Calculus Notes

Leonardo Tiditada Pedersen

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Part I
Differentiation

1 The Concept of the Tangent Line

The idea of a tangent line is fundamental to differential calculus. Intuitively, the tangent line to a curve at a point is the straight line that "just touches" the curve at that point and matches the curve's direction there.

1.1 Slope of the Tangent Line

Suppose $f(x)$ is a continuous function defined on an interval containing a point a. We are interested in finding the equation of the line tangent to the graph of $y = f(x)$ at the point $P(a, f(a))$. The primary challenge is determining the slope of this tangent line.

1.1.1 Linear Functions

For a linear function, $f(x) = mx + c$, the graph is a straight line. The slope m is constant for all points on the line. This slope can be computed using any two distinct points $(a, f(a))$ and $(b, f(b))$ on the line:

$$m = \frac{f(b) - f(a)}{b - a}$$

This formula gives the same value m regardless of the choice of a and b ($a \neq b$).

1.1.2 General Functions

For a general (non-linear) continuous function $f(x)$, the "slope" of the curve changes from point to point. The slope of the tangent line at a specific point $x = a$ represents the instantaneous rate of change of the function at that point.

We can approximate the slope of the tangent line at $P(a, f(a))$ by considering a nearby point $Q(x, f(x))$ on the curve, where $x \neq a$. The line passing through P and Q is called a **secant line**. The slope of this secant line is given by:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

As the point Q gets closer to P (i.e., as x approaches a), the slope of the secant line, m_{sec}, provides a better approximation to the slope of the tangent line at P. [Image showing secant line approaching tangent line]

This leads to the formal definition of the slope of the tangent line using the concept of a limit.

Definition 1.1 (Slope of the Tangent Line). *Let $y = f(x)$ be a function continuous at $x = a$. The slope m of the tangent line to the graph of f at the point $(a, f(a))$ is defined as the limit of the slopes of the secant lines, provided the limit exists:*

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

If this limit exists, the tangent line to the graph of f at $(a, f(a))$ is the line passing through the point $(a, f(a))$ with this slope m.

1.1.3 Alternative Limit Form

An equivalent way to express the slope m is by introducing a variable $h = x - a$. As $x \to a$, we have $h \to 0$. Substituting $x = a + h$ into Equation (1), we get:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2}$$

This form is often more convenient for calculations.

Example 1.1 (Finding Tangent Slope using Limit Definition)**.** Find the slope of the tangent line to the graph of $f(x) = \frac{2}{x}$ at $x = 2$. Determine the equation of this tangent line. *Solution:* We use the limit definition with $a = 2$:

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

First, find $f(2+h)$ and $f(2)$: $f(2+h) = \frac{2}{2+h}$ $f(2) = \frac{2}{2} = 1$

Now substitute into the limit formula:

$$\begin{aligned} m &= \lim_{h \to 0} \frac{\frac{2}{2+h} - 1}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \left(\frac{2 - (2+h)}{2+h} \right) \\ &= \lim_{h \to 0} \frac{1}{h} \left(\frac{-h}{2+h} \right) \\ &= \lim_{h \to 0} \frac{-1}{2+h} \\ &= \frac{-1}{2+0} = -\frac{1}{2} \end{aligned}$$

The slope of the tangent line at $x = 2$ is $m = -1/2$.

The point of tangency is $(a, f(a)) = (2, f(2)) = (2, 1)$. Using the point-slope form of a line equation, $y - y_0 = m(x - x_0)$:

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$y - 1 = -\frac{1}{2}x + 1$$

$$y = -\frac{1}{2}x + 2$$

The equation of the tangent line is $y = -\frac{1}{2}x + 2$.

Exercise 1.1. Find the slope of the tangent line to the graph $y = f(x)$ at $x = a$ and find an equation of the tangent line for $f(x)$ at $x = a$ for each of the following functions using the limit definition (Equation (2)).

1. $f(x) = x^2 + 2$, $a = 1$
2. $f(x) = \sqrt{x - 1}$, $a = 5$

1.2 Types of Tangent Lines

The limit defining the slope m (Equation (2)) might result in different scenarios.

1.2.1 Tangent Line with a Finite Slope

If the limit $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists and is a finite real number ($m \in \mathbb{R}$), the tangent line has a constant slope m. The equation is given by $y - f(a) = m(x-a)$.

A special case is a **horizontal tangent line**, which occurs when the slope $m = 0$. The equation simplifies to $y - f(a) = 0$, or $y = f(a)$. This happens at points where the instantaneous rate of change is zero, often corresponding to local maxima or minima.

1.2.2 Vertical Tangent Line

If the limit defining the slope results in infinity, i.e.,

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \infty \quad \text{or} \quad \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = -\infty$$

then the tangent line at $(a, f(a))$ is a **vertical line**. The equation of a vertical tangent line is simply $x = a$. This indicates an infinite instantaneous rate of change at that point.

1.2.3 No Tangent Line

A tangent line may not exist at a point $(a, f(a))$ if the limit defining the slope does not exist. This typically happens in two main situations:

1. **The limit is different from the left and the right:** The left-hand limit and the right-hand limit of the difference quotient are not equal:

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \neq \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

   This often occurs at "sharp corners" or "cusps" in the graph.

2. **The function is discontinuous at $x = a$:** If $f(x)$ is not continuous at $x = a$, the limit defining the slope cannot exist in the standard sense. Tangency requires the line and curve to approach the same point smoothly.

Example 1.2 (Vertical Tangent and No Tangent)**.** Investigate the slope of the tangent line for the following functions at $x = 0$ using the limit definition.

(a) $f(x) = x^{1/3}$

(b) $f(x) = |x|$

Solution: (a) For $f(x) = x^{1/3}$, we calculate the slope at $a = 0$:

$$m = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{1/3}-0^{1/3}}{h} = \lim_{h\to 0} \frac{h^{1/3}}{h}$$

$$m = \lim_{h\to 0} h^{1/3-1} = \lim_{h\to 0} h^{-2/3} = \lim_{h\to 0} \frac{1}{h^{2/3}}$$

As $h \to 0$, $h^{2/3}$ approaches 0 through positive values (since $h^2 \geq 0$). Therefore, the limit is:

$$m = +\infty$$

Since the limit is infinite, $f(x) = x^{1/3}$ has a **vertical tangent line** at $x = 0$. The equation of this tangent line is $x = 0$. (b) For $f(x) = |x|$, we calculate the slope at $a = 0$:

$$m = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

To evaluate this limit, we must consider the one-sided limits, because the definition of $|h|$ changes at $h = 0$:

$$|h| = \begin{cases} -h & \text{if } h < 0 \\ h & \text{if } h \geq 0 \end{cases}$$

Left-hand limit ($h \to 0^-$):

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Right-hand limit ($h \to 0^+$):

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} (1) = 1$$

Since the left-hand limit (-1) is not equal to the right-hand limit $(+1)$, the overall limit $\lim_{h \to 0} \frac{|h|}{h}$ does not exist. Therefore, $f(x) = |x|$ has **no tangent line** at $x = 0$. This corresponds to the sharp corner at the origin.

2 Rate of Change

The concept of the slope of a tangent line is closely related to the idea of the rate of change of a function.

2.1 Average Rate of Change

The average rate of change of a function $y = f(x)$ over an interval $[a, b]$ is the ratio of the change in y to the change in x. It's geometrically represented by the slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$.

$$\text{Average Rate of Change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Example 2.1 (Average Velocity)**.** Consider the motion of an object. If $s(t)$ represents the position of the object at time t, the average velocity over the time interval $[t_1, t_2]$ is:

$$v_{avg} = \frac{\text{change in position}}{\text{change in time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

For instance, a runner finishing a 10 km race in 1 hour 15 minutes (1.25 hours) has an average velocity:

$$v_{avg} = \frac{10 \text{ km} - 0 \text{ km}}{1.25 \text{ h} - 0 \text{ h}} = \frac{10}{1.25} \text{ km/h} = 8 \text{ km/h}$$

If the runner is 5 km from the start at $t = 0.5$ h and 5.7 km from the start at $t = 0.6$ h, the average velocity during the interval $[0.5, 0.6]$ is:

$$v_{avg} = \frac{5.7 \text{ km} - 5 \text{ km}}{0.6 \text{ h} - 0.5 \text{ h}} = \frac{0.7}{0.1} \text{ km/h} = 7 \text{ km/h}$$

2.2 Instantaneous Rate of Change

The instantaneous rate of change of a function $f(x)$ at a specific point $x = a$ describes how the function is changing at that exact moment. It is defined as the limit of the average rates of change over progressively smaller intervals around a. This limit is precisely the slope of the tangent line at $x = a$.

Definition 2.1 (Instantaneous Rate of Change)**.** *The instantaneous rate of change of $y = f(x)$ with respect to x at $x = a$ is the limit:*

$$\textit{Instantaneous Rate of Change} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

Definition 2.2 (Instantaneous Velocity)**.** *Let $s = s(t)$ be the position function of an object moving in a straight line. The instantaneous velocity at time $t = t_0$ is the instantaneous rate of change of position with respect to time:*

$$v(t_0) = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

This is equivalent to the derivative of the position function evaluated at t_0.

Example 2.2 (Calculating Instantaneous Velocity)**.** An object moves in a straight line described by the position function $s(t) = -t^2 + 6$. Find the instantaneous velocity of this object at $t = 2$. *Solution:* We need to compute the limit:

$$v(2) = \lim_{\Delta t \to 0} \frac{s(2 + \Delta t) - s(2)}{\Delta t}$$

First, find $s(2 + \Delta t)$ and $s(2)$: $s(2 + \Delta t) = -(2 + \Delta t)^2 + 6 = -(4 + 4\Delta t + (\Delta t)^2) + 6 = -4 - 4\Delta t - (\Delta t)^2 + 6 = 2 - 4\Delta t - (\Delta t)^2$ $s(2) = -(2)^2 + 6 = -4 + 6 = 2$

Now substitute into the limit formula:

$$\begin{aligned} v(2) &= \lim_{\Delta t \to 0} \frac{(2 - 4\Delta t - (\Delta t)^2) - 2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{-4\Delta t - (\Delta t)^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\Delta t(-4 - \Delta t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} (-4 - \Delta t) \\ &= -4 - 0 = -4 \end{aligned}$$

The instantaneous velocity at $t = 2$ is -4 (units of distance per unit of time).

3 The Derivative

3.1 Definition of the Derivative

The limit we used to define the slope of a tangent line and the instantaneous rate of change is of fundamental importance and is given a special name: the derivative.

Definition 3.1 (Derivative)**.** *The derivative of a function $f(x)$ with respect to x, denoted by $f'(x)$, is the function defined by:*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{3}$$

*The domain of $f'(x)$ consists of all x for which this limit exists. If $f'(x)$ exists at a particular $x = a$, we say that f is **differentiable** at a. The value $f'(a)$ represents the slope of the tangent line to the graph of f at the point $(a, f(a))$.*

Remark 3.1. The process of finding the derivative is called **differentiation**. The derivative $f'(x)$ gives the instantaneous rate of change of $f(x)$ with respect to x.

Example 3.1 (Derivative using Limit Definition)**.** Find the derivative of $f(x) = x^3 + 2x^2$ using the limit definition. *Solution:* We apply Definition 3.1:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

First, find $f(x+h)$:

$$\begin{aligned} f(x+h) &= (x+h)^3 + 2(x+h)^2 \\ &= (x^3 + 3x^2h + 3xh^2 + h^3) + 2(x^2 + 2xh + h^2) \\ &= x^3 + 3x^2h + 3xh^2 + h^3 + 2x^2 + 4xh + 2h^2 \end{aligned}$$

Now, compute $f(x+h) - f(x)$:

$$\begin{aligned} f(x+h) - f(x) &= (x^3 + 3x^2h + 3xh^2 + h^3 + 2x^2 + 4xh + 2h^2) - (x^3 + 2x^2) \\ &= 3x^2h + 3xh^2 + h^3 + 4xh + 2h^2 \end{aligned}$$

Finally, compute the limit:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 + 4xh + 2h^2}{h} \\ &= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 + 4x + 2h)}{h} \\ &= \lim_{h \to 0} (3x^2 + 3xh + h^2 + 4x + 2h) \\ &= 3x^2 + 3x(0) + (0)^2 + 4x + 2(0) \\ &= 3x^2 + 4x \end{aligned}$$

Thus, the derivative of $f(x) = x^3 + 2x^2$ is $f'(x) = 3x^2 + 4x$.

Example 3.2 (Derivative of $\sqrt{x}$)**.** Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$ using the limit definition. Find $f'(1)$. *Hint:* Use rationalization by multiplying by the conjugate. *Solution:*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

This limit is of the indeterminate form $\frac{0}{0}$. We multiply the numerator and denominator by the conjugate of the numerator, $\sqrt{x+h} + \sqrt{x}$:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \quad \text{(since } h \neq 0) \\ &= \frac{1}{\sqrt{x+0} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

The derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$ for $x > 0$.

To find $f'(1)$, substitute $x = 1$:

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

3.2 One-Sided Derivatives and Differentiability

Similar to limits, we can define one-sided derivatives.

Definition 3.2 (One-Sided Derivatives). *Let $f(x)$ be a function defined on an interval containing x.*

- *The **right-hand derivative** of f at x is:*

$$f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \tag{4}$$

  provided this limit exists.

- *The **left-hand derivative** of f at x is:*

$$f'_-(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \tag{5}$$

  provided this limit exists.

The existence of the derivative $f'(x)$ at a point requires the existence and equality of both one-sided derivatives.

Theorem 3.1 (Condition for Differentiability). *A function $f(x)$ is differentiable at $x = a$ if and only if both the left-hand derivative $f'_-(a)$ and the right-hand derivative $f'_+(a)$ exist and are equal. In this case, $f'(a) = f'_-(a) = f'_+(a)$.*

Definition 3.3 (Differentiability on Intervals).

- *A function $f(x)$ is **differentiable on an open interval** (a, b) if it is differentiable at every point $x \in (a, b)$.*
- *A function $f(x)$ is **differentiable everywhere** if it is differentiable on $(-\infty, \infty)$.*
- *A function $f(x)$ is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the open interval (a, b), and the right-hand derivative $f'_+(a)$ exists at a, and the left-hand derivative $f'_-(b)$ exists at b.*

Exercise 3.1 (Non-Differentiability of $|x|$ at $x = 0$). Show explicitly using one-sided derivatives that $f(x) = |x|$ is not differentiable at $x = 0$. *Solution Steps:*

- For $x \neq 0$, find $f'(x)$.
  - For $x > 0$, $f(x) = x$, so $f'(x) = \frac{d}{dx}(x) = 1$.
  - For $x < 0$, $f(x) = -x$, so $f'(x) = \frac{d}{dx}(-x) = -1$.
- Calculate the one-sided derivatives at $x = 0$:
  - $f'_+(0) = \lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$.
  - $f'_-(0) = \lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$.
- Since $f'_-(0) = -1 \neq 1 = f'_+(0)$, the function $f(x) = |x|$ is not differentiable at $x = 0$.

3.3 Differentiability and Continuity

There is a crucial relationship between differentiability and continuity.

Theorem 3.2 (Differentiability Implies Continuity)**.** *If a function f is differentiable at $x = a$, then f is continuous at $x = a$.*

Proof. To show that f is continuous at a, we need to show that $\lim_{x\to a} f(x) = f(a)$. This is equivalent to showing that $\lim_{x\to a}(f(x) - f(a)) = 0$. Assume f is differentiable at a. This means that the limit

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. Consider the limit of $f(x) - f(a)$ as $x \to a$:

$$\begin{aligned}
\lim_{x\to a}(f(x) - f(a)) &= \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a} \cdot (x - a)\right) \\
&= \left(\lim_{x\to a} \frac{f(x) - f(a)}{x - a}\right) \cdot \left(\lim_{x\to a}(x - a)\right) \quad \text{(Limit product rule)} \\
&= f'(a) \cdot (a - a) \\
&= f'(a) \cdot 0 = 0
\end{aligned}$$

Since $\lim_{x\to a}(f(x) - f(a)) = 0$, it follows that $\lim_{x\to a} f(x) = f(a)$. Therefore, f is continuous at a. □

Remark 3.2 (Continuity does NOT imply Differentiability)**.** The converse of the theorem is false. A function can be continuous at a point but not differentiable there.

Example 3.3 (Continuous but not Differentiable)**.** The functions $f(x) = |x|$ and $g(x) = x^{1/3}$ are both continuous everywhere, including at $x = 0$.

- $f(x) = |x|$: We showed earlier that $f'(0)$ does not exist (due to the corner), so f is not differentiable at $x = 0$.
- $g(x) = x^{1/3}$: We showed earlier that the limit defining the slope at $x = 0$ is ∞, meaning $g'(0)$ does not exist as a finite number (vertical tangent), so g is not differentiable at $x = 0$.

These examples illustrate that continuity at a point is a necessary condition for differentiability, but it is not sufficient. [Image comparing graphs of —x— and x$^($1/3)*attheorigin*]

3.4 Situations where Differentiability Fails

A function f is not differentiable at $x = a$ if any of the following occur:

1. **Discontinuity:** The function is not continuous at $x = a$. (Since differentiability implies continuity).

   [Image of a jump discontinuity]

2. **Corner or Cusp:** The graph of f has a sharp corner or cusp at $(a, f(a))$. (Left and right derivatives differ). [Image comparing a corner and a cusp]

3. **Vertical Tangent Line:** The graph of f has a vertical tangent line at $(a, f(a))$. (The limit defining the derivative is ∞ or $-\infty$).

3.5 Notations for the Derivative

Several notations are commonly used for the derivative of $y = f(x)$ with respect to x:

- **Lagrange's notation:** $f'(x)$, y'
- **Leibniz's notation:** $\frac{dy}{dx}$, $\frac{d}{dx}f(x)$
- **Operator notation:** $D_x y$, $Df(x)$, $D_x f(x)$

When evaluating the derivative at a specific point $x = a$, the notations become:

- $f'(a)$
- $\frac{dy}{dx}\Big|_{x=a}$
- $y'(a)$
- $D_x y|_{x=a}$

Remark 3.3 (Tangent Line Equation using Derivative Notation)**.** The slope m of the tangent line to $y = f(x)$ at $x = a$ is given by $m = f'(a)$. The equation of the tangent line is:

$$y - f(a) = f'(a)(x - a) \quad \text{or} \quad y = f(a) + f'(a)(x - a)$$

The function $L(x) = f(a) + f'(a)(x - a)$ is called the linearization of f at a.

A **normal line** at a point $(a, f(a))$ is the line perpendicular to the tangent line at that point. If the tangent slope $m = f'(a) \neq 0$, the normal slope is $m_n = -1/m = -1/f'(a)$. The equation of the normal line is $y - f(a) = -\frac{1}{f'(a)}(x - a)$. If the tangent is horizontal ($f'(a) = 0$), the normal line is vertical ($x = a$). If the tangent is vertical, the normal line is horizontal ($y = f(a)$).

Exercise 3.2. 1. Let $f(x) = |x|$. Find $\lim_{x\to 0} f(x)$ and discuss the existence of $f'(0)$.

2. Find the derivative $f'(x)$ of the following functions using the definition of the derivative (Equation (3)).

   (a) $f(x) = 2x^2 + x + 1$ (Verify Example)

   (b) $f(x) = \sqrt{5x - 8}$

3. Show that $f(x) = x^2$ has a horizontal tangent line at $x = 0$.

4 Basic Differentiation Rules

Calculating derivatives directly from the limit definition can be tedious. Fortunately, there are rules that simplify this process for common types of functions.

4.1 The Power Rule and Constant Rule

Theorem 4.1 (Constant Rule)**.** *If $f(x) = c$, where c is a constant, then $f'(x) = 0$. That is,*

$$\frac{d}{dx}(c) = 0$$

Proof. Using the limit definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

□

Geometrically, the graph of $y = c$ is a horizontal line, which has a slope of 0 everywhere.

Theorem 4.2 (Power Rule)**.** *For any real number n, the function $f(x) = x^n$ is differentiable and*

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

This rule applies wherever x^{n-1} is defined. For example, if $n = 1/2$, the rule holds for $x > 0$. If $n = -1$, it holds for $x \neq 0$.

Partial Proof for Positive Integer n. The proof for an arbitrary real number n requires more advanced techniques (like logarithmic differentiation or the generalized binomial theorem). We can prove it for positive integers n using the binomial theorem: $(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n) - x^n}{h} \\ &= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n}{h} \\ &= \lim_{h\to 0} \left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1} \right) \end{aligned}$$

As $h \to 0$, all terms except the first one vanish:

$$f'(x) = nx^{n-1} + 0 + \cdots + 0 = nx^{n-1}$$

□

Example 4.1 (Applying Power and Constant Rules)**.** Differentiate the following:

1. $f(x) = x^{13} \implies f'(x) = 13x^{13-1} = 13x^{12}$
2. $f(x) = x = x^1 \implies f'(x) = 1x^{1-1} = 1x^0 = 1$ (for $x \neq 0$, but the limit definition shows $f'(x) = 1$ for all x)
3. $f(x) = x^{\sqrt{3}} \implies f'(x) = \sqrt{3}x^{\sqrt{3}-1}$ (for $x > 0$)
4. $f(x) = 1 = x^0 \implies f'(x) = 0x^{0-1} = 0$. This confirms the Constant Rule.
5. $f(x) = \frac{1}{x^2} = x^{-2} \implies f'(x) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$ (for $x \neq 0$)
6. $f(x) = \sqrt{x} = x^{1/2} \implies f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ (for $x > 0$)

Example 4.2 (Case Study: $f(x) = x^{2/3}$)**.** Differentiate $y = x^{2/3}$. Find the tangent line equation at $x = 0$. *Solution:* Using the Power Rule (formally, for $x \neq 0$):

$$\frac{dy}{dx} = f'(x) = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

This derivative is defined for all $x \neq 0$. At $x = 0$, the formula is undefined. Let's check the limit definition at $a = 0$:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{2/3} - 0}{h} = \lim_{h\to 0} \frac{h^{2/3}}{h}$$

$$= \lim_{h\to 0} h^{2/3-1} = \lim_{h\to 0} h^{-1/3} = \lim_{h\to 0} \frac{1}{h^{1/3}}$$

This limit does not exist as a finite number. Let's check one-sided limits:

$$\lim_{h\to 0^+} \frac{1}{h^{1/3}} = +\infty \quad (\text{since } h > 0 \implies h^{1/3} > 0)$$

$$\lim_{h\to 0^-} \frac{1}{h^{1/3}} = -\infty \quad (\text{since } h < 0 \implies h^{1/3} < 0)$$

Since the limit defining the derivative is infinite, $f(x) = x^{2/3}$ is not differentiable at $x = 0$. However, it has a **vertical tangent line** at $x = 0$. The equation of this tangent is $x = 0$. The graph of $y = x^{2/3}$ has a cusp at the origin.

4.2 Higher-Order Derivatives

Since the derivative $f'(x)$ is itself a function, we can differentiate it again to obtain the **second derivative**, differentiate the result to get the **third derivative**, and so on. These are collectively known as **higher-order derivatives**.

Definition 4.1 (Higher-Order Derivatives)**.** *Let $y = f(x)$ be a differentiable function.*

- *The **second derivative**, denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$, is the derivative of the first derivative:*

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

  Other notations include y'', $D_x^2 y$, $\frac{d^2}{dx^2}f(x)$, $D^2 f(x)$.

- *The **third derivative**, denoted by $f'''(x)$ or $\frac{d^3y}{dx^3}$, is the derivative of the second derivative:*

$$f'''(x) = \frac{d}{dx}(f''(x)) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

  Other notations include y''', $D_x^3 y$, $\frac{d^3}{dx^3}f(x)$, $D^3 f(x)$.

- *The **n-th derivative**, denoted by $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$, is the derivative obtained by differentiating $f(x)$ n times successively. It is defined recursively as:*

$$f^{(n)}(x) = \frac{d}{dx}(f^{(n-1)}(x)) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$$

  where $f^{(0)}(x) = f(x)$. For $n \geq 4$, the notation $f^{(n)}(x)$ is preferred over $f''''...(x)$. Other notations include $y^{(n)}$, $D_x^n y$, $\frac{d^n}{dx^n}f(x)$, $D^n f(x)$.

Example 4.3 (Calculating Higher-Order Derivatives)**.** Let $f(x) = x^{10}$.

(a) Find the second ($f''(x)$) and the fourth ($f^{(4)}(x)$) derivatives.

(b) Find the eleventh derivative ($f^{(11)}(x)$).

Solution: (a) We differentiate successively using the Power Rule: $f'(x) = 10x^9$ $f''(x) = \frac{d}{dx}(10x^9) = 10 \cdot 9x^8 = 90x^8$ $f'''(x) = \frac{d}{dx}(90x^8) = 90 \cdot 8x^7 = 720x^7$ $f^{(4)}(x) = \frac{d}{dx}(720x^7) = 720 \cdot 7x^6 = 5040x^6$

(b) Observe the pattern. Each differentiation reduces the power by one and multiplies by the current power. $f^{(5)}(x) = 5040 \cdot 6x^5$ ... $f^{(10)}(x) = (10 \cdot 9 \cdot 8 \cdot \cdots \cdot 1)x^0 = 10! \cdot 1$ The 10th derivative is the constant 10!. Therefore, the eleventh derivative is the derivative of a constant:

$$f^{(11)}(x) = \frac{d}{dx}(f^{(10)}(x)) = \frac{d}{dx}(10!) = 0$$

In general, for $f(x) = x^n$ where n is a positive integer, $f^{(k)}(x) = 0$ for all $k > n$.

4.3 Constant Multiple, Sum, and Difference Rules

These rules follow directly from the properties of limits.

Theorem 4.3 (Constant Multiple Rule)**.** *If c is a constant and f is a differentiable function at x, then the function cf is also differentiable at x, and*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] = cf'(x)$$

Proof. Let $F(x) = cf(x)$.

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\
&= \lim_{h\to 0} c\left(\frac{f(x+h)-f(x)}{h}\right) \\
&= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{(Limit property)} \\
&= cf'(x)
\end{aligned}$$

□

Theorem 4.4 (Sum and Difference Rules)**.** *If f and g are differentiable functions at x, then their sum $f+g$ and difference $f-g$ are also differentiable at x, and*

1. $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$
2. $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) = f'(x) - g'(x)$

These rules extend to the sum or difference of any finite number of differentiable functions.

Proof of the Sum Rule. Let $S(x) = f(x) + g(x)$.

$$\begin{aligned}
S'(x) &= \lim_{h\to 0}\frac{S(x+h)-S(x)}{h} \\
&= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right) \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \quad \text{(Limit sum rule)} \\
&= f'(x) + g'(x)
\end{aligned}$$

The proof for the difference rule is analogous. □

Example 4.4 (Applying Sum/Difference/Constant Multiple Rules)**.** Differentiate the following functions:

(a) $y = -3(\sqrt[3]{x^7} + e^{\pi} + \frac{4}{x\sqrt{x}}) + 9$

(b) $y = \frac{x^5 - x^{3/2} + 1}{\sqrt{x}}$

(c) $y = \frac{x^2+3x+2}{x+1}$, for $x \neq -1$

Solution: (a) First, rewrite the function using powers of x: $\sqrt[3]{x^7} = x^{7/3}$ e^{π} is a constant. $\frac{4}{x\sqrt{x}} = \frac{4}{x \cdot x^{1/2}} = \frac{4}{x^{3/2}} = 4x^{-3/2}$ So, $y = -3(x^{7/3} + e^{\pi} + 4x^{-3/2}) + 9 = -3x^{7/3} - 3e^{\pi} - 12x^{-3/2} + 9$. Now differentiate term by term:

$$\begin{aligned} y' &= \frac{d}{dx}(-3x^{7/3}) + \frac{d}{dx}(-3e^{\pi}) + \frac{d}{dx}(-12x^{-3/2}) + \frac{d}{dx}(9) \\ &= -3 \cdot \frac{7}{3}x^{7/3-1} + 0 - 12 \cdot (-\frac{3}{2})x^{-3/2-1} + 0 \\ &= -7x^{4/3} + 18x^{-5/2} \\ &= -7x^{4/3} + \frac{18}{x^{5/2}} \end{aligned}$$

(b) First, simplify the expression by dividing each term in the numerator by $\sqrt{x} = x^{1/2}$:

$$y = \frac{x^5}{x^{1/2}} - \frac{x^{3/2}}{x^{1/2}} + \frac{1}{x^{1/2}} = x^{5-1/2} - x^{3/2-1/2} + x^{-1/2}$$

$$y = x^{9/2} - x^1 + x^{-1/2}$$

Now differentiate:

$$\begin{aligned} y' &= \frac{d}{dx}(x^{9/2}) - \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1/2}) \\ &= \frac{9}{2}x^{9/2-1} - 1x^{1-1} + (-\frac{1}{2})x^{-1/2-1} \\ &= \frac{9}{2}x^{7/2} - 1 - \frac{1}{2}x^{-3/2} \\ &= \frac{9}{2}x^{7/2} - 1 - \frac{1}{2x^{3/2}} \end{aligned}$$

(c) First, simplify the expression for $x \neq -1$: Factor the numerator: $x^2 + 3x + 2 = (x + 1)(x + 2)$.

$$y = \frac{(x+1)(x+2)}{x+1} = x + 2 \quad (\text{for } x \neq -1)$$

Now differentiate:

$$y' = \frac{d}{dx}(x+2) = \frac{d}{dx}(x) + \frac{d}{dx}(2) = 1 + 0 = 1$$

The derivative is $y' = 1$ for all $x \neq -1$.

5 Product and Quotient Rules

How do we differentiate the product or quotient of two functions? It's not as simple as multiplying or dividing the individual derivatives.

Theorem 5.1 (Product Rule)**.** *If f and g are differentiable functions at x, then their product fg is also differentiable at x, and*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Or, in prime notation:

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x)$$

In words: "The derivative of a product of two functions is the first function times the derivative of the second, plus the second function times the derivative of the first."

Proof. Let $P(x) = f(x)g(x)$.

$$\begin{aligned} P'(x) &= \lim_{h\to 0} \frac{P(x+h) - P(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \end{aligned}$$

We use a common trick: subtract and add $f(x+h)g(x)$ in the numerator:

$$\begin{aligned} P'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(x+h)[g(x+h) - g(x)]}{h} + \frac{g(x)[f(x+h) - f(x)]}{h} \right) \\ &= \lim_{h\to 0} \left(f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + g(x) \cdot \frac{f(x+h) - f(x)}{h} \right) \end{aligned}$$

Since f is differentiable at x, it is also continuous at x, so $\lim_{h\to 0} f(x+h) = f(x)$. Also, $g(x)$ is treated as a constant with respect to h. Using limit properties:

$$\begin{aligned} P'(x) &= \left(\lim_{h\to 0} f(x+h)\right) \left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) + \left(\lim_{h\to 0} g(x)\right) \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) \\ &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \end{aligned}$$

□

Warning. *The derivative of a product is NOT the product of the derivatives:* $(fg)' \neq f'g'$.

Example 5.1 (Applying the Product Rule). (a) Differentiate $y = (2 + \sqrt{x} + x)(x^3 - 3x + 1)$.

(b) Find the tangent line to the graph from part (a) at $x = 1$.

Solution: (a) Let $f(x) = 2 + x^{1/2} + x$ and $g(x) = x^3 - 3x + 1$. $f'(x) = 0 + \frac{1}{2}x^{-1/2} + 1 = \frac{1}{2\sqrt{x}} + 1$ $g'(x) = 3x^2 - 3 + 0 = 3x^2 - 3$ Using the product rule, $y' = f(x)g'(x) + g(x)f'(x)$:

$$y' = (2 + \sqrt{x} + x)(3x^2 - 3) + (x^3 - 3x + 1)\left(\frac{1}{2\sqrt{x}} + 1\right)$$

(b) To find the tangent line at $x = 1$, we need the point $(1, y(1))$ and the slope $y'(1)$. Point: $y(1) = (2 + \sqrt{1} + 1)(1^3 - 3(1) + 1) = (2 + 1 + 1)(1 - 3 + 1) = (4)(-1) = -4$. The point is $(1, -4)$. Slope: Substitute $x = 1$ into the derivative y' found in part (a).

$$\begin{aligned} y'(1) &= (2 + \sqrt{1} + 1)(3(1)^2 - 3) + (1^3 - 3(1) + 1)\left(\frac{1}{2\sqrt{1}} + 1\right) \\ &= (4)(3 - 3) + (1 - 3 + 1)\left(\frac{1}{2} + 1\right) \\ &= (4)(0) + (-1)\left(\frac{3}{2}\right) \\ &= -\frac{3}{2} \end{aligned}$$

The slope is $m = -3/2$. Equation of the tangent line: $y - y_0 = m(x - x_0)$

$$\begin{aligned} y - (-4) &= -\frac{3}{2}(x - 1) \\ y + 4 &= -\frac{3}{2}x + \frac{3}{2} \\ y &= -\frac{3}{2}x + \frac{3}{2} - 4 \\ y &= -\frac{3}{2}x - \frac{5}{2} \end{aligned}$$

Example 5.2 (Product Rule for Three Functions)**.** Differentiate $y = (2+x)(x^3+1)(x^2-7x)$. *Solution:* We can group the first two factors and apply the product rule twice. Let $F(x) = (2+x)(x^3+1)$ and $G(x) = (x^2-7x)$. Then $y = F(x)G(x)$.

$$y' = F(x)G'(x) + G(x)F'(x)$$

First find $F'(x)$ using the product rule: $F'(x) = (2+x)\frac{d}{dx}(x^3+1) + (x^3+1)\frac{d}{dx}(2+x)$ $F'(x) = (2+x)(3x^2) + (x^3+1)(1)$ Now find $G'(x)$: $G'(x) = \frac{d}{dx}(x^2-7x) = 2x-7$. Substitute into the formula for y':

$$y' = [(2+x)(x^3+1)](2x-7) + (x^2-7x)[(2+x)(3x^2) + (x^3+1)]$$

This can be generalized: $(fgh)' = f'gh + fg'h + fgh'$.

