Notes on Special Relativity

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1 Postulates of Special Relativity

Special Relativity (SR) is built on two fundamental postulates:

1. **The Principle of Relativity:** The laws of physics are the same in all inertial (non-accelerating) frames of reference.
2. **The Constancy of the Speed of Light:** The speed of light in a vacuum, $c \approx 299,792,458$ m/s, is the same for all observers in inertial frames, regardless of the motion of the light source or the observer.

2 Spacetime and Transformations

2.1 Galilean Transformation (Classical)

Consider two inertial frames, S and S'. Frame S' moves with a constant velocity $\boldsymbol{v}$ (assumed to be along the x-axis) relative to S. Classically, the coordinates of an event (t, x, y, z) in S are related to the coordinates (t', x', y', z') in S' by the **Galilean transformation**:

$$
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

This transformation assumes an absolute, universal time ($t' = t$).

2.2 Lorentz Transformation (Relativistic)

The Galilean transformation is incompatible with the second postulate of SR. The correct transformation that preserves the speed of light is the **Lorentz transformation**. For motion along the x-axis:

$$
\begin{aligned}
ct' &= \gamma\left(ct - \frac{v}{c}x\right) \\
x' &= \gamma\left(x - \frac{v}{c}(ct)\right) \\
y' &= y \\
z' &= z
\end{aligned}
$$

where γ is the **Lorentz factor**:

$$
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
$$

Note that as $v \to 0$ (or $v \ll c$), $\gamma \to 1$ and the Lorentz transformation reduces to the Galilean transformation.

2.3 The Spacetime Interval

In SR, we unite space and time into a 4-dimensional **spacetime**. The separation between two infinitesimally close events is given by the **spacetime interval**, ds^2:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

The spacetime interval ds^2 is an **invariant** quantity, meaning it has the same value in all inertial frames ($ds^2 = ds'^2$).

The causal relationship between two events is determined by the sign of their interval:

- $ds^2 < 0$: The interval is **time-like**. One event can causally affect the other. This is the path taken by massive particles.
- $ds^2 = 0$: The interval is **light-like** (or null). The events are connected only by a light signal.
- $ds^2 > 0$: The interval is **space-like**. The events are causally disconnected; neither can affect the other.

3 Relativistic Kinematics

3.1 Spacetime Diagrams

A spacetime diagram (or Minkowski diagram) plots time (ct) on the vertical axis against one spatial dimension (x) on the horizontal axis.

- Light travels at 45° lines ($x = \pm ct$), forming the **light cone**.
- The interior of the cone (time-like region) represents the causal future and past.
- The exterior of the cone (space-like region) is causally inaccessible.

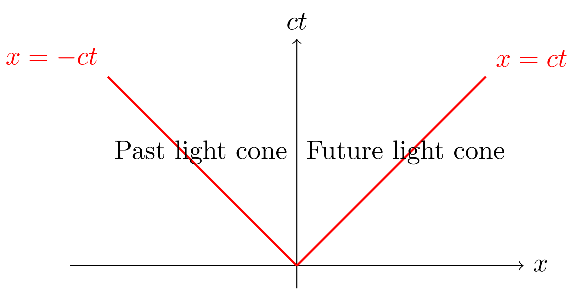


Figure 1: Minkowski diagram showing light cones in spacetime.

3.2 Time Dilation

Time dilation is the phenomenon where a moving clock is measured to run slower than a stationary clock.

