{\varphi(X)}$.

$\preceq;\equiv$ IS TRANSITIVE

Let $\preceq;\equiv$ be the composition of $\preceq$ and $\equiv$.

Lemma

If $M = \langle W, \preceq, \equiv, V \rangle$ is a birelational model, then $\preceq;\equiv$ is transitive.

Proof.

Suppose $w \preceq;\equiv v \preceq;\equiv u$. Then:

$$w \xrightarrow{\preceq} \cdot \xrightarrow{\equiv} v \xrightarrow{\preceq} \cdot \xrightarrow{\equiv} u$$

$\square$

$\preceq;\equiv$ is transitive

Let $\preceq;\equiv$ be the composition of $\preceq$ and $\equiv$.

Lemma

If $M = \langle W, \preceq, \equiv, V\rangle$ is a birelational model, then $\preceq;\equiv$ is transitive.

Proof.

Suppose $w \preceq;\equiv v \preceq;\equiv u$. Then:

$$\begin{tikzcd}
w \arrow[r, "\preceq"] & \cdot \arrow[r, "\equiv"] \arrow[dr, "\preceq"'] & v \arrow[r, "\preceq"] & \cdot \arrow[r, "\equiv"] & u \\
 & & \cdot \arrow[ur, "\equiv"'] & &
\end{tikzcd}$$

$\square$

$\preceq;\equiv$ IS TRANSITIVE

Let $\preceq;\equiv$ be the composition of $\preceq$ and $\equiv$.

Lemma

If $M = \langle W, \preceq, \equiv, V\rangle$ is a birelational model, then $\preceq;\equiv$ is transitive.

Proof.

Suppose $w \preceq;\equiv v \preceq;\equiv u$. Then:

$$\begin{tikzcd}
w \arrow[r, "\preceq"] \arrow[drr, bend right, "\preceq"'] & \cdot \arrow[r, "\equiv"] \arrow[dr, "\preceq"] & v \arrow[r, "\preceq"] & \cdot \arrow[r, "\equiv"] & u \\
& & \cdot \arrow[ur, "\equiv"'] \arrow[urr, bend right, "\equiv"'] & &
\end{tikzcd}$$

$\square$

Key lemma

Lemma (Alberucci, Facchini)

Let $M = \langle W, R, V \rangle$ be a transitive Kripke model, w' be a member of the strongly connected component of w, φ be a μ-formula, and $\triangle \in \{\Box, \Diamond\}$. Then $w \in \|\triangle\varphi\|^M$ iff $w' \in \|\triangle\varphi\|^M$.

This lemma does not generalize to intuitionistic semantics, but we can get a good enough version:

Lemma

Let $M = \langle W, \preceq, \equiv, V \rangle$ be a bi-relational model and $w \preceq ; \equiv w'$. Then

$$M, w \models \triangle\varphi \text{ implies } M, w' \models \triangle\varphi,$$

where $\triangle \in \{\Box, \Diamond\}$.

Key lemma — Proof

- Suppose $w \preceq;\equiv w'$ and $M, w \models \Diamond\varphi$.
- For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
- Let $v' \succeq w'$, then:

$$w \xrightarrow{\preceq} \cdot \xrightarrow{\equiv} w' \xrightarrow{\preceq} v'$$

- So $M, w' \models \Diamond\varphi$. $\square$

KEY LEMMA — PROOF

- Suppose $w \preceq ; \equiv w'$ and $M, w \models \Diamond\varphi$.
- For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
- Let $v' \succeq w'$, then:

$$\begin{tikzcd} w \arrow[r, "\preceq"] & \cdot \arrow[r, "\equiv"] \arrow[dr, "\preceq"'] & w' \arrow[r, "\preceq"] & v' \\ & & \cdot \arrow[ur, "\equiv"'] & \end{tikzcd}$$

- So $M, w' \models \Diamond\varphi$. $\square$

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- Suppose $w \preceq;\equiv w'$ and $M, w \models \Diamond\varphi$.
- For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
- Let $v' \succeq w'$, then:

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w \arrow[r, "\preceq"] \arrow[drr, bend right, "\preceq"'] & \cdot \arrow[r, "\equiv"] \arrow[dr, "\preceq"] & w' \arrow[r, "\preceq"] & v' \\
 & & \cdot \arrow[ur, "\equiv"'] &
\end{tikzcd}$$

- So $M, w' \models \Diamond\varphi$. $\square$

Key lemma — Proof

- Suppose $w \preceq;\equiv w'$ and $M, w \models \Diamond\varphi$.
- For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
- Let $v' \succeq w'$, then:

$$
\begin{tikzcd}
w \arrow[r, "\preceq"] \arrow[drr, bend right, "\preceq"'] & \cdot \arrow[r, "\equiv"] \arrow[dr, "\preceq"] & w' \arrow[r, "\preceq"] & v' \\
 & & \cdot \arrow[ur, "\equiv"'] \arrow[dr, "\equiv"'] & \\
 & & & u'
\end{tikzcd}
$$