Theorem 5.2 (Quotient Rule)**.** *If f and g are differentiable functions at x and $g(x) \neq 0$, then the quotient f/g is differentiable at x, and*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Or, in prime notation:

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

In words: "Low dHigh minus High dLow, square the bottom and away we go."

Proof. Let $Q(x) = f(x)/g(x)$.

$$\begin{aligned}
Q'(x) &= \lim_{h\to 0} \frac{Q(x+h) - Q(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}
\end{aligned}$$

Subtract and add $f(x)g(x)$ in the numerator:

$$\begin{aligned}
Q'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)}
\end{aligned}$$

Using limit properties and continuity of g (since it's differentiable):

$$\begin{aligned}
Q'(x) &= \frac{\lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}{\lim_{h\to 0} g(x+h) \lim_{h\to 0} g(x)} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}$$

□

Warning. *The derivative of a quotient is NOT the quotient of the derivatives: $(f/g)' \neq f'/g'$.*

Example 5.3 (Applying the Quotient Rule)**.** Differentiate $y = \frac{2+x\sqrt{x}}{x^3+1}$. *Solution:* Let $f(x) = 2 + x\sqrt{x} = 2 + x^{3/2}$ and $g(x) = x^3 + 1$. $f'(x) = 0 + \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$ $g'(x) = 3x^2 + 0 = 3x^2$ Using the quotient rule, $y' = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$:

$$y' = \frac{(x^3+1)(\frac{3}{2}x^{1/2}) - (2+x^{3/2})(3x^2)}{[x^3+1]^2}$$

$$y' = \frac{\frac{3}{2}x^{7/2} + \frac{3}{2}x^{1/2} - 6x^2 - 3x^{7/2}}{(x^3+1)^2}$$

$$y' = \frac{-\frac{3}{2}x^{7/2} - 6x^2 + \frac{3}{2}x^{1/2}}{(x^3+1)^2}$$

Example 5.4 (Quotient Rule with Product in Denominator)**.** Differentiate $y = \frac{2+x}{(x^3+1)(x^2-7x)}$. *Solution:* Let $f(x) = 2 + x$ and $g(x) = (x^3 + 1)(x^2 - 7x)$. $f'(x) = 1$. We need $g'(x)$. From Example 1.1(b), we found the derivative of this product: $g'(x) = (x^3+1)(2x-7)+(x^2-7x)(3x^2)$. Now apply the quotient rule:

$$y' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{[(x^3+1)(x^2-7x)](1) - (2+x)[(x^3+1)(2x-7) + (x^2-7x)(3x^2)]}{[(x^3+1)(x^2-7x)]^2}$$

Example 5.5 (Using Given Derivative Values)**.** Let $h(x) = \frac{x^2+1}{g(x)}$, where $g(x)$ is a non-zero differentiable function with $g(1) = 2$ and $g'(1) = 4$. Find an equation of the tangent line to the graph of $y = h(x)$ at $x = 1$. *Solution:* We need the point $(1, h(1))$ and the slope $h'(1)$. Point: $h(1) = \frac{1^2+1}{g(1)} = \frac{2}{2} = 1$. The point is $(1, 1)$. Slope: Find $h'(x)$ using the quotient rule. Let $f(x) = x^2 + 1$, $f'(x) = 2x$.

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} = \frac{g(x)(2x) - (x^2+1)g'(x)}{[g(x)]^2}$$

Now evaluate at $x = 1$:

$$h'(1) = \frac{g(1)(2 \cdot 1) - (1^2+1)g'(1)}{[g(1)]^2}$$

$$h'(1) = \frac{(2)(2) - (2)(4)}{[2]^2} = \frac{4-8}{4} = \frac{-4}{4} = -1$$

The slope is $m = -1$. Equation of the tangent line: $y - y_0 = m(x - x_0)$

$$y - 1 = -1(x - 1)$$

$$y - 1 = -x + 1$$

$$y = -x + 2$$

6 Derivatives of Trigonometric Functions

We now extend our differentiation rules to trigonometric functions.

6.1 Derivatives of Sine and Cosine

Theorem 6.1 (Derivatives of Sine and Cosine)**.** *For any real number x,*

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$

Proof of $\frac{d}{dx}(\sin x) = \cos x$. We use the limit definition of the derivative (Equation (3)) and the angle addition formula for sine: $\sin(A+B) = \sin A \cos B + \cos A \sin B$. Let $f(x) = \sin x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right) \end{aligned}$$

Using the limit sum rule and the fact that $\sin x$ and $\cos x$ are constants with respect to h:

$$f'(x) = (\sin x)\left(\lim_{h\to 0} \frac{\cos h - 1}{h}\right) + (\cos x)\left(\lim_{h\to 0} \frac{\sin h}{h}\right)$$

To complete the proof, we need the values of two fundamental trigonometric limits:

1. $\lim_{h\to 0} \frac{\sin h}{h} = 1$
2. $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$

These limits can be established using geometric arguments (based on the unit circle) or L'Hôpital's Rule (though using L'Hôpital's Rule here would be circular reasoning if we haven't yet established the derivative of sine). Assuming these limits hold:

$$f'(x) = (\sin x)(0) + (\cos x)(1) = \cos x$$

Thus, $\frac{d}{dx}(\sin x) = \cos x$.

The proof for $\frac{d}{dx}(\cos x) = -\sin x$ is analogous, using the angle addition formula $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and the same two fundamental limits. □

Remark 6.1 (Fundamental Trigonometric Limits Visualization)**.** The limits $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ are crucial. Their graphs near $h = 0$ illustrate this behavior.

6.2 Derivatives of Other Trigonometric Functions

Using the derivatives of sine and cosine, along with the quotient rule, we can find the derivatives of tangent, cotangent, secant, and cosecant.

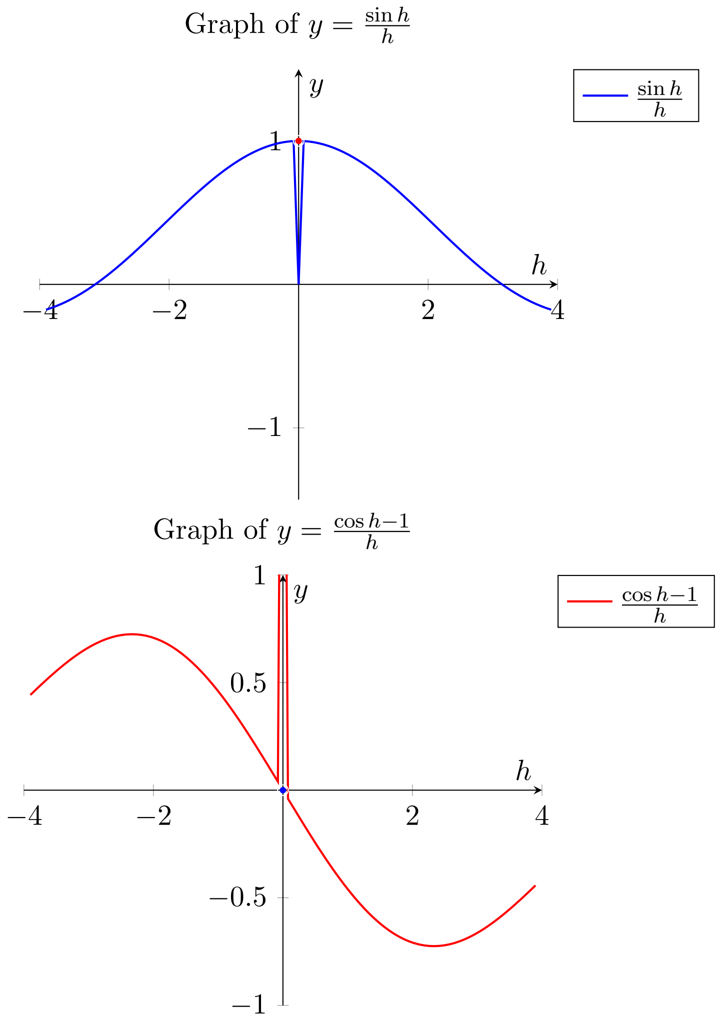


Figure 1: Graphs illustrating the fundamental trigonometric limits near $h = 0$.

6.2.1 Tangent and Cotangent

Recall $\tan x = \frac{\sin x}{\cos x}$. Applying the quotient rule:

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{(\cos x)\frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(\cos x)}{(\cos x)^2} \\
&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \quad (\text{using } \sin^2 x + \cos^2 x = 1) \\
&= \sec^2 x \quad (\text{since } \sec x = 1/\cos x)
\end{aligned}$$

Thus, $\frac{d}{dx}(\tan x) = \sec^2 x$.

Similarly, recall $\cot x = \frac{\cos x}{\sin x}$. Applying the quotient rule:

$$\begin{aligned}
\frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\
&= \frac{(\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(\sin x)}{(\sin x)^2} \\
&= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} \\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\
&= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\
&= \frac{-1}{\sin^2 x} \quad (\text{using } \sin^2 x + \cos^2 x = 1) \\
&= -\csc^2 x \quad (\text{since } \csc x = 1/\sin x)
\end{aligned}$$

Thus, $\frac{d}{dx}(\cot x) = -\csc^2 x$.

6.2.2 Secant and Cosecant

Recall $\sec x = \frac{1}{\cos x}$. Applying the quotient rule (or the reciprocal rule, a special case):

$$\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) \\
&= \frac{(\cos x)\frac{d}{dx}(1) - (1)\frac{d}{dx}(\cos x)}{(\cos x)^2} \\
&= \frac{(\cos x)(0) - (1)(-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \\
&= \sec x \tan x
\end{aligned}$$

Thus, $\frac{d}{dx}(\sec x) = \sec x \tan x$.

Similarly, recall $\csc x = \frac{1}{\sin x}$. Applying the quotient rule:

$$\begin{aligned}
\frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) \\
&= \frac{(\sin x)\frac{d}{dx}(1) - (1)\frac{d}{dx}(\sin x)}{(\sin x)^2} \\
&= \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x} \\
&= \frac{-\cos x}{\sin^2 x} \\
&= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \\
&= -\csc x \cot x
\end{aligned}$$

Thus, $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

6.3 Summary of Trigonometric Derivatives

Theorem 6.2 (Derivatives of Trigonometric Functions)**.** *The derivatives of the six basic trigonometric functions are:*

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Example 6.1 (Differentiating Combinations)**.** Find the first derivatives of the following functions.

(a) $f(x) = 3\sec x - 10\cot x$

(b) $f(x) = \frac{\sin x + x\tan x}{3 - 2\cos x}$

Solution: (a) Use the constant multiple and difference rules, along with Theorem 6.2:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3\sec x) - \frac{d}{dx}(10\cot x) \\ &= 3\frac{d}{dx}(\sec x) - 10\frac{d}{dx}(\cot x) \\ &= 3(\sec x \tan x) - 10(-\csc^2 x) \\ &= 3\sec x \tan x + 10\csc^2 x \end{aligned}$$

(b) This requires the quotient rule. Let $N(x) = \sin x + x\tan x$ (numerator) and $D(x) = 3 - 2\cos x$ (denominator). First, find $N'(x)$ using the sum rule and the product rule for $x\tan x$:

$$\begin{aligned} N'(x) &= \frac{d}{dx}(\sin x) + \frac{d}{dx}(x\tan x) \\ &= \cos x + \left(x\frac{d}{dx}(\tan x) + \tan x\frac{d}{dx}(x)\right) \\ &= \cos x + (x\sec^2 x + \tan x \cdot 1) \\ &= \cos x + x\sec^2 x + \tan x \end{aligned}$$

Next, find $D'(x)$:

$$D'(x) = \frac{d}{dx}(3 - 2\cos x) = 0 - 2(-\sin x) = 2\sin x$$

Now apply the quotient rule: $f'(x) = \frac{D(x)N'(x) - N(x)D'(x)}{[D(x)]^2}$

$$f'(x) = \frac{(3 - 2\cos x)(\cos x + x\sec^2 x + \tan x) - (\sin x + x\tan x)(2\sin x)}{(3 - 2\cos x)^2}$$

Further simplification is possible but often not required unless specified.

Example 6.2 (Higher Derivatives involving Trig Functions)**.** Let $f(x) = x \sin x$. Find the second derivative $f''(x)$. *Solution:* First, find the first derivative using the product rule:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(x \sin x) \\
&= x\frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x) \\
&= x(\cos x) + (\sin x)(1) \\
&= x \cos x + \sin x
\end{aligned}$$

Now, differentiate $f'(x)$ to find $f''(x)$, using the sum rule and the product rule for $x \cos x$:

$$\begin{aligned}
f''(x) &= \frac{d}{dx}(x \cos x + \sin x) \\
&= \frac{d}{dx}(x \cos x) + \frac{d}{dx}(\sin x) \\
&= \left(x\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x) \right) + (\cos x) \\
&= (x(-\sin x) + \cos x \cdot 1) + \cos x \\
&= -x \sin x + \cos x + \cos x \\
&= -x \sin x + 2 \cos x
\end{aligned}$$

Exercise 6.1. Let $f(x) = \sin x$. Find a formula for the n-th derivative $f^{(n)}(x)$. Evaluate $f^{(4)}(x)$, $f^{(18)}(x)$, and $f^{(100)}(x)$. *Hint:* Calculate the first few derivatives and look for a pattern. $f'(x) = \cos x$ $f''(x) = -\sin x$ $f'''(x) = -\cos x$ $f^{(4)}(x) = \sin x$ The pattern repeats every 4 derivatives.

7 The Chain Rule

Many functions are compositions of simpler functions. For example, $F(x) = \sin(x^2 + 1)$ is a composition where the "outer" function is $f(u) = \sin u$ and the "inner" function is $g(x) = u = x^2 + 1$. The Chain Rule provides a way to differentiate such composite functions.

Theorem 7.1 (The Chain Rule)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x, and its derivative is given by:*

$$F'(x) = f'(g(x)) \cdot g'(x)$$

Remark 7.1 (Leibniz Notation for the Chain Rule)**.** If we let $y = f(u)$ and $u = g(x)$, then $y = f(g(x)) = F(x)$. The Chain Rule can be expressed in Leibniz notation as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Here, $\frac{dy}{du}$ is evaluated at $u = g(x)$, and $\frac{du}{dx}$ is evaluated at x. This notation provides a helpful mnemonic, as if the du terms "cancel".

7.1 Special Case: The Power Rule for Functions

A common application of the Chain Rule is when the outer function is a power function, $f(u) = u^n$.

Theorem 7.2 (Power Rule for Functions / Generalized Power Rule)**.** *Let n be any real number. If $u = g(x)$ is a differentiable function, then*

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Or, using Leibniz notation:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Proof. Let $y = f(u) = u^n$. Then $\frac{dy}{du} = nu^{n-1}$. Let $u = g(x)$, so $\frac{du}{dx} = g'(x)$. By the Chain Rule:

$$\frac{d}{dx}[g(x)]^n = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (nu^{n-1}) \cdot g'(x) = n[g(x)]^{n-1}g'(x)$$

□

Example 7.1 (Applying the Power Rule for Functions)**.** Differentiate $F(x) = (1 + 3x + 5x^5 + 4x^{10})^7$. *Solution:* Here, the inner function is $u = g(x) = 1 + 3x + 5x^5 + 4x^{10}$, and the outer function is $f(u) = u^7$. First, find $g'(x)$:

$$g'(x) = \frac{d}{dx}(1 + 3x + 5x^5 + 4x^{10}) = 0 + 3 + 25x^4 + 40x^9$$

Now apply the Power Rule for Functions: $F'(x) = n[g(x)]^{n-1}g'(x)$ with $n = 7$.

$$F'(x) = 7(1 + 3x + 5x^5 + 4x^{10})^{7-1} \cdot (3 + 25x^4 + 40x^9)$$

$$F'(x) = 7(1 + 3x + 5x^5 + 4x^{10})^6(3 + 25x^4 + 40x^9)$$

7.2 Combining the Chain Rule with Other Rules

The Chain Rule often needs to be used in conjunction with the sum, product, quotient, and trigonometric differentiation rules.

Example 7.2 (Chain Rule with Trig Functions)**.** Differentiate the following:

(a) $y = \sin^{10}(2x + 1)$

(b) $F(x) = (x^3 + \sin^3 x)^{100}$

(c) $G(x) = \sin(\sin(2x))$

Solution: (a) Rewrite $y = [\sin(2x + 1)]^{10}$. This involves multiple compositions. Let $u = \sin(2x + 1)$ and $v = 2x + 1$. Then $y = u^{10}$, $u = \sin v$. Using the Chain Rule multiple times (or nested):

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$\frac{dy}{du} = 10u^9 = 10[\sin(2x + 1)]^9$ $\frac{du}{dv} = \frac{d}{dv}(\sin v) = \cos v = \cos(2x + 1)$ $\frac{dv}{dx} = \frac{d}{dx}(2x + 1) = 2$ Combining these:

$$\frac{dy}{dx} = 10[\sin(2x + 1)]^9 \cdot \cos(2x + 1) \cdot 2$$

$$\frac{dy}{dx} = 20\sin^9(2x + 1)\cos(2x + 1)$$

Alternatively, think step-by-step: derivative of outer power function, times derivative of inner sine function, times derivative of innermost linear function. $y' = 10[\sin(2x+1)]^9 \cdot \frac{d}{dx}[\sin(2x+1)]$ $y' = 10[\sin(2x + 1)]^9 \cdot [\cos(2x + 1) \cdot \frac{d}{dx}(2x + 1)]$ $y' = 10[\sin(2x + 1)]^9 \cdot \cos(2x + 1) \cdot 2$

(b) Let $u = g(x) = x^3 + \sin^3 x = x^3 + (\sin x)^3$. Then $F(x) = u^{100}$. $F'(x) = 100u^{99} \cdot \frac{du}{dx} = 100[x^3 + \sin^3 x]^{99} \cdot \frac{d}{dx}[x^3 + (\sin x)^3]$ We need the derivative of $x^3 + (\sin x)^3$: $\frac{d}{dx}(x^3) = 3x^2$ $\frac{d}{dx}(\sin x)^3 = 3(\sin x)^2 \cdot \frac{d}{dx}(\sin x) = 3\sin^2 x \cos x$ (using Power Rule for Functions) So, $\frac{du}{dx} = 3x^2 + 3\sin^2 x \cos x$. Substituting back:

$$F'(x) = 100[x^3 + \sin^3 x]^{99}(3x^2 + 3\sin^2 x \cos x)$$

(c) This is $\sin(u)$ where $u = \sin(2x)$. $G'(x) = \frac{d}{dx}[\sin(u)] = \cos(u) \cdot \frac{du}{dx}$ $G'(x) = \cos(\sin(2x)) \cdot \frac{d}{dx}[\sin(2x)]$ Now find the derivative of $\sin(2x)$. Let $v = 2x$. $\frac{d}{dx}[\sin(2x)] = \frac{d}{dv}(\sin v) \cdot \frac{dv}{dx} = \cos(v) \cdot 2 = \cos(2x) \cdot 2$ Substituting back:

$$G'(x) = \cos(\sin(2x)) \cdot [2\cos(2x)] = 2\cos(2x)\cos(\sin(2x))$$

7.3 General Derivatives of Trigonometric Functions

Combining the Chain Rule with the basic trigonometric derivatives (Theorem 6.2) gives the following general forms, where $u = g(x)$ is a differentiable function:

- $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$
- $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$
- $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$
- $\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}$
- $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$
- $\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$

Example 7.3 (Chain Rule with Nested Cosines)**.** Differentiate $F(x) = \cos^3(\sin x)$. *Solution:* Rewrite as $F(x) = [\cos(\sin x)]^3$. This involves three layers: $y = u^3$, $u = \cos v$, $v = \sin x$.

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$\frac{dy}{du} = 3u^2 = 3[\cos(\sin x)]^2$ $\frac{du}{dv} = \frac{d}{dv}(\cos v) = -\sin v = -\sin(\sin x)$ $\frac{dv}{dx} = \frac{d}{dx}(\sin x) = \cos x$ Combining:

$$F'(x) = 3[\cos(\sin x)]^2 \cdot [-\sin(\sin x)] \cdot (\cos x)$$
$$F'(x) = -3\cos^2(\sin x)\sin(\sin x)\cos x$$

Exercise 7.1. Differentiate the following complex functions involving the Chain Rule.

1. $y = \left(\frac{\sqrt{x+\tan(\pi x/2)}}{\sqrt[3]{4x^2+1}+\ln 2}\right)^{3/5}$

2. $y = \cos^3\left(\frac{x^3}{x^3+\sin(6x)}\right)$

8 Implicit Differentiation

8.1 Implicit vs. Explicit Functions

Definition 8.1 (Explicit Function)**.** *An **explicit function** is a function where the dependent variable (usually y) is expressed solely in terms of the independent variable (usually x). It is written in the form $y = f(x)$. Examples: $y = x^3 + 1$, $y = \sqrt{2x - 1}$.*

Definition 8.2 (Implicit Function)**.** *An **implicit function** is defined by an equation relating x and y, typically written in the form $F(x, y) = 0$ or $F(x, y) = G(x, y)$. It may be difficult or impossible to solve this equation explicitly for y in terms of x. However, such an equation often still defines y as one or more functions of x locally (i.e., on some interval). We say that y is defined **implicitly** as a function of x. Examples: $x^2 + y^2 = 2$, $x^3 + y^2 + xy = 0$, $x^4 + x^2y^3 - 2x - \sin(y) = y^5$.*

Even when we cannot solve for y explicitly, we can often still find the derivative $\frac{dy}{dx}$ using a technique called **implicit differentiation**.

8.2 The Technique of Implicit Differentiation

To find $\frac{dy}{dx}$ for an equation defining y implicitly as a function of x:

1. **Differentiate both sides:** Differentiate both sides of the equation with respect to x. Remember that y is treated as a function of x, so the Chain Rule must be used whenever differentiating a term involving y. For example:
   - $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$
   - $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$
   - $\frac{d}{dx}(xy) = x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = x\frac{dy}{dx} + y$ (Product Rule)
2. **Solve for $\frac{dy}{dx}$:** The resulting equation will contain x, y, and $\frac{dy}{dx}$. Algebraically rearrange the equation to isolate $\frac{dy}{dx}$. The result will typically be an expression involving both x and y.

Example 8.1 (Implicit vs. Explicit Differentiation)**.** Consider the equation $x^2 + y^2 = 2$.

(i) Find $\frac{dy}{dx}$ using implicit differentiation.

(ii) Solve for y explicitly in terms of x and differentiate to find $\frac{dy}{dx}$.

(iii) Show that the answers are equivalent.

Solution: (i) *Implicit Differentiation:* Differentiate both sides of $x^2 + y^2 = 2$ with respect to x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(2)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{(using Chain Rule for } y^2\text{)}$$

Now, solve for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

So, implicitly, $\frac{dy}{dx} = -\frac{x}{y}$.

(ii) *Explicit Differentiation:* Solve $x^2 + y^2 = 2$ for y:

$$y^2 = 2 - x^2$$

$$y = \pm\sqrt{2 - x^2}$$

This implicit equation defines two explicit functions: $f(x) = \sqrt{2-x^2}$ (upper semicircle) and $g(x) = -\sqrt{2-x^2}$ (lower semicircle). We differentiate each separately using the Chain Rule (Power Rule for Functions with exponent 1/2): For $y = f(x) = (2-x^2)^{1/2}$ (where $y > 0$):

$$\frac{dy}{dx} = \frac{1}{2}(2-x^2)^{-1/2} \cdot \frac{d}{dx}(2-x^2) = \frac{1}{2\sqrt{2-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{2-x^2}}$$

For $y = g(x) = -(2-x^2)^{1/2}$ (where $y < 0$):

$$\frac{dy}{dx} = -\frac{1}{2}(2-x^2)^{-1/2} \cdot (-2x) = \frac{x}{\sqrt{2-x^2}}$$

(iii) *Equivalence:* Compare the results. From (i), $\frac{dy}{dx} = -\frac{x}{y}$. From (ii), if $y > 0$, then $y = \sqrt{2-x^2}$, and $\frac{dy}{dx} = \frac{-x}{\sqrt{2-x^2}} = -\frac{x}{y}$. This matches. From (ii), if $y < 0$, then $y = -\sqrt{2-x^2}$, so $\sqrt{2-x^2} = -y$. The derivative is $\frac{dy}{dx} = \frac{x}{\sqrt{2-x^2}} = \frac{x}{-y} = -\frac{x}{y}$. This also matches. The implicit differentiation result covers both cases simultaneously.

Example 8.2 (Tangent Slope using Implicit Differentiation)**.** Find the slopes of the tangent lines to the graph $x^2 + y^2 = 2$ at the points $(1, 1)$ and $(1, -1)$. *Solution:* From the previous example, we found $\frac{dy}{dx} = -\frac{x}{y}$. At the point $(x, y) = (1, 1)$:

$$\text{Slope} = \left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{1}{1} = -1$$

The tangent line equation is $y - 1 = -1(x - 1)$, or $y = -x + 2$.

At the point $(x, y) = (1, -1)$:

$$\text{Slope} = \left.\frac{dy}{dx}\right|_{(1,-1)} = -\frac{1}{-1} = 1$$

The tangent line equation is $y - (-1) = 1(x - 1)$, or $y = x - 2$.

Exercise 8.1. Consider $x^2 + y^2 = 4$.

1. Find $\frac{dy}{dx}$ using implicit differentiation.

2. Find the slopes of the tangent lines at the points corresponding to $x = 1$. (Note: When $x = 1$, $1^2 + y^2 = 4 \implies y^2 = 3 \implies y = \pm\sqrt{3}$).

Exercise 8.2. Find $y' = \frac{dy}{dx}$ if $\sin(y) = y\cos(2x)$. *Solution Hint:* Differentiate both sides wrt x. Use Chain Rule for $\sin y$ and Product/Chain Rules for $y\cos(2x)$.

$$\cos(y)\frac{dy}{dx} = y\frac{d}{dx}(\cos(2x)) + \cos(2x)\frac{d}{dx}(y)$$

$$\cos(y)\frac{dy}{dx} = y(-\sin(2x) \cdot 2) + \cos(2x)\frac{dy}{dx}$$

Now solve for $\frac{dy}{dx}$.

$$(\cos y - \cos(2x))\frac{dy}{dx} = -2y\sin(2x)$$

$$\frac{dy}{dx} = \frac{-2y\sin(2x)}{\cos y - \cos(2x)}$$

Example 8.3. Find $\frac{dy}{dx}$ if $x^4 + x^2y^3 - y^5 = 2x + 1$. *Solution:* Differentiate both sides with respect to x:

$$\frac{d}{dx}(x^4 + x^2y^3 - y^5) = \frac{d}{dx}(2x + 1)$$

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(x^2y^3) - \frac{d}{dx}(y^5) = \frac{d}{dx}(2x) + \frac{d}{dx}(1)$$

Apply rules term by term: $\frac{d}{dx}(x^4) = 4x^3$ $\frac{d}{dx}(x^2y^3) = x^2\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(x^2)$ (Product Rule) $= x^2(3y^2\frac{dy}{dx}) + y^3(2x) = 3x^2y^2\frac{dy}{dx} + 2xy^3$ $\frac{d}{dx}(y^5) = 5y^4\frac{dy}{dx}$ (Chain Rule) $\frac{d}{dx}(2x) = 2$ $\frac{d}{dx}(1) = 0$ Substitute these back into the differentiated equation:

$$4x^3 + (3x^2y^2\frac{dy}{dx} + 2xy^3) - 5y^4\frac{dy}{dx} = 2 + 0$$

Now, group terms with $\frac{dy}{dx}$ and solve:

$$(3x^2y^2 - 5y^4)\frac{dy}{dx} = 2 - 4x^3 - 2xy^3$$

$$\frac{dy}{dx} = \frac{2 - 4x^3 - 2xy^3}{3x^2y^2 - 5y^4}$$

This derivative is valid provided $3x^2y^2 - 5y^4 \neq 0$.

Example 8.4 (Tangent Line Equation from Implicit Differentiation)**.** Find an equation of the tangent line to the graph $\cos(xy^2) = y^2 + x$ at the point $(0, 1)$. *Solution:* First, check if $(0, 1)$ is on the curve: $\cos(0 \cdot 1^2) = \cos(0) = 1$. And $y^2 + x = 1^2 + 0 = 1$. Yes, the point is on the curve. Now, find $\frac{dy}{dx}$ using implicit differentiation. Differentiate both sides wrt x:

$$\frac{d}{dx}[\cos(xy^2)] = \frac{d}{dx}(y^2 + x)$$

Left side (Chain Rule twice):

$$-\sin(xy^2) \cdot \frac{d}{dx}(xy^2) = -\sin(xy^2) \cdot [x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x)]$$

$$= -\sin(xy^2) \cdot [x(2y\frac{dy}{dx}) + y^2(1)] = -\sin(xy^2)(2xy\frac{dy}{dx} + y^2)$$

Right side:

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x) = 2y\frac{dy}{dx} + 1$$

Equating the derivatives:

$$-\sin(xy^2)(2xy\frac{dy}{dx} + y^2) = 2y\frac{dy}{dx} + 1$$

$$-2xy\sin(xy^2)\frac{dy}{dx} - y^2\sin(xy^2) = 2y\frac{dy}{dx} + 1$$

Group $\frac{dy}{dx}$ terms:

$$-y^2\sin(xy^2) - 1 = 2y\frac{dy}{dx} + 2xy\sin(xy^2)\frac{dy}{dx}$$

$$-1 - y^2\sin(xy^2) = (2y + 2xy\sin(xy^2))\frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-1 - y^2\sin(xy^2)}{2y + 2xy\sin(xy^2)} = -\frac{1 + y^2\sin(xy^2)}{2y(1 + x\sin(xy^2))}$$

Now, evaluate the slope m at $(x, y) = (0, 1)$:

$$m = \left.\frac{dy}{dx}\right|_{(0,1)} = -\frac{1 + (1)^2 \sin(0 \cdot 1^2)}{2(1)(1 + 0 \cdot \sin(0 \cdot 1^2))} = -\frac{1 + 1\sin(0)}{2(1 + 0)} = -\frac{1 + 0}{2} = -\frac{1}{2}$$

The slope is $m = -1/2$. The point is $(0, 1)$. Equation of the tangent line: $y - 1 = -\frac{1}{2}(x - 0)$

$$y = -\frac{1}{2}x + 1$$

8.3 Higher-Order Derivatives via Implicit Differentiation

We can find second, third, and higher derivatives of implicitly defined functions by differentiating the expression for the previous derivative implicitly.

Recall the definitions:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

and so on. When $\frac{dy}{dx}$ is expressed in terms of x and y, differentiating it with respect to x will require the quotient rule (usually) and the chain rule whenever y appears, leading to another $\frac{dy}{dx}$ term, which must then be substituted with its known expression in terms of x and y.

Example 8.5 (Second and Third Implicit Derivatives)**.** Consider the equation $x^2 + y^2 = 2$. Find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$. *Solution:* From Example 7.2, we found the first derivative:

$$\frac{dy}{dx} = -\frac{x}{y}$$

To find the second derivative, we differentiate $\frac{dy}{dx}$ with respect to x using the quotient rule:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{x}{y}\right)\\ &= -\frac{y\frac{d}{dx}(x) - x\frac{d}{dx}(y)}{y^2}\\ &= -\frac{y(1) - x(\frac{dy}{dx})}{y^2}\end{aligned}$$

Now, substitute the expression for $\frac{dy}{dx}$:

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{y - x(-\frac{x}{y})}{y^2}\\ &= -\frac{y + \frac{x^2}{y}}{y^2}\\ &= -\frac{\frac{y^2+x^2}{y}}{y^2}\\ &= -\frac{y^2 + x^2}{y^3}\end{aligned}$$

Since the original equation is $x^2 + y^2 = 2$, we can substitute this into the numerator:

$$\frac{d^2y}{dx^2} = -\frac{2}{y^3}$$

To find the third derivative, we differentiate $\frac{d^2y}{dx^2}$ with respect to x:

$$\begin{aligned}\frac{d^3y}{dx^3} &= \frac{d}{dx}\left(-\frac{2}{y^3}\right) = \frac{d}{dx}(-2y^{-3}) \\ &= -2(-3)y^{-4}\frac{dy}{dx} \quad \text{(Chain Rule)} \\ &= 6y^{-4}\frac{dy}{dx} = \frac{6}{y^4}\frac{dy}{dx}\end{aligned}$$

Now, substitute the expression for $\frac{dy}{dx}$:

$$\frac{d^3y}{dx^3} = \frac{6}{y^4}\left(-\frac{x}{y}\right) = -\frac{6x}{y^5}$$

Thus, $\frac{d^2y}{dx^2} = -\frac{2}{y^3}$ and $\frac{d^3y}{dx^3} = -\frac{6x}{y^5}$.

