A Simple Summary of Orbital Mechanics

Leonardo Tiditada Pedersen

8 October 2024

Contents

1 Derivation of Kepler's First Law

This is the derivation for the equation of an orbit, starting from Newton's Law of Gravitation.

1.1 Step 1: Acceleration in Polar Coordinates

We work in a 2D plane. The position vector $\vec{r}$ is given by:

$$\vec{r} = r\hat{r}$$

where $\hat{r}$ is the unit vector in the radial direction and $\hat{\theta}$ is the unit vector in the tangential (angular) direction.

The time derivatives of the unit vectors are:

$$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\hat{r}$$

The **velocity** $\vec{v}$ is the first time derivative of position:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

The **acceleration** $\vec{a}$ is the second time derivative:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$\vec{a} = (\ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt}) + (\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt})$$

$$\vec{a} = (\ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta}) + (\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r})$$

Grouping the $\hat{r}$ and $\hat{\theta}$ terms gives the acceleration in polar coordinates:

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

1.2 Step 2: Newton's Law of Gravitation

Newton's Second Law is $\vec{F} = m\vec{a}$. The gravitational force between a large mass M (like the sun) and a small mass m (like a planet) is a central force:

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

Now we set $\vec{F} = m\vec{a}$:

$$-\frac{GMm}{r^2}\hat{r} = m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

Equating the components gives two differential equations:

$$\text{(Radial)} \quad \ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2} \tag{1}$$

$$\text{(Tangential)} \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{2}$$

1.3 Step 3: Solving the Differential Equation

First, look at the tangential equation (2). It can be rewritten as:

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \implies \frac{d}{dt}(r^2\dot{\theta}) = 0$$

This means $r^2\dot{\theta}$ is a constant. We call this constant h, the specific angular momentum ($h = L/m$).

$$h = r^2\dot{\theta} = \text{constant}$$

This is **Kepler's Second Law** (a body sweeps out equal areas in equal times, since $dA/dt = \frac{1}{2}r^2\dot{\theta} = h/2$).

Now, we solve the radial equation (1). It's hard to solve with respect to time (t). We change the variable from r to $u = 1/r$ and from t to θ.

- $r = 1/u$
- $\dot{\theta} = \frac{h}{r^2} = hu^2$
- $\dot{r} = \frac{d}{dt}(\frac{1}{u}) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}(hu^2) = -h\frac{du}{d\theta}$
- $\ddot{r} = \frac{d}{dt}(-h\frac{du}{d\theta}) = \frac{d}{d\theta}(-h\frac{du}{d\theta})\frac{d\theta}{dt} = (-h\frac{d^2u}{d\theta^2})(hu^2) = -h^2u^2\frac{d^2u}{d\theta^2}$

Substitute these into the radial equation (1):

$$(\ddot{r}) - (r)(\dot{\theta})^2 = -\frac{GM}{r^2}$$

$$(-h^2u^2\frac{d^2u}{d\theta^2}) - (\frac{1}{u})(hu^2)^2 = -GMu^2$$

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -GMu^2$$

Divide the entire equation by $-h^2u^2$:

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

This is a standard second-order linear inhomogeneous differential equation.

1.4 Step 4: The Orbit Equation and Eccentricity

The general solution to $\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$ is the sum of the homogeneous solution (u_h) and the particular solution (u_p).

- **Particular solution (u_p):** A constant C. $0 + C = GM/h^2 \implies u_p = GM/h^2$.
- **Homogeneous solution (u_h):** Solution to $u'' + u = 0$. $u_h = A\cos(\theta - \theta_0)$. We can set the phase $\theta_0 = 0$ by rotating our coordinate system so that $\theta = 0$ is the point of closest approach (periapsis). So, $u_h = A\cos(\theta)$.

