Summary Sheet: Continuous Random Variables

(Based on Chapter 4)

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Contents

Notation cheat-sheet

- Y (or X): continuous random variable.
- $f(y)$: probability density function (PDF).
- $F(y)$: cumulative distribution function (CDF), $F(y) = P(Y \leq y)$.
- $E[\cdot]$: expectation (mean).
- $V(\cdot)$ or $\text{Var}(\cdot)$: variance.
- $\mathbb{R}$: set of real numbers.

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1 1. Continuous Random Variables (CRV)

Definition and intuition

A random variable Y is *continuous* if its set of possible values forms an interval (finite or infinite). Intuitively, probabilities are represented by *areas under a curve* rather than by sums of point-masses.

Key property: For any specific number a,

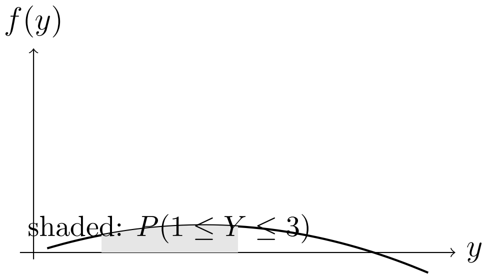
$$P(Y = a) = 0.$$

That is, single points have zero probability; only intervals have non-zero probability.

Because of this,

$$P(a \le Y \le b) = P(a < Y \le b) = P(a \le Y < b) = P(a < Y < b).$$

Visual intuition (sketch)



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2 2. PDF and CDF: definitions and connection

Probability Density Function (PDF)

The PDF $f(y)$ is a nonnegative function with total area 1:

1. $f(y) \ge 0$ for all y.
2. $\displaystyle\int_{-\infty}^{\infty} f(y)\,dy = 1.$

Probabilities for intervals are computed by integrals:

$$P(a \le Y \le b) = \int_a^b f(y)\,dy.$$

Cumulative Distribution Function (CDF)

The CDF is

$$F(y) = P(Y \le y) = \int_{-\infty}^{y} f(t)\,dt.$$

Differentiation gives back the density (where differentiable):

$$f(y) = F'(y).$$

$$\boxed{F(y) = \int_{-\infty}^{y} f(t)\,dt \quad \text{and} \quad f(y) = \frac{d}{dy}F(y).}$$

Useful identities

$$P(Y > a) = 1 - F(a), \qquad P(a < Y \le b) = F(b) - F(a).$$

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3 3. Expectation (Mean) and Variance

Expectation

The expected value (mean) of Y is

$$\mu = E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy,$$

provided the integral converges.

More generally, for any measurable function g,

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y)\, f(y)\, dy.$$

Variance

Variance measures spread:

$$\mathrm{Var}(Y) = E[(Y - \mu)^2] = \int_{-\infty}^{\infty} (y - \mu)^2 f(y)\, dy.$$

Computational formula (often easier):

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2, \qquad E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\, dy.$$

Linearity and scaling properties

For constant c and random variable Y:

$$\begin{aligned} E[c] &= c, & E[cY] &= cE[Y], \\ \mathrm{Var}(c) &= 0, & \mathrm{Var}(Y + c) &= \mathrm{Var}(Y), \\ \mathrm{Var}(cY) &= c^2\,\mathrm{Var}(Y). \end{aligned}$$

Example: compute mean and variance

Let $f(y) = \frac{1}{2}$ for $y \in [0, 2]$ (Uniform(0,2)). Then

$$E(Y) = \int_0^2 y \cdot \frac{1}{2}\, dy = \frac{1}{2} \left[\frac{y^2}{2}\right]_0^2 = \frac{1}{2} \cdot \frac{4}{2} = 1.$$

$$E(Y^2) = \frac{1}{2} \int_0^2 y^2\, dy = \frac{1}{2} \cdot \frac{8}{3} = \frac{4}{3}.$$

Hence

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{4}{3} - 1^2 = \frac{1}{3}.$$

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4 4. Worked example (detailed): general PDF

Problem statement

Let

$$f(y) = \begin{cases} k\, y(2-y), & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find k, $P(0.5 \le Y \le 1)$, and $E[Y]$.

(A) Find k

Set total area to 1:

$$1 = \int_0^2 k(2y - y^2)\, dy = k \left[y^2 - \frac{y^3}{3}\right]_0^2 = k \left(4 - \frac{8}{3}\right) = k \cdot \frac{4}{3}.$$

Thus $k = \dfrac{3}{4}$.