Exercise 8.3. Find $\frac{d^2y}{dx^2}$ if $x^2+y^2 = 4$. Compare the result with the previous example. *Answer:* $\frac{d^2y}{dx^2} = -4/y^3$.

Example 8.6 (Evaluating Second Implicit Derivative at a Point)**.** Find $\frac{d^2y}{dx^2}$ at the point $(0,1)$ if $y^4 - xy = 4$. *Solution:* First, find $\frac{dy}{dx}$ using implicit differentiation:

$$\begin{aligned}\frac{d}{dx}(y^4 - xy) &= \frac{d}{dx}(4) \\ 4y^3\frac{dy}{dx} - \left[x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right] &= 0 \\ 4y^3\frac{dy}{dx} - \left[x\frac{dy}{dx} + y(1)\right] &= 0 \\ 4y^3\frac{dy}{dx} - x\frac{dy}{dx} - y &= 0 \\ (4y^3 - x)\frac{dy}{dx} &= y \\ \frac{dy}{dx} &= \frac{y}{4y^3 - x}\end{aligned}$$

Now, differentiate this expression with respect to x using the quotient rule to find $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = \frac{(4y^3 - x)\frac{d}{dx}(y) - y\frac{d}{dx}(4y^3 - x)}{(4y^3 - x)^2}$$

We need $\frac{d}{dx}(y) = \frac{dy}{dx}$ and $\frac{d}{dx}(4y^3 - x) = 4(3y^2\frac{dy}{dx}) - 1 = 12y^2\frac{dy}{dx} - 1$.

$$\frac{d^2y}{dx^2} = \frac{(4y^3 - x)(\frac{dy}{dx}) - y(12y^2\frac{dy}{dx} - 1)}{(4y^3 - x)^2}$$

Substitute the expression for $\frac{dy}{dx}$:

$$\frac{d^2y}{dx^2} = \frac{(4y^3 - x)\left(\frac{y}{4y^3-x}\right) - y\left[12y^2\left(\frac{y}{4y^3-x}\right) - 1\right]}{(4y^3 - x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{y - y\left[\frac{12y^3}{4y^3-x} - 1\right]}{(4y^3-x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{y - \frac{12y^4}{4y^3-x} + y}{(4y^3-x)^2} = \frac{2y - \frac{12y^4}{4y^3-x}}{(4y^3-x)^2}$$

Now, evaluate this expression at $(x, y) = (0, 1)$:

$$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = \frac{2(1) - \frac{12(1)^4}{4(1)^3-0}}{(4(1)^3-0)^2}$$

$$= \frac{2 - \frac{12}{4}}{(4)^2} = \frac{2-3}{16} = -\frac{1}{16}$$

9 Derivatives of Inverse Functions

Definition 9.1 (Inverse Function). *Let f be a **one-to-one** function with domain X and range Y. The **inverse function** f^{-1} has domain Y and range X. It is defined by the property:*

$$f^{-1}(y) = x \iff f(x) = y$$

for any $y \in Y$ and $x \in X$. Equivalently, the inverse function satisfies the cancellation equations:

$$f(f^{-1}(x)) = x \quad \textit{for every } x \in Y$$
$$f^{-1}(f(x)) = x \quad \textit{for every } x \in X$$

Remark 9.1 (Graphical Relationship). The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$. If (a, b) is a point on the graph of f, then (b, a) is a point on the graph of f^{-1}. [Image showing a function and its inverse reflected across y=x]

Remark 9.2 (Existence of Inverse). A function f has an inverse function f^{-1} if and only if f is one-to-one. A function is one-to-one if it never takes on the same value twice; that is, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. The **Horizontal Line Test** states that a function is one-to-one if and only if no horizontal line intersects its graph more than once. If a function is strictly monotonic (always increasing or always decreasing) on its domain, it is one-to-one.

Theorem 9.1 (Derivative of an Inverse Function). *Let f be a function that satisfies the following conditions:*

1. *f is differentiable on an open interval I.*
2. *$f'(x) \neq 0$ for all $x \in I$.*
3. *f has an inverse function f^{-1}. (This is guaranteed if $f'(x)$ is always positive or always negative on I, implying f is monotonic).*

Then f^{-1} is differentiable at any y in the range of f for which $x = f^{-1}(y)$ is in I, and its derivative is given by:

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Or, if we let $y = f(x)$, so $x = f^{-1}(y)$, then:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

provided $\frac{dy}{dx} \neq 0$.

Intuitive Derivation using Implicit Differentiation. Let $y = f^{-1}(x)$. By definition of the inverse, this means $f(y) = x$. Now, differentiate both sides of $f(y) = x$ with respect to x, treating y as a function of x:

$$\frac{d}{dx}[f(y)] = \frac{d}{dx}(x)$$

Using the Chain Rule on the left side:

$$f'(y) \cdot \frac{dy}{dx} = 1$$

Solving for $\frac{dy}{dx}$, which is $(f^{-1})'(x)$:

$$\frac{dy}{dx} = \frac{1}{f'(y)}$$

Substituting back $y = f^{-1}(x)$:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

This matches the theorem statement (with x playing the role of y from the theorem). $\square$

Example 9.1 (Derivative of an Inverse: Algebraic Function)**.** Consider $f(x) = x^2 + 1$ for $x \geq 0$. Find $f^{-1}(x)$, $(f^{-1})'(x)$, and evaluate $(f^{-1})'(5)$ using both direct differentiation of f^{-1} and Theorem 9.1. *Solution:* 1. *Find $f^{-1}(x)$:* Let $y = x^2 + 1$. To find the inverse, swap x and y and solve for y: $x = y^2 + 1$ $y^2 = x - 1$ $y = \pm\sqrt{x-1}$ Since the domain of f was $x \geq 0$, its range is $y \geq 1$. The domain of f^{-1} is therefore $x \geq 1$, and its range must be $y \geq 0$. Thus, we choose the positive root: $f^{-1}(x) = \sqrt{x-1}$ for $x \geq 1$.

2. *Direct differentiation of $f^{-1}(x)$:*

$$(f^{-1})'(x) = \frac{d}{dx}(\sqrt{x-1}) = \frac{d}{dx}(x-1)^{1/2} = \frac{1}{2}(x-1)^{-1/2} \cdot 1 = \frac{1}{2\sqrt{x-1}}$$

Now evaluate at $x = 5$:

$$(f^{-1})'(5) = \frac{1}{2\sqrt{5-1}} = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4}$$

3. *Using Theorem 9.1:* The formula is $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$. We want $(f^{-1})'(5)$. First find $f'(x)$: $f(x) = x^2 + 1 \implies f'(x) = 2x$. Next find the value x such that $f(x) = 5$: $x^2 + 1 = 5 \implies x^2 = 4$. Since the domain is $x \geq 0$, we have $x = 2$. So, $f^{-1}(5) = 2$. Now apply the formula:

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(2)}$$

Calculate $f'(2) = 2(2) = 4$.

$$(f^{-1})'(5) = \frac{1}{4}$$

Both methods yield the same result. The theorem is useful when finding $f^{-1}(x)$ explicitly is difficult or impossible.

Example 9.2 (Slope of Inverse using Implicit Differentiation)**.** Consider the function $f(x) = 5x^3 + 8x - 9$. Find the slope of the graph of the inverse function $f^{-1}(x)$ at the point $(4, 1)$. *Solution:* First, verify that $(4, 1)$ is indeed on the graph of f^{-1}. This means $(1, 4)$ must be on the graph of f. Let's check: $f(1) = 5(1)^3 + 8(1) - 9 = 5 + 8 - 9 = 4$. Yes, it is.

Let $y = f^{-1}(x)$. Then $x = f(y) = 5y^3 + 8y - 9$. We want to find the slope $\frac{dy}{dx}$ at the point $(x, y) = (4, 1)$. We differentiate the equation $x = 5y^3 + 8y - 9$ implicitly with respect to x:

$$\frac{d}{dx}(x) = \frac{d}{dx}(5y^3 + 8y - 9)$$

$$1 = 5(3y^2 \frac{dy}{dx}) + 8(\frac{dy}{dx}) - 0$$
$$1 = (15y^2 + 8)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{15y^2 + 8}$$

This is the derivative $(f^{-1})'(x)$ expressed in terms of y. Now evaluate at the point $(4, 1)$, which means using $y = 1$:

$$\left.\frac{dy}{dx}\right|_{(4,1)} = \frac{1}{15(1)^2 + 8} = \frac{1}{15 + 8} = \frac{1}{23}$$

The slope of $f^{-1}(x)$ at $(4, 1)$ is $1/23$. Note that $f'(x) = 15x^2 + 8$, so $f'(1) = 15(1)^2 + 8 = 23$. The result confirms $(f^{-1})'(4) = 1/f'(1)$.

Example 9.3 (Derivative of $\arcsin x$)**.** Find the derivative of $f(x) = \sin^{-1} x = \arcsin x$. *Solution:* Let $y = \sin^{-1} x$. By definition of the inverse function, this means:

$$\sin y = x$$

We assume y is in the range of $\arcsin x$, which is $[-\pi/2, \pi/2]$. Now differentiate $\sin y = x$ implicitly with respect to x:

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$
$$\cos y \cdot \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$

We need to express $\cos y$ in terms of x. We know $\sin^2 y + \cos^2 y = 1$, so $\cos^2 y = 1 - \sin^2 y$. Since $\sin y = x$, we have $\cos^2 y = 1 - x^2$. Therefore, $\cos y = \pm\sqrt{1 - x^2}$. Which sign should we choose? Since y is in $[-\pi/2, \pi/2]$, $\cos y$ is always non-negative ($\cos y \geq 0$). Thus, we take the positive root:

$$\cos y = \sqrt{1 - x^2}$$

Substituting this back into the expression for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

So, $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$. This is valid for $-1 < x < 1$ (where $\cos y \neq 0$).

9.1 Derivatives of Other Inverse Trigonometric Functions

Using similar implicit differentiation techniques (or identities relating the inverse functions), we can derive the derivatives for the other five inverse trigonometric functions. The results, combined with the Chain Rule where $u = g(x)$, are summarized below.

Theorem 9.2 (Derivatives of Inverse Trigonometric Functions)**.** *Let $u = g(x)$ be a differentiable function.*

- $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u| < 1$
- $\frac{d}{dx}(\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u| < 1$
- $\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \frac{du}{dx}$

- $\frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2}\frac{du}{dx}$
- $\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$
- $\frac{d}{dx}(\csc^{-1} u) = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$

Remark 9.3 (Proof Strategy)**.** The proofs for $\cos^{-1} u$, $\tan^{-1} u$, $\cot^{-1} u$, $\sec^{-1} u$, and $\csc^{-1} u$ follow the same pattern as the proof for $\sin^{-1} u$ (Example 9.3): 1. Start with $y = \text{invTrig}(x)$. 2. Rewrite as $x = \text{Trig}(y)$. 3. Differentiate implicitly with respect to x: $1 = \text{Trig}'(y) \cdot \frac{dy}{dx}$. 4. Solve for $\frac{dy}{dx} = \frac{1}{\text{Trig}'(y)}$. 5. Use trigonometric identities (often $\sin^2 y + \cos^2 y = 1$ or $1 + \tan^2 y = \sec^2 y$) to express $\text{Trig}'(y)$ in terms of $x = \text{Trig}(y)$. Pay attention to the range of y to determine correct signs for square roots. 6. Apply the Chain Rule for the general case with $u = g(x)$.

Example 9.4 (Applying Inverse Trig Derivative Rules)**.** Differentiate:

(a) $y = \sin^{-1}(2x) + \cos^{-1}(\sqrt{x+2})$

(b) $y = \sec^{-1}(x^2 + 1)$

Solution: (a) Differentiate term by term:

$$\frac{d}{dx}(\sin^{-1}(2x)) = \frac{1}{\sqrt{1-(2x)^2}} \cdot \frac{d}{dx}(2x) = \frac{1}{\sqrt{1-4x^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

For the second term, let $u = \sqrt{x+2} = (x+2)^{1/2}$. Then $\frac{du}{dx} = \frac{1}{2}(x+2)^{-1/2} = \frac{1}{2\sqrt{x+2}}$.

$$\frac{d}{dx}(\cos^{-1}(\sqrt{x+2})) = -\frac{1}{\sqrt{1-(\sqrt{x+2})^2}} \cdot \frac{du}{dx}$$

$$= -\frac{1}{\sqrt{1-(x+2)}} \cdot \frac{1}{2\sqrt{x+2}} = -\frac{1}{\sqrt{-x-1}} \cdot \frac{1}{2\sqrt{x+2}}$$

This term is only defined if $-x-1 > 0 \implies x < -1$ and $x+2 \geq 0 \implies x \geq -2$. Combining these, the domain for this part is $[-2, -1)$. Combining the derivatives:

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}} - \frac{1}{2\sqrt{-x-1}\sqrt{x+2}}$$

(b) Let $u = x^2 + 1$. Then $\frac{du}{dx} = 2x$. Note that $u = x^2 + 1 \geq 1$. Since the formula for $\sec^{-1} u$ requires $|u| > 1$, we assume $x \neq 0$. Also, since $u \geq 1$, $|u| = u$.

$$\frac{dy}{dx} = \frac{d}{dx}(\sec^{-1}(x^2+1)) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

$$= \frac{1}{(x^2+1)\sqrt{(x^2+1)^2-1}} \cdot (2x)$$

Simplify the term under the square root: $(x^2+1)^2 - 1 = (x^4+2x^2+1) - 1 = x^4+2x^2 = x^2(x^2+2)$. So, $\sqrt{(x^2+1)^2-1} = \sqrt{x^2(x^2+2)} = \sqrt{x^2}\sqrt{x^2+2} = |x|\sqrt{x^2+2}$.

$$\frac{dy}{dx} = \frac{1}{(x^2+1)|x|\sqrt{x^2+2}} \cdot (2x)$$

If $x > 0$, $|x| = x$, so $\frac{dy}{dx} = \frac{2x}{(x^2+1)x\sqrt{x^2+2}} = \frac{2}{(x^2+1)\sqrt{x^2+2}}$. If $x < 0$, $|x| = -x$, so $\frac{dy}{dx} = \frac{2x}{(x^2+1)(-x)\sqrt{x^2+2}} = -\frac{2}{(x^2+1)\sqrt{x^2+2}}$. We can write this compactly as:

$$\frac{dy}{dx} = \frac{2x}{|x|(x^2+1)\sqrt{x^2+2}}, \quad x \neq 0$$

Example 9.5 (Tangent Line with Inverse Trig Function)**.** Find the tangent line equation for $y = \arctan(2x) + 1$ (Note: $\arctan x = \tan^{-1} x$) at the point $(0, 1)$. *Solution:* First, check the point: $y(0) = \arctan(2 \cdot 0) + 1 = \arctan(0) + 1 = 0 + 1 = 1$. The point $(0, 1)$ is on the graph. Next, find the slope $m = \frac{dy}{dx}$ at $x = 0$.

$$\frac{dy}{dx} = \frac{d}{dx}(\arctan(2x) + 1) = \frac{d}{dx}(\arctan(2x)) + \frac{d}{dx}(1)$$

Using the rule for $\arctan u$ with $u = 2x$, $\frac{du}{dx} = 2$:

$$\frac{dy}{dx} = \frac{1}{1 + u^2}\frac{du}{dx} + 0 = \frac{1}{1 + (2x)^2} \cdot 2 = \frac{2}{1 + 4x^2}$$

Evaluate the slope at $x = 0$:

$$m = \left.\frac{dy}{dx}\right|_{x=0} = \frac{2}{1 + 4(0)^2} = \frac{2}{1} = 2$$

The slope is $m = 2$. The point is $(0, 1)$. Tangent line equation: $y - y_0 = m(x - x_0)$

$$y - 1 = 2(x - 0)$$

$$y = 2x + 1$$

10 Derivatives of Exponential and Logarithmic Functions

10.1 Exponential Functions

Recall the natural exponential function e^x and the general exponential function b^x (where $b > 0, b \neq 1$).

Theorem 10.1 (Derivatives of Exponential Functions)**.** *Let $u = g(x)$ be a differentiable function. Let b be a positive constant, $b \neq 1$.*

1. $\frac{d}{dx}(e^x) = e^x$
2. $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$ *(Chain Rule version)*
3. $\frac{d}{dx}(b^x) = b^x \ln b$
4. $\frac{d}{dx}(b^u) = b^u \ln b \frac{du}{dx}$ *(Chain Rule version)*

Derivation of $\frac{d}{dx}(e^x) = e^x$. Using the limit definition of the derivative:

$$\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$

Since e^x does not depend on h, we can pull it out of the limit:

$$\frac{d}{dx}(e^x) = e^x \left(\lim_{h \to 0} \frac{e^h - 1}{h} \right)$$

The limit $\lim_{h \to 0} \frac{e^h - 1}{h}$ is the derivative of $f(t) = e^t$ evaluated at $t = 0$. One fundamental definition of the number e (Euler's number, $e \approx 2.71828$) is that it is the unique base for which this limit equals 1.

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Therefore,

$$\frac{d}{dx}(e^x) = e^x \cdot 1 = e^x$$

□

Derivation of $\frac{d}{dx}(b^x) = b^x \ln b$. We can rewrite b^x using the natural exponential and logarithm: $b = e^{\ln b}$, so $b^x = (e^{\ln b})^x = e^{(\ln b)x}$. Now we differentiate $e^{(\ln b)x}$ using the Chain Rule (Part 2 of the theorem) with $u = (\ln b)x$. Note that $\ln b$ is a constant.

$$\frac{du}{dx} = \frac{d}{dx}((\ln b)x) = \ln b$$

$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^u) = e^u \frac{du}{dx} = e^{(\ln b)x} \cdot (\ln b)$$

Substituting back $e^{(\ln b)x} = b^x$:

$$\frac{d}{dx}(b^x) = b^x \ln b$$

$\square$

Example 10.1 (Differentiating Exponential Functions)**.** Differentiate:

(a) $y = e^{-x^2}$

(b) $y = 10^{3x^{-3}}$

Solution: (a) Use the rule $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$ with $u = -x^2$. $\frac{du}{dx} = \frac{d}{dx}(-x^2) = -2x$.

$$\frac{dy}{dx} = e^{-x^2} \cdot (-2x) = -2xe^{-x^2}$$

(b) Use the rule $\frac{d}{dx}(b^u) = b^u \ln b \frac{du}{dx}$ with $b = 10$ and $u = 3x^{-3}$. $\frac{du}{dx} = \frac{d}{dx}(3x^{-3}) = 3(-3)x^{-4} = -9x^{-4}$.

$$\frac{dy}{dx} = 10^{3x^{-3}} \cdot \ln(10) \cdot (-9x^{-4})$$

$$\frac{dy}{dx} = -9x^{-4}(\ln 10)10^{3/x^3}$$

10.2 Logarithmic Functions

Recall the natural logarithm $\ln x = \log_e x$ (defined for $x > 0$) and the general logarithm $\log_b x$ (defined for $x > 0$, $b > 0, b \neq 1$).

Theorem 10.2 (Derivatives of Logarithmic Functions)**.** *Let $u = g(x)$ be a differentiable function such that $u \neq 0$ for $\ln|u|$ and $u > 0$ for $\log_b u$. Let b be a positive constant, $b \neq 1$.*

1. $\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0$
2. $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}, \quad u > 0$ *(Chain Rule version)*
3. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \quad x \neq 0$
4. $\frac{d}{dx}(\ln|u|) = \frac{1}{u}\frac{du}{dx}, \quad u \neq 0$ *(Chain Rule version)*
5. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}, \quad x > 0$
6. $\frac{d}{dx}(\log_b u) = \frac{1}{u \ln b}\frac{du}{dx}, \quad u > 0$ *(Chain Rule version)*

Derivation of $\frac{d}{dx}(\ln x) = \frac{1}{x}$. Let $y = \ln x$. By definition of the natural logarithm as the inverse of the exponential function, this is equivalent to $e^y = x$. Now we differentiate this equation implicitly with respect to x:

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

Using the Chain Rule for e^y:

$$e^y \frac{dy}{dx} = 1$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{e^y}$$

Since $e^y = x$, we substitute back:

$$\frac{dy}{dx} = \frac{1}{x}$$

Thus, $\frac{d}{dx}(\ln x) = \frac{1}{x}$. This holds for $x > 0$. $\square$

Derivation of $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$. We consider two cases for $x \neq 0$: Case 1: $x > 0$. Then $|x| = x$.

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Case 2: $x < 0$. Then $|x| = -x$. Note that $-x > 0$.

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x))$$

Use the Chain Rule with $u = -x$. $\frac{du}{dx} = -1$.

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$$

In both cases, the derivative is $\frac{1}{x}$. $\square$

Derivation of $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$. We use the change of base formula for logarithms: $\log_b x = \frac{\ln x}{\ln b}$. Since $\ln b$ is a constant:

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b}\frac{d}{dx}(\ln x) = \frac{1}{\ln b} \cdot \frac{1}{x} = \frac{1}{x \ln b}$$

$\square$

Example 10.2 (Differentiating Logarithmic Functions)**.** Differentiate:

(a) $f(x) = \ln(\tan x)$

(b) $y = \ln(\ln x)$

(c) $y = \log_2(3x^5)$ (Not in notes, added for illustration)

(d) $y = \ln|2x + 5|$

Solution: (a) Use $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ with $u = \tan x$. $\frac{du}{dx} = \sec^2 x$.

$$f'(x) = \frac{1}{\tan x} \cdot \sec^2 x = \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x} = \frac{1}{\sin x \cos x}$$

(Alternatively: $\cot x \sec^2 x$. Also $1/(\sin x \cos x) = 2/(2 \sin x \cos x) = 2/\sin(2x) = 2\csc(2x)$.)
(b) Use $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ with $u = \ln x$. $\frac{du}{dx} = \frac{1}{x}$.

$$y' = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}$$

(c) Use $\frac{d}{dx}(\log_b u) = \frac{1}{u \ln b}\frac{du}{dx}$ with $b = 2$, $u = 3x^5$. $\frac{du}{dx} = 15x^4$.

$$y' = \frac{1}{(3x^5)\ln 2} \cdot (15x^4) = \frac{15x^4}{3x^5 \ln 2} = \frac{5}{x \ln 2}$$

Alternatively, use log properties first: $y = \log_2 3 + \log_2(x^5) = \log_2 3 + 5\log_2 x$. $y' = 0 + 5\frac{d}{dx}(\log_2 x) = 5 \cdot \frac{1}{x \ln 2}$.

(d) Use $\frac{d}{dx}(\ln |u|) = \frac{1}{u}\frac{du}{dx}$ with $u = 2x + 5$. $\frac{du}{dx} = 2$.

$$y' = \frac{1}{2x+5} \cdot 2 = \frac{2}{2x+5}$$

11 Logarithmic Differentiation

The Power Rule $\frac{d}{dx}(x^n) = nx^{n-1}$ applies when the base is a variable and the exponent is a constant. The Exponential Rule $\frac{d}{dx}(b^x) = b^x \ln b$ applies when the base is a constant and the exponent is a variable. Neither rule applies directly to functions where both the base and the exponent are variables, such as $y = x^x$ or $y = (\sin x)^x$.

Logarithmic differentiation is a technique used to differentiate such functions, or functions that involve complex products, quotients, and powers, by simplifying them using logarithm properties before differentiating.

11.1 Steps for Logarithmic Differentiation

To differentiate $y = f(x)$ using logarithmic differentiation:

1. **Take Natural Logarithm:** Take the natural logarithm (ln) of both sides of the equation $y = f(x)$:

   $$\ln y = \ln(f(x))$$

   Assume $y > 0$. If y could be negative, use $\ln |y| = \ln |f(x)|$.

2. **Simplify:** Use properties of logarithms to simplify the right-hand side, $\ln(f(x))$. Key properties include:

   - $\ln(a^b) = b \ln a$
   - $\ln(ab) = \ln a + \ln b$
   - $\ln(a/b) = \ln a - \ln b$

   This step often converts products into sums, quotients into differences, and powers/roots into coefficients.

3. **Differentiate Implicitly:** Differentiate both sides of the simplified equation with respect to x. Remember to use the Chain Rule for the left side: $\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$. Differentiate the right side using standard rules.

4. **Solve for $\frac{dy}{dx}$:** Algebraically solve the resulting equation for $\frac{dy}{dx}$.

5. **Substitute Back:** Replace y with the original function $f(x)$ to express $\frac{dy}{dx}$ entirely in terms of x.

Remark 11.1. The final result can often be expressed as:

$$\frac{dy}{dx} = y \cdot \frac{d}{dx}[\ln(f(x))] = f(x) \cdot \frac{d}{dx}[\ln(f(x))]$$

This shows that logarithmic differentiation essentially involves multiplying the original function by the derivative of its natural logarithm.

Example 11.1 (Differentiating $y = x^x$)**.** Find the derivative of $y = x^x$ for $x > 0$. *Solution:* This function has a variable base and a variable exponent. 1. *Take Logarithm:* $\ln y = \ln(x^x)$. 2. *Simplify:* $\ln y = x \ln x$. 3. *Differentiate Implicitly:* Differentiate both sides with respect to x. Use the Product Rule on the right side.

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(x \ln x) \\
\frac{1}{y}\frac{dy}{dx} &= x\frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(x) \\
\frac{1}{y}\frac{dy}{dx} &= x\left(\frac{1}{x}\right) + (\ln x)(1) \\
\frac{1}{y}\frac{dy}{dx} &= 1 + \ln x
\end{aligned}$$

4. *Solve for* $\frac{dy}{dx}$*:*

$$\frac{dy}{dx} = y(1 + \ln x)$$

5. *Substitute Back:* Replace y with x^x.

$$\frac{dy}{dx} = x^x(1 + \ln x)$$

Thus, $\frac{d}{dx}(x^x) = x^x(1 + \ln x)$.

Example 11.2 (Simplifying Complex Fractions before Differentiating)**.** Differentiate $y = \ln\left(\frac{x^2}{\sqrt{(x-1)^3(2x+1)^5}}\right)$. (Note: Original notes had $(2x+1)^2$, using 5 for illustration). *Solution:* While we could use the Chain Rule directly, it's much easier to simplify using log properties first.

$$\begin{aligned}
y &= \ln(x^2) - \ln(\sqrt{(x-1)^3(2x+1)^5}) \\
&= 2\ln x - \ln((x-1)^{3/2}(2x+1)^{5/2}) \\
&= 2\ln x - \left[\ln((x-1)^{3/2}) + \ln((2x+1)^{5/2})\right] \\
&= 2\ln x - \frac{3}{2}\ln(x-1) - \frac{5}{2}\ln(2x+1)
\end{aligned}$$

This expression is much simpler to differentiate:

$$\begin{aligned}
\frac{dy}{dx} &= 2\frac{d}{dx}(\ln x) - \frac{3}{2}\frac{d}{dx}(\ln(x-1)) - \frac{5}{2}\frac{d}{dx}(\ln(2x+1)) \\
&= 2\left(\frac{1}{x}\right) - \frac{3}{2}\left(\frac{1}{x-1}\cdot 1\right) - \frac{5}{2}\left(\frac{1}{2x+1}\cdot 2\right) \\
&= \frac{2}{x} - \frac{3}{2(x-1)} - \frac{5}{2x+1}
\end{aligned}$$

Example 11.3 (Logarithmic Differentiation for Complex Products/Quotients)**.** Let a, b, c, d be constants. Differentiate $f(x) = \frac{(x-2)^a(x-3)^b}{(x+4)^c(x+5)^d}$. *Solution:* Direct differentiation using quotient and product rules would be very complex. Use logarithmic differentiation. Let $y = f(x)$. 1. *Take Logarithm:* $\ln y = \ln\left(\frac{(x-2)^a(x-3)^b}{(x+4)^c(x+5)^d}\right)$. 2. *Simplify:*

$$\ln y = \ln((x-2)^a) + \ln((x-3)^b) - \ln((x+4)^c) - \ln((x+5)^d)$$

$$\ln y = a\ln(x-2) + b\ln(x-3) - c\ln(x+4) - d\ln(x+5)$$

3. *Differentiate Implicitly:*

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[a\ln(x-2) + b\ln(x-3) - c\ln(x+4) - d\ln(x+5)]$$

$$\frac{1}{y}\frac{dy}{dx} = a\left(\frac{1}{x-2}\right) + b\left(\frac{1}{x-3}\right) - c\left(\frac{1}{x+4}\right) - d\left(\frac{1}{x+5}\right)$$

4. *Solve for $\frac{dy}{dx}$:*

$$\frac{dy}{dx} = y\left[\frac{a}{x-2} + \frac{b}{x-3} - \frac{c}{x+4} - \frac{d}{x+5}\right]$$

5. *Substitute Back:*

$$\frac{dy}{dx} = \frac{(x-2)^a(x-3)^b}{(x+4)^c(x+5)^d}\left[\frac{a}{x-2} + \frac{b}{x-3} - \frac{c}{x+4} - \frac{d}{x+5}\right]$$

Example 11.4 (Tangent Line using Logarithmic Differentiation)**.** Find the tangent line to the graph of $y = x(\ln x)^x$ at $x = e$. *Solution:* First, find the point: $y(e) = e(\ln e)^e = e(1)^e = e$. The point is (e, e). Next, find the slope $\frac{dy}{dx}$ at $x = e$ using logarithmic differentiation. 1. *Take Logarithm:* $\ln y = \ln(x(\ln x)^x)$. 2. *Simplify:* $\ln y = \ln x + \ln((\ln x)^x) = \ln x + x\ln(\ln x)$. 3. *Differentiate Implicitly:*

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\ln x) + \frac{d}{dx}(x\ln(\ln x))$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \left[x\frac{d}{dx}(\ln(\ln x)) + \ln(\ln x)\frac{d}{dx}(x)\right]$$

For $\frac{d}{dx}(\ln(\ln x))$, let $u = \ln x$, $\frac{du}{dx} = 1/x$. $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x}$.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \left[x\left(\frac{1}{x\ln x}\right) + \ln(\ln x) \cdot 1\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\ln x} + \ln(\ln x)$$

4. *Solve for $\frac{dy}{dx}$:*

$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{1}{\ln x} + \ln(\ln x)\right]$$

5. *Substitute Back:*

$$\frac{dy}{dx} = x(\ln x)^x\left[\frac{1}{x} + \frac{1}{\ln x} + \ln(\ln x)\right]$$

Now, evaluate the slope at $x = e$. Recall $\ln e = 1$.

$$m = \left.\frac{dy}{dx}\right|_{x=e} = e(\ln e)^e\left[\frac{1}{e} + \frac{1}{\ln e} + \ln(\ln e)\right]$$

$$m = e(1)^e\left[\frac{1}{e} + \frac{1}{1} + \ln(1)\right] = e\left[\frac{1}{e} + 1 + 0\right] = e(\frac{1}{e}) + e(1) = 1 + e$$

The slope is $m = 1 + e$. The point is (e, e). Tangent line equation: $y - y_0 = m(x - x_0)$

$$y - e = (1+e)(x-e)$$

$$y = (1+e)x - (1+e)e + e = (1+e)x - e - e^2 + e$$

$$y = (1+e)x - e^2$$

Exercise 11.1. Differentiate $y = \frac{(\sin x+1)^x}{x^3}$ using logarithmic differentiation.

12 Derivatives of Hyperbolic Functions

Hyperbolic functions are analogs of trigonometric functions defined using the exponential function e^x.

Definition 12.1 (Hyperbolic Sine and Cosine)**.** *For any real number x:*

- *Hyperbolic sine:* $\sinh x = \frac{e^x - e^{-x}}{2}$
- *Hyperbolic cosine:* $\cosh x = \frac{e^x + e^{-x}}{2}$

Remark 12.1.