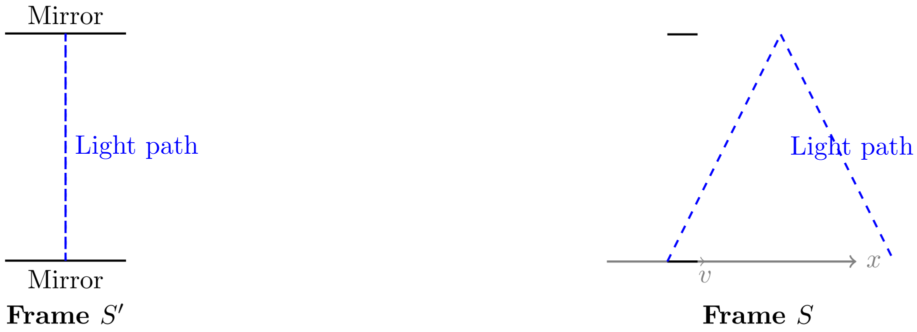
Consider a "light clock" in frame S' (moving at velocity v) where light bounces between two mirrors separated by a distance d.

- In frame S', the time for one tick (proper time, $\Delta t'$) is:

$$\Delta t' = \frac{2d}{c}$$

- In frame S, the light travels a longer, diagonal path. By the Pythagorean theorem:

$$\left(\frac{c\Delta t}{2}\right)^2 = \left(\frac{v\Delta t}{2}\right)^2 + d^2$$



(a) Rest frame S' (light moves vertically).

(b) Ground frame S (light travels diagonally).

Figure 2: Comparison of the light clock in (a) its rest frame S' and (b) the ground frame S. The diagonal path in S explains time dilation, since light travels a longer path between ticks.

- Solving for Δt:

$$(c\Delta t)^2 = (v\Delta t)^2 + (2d)^2 \implies \Delta t^2(c^2 - v^2) = (2d)^2$$

$$\Delta t^2 = \frac{(2d)^2}{c^2(1 - v^2/c^2)} = \frac{(2d/c)^2}{1 - v^2/c^2}$$

- Substituting $\Delta t' = 2d/c$:

$$\Delta t^2 = \frac{(\Delta t')^2}{1 - v^2/c^2} \implies \Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}} = \gamma \Delta t'$$

Since $\gamma \geq 1$, the time measured in frame S (Δt) is always greater than the proper time ($\Delta t'$), the time measured in the clock's own rest frame.

Example (Muon Decay): Muons are unstable particles with a proper lifetime $\Delta t' \approx 2.2\,\mu$s. If they travel at $v = 0.99c$, $\gamma \approx 7.1$.

- Classically: They would travel $d = v\Delta t' \approx (0.99)(3 \times 10^8 \text{ m/s})(2.2 \times 10^{-6} \text{ s}) \approx 650$ m. They should not reach Earth's surface.
- Relativistically (Earth frame): Their lifetime is dilated to $\Delta t = \gamma \Delta t' \approx (7.1)(2.2\,\mu\text{s}) \approx 15.6\,\mu$s. They travel $d = v\Delta t \approx (0.99c)(15.6\,\mu\text{s}) \approx 4600$ m. This allows them to be detected on Earth.

3.3 Length Contraction

An object of proper length L_p (its length in its rest frame) is measured to be shorter in a frame where it is moving.

- In the Earth frame (S), the distance to the atmosphere is L_p. The time for the muon is $\Delta t = L_p/v$.
- In the muon's frame (S'), the Earth is moving towards it. The distance is L, and the time is $\Delta t'$.

$$L = v\Delta t'$$

- We know $\Delta t' = \Delta t/\gamma = (L_p/v)/\gamma$.
- Substituting this into the equation for L:

$$L = v\left(\frac{L_p}{v\gamma}\right) = \frac{L_p}{\gamma}$$

The object's length is contracted in the direction of its motion.

4 Relativistic Dynamics

4.1 Relativistic Momentum

The classical definition of momentum ($\boldsymbol{p} = m\boldsymbol{u}$) is not conserved in SR. The correct **relativistic momentum** is:

$$\boldsymbol{p} = \gamma m \boldsymbol{u} = \frac{m\boldsymbol{u}}{\sqrt{1 - u^2/c^2}}$$

4.2 Relativistic Energy

From the Work-Energy theorem, the **relativistic kinetic energy** (K) is found to be:

$$K = \frac{mc^2}{\sqrt{1 - u^2/c^2}} - mc^2 = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2$$

This leads to the following definitions:

- $E_0 = mc^2$: **Rest Energy**, the intrinsic energy of a particle at rest.
- $E = \gamma mc^2$: **Total Relativistic Energy**.