(B) Probability on an interval

$$P(0.5 \le Y \le 1) = \int_{0.5}^1 \frac{3}{4}(2y - y^2)\, dy = \frac{3}{4} \left[y^2 - \frac{y^3}{3}\right]_{0.5}^1 = \frac{11}{32} \quad \text{(exact)}.$$

(C) Expectation $E[Y]$

$$E[Y] = \int_0^2 y \cdot \frac{3}{4}(2y - y^2)\, dy = \frac{3}{4} \int_0^2 (2y^2 - y^3)\, dy.$$

Compute:

$$\int_0^2 2y^2 - y^3\, dy = \left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = \frac{16}{3} - 4 = \frac{4}{3}.$$

So

$$E[Y] = \frac{3}{4} \cdot \frac{4}{3} = 1.$$

Interpretation: mean at 1 (center of mass), consistent with the support $0 < y < 2$.

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5 5. The Normal Distribution (in detail)

Definition and parameters

A random variable Y has a normal distribution with mean μ and variance σ^2 if its PDF is

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right), \qquad y \in \mathbb{R}.$$

Notation: $Y \sim N(\mu, \sigma^2)$.

Properties and intuition

- Symmetric about μ. Mean = median = mode = μ.
- Empirical rule (approx): about 68% of mass within $\mu \pm \sigma$, 95% within $\mu \pm 2\sigma$ (approx), 99.7% within $\mu \pm 3\sigma$.
- Many sample means converge to normality (Central Limit Theorem); hence normality is very common in practice.

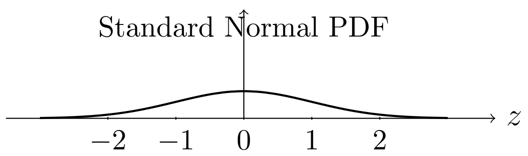
Standard normal

Let $Z \sim N(0, 1)$. To standardize $X \sim N(\mu, \sigma^2)$,

$$Z = \frac{X-\mu}{\sigma}.$$

Then $P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$.

Sketch: normal PDF



Worked problems

1. Probability example. Let $X \sim N(50, 10^2)$. Find $P(45 < X < 60)$.

Standardize:

$$z_1 = \frac{45-50}{10} = -0.5, \quad z_2 = \frac{60-50}{10} = 1.$$

So $P(-0.5 < Z < 1) = \Phi(1) - \Phi(-0.5) \approx 0.84134 - 0.30854 = 0.5328$ (using tables/software).

2. Quantile example. For $X \sim N(40, 5^2)$, find x with $P(X > x) = 0.05$. We want k with $P(Z > k) = 0.05$ so $k \approx 1.645$. Then solve

$$1.645 = \frac{x - 40}{5} \implies x \approx 40 + 1.645 \cdot 5 = 48.225.$$

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6 6. Practical tips & common mistakes

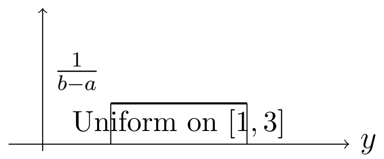
- Always check whether a variable is discrete or continuous before integrating/summing.
- Remember $P(Y = a) = 0$ for continuous variables — don't try to calculate point probabilities.
- Watch limits of integration: PDF support matters.
- When using normal tables, be careful whether they give left-tail, right-tail, or area from mean.
- Use continuity correction when approximating discrete distributions with continuous ones (Binomial $\to$ Normal).

Optional: Extra Distributions (not required on exam)

Uniform distribution (continuous) $U(a,b)$

$$f(y) = \begin{cases} \dfrac{1}{b-a}, & a \le y \le b, \\ 0, & \text{otherwise.} \end{cases}$$

Properties: $E[Y] = \dfrac{a+b}{2}$, $\mathrm{Var}(Y) = \dfrac{(b-a)^2}{12}$. **Example:** $U(0,2)$ computed earlier.

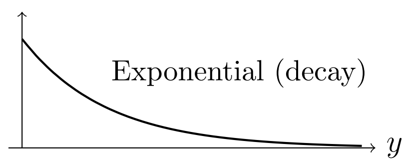


Exponential distribution

PDF:

$$f(y) = \lambda e^{-\lambda y}, \quad y \ge 0,\ \lambda > 0.$$

Properties: $E[Y] = 1/\lambda$, $\mathrm{Var}(Y) = 1/\lambda^2$. Memoryless property: $P(Y > s+t \mid Y > s) = P(Y > t)$.



Gamma and Beta (brief)

- **Gamma** $Y \sim \Gamma(\alpha, \beta)$: flexible family (includes Exponential as $\alpha = 1$), used for waiting times and shapes.
- **Beta** on $[0,1]$: flexible for modeling proportions; parameters $\alpha, \beta > 0$ shape the density.