- The domain for both $\sinh x$ and $\cosh x$ is $\mathbb{R} = (-\infty, \infty)$.
- $\sinh(0) = (e^0 - e^0)/2 = 0$.
- $\cosh(0) = (e^0 + e^0)/2 = (1+1)/2 = 1$.
- $\cosh x \geq 1$ for all x. ($\cosh x = \sqrt{1 + \sinh^2 x}$)

Definition 12.2 (Other Hyperbolic Functions)**.** *Analogous to trigonometric functions:*

- *Hyperbolic tangent:* $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
- *Hyperbolic cotangent:* $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \neq 0$
- *Hyperbolic secant: sech* $x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
- *Hyperbolic cosecant: csch* $x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad x \neq 0$

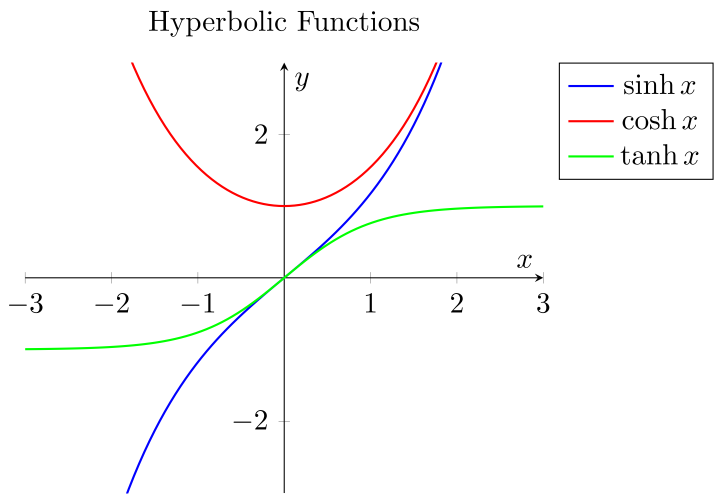


Figure 2: Graphs of $\sinh x$, $\cosh x$, and $\tanh x$.

12.1 Identities for Hyperbolic Functions

Hyperbolic functions satisfy identities similar, but not identical, to trigonometric identities.

- $\cosh^2 x - \sinh^2 x = 1$ (Compare to $\cos^2 x + \sin^2 x = 1$)
- $1 - \tanh^2 x = \text{sech}^2 x$ (Compare to $1 + \tan^2 x = \sec^2 x$)
- $\coth^2 x - 1 = \text{csch}^2 x$ (Compare to $\cot^2 x + 1 = \csc^2 x$)

- $\sinh(-x) = -\sinh x$ (Odd function)
- $\cosh(-x) = \cosh x$ (Even function)
- $\tanh(-x) = -\tanh x$ (Odd function)
- $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
- $\sinh(2x) = 2 \sinh x \cosh x$
- $\cosh(2x) = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1$

Proof of $\cosh^2 x - \sinh^2 x = 1$.

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} \\
&= \frac{4}{4} = 1
\end{aligned}$$

□

12.2 Derivatives of Hyperbolic Functions

The derivatives can be found directly from the definitions using the derivative of e^x.

Example 12.1 (Derivative of $\sinh x$)**.** Differentiate $y = \sinh x$. *Solution:*

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left(\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right)$$

$$= \frac{1}{2}(e^x - (e^{-x} \cdot (-1))) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

So, $\frac{d}{dx}(\sinh x) = \cosh x$.

Theorem 12.1 (Derivatives of Hyperbolic Functions)**.** *Let $u = g(x)$ be a differentiable function.*

- $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$
- $\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$
- $\frac{d}{dx}(\tanh u) = sech^2 u \frac{du}{dx}$
- $\frac{d}{dx}(\coth u) = -csch^2 u \frac{du}{dx}$
- $\frac{d}{dx}(sech\, u) = -sech\, u \tanh u \frac{du}{dx}$
- $\frac{d}{dx}(csch\, u) = -csch\, u \coth u \frac{du}{dx}$

Remark 12.2 (Comparison with Trigonometric Derivatives)**.** Notice the similarities and differences in signs compared to trigonometric derivatives. For example, $\frac{d}{dx}(\cosh x) = \sinh x$ (no minus sign), while $\frac{d}{dx}(\cos x) = -\sin x$.

Derivation of $\frac{d}{dx}(\tanh x)$*.* Using the quotient rule and the derivatives of $\sinh x$ and $\cosh x$:

$$\begin{aligned}\frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) \\ &= \frac{\cosh x \frac{d}{dx}(\sinh x) - \sinh x \frac{d}{dx}(\cosh x)}{\cosh^2 x} \\ &= \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\ &= \frac{1}{\cosh^2 x} \quad (\text{using } \cosh^2 x - \sinh^2 x = 1) \\ &= \text{sech}^2 x\end{aligned}$$

The Chain Rule version follows directly. □

Example 12.2 (Applying Hyperbolic Derivative Rules)**.** Differentiate:

(a) $y = \sinh(\sqrt{2x+1})$

(b) $y = \coth(x^3)$

Solution: (a) Use $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$ with $u = \sqrt{2x+1} = (2x+1)^{1/2}$. $\frac{du}{dx} = \frac{1}{2}(2x+1)^{-1/2} \cdot 2 = (2x+1)^{-1/2} = \frac{1}{\sqrt{2x+1}}$.

$$\frac{dy}{dx} = \cosh(\sqrt{2x+1}) \cdot \frac{1}{\sqrt{2x+1}}$$

(b) Use $\frac{d}{dx}(\coth u) = -\text{csch}^2 u \frac{du}{dx}$ with $u = x^3$. $\frac{du}{dx} = 3x^2$.

$$\frac{dy}{dx} = -\text{csch}^2(x^3) \cdot (3x^2) = -3x^2\text{csch}^2(x^3)$$

12.3 Inverse Hyperbolic Functions

Since $\sinh x$ and $\tanh x$ are strictly increasing, they are one-to-one and have inverse functions, denoted $\sinh^{-1} x$ (or arsinh x) and $\tanh^{-1} x$ (or artanh x). The function $\cosh x$ is not one-to-one on $\mathbb{R}$, but if we restrict its domain to $x \geq 0$ (where its range is $y \geq 1$), it becomes one-to-one and has an inverse $\cosh^{-1} x$ (or arcosh x) defined for $x \geq 1$. Similarly, inverses for $\coth x$, sech x, and csch x can be defined on appropriate restricted domains.

Theorem 12.2 (Logarithmic Forms of Inverse Hyperbolic Functions)**.** *Inverse hyperbolic functions can be expressed using natural logarithms:*

- $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ *for all* $x \in \mathbb{R}$
- $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ *for* $x \geq 1$
- $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ *for* $|x| < 1$
- $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ *for* $|x| > 1$
- $sech^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ *for* $0 < x \leq 1$
- $csch^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$ *for* $x \neq 0$

Derivation of $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$. Let $y = \sinh^{-1} x$. Then $x = \sinh y$. By definition, $x = \frac{e^y - e^{-y}}{2}$. We want to solve for y. Multiply by $2e^y$ to clear denominators and negative exponents:

$$2xe^y = (e^y - e^{-y})e^y$$
$$2xe^y = (e^y)^2 - e^0 = (e^y)^2 - 1$$

Rearrange into a quadratic equation in terms of e^y:

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Let $Z = e^y$. The equation is $Z^2 - 2xZ - 1 = 0$. Use the quadratic formula to solve for Z:

$$Z = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)} = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$Z = \frac{2x \pm \sqrt{4(x^2+1)}}{2} = \frac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1}$$

Since $Z = e^y$ must be positive, and $\sqrt{x^2+1} > \sqrt{x^2} = |x|$, the term $x - \sqrt{x^2+1}$ is always negative (if $x \geq 0$, $x - \sqrt{x^2+1} < 0$; if $x < 0$, $x - \sqrt{x^2+1} < 0$). Therefore, we must choose the positive sign:

$$e^y = x + \sqrt{x^2+1}$$

Taking the natural logarithm of both sides gives:

$$y = \ln(x + \sqrt{x^2+1})$$

Thus, $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$. □

12.4 Derivatives of Inverse Hyperbolic Functions

The derivatives can be found either by differentiating the logarithmic forms or by using implicit differentiation.

Theorem 12.3 (Derivatives of Inverse Hyperbolic Functions)**.** *Let $u = g(x)$ be a differentiable function.*

- $\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$
- $\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$
- $\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$
- $\frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$
- $\frac{d}{dx}(sech^{-1} u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$
- $\frac{d}{dx}(csch^{-1} u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$

Proof of $\frac{d}{dx}(\sinh^{-1} x)$ using Implicit Differentiation. Let $y = \sinh^{-1} x$. Then $x = \sinh y$. Differentiate both sides with respect to x:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sinh y)$$

$$1 = \cosh y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

Use the identity $\cosh^2 y - \sinh^2 y = 1$, which gives $\cosh^2 y = 1 + \sinh^2 y$. Since $\cosh y$ is always positive (its range is $[1, \infty)$), $\cosh y = \sqrt{1 + \sinh^2 y}$. Substitute $x = \sinh y$: $\cosh y = \sqrt{1 + x^2}$. Therefore,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$$

The Chain Rule version follows directly. □

Proof of $\frac{d}{dx}(\tanh^{-1} x)$ using Logarithmic Form. From the logarithmic form, $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}[\ln(1 + x) - \ln(1 - x)]$. Differentiate with respect to x for $|x| < 1$:

$$\begin{aligned}
\frac{d}{dx}(\tanh^{-1} x) &= \frac{1}{2}\left[\frac{d}{dx}(\ln(1 + x)) - \frac{d}{dx}(\ln(1 - x))\right] \\
&= \frac{1}{2}\left[\frac{1}{1 + x} \cdot 1 - \frac{1}{1 - x} \cdot (-1)\right] \\
&= \frac{1}{2}\left[\frac{1}{1 + x} + \frac{1}{1 - x}\right] \\
&= \frac{1}{2}\left[\frac{(1 - x) + (1 + x)}{(1 + x)(1 - x)}\right] \\
&= \frac{1}{2}\left[\frac{2}{1 - x^2}\right] = \frac{1}{1 - x^2}
\end{aligned}$$

The Chain Rule version follows directly. □

Example 12.3 (Applying Inverse Hyperbolic Derivative Rules)**.** Differentiate $y = \sinh^{-1}(e^x + \sin(2x))$. *Solution:* Use $\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{u^2+1}} \frac{du}{dx}$ with $u = e^x + \sin(2x)$. First find $\frac{du}{dx}$:

$$\frac{du}{dx} = \frac{d}{dx}(e^x + \sin(2x)) = e^x + \cos(2x) \cdot 2 = e^x + 2\cos(2x)$$

Now apply the formula:

$$\frac{dy}{dx} = \frac{1}{\sqrt{(e^x + \sin(2x))^2 + 1}} \cdot (e^x + 2\cos(2x))$$

$$\frac{dy}{dx} = \frac{e^x + 2\cos(2x)}{\sqrt{(e^x + \sin(2x))^2 + 1}}$$

Exercise 12.1. Differentiate $y = (\tanh^{-1}(x^x))^{-1/3}$. *Hint:* Use Power Rule for Functions, Chain Rule for $\tanh^{-1} u$, and the derivative of x^x found earlier using logarithmic differentiation.

13 Linear Approximation and Differentials

One of the primary applications of derivatives is to approximate the value of a function near a known point using its tangent line.

13.1 Linearization (Linear Approximation)

Recall the equation of the tangent line to $y = f(x)$ at the point $(a, f(a))$, assuming f is differentiable at a:

$$y - f(a) = f'(a)(x - a)$$
$$y = f(a) + f'(a)(x - a)$$

This tangent line provides a good approximation of the function $f(x)$ for values of x that are close to a. [Image showing a curve and its tangent line at point 'a', illustrating the approximation]

Definition 13.1 (Linearization)**.** *If f is differentiable at $x = a$, the **linearization** of f at a is the linear function $L(x)$ defined by:*

$$L(x) = f(a) + f'(a)(x - a)$$

*For x near a, we have the **linear approximation** (or **tangent line approximation**):*

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

*This is also called the **local linear approximation** of f at a.*

Example 13.1 (Finding a Linearization)**.** Find the linearization of $f(x) = \sin(2x + x^2)$ at $a = 0$. *Solution:* We need $f(0)$ and $f'(0)$. $f(0) = \sin(2(0) + 0^2) = \sin(0) = 0$. To find $f'(x)$, use the Chain Rule: $f'(x) = \cos(2x + x^2) \cdot \frac{d}{dx}(2x + x^2) = \cos(2x + x^2) \cdot (2 + 2x)$. Now evaluate at $a = 0$: $f'(0) = \cos(0) \cdot (2 + 0) = 1 \cdot 2 = 2$. The linearization is $L(x) = f(0) + f'(0)(x - 0)$:

$$L(x) = 0 + 2(x - 0) = 2x$$

So, for x near 0, $\sin(2x + x^2) \approx 2x$.

Example 13.2 (Using Linearization to Approximate a Value)**.** Use a linear approximation to estimate $\sqrt{4.01}$. *Solution:* We want to approximate $f(x) = \sqrt{x}$ at $x = 4.01$. We should choose a nearby point a where $f(a)$ and $f'(a)$ are easy to compute. A natural choice is $a = 4$. We need the linearization $L(x) = f(a)+f'(a)(x-a)$ at $a = 4$. $f(x) = \sqrt{x} = x^{1/2} \implies f(a) = f(4) = \sqrt{4} = 2$. $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \implies f'(a) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. The linearization at $a = 4$ is:

$$L(x) = f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$$

Now, approximate $f(4.01) = \sqrt{4.01}$ using $L(4.01)$:

$$\sqrt{4.01} \approx L(4.01) = 2 + \frac{1}{4}(4.01 - 4) = 2 + \frac{1}{4}(0.01)$$
$$= 2 + 0.0025 = 2.0025$$

The linear approximation gives $\sqrt{4.01} \approx 2.0025$. (The actual value is approximately 2.002498, so the approximation is quite good).

Example 13.3 (Linearization and Approximation)**.** (a) Find the linearization of $f(x) = \sqrt{x+1}$ at $a = 3$.

(b) Use the linearization to approximate $\sqrt{3.95}$ and $\sqrt{4.01}$.

Solution: (a) Find $f(3)$ and $f'(3)$. $f(x) = (x + 1)^{1/2} \implies f(3) = \sqrt{3+1} = \sqrt{4} = 2$. $f'(x) = \frac{1}{2}(x + 1)^{-1/2} \cdot 1 = \frac{1}{2\sqrt{x+1}}$. $f'(3) = \frac{1}{2\sqrt{3+1}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. The linearization at $a = 3$ is:

$$L(x) = f(3) + f'(3)(x - 3) = 2 + \frac{1}{4}(x - 3)$$

(b) Approximate $\sqrt{3.95}$ and $\sqrt{4.01}$. Note that $f(x) = \sqrt{x+1}$. To approximate $\sqrt{3.95}$, we need $x + 1 = 3.95$, which means $x = 2.95$. This value is close to $a = 3$.

$$\sqrt{3.95} = f(2.95) \approx L(2.95) = 2 + \frac{1}{4}(2.95 - 3) = 2 + \frac{1}{4}(-0.05)$$
$$= 2 - 0.0125 = 1.9875$$

To approximate $\sqrt{4.01}$, we need $x + 1 = 4.01$, which means $x = 3.01$. This value is close to $a = 3$.

$$\sqrt{4.01} = f(3.01) \approx L(3.01) = 2 + \frac{1}{4}(3.01 - 3) = 2 + \frac{1}{4}(0.01)$$
$$= 2 + 0.0025 = 2.0025$$

Example 13.4 (Approximating Logarithm)**.** Use linearization to approximate $\ln(1.1)$. *Solution:* Let $f(x) = \ln x$. We want to approximate $f(1.1)$. Choose a nearby point where f and f' are easy to compute: $a = 1$. $f(x) = \ln x \implies f(1) = \ln 1 = 0$. $f'(x) = \frac{1}{x} \implies f'(1) = \frac{1}{1} = 1$. The linearization at $a = 1$ is:

$$L(x) = f(1) + f'(1)(x - 1) = 0 + 1(x - 1) = x - 1$$

Now approximate $\ln(1.1) = f(1.1)$ using $L(1.1)$:

$$\ln(1.1) \approx L(1.1) = 1.1 - 1 = 0.1$$

(The actual value is approximately 0.0953).

Remark 13.1 (Linearization and Taylor Polynomials)**.** The linearization $L(x)$ is the first-degree Taylor polynomial of $f(x)$ centered at a. Higher-degree Taylor polynomials provide better approximations over larger intervals.

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

The linear approximation is just the first two terms of this series.

13.2 Differentials

Differentials provide an alternative notation and perspective for linear approximation, focusing on the changes in x and y.

Let $y = f(x)$ be a differentiable function. Consider a change in x from x to $x + \Delta x$.

- The **increment** in x is Δx.
- The corresponding exact change in y is $\Delta y = f(x + \Delta x) - f(x)$.

[Image showing dx=x and dy approximating y on a graph]

We define two new quantities, the differentials dx and dy.

Definition 13.2 (Differentials)**.** *Let $y = f(x)$ be a differentiable function.*

- *The **differential of x**, denoted dx, is defined as $dx = \Delta x$, where Δx is any non-zero real number representing the change in x.*
- *The **differential of y**, denoted dy, is defined as:*

$$dy = f'(x)dx$$

Remark 13.2 (Interpreting Differentials)**.** • dx is simply the change in the independent variable, Δx.

- dy represents the change in y along the tangent line when x changes by $dx = \Delta x$.
- Comparing dy with Δy:
$$\Delta y = f(x + \Delta x) - f(x)$$
$$dy = f'(x)\Delta x$$
Since $f(x + \Delta x) \approx L(x + \Delta x) = f(x) + f'(x)\Delta x$, we have
$$f(x + \Delta x) - f(x) \approx f'(x)\Delta x$$
Therefore, $\Delta y \approx dy$.
- The differential dy provides a linear approximation to the actual change Δy. The approximation $\Delta y \approx dy$ is good when $dx = \Delta x$ is small.
- The notation $\frac{dy}{dx} = f'(x)$ can now be interpreted as the ratio of the differentials dy and dx (when $dx \neq 0$).

Example 13.5 (Finding the Differential dy)**.** Find the differential dy for $y = x^2 \cos(3x)$. *Solution:* First, find the derivative $\frac{dy}{dx}$ using the product rule:
$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\cos(3x)) + \cos(3x)\frac{d}{dx}(x^2)$$
$$= x^2(-\sin(3x) \cdot 3) + \cos(3x)(2x)$$
$$= -3x^2 \sin(3x) + 2x \cos(3x)$$
Now, use the definition $dy = f'(x)dx$:
$$dy = (-3x^2 \sin(3x) + 2x \cos(3x))dx$$

Example 13.6. Find dy for $y = x^2 + e^{\sin(2x)}$. *Solution:* Find $\frac{dy}{dx}$:
$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(e^{\sin(2x)})$$
$$= 2x + e^{\sin(2x)} \cdot \frac{d}{dx}(\sin(2x)) \quad \text{(Chain Rule)}$$
$$= 2x + e^{\sin(2x)} \cdot \cos(2x) \cdot \frac{d}{dx}(2x)$$
$$= 2x + e^{\sin(2x)} \cos(2x) \cdot 2$$
$$= 2x + 2\cos(2x)e^{\sin(2x)}$$
Then, $dy = \frac{dy}{dx}dx$:
$$dy = (2x + 2\cos(2x)e^{\sin(2x)})dx$$

Example 13.7 (Comparing Δy and dy)**.** Let $f(x) = 5x^2 + 4x + 1$.

(a) Find Δy and dy.

(b) Compare their values for $x = 6$ and $\Delta x = 0.02$.

Solution: (a) Find $\Delta y = f(x + \Delta x) - f(x)$:

$$\begin{aligned}
\Delta y &= [5(x + \Delta x)^2 + 4(x + \Delta x) + 1] - [5x^2 + 4x + 1] \\
&= [5(x^2 + 2x\Delta x + (\Delta x)^2) + 4x + 4\Delta x + 1] - 5x^2 - 4x - 1 \\
&= 5x^2 + 10x\Delta x + 5(\Delta x)^2 + 4x + 4\Delta x + 1 - 5x^2 - 4x - 1 \\
&= 10x\Delta x + 4\Delta x + 5(\Delta x)^2 \\
&= (10x + 4)\Delta x + 5(\Delta x)^2
\end{aligned}$$

Find $dy = f'(x)dx$: $f'(x) = 10x + 4$. Since $dx = \Delta x$,

$$dy = (10x + 4)dx = (10x + 4)\Delta x$$

(b) Compare for $x = 6$ and $\Delta x = 0.02$.

$$\Delta y = (10(6) + 4)(0.02) + 5(0.02)^2 = (64)(0.02) + 5(0.0004)$$

$$= 1.28 + 0.002 = 1.282$$

$$dy = (10(6) + 4)(0.02) = (64)(0.02) = 1.28$$

The values $\Delta y = 1.282$ and $dy = 1.28$ are very close, as expected for a small Δx. The difference $\Delta y - dy = 5(\Delta x)^2$ represents the error in the linear approximation over the interval Δx.

13.3 Error Propagation using Differentials

Differentials are useful for estimating the error in a calculated quantity (y) that results from a small error in a measurement (x).

If $y = f(x)$, and x is measured with a possible error of $\Delta x \approx dx$, then the resulting propagated error in y is $\Delta y \approx dy$.

$$\text{Propagated Error:} \quad \Delta y \approx dy = f'(x)dx$$

The **relative error** in x is $\frac{\Delta x}{x} \approx \frac{dx}{x}$. The relative error in y is $\frac{\Delta y}{y} \approx \frac{dy}{y}$. The **percentage error** is the relative error multiplied by 100

Example 13.8 (Error Estimation for Volume of a Cube)**.** A side of a cube is measured to be $x = 30$ cm with a possible measurement error of $dx = \pm 0.02$ cm. Estimate the maximum possible error (ΔV) in calculating the volume $V = x^3$. Also find the relative error and percentage error. *Solution:* The volume is $V(x) = x^3$. The propagated error ΔV can be approximated by the differential dV.

$$dV = V'(x)dx$$

First, find $V'(x) = \frac{d}{dx}(x^3) = 3x^2$. Now, evaluate dV at $x = 30$ with $dx = \pm 0.02$:

$$\Delta V \approx dV = (3x^2)dx = (3 \cdot (30)^2)(\pm 0.02)$$

$$= (3 \cdot 900)(\pm 0.02) = (2700)(\pm 0.02) = \pm 54$$

The maximum possible error in the volume is approximately ± 54 cm^3.

The calculated volume at $x = 30$ is $V(30) = 30^3 = 27000$ cm^3. The relative error is:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{\pm 54}{27000} = \pm\frac{54}{27000} = \pm\frac{1}{500} = \pm 0.002$$

The percentage error is:

$$\text{Percentage Error} = \text{Relative Error} \times 100\% = (\pm 0.002) \times 100\% = \pm 0.2\%$$

So, a $0.02/30 \approx 0.067\%$ error in measuring the side leads to an approximate 0.2% error in the calculated volume.

Exercise 13.1 (Error Propagation for Circle Area)**.** The area of a circle is $A = \pi r^2$.

1. If the radius changes from $r = 4$ cm to $r = 5$ cm ($\Delta r = 1$), find the exact change ΔA.
2. Use differentials to approximate the change dA when $r = 4$ and $dr = \Delta r = 1$.
3. Compare the results. Why might the approximation be less accurate here compared to the cube example? (Hint: Consider the relative size of Δr).

Partial Solution: (1) $\Delta A = A(5) - A(4) = \pi(5^2) - \pi(4^2) = 25\pi - 16\pi = 9\pi$. (2) $A'(r) = 2\pi r$. $dA = A'(r)dr = (2\pi r)dr$. At $r = 4$ with $dr = 1$, $dA = (2\pi(4))(1) = 8\pi$. (3) The exact change is 9π, the approximation is 8π. The difference is π. The approximation is less accurate because $\Delta r = 1$ is a relatively large change compared to the initial radius $r = 4$. Differentials give better approximations for smaller increments dr.

Part II
Applications of Differentiation

Derivatives provide powerful tools for analyzing the behavior of functions, including finding maximum and minimum values, determining intervals of increase and decrease, understanding concavity, and solving optimization problems.

14 Mean Value Theorem (MVT)

The Mean Value Theorem is a cornerstone result in differential calculus, linking the average rate of change of a function over an interval to its instantaneous rate of change at some point within that interval. We first introduce a special case, Rolle's Theorem.

14.1 Rolle's Theorem

Theorem 14.1 (Rolle's Theorem)**.** *Suppose a function $f(x)$ satisfies the following three conditions:*

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*
3. *$f(a) = f(b)$.*

Then there exists at least one number c in the open interval (a, b) such that $f'(c) = 0$.

Proof. Since f is continuous on the closed interval $[a, b]$, by the Extreme Value Theorem (Theorem 15.1 below), f must attain an absolute maximum value and an absolute minimum value on $[a, b]$. Let $f(c_{max})$ be the absolute maximum and $f(c_{min})$ be the absolute minimum.

Case 1: The absolute maximum and minimum occur at the endpoints. Since $f(a) = f(b)$, both the maximum and minimum values are equal to $f(a)$. This means $f(x)$ must be a constant function on $[a, b]$, i.e., $f(x) = f(a)$ for all $x \in [a, b]$. For a constant function, the derivative is zero everywhere, so $f'(x) = 0$ for all $x \in (a, b)$. We can choose any $c \in (a, b)$, and we will have $f'(c) = 0$.

Case 2: At least one of the absolute extrema occurs at an interior point $c \in (a, b)$. Suppose the absolute maximum occurs at $c \in (a, b)$. Since $f(c)$ is an absolute maximum, $f(x) \leq f(c)$ for all x near c. By Fermat's Theorem (Theorem 15.2 below), if f has a local extremum at an interior point c and $f'(c)$ exists, then $f'(c) = 0$. Since f is differentiable on (a, b), $f'(c)$ exists.

Therefore, $f'(c) = 0$. A similar argument applies if the absolute minimum occurs at an interior point c.

In either case, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$. □

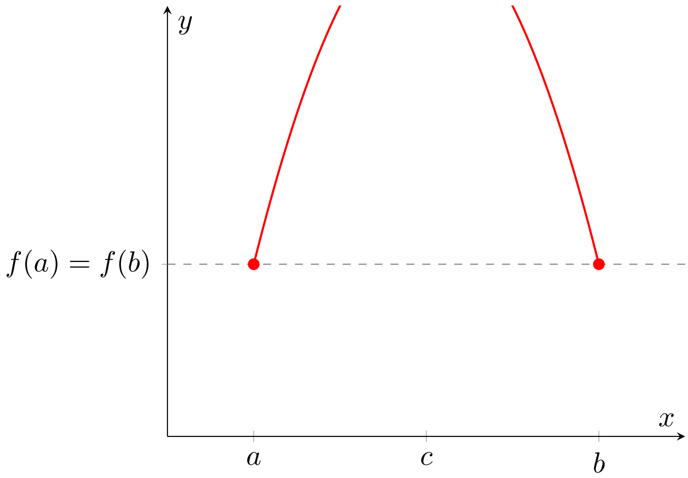


Figure 3: Illustration of Rolle's Theorem. Since $f(a) = f(b)$, there is a point c where the tangent is horizontal.

Example 14.1 (Applying Rolle's Theorem)**.** Determine whether $f(x) = x^3 + x^2$ satisfies the hypotheses of Rolle's Theorem on the interval $[-1, 0]$. If so, find all values c in $(-1, 0)$ that satisfy the conclusion. *Solution:* 1. **Continuity:** $f(x)$ is a polynomial, so it is continuous everywhere, including on $[-1, 0]$. 2. **Differentiability:** $f'(x) = 3x^2 + 2x$. This exists for all x, so f is differentiable everywhere, including on $(-1, 0)$. 3. **Endpoint Values:** $f(-1) = (-1)^3 + (-1)^2 = -1 + 1 = 0$. $f(0) = 0^3 + 0^2 = 0$. So, $f(-1) = f(0)$.

All hypotheses are satisfied. Rolle's Theorem guarantees at least one $c \in (-1, 0)$ such that $f'(c) = 0$. To find c, set $f'(c) = 0$:

$$3c^2 + 2c = 0$$

$$c(3c + 2) = 0$$

The solutions are $c = 0$ and $c = -2/3$. We need c to be in the *open* interval $(-1, 0)$. Only $c = -2/3$ satisfies this condition. Thus, $c = -2/3$ is the value guaranteed by Rolle's Theorem.

Exercise 14.1. Determine whether the given function satisfies the hypotheses of Rolle's Theorem on the indicated interval. If so, find all values c that satisfy the conclusion.

1. $f(x) = -x^3 + x$ on $[-1, 1]$.
2. $f(x) = x - 4x^{1/3}$ on $[-8, 8]$. (Check differentiability carefully).
3. $f(x) = 1 - x^{2/3}$ on $[-1, 1]$. (Check differentiability carefully).

14.2 Statement and Interpretation of the MVT

The Mean Value Theorem generalizes Rolle's Theorem by removing the condition that $f(a) = f(b)$.

Theorem 14.2 (Mean Value Theorem (MVT))**.** *Suppose a function $f(x)$ satisfies the following two conditions:*

1. *f is continuous on the closed interval $[a, b]$.*

2. *f is differentiable on the open interval (a, b).*

Then there exists at least one number c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. The proof involves constructing an auxiliary function $h(x)$ that satisfies the conditions of Rolle's Theorem. The equation of the secant line connecting $(a, f(a))$ and $(b, f(b))$ is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Let $h(x)$ be the vertical difference between the function $f(x)$ and the secant line:

$$h(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right]$$

Now check the conditions of Rolle's Theorem for $h(x)$: 1. **Continuity:** Since $f(x)$ is continuous on $[a, b]$ and the linear function representing the secant line is also continuous, their difference $h(x)$ is continuous on $[a, b]$. 2. **Differentiability:** Similarly, since $f(x)$ and the linear function are differentiable on (a, b), their difference $h(x)$ is differentiable on (a, b). Its derivative is:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

3. **Endpoint Values:** $h(a) = f(a) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(a - a)\right] = f(a) - f(a) = 0$. $h(b) = f(b) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(b - a)\right] = f(b) - [f(a) + (f(b) - f(a))] = f(b) - f(b) = 0$. So, $h(a) = h(b) = 0$.

Since $h(x)$ satisfies all conditions of Rolle's Theorem, there exists at least one $c \in (a, b)$ such that $h'(c) = 0$. Substituting c into the expression for $h'(x)$:

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

Rearranging gives the conclusion of the MVT:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

□

Remark 14.1 (Interpretations of MVT).

- **Geometric Interpretation:** The MVT guarantees that there is at least one point c between a and b where the tangent line to the graph is parallel to the secant line connecting the endpoints $(a, f(a))$ and $(b, f(b))$.
- **Physical Interpretation:** If $f(t)$ represents the position of an object at time t, then $\frac{f(b)-f(a)}{b-a}$ is the average velocity over the time interval $[a, b]$, and $f'(c)$ is the instantaneous velocity at time c. The MVT states that the average velocity over an interval is equal to the instantaneous velocity at some specific time c within that interval.
- Rolle's Theorem is the special case of MVT where the average rate of change $\frac{f(b)-f(a)}{b-a}$ is zero because $f(a) = f(b)$.

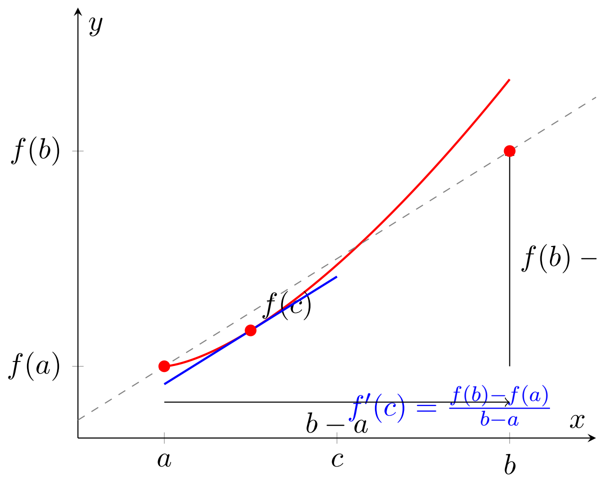


Figure 4: Illustration of the Mean Value Theorem. The slope of the tangent line at c is equal to the slope of the secant line connecting $(a, f(a))$ and $(b, f(b))$.

- The MVT guarantees the *existence* of such a point c, but it does not provide a method to find it, nor does it say that c is unique. There might be multiple such points within (a, b).

Example 14.2 (Applying MVT)**.** Consider $f(x) = x^3 - 12x$ on the interval $[-1, 3]$. Verify the hypotheses of the MVT and find all values $c \in (-1, 3)$ that satisfy the conclusion. *Solution:* 1. **Continuity:** $f(x)$ is a polynomial, so it is continuous everywhere, including on $[-1, 3]$. 2. **Differentiability:** $f'(x) = 3x^2 - 12$. This exists for all x, so f is differentiable everywhere, including on $(-1, 3)$.

