Derivation of Newtonian Gravity from Einstein's Field Equations

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1 The Einstein Field Equations (EFE)

The Einstein Field Equations (EFE) are the foundation of general relativity. They relate the geometry of spacetime, encoded in the Einstein tensor G_{ab}, to the distribution of matter and energy, encoded in the stress-energy tensor T_{ab}.

The equation is built from several components:

- R_{ab}: The Ricci curvature tensor, which describes how the volume of matter changes.
- R: The Ricci scalar ($R = g^{ab}R_{ab}$), the trace of the Ricci tensor.
- g_{ab}: The metric tensor, which defines distances and angles in spacetime.
- T_{ab}: The stress-energy tensor, which represents the density and flux of energy and momentum.

A crucial requirement is the conservation of energy and momentum, which is mathematically expressed as the covariant derivative of the stress-energy tensor being zero:

$$\nabla^a T_{ab} = 0$$

The Einstein tensor G_{ab} is constructed to automatically satisfy this conservation law, thanks to the contracted Bianchi identity:

$$\nabla^a G_{ab} = \nabla^a \left(R_{ab} - \frac{1}{2} R g_{ab} \right) = 0$$

This leads to the final form of the Einstein Field Equations, where k is a constant of proportionality to be determined:

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = k T_{ab}$$

Our goal is to show that this complex, non-linear equation reduces to Newton's law of gravitation in the appropriate limit. Specifically, we aim to recover the Poisson equation for the Newtonian potential Φ:

$$\nabla^2 \Phi = 4\pi G \rho$$

where G is the gravitational constant and ρ is the mass density.

2 The Newtonian Limit: Assumptions

To bridge the gap between general relativity and Newtonian gravity, we must apply three simplifying assumptions, known as the "Newtonian limit":

1. **Weak Gravitational Field:** The spacetime is nearly flat. We can express the metric g_{ab} as a small perturbation h_{ab} (where $|h_{ab}| \ll 1$) added to the flat Minkowski metric η_{ab}:

   $$g_{ab} = \eta_{ab} + h_{ab}$$

   In our convention (setting $c = 1$), $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. The inverse metric is $g^{ab} \approx \eta^{ab} - h^{ab}$.

2. **Static Field:** The gravitational field is not changing with time. This means all time derivatives of the metric perturbation are zero:

   $$\partial_0 h_{ab} = 0$$

3. **Non-Relativistic (Slow-Moving) Particles:** Particles are moving much slower than the speed of light ($v \ll c$). This implies that the spatial components of their 4-velocity are negligible compared to the time component:
$$\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \quad (\text{where } x^0 = t)$$

3 Step 1: The Geodesic Equation (Equation of Motion)

In general relativity, a test particle follows a geodesic, described by:

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0$$

where Γ^a_{bc} are the Christoffel symbols (representing the gravitational field).

Applying the non-relativistic assumption (3), the only significant term in the sum is when $b = c = 0$. The equation simplifies to:
$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$
Now, we calculate the Christoffel symbol Γ^a_{00} under our assumptions:

$$\Gamma^a_{00} = \frac{1}{2}g^{ad}(\partial_0 g_{d0} + \partial_0 g_{0d} - \partial_d g_{00})$$

The static assumption (2) makes the first two terms zero:

$$\Gamma^a_{00} = -\frac{1}{2}g^{ad}\partial_d g_{00}$$

Applying the weak-field assumption (1):

$$\Gamma^a_{00} \approx -\frac{1}{2}(\eta^{ad} - h^{ad})\partial_d(\eta_{00} + h_{00})$$

Since $\eta_{00} = -1$ is constant, $\partial_d\eta_{00} = 0$. We also drop the $h^{ad}\partial_d h_{00}$ term, as it is second-order small ($O(h^2)$).
$$\Gamma^a_{00} \approx -\frac{1}{2}\eta^{ad}\partial_d h_{00}$$

3.1 Analyzing the Geodesic Components

For a = 0 (time component):

$$\Gamma^0_{00} \approx -\frac{1}{2}\eta^{0d}\partial_d h_{00} = -\frac{1}{2}\eta^{00}\partial_0 h_{00} = -\frac{1}{2}(-1)(0) = 0$$

The geodesic equation becomes $\frac{d^2x^0}{d\tau^2} = 0$, or $\frac{d^2t}{d\tau^2} = 0$. This means $\frac{dt}{d\tau}$ is constant, so proper time τ is proportional to coordinate time t.