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7 7. Exercises (3–5 worked examples with solutions)

Exercise 1 (Normalisation & moments)

Let

$$f(y) = \begin{cases} k\,y^2, & 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find k. (b) Compute $E[Y]$. (c) Compute $\mathrm{Var}(Y)$.

Solution. (a) Normalisation:

$$1 = \int_0^2 ky^2\,dy = k\left[\frac{y^3}{3}\right]_0^2 = k\cdot\frac{8}{3} \quad\Rightarrow\quad k = \frac{3}{8}.$$

(b) Mean:

$$E[Y] = \int_0^2 y\cdot\frac{3}{8}y^2\,dy = \frac{3}{8}\int_0^2 y^3\,dy = \frac{3}{8}\left[\frac{y^4}{4}\right]_0^2 = \frac{3}{8}\cdot\frac{16}{4} = \frac{3}{8}\cdot 4 = \frac{3}{2} = 1.5.$$

(c) Second moment and variance:

$$E[Y^2] = \frac{3}{8}\int_0^2 y^4\,dy = \frac{3}{8}\left[\frac{y^5}{5}\right]_0^2 = \frac{3}{8}\cdot\frac{32}{5} = \frac{12}{5} = 2.4.$$

$$\mathrm{Var}(Y) = E[Y^2] - [E[Y]]^2 = 2.4 - (1.5)^2 = 2.4 - 2.25 = 0.15 = \frac{3}{20}.$$

Exercise 2 (Normal probability)

Let $X \sim N(100, 15^2)$. Compute $P(85 < X < 125)$.

Solution. Standardize:

$$z_1 = \frac{85-100}{15} = -1.0, \qquad z_2 = \frac{125-100}{15} = \frac{25}{15} \approx 1.6667.$$

Using standard normal table or software:

$$P(-1 < Z < 1.6667) = \Phi(1.6667) - \Phi(-1).$$

Numerical values: $\Phi(1.6667) \approx 0.9525$, $\Phi(-1) = 0.15866$. Therefore

$$P(85 < X < 125) \approx 0.9525 - 0.15866 = 0.79384 \approx 0.7938.$$

Exercise 3 (Normal quantile)

Let $X \sim N(50, 8^2)$. Find x such that $P(X < x) = 0.90$.

Solution. Find $z_{0.90}$ with $\Phi(z_{0.90}) = 0.90$. From tables/software $z_{0.90} \approx 1.2816$. Unstandardize:

$$x = 50 + 1.2816\cdot 8 \approx 50 + 10.2528 = 60.2528.$$

Exercise 4 (Normal approximation to Binomial with continuity correction)

Let $X \sim \text{Binomial}(n = 100, p = 0.3)$. Approximate $P(X \leq 40)$ using the normal approximation (with continuity correction).

Solution. First compute mean and standard deviation:

$$\mu = np = 100 \cdot 0.3 = 30, \qquad \sigma = \sqrt{npq} = \sqrt{100 \cdot 0.3 \cdot 0.7} = \sqrt{21} \approx 4.5826.$$

Continuity correction: $P(X \leq 40) \approx P(Y < 40.5)$ where $Y \sim N(\mu, \sigma^2)$. Standardize:

$$z = \frac{40.5 - 30}{4.5826} \approx \frac{10.5}{4.5826} \approx 2.2913.$$

Therefore $P(X \leq 40) \approx \Phi(2.2913) \approx 0.9890$.

Exercise 5 (CDF inversion — median of exponential)

Let the CDF be $F(y) = 1 - e^{-y/2}$ for $y \geq 0$. Find the median m (value with 50% probability to the left).

Solution. Solve $F(m) = 0.5$:

$$1 - e^{-m/2} = 0.5 \quad \Rightarrow \quad e^{-m/2} = 0.5 \quad \Rightarrow \quad -\frac{m}{2} = \ln(0.5).$$

Hence $m = -2\ln(0.5) = 2\ln 2 \approx 2 \cdot 0.6931 = 1.3862$.

Summary (one-page quick facts)

Concept	Formula / Notes
PDF	$f(y) \geq 0,\ \int f(y)\,dy = 1$
CDF	$F(y) = \int_{-\infty}^{y} f(t)\,dt$
Mean	$E[Y] = \int y f(y)\,dy$
Variance	$\text{Var}(Y) = E[Y^2] - [E[Y]]^2$
Normal	$N(\mu, \sigma^2)$, standardize via $Z = (X - \mu)/\sigma$
Uniform	$E = (a+b)/2$, $\text{Var} = (b-a)^2/12$
Exponential	$E = 1/\lambda$, memoryless

Final tip: practise converting between PDF/CDF, standardising normals, and integrating simple polynomials — those are the operations you will use most often.