Both hypotheses are satisfied. The MVT guarantees at least one $c \in (-1, 3)$ such that $f'(c) = \frac{f(3)-f(-1)}{3-(-1)}$. Calculate the values: $f(3) = 3^3 - 12(3) = 27 - 36 = -9$. $f(-1) = (-1)^3 - 12(-1) = -1 + 12 = 11$. The slope of the secant line is:

$$\frac{f(3) - f(-1)}{3 - (-1)} = \frac{-9 - 11}{4} = \frac{-20}{4} = -5$$

Now, find c such that $f'(c) = -5$:

$$3c^2 - 12 = -5$$
$$3c^2 = 7$$
$$c^2 = \frac{7}{3}$$
$$c = \pm\sqrt{\frac{7}{3}}$$

We need to check which of these values lie in the open interval $(-1, 3)$. $\sqrt{7/3} \approx \sqrt{2.33}$. Since $1^2 = 1$ and $2^2 = 4$, we have $1 < \sqrt{7/3} < 2$. Specifically, $\sqrt{7/3} \approx 1.528$. This value is in $(-1, 3)$. $-\sqrt{7/3} \approx -1.528$. This value is *not* in $(-1, 3)$. Therefore, the only value of c that satisfies the conclusion of the MVT is $c = \sqrt{\frac{7}{3}}$.

Exercise 14.2. Determine whether the given function satisfies the hypotheses of the Mean Value Theorem on the indicated interval. If so, find all values c that satisfy the conclusion.

1. $f(x) = 1 + \sqrt{x}$ on $[0, 9]$.
2. $f(x) = \frac{1}{x}$ on $[-10, 10]$. (Check continuity).

14.3 Consequences of the MVT

The MVT is a powerful tool for proving other important theorems.

Theorem 14.3 (Constant Function Theorem)**.** *Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x)$ is a constant function on $[a, b]$.*

Proof. Let x_1 and x_2 be any two distinct points in $[a, b]$ such that $a \leq x_1 < x_2 \leq b$. Since f is continuous on $[a, b]$ and differentiable on (a, b), it must also be continuous on the subinterval $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Mean Value Theorem applied to the interval $[x_1, x_2]$, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

By hypothesis, $f'(x) = 0$ for all $x \in (a, b)$. Since $c \in (x_1, x_2) \subseteq (a, b)$, we have $f'(c) = 0$. Therefore,

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Since $x_1 \neq x_2$, the denominator $x_2 - x_1$ is non-zero. This implies the numerator must be zero:

$$f(x_2) - f(x_1) = 0 \implies f(x_2) = f(x_1)$$

Since x_1 and x_2 were arbitrary points in $[a, b]$, this shows that the function value $f(x)$ is the same for all $x \in [a, b]$. Hence, $f(x)$ is a constant function on $[a, b]$. ☐

Corollary 14.4 (Functions with the Same Derivative)**.** *If $f'(x) = g'(x)$ for all x in an interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$.*

Proof. Consider the function $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a, b)$. By the Constant Function Theorem, $h(x)$ must be a constant, say C. Thus, $f(x) - g(x) = C$, or $f(x) = g(x) + C$. ☐

This corollary is fundamental to the concept of indefinite integration.

Example 14.3 (Application of MVT: Speeding Ticket)**.** Suppose a car travels between two toll plazas, A (mile marker 226) and B (mile marker 326), covering the 100 miles in 1 hour and 15 minutes (1.25 hours). The speed limit is 65 mph. Can the police issue a speeding ticket based on this information? *Solution:* Let $P(t)$ be the position (mile marker) of the car at time t (in hours), with $t = 0$ at plaza A and $t = 1.25$ at plaza B. $P(0) = 226$ $P(1.25) = 326$ Assume the position function $P(t)$ is continuous and differentiable (reasonable for car motion). By the Mean Value Theorem, there exists some time $c \in (0, 1.25)$ such that the instantaneous velocity $P'(c)$ equals the average velocity:

$$P'(c) = \frac{P(1.25) - P(0)}{1.25 - 0} = \frac{326 - 226}{1.25} = \frac{100}{1.25} = 80 \text{ mph}$$

Since the MVT guarantees that the car's instantaneous speed was exactly 80 mph at some time c during the trip, and 80 mph ¿ 65 mph, the police know the driver exceeded the speed limit at least once.

15 Extrema of Functions

Finding the maximum and minimum values of functions is a central theme in optimization problems.

15.1 Definitions of Extrema

Definition 15.1 (Absolute and Relative Extrema). *Let f be a function defined on a domain $\mathcal{D}$. Let c be a point in $\mathcal{D}$.*

- *$f(c)$ is the **absolute maximum** (or **global maximum**) value of f on $\mathcal{D}$ if $f(c) \geq f(x)$ for all $x \in \mathcal{D}$.*
- *$f(c)$ is the **absolute minimum** (or **global minimum**) value of f on $\mathcal{D}$ if $f(c) \leq f(x)$ for all $x \in \mathcal{D}$.*
- *$f(c)$ is a **relative maximum** (or **local maximum**) value of f if there exists an open interval I containing c such that $f(c) \geq f(x)$ for all $x \in I \cap \mathcal{D}$.*
- *$f(c)$ is a **relative minimum** (or **local minimum**) value of f if there exists an open interval I containing c such that $f(c) \leq f(x)$ for all $x \in I \cap \mathcal{D}$.*

*The term **extremum** (plural: **extrema**) refers to either a maximum or a minimum value.*

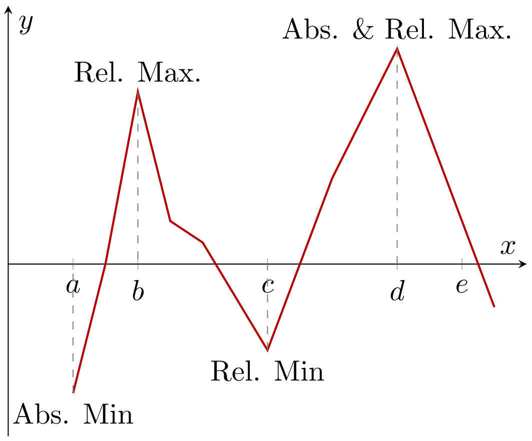


Figure 5: Illustration of absolute and relative extrema on the domain $[a, e]$.

Remark 15.1.

- An absolute extremum is automatically a relative extremum, provided it does not occur at an endpoint of the domain $\mathcal{D}$ (if $\mathcal{D}$ is an interval).
- Relative extrema occur at "peaks" and "valleys" of the graph.

15.2 The Extreme Value Theorem (EVT)

Does a function always have absolute extrema on a given domain? Not necessarily. However, if the domain is a closed interval and the function is continuous, the existence is guaranteed.

Theorem 15.1 (Extreme Value Theorem (EVT)). *If a function f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value and an absolute minimum value on $[a, b]$.*

Remark 15.2 (Importance of Conditions). Both conditions (continuity and closed interval) are essential for the EVT to hold.

- **Continuity is required:** The function $f(x) = \begin{cases} 1/x & -2 \leq x < 0 \text{ or } 0 < x \leq 2 \\ 0 & x = 0 \end{cases}$ is defined on the closed interval $[-2, 2]$ but is discontinuous at $x = 0$. It has no absolute maximum (approaches ∞ near $x = 0^+$) and no absolute minimum (approaches $-\infty$ near $x = 0^-$).

- **Closed interval is required:** The function $f(x) = x^2$ is continuous on the open interval $(1, 2)$. The function values get arbitrarily close to $f(1) = 1$ (from above) and $f(2) = 4$ (from below) but never reach them, as x cannot equal 1 or 2. Thus, f has no absolute minimum or maximum on $(1, 2)$.

- However, a continuous function *can* have absolute extrema on an open interval. For example, $f(x) = \sin x$ on $(-\infty, \infty)$ has an absolute maximum of 1 (at $x = \pi/2 + 2n\pi$) and an absolute minimum of -1 (at $x = 3\pi/2 + 2n\pi$).

Definition 15.2 (Endpoint Extremum). *When an absolute extremum of a function f on a closed interval $[a, b]$ occurs at $x = a$ or $x = b$, it is called an **endpoint extremum**.*

Example 15.1 (EVT Examples). (a) $f(x) = x^2$ on $[1, 2]$. Continuous on a closed interval. EVT applies. Absolute minimum is $f(1) = 1$ (endpoint extremum). Absolute maximum is $f(2) = 4$ (endpoint extremum).

(b) $f(x) = x^2$ on $[-1, 2]$. Continuous on a closed interval. EVT applies. Absolute maximum is $f(2) = 4$ (endpoint extremum). Absolute minimum is $f(0) = 0$. Since $0 \in (-1, 2)$, this minimum occurs at an interior point.

15.3 Critical Numbers and Fermat's Theorem

If an extremum occurs at an interior point of the domain, the derivative often provides information.

Theorem 15.2 (Fermat's Theorem on Local Extrema). *If f has a relative (local) maximum or minimum at an interior point c of its domain, and if $f'(c)$ exists, then $f'(c) = 0$.*

Intuitive Argument. Suppose f has a local maximum at c. For h small and positive, $f(c+h) \leq f(c)$, so $\frac{f(c+h)-f(c)}{h} \leq 0$. Taking the limit as $h \to 0^+$, we expect $f'(c) \leq 0$. For h small and negative ($h < 0$), $f(c + h) \leq f(c)$, so $\frac{f(c+h)-f(c)}{h} \geq 0$ (denominator is negative). Taking the limit as $h \to 0^-$, we expect $f'(c) \geq 0$. Since $f'(c)$ exists, the left and right limits must be equal. The only way $f'(c) \leq 0$ and $f'(c) \geq 0$ can both be true is if $f'(c) = 0$. A similar argument holds for a local minimum. [Image showing slopes near a local max/min] □

Warning.
- *Fermat's Theorem only applies to interior points. Extrema can occur at endpoints where the derivative is not necessarily zero (e.g., $f(x) = x^2$ on $[1, 2]$).*

- *Fermat's Theorem only applies if $f'(c)$ exists. Extrema can occur where the derivative does not exist (e.g., $f(x) = |x|$ has a minimum at $x = 0$, but $f'(0)$ DNE).*

- *The converse of Fermat's Theorem is false: $f'(c) = 0$ does *not* guarantee a local extremum at c. For example, $f(x) = x^3$ has $f'(0) = 0$, but f has neither a max nor a min at $x = 0$.*

Fermat's Theorem motivates the definition of critical numbers, which are the only possible locations for interior extrema.

Definition 15.3 (Critical Number/Point). *Let f be a function defined on a domain $\mathcal{D}$. A number c in the domain $\mathcal{D}$ (i.e., $c \in \mathcal{D}$) is called a **critical number** (or **critical point**) of f if either:*

1. *$f'(c) = 0$, or*

2. *$f'(c)$ does not exist.*

¡/enddefinition

Remark 15.3. By Fermat's Theorem, if f has a relative extremum at an interior point c, then c must be a critical number. However, not all critical numbers correspond to relative extrema.

Example 15.2 (Finding Critical Numbers)**.** Find the critical numbers of $f(x) = (x-1)^{2/3}$. *Solution:* 1. **Domain:** The function $f(x) = \sqrt[3]{(x-1)^2}$ is defined for all real numbers. $\mathcal{D} = \mathbb{R}$. 2. **Derivative:** Use the Power Rule for Functions:

$$f'(x) = \frac{2}{3}(x-1)^{2/3-1} \cdot \frac{d}{dx}(x-1) = \frac{2}{3}(x-1)^{-1/3} \cdot 1 = \frac{2}{3(x-1)^{1/3}}$$

3. **Check Conditions:**

- $f'(c) = 0$: Can $\frac{2}{3(c-1)^{1/3}} = 0$? The numerator is never zero, so this condition yields no critical numbers.
- $f'(c)$ DNE: The derivative $f'(x)$ is undefined when the denominator is zero, which occurs when $(x-1)^{1/3} = 0$, i.e., $x - 1 = 0$, or $x = 1$.

4. **Verify Domain:** Is $x = 1$ in the domain $\mathcal{D} = \mathbb{R}$? Yes. Therefore, the only critical number is $c = 1$.

Example 15.3 (Finding Critical Numbers of a Rational Function)**.** Find the critical numbers of $f(x) = \frac{x^2}{x-1}$. *Solution:* 1. **Domain:** The function is undefined when $x - 1 = 0$, so $x = 1$. The domain is $\mathcal{D} = \{x \in \mathbb{R} \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$. 2. **Derivative:** Use the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(x-1)}{(x-1)^2} = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} \\ &= \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2} \end{aligned}$$

3. **Check Conditions:**

- $f'(c) = 0$: The derivative is zero when the numerator is zero: $c(c-2) = 0$. This occurs at $c = 0$ and $c = 2$.
- $f'(c)$ DNE: The derivative is undefined when the denominator is zero: $(c-1)^2 = 0$. This occurs at $c = 1$.

4. **Verify Domain:** We must check if these potential critical numbers $(0, 2, 1)$ are in the domain $\mathcal{D}$. $c = 0$ is in $\mathcal{D}$. $c = 2$ is in $\mathcal{D}$. $c = 1$ is NOT in $\mathcal{D}$. Therefore, the critical numbers are $c = 0$ and $c = 2$. The point $x = 1$ is where the derivative DNE, but it's not a critical number because it's not in the function's domain.

Exercise 15.1. Find the critical numbers of:

1. $f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + 4$.
2. $f(x) = x \ln x$. (Remember the domain of $\ln x$).

15.4 Finding Absolute Extrema on a Closed Interval

The combination of the Extreme Value Theorem (guaranteeing existence) and Fermat's Theorem (narrowing down interior possibilities) leads to a systematic method for finding the absolute maximum and minimum values of a continuous function on a closed interval.

Theorem 15.3 (Location of Absolute Extrema)**.** *If f is continuous on a closed interval $[a, b]$, then its absolute extrema must occur at either:*

1. *The endpoints of the interval ($x = a$ or $x = b$).*

2. *The critical numbers of f that lie within the open interval (a, b).*

Proof. By the EVT, f must attain an absolute maximum and an absolute minimum on $[a, b]$. Let c be a point where an absolute extremum occurs. If c is one of the endpoints ($c = a$ or $c = b$), the first possibility holds. If c is an interior point ($c \in (a, b)$), then $f(c)$ is also a relative extremum. If $f'(c)$ exists, then by Fermat's Theorem, $f'(c) = 0$. If $f'(c)$ does not exist, then c is a critical number by definition. In either case, if c is an interior point where an extremum occurs, c must be a critical number. Therefore, any absolute extremum must occur at an endpoint or an interior critical number. □

15.4.1 Procedure for Finding Absolute Extrema on $[a, b]$

Based on Theorem 15.3, we can use the following steps to find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. ***Find Critical Numbers:** Find all critical numbers of f by solving $f'(x) = 0$ and finding where $f'(x)$ does not exist.*

2. ***Select Interior Critical Numbers:** Identify the critical numbers that lie within the open interval (a, b). Discard any critical numbers outside this interval.*

3. ***Evaluate Function:** Evaluate the function $f(x)$ at:*
   - *The endpoints a and b.*
   - *All critical numbers selected in Step 2.*

4. ***Compare Values:** The largest value from Step 3 is the absolute maximum value of f on $[a, b]$, and the smallest value is the absolute minimum value.*

*This method is sometimes called the **Closed Interval Method**.*

Example 15.4 (Finding Absolute Extrema)**.** Find the absolute maximum and minimum values of $f(x) = x^3 - 3x^2 - 24x + 2$ on the following intervals:

(a) $[-3, 1]$

(b) $[-3, 8]$

Solution: First, note that $f(x)$ is a polynomial, so it is continuous and differentiable everywhere.
1. *Find Critical Numbers:*

$$f'(x) = 3x^2 - 6x - 24$$

Set $f'(x) = 0$:

$$3x^2 - 6x - 24 = 0$$
$$3(x^2 - 2x - 8) = 0$$
$$3(x - 4)(x + 2) = 0$$

The solutions are $x = 4$ and $x = -2$. Does $f'(x)$ ever not exist? No, because it is a polynomial. The critical numbers are $x = -2$ and $x = 4$.
2. *Apply to Interval (a):* $[-3, 1]$

- Interior Critical Numbers: Check which critical numbers $(-2, 4)$ lie in $(-3, 1)$. Only $x = -2$ is in this interval.

- Evaluate Function:
  - Endpoints: $f(-3) = (-3)^3 - 3(-3)^2 - 24(-3) + 2 = -27 - 3(9) + 72 + 2 = -27 - 27 + 72 + 2 = 20$. $f(1) = (1)^3 - 3(1)^2 - 24(1) + 2 = 1 - 3 - 24 + 2 = -24$.
  - Interior Critical Number: $f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 2 = -8 - 3(4) + 48 + 2 = -8 - 12 + 48 + 2 = 30$.
- Compare Values: The values are $20, -24, 30$. The absolute maximum is 30, occurring at $x = -2$. The absolute minimum is -24, occurring at $x = 1$.

3. *Apply to Interval (b):* $[-3, 8]$

- Interior Critical Numbers: Check which critical numbers $(-2, 4)$ lie in $(-3, 8)$. Both $x = -2$ and $x = 4$ are in this interval.
- Evaluate Function:
  - Endpoints: $f(-3) = 20$ (from part a). $f(8) = (8)^3 - 3(8)^2 - 24(8) + 2 = 512 - 3(64) - 192 + 2 = 512 - 192 - 192 + 2 = 130$.
  - Interior Critical Numbers: $f(-2) = 30$ (from part a). $f(4) = (4)^3 - 3(4)^2 - 24(4) + 2 = 64 - 3(16) - 96 + 2 = 64 - 48 - 96 + 2 = -78$.
- Compare Values: The values are $20, 130, 30, -78$. The absolute maximum is 130, occurring at $x = 8$. The absolute minimum is -78, occurring at $x = 4$.

15.5 Increasing and Decreasing Functions

The sign of the first derivative tells us whether a function is increasing or decreasing.

Definition 15.4 (Increasing/Decreasing Function). *Let f be a function defined on an interval I.*

- *f is **increasing** on I if for any two numbers x_1, x_2 in I,*

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

- *f is **decreasing** on I if for any two numbers x_1, x_2 in I,*

$$x_1 < x_2 \implies f(x_1) > f(x_2)$$

- *f is **monotonic** on I if it is either increasing on I or decreasing on I.*

[Image showing graphs of increasing and decreasing functions]

Theorem 15.4 (Test for Monotonicity). *Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b).*

1. *If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.*
2. *If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.*
3. *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$. (This is Theorem 14.3)*

Proof. Let x_1, x_2 be any two points in $[a, b]$ such that $x_1 < x_2$. Since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), by the Mean Value Theorem (Theorem 14.2), there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Since $x_1 < x_2$, the denominator $x_2 - x_1$ is positive. 1. If $f'(x) > 0$ for all $x \in (a, b)$, then $f'(c) > 0$. This implies $\frac{f(x_2)-f(x_1)}{x_2-x_1} > 0$. Since $x_2 - x_1 > 0$, we must have $f(x_2) - f(x_1) > 0$, or $f(x_1) < f(x_2)$. Since this holds for any $x_1 < x_2$ in $[a, b]$, f is increasing on $[a, b]$. 2. If $f'(x) < 0$ for all $x \in (a, b)$, then $f'(c) < 0$. This implies $\frac{f(x_2)-f(x_1)}{x_2-x_1} < 0$. Since $x_2 - x_1 > 0$, we must have $f(x_2) - f(x_1) < 0$, or $f(x_1) > f(x_2)$. Since this holds for any $x_1 < x_2$ in $[a, b]$, f is decreasing on $[a, b]$. □

Remark 15.4 (Finding Intervals of Increase/Decrease)**.** To find the intervals where a function f is increasing or decreasing:

1. **Find Partition Points:** Determine the values of x for which $f'(x) = 0$ or $f'(x)$ is undefined. These are the critical numbers, but also include points where f itself might be undefined (like vertical asymptotes, although technically the function isn't increasing/decreasing *at* the asymptote). These points partition the domain of f into open intervals.

2. **Test Intervals:** Choose a test value x^* within each interval found in Step 1.

3. **Determine Sign of $f'(x^*)$:** Evaluate the sign of $f'(x^*)$ at the test value.

   - If $f'(x^*) > 0$, then f is increasing on that entire interval.
   - If $f'(x^*) < 0$, then f is decreasing on that entire interval.

   (This works because if f' were to change sign within an interval, it would have to pass through 0 or be undefined, but we have already identified all such points in Step 1).

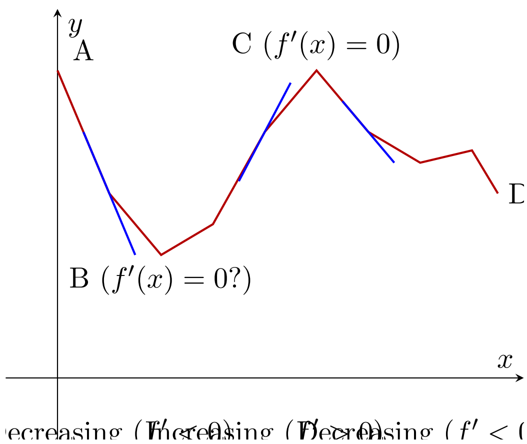


Figure 6: Relationship between the sign of $f'(x)$ and the monotonicity of $f(x)$.

Example 15.5 (Finding Intervals of Increase/Decrease)**.** Find the intervals on which the following functions are increasing and decreasing.

(a) $f(x) = x^3 - 3x^2 - 24x$

(b) $f(x) = \frac{x^2-3}{x^2+1}$

Solution: (a) $f(x) = x^3 - 3x^2 - 24x$. 1. *Find $f'(x)$ and Partition Points:*

$$f'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) = 3(x-4)(x+2)$$

Set $f'(x) = 0 \implies x = 4, x = -2$. $f'(x)$ exists everywhere. The partition points are -2 and 4. The intervals to test are $(-\infty, -2)$, $(-2, 4)$, and $(4, \infty)$. 2. *Test Intervals:*

- Interval $(-\infty, -2)$: Choose $x^* = -3$. $f'(-3) = 3(-3-4)(-3+2) = 3(-7)(-1) = 21 > 0$. $\implies f$ is increasing.
- Interval $(-2, 4)$: Choose $x^* = 0$. $f'(0) = 3(0-4)(0+2) = 3(-4)(2) = -24 < 0$. $\implies f$ is decreasing.
- Interval $(4, \infty)$: Choose $x^* = 5$. $f'(5) = 3(5-4)(5+2) = 3(1)(7) = 21 > 0$. $\implies f$ is increasing.

3. *Conclusion:* f is increasing on $(-\infty, -2]$ and $[4, \infty)$. f is decreasing on $[-2, 4]$. (We use closed brackets because f is continuous everywhere).

Sign Chart Summary:

$+$ $\quad f'(x)$ $\quad -$ $\quad +$

x: -2, 4

Inc. $\quad$ Dec. $\quad$ Inc.

(b) $f(x) = \frac{x^2-3}{x^2+1}$. Domain is $\mathbb{R}$. 1. *Find $f'(x)$ and Partition Points:* Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(x^2+1)\frac{d}{dx}(x^2-3) - (x^2-3)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\ &= \frac{(x^2+1)(2x) - (x^2-3)(2x)}{(x^2+1)^2} \\ &= \frac{2x^3 + 2x - (2x^3 - 6x)}{(x^2+1)^2} \\ &= \frac{2x^3 + 2x - 2x^3 + 6x}{(x^2+1)^2} = \frac{8x}{(x^2+1)^2} \end{aligned}$$

Set $f'(x) = 0 \implies 8x = 0 \implies x = 0$. Does $f'(x)$ ever not exist? The denominator $(x^2+1)^2$ is always positive (since $x^2 \geq 0 \implies x^2 + 1 \geq 1$), so it's never zero. $f'(x)$ exists everywhere. The only partition point is $x = 0$. The intervals are $(-\infty, 0)$ and $(0, \infty)$. 2. *Test Intervals:*

- Interval $(-\infty, 0)$: Choose $x^* = -1$. $f'(-1) = \frac{8(-1)}{((-1)^2+1)^2} = \frac{-8}{(2)^2} = -2 < 0$. $\implies f$ is decreasing.
- Interval $(0, \infty)$: Choose $x^* = 1$. $f'(1) = \frac{8(1)}{((1)^2+1)^2} = \frac{8}{(2)^2} = 2 > 0$. $\implies f$ is increasing.

3. *Conclusion:* f is decreasing on $(-\infty, 0]$. f is increasing on $[0, \infty)$.

Sign Chart Summary:

$-$ $\quad f'(x)$ $+$

x: 0

Dec. $\quad$ Inc.

16 Connecting Derivatives to Graph Shape: Extrema and Concavity

We now combine the concepts of critical numbers and monotonicity to identify relative extrema, and introduce the second derivative to analyze the curvature (concavity) of the graph.

16.1 The First Derivative Test for Relative Extrema

Knowing where a function increases or decreases allows us to identify local peaks and valleys.

Theorem 16.1 (First Derivative Test for Relative Extrema)**.** *Let c be a critical number of a function f that is continuous on an open interval I containing c. Assume f is differentiable on I, except possibly at c.*

1. *If $f'(x)$ changes sign from positive to negative at c (i.e., $f'(x) > 0$ for $x < c$ in I and $f'(x) < 0$ for $x > c$ in I), then f has a **relative maximum** at c.*
2. *If $f'(x)$ changes sign from negative to positive at c (i.e., $f'(x) < 0$ for $x < c$ in I and $f'(x) > 0$ for $x > c$ in I), then f has a **relative minimum** at c.*
3. *If $f'(x)$ does not change sign at c (i.e., $f'(x)$ has the same sign on both sides of c in I), then f has **no relative extremum** at c. (This could be a horizontal inflection point, like at $x = 0$ for $f(x) = x^3$).*

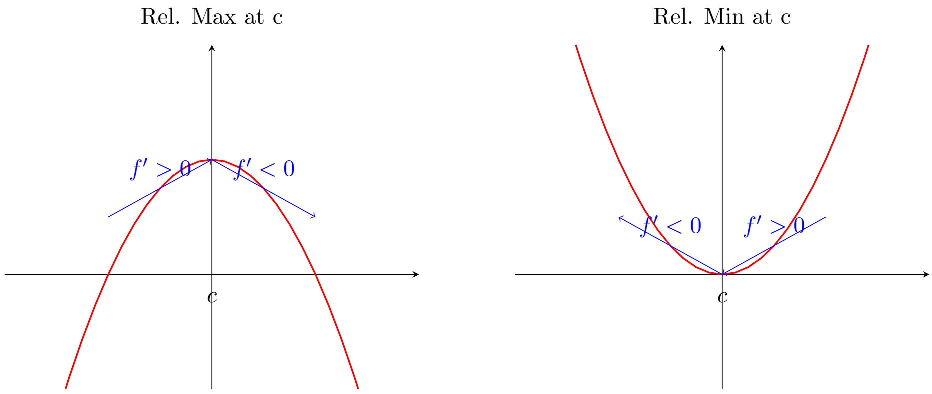
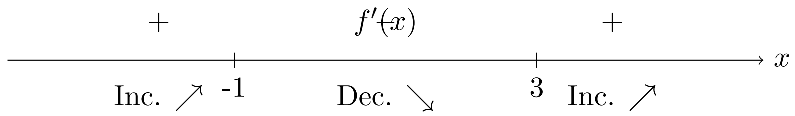


Figure 7: Illustration of the First Derivative Test.

Example 16.1 (Using the First Derivative Test)**.** Find the relative extrema of $f(x) = x^3 - 3x^2 - 9x + 2$. *Solution:* From Example 2.10, we found the critical numbers are $x = -1$ and $x = 3$. We also determined the sign of $f'(x)$ in the intervals $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$.

Sign Chart for $f'(x) = 3(x - 3)(x + 1)$:



Applying the First Derivative Test:

- At $x = -1$: $f'(x)$ changes from $+$ to $-$. Therefore, f has a **relative maximum** at $x = -1$. The maximum value is $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 2 = -1 - 3 + 9 + 2 = 7$.
- At $x = 3$: $f'(x)$ changes from $-$ to $+$. Therefore, f has a **relative minimum** at $x = 3$. The minimum value is $f(3) = (3)^3 - 3(3)^2 - 9(3) + 2 = 27 - 27 - 27 + 2 = -25$.

Theorem 16.2 (The Sole Critical Number Test)**.** *Suppose f is continuous on an interval I and c is the *only* critical number of f in the interior of I.*

1. *If f has a relative maximum at c, then $f(c)$ is the absolute maximum value of f on I.*
2. *If f has a relative minimum at c, then $f(c)$ is the absolute minimum value of f on I.*

This is particularly useful in optimization problems where the domain is an open interval or the entire real line.

Exercise 16.1. Use the First Derivative Test to find all relative extrema of the following functions.

1. $f(x) = x^2 + x - \ln x$ (Domain $x > 0$)
2. $f(x) = -x^{5/3} + 5x^{2/3}$ (Critical numbers at $x = 0, 2$)
3. $f(x) = \sqrt{x}e^{-x/2}$ (Domain $x \geq 0$)

16.2 Concavity and the Second Derivative Test

*While the first derivative tells us about the slope (increasing/decreasing), the second derivative tells us how the slope is changing, which relates to the curvature or **concavity** of the graph.*

16.2.1 Concavity

Definition 16.1 (Concavity). *Let f be differentiable on an open interval (a, b).*

- *The graph of f is **concave upward** (or **convex**) on (a, b) if f' (the slope) is an increasing function on (a, b). This means the tangent lines lie below the curve.*
- *The graph of f is **concave downward** (or **concave**) on (a, b) if f' (the slope) is a decreasing function on (a, b). This means the tangent lines lie above the curve.*

How can we easily determine if f' is increasing or decreasing? By looking at its derivative, which is f''. This leads to the test for concavity.

Theorem 16.3 (Test for Concavity). *Let f be a function for which the second derivative f'' exists on an open interval (a, b).*

1. *If $f''(x) > 0$ for all $x \in (a, b)$, then the graph of f is **concave upward** on (a, b).*
2. *If $f''(x) < 0$ for all $x \in (a, b)$, then the graph of f is **concave downward** on (a, b).*

Proof. This follows directly from applying the Test for Monotonicity (Theorem 15.4) to the function f'. 1. If $f''(x) > 0$ on (a, b), then $f'' = (f')' > 0$ on (a, b). By Theorem 15.4, f' is increasing on (a, b). By Definition 16.1, f is concave upward on (a, b). 2. If $f''(x) < 0$ on (a, b), then $f'' = (f')' < 0$ on (a, b). By Theorem 15.4, f' is decreasing on (a, b). By Definition 16.1, f is concave downward on (a, b). $\square$

16.2.2 Points of Inflection

A point where the concavity changes is called a point of inflection.

Definition 16.2 (Point of Inflection). *Let f be continuous on an open interval containing c. A point $(c, f(c))$ on the graph of f is a **point of inflection** if the graph changes concavity at this point (from upward to downward, or vice versa). This requires that the graph has a tangent line (possibly vertical) at $(c, f(c))$.*

Since concavity is related to the sign of f'', changes in concavity are likely to occur where f'' is zero or undefined.

Theorem 16.4 (Location of Potential Inflection Points). *If $(c, f(c))$ is a point of inflection for the graph of a function f, and if f'' exists in an open interval containing c (except possibly at c itself), then either:*

1. $f''(c) = 0$, *or*
2. $f''(c)$ *does not exist.*

Warning. *The converse is not true. A point where $f''(c) = 0$ or $f''(c)$ DNE is not necessarily an inflection point. The concavity *must* change sign around c. For example, if $f(x) = x^4$, then $f''(x) = 12x^2$. $f''(0) = 0$, but $f''(x) > 0$ for both $x < 0$ and $x > 0$, so the concavity does not change at $x = 0$, and $(0, 0)$ is not an inflection point.*

Remark 16.1 (Finding Inflection Points and Concavity Intervals)**.** To find intervals of concavity and inflection points:

1. **Find Second Derivative:** Calculate $f'(x)$ and $f''(x)$.
2. **Find Potential Inflection Points:** Determine the values of x for which $f''(x) = 0$ or $f''(x)$ is undefined. These are the candidates for the x-coordinates of inflection points.
3. **Test Intervals:** These candidate points partition the domain of f into open intervals. Choose a test value x^* in each interval.
4. **Determine Sign of $f''(x^*)$:** Evaluate the sign of $f''(x^*)$.
   - If $f''(x^*) > 0$, then f is concave upward (CU) on that interval.
   - If $f''(x^*) < 0$, then f is concave downward (CD) on that interval.
5. **Identify Inflection Points:** If the concavity changes sign across a candidate point c (where $f(c)$ is defined), then $(c, f(c))$ is an inflection point.

