Derivation of the Friedmann Equations from Einstein's Field Equations

Leonardo Tiditada Pedersen

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1 Introduction: The Starting Point

This document outlines the derivation of the key equations in cosmology, starting from Einstein's Field Equations (EFE).

1.1 The Einstein Field Equations (EFE)

The EFE relate the geometry of spacetime (left) to its matter and energy content (right). Including the cosmological constant Λ, the equation is:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1}$$

This can be rewritten using the Einstein Tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2}$$

For simplicity in the derivation, we will use natural units ($c = 1$) and set $\kappa = 8\pi G$. We will re-insert c and G in the final results.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3}$$

1.2 The Friedmann-Lemaître-Robertson-Walker (FLRW) Metric

We do not derive the metric *from* the EFE. Instead, we *posit* a metric based on the **Cosmological Principle**, which states that on large scales, the universe is homogeneous (the same everywhere) and isotropic (the same in all directions). The most general metric satisfying these conditions is the FLRW metric:

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4}$$

where:

- $a(t)$ is the **scale factor**, describing the expansion of the universe.
- k is the **curvature parameter**: $k = +1$ for a closed universe, $k = 0$ for a flat universe, and $k = -1$ for an open universe.

The corresponding metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are diagonal:

$$g_{\mu\nu} = \text{diag}\left(-1, \frac{a(t)^2}{1-kr^2}, a(t)^2r^2, a(t)^2r^2\sin^2\theta\right) \tag{5}$$

$$g^{\mu\nu} = \text{diag}\left(-1, \frac{1-kr^2}{a(t)^2}, \frac{1}{a(t)^2r^2}, \frac{1}{a(t)^2r^2\sin^2\theta}\right) \tag{6}$$

1.3 The Stress-Energy Tensor

We model the contents of the universe as a **perfect fluid**, which is also homogeneous and isotropic. Its stress-energy tensor is:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{7}$$

In comoving coordinates (where the fluid is at rest), the four-velocity is $u^\mu = (1, 0, 0, 0)$ and $u_\mu = (-1, 0, 0, 0)$. This simplifies $T_{\mu\nu}$ to a diagonal matrix:

$$T_{\mu\nu} = \text{diag}\left(\rho, \frac{pa^2}{1-kr^2}, pa^2r^2, pa^2r^2\sin^2\theta\right) \tag{8}$$

It is often simpler to work with the mixed-index tensor $T^\mu_{\ \nu} = g^{\mu\alpha}T_{\alpha\nu}$:

$$T^\mu_{\ \nu} = \text{diag}\left(-\rho, p, p, p\right) \tag{9}$$

Here, ρ is the energy density and p is the pressure.

2 Deriving the Friedmann Equations

To solve the EFE, we first compute the Einstein tensor $G_{\mu\nu}$ for the FLRW metric. This is a standard but lengthy calculation involving the Christoffel symbols and the Ricci tensor. The non-zero components of the Einstein tensor $G^\mu_{\ \nu}$ are:

$$G^0_{\ 0} = -3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] \tag{10}$$

$$G^i_{\ j} = -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]\delta^i_{\ j} \quad \text{(for } i, j \in \{1, 2, 3\}) \tag{11}$$

where $\dot{a} = da/dt$ and $\ddot{a} = d^2a/dt^2$.

We now equate the components of $G^\mu_{\ \nu} + \Lambda\delta^\mu_{\ \nu} = 8\pi G T^\mu_{\ \nu}$.

2.1 The First Friedmann Equation (00-component)

We use the 00-component (for $\mu = \nu = 0$):

$$G^0_{\ 0} + \Lambda\delta^0_{\ 0} = 8\pi G T^0_{\ 0} \tag{12}$$

Substituting the components we found:

$$-3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] + \Lambda = 8\pi G(-\rho) \tag{13}$$

Dividing by -3 and rearranging gives the **First Friedmann Equation**. Re-inserting c:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3}} \tag{14}$$

Defining the Hubble parameter $H = \dot{a}/a$, this is often written as:

$$H^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2} \tag{15}$$

2.2 The Second Friedmann Equation (ii-component)

We use the spatial components (e.g., $\mu = \nu = 1$, so $G^1{}_1 = G^i{}_j \delta^1{}_1$):

$$G^i{}_j + \Lambda \delta^i{}_j = 8\pi G T^i{}_j \tag{16}$$

$$-\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] + \Lambda = 8\pi G(p) \tag{17}$$

We now have a system of two equations. We can simplify this by substituting the First Friedmann Equation. From (14) (with $c = 1$ for now):