- So $M, w' \models \Diamond\varphi$. $\square$

Key lemma — Proof

- Suppose $w \preceq ; \equiv w'$ and $M, w \models \Diamond\varphi$.
- For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$.
- Let $v' \succeq w'$, then:

$$\begin{tikzcd}
w \arrow[r, "\preceq"] \arrow[drr, bend right, "\preceq"'] & \cdot \arrow[r, "\equiv"] \arrow[dr, "\preceq"] & w' \arrow[r, "\preceq"] & v' \arrow[dd, "\equiv"] \\
& & \cdot \arrow[ur, "\equiv"'] \arrow[dr, "\equiv"] & \\
& & & u'
\end{tikzcd}$$

- So $M, w' \models \Diamond\varphi$. $\square$

GAME SEMANTICS — INTUITIONISTIC MODAL LOGIC

We define the evaluation game $\mathcal{G}(M, w \models \varphi)$:

- Two players: Verifier and Refuter.
- At $\langle v, P\rangle$, V wins iff $v \in V(P)$.
- At $\langle v, \varphi \vee \psi\rangle$, V chooses to move to $\langle v, \varphi\rangle$ or $\langle v, \psi\rangle$.
- $\dots$
- At $\langle v, \Diamond\varphi\rangle$, R chooses $v \succeq w$ and V chooses $u \equiv v$. The players move to $\langle u, \varphi\rangle$.
- At $\langle v, \Box\varphi\rangle$, R chooses $v \succeq w$ and $u \equiv v$. The players move to $\langle u, \varphi\rangle$.

Defining the fixed-points

Lemma

Let $M = \langle W, \preceq, \equiv, V\rangle$ be a birelational model and φ be a formula where X is positive. Then

$$\|\varphi(\varphi(\top))\| = \|\varphi(\varphi(\varphi(\top)))\| \text{ and } \|\varphi(\varphi(\bot))\| = \|\varphi(\varphi(\varphi(\bot)))\|.$$

- $\|\varphi(\varphi(\varphi(\top)))\| \subseteq \|\varphi(\varphi(\top))\|$ as X is positive in φ.
- We show $\|\varphi(\varphi(\top))\| \subseteq \|\varphi(\varphi(\varphi(\top)))\|$.
- Suppose $M, w \models \varphi(\varphi(\top))$ and $M, w \not\models \varphi(\varphi(\varphi(\top)))$.
- V has a winning strategy σ for $\mathcal{G}(M, w \models \varphi(\varphi(\top)))$ and R has a winning strategy τ for $\mathcal{G}(M, w \models \varphi(\varphi(\varphi(\top))))$.

DEFINING THE FIXED-POINTS — CONT.

V and R play $\mathcal{G}(M, w \models \varphi(\varphi(\top)))$ and $\mathcal{G}(M, w \models \varphi(\varphi(\varphi(\top))))$ simultaneously, using analogous strategies σ' and τ'.
For example:

$$\mathcal{G}(M, w \models \varphi(\varphi(\top))) : \to^* \langle v, (\psi \vee \theta)(\top)\rangle \to \langle v, \psi(\top)\rangle$$
$$\mathcal{G}(M, w \models \varphi(\varphi(\varphi(\top)))) : \to^* \langle v, (\psi \vee \theta)(\varphi(\top))\rangle \to \langle v, \psi(\varphi(\top))\rangle$$

Eventually, the players will reach positions as follows:

$$\mathcal{G}(M, w \models \varphi(\varphi(\top))) : \to^* \langle w', \triangle\psi(\varphi(\top))\rangle \to^* \langle w'', \triangle\psi(\top)\rangle$$
$$\mathcal{G}(M, w \models \varphi(\varphi(\varphi(\top)))) : \to^* \langle w', \triangle\psi(\varphi(\varphi(\top)))\rangle \to^* \langle w'', \triangle\psi(\varphi(\top))\rangle$$

By the lemma we proved above, $M, w'' \models \triangle\psi(\varphi(\top))$; since $w' \preceq;\equiv w''$ and $M, w' \models \triangle\psi(\varphi(\top))$. $\square$

THE COLLAPSE

Theorem

Over birelational models of IS5*, every μ-formula is equivalent to a modal formula.*

Proof.

Let φ be a μ-formula and ψ be an equivalent modal formula. Then

$$\mu X.\varphi \equiv \mu X.\psi \equiv \psi(\psi(\bot)),$$

and

$$\nu X.\varphi \equiv \nu X.\psi \equiv \psi(\psi(\top)). \qquad \square$$

Thanks!

A QUESTION — INFLATIONARY μ-CALCULUS

The μ-calculus allows only positive fixed-point operators. What happens on equivalence relations if we allow non-positive fixed-points operators?

	positive fp	non-positive fp
GL	μ-calc $\equiv$ ML	μ-calc $\equiv$ ML
S5	μ-calc $\equiv$ ML	???