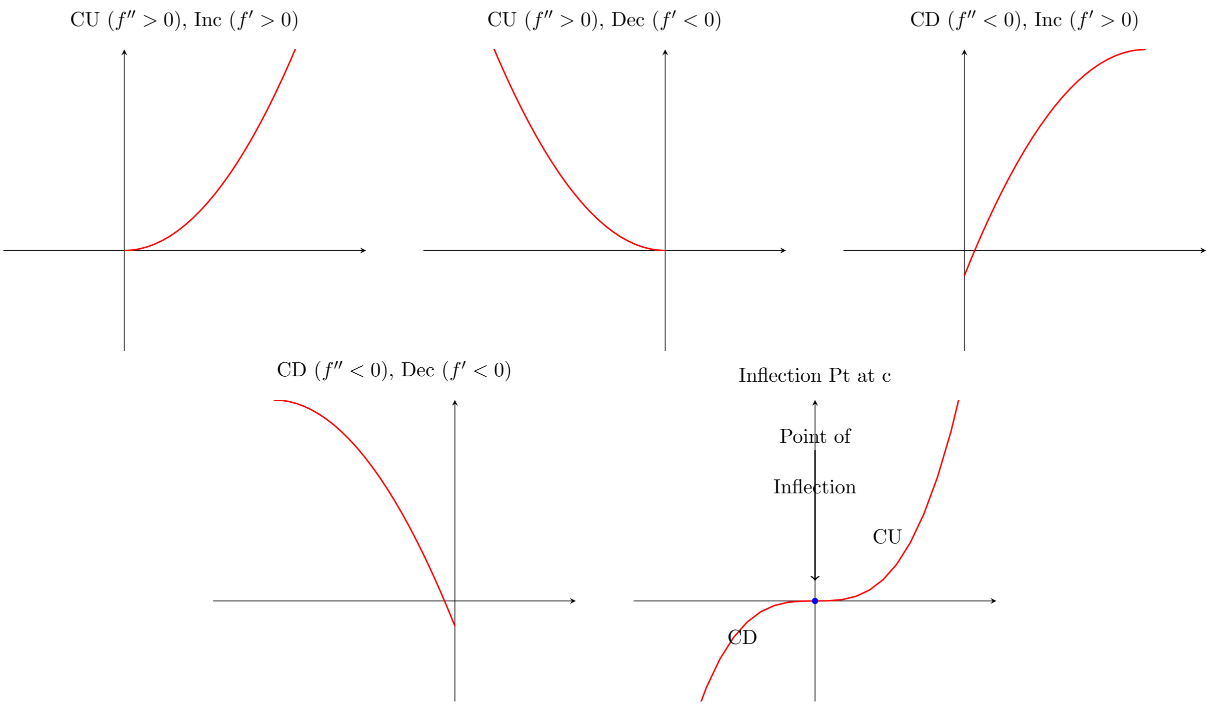


Figure 8: Illustrations of concavity based on signs of f' and f'', and an inflection point.

16.2.3 The Second Derivative Test for Relative Extrema

Concavity provides another way to classify critical numbers where $f'(c) = 0$. If the graph is concave upward at such a point, it must be a local minimum. If it's concave downward, it must be a local maximum.

Theorem 16.5 (Second Derivative Test for Relative Extrema)**.** *Let f be a function such that $f'(c) = 0$ and the second derivative f'' exists on an open interval containing c.*

1. *If $f''(c) > 0$, then f has a **relative minimum** at c. (Graph is CU like $\cup$ near c).*
2. *If $f''(c) < 0$, then f has a **relative maximum** at c. (Graph is CD like $\cap$ near c).*
3. *If $f''(c) = 0$, the test is **inconclusive**. f may have a relative maximum, a relative minimum, or neither at c. In this case, the First Derivative Test (Theorem 16.1) must be used.*

Intuitive Explanation. 1. If $f'(c) = 0$ (horizontal tangent) and $f''(c) > 0$ (concave upward), the function's slope is increasing as it passes through c. It must go from negative (decreasing f) to zero (at c) to positive (increasing f). This pattern corresponds to a relative minimum. 2. If $f'(c) = 0$ (horizontal tangent) and $f''(c) < 0$ (concave downward), the function's slope is decreasing as it passes through c. It must go from positive (increasing f) to zero (at c) to negative (decreasing f). This pattern corresponds to a relative maximum. 3. If $f''(c) = 0$, the concavity at c doesn't immediately tell us the behavior. Consider $f(x) = x^4$ (rel min at $x = 0$), $f(x) = -x^4$ (rel max at $x = 0$), and $f(x) = x^3$ (neither at $x = 0$). All have $f'(0) = 0$ and $f''(0) = 0$. □

Example 16.2 (Applying Concavity Tests and Second Derivative Test)**.** Let $f(x) = x^3 - 3x^2 - 9x + 2$. Determine the intervals of concavity, find any inflection points, and use the Second Derivative Test to classify the relative extrema. *Solution:* From previous examples (Example 2.10, Example **??**), we know: $f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1)$. Critical numbers ($f'(x) = 0$): $x = -1, x = 3$.

1. *Find $f''(x)$ and Potential Inflection Points:*

$$f''(x) = \frac{d}{dx}(3x^2 - 6x - 9) = 6x - 6$$

Set $f''(x) = 0$: $6x - 6 = 0 \implies x = 1$. Does $f''(x)$ ever not exist? No. The only potential inflection point is at $x = 1$.

2. *Determine Concavity Intervals:* The point $x = 1$ divides the number line into $(-\infty, 1)$ and $(1, \infty)$.

- Interval $(-\infty, 1)$: Choose $x^* = 0$. $f''(0) = 6(0) - 6 = -6 < 0$. $\implies$ f is concave downward (CD).
- Interval $(1, \infty)$: Choose $x^* = 2$. $f''(2) = 6(2) - 6 = 6 > 0$. $\implies$ f is concave upward (CU).

Conclusion: f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.

3. *Identify Inflection Points:* Since the concavity changes sign at $x = 1$ (from CD to CU) and f is continuous at $x = 1$, there is an inflection point at $x = 1$. The coordinates are $(1, f(1))$. $f(1) = 1^3 - 3(1)^2 - 9(1) + 2 = 1 - 3 - 9 + 2 = -9$. The inflection point is $(1, -9)$.

4. *Apply Second Derivative Test to Critical Numbers:* We test the critical numbers $x = -1$ and $x = 3$ using $f''(x) = 6x - 6$.

- At $x = -1$: $f''(-1) = 6(-1) - 6 = -12$. Since $f''(-1) < 0$, f has a **relative maximum** at $x = -1$. (Value $f(-1) = 7$).

- At $x = 3$: $f''(3) = 6(3) - 6 = 12$. Since $f''(3) > 0$, f has a **relative minimum** at $x = 3$. (Value $f(3) = -25$).

These results match the classification obtained using the First Derivative Test.

Exercise 16.2. Let $f(x) = x^4 - 4x^3 + 10$.

1. Find the intervals of increase/decrease and classify relative extrema using the First Derivative Test.

2. Find the intervals of concavity and any inflection points.

3. Classify the relative extrema using the Second Derivative Test where possible. Compare with part (a).

Hints: $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$. Critical numbers are $x = 0, x = 3$. $f''(x) = 12x^2 - 24x = 12x(x-2)$. Potential inflection points at $x = 0, x = 2$. Check $f''(3)$ (positive, rel min). Check $f''(0)$ (zero, inconclusive). Use First Deriv Test at $x = 0$ (no sign change, no extremum). Inflection points occur at $x = 0$ and $x = 2$. Concave up on $(-\infty, 0) \cup (2, \infty)$, concave down on $(0, 2)$.

17 Graphing Functions Using Calculus

Calculus provides a systematic way to analyze the key features of a function's graph. Combining information from the function itself, its first derivative, and its second derivative allows for accurate sketching.

17.1 Checklist for Graphing $y = f(x)$

To sketch the graph of f, consider the following aspects:

1. ***Domain:** Determine the set of all x-values for which $f(x)$ is defined. Note any points or intervals excluded.*

2. ***Intercepts:***

   - ***y-intercept:** Evaluate $f(0)$, if 0 is in the domain. The point is $(0, f(0))$.*
   - ***x-intercepts:** Solve the equation $f(x) = 0$ for x. These are the roots or zeros of the function. The points are $(x, 0)$.*

3. ***Symmetry:** (Optional, but helpful)*

   - ***Even function:** $f(-x) = f(x)$ for all x. Symmetric about the y-axis.*
   - ***Odd function:** $f(-x) = -f(x)$ for all x. Symmetric about the origin.*
   - ***Periodic function:** $f(x + P) = f(x)$ for some period P.*

4. ***Asymptotes:***

   - ***Vertical Asymptotes (VA):** Look for values $x = a$ where $\lim_{x \to a^\pm} f(x) = \pm\infty$. Often occur where the denominator of a rational function is zero (after simplification).*
   - ***Horizontal Asymptotes (HA):** Evaluate $\lim_{x \to \infty} f(x) = L_1$ and $\lim_{x \to -\infty} f(x) = L_2$. If L_1 or L_2 is finite, then $y = L_1$ or $y = L_2$ are horizontal asymptotes.*

- ***Slant (Oblique) Asymptotes:** If $f(x)$ is a rational function where the degree of the numerator is exactly one greater than the degree of the denominator, perform polynomial long division $f(x) = (mx+b) + \frac{R(x)}{Q(x)}$. The line $y = mx + b$ is a slant asymptote if $\lim_{x\to\pm\infty} \frac{R(x)}{Q(x)} = 0$. [Image illustrating VA, HA, and Slant Asymptotes]*

5. ***Intervals of Increase/Decrease:***
   - *Find $f'(x)$.*
   - *Find critical numbers ($f'(x) = 0$ or DNE) and partition points.*
   - *Create a sign chart for $f'(x)$ to determine intervals where f is increasing ($f' > 0$) and decreasing ($f' < 0$).*
6. ***Relative Maxima and Minima:***
   - *Use the sign changes from the f' chart (First Derivative Test) or evaluate f'' at critical numbers (Second Derivative Test) to classify local extrema.*
   - *Calculate the y-values $f(c)$ for each relative extremum c.*
7. ***Intervals of Concavity and Inflection Points:***
   - *Find $f''(x)$.*
   - *Find potential inflection points ($f''(x) = 0$ or DNE).*
   - *Create a sign chart for $f''(x)$ to determine intervals where f is concave upward ($f'' > 0$) and concave downward ($f'' < 0$).*
   - *Identify inflection points where concavity changes. Calculate the y-values $f(c)$ for each inflection point c.*
8. ***Sketch the Graph:***
   - *Draw dashed lines for all asymptotes.*
   - *Plot intercepts, relative extrema, and inflection points.*
   - *Sketch the curve through these points, ensuring it follows the correct increasing/decreasing behavior and concavity determined by the sign charts.*
   - *Check that the behavior near asymptotes matches the limits.*

Example 17.1 (Review Asymptotes/Intercepts)**.** (Exercise from notes) Consider $f(x) = \frac{x^2+5x+4}{x^2}$.

1. Find the domain.
2. Find x-intercepts and y-intercepts.
3. Find vertical and horizontal asymptotes.

Solution Steps: 1. Domain: Denominator $x^2 = 0$ only at $x = 0$. Domain is $(-\infty, 0) \cup (0, \infty)$. 2. Intercepts: - y-intercept: $x = 0$ is not in the domain, so there is no y-intercept. - x-intercepts: Set $f(x) = 0$. $\frac{x^2+5x+4}{x^2} = 0 \implies x^2 + 5x + 4 = 0$. Factor: $(x+1)(x+4) = 0$. Solutions $x = -1, x = -4$. Intercepts are $(-1, 0)$ and $(-4, 0)$. 3. Asymptotes: - Vertical: Check behavior near $x = 0$ (where denominator was zero). $f(x) = \frac{(x+1)(x+4)}{x^2}$. As $x \to 0$, numerator $\to (1)(4) = 4$. Denominator $x^2 \to 0$ through positive values. $\lim_{x\to 0} f(x) = \frac{4}{0^+} = +\infty$. So, $x = 0$ (the y-axis) is a vertical asymptote. - Horizontal: Check limits as $x \to \pm\infty$. Use highest powers: $\lim_{x\to\infty} \frac{x^2+5x+4}{x^2} = \lim_{x\to\infty} \frac{x^2(1+5/x+4/x^2)}{x^2(1)} = \lim_{x\to\infty}(1 + 5/x + 4/x^2) = 1 + 0 + 0 = 1$. $\lim_{x\to-\infty} \frac{x^2+5x+4}{x^2} = \cdots = 1$. So, $y = 1$ is a horizontal asymptote (for both $x \to \infty$ and $x \to -\infty$).

Warning

1 Graphing Examples

Let's apply the checklist to sketch the graphs of some functions.

1.1 Example 1: $f(x) = 4x^4 - 4x^2$

Solution:

1. **Domain:** All real numbers, $(-\infty, \infty)$.
2. **Intercepts:**
   - y-intercept: $(0, 0)$
   - x-intercepts: Solve $4x^4 - 4x^2 = 0 \implies x = 0, \pm 1$; points $(0, 0), (1, 0), (-1, 0)$
3. **Symmetry:** Even function, symmetric about the y-axis.
4. **Asymptotes:** None (polynomial), $\lim_{x\to\pm\infty} f(x) = \infty$.
5. **Intervals of Increase/Decrease:**

   $$f'(x) = 16x^3 - 8x = 8x(2x^2 - 1) = 8x(\sqrt{2}x - 1)(\sqrt{2}x + 1)$$

   Critical points: $x = 0, \pm 1/\sqrt{2} \approx \pm 0.707$
   - Increasing: $[-1/\sqrt{2}, 0] \cup [1/\sqrt{2}, \infty)$
   - Decreasing: $(-\infty, -1/\sqrt{2}] \cup [0, 1/\sqrt{2}]$
6. **Relative Extrema:**
   - Rel min: $(-1/\sqrt{2}, -1), (1/\sqrt{2}, -1)$
   - Rel max: $(0, 0)$
7. **Concavity and Inflection Points:**

   $$f''(x) = 48x^2 - 8 = 8(6x^2 - 1)$$

   Inflection points: $x = \pm 1/\sqrt{6} \approx \pm 0.408$, $f(\pm 1/\sqrt{6}) \approx -0.556$. Concave up: $(-\infty, -1/\sqrt{6}) \cup (1/\sqrt{6}, \infty)$
   Concave down: $(-1/\sqrt{6}, 1/\sqrt{6})$

1.2 Example 2: $f(x) = 3x^{2/3} - x$

Solution:

1. Domain: $(-\infty, \infty)$
2. Intercepts:
   - $(0, 0)$ (cusp)
   - $(27, 0)$
3. Symmetry: None

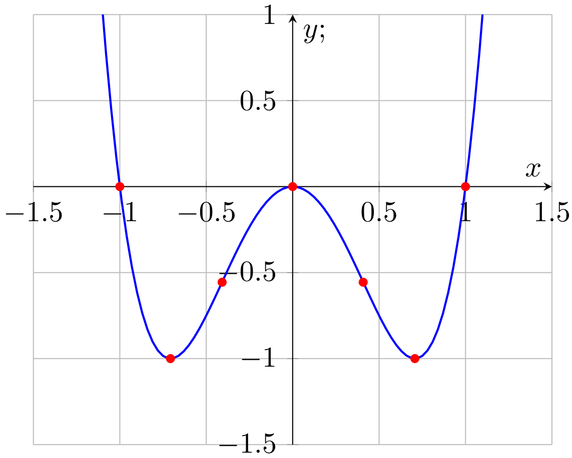


Figure 1: Sketch of $f(x) = 4x^4 - 4x^2$.

4. Asymptotes: None; $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$
5. Intervals of Increase/Decrease:

$$f'(x) = 2x^{-1/3} - 1$$

   Critical points: $x = 0$ (DNE), $x = 8$

   - Increasing: $(0, 8)$
   - Decreasing: $(-\infty, 0) \cup (8, \infty)$

6. Relative Extrema:
   - Rel min: $(0, 0)$ (cusp)
   - Rel max: $(8, 4)$

7. Concavity: $f''(x) = -\frac{2}{3x^{4/3}} < 0$ for $x \neq 0$; concave down everywhere; no inflection points

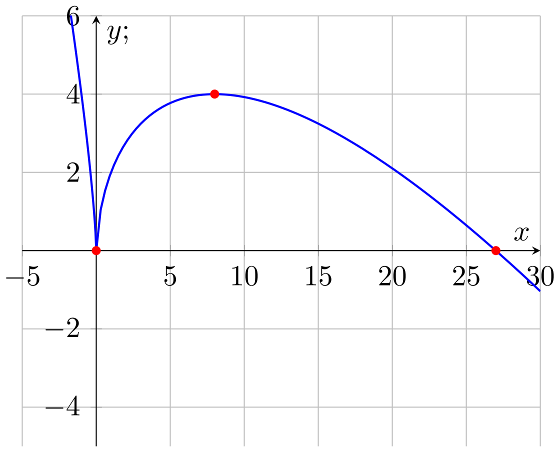


Figure 2: Sketch of $f(x) = 3x^{2/3} - x$.

1.3 Example 3: $f(x) = \frac{x^2-3}{x^2-1}$

Solution:

1. Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
2. Intercepts:
   - y-intercept: $(0, 3)$
   - x-intercepts: $(\pm\sqrt{3}, 0)$
3. Symmetry: Even function, symmetric about y-axis
4. Asymptotes:
   - Vertical: $x = -1, 1$
   - Horizontal: $y = 1$
5. Intervals of Increase/Decrease:
$$f'(x) = \frac{4x}{(x^2-1)^2}$$
   Critical points: $x = 0$; $f'(x)$ DNE at $x = \pm 1$
   - Increasing: $(0, 1), (1, \infty)$
   - Decreasing: $(-\infty, -1), (-1, 0)$
6. Relative Extrema: $(0, 3)$ minimum
7. Concavity:
$$f''(x) = \frac{-4(3x^2+1)}{(x^2-1)^3}$$
   Concave up: $(-1, 1)$, concave down: $(-\infty, -1) \cup (1, \infty)$; no inflection points

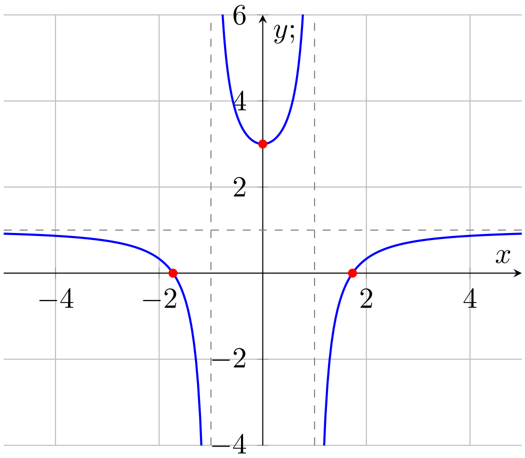


Figure 3: Sketch of $f(x) = \frac{x^2-3}{x^2-1}$.

2 Exercises

Sketch the graphs of the following functions using the analysis techniques:

1. $f(x) = \frac{x^2-3}{x^2+1}$
2. $f(x) = x^2 + x - \ln(|x|)$
3. $f(x) = -x^{5/3} + 5x^{2/3}$
4. $f(x) = x^4 - 4x^3 + 10$
5. $f(x) = 2\cos x - \cos(2x)$, $x \in [0, 2\pi]$

3 Logarithmic Differentiation

The Power Rule $\frac{d}{dx}(x^n) = nx^{n-1}$ applies when the base is a variable and the exponent is a constant. The Exponential Rule $\frac{d}{dx}(b^x) = b^x \ln b$ applies when the base is a constant and the exponent is a variable. Neither rule applies directly to functions where both the base and the exponent are variables, such as $y = x^x$ or $y = (\sin x)^x$.

Logarithmic differentiation is a technique used to differentiate such functions, or functions that involve complex products, quotients, and powers, by simplifying them using logarithm properties before differentiating.

3.1 Steps for Logarithmic Differentiation

To differentiate $y = f(x)$ using logarithmic differentiation:

1. **Take Natural Logarithm:** Take the natural logarithm (ln) of both sides of the equation $y = f(x)$:
$$\ln y = \ln(f(x))$$
Assume $y > 0$. If y could be negative, use $\ln|y| = \ln|f(x)|$.

2. **Simplify:** Use properties of logarithms to simplify the right-hand side, $\ln(f(x))$. Key properties include:
   - $\ln(a^b) = b \ln a$
   - $\ln(ab) = \ln a + \ln b$
   - $\ln(a/b) = \ln a - \ln b$

   This step often converts products into sums, quotients into differences, and powers/roots into coefficients.

3. **Differentiate Implicitly:** Differentiate both sides of the simplified equation with respect to x. Remember to use the Chain Rule for the left side: $\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$. Differentiate the right side using standard rules.

4. **Solve for $\frac{dy}{dx}$:** Algebraically solve the resulting equation for $\frac{dy}{dx}$.

5. **Substitute Back:** Replace y with the original function $f(x)$ to express $\frac{dy}{dx}$ entirely in terms of x.

Remark 3.1. The final result can often be expressed as:

$$\frac{dy}{dx} = y \cdot \frac{d}{dx}[\ln(f(x))] = f(x) \cdot \frac{d}{dx}[\ln(f(x))]$$

This shows that logarithmic differentiation essentially involves multiplying the original function by the derivative of its natural logarithm.

Example 3.1 (Differentiating $y = x^x$)**.** Find the derivative of $y = x^x$ for $x > 0$. *Solution:* This function has a variable base and a variable exponent. 1. *Take Logarithm:* $\ln y = \ln(x^x)$. 2. *Simplify:* $\ln y = x \ln x$. 3. *Differentiate Implicitly:* Differentiate both sides with respect to x. Use the Product Rule on the right side.

$$\begin{aligned}\frac{d}{dx}(\ln y) &= \frac{d}{dx}(x \ln x)\\ \frac{1}{y}\frac{dy}{dx} &= x\frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(x)\\ \frac{1}{y}\frac{dy}{dx} &= x\left(\frac{1}{x}\right) + (\ln x)(1)\\ \frac{1}{y}\frac{dy}{dx} &= 1 + \ln x\end{aligned}$$

4. *Solve for $\frac{dy}{dx}$:*

$$\frac{dy}{dx} = y(1 + \ln x)$$

5. *Substitute Back:* Replace y with x^x.

$$\frac{dy}{dx} = x^x(1 + \ln x)$$

Thus, $\frac{d}{dx}(x^x) = x^x(1 + \ln x)$.

Example 3.2 (Simplifying Complex Fractions before Differentiating)**.** Differentiate $y = \ln\left(\frac{x^2}{\sqrt{(x-1)^3(2x+1)^5}}\right)$. *Solution:* While we could use the Chain Rule directly, it's much easier to simplify using log properties first.

$$\begin{aligned}y &= \ln(x^2) - \ln(\sqrt{(x-1)^3(2x+1)^5})\\ &= 2\ln x - \ln((x-1)^{3/2}(2x+1)^{5/2})\\ &= 2\ln x - \left[\ln((x-1)^{3/2}) + \ln((2x+1)^{5/2})\right]\\ &= 2\ln x - \frac{3}{2}\ln(x-1) - \frac{5}{2}\ln(2x+1)\end{aligned}$$

This expression is much simpler to differentiate:

$$\begin{aligned}\frac{dy}{dx} &= 2\frac{d}{dx}(\ln x) - \frac{3}{2}\frac{d}{dx}(\ln(x-1)) - \frac{5}{2}\frac{d}{dx}(\ln(2x+1))\\ &= 2\left(\frac{1}{x}\right) - \frac{3}{2}\left(\frac{1}{x-1} \cdot 1\right) - \frac{5}{2}\left(\frac{1}{2x+1} \cdot 2\right)\\ &= \frac{2}{x} - \frac{3}{2(x-1)} - \frac{5}{2x+1}\end{aligned}$$

Example 3.3 (Logarithmic Differentiation for Complex Products/Quotients)**.** Let a, b, c, d be constants. Differentiate $f(x) = \frac{(x-2)^a(x-3)^b}{(x+4)^c(x+5)^d}$. *Solution:* Direct differentiation using quotient and product rules would be very complex. Use logarithmic differentiation. Let $y = f(x)$. 1. *Take Logarithm:* $\ln y = \ln\left(\frac{(x-2)^a(x-3)^b}{(x+4)^c(x+5)^d}\right)$. 2. *Simplify:*

$$\ln y = \ln((x-2)^a) + \ln((x-3)^b) - \ln((x+4)^c) - \ln((x+5)^d)$$

$$\ln y = a\ln(x-2) + b\ln(x-3) - c\ln(x+4) - d\ln(x+5)$$

3. *Differentiate Implicitly:*

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[a\ln(x-2) + b\ln(x-3) - c\ln(x+4) - d\ln(x+5)]$$

$$\frac{1}{y}\frac{dy}{dx} = a\left(\frac{1}{x-2}\right) + b\left(\frac{1}{x-3}\right) - c\left(\frac{1}{x+4}\right) - d\left(\frac{1}{x+5}\right)$$

4. *Solve for* $\frac{dy}{dx}$*:*

$$\frac{dy}{dx} = y\left[\frac{a}{x-2} + \frac{b}{x-3} - \frac{c}{x+4} - \frac{d}{x+5}\right]$$

5. *Substitute Back:*

$$\frac{dy}{dx} = \frac{(x-2)^a(x-3)^b}{(x+4)^c(x+5)^d}\left[\frac{a}{x-2} + \frac{b}{x-3} - \frac{c}{x+4} - \frac{d}{x+5}\right]$$

Example 3.4 (Tangent Line using Logarithmic Differentiation)**.** Find the tangent line to the graph of $y = x(\ln x)^x$ at $x = e$. *Solution:* First, find the point: $y(e) = e(\ln e)^e = e(1)^e = e$. The point is (e, e). Next, find the slope $\frac{dy}{dx}$ at $x = e$ using logarithmic differentiation. 1. *Take Logarithm:* $\ln y = \ln(x(\ln x)^x)$. 2. *Simplify:* $\ln y = \ln x + \ln((\ln x)^x) = \ln x + x\ln(\ln x)$. 3. *Differentiate Implicitly:*

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\ln x) + \frac{d}{dx}(x\ln(\ln x))$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \left[x\frac{d}{dx}(\ln(\ln x)) + \ln(\ln x)\frac{d}{dx}(x)\right]$$

For $\frac{d}{dx}(\ln(\ln x))$, let $u = \ln x$, $\frac{du}{dx} = 1/x$. $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x}$.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \left[x\left(\frac{1}{x\ln x}\right) + \ln(\ln x) \cdot 1\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\ln x} + \ln(\ln x)$$

4. *Solve for* $\frac{dy}{dx}$*:*

$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{1}{\ln x} + \ln(\ln x)\right]$$

5. *Substitute Back:*

$$\frac{dy}{dx} = x(\ln x)^x\left[\frac{1}{x} + \frac{1}{\ln x} + \ln(\ln x)\right]$$

Now, evaluate the slope at $x = e$. Recall $\ln e = 1$.

$$m = \left.\frac{dy}{dx}\right|_{x=e} = e(\ln e)^e\left[\frac{1}{e} + \frac{1}{\ln e} + \ln(\ln e)\right]$$

$$m = e(1)^e \left[\frac{1}{e} + \frac{1}{1} + \ln(1)\right] = e\left[\frac{1}{e} + 1 + 0\right] = e(\frac{1}{e}) + e(1) = 1 + e$$

The slope is $m = 1 + e$. The point is (e, e). Tangent line equation: $y - y_0 = m(x - x_0)$

$$y - e = (1 + e)(x - e)$$

$$y = (1 + e)x - (1 + e)e + e = (1 + e)x - e - e^2 + e$$
$$y = (1 + e)x - e^2$$

Exercise 3.1. Differentiate $y = \frac{(\sin x+1)^x}{x^3}$ using logarithmic differentiation.

4 Derivatives of Hyperbolic Functions

Hyperbolic functions are analogs of trigonometric functions defined using the exponential function e^x.

Definition 4.1 (Hyperbolic Sine and Cosine)**.** *For any real number x:*

- *Hyperbolic sine:* $\sinh x = \frac{e^x - e^{-x}}{2}$
- *Hyperbolic cosine:* $\cosh x = \frac{e^x + e^{-x}}{2}$

Remark 4.1.
- The domain for both $\sinh x$ and $\cosh x$ is $\mathbb{R} = (-\infty, \infty)$.
- $\sinh(0) = (e^0 - e^0)/2 = 0$.
- $\cosh(0) = (e^0 + e^0)/2 = (1 + 1)/2 = 1$.
- $\cosh x \geq 1$ for all x. ($\cosh x = \sqrt{1 + \sinh^2 x}$)

Definition 4.2 (Other Hyperbolic Functions)**.** *Analogous to trigonometric functions:*

- *Hyperbolic tangent:* $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
- *Hyperbolic cotangent:* $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \neq 0$
- *Hyperbolic secant: sech* $x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
- *Hyperbolic cosecant: csch* $x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad x \neq 0$

4.1 Identities for Hyperbolic Functions

Hyperbolic functions satisfy identities similar, but not identical, to trigonometric identities.

- $\cosh^2 x - \sinh^2 x = 1$
- $1 - \tanh^2 x = \text{sech}^2 x$
- $\coth^2 x - 1 = \text{csch}^2 x$
- $\sinh(-x) = -\sinh x$ (Odd function)
- $\cosh(-x) = \cosh x$ (Even function)
- $\tanh(-x) = -\tanh x$ (Odd function)
- $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

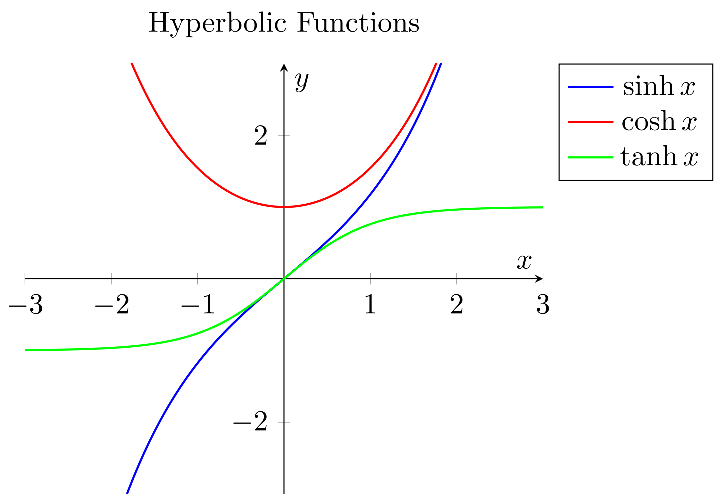


Figure 4: Graphs of $\sinh x$, $\cosh x$, and $\tanh x$.

- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
- $\sinh(2x) = 2 \sinh x \cosh x$
- $\cosh(2x) = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1$

Proof of $\cosh^2 x - \sinh^2 x = 1$.

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} \\
&= \frac{4}{4} = 1
\end{aligned}$$

□

4.2 Derivatives of Hyperbolic Functions

The derivatives can be found directly from the definitions using the derivative of e^x.

Example 4.1 (Derivative of $\sinh x$)**.** Differentiate $y = \sinh x$. *Solution:*

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left(\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right)$$

$$= \frac{1}{2}(e^x - (e^{-x} \cdot (-1))) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

So, $\frac{d}{dx}(\sinh x) = \cosh x$.

Theorem 4.1 (Derivatives of Hyperbolic Functions)**.** *Let $u = g(x)$ be a differentiable function.*

- $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$

- $\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$
- $\frac{d}{dx}(\tanh u) = sech^2 u \frac{du}{dx}$
- $\frac{d}{dx}(\coth u) = -csch^2 u \frac{du}{dx}$
- $\frac{d}{dx}(sech\, u) = -sech\, u \tanh u \frac{du}{dx}$
- $\frac{d}{dx}(csch\, u) = -csch\, u \coth u \frac{du}{dx}$

Remark 4.2 (Comparison with Trigonometric Derivatives)**.** Notice the similarities and differences in signs compared to trigonometric derivatives. For example, $\frac{d}{dx}(\cosh x) = \sinh x$ (no minus sign), while $\frac{d}{dx}(\cos x) = -\sin x$.

Derivation of $\frac{d}{dx}(\tanh x)$. Using the quotient rule and the derivatives of $\sinh x$ and $\cosh x$:

$$\begin{aligned}
\frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) \\
&= \frac{\cosh x \frac{d}{dx}(\sinh x) - \sinh x \frac{d}{dx}(\cosh x)}{\cosh^2 x} \\
&= \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x} \\
&= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\
&= \frac{1}{\cosh^2 x} \quad (\text{using } \cosh^2 x - \sinh^2 x = 1) \\
&= \text{sech}^2 x
\end{aligned}$$

The Chain Rule version follows directly. □

Example 4.2 (Applying Hyperbolic Derivative Rules)**.** Differentiate:

(a) $y = \sinh(\sqrt{2x+1})$

(b) $y = \coth(x^3)$

Solution: (a) Use $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$ with $u = \sqrt{2x+1} = (2x+1)^{1/2}$. $\frac{du}{dx} = \frac{1}{2}(2x+1)^{-1/2} \cdot 2 = (2x+1)^{-1/2} = \frac{1}{\sqrt{2x+1}}$.

$$\frac{dy}{dx} = \cosh(\sqrt{2x+1}) \cdot \frac{1}{\sqrt{2x+1}}$$

(b) Use $\frac{d}{dx}(\coth u) = -\text{csch}^2 u \frac{du}{dx}$ with $u = x^3$. $\frac{du}{dx} = 3x^2$.

$$\frac{dy}{dx} = -\text{csch}^2(x^3) \cdot (3x^2) = -3x^2 \text{csch}^2(x^3)$$

4.3 Inverse Hyperbolic Functions

Since $\sinh x$ and $\tanh x$ are strictly increasing, they are one-to-one and have inverse functions, denoted $\sinh^{-1} x$ (or arsinh x) and $\tanh^{-1} x$ (or artanh x). The function $\cosh x$ is not one-to-one on $\mathbb{R}$, but if we restrict its domain to $x \geq 0$ (where its range is $y \geq 1$), it becomes one-to-one and has an inverse $\cosh^{-1} x$ (or arcosh x) defined for $x \geq 1$. Similarly, inverses for $\coth x$, sech x, and csch x can be defined on appropriate restricted domains.