The full solution is $u = u_p + u_h$:

$$u(\theta) = \frac{GM}{h^2} + A\cos(\theta) = \frac{GM}{h^2}\left(1 + \frac{Ah^2}{GM}\cos(\theta)\right)$$

We define the **eccentricity** (e) as $e = \frac{Ah^2}{GM}$.

$$u(\theta) = \frac{GM}{h^2}(1 + e\cos(\theta))$$

Since $u = 1/r$, we flip the equation to get the polar form of the orbit:

$$r(\theta) = \frac{h^2/GM}{1 + e\cos(\theta)}$$

This is **Kepler's First Law**: The path of a body in orbit is a conic section (circle, ellipse, parabola, or hyperbola) with the central body (M) at one focus.

2 Energy, Eccentricity, and Orbit Types

The total energy of the system (E_{tot}) determines the shape of the orbit. The total energy is the sum of kinetic (K) and potential (U) energy.

$$E_{tot} = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Through a more involved derivation (substituting $v^2 = \dot{r}^2 + (r\dot{\theta})^2$ using the solutions from section 1), this energy can be shown to be constant and related to the eccentricity e and specific angular momentum h

$$E_{tot} = \frac{G^2M^2m}{2h^2}(e^2 - 1)$$

2.1 Classifying Orbits

We can also relate the energy to the **semi-major axis (a)**. For a bound elliptical orbit, $2a = r_p + r_a$, where r_p (periapsis) and r_a (apoapsis) are the closest and farthest points.

- $r_p = r(\theta = 0) = \frac{h^2/GM}{1+e}$
- $r_a = r(\theta = \pi) = \frac{h^2/GM}{1-e}$

$$2a = r_p + r_a = \frac{h^2}{GM}\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{h^2}{GM}\left(\frac{1-e+1+e}{1-e^2}\right) = \frac{2h^2}{GM(1-e^2)}$$

This gives a key relationship: $h^2 = GMa(1-e^2)$. Substituting this h^2 into the energy equation:

$$E_{tot} = \frac{G^2M^2m}{2(GMa(1-e^2))}(e^2 - 1) = \frac{GMm}{2a(1-e^2)}(-(1-e^2))$$

$$E_{tot} = -\frac{GMm}{2a}$$

This fundamental equation links total energy to the size of the orbit. The type of orbit depends directly on the total energy and eccentricity

Orbit Type	**Eccentricity (e)**	**Total Energy (E_{tot})**	**Description**
Circular	$e = 0$	$E_{tot} = -\frac{GMm}{2R}$	Bound, minimum energy
Ellipse	$0 < e < 1$	$E_{tot} = -\frac{GMm}{2a} < 0$	Bound, closed orbit
Parabola	$e = 1$	$E_{tot} = 0$	Unbound, open (escape orbit)
Hyperbola	$e > 1$	$E_{tot} = +\frac{GMm}{2a} > 0$	Unbound, open

Note: For hyperbolas, a is defined as $a = \frac{h^2}{GM(e^2-1)}$, which is why E_{tot} becomes positive.

3 The Vis-Viva Equation

The **vis-viva equation** (living force) gives the velocity v of an orbiting body at any distance r from the center. It is derived directly from the energy conservation equation.

$$E_{tot} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

3.1 For Ellipses and Circles ($E_{tot} \leq 0$)

We use $E_{tot} = -\frac{GMm}{2a}$[cite: 2]:

$$-\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$\frac{GMm}{r} - \frac{GMm}{2a} = \frac{1}{2}mv^2$$

Divide by $m/2$:

$$2(\frac{GM}{r} - \frac{GM}{2a}) = v^2 \implies v^2 = GM(\frac{2}{r} - \frac{1}{a})$$

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

This is the vis-viva equation for bound orbits.

