Fixed-points in epistemic logic

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Some axioms for epistemic logic

Knowledge

- $K\varphi \to \varphi$;
- $K\varphi \to KK\varphi$.

Belief

- $\neg B\bot$;
- $B\varphi \to BB\varphi$;
- $\neg B\varphi \to B\neg B\varphi$.

Interaction axioms

- $K\varphi \to B\varphi$;
- $B\varphi \to KB\varphi$;
- $\neg B\varphi \to K\neg B\varphi$.

FIXED-POINTS

Let E be the "every one knows" modality:

$$E\varphi := K_1\varphi \wedge K_2\varphi \wedge \cdots \wedge K_n\varphi.$$

Common knowledge is defined as:

$$C\varphi := \varphi \wedge E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge \cdots$$

We can give a finitary definition of common knowledge using a greatest fixed-point operator:

$$C\varphi := \nu X.\varphi \wedge EX.$$

Two epistemic logics

We will study the effect of fixed-point operators on:

- S4.3 — knowledge is justified true belief:

$$K(KP \to Q) \vee K(KQ \to P).$$

- S4.4 — knowledge is true belief:

$$K\varphi \leftrightarrow \varphi \wedge B\varphi.$$

Ignorance

Van der Hoek and Lomuscio defined the ignorance modality:

$$I\varphi := \neg K\varphi \wedge \neg K\neg\varphi.$$

Fine proved that ignorance about ignorance is unobtainable:

$$\mathsf{S4} \models \neg II\varphi \text{ for all } \varphi.$$

We will generalize the ignorance modality in another direction.

DEGREES OF IGNORANCE

Given φ, define formulas:

- $\alpha_\varphi(X) := \hat{K}(\varphi \wedge X) \wedge \hat{K}(\neg\varphi \wedge X)$;
- $\alpha_\varphi^0 := \top$;
- $\alpha_\varphi^{n+1} := \alpha_\varphi(\alpha_\varphi^n)$; and
- $\alpha_\varphi^\infty := \nu X.\alpha_\varphi(X)$.

α_φ^n means "the agent has nth degree ignorance whether φ".

DEGREES OF IGNORANCE IN S4.3 AND S4.4

Theorem (P., Tanaka [5])

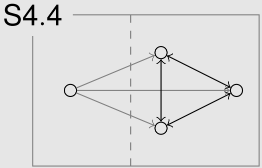
Over S4.4, *every modal formula with fixed-point operators is equivalent to a formula without fixed-point operators. The same does not happen over* S4.3.

- S4.3 has infinitely many degrees of ignorance:
  - $\alpha^0_\varphi \wedge \neg\alpha^1_\varphi \equiv$ knowledge
  - $\alpha^1_\varphi \wedge \neg\alpha^2_\varphi \equiv$ false belief, while believing knowledge;
  - $\alpha^2_\varphi \wedge \neg\alpha^3_\varphi \equiv$ true belief, considers false belief possible;
  - $\alpha^3_\varphi \wedge \neg\alpha^4_\varphi \equiv$ false belief, considers true belief (with doubts) possible;
  - $\ldots$
- S4.4 has two degrees of ignorance:
  - $\alpha^0_\varphi \wedge \neg\alpha^1_\varphi \equiv$ knowledge;
  - $\alpha^1_\varphi \wedge \neg\alpha^2_\varphi \equiv$ false belief;
  - $\alpha^2_\varphi \wedge \neg\alpha^3_\varphi \equiv$ no belief.

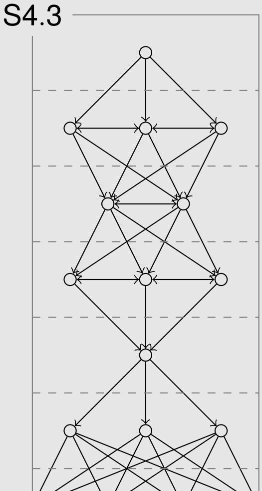
Models of S4.3 and S4.4

(The models below are reflexive and transitive.)

Collapse to modal logic



Non-collapse to modal logic



Multiple agents and fixed-points

With more that one agent, fixed-points are *very* expressive:

Theorem

Consider a bimodal logic where both modalities satisfy S5. *For all n, there is a (bimodal) formula φ without fixed-points such that*

$$\nu X_0 \mu X_1 \nu X_2 \mu X_3 \ldots \eta X_n . \varphi$$

is not equivalent to any formula with less fixed-point operators.

Epistemic logic with common knowledge is much less expressive than epistemic logic with arbitrary fixed-point operators.

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[4] W. van der Hoek, A. Lomuscio, "A logic for ignorance", 2004.

[5] L. Pacheco, K. Tanaka, "The Alternation Hierarchy of the μ-calculus over Weakly Transitive Frames", 2022.