Theorem 4.2 (Logarithmic Forms of Inverse Hyperbolic Functions)**.** *Inverse hyperbolic functions can be expressed using natural logarithms:*

- $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ *for all* $x \in \mathbb{R}$
- $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ *for* $x \geq 1$
- $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ *for* $|x| < 1$
- $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ *for* $|x| > 1$
- $sech^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ *for* $0 < x \leq 1$
- $csch^{-1}x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$ *for* $x \neq 0$

Derivation of $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$. Let $y = \sinh^{-1} x$. Then $x = \sinh y$. By definition, $x = \frac{e^y - e^{-y}}{2}$. We want to solve for y. Multiply by $2e^y$ to clear denominators and negative exponents:

$$2xe^y = (e^y - e^{-y})e^y$$
$$2xe^y = (e^y)^2 - e^0 = (e^y)^2 - 1$$

Rearrange into a quadratic equation in terms of e^y:

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Let $Z = e^y$. The equation is $Z^2 - 2xZ - 1 = 0$. Use the quadratic formula to solve for Z:

$$Z = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)} = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$Z = \frac{2x \pm \sqrt{4(x^2+1)}}{2} = \frac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1}$$

Since $Z = e^y$ must be positive, and $\sqrt{x^2+1} > \sqrt{x^2} = |x|$, the term $x - \sqrt{x^2+1}$ is always negative (if $x \geq 0$, $x - \sqrt{x^2+1} < 0$; if $x < 0$, $x - \sqrt{x^2+1} < 0$). Therefore, we must choose the positive sign:

$$e^y = x + \sqrt{x^2+1}$$

Taking the natural logarithm of both sides gives:

$$y = \ln(x + \sqrt{x^2+1})$$

Thus, $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$. □

4.4 Derivatives of Inverse Hyperbolic Functions

The derivatives can be found either by differentiating the logarithmic forms or by using implicit differentiation.

Theorem 4.3 (Derivatives of Inverse Hyperbolic Functions)**.** *Let $u = g(x)$ be a differentiable function.*

- $\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$
- $\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$
- $\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$

- $\frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$
- $\frac{d}{dx}(sech^{-1}u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$
- $\frac{d}{dx}(csch^{-1}u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$

Proof of $\frac{d}{dx}(\sinh^{-1} x)$ using Implicit Differentiation. Let $y = \sinh^{-1} x$. Then $x = \sinh y$. Differentiate both sides with respect to x:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sinh y)$$

$$1 = \cosh y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

Use the identity $\cosh^2 y - \sinh^2 y = 1$, which gives $\cosh^2 y = 1 + \sinh^2 y$. Since $\cosh y$ is always positive (its range is $[1, \infty)$), $\cosh y = \sqrt{1 + \sinh^2 y}$. Substitute $x = \sinh y$: $\cosh y = \sqrt{1 + x^2}$. Therefore,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$$

The Chain Rule version follows directly. □

Proof of $\frac{d}{dx}(\tanh^{-1} x)$ using Logarithmic Form. From the logarithmic form, $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}[\ln(1 + x) - \ln(1 - x)]$. Differentiate with respect to x for $|x| < 1$:

$$\begin{aligned}
\frac{d}{dx}(\tanh^{-1} x) &= \frac{1}{2}\left[\frac{d}{dx}(\ln(1 + x)) - \frac{d}{dx}(\ln(1 - x))\right] \\
&= \frac{1}{2}\left[\frac{1}{1 + x} \cdot 1 - \frac{1}{1 - x} \cdot (-1)\right] \\
&= \frac{1}{2}\left[\frac{1}{1 + x} + \frac{1}{1 - x}\right] \\
&= \frac{1}{2}\left[\frac{(1 - x) + (1 + x)}{(1 + x)(1 - x)}\right] \\
&= \frac{1}{2}\left[\frac{2}{1 - x^2}\right] = \frac{1}{1 - x^2}
\end{aligned}$$

The Chain Rule version follows directly. □

Example 4.3 (Applying Inverse Hyperbolic Derivative Rules)**.** Differentiate $y = \sinh^{-1}(e^x + \sin(2x))$. *Solution:* Use $\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$ with $u = e^x + \sin(2x)$. First find $\frac{du}{dx}$:

$$\frac{du}{dx} = \frac{d}{dx}(e^x + \sin(2x)) = e^x + \cos(2x) \cdot 2 = e^x + 2\cos(2x)$$

Now apply the formula:

$$\frac{dy}{dx} = \frac{1}{\sqrt{(e^x + \sin(2x))^2 + 1}} \cdot (e^x + 2\cos(2x))$$

$$\frac{dy}{dx} = \frac{e^x + 2\cos(2x)}{\sqrt{(e^x + \sin(2x))^2 + 1}}$$

Exercise 4.1. Differentiate $y = (\tanh^{-1}(x^x))^{-1/3}$. *Hint:* Use Power Rule for Functions, Chain Rule for $\tanh^{-1} u$, and the derivative of x^x found earlier using logarithmic differentiation. Let $v = \tanh^{-1}(x^x)$, so $y = v^{-1/3}$. $\frac{dy}{dx} = -\frac{1}{3}v^{-4/3}\frac{dv}{dx}$. Let $u = x^x$, so $v = \tanh^{-1}(u)$. $\frac{dv}{dx} = \frac{1}{1-u^2}\frac{du}{dx} = \frac{1}{1-(x^x)^2}\frac{d}{dx}(x^x)$. Recall $\frac{d}{dx}(x^x) = x^x(1+\ln x)$. Combine: $\frac{dy}{dx} = -\frac{1}{3}(\tanh^{-1}(x^x))^{-4/3} \cdot \frac{1}{1-x^{2x}} \cdot x^x(1+\ln x)$.

5 Linear Approximation and Differentials

One of the primary applications of derivatives is to approximate the value of a function near a known point using its tangent line.

5.1 Linearization (Linear Approximation)

Recall the equation of the tangent line to $y = f(x)$ at the point $(a, f(a))$, assuming f is differentiable at a:

$$y - f(a) = f'(a)(x - a)$$
$$y = f(a) + f'(a)(x - a)$$

This tangent line provides a good approximation of the function $f(x)$ for values of x that are close to a.

Definition 5.1 (Linearization)**.** *If f is differentiable at $x = a$, the **linearization** of f at a is the linear function $L(x)$ defined by:*

$$L(x) = f(a) + f'(a)(x - a)$$

*For x near a, we have the **linear approximation** (or **tangent line approximation**):*

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

*This is also called the **local linear approximation** of f at a.*

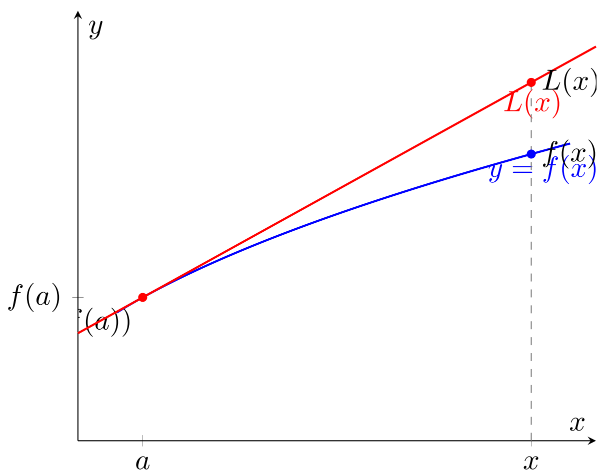


Figure 5: Linear approximation $L(x)$ of $f(x)$ near $x = a$.

Example 5.1 (Finding a Linearization)**.** Find the linearization of $f(x) = \sin(2x + x^2)$ at $a = 0$. *Solution:* We need $f(0)$ and $f'(0)$. $f(0) = \sin(2(0) + 0^2) = \sin(0) = 0$. To find $f'(x)$, use the

Chain Rule: $f'(x) = \cos(2x+x^2) \cdot \frac{d}{dx}(2x+x^2) = \cos(2x+x^2) \cdot (2+2x)$. Now evaluate at $a = 0$: $f'(0) = \cos(0) \cdot (2+0) = 1 \cdot 2 = 2$. The linearization is $L(x) = f(0) + f'(0)(x-0)$:

$$L(x) = 0 + 2(x-0) = 2x$$

So, for x near 0, $\sin(2x + x^2) \approx 2x$.

Example 5.2 (Using Linearization to Approximate a Value)**.** Use a linear approximation to estimate $\sqrt{4.01}$. *Solution:* We want to approximate $f(x) = \sqrt{x}$ at $x = 4.01$. We should choose a nearby point a where $f(a)$ and $f'(a)$ are easy to compute. A natural choice is $a = 4$. We need the linearization $L(x) = f(a)+f'(a)(x-a)$ at $a = 4$. $f(x) = \sqrt{x} = x^{1/2} \implies f(a) = f(4) = \sqrt{4} = 2$. $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \implies f'(a) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. The linearization at $a = 4$ is:

$$L(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4)$$

Now, approximate $f(4.01) = \sqrt{4.01}$ using $L(4.01)$:

$$\sqrt{4.01} \approx L(4.01) = 2 + \frac{1}{4}(4.01-4) = 2 + \frac{1}{4}(0.01)$$

$$= 2 + 0.0025 = 2.0025$$

The linear approximation gives $\sqrt{4.01} \approx 2.0025$.

Example 5.3 (Linearization and Approximation)**.** (a) Find the linearization of $f(x) = \sqrt{x+1}$ at $a = 3$.

(b) Use the linearization to approximate $\sqrt{3.95}$ and $\sqrt{4.01}$.

Solution: (a) Find $f(3)$ and $f'(3)$. $f(x) = (x+1)^{1/2} \implies f(3) = \sqrt{3+1} = \sqrt{4} = 2$. $f'(x) = \frac{1}{2}(x+1)^{-1/2} \cdot 1 = \frac{1}{2\sqrt{x+1}}$. $f'(3) = \frac{1}{2\sqrt{3+1}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. The linearization at $a = 3$ is:

$$L(x) = f(3) + f'(3)(x-3) = 2 + \frac{1}{4}(x-3)$$

(b) Approximate $\sqrt{3.95}$ and $\sqrt{4.01}$. Note that $f(x) = \sqrt{x+1}$. To approximate $\sqrt{3.95}$, we need $x + 1 = 3.95$, which means $x = 2.95$. This value is close to $a = 3$.

$$\sqrt{3.95} = f(2.95) \approx L(2.95) = 2 + \frac{1}{4}(2.95-3) = 2 + \frac{1}{4}(-0.05)$$

$$= 2 - 0.0125 = 1.9875$$

To approximate $\sqrt{4.01}$, we need $x + 1 = 4.01$, which means $x = 3.01$. This value is close to $a = 3$.

$$\sqrt{4.01} = f(3.01) \approx L(3.01) = 2 + \frac{1}{4}(3.01-3) = 2 + \frac{1}{4}(0.01)$$

$$= 2 + 0.0025 = 2.0025$$

Example 5.4 (Approximating Logarithm)**.** Use linearization to approximate $\ln(1.1)$. *Solution:* Let $f(x) = \ln x$. We want to approximate $f(1.1)$. Choose a nearby point where f and f' are easy to compute: $a = 1$. $f(x) = \ln x \implies f(1) = \ln 1 = 0$. $f'(x) = \frac{1}{x} \implies f'(1) = \frac{1}{1} = 1$. The linearization at $a = 1$ is:

$$L(x) = f(1) + f'(1)(x-1) = 0 + 1(x-1) = x - 1$$

Now approximate $\ln(1.1) = f(1.1)$ using $L(1.1)$:

$$\ln(1.1) \approx L(1.1) = 1.1 - 1 = 0.1$$

Remark 5.1 (Linearization and Taylor Polynomials)**.** The linearization $L(x)$ is the first-degree Taylor polynomial of $f(x)$ centered at a. Higher-degree Taylor polynomials provide better approximations over larger intervals.

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The linear approximation is just the first two terms of this series.

5.2 Differentials

Differentials provide an alternative notation and perspective for linear approximation, focusing on the changes in x and y.

Let $y = f(x)$ be a differentiable function. Consider a change in x from x to $x + \Delta x$.

- The **increment** in x is Δx.
- The corresponding exact change in y is $\Delta y = f(x + \Delta x) - f(x)$.

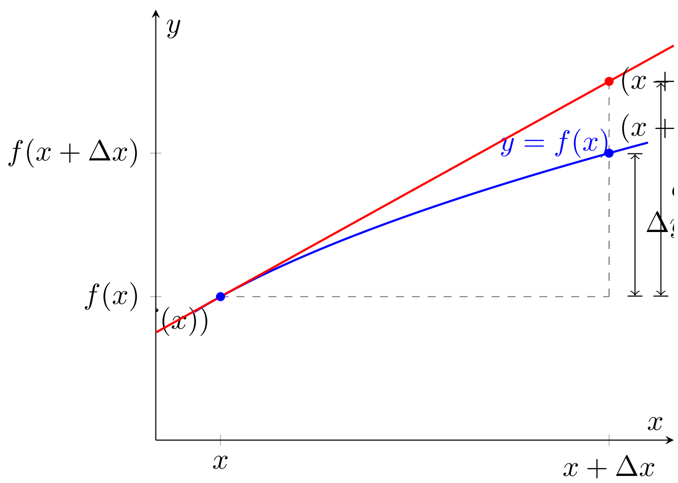


Figure 6: Geometric interpretation of increments (Δx, Δy) and differentials (dx, dy). dy is the change along the tangent line, approximating the true change Δy.

We define two new quantities, the differentials dx and dy.

Definition 5.2 (Differentials)**.** *Let $y = f(x)$ be a differentiable function.*

- *The **differential of x**, denoted dx, is defined as an independent variable that can take any non-zero real value. It is often set equal to the increment Δx: $dx = \Delta x$.*
- *The **differential of y**, denoted dy, is defined as a dependent variable (depending on both x and dx):*

$$dy = f'(x)dx$$

Remark 5.2 (Interpreting Differentials)**.**
- dx represents a small change in the independent variable x.
- dy represents the corresponding change in y measured along the tangent line at the point $(x, f(x))$.

- Comparing dy with Δy:

$$\Delta y = f(x + \Delta x) - f(x) \quad \text{(Exact change in } f\text{)}$$

$$dy = f'(x)\Delta x \quad \text{(Approximate change in } f\text{, using tangent)}$$

  Since the linearization is $L(x+\Delta x) = f(x)+f'(x)\Delta x$, we see that $dy = L(x+\Delta x)-f(x)$. Because $f(x+\Delta x) \approx L(x+\Delta x)$ for small Δx, we have $f(x+\Delta x)-f(x) \approx L(x+\Delta x)-f(x)$, which means $\Delta y \approx dy$.

- The differential dy provides a linear approximation to the actual change Δy. The approximation $\Delta y \approx dy$ improves as $dx = \Delta x$ approaches zero.

- The notation $\frac{dy}{dx} = f'(x)$ can be consistently interpreted as the ratio of the differentials dy and dx (when $dx \neq 0$).

Example 5.5 (Finding the Differential dy)**.** Find the differential dy for $y = x^2\cos(3x)$. *Solution:* First, find the derivative $\frac{dy}{dx}$ using the product rule:

$$\begin{aligned}\frac{dy}{dx} &= x^2\frac{d}{dx}(\cos(3x)) + \cos(3x)\frac{d}{dx}(x^2)\\ &= x^2(-\sin(3x)\cdot 3) + \cos(3x)(2x)\\ &= -3x^2\sin(3x) + 2x\cos(3x)\end{aligned}$$

Now, use the definition $dy = f'(x)dx$:

$$dy = (-3x^2\sin(3x) + 2x\cos(3x))dx$$

Example 5.6. Find dy for $y = x^2 + e^{\sin(2x)}$. *Solution:* Find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(e^{\sin(2x)})$$

$$= 2x + e^{\sin(2x)}\cdot\frac{d}{dx}(\sin(2x)) \quad \text{(Chain Rule)}$$

$$\begin{aligned}&= 2x + e^{\sin(2x)}\cdot\cos(2x)\cdot\frac{d}{dx}(2x)\\ &= 2x + e^{\sin(2x)}\cos(2x)\cdot 2\\ &= 2x + 2\cos(2x)e^{\sin(2x)}\end{aligned}$$

Then, $dy = \frac{dy}{dx}dx$:

$$dy = (2x + 2\cos(2x)e^{\sin(2x)})dx$$

Example 5.7 (Comparing Δy and dy)**.** Let $f(x) = 5x^2 + 4x + 1$.

(a) Find Δy and dy.

(b) Compare their values for $x = 6$ and $\Delta x = 0.02$.

Solution: (a) Find $\Delta y = f(x + \Delta x) - f(x)$:

$$\begin{aligned}\Delta y &= [5(x+\Delta x)^2 + 4(x+\Delta x) + 1] - [5x^2 + 4x + 1]\\ &= [5(x^2 + 2x\Delta x + (\Delta x)^2) + 4x + 4\Delta x + 1] - 5x^2 - 4x - 1\\ &= 5x^2 + 10x\Delta x + 5(\Delta x)^2 + 4x + 4\Delta x + 1 - 5x^2 - 4x - 1\\ &= 10x\Delta x + 4\Delta x + 5(\Delta x)^2\\ &= (10x+4)\Delta x + 5(\Delta x)^2\end{aligned}$$

Find $dy = f'(x)dx$: $f'(x) = 10x + 4$. Since $dx = \Delta x$,

$$dy = (10x + 4)dx = (10x + 4)\Delta x$$

(b) Compare for $x = 6$ and $\Delta x = 0.02$.

$$\Delta y = (10(6) + 4)(0.02) + 5(0.02)^2 = (64)(0.02) + 5(0.0004)$$

$$= 1.28 + 0.002 = 1.282$$

$$dy = (10(6) + 4)(0.02) = (64)(0.02) = 1.28$$

The values $\Delta y = 1.282$ and $dy = 1.28$ are very close, as expected for a small Δx. The difference $\Delta y - dy = 5(\Delta x)^2$ represents the error in the linear approximation over the interval Δx.

Part I
Integration

Integration arises from two main problems: finding the antiderivative of a function (the inverse operation of differentiation) and calculating the area under a curve. The Fundamental Theorem of Calculus establishes a profound connection between these two concepts.

The Definite Integral and Area

6 The Area Problem

One of the motivating problems for integration is finding the area of a region bounded by the graph of a function, the x-axis, and vertical lines.

6.1 Area Under a Curve

Consider the problem: Find the area A of the region bounded by the x-axis, the graph of a continuous, non-negative function $y = f(x)$, and the vertical lines $x = a$ and $x = b$ (where $a < b$).

6.2 Approximation using Rectangles: Riemann Sums

We can approximate this area by dividing the interval $[a, b]$ into smaller subintervals and constructing rectangles on these subintervals whose heights are determined by the function $f(x)$. The sum of the areas of these rectangles provides an approximation of the area under the curve. This sum is known as a **Riemann Sum**.

Definition 6.1 (Partition and Riemann Sum)**.** *Let f be a function defined on a closed interval $[a, b]$.*

1. *A **partition** P of $[a, b]$ is a set of points $\{x_0, x_1, x_2, \ldots, x_n\}$ such that*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

This partition divides $[a, b]$ into n subintervals $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$.

2. *The **width** of the k-th subinterval is $\Delta x_k = x_k - x_{k-1}$.*

3. *The **norm** (or **mesh**) of the partition P, denoted $||P||$, is the width of the widest subinterval:*

$$||P|| = \max_{1 \le k \le n} \{\Delta x_k\}$$

4. *A **sample point** x_k^* is any point chosen within the k-th subinterval, i.e., $x_{k-1} \le x_k^* \le x_k$.*

5. *The **Riemann Sum** of f for the partition P and the sample points $\{x_k^*\}$ is:*

$$S(f, P, \{x_k^*\}) = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

$$= f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

Geometrically, if $f(x) \ge 0$, the Riemann sum represents the sum of the areas of n rectangles with base Δx_k and height $f(x_k^*)$.

Remark 6.1 (Types of Riemann Sums)**.** Common choices for sample points x_k^* lead to specific types of Riemann sums, especially when using a **regular partition** where all subintervals have equal width $\Delta x = \frac{b-a}{n}$. In this case, $x_k = a + k\Delta x$.

- **Left Riemann Sum:** $x_k^* = x_{k-1} = a + (k-1)\Delta x$. Sum: $\sum_{k=1}^{n} f(x_{k-1})\Delta x$.
- **Right Riemann Sum:** $x_k^* = x_k = a + k\Delta x$. Sum: $\sum_{k=1}^{n} f(x_k)\Delta x$.
- **Midpoint Riemann Sum:** $x_k^* = \frac{x_{k-1}+x_k}{2} = a+(k-1/2)\Delta x$. Sum: $\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x$.

[Image showing Left, Right, and Midpoint Riemann Sum approximations] As the number of subintervals n increases (and thus the norm $||P||$ decreases), the Riemann sum generally provides a better approximation to the net area under the curve.

7 The Definite Integral

The definite integral is defined as the limit of Riemann sums as the width of the subintervals shrinks to zero.

Definition 7.1 (The Definite Integral)**.** *Let f be a function defined on $[a, b]$. If the limit*

$$\lim_{||P|| \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

*exists and gives the same value for all possible choices of partitions P and sample points x_k^*, then we say that f is **integrable** on $[a, b]$. The value of the limit is called the **definite integral** of f from a to b, denoted by $\int_a^b f(x)\,dx$.*

$$\int_a^b f(x)\,dx = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

In this notation:

- *a is the **lower limit of integration**.*
- *b is the **upper limit of integration**.*
- *$f(x)$ is the **integrand**.*
- *x is the **variable of integration**.*

Remark 7.1 (Definite Integral using Regular Partitions)**.** If f is integrable, the definite integral can often be computed using the simpler limit formulation based on regular partitions and a consistent choice of sample points (e.g., right endpoints): Let $\Delta x = \frac{b-a}{n}$ and $x_k^* = x_k = a + k\Delta x = a + k\frac{b-a}{n}$. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)$$

This form is particularly useful when evaluating integrals from the definition using summation formulas.

Remark 7.2 (Integrability Conditions)**.** While the definition requires the limit to exist for *all* partitions and sample points, certain conditions on f guarantee integrability:

- **Continuity:** If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.
- **Boundedness and Finite Discontinuities:** If f is bounded on $[a, b]$ (i.e., there exists M such that $|f(x)| \le M$ for all $x \in [a, b]$) and has only a finite number of discontinuities on $[a, b]$, then f is integrable on $[a, b]$.

Most functions encountered in introductory calculus meet these conditions.

Remark 7.3 (Definite Integral and Net Area)**.** The definite integral $\int_a^b f(x)\,dx$ represents the **net signed area** between the graph of $y = f(x)$ and the x-axis over $[a, b]$.

- Area above the x-axis counts positively.
- Area below the x-axis counts negatively.

[Image showing net signed area for a function above and below x-axis]

$$\int_a^b f(x)\,dx = (\text{Area above x-axis}) - (\text{Area below x-axis})$$

If $f(x) \ge 0$ for all $x \in [a, b]$, then the definite integral equals the geometric area under the curve:

$$A = \int_a^b f(x)\,dx \quad (\text{if } f(x) \ge 0)$$

The **total area** bounded by $y = f(x)$ and the x-axis on $[a, b]$ (treating all areas as positive) is given by integrating the absolute value:

$$\text{Total Area} = \int_a^b |f(x)|\,dx$$

Example 7.1 (Evaluating Definite Integral from Definition)**.** Evaluate $\int_{-2}^{1} x^3\,dx$ using the limit of Riemann sums with regular partitions and right endpoints. *Solution:* Here $a = -2$, $b = 1$, $f(x) = x^3$. $\Delta x = \frac{b-a}{n} = \frac{1-(-2)}{n} = \frac{3}{n}$. Right endpoint: $x_k = a + k\Delta x = -2 + k\frac{3}{n}$. $f(x_k) = x_k^3 = \left(-2 + \frac{3k}{n}\right)^3$. Expand the cube: $(p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$. Let $p = -2, q = 3k/n$. $f(x_k) = (-2)^3 + 3(-2)^2(\frac{3k}{n}) + 3(-2)(\frac{3k}{n})^2 + (\frac{3k}{n})^3$ $f(x_k) = -8 + 3(4)(\frac{3k}{n}) - 6(\frac{9k^2}{n^2}) + \frac{27k^3}{n^3}$ $f(x_k) = -8 + \frac{36k}{n} - \frac{54k^2}{n^2} + \frac{27k^3}{n^3}$

The Riemann sum is $\sum_{k=1}^{n} f(x_k)\Delta x$:

$$\sum_{k=1}^{n}\left(-8 + \frac{36k}{n} - \frac{54k^2}{n^2} + \frac{27k^3}{n^3}\right)\left(\frac{3}{n}\right)$$

$$= \frac{3}{n}\left[\sum_{k=1}^{n}(-8) + \sum_{k=1}^{n}\frac{36k}{n} - \sum_{k=1}^{n}\frac{54k^2}{n^2} + \sum_{k=1}^{n}\frac{27k^3}{n^3}\right]$$

$$= \frac{3}{n}\left[-8n + \frac{36}{n}\sum_{k=1}^{n}k - \frac{54}{n^2}\sum_{k=1}^{n}k^2 + \frac{27}{n^3}\sum_{k=1}^{n}k^3\right]$$

Use summation formulas: $\sum_{k=1}^{n} 1 = n$ $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$

Substitute these into the sum:

$$= \frac{3}{n}\left[-8n + \frac{36}{n}\frac{n(n+1)}{2} - \frac{54}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{27}{n^3}\frac{n^2(n+1)^2}{4}\right]$$

$$= \frac{3}{n}\left[-8n + 18(n+1) - 9\frac{(n+1)(2n+1)}{n} + \frac{27}{4}\frac{(n+1)^2}{n}\right]$$

Distribute the $3/n$:

$$= -24 + 54\frac{n+1}{n} - 27\frac{(n+1)(2n+1)}{n^2} + \frac{81}{4}\frac{(n+1)^2}{n^2}$$

Now take the limit as $n \to \infty$: Recall $\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}(1+1/n) = 1$. $\lim_{n\to\infty} \frac{(n+1)(2n+1)}{n^2} = \lim_{n\to\infty} \frac{2n^2+3n+1}{n^2} = \lim_{n\to\infty}(2 + 3/n + 1/n^2) = 2$. $\lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = 1^2 = 1$.

So the limit is:

$$\int_{-2}^{1} x^3\,dx = -24 + 54(1) - 27(2) + \frac{81}{4}(1)$$

$$= -24 + 54 - 54 + \frac{81}{4} = -24 + \frac{81}{4} = \frac{-96+81}{4} = -\frac{15}{4}$$

The definite integral is $-15/4$. Since the function x^3 is negative on $[-2, 0)$ and positive on $(0, 1]$, this represents the net signed area.

7.1 Properties of Definite Integrals

Definite integrals share linearity properties with derivatives and indefinite integrals, and have additional properties related to the limits of integration. Assume f and g are integrable on the relevant intervals, and c is a constant.

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (Reversing limits negates the integral).
2. $\int_a^a f(x)\,dx = 0$ (Integral over a zero-width interval is zero).
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ (Constant factor pulls out).
4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ (Integral of sum/difference is sum/difference of integrals).
5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ (Additivity over intervals). This holds for any c, even if c is not between a and b.
6. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ (Variable of integration is a dummy variable). The value depends only on the integrand function and the limits.
7. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

8. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

9. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$ (Bounds for integral).

Example 7.2 (Using Properties)**.** Given $\int_{-1}^2 f(x)\,dx = 5$ and $\int_{-1}^2 g(x)\,dx = -3$. Find:

1. $\int_2^{-1} f(x)\,dx = -\int_{-1}^2 f(x)\,dx = -5$. (Property 1)

2. $\int_{-1}^2 4f(x)\,dx = 4\int_{-1}^2 f(x)\,dx = 4(5) = 20$. (Property 3)

3. $\int_{-1}^2 [2f(x) + 3g(x)]\,dx = 2\int_{-1}^2 f(x)\,dx + 3\int_{-1}^2 g(x)\,dx = 2(5) + 3(-3) = 10 - 9 = 1$. (Properties 3, 4)

4. $\int_{-1}^2 [f(x)^2 + \frac{g(x)}{|f(x)|+1}]\,dx$. Cannot evaluate without knowing $f(x)$ and $g(x)$. Properties do not distribute over products or quotients in this way.

5. $\int_2^2 [f(x)^2 + \frac{g(x)}{|f(x)|+1}]\,dx = 0$. (Property 2)

8 The Fundamental Theorem of Calculus (FTC)

This theorem provides the essential link between differentiation and definite integration, allowing us to compute definite integrals much more easily than using the limit of Riemann sums. It has two parts.

8.1 FTC Part 1: Derivative of an Integral

This part states that the process of integrating f and then differentiating the result returns the original function f.

Theorem 8.1 (Fundamental Theorem of Calculus, Part 1)**.** *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)\,dt, \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and its derivative is

$$g'(x) = f(x)$$

That is,

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

Idea of Proof. We apply the definition of the derivative to $g(x)$:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

Using the additivity property $\int_a^{x+h} = \int_a^x + \int_x^{x+h}$, this simplifies to:

$$g'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

The expression $\frac{1}{h}\int_x^{x+h} f(t)\,dt$ represents the average value of f on the interval $[x, x+h]$ (or $[x+h, x]$ if $h < 0$). By the Mean Value Theorem for Integrals (related to MVT for derivatives), this average value equals $f(c)$ for some c between x and $x+h$. As $h \to 0$, $c \to x$. Since f is continuous, $\lim_{h\to 0} f(c) = f(x)$. Thus, $g'(x) = f(x)$. □

Remark 8.1. FTC Part 1 shows that every continuous function f has an antiderivative, namely $g(x) = \int_a^x f(t)\,dt$.

Example 8.1 (Applying FTC Part 1)**.** Find the derivatives:

1. $\frac{d}{dx}\int_{-1}^{x} e^t t^\pi\,dt = e^x x^\pi$ (Direct application with $f(t) = e^t t^\pi$)
2. $\frac{d}{dx}\int_{-3}^{x} \frac{1}{\sqrt[3]{t^4+2}}\,dt = \frac{1}{\sqrt[3]{x^4+2}}$ (Direct application)

Remark 8.2 (Variations of FTC Part 1)**.**

- **Variable lower limit:** Use property 1 of definite integrals:
$$\frac{d}{dx}\int_x^b f(t)\,dt = \frac{d}{dx}\left(-\int_b^x f(t)\,dt\right) = -f(x)$$
Example: $\frac{d}{dx}\int_x^0 \frac{\cos t}{t^2+1}\,dt = -\frac{\cos x}{x^2+1}$.

- **Function in upper limit:** Use the Chain Rule. Let $u = h(x)$.
$$\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = \frac{d}{dx}\int_a^{u} f(t)\,dt$$
Let $g(u) = \int_a^u f(t)\,dt$. Then we want $\frac{d}{dx}g(u)$. By Chain Rule:
$$\frac{d}{dx}g(u) = \frac{dg}{du}\cdot\frac{du}{dx}$$
By FTC Part 1, $\frac{dg}{du} = f(u)$. So,
$$\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = f(u)\cdot\frac{du}{dx} = f(h(x))\cdot h'(x)$$
Example: $\frac{d}{dx}\int_\pi^{x^3}\cos t\,dt$. Here $f(t) = \cos t$, $h(x) = x^3$, $h'(x) = 3x^2$. Result: $f(h(x))h'(x) = \cos(x^3)\cdot 3x^2$.

- **Functions in both limits:** Use additivity and the above rules:
$$\begin{aligned}\int_{k(x)}^{h(x)} f(t)\,dt &= \int_{k(x)}^{a} f(t)\,dt + \int_a^{h(x)} f(t)\,dt \\ &= -\int_a^{k(x)} f(t)\,dt + \int_a^{h(x)} f(t)\,dt\end{aligned}$$
Differentiating gives:
$$\frac{d}{dx}\int_{k(x)}^{h(x)} f(t)\,dt = -f(k(x))k'(x) + f(h(x))h'(x)$$
Example: $\frac{d}{dx}\int_{e^x}^{x^2} t^2\sin t\,dt$. Here $f(t) = t^2\sin t$, $h(x) = x^2$, $k(x) = e^x$. $h'(x) = 2x$, $k'(x) = e^x$. Result: $f(h(x))h'(x) - f(k(x))k'(x) = (x^2)^2\sin(x^2)\cdot(2x) - (e^x)^2\sin(e^x)\cdot(e^x)$ $= 2x^5\sin(x^2) - e^{3x}\sin(e^x)$.