The total energy is the sum of the rest energy and the kinetic energy: $E = K + E_0$.

Low-Velocity Limit: For $u \ll c$, we can use the binomial approximation $\gamma = (1 - u^2/c^2)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2}$.

$$K \approx \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right) mc^2 - mc^2 = mc^2 + \frac{1}{2}mu^2 - mc^2 = \frac{1}{2}mu^2$$

This correctly reduces to the classical kinetic energy.

4.3 Energy-Momentum Relation

A very useful formula relates total energy, momentum, and rest mass.

$$E^2 = (\gamma mc^2)^2 = \gamma^2 m^2 c^4$$

$$p^2c^2 = (\gamma mu)^2 c^2 = \gamma^2 m^2 u^2 c^2$$

Subtracting the two:

$$\begin{aligned}
E^2 - p^2c^2 &= \gamma^2 m^2 c^4 - \gamma^2 m^2 u^2 c^2 \\
&= \gamma^2 m^2 c^2 (c^2 - u^2) \\
&= \left(\frac{1}{1 - u^2/c^2}\right) m^2 c^2 \left(\frac{c^2(1 - u^2/c^2)}{c^2}\right)(c^2 - u^2) \\
&= \left(\frac{1}{1 - u^2/c^2}\right) m^2 c^4 (1 - u^2/c^2) \\
&= m^2 c^4
\end{aligned}$$

This gives the **energy-momentum relation**:

$$E^2 = (pc)^2 + (mc^2)^2$$

This relation is a Lorentz invariant.

- For a particle at rest ($p = 0$): $E^2 = (mc^2)^2 \implies E = mc^2$.
- For a massless particle ($m = 0$), like a photon: $E^2 = (pc)^2 \implies E = pc$.

5 4-Vectors and Minkowski Spacetime

5.1 4-Vector Notation

We can formalize SR using **4-vectors**. The position 4-vector x^μ is:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

where $\mu \in \{0, 1, 2, 3\}$. A Lorentz transformation is a linear transformation that relates x^μ in S to x'^μ in S':

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu$$

where Λ^μ_ν (or $\frac{\partial x'^\mu}{\partial x^\nu}$) is the Lorentz transformation matrix.

5.2 The Metric Tensor

The invariant spacetime interval ds^2 can be written using the **Minkowski metric tensor**, $\eta_{\mu\nu}$:

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu$$

(Einstein summation convention is implied). For $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$, the metric is:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(This is the ”mostly plus” signature. The ”mostly minus” $(1, -1, -1, -1)$ signature is also common).

The metric is used to ”lower” indices, converting a contravariant vector v^μ to a covariant vector v_μ:

$$v_\mu = \eta_{\mu\nu} v^\nu$$

For example, $v_0 = \eta_{0\nu} v^\nu = \eta_{00} v^0 = -v^0$.

The **invariant inner product** (or ”dot product”) of two 4-vectors A^μ and B^μ is:

$$A \cdot B = A^\mu B_\mu = \eta_{\mu\nu} A^\mu B^\nu = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$$

The squared magnitude of a 4-vector v^μ is $v^\mu v_\mu$, which is a Lorentz invariant.

5.3 4-Velocity and 4-Momentum

The 4-velocity is defined as $u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \boldsymbol{u})$, and the 4-momentum as $p^\mu = mu^\mu = (\frac{E}{c}, \boldsymbol{p})$. Its invariant magnitude gives $p^\mu p_\mu = -m^2 c^2$, which recovers the energy–momentum relation $E^2 = (pc)^2 + (mc^2)^2$.

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