3.2 For Hyperbolas ($E_{tot} > 0$)

We use $E_{tot} = +\frac{GMm}{2a}$[cite: 3]:

$$+\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$\frac{GMm}{r} + \frac{GMm}{2a} = \frac{1}{2}mv^2$$

$$v = \sqrt{GM\left(\frac{2}{r} + \frac{1}{a}\right)}$$

3.3 Special Case Velocities

- **Circular Velocity (v_{circ}):** For a circle, $r = a = R$.

$$v_{circ} = \sqrt{GM\left(\frac{2}{R} - \frac{1}{R}\right)} = \sqrt{\frac{GM}{R}}$$

- **Escape Velocity (v_{esc}):** This is the speed needed to reach infinity, which is a parabolic orbit ($E_{tot} = 0$, so $a \to \infty$).

$$v_{esc} = \sqrt{GM\left(\frac{2}{r} - \frac{1}{\infty}\right)} = \sqrt{\frac{2GM}{r}}$$

- **Relationship:** Notice that $v_{esc} = \sqrt{2} \times v_{circ}$. To escape from a circular orbit, you must multiply your speed by $\sqrt{2}$.

4 Perihelion and Aphelion (Key Points)

These are the points of closest and farthest approach, respectively. (The general terms are periapsis and apoapsis; "helion" is specific to the Sun).

4.1 Distances

As shown in Section 2.1, for an ellipse:

- **Perihelion distance (r_p):** $r_p = a(1 - e)$
- **Aphelion distance (r_a):** $r_a = a(1 + e)$

For a parabola ($e = 1$):

- $r_p = \frac{h^2/GM}{1+1} = \frac{h^2}{2GM}$
- $r_a = \infty$ (it never returns)

For a hyperbola ($e > 1$):

- $r_p = a(e - 1)$ (using the hyperbola definition of a)

4.2 Velocities

We find the velocities at these points by plugging the distances into the vis-viva equation. For an ellipse:

- **Perihelion velocity (v_p, max speed):**

$$v_p^2 = GM\left(\frac{2}{r_p} - \frac{1}{a}\right) = GM\left(\frac{2}{a(1-e)} - \frac{1}{a}\right)$$

$$v_p^2 = \frac{GM}{a}\left(\frac{2-(1-e)}{1-e}\right) = \frac{GM}{a}\left(\frac{1+e}{1-e}\right)$$

$$v_p = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$$

- **Aphelion velocity (v_a, min speed):**

$$v_a^2 = GM\left(\frac{2}{r_a} - \frac{1}{a}\right) = GM\left(\frac{2}{a(1+e)} - \frac{1}{a}\right)$$

$$v_a^2 = \frac{GM}{a}\left(\frac{2-(1+e)}{1+e}\right) = \frac{GM}{a}\left(\frac{1-e}{1+e}\right)$$

$$v_a = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$$

These also satisfy the conservation of angular momentum: $v_p r_p = v_a r_a = h$.

5 Kepler's Other Two Laws

5.1 Second Law: Law of Equal Areas

As shown in the derivation (Section 1.3), the tangential equation of motion implies:

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2} = \text{constant}$$

A line joining a planet and the Sun sweeps out equal areas in equal intervals of time. This means the planet moves **fastest** at perihelion (closest) and **slowest** at aphelion (farthest).

5.2 Third Law: Law of Periods

The area of an ellipse is $A = \pi ab$, where $b = a\sqrt{1-e^2}$ is the semi-minor axis.

$$A = \pi a^2\sqrt{1-e^2}$$

The total area is swept out in one period, T.

$$A = \int_0^T \frac{dA}{dt}dt = \int_0^T \frac{h}{2}dt = \frac{hT}{2}$$

Equating the two expressions for area:

$$\frac{hT}{2} = \pi a^2\sqrt{1-e^2} \implies T = \frac{2\pi a^2\sqrt{1-e^2}}{h}$$

Square both sides:

$$T^2 = \frac{4\pi^2 a^4(1-e^2)}{h^2}$$

From Section 2.1, we know $h^2 = GMa(1-e^2)$. Substitute this in:

$$T^2 = \frac{4\pi^2 a^4(1-e^2)}{GMa(1-e^2)}$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$

The square of the orbital period (T) is directly proportional to the cube of the semi-major axis (a).