For a = i (spatial components):

$$\Gamma^i_{00} \approx -\frac{1}{2}\eta^{id}\partial_d h_{00} = -\frac{1}{2}(\eta^{i0}\partial_0 h_{00} + \eta^{ij}\partial_j h_{00})$$

Since $\eta^{i0} = 0$ and $\eta^{ij} = \delta^{ij}$ (the Kronecker delta):

$$\Gamma^i_{00} \approx -\frac{1}{2}\delta^{ij}\partial_j h_{00} = -\frac{1}{2}\partial^i h_{00}$$

Substitute this back into the geodesic equation:

$$\frac{d^2x^i}{d\tau^2} - \frac{1}{2}\partial^i h_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

Using the chain rule, $\frac{d^2x^i}{d\tau^2} = \frac{d^2x^i}{dt^2}(\frac{dt}{d\tau})^2 + \frac{dx^i}{dt}\frac{d^2t}{d\tau^2}$. Since $\frac{d^2t}{d\tau^2} = 0$:

$$\frac{d^2x^i}{dt^2}\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{2}\partial^i h_{00}\left(\frac{dt}{d\tau}\right)^2$$

The particle's acceleration $a^i = \frac{d^2x^i}{dt^2}$ is therefore:

$$a^i = \frac{1}{2}\partial^i h_{00}$$

We compare this to Newton's law of gravity, $\vec{a} = -\nabla\Phi$, or in index notation:

$$a^i = -\partial^i\Phi$$

Equating the two expressions for acceleration gives:

$$\frac{1}{2}\partial^i h_{00} = -\partial^i\Phi \implies h_{00} = -2\Phi + \text{const}$$

Setting the constant to zero, we find the link between the metric and the Newtonian potential:

$$h_{00} = -2\Phi$$

This means the g_{00} component of the metric is $g_{00} = \eta_{00} + h_{00} = -1 - 2\Phi = -(1 + 2\Phi)$.

4 Step 2: The Field Equation (Poisson's Equation)

Now we analyze the EFE itself. First, we contract the EFE:

$$g^{ab}\left(R_{ab} - \frac{1}{2}Rg_{ab} = kT_{ab}\right) \implies R - \frac{1}{2}R(4) = kT \implies -R = kT$$

So, $R = -kT$. Substitute this back into the original EFE:

$$R_{ab} - \frac{1}{2}(-kT)g_{ab} = kT_{ab} \implies R_{ab} = k\left(T_{ab} - \frac{1}{2}Tg_{ab}\right)$$

We only need to analyze the R_{00} component. In our non-relativistic "dust" limit (where pressure is zero), the stress-energy tensor is dominated by the mass-energy density, $T_{00} \approx \rho$. The trace $T = g^{ab}T_{ab} \approx \eta^{00}T_{00} \approx (-1)\rho = -\rho$.

Plugging these into the R_{00} equation:

$$R_{00} \approx k\left(T_{00} - \frac{1}{2}g_{00}T\right) \approx k\left(\rho - \frac{1}{2}(\eta_{00} + h_{00})(-\rho)\right)$$

To first order, $g_{00} \approx \eta_{00} = -1$:

$$R_{00} \approx k\left(\rho - \frac{1}{2}(-1)(-\rho)\right) = k\left(\rho - \frac{1}{2}\rho\right)$$

$$R_{00} = \frac{k\rho}{2}$$

4.1 Calculating R_{00} from Geometry

Now we calculate R_{00} from the metric in the weak, static limit. The full formula for the Ricci tensor is $R_{ab} = \partial_c\Gamma^c_{ab} - \partial_b\Gamma^c_{ac} + \Gamma^c_{cd}\Gamma^d_{ab} - \Gamma^c_{bd}\Gamma^d_{ac}$. In the weak-field limit, all terms with two Christoffel symbols ($\Gamma\Gamma$) are $O(h^2)$ and can be ignored.

$$R_{ab} \approx \partial_c\Gamma^c_{ab} - \partial_b\Gamma^c_{ac}$$

For R_{00}:

$$R_{00} \approx \partial_c\Gamma^c_{00} - \partial_0\Gamma^c_{0c}$$

The static assumption ($\partial_0 = 0$) makes the second term zero. Since $\Gamma^0_{00} = 0$:

$$R_{00} \approx \partial_i \Gamma^i_{00}$$

We already found $\Gamma^i_{00} \approx -\frac{1}{2}\partial^i h_{00}$. Substituting this in:

$$R_{00} \approx \partial_i \left(-\frac{1}{2}\partial^i h_{00} \right) = -\frac{1}{2}\partial_i \partial^i h_{00}$$

$\partial_i \partial^i$ is the spatial Laplacian, ∇^2.

$$R_{00} \approx -\frac{1}{2}\nabla^2 h_{00}$$

Now, use our result from Step 1: $h_{00} = -2\Phi$.

$$R_{00} \approx -\frac{1}{2}\nabla^2(-2\Phi) = \nabla^2 \Phi$$

5 Step 3: Combining Results and Finding k

We have derived two expressions for R_{00}:

1. From the Stress-Energy Tensor: $R_{00} = \frac{k\rho}{2}$
2. From Spacetime Geometry: $R_{00} = \nabla^2 \Phi$

Equating these gives:

$$\nabla^2 \Phi = \frac{k\rho}{2}$$

We compare this directly to Newton's Poisson equation:

$$\nabla^2 \Phi = 4\pi G \rho$$

This immediately gives us the value of the constant k:

$$\frac{k}{2} = 4\pi G \implies k = 8\pi G$$

(This derivation was done with $c = 1$. If we restore c, $k = \frac{8\pi G}{c^4}$.)

By substituting this constant back into the original equation, we arrive at the complete form of the Einstein Field Equations:

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab}$$