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} \tag{18}$$

Substitute this into the spatial equation:

$$-\left[2\frac{\ddot{a}}{a} + \left(\frac{8\pi G\rho}{3} + \frac{\Lambda}{3}\right)\right] + \Lambda = 8\pi G p \tag{19}$$

Now, we just solve for $\ddot{a}$:

$$-2\frac{\ddot{a}}{a} - \frac{8\pi G\rho}{3} - \frac{\Lambda}{3} + \Lambda = 8\pi G p \tag{20}$$

$$-2\frac{\ddot{a}}{a} + \frac{2\Lambda}{3} = 8\pi G p + \frac{8\pi G\rho}{3} \tag{21}$$

$$-2\frac{\ddot{a}}{a} = 8\pi G\left(p + \frac{\rho}{3}\right) - \frac{2\Lambda}{3} \tag{22}$$

Dividing by -2 gives the **Second Friedmann Equation**, also known as the **Acceleration Equation**. Re-inserting c:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}} \tag{23}$$

This equation is crucial as it shows that a positive cosmological constant ($\Lambda > 0$) or a component with negative pressure ($p < -\rho c^2/3$) can cause accelerated expansion ($\ddot{a} > 0$).

3 Deriving the Fluid (Continuity) Equation

The fluid equation describes the conservation of energy and can be derived from the Bianchi identity ($\nabla_\mu G^{\mu\nu} = 0$), which implies $\nabla_\mu T^{\mu\nu} = 0$.

Alternatively, and more simply, we can derive it by combining the two Friedmann equations. This shows that the three equations are not independent; any two imply the third.

1. Start with the First Friedmann Equation (14). For simplicity, let's multiply by a^2 and ignore Λ and k for a moment (they will cancel out).

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 \tag{24}$$

Differentiate with respect to time t:

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + \rho \cdot 2a\dot{a}) \tag{25}$$

Divide by $2\dot{a}a$:

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\left(\frac{\dot{\rho}a}{2\dot{a}} + \rho\right) \quad \text{This is getting complicated.} \tag{26}$$

Let's try a cleaner method. 1. Start with the First Friedmann Equation (14), solved for ρ:

$$\rho = \frac{3}{8\pi G}\left(H^2 + \frac{kc^2}{a^2}\right) - \frac{\Lambda c^2}{8\pi G} \tag{27}$$

2. Differentiate ρ with respect to time t:

$$\dot{\rho} = \frac{3}{8\pi G}\left(2H\dot{H} - \frac{2kc^2\dot{a}}{a^3}\right) \tag{28}$$

Using $H = \dot{a}/a$, $\dot{H} = \ddot{a}/a - \dot{a}^2/a^2 = \ddot{a}/a - H^2$, and $\dot{a} = Ha$:

$$\dot{\rho} = \frac{3}{8\pi G}\left(2H\left(\frac{\ddot{a}}{a} - H^2\right) - \frac{2kc^2H}{a^2}\right) \tag{29}$$

$$\dot{\rho} = \frac{3H}{4\pi G}\left(\frac{\ddot{a}}{a} - H^2 - \frac{kc^2}{a^2}\right) \tag{30}$$

3. Now substitute the two Friedmann equations.

- From (23): $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$
- From (14): $-H^2 - \frac{kc^2}{a^2} = -\left(\frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3}\right)$

Substitute these into the expression for $\dot{\rho}$:

$$\dot{\rho} = \frac{3H}{4\pi G}\left[\left(-\frac{4\pi G}{3}(\rho + \frac{3p}{c^2}) + \frac{\Lambda c^2}{3}\right) - \left(\frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3}\right)\right] \tag{31}$$

$$\dot{\rho} = \frac{3H}{4\pi G}\left[-\frac{4\pi G\rho}{3} - \frac{12\pi Gp}{3c^2} + \frac{\Lambda c^2}{3} - \frac{8\pi G\rho}{3} - \frac{\Lambda c^2}{3}\right] \tag{32}$$

$$\dot{\rho} = \frac{3H}{4\pi G}\left[-\frac{12\pi G\rho}{3} - \frac{12\pi Gp}{3c^2}\right] \tag{33}$$

$$\dot{\rho} = \frac{3H}{4\pi G}\left[-4\pi G\left(\rho + \frac{p}{c^2}\right)\right] \tag{34}$$

$$\dot{\rho} = -3H\left(\rho + \frac{p}{c^2}\right) \tag{35}$$

Rearranging gives the **Fluid Equation** or **Continuity Equation**:

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0} \tag{36}$$

This equation describes the conservation of energy in an expanding volume. The $\dot{\rho}$ term is the change in energy density, and the $3H(\rho + p/c^2)$ term represents the dilution of energy (and work done by pressure) due to the expansion.