Exercise 8.1. Use the Fundamental Theorem of Calculus (derivative form) to find the indicated derivative.

1. $\frac{d}{dx}\int_2^x [t^2e^t + \ln(|\sin t|)]\,dt$
2. $\frac{d}{dx}\int_{e^x}^{\pi}\ln(t)\tan(2t)\,dt$
3. $\frac{d}{dx}\int_{\ln x}^{\sin x}\frac{1}{1+t^5}\,dt$
4. $\frac{d}{dx}\int_\pi^{e^\pi}\ln(t)\,dt$ (Derivative of a constant is zero)

8.2 FTC Part 2: Evaluating Definite Integrals

This part provides the standard method for evaluating definite integrals using antiderivatives.

Theorem 8.2 (Fundamental Theorem of Calculus, Part 2 (Evaluation Theorem))**.** *If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$ (i.e., $F'(x) = f(x)$), then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof. Let $g(x) = \int_a^x f(t)\,dt$. By FTC Part 1, $g'(x) = f(x)$, so $g(x)$ is an antiderivative of $f(x)$. Since $F(x)$ is also an antiderivative of $f(x)$, they must differ by a constant (Corollary to MVT): $F(x) = g(x)+C$ for some constant C. To find C, let $x = a$. $F(a) = g(a)+C = \int_a^a f(t)\,dt+C = 0 + C = C$. So $C = F(a)$. Thus, $F(x) = g(x) + F(a)$. Now let $x = b$:

$$F(b) = g(b) + F(a) = \int_a^b f(t)\,dt + F(a)$$

Rearranging gives the result:

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Replacing the dummy variable t with x yields the theorem statement. □

[Evaluation Notation] The difference $F(b)-F(a)$ is often denoted using brackets or a vertical bar:

$$F(b) - F(a) = [F(x)]_a^b = F(x)|_a^b$$

So, FTC Part 2 can be written as $\int_a^b f(x)\,dx = [F(x)]_a^b$.

Example 8.2 (Evaluating Definite Integrals using FTC Part 2)**.** 1. Evaluate $\int_{-1}^2 x^3\,dx$. (Compare with result from Riemann sum definition) An antiderivative of $f(x) = x^3$ is $F(x) = \frac{x^4}{4}$.

$$\int_{-1}^2 x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^2 = F(2) - F(-1) = \frac{(2)^4}{4} - \frac{(-1)^4}{4}$$
$$= \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$$

(Wait, the Riemann sum gave $-15/4$. Let's recheck the Riemann sum calculation... Ah, the term $\frac{81}{4}\frac{(n+1)^2}{n^2}$ in the sum expansion should be correct. Let's recheck the limit: $-24+54-27(2)+\frac{81}{4} = -24+54-54+81/4 = -24+20.25 = -3.75 = -15/4$. The Riemann sum calculation was correct. The FTC evaluation is also correct. Let's re-read the problem statement for the Riemann sum: $\int_{-2}^1 x^3 dx$. Ah, the limits were -2 to 1. My FTC example used -1 to 2. Let's redo the FTC example with correct limits.)

Evaluate $\int_{-2}^1 x^3\,dx$.

$$\int_{-2}^1 x^3\,dx = \left[\frac{x^4}{4}\right]_{-2}^1 = F(1) - F(-2) = \frac{(1)^4}{4} - \frac{(-2)^4}{4}$$
$$= \frac{1}{4} - \frac{16}{4} = -\frac{15}{4}$$

This matches the result from the Riemann sum definition.

2. Evaluate $\int_{-1}^{1}(x^4 + 3x^1 + 1 - e^x)\,dx$. Antiderivative: $F(x) = \frac{x^5}{5} + \frac{3x^2}{2} + x - e^x$.

$$\int_{-1}^{1}(x^4 + 3x + 1 - e^x)\,dx = \left[\frac{x^5}{5} + \frac{3x^2}{2} + x - e^x\right]_{-1}^{1}$$
$$= \left(\frac{1^5}{5} + \frac{3(1)^2}{2} + 1 - e^1\right) - \left(\frac{(-1)^5}{5} + \frac{3(-1)^2}{2} + (-1) - e^{-1}\right)$$
$$= \left(\frac{1}{5} + \frac{3}{2} + 1 - e\right) - \left(-\frac{1}{5} + \frac{3}{2} - 1 - e^{-1}\right)$$
$$= \frac{1}{5} + \frac{3}{2} + 1 - e + \frac{1}{5} - \frac{3}{2} + 1 + e^{-1}$$
$$= \frac{2}{5} + 2 - e + e^{-1}$$

3. Evaluate $\int_{0}^{\pi}(\cos x - \sin x)\,dx$. Antiderivative: $F(x) = \sin x - (-\cos x) = \sin x + \cos x$.

$$\int_{0}^{\pi}(\cos x - \sin x)\,dx = [\sin x + \cos x]_0^{\pi}$$
$$= (\sin\pi + \cos\pi) - (\sin 0 + \cos 0)$$
$$= (0 + (-1)) - (0 + 1) = -1 - 1 = -2$$

8.3 Integrating Piecewise Continuous Functions

If f is piecewise continuous on $[a, b]$, meaning it is continuous except at a finite number of points $c_1, c_2, \dots, c_{n-1}$ where $a = c_0 < c_1 < \cdots < c_n = b$, we can integrate f by splitting the integral at the points of discontinuity using the additivity property:

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_{n-1}}^{b} f(x)\,dx$$

On each subinterval (c_{k-1}, c_k), f behaves like a continuous function, allowing the use of FTC Part 2. (Note: The value of f at the discontinuity points c_k does not affect the value of the definite integral).

Example 8.3 (Integrating a Piecewise Function)**.** Let $f(x) = \begin{cases} x - 1, & x < 0 \\ x^2, & 0 \le x < 2 \\ 3x + 1, & x \ge 2 \end{cases}$. Evaluate $\int_{-1}^{3} f(x)\,dx$. *Solution:* The function definition changes at $x = 0$ and $x = 2$, both of which are within the integration interval $[-1, 3]$. We split the integral:

$$\int_{-1}^{3} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$$

Now use the appropriate definition of $f(x)$ for each interval:

$$= \int_{-1}^{0}(x - 1)\,dx + \int_{0}^{2}(x^2)\,dx + \int_{2}^{3}(3x + 1)\,dx$$

Evaluate each integral using FTC Part 2:

$$\int_{-1}^{0}(x - 1)\,dx = \left[\frac{x^2}{2} - x\right]_{-1}^{0}$$
$$= \left(\frac{0^2}{2} - 0\right) - \left(\frac{(-1)^2}{2} - (-1)\right) = 0 - (\frac{1}{2} + 1) = -\frac{3}{2}$$

$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$$

$$\begin{aligned}\int_2^3 (3x+1)\,dx &= \left[\frac{3x^2}{2} + x\right]_2^3 \\ &= \left(\frac{3(3)^2}{2} + 3\right) - \left(\frac{3(2)^2}{2} + 2\right) = \left(\frac{27}{2} + 3\right) - (6+2) \\ &= \frac{33}{2} - 8 = \frac{33-16}{2} = \frac{17}{2}\end{aligned}$$

Combine the results:

$$\int_{-1}^3 f(x)\,dx = -\frac{3}{2} + \frac{8}{3} + \frac{17}{2} = \frac{-9+16+51}{6} = \frac{58}{6} = \frac{29}{3}$$

Example 8.4 (Integral involving Absolute Value)**.** Evaluate $\int_{-1}^3 |x-1|\,dx$. *Solution:* The absolute value function changes definition at the point where its argument is zero. $x-1=0$ when $x=1$.

$$|x-1| = \begin{cases} -(x-1) = 1-x, & x<1 \\ x-1, & x \geq 1 \end{cases}$$

Since $x=1$ is within the integration interval $[-1,3]$, we split the integral at $x=1$:

$$\int_{-1}^3 |x-1|\,dx = \int_{-1}^1 |x-1|\,dx + \int_1^3 |x-1|\,dx$$

$$= \int_{-1}^1 (1-x)\,dx + \int_1^3 (x-1)\,dx$$

Evaluate each integral:

$$\begin{aligned}\int_{-1}^1 (1-x)\,dx &= \left[x - \frac{x^2}{2}\right]_{-1}^1 \\ &= \left(1 - \frac{1^2}{2}\right) - \left(-1 - \frac{(-1)^2}{2}\right) = (\frac{1}{2}) - (-\frac{3}{2}) = 2\end{aligned}$$

$$\begin{aligned}\int_1^3 (x-1)\,dx &= \left[\frac{x^2}{2} - x\right]_1^3 \\ &= \left(\frac{3^2}{2} - 3\right) - \left(\frac{1^2}{2} - 1\right) = (\frac{9}{2} - 3) - (\frac{1}{2} - 1) \\ &= (\frac{3}{2}) - (-\frac{1}{2}) = 2\end{aligned}$$

Combine the results:

$$\int_{-1}^3 |x-1|\,dx = 2 + 2 = 4$$

Geometrically, this represents the sum of the areas of two triangles, one with base from -1 to 1 (length 2) and height $|-1-1| = 2$, area $\frac{1}{2}(2)(2) = 2$, and one with base from 1 to 3 (length 2) and height $|3-1| = 2$, area $\frac{1}{2}(2)(2) = 2$. Total area is $2+2=4$.

8.4 Substitution Rule for Definite Integrals

We can combine u-substitution with the FTC Part 2.

Theorem 8.3 (Substitution Rule for Definite Integrals)**.** *If $u = g(x)$ has a continuous derivative on $[a, b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Note that the limits of integration change from x-values (a, b) to corresponding u-values $(g(a), g(b))$.

Proof. Let F be an antiderivative of f, so $F'(u) = f(u)$. By the Chain Rule, $\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$. Therefore, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$. Using FTC Part 2 on the left side:

$$\int_a^b f(g(x))g'(x)\,dx = [F(g(x))]_a^b = F(g(b)) - F(g(a))$$

Using FTC Part 2 on the right side with limits $u_{lower} = g(a)$ and $u_{upper} = g(b)$:

$$\int_{g(a)}^{g(b)} f(u)\,du = [F(u)]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

Since both sides equal $F(g(b)) - F(g(a))$, they are equal to each other. □

Remark 8.3 (Applying Substitution for Definite Integrals)**.** Two methods: 1. **Change Limits:** Perform the substitution $u = g(x)$, find $du = g'(x)dx$, and calculate the new limits $u_{lower} = g(a)$ and $u_{upper} = g(b)$. Evaluate the integral $\int_{g(a)}^{g(b)} f(u)\,du$ entirely in terms of u. Do NOT substitute back to x. 2. **Integrate Indefinitely First:** Find the indefinite integral $\int f(g(x))g'(x)\,dx = F(g(x)) + C$ using substitution and substituting back to x. Then evaluate using the original limits: $[F(g(x))]_a^b = F(g(b)) - F(g(a))$. Method 1 is usually more direct.

Example 8.5 (Definite Integral Substitution)**.** Evaluate $\int_1^3 x\sqrt[3]{x^2 - 1}\,dx$. *Solution (Method 1: Change Limits):* 1. Choose $u = x^2 - 1$. 2. Find $du = 2x\,dx \implies x\,dx = \frac{1}{2}du$. 3. Change Limits: When $x = 1$ (lower limit), $u = 1^2 - 1 = 0$. When $x = 3$ (upper limit), $u = 3^2 - 1 = 8$. 4. Substitute and Integrate:

$$\int_1^3 \sqrt[3]{x^2 - 1}(x\,dx) = \int_0^8 \sqrt[3]{u}(\frac{1}{2}du) = \frac{1}{2}\int_0^8 u^{1/3}\,du$$

$$= \frac{1}{2}\left[\frac{u^{1/3+1}}{1/3 + 1}\right]_0^8 = \frac{1}{2}\left[\frac{u^{4/3}}{4/3}\right]_0^8 = \frac{1}{2}\left[\frac{3}{4}u^{4/3}\right]_0^8$$

$$= \frac{3}{8}[u^{4/3}]_0^8 = \frac{3}{8}(8^{4/3} - 0^{4/3})$$

Calculate $8^{4/3} = (\sqrt[3]{8})^4 = (2)^4 = 16$.

$$= \frac{3}{8}(16 - 0) = \frac{3 \times 16}{8} = 3 \times 2 = 6$$

The value is 6.

Example 8.6. Evaluate $\int_3^5 \frac{4t}{2-8t^2}\,dt$. *Solution:* 1. Choose $u = 2 - 8t^2$. 2. Find $du = -16t\,dt$. 3. We have $4t\,dt$. Since $du = -4(4t\,dt)$, we have $4t\,dt = -\frac{1}{4}du$. 4. Change Limits: When $t = 3$, $u = 2-8(3^2) = 2-8(9) = 2-72 = -70$. When $t = 5$, $u = 2-8(5^2) = 2-8(25) = 2-200 = -198$. 5. Substitute and Integrate:

$$\int_3^5 \frac{1}{2-8t^2}(4t\,dt) = \int_{-70}^{-198} \frac{1}{u}\left(-\frac{1}{4}du\right) = -\frac{1}{4}\int_{-70}^{-198} \frac{1}{u}\,du$$

$$= -\frac{1}{4}[\ln|u|]_{-70}^{-198} = -\frac{1}{4}(\ln|-198| - \ln|-70|)$$

$$= -\frac{1}{4}(\ln 198 - \ln 70) = -\frac{1}{4}\ln\left(\frac{198}{70}\right) = -\frac{1}{4}\ln\left(\frac{99}{35}\right)$$

(Alternatively, $\frac{1}{4}\ln\left(\frac{70}{198}\right) = \frac{1}{4}\ln\left(\frac{35}{99}\right)$.)

8.5 Integrals of Even and Odd Functions

Symmetry can simplify definite integrals over symmetric intervals $[-a, a]$.

Definition 8.1 (Even and Odd Functions)**.** *Let f be defined on an interval symmetric about the origin (e.g., $[-a, a]$).*

- *f is **even** if $f(-x) = f(x)$ for all x in the domain. (Symmetric about y-axis).*
- *f is **odd** if $f(-x) = -f(x)$ for all x in the domain. (Symmetric about origin).*

Examples: $\cos x, x^2, x^4, |x|$ *are even.* $\sin x, x, x^3$ *are odd.*

Theorem 8.4 (Integrals of Even and Odd Functions)**.** *Let f be integrable on $[-a, a]$.*

1. *If f is **even**, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.*
2. *If f is **odd**, then $\int_{-a}^a f(x)\,dx = 0$.*

Proof. Use additivity: $\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx$. Consider the first integral $\int_{-a}^0 f(x)\,dx$. Let $x = -u$, then $dx = -du$. When $x = -a$, $u = a$. When $x = 0$, $u = 0$.

$$\int_{x=-a}^{x=0} f(x)\,dx = \int_{u=a}^{u=0} f(-u)(-du) = -\int_a^0 f(-u)\,du = \int_0^a f(-u)\,du$$

1. If f is even, $f(-u) = f(u)$.

$$\int_{-a}^0 f(x)\,dx = \int_0^a f(u)\,du = \int_0^a f(x)\,dx$$

So, $\int_{-a}^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$. 2. If f is odd, $f(-u) = -f(u)$.

$$\int_{-a}^0 f(x)\,dx = \int_0^a -f(u)\,du = -\int_0^a f(x)\,dx$$

So, $\int_{-a}^a f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0$. □

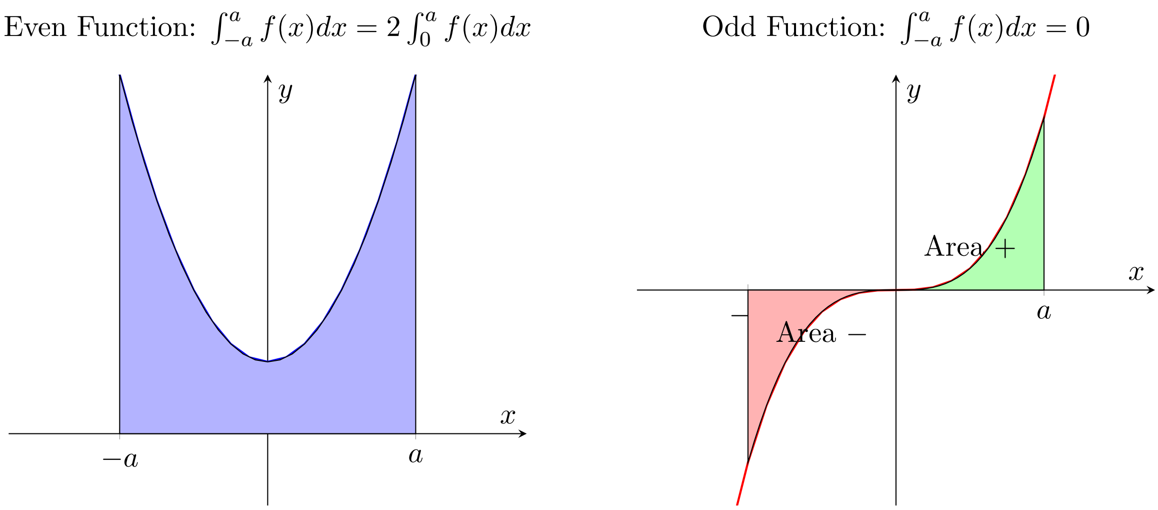


Figure 7: Integrals of even and odd functions over $[-a, a]$.

Example 8.7 (Using Odd Function Property)**.** Evaluate $\int_{-\pi}^{\pi}(x^5 + \sin x)^3\,dx$. *Solution:* Let $f(x) = (x^5 + \sin x)^3$. Check if f is odd or even. $f(-x) = ((-x)^5 + \sin(-x))^3 = (-x^5 - \sin x)^3$ Recall $\sin(-x) = -\sin x$. $f(-x) = (-(x^5 + \sin x))^3 = (-1)^3(x^5 + \sin x)^3 = -(x^5 + \sin x)^3 = -f(x)$. Since $f(-x) = -f(x)$, the function is odd. The interval of integration $[-\pi, \pi]$ is symmetric about the origin. Therefore, by the property of odd functions:

$$\int_{-\pi}^{\pi}(x^5 + \sin x)^3\,dx = 0$$

Example 8.8 (Combining Odd/Even Properties)**.** Evaluate $\int_{-1}^{1}\left(\frac{x^5+x}{(x^4+4x^2+4)^3} + |x|\right)dx$. *Solution:* Split the integral:

$$I = \int_{-1}^{1}\frac{x^5 + x}{(x^4 + 4x^2 + 4)^3}\,dx + \int_{-1}^{1}|x|\,dx$$

Consider the first integrand: $g(x) = \frac{x^5+x}{(x^4+4x^2+4)^3}$. $g(-x) = \frac{(-x)^5+(-x)}{((-x)^4+4(-x)^2+4)^3} = \frac{-x^5-x}{(x^4+4x^2+4)^3} = -\frac{x^5+x}{(x^4+4x^2+4)^3} = -g(x)$. So $g(x)$ is an odd function. Since the interval $[-1, 1]$ is symmetric,

$$\int_{-1}^{1} g(x)\,dx = 0$$

Consider the second integrand: $h(x) = |x|$. $h(-x) = |-x| = |x| = h(x)$. So $h(x)$ is an even function.

$$\int_{-1}^{1}|x|\,dx = 2\int_{0}^{1}|x|\,dx$$

For $x \in [0, 1]$, $|x| = x$.

$$= 2\int_{0}^{1} x\,dx = 2\left[\frac{x^2}{2}\right]_0^1 = 2\left(\frac{1^2}{2} - \frac{0^2}{2}\right) = 2(\frac{1}{2}) = 1$$

Combining the results: $I = 0 + 1 = 1$.

Techniques of Integration

While the Fundamental Theorem of Calculus provides a powerful way to evaluate definite integrals using antiderivatives, finding those antiderivatives is not always straightforward. This chapter explores several advanced techniques for finding indefinite integrals (antiderivatives).

9 Integration by Substitution (Review)

As covered briefly in the context of the indefinite integral (Section **??**), the substitution rule reverses the Chain Rule:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \quad \text{where } u = g(x)$$

This technique is fundamental and often the first approach to try when the integrand is a composition of functions or when a function and its derivative (up to a constant) appear.

Example 9.1 (Completing the Square with Substitution)**.** Evaluate $\int \frac{x+4}{x^2+6x+18}\,dx$. *Solution:* The denominator $x^2+6x+18$ has no simple factor $g'(x)$ in the numerator. The derivative is $2x+6$. We can try to create this in the numerator. Also, notice the denominator is an irreducible quadratic ($b^2-4ac = 6^2-4(1)(18) = 36-72 < 0$). We can complete the square: $x^2+6x+18 = (x^2+6x+9)+9 = (x+3)^2+3^2$. The integral becomes $\int \frac{x+4}{(x+3)^2+9}\,dx$. Let $u = x+3$. Then $x = u-3$ and $dx = du$. Substitute:

$$\int \frac{(u-3)+4}{u^2+9}\,du = \int \frac{u+1}{u^2+9}\,du$$

Split the integral:

$$= \int \frac{u}{u^2+9}\,du + \int \frac{1}{u^2+9}\,du$$

For the first integral, let $v = u^2+9$, $dv = 2u\,du \implies u\,du = \frac{1}{2}dv$.

$$\int \frac{u}{u^2+9}\,du = \int \frac{1/2\,dv}{v} = \frac{1}{2}\ln|v| = \frac{1}{2}\ln(u^2+9)$$

(Absolute value dropped as $u^2+9 > 0$). For the second integral, use the arctangent form $\int \frac{1}{u^2+a^2}du = \frac{1}{a}\tan^{-1}(\frac{u}{a})$ with $a = 3$.

$$\int \frac{1}{u^2+3^2}\,du = \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)$$

Combine the results (adding one constant C):

$$\frac{1}{2}\ln(u^2+9) + \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

Substitute back $u = x+3$. Recall $u^2+9 = (x+3)^2+9 = x^2+6x+18$.

$$\int \frac{x+4}{x^2+6x+18}\,dx = \frac{1}{2}\ln(x^2+6x+18) + \frac{1}{3}\tan^{-1}\left(\frac{x+3}{3}\right) + C$$

Exercise 9.1 (Substitution involving algebraic manipulation)**.** Evaluate $\int x^2\sqrt{2x+1}\,dx$. *Hint:* Let $u = 2x+1$. Then $x = (u-1)/2$, $dx = du/2$. Substitute for x^2, $\sqrt{2x+1}$, and dx in terms of u. Expand and integrate the resulting polynomial in u. Substitute back. *Result:* $\frac{1}{28}(2x+1)^{7/2} - \frac{1}{10}(2x+1)^{5/2} + \frac{1}{12}(2x+1)^{3/2} + C$.

Exercise 9.2 (Substitution with radicals)**.** Evaluate $\int \frac{1}{1+\sqrt{x}}\,dx$. *Hint:* Let $u = \sqrt{x}$. Then $x = u^2$, $dx = 2u\,du$. Substitute and perform polynomial long division or algebraic manipulation on $\frac{2u}{1+u}$. *Result:* $2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$.

10 Integration by Parts

Integration by Parts is a technique based on reversing the Product Rule for differentiation. The Product Rule states:

$$\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + v(x)u'(x)$$

Integrating both sides with respect to x:

$$\int \frac{d}{dx}[u(x)v(x)]\,dx = \int u(x)v'(x)\,dx + \int v(x)u'(x)\,dx$$

$$u(x)v(x) = \int u(x)v'(x)\,dx + \int v(x)u'(x)\,dx$$

Rearranging gives the formula for integration by parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

Using the differential notation $du = u'(x)\,dx$ and $dv = v'(x)\,dx$, the formula becomes more compact.

Theorem 10.1 (Integration by Parts Formula)**.** *Let u and v be differentiable functions of x. Then*

$$\int u\,dv = uv - \int v\,du$$

For definite integrals:

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

where $[uv]_a^b = u(b)v(b) - u(a)v(a)$.

10.1 Guidelines for Integration by Parts

The goal is to choose u and dv such that the integral $\int v\,du$ is simpler to evaluate than the original integral $\int u\,dv$.

1. **Choose dv:** Identify a factor in the integrand that is easily integrable. This factor, along with dx, becomes dv.
2. **Choose u:** The remaining factor(s) in the integrand become u.
3. **Calculate du and v:** Differentiate u to find du. Integrate dv to find v (no constant of integration is needed here).
4. **Apply Formula:** Substitute u, v, du into $\int u\,dv = uv - \int v\,du$.
5. **Evaluate $\int v\,du$:** Calculate the new integral. This might require another application of integration by parts or a different technique.

Remark 10.1 (Choosing u - The LIATE Rule)**.** A common heuristic for choosing u (when the integrand is a product of different function types) is the LIATE principle. Choose u as the function type that appears first in this list:

- **L**ogarithmic functions (e.g., $\ln x, \log_b x$)
- **I**nverse trigonometric functions (e.g., $\arcsin x, \arctan x$)
- **A**lgebraic functions (polynomials, roots, e.g., $x^2, \sqrt{x}$)

- **T**rigonometric functions (e.g., $\sin x, \cos x, \sec x$)
- **E**xponential functions (e.g., e^x, b^x)

The function chosen as u is typically one that simplifies upon differentiation, while the remaining part dv should be readily integrable.

Example 10.1 (Integrating $\ln x$)**.** Evaluate $\int \ln x \, dx$. *Solution:* This doesn't look like a product, but we can write it as $\int (\ln x)(1 \, dx)$. Using LIATE, Logarithmic comes first. 1. Choose $u = \ln x$. 2. Choose $dv = 1 \, dx$. 3. Calculate $du = \frac{1}{x} \, dx$. Calculate $v = \int dv = \int 1 \, dx = x$. 4. Apply Formula:

$$\int \ln x \, dx = (\ln x)(x) - \int x \left(\frac{1}{x} \, dx \right)$$

5. Evaluate $\int v \, du$:

$$= x \ln x - \int 1 \, dx = x \ln x - x + C$$

Therefore, $\int \ln x \, dx = x \ln x - x + C$.

Exercise 10.1. Evaluate $\int x^3 \ln x \, dx$. *Hint:* Use LIATE. Let $u = \ln x, dv = x^3 dx$. *Result:* $\frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + C$.

Example 10.2 (Integration by Parts for Definite Integrals)**.** Find the area under the graph of $f(x) = xe^x$ on the interval $[0, 1]$. *Solution:* Since $xe^x \geq 0$ for $x \in [0, 1]$, the area is $A = \int_0^1 xe^x \, dx$. We use integration by parts. Using LIATE, Algebraic comes before Exponential. 1. Choose $u = x$. 2. Choose $dv = e^x \, dx$. 3. Calculate $du = dx$. Calculate $v = \int e^x \, dx = e^x$. 4. Apply Formula for definite integrals:

$$A = \int_0^1 u \, dv = [uv]_0^1 - \int_0^1 v \, du$$

$$A = [xe^x]_0^1 - \int_0^1 e^x \, dx$$

5. Evaluate:

$$A = (1 \cdot e^1 - 0 \cdot e^0) - [e^x]_0^1$$
$$A = (e - 0) - (e^1 - e^0) = e - (e - 1) = e - e + 1 = 1$$

The area is exactly 1.

Exercise 10.2. Evaluate $\int \frac{x}{\sqrt{x+1}} \, dx$. *Hint:* Let $u = x, dv = (x + 1)^{-1/2} dx$. *Result:* $2x(x + 1)^{1/2} - \frac{4}{3}(x + 1)^{3/2} + C$.

Exercise 10.3. Evaluate $\int x \tan^{-1} x \, dx$. *Hint:* Use LIATE (Inverse Trig before Algebraic). Let $u = \tan^{-1} x, dv = x dx$. Use $\frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2} = 1 - \frac{1}{1+x^2}$ to evaluate $\int v du$. *Result:* $\frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C$.

10.2 Repeated Integration by Parts

Sometimes, the integral $\int v \, du$ is still not directly integrable but is simpler than the original. It might require another application of integration by parts. This often occurs for integrals of the form:

$$\int p(x) \sin(kx) \, dx, \quad \int p(x) \cos(kx) \, dx, \quad \int p(x) e^{kx} \, dx$$

where $p(x)$ is a polynomial of degree $m \geq 1$. Each application of integration by parts (with $u = p(x)$ or its derivative) reduces the degree of the polynomial factor.

Example 10.3 (Repeated Integration by Parts)**.** Evaluate $\int x^2 \sin x \, dx$. *Solution:* Use LIATE: Algebraic before Trig. 1. First application: $u = x^2$, $dv = \sin x \, dx$. $du = 2x \, dx$, $v = -\cos x$.

$$\int x^2 \sin x \, dx = x^2(-\cos x) - \int (-\cos x)(2x \, dx)$$

$$= -x^2 \cos x + 2 \int x \cos x \, dx$$

2. Second application (for the remaining integral): $u = x$, $dv = \cos x \, dx$. $du = dx$, $v = \sin x$.

$$\int x \cos x \, dx = x(\sin x) - \int (\sin x)(dx)$$

$$= x \sin x - (-\cos x) = x \sin x + \cos x$$

3. Substitute back:

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2(x \sin x + \cos x) + C$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

10.3 Integration by Parts: Loop / Solving for the Integral

For integrals like $\int e^{ax} \sin(bx) \, dx$ or $\int e^{ax} \cos(bx) \, dx$, applying integration by parts twice often leads back to the original integral (multiplied by a constant). This allows solving for the integral algebraically.

Example 10.4 (Solving for the Integral)**.** Evaluate $I = \int e^{2x} \cos(3x) \, dx$. *Solution:* Use LIATE: Trig before Exponential (or vice versa, the choice here is less critical, but consistency helps). Let's follow LIATE. 1. First application: $u = \cos(3x)$, $dv = e^{2x} \, dx$. $du = -3 \sin(3x) \, dx$, $v = \frac{1}{2} e^{2x}$.

$$I = (\cos(3x))(\frac{1}{2} e^{2x}) - \int (\frac{1}{2} e^{2x})(-3 \sin(3x) \, dx)$$

$$I = \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{2} \int e^{2x} \sin(3x) \, dx$$

2. Second application (for the remaining integral): Keep the same type for u. $u = \sin(3x)$, $dv = e^{2x} \, dx$. $du = 3 \cos(3x) \, dx$, $v = \frac{1}{2} e^{2x}$.

$$\int e^{2x} \sin(3x) \, dx = (\sin(3x))(\frac{1}{2} e^{2x}) - \int (\frac{1}{2} e^{2x})(3 \cos(3x) \, dx)$$

$$= \frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} \int e^{2x} \cos(3x) \, dx$$

Notice the original integral I reappears. 3. Substitute back and solve for I:

$$I = \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{2} \left[\frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} I \right]$$

$$I = \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x) - \frac{9}{4} I$$

Move all terms with I to the left side:

$$I + \frac{9}{4} I = \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x)$$

$$\left(1 + \frac{9}{4}\right) I = \frac{13}{4} I = \frac{1}{4} e^{2x}(2\cos(3x) + 3\sin(3x))$$

Multiply by 4/13:

$$I = \int e^{2x}\cos(3x)\,dx = \frac{1}{13}e^{2x}(2\cos(3x) + 3\sin(3x)) + C$$

(Remember to add the constant C at the end).

11 Powers of Trigonometric Functions

This section deals with integrals of the form $\int \sin^m x \cos^n x\,dx$ and $\int \tan^m x \sec^n x\,dx$. The strategy depends on whether the powers m and n are odd or even.

11.1 Integrals of $\sin^m x \cos^n x$

Case 1: At least one of m or n is an odd positive integer.

- If m is odd ($m = 2k+1$): Save one $\sin x$ factor ($d(\cos x) = -\sin x dx$). Convert the remaining even power $\sin^{2k} x = (\sin^2 x)^k$ to cosines using $\sin^2 x = 1 - \cos^2 x$. The integral becomes $\int(\text{polynomial in } \cos x)\sin x\,dx$. Substitute $u = \cos x$.
- If n is odd ($n = 2k+1$): Save one $\cos x$ factor ($d(\sin x) = \cos x dx$). Convert the remaining even power $\cos^{2k} x = (\cos^2 x)^k$ to sines using $\cos^2 x = 1 - \sin^2 x$. The integral becomes $\int(\text{polynomial in } \sin x)\cos x\,dx$. Substitute $u = \sin x$.
- If both are odd, either method works.

Example 11.1 (Odd Power Case)**.** Evaluate $\int \sin^3 x \cos^2 x\,dx$. *Solution:* The power of sine ($m = 3$) is odd. Save one $\sin x$. Convert $\sin^2 x$.

$$\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x(\sin x\,dx)$$

$$= \int (1 - \cos^2 x)\cos^2 x(\sin x\,dx)$$

Let $u = \cos x$, $du = -\sin x\,dx \implies \sin x\,dx = -du$.

$$= \int (1 - u^2)u^2(-du) = \int (u^4 - u^2)\,du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

Substitute back $u = \cos x$:

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

Case 2: Both m and n are non-negative even integers. Use the half-angle (power-reducing) identities repeatedly:

$$\sin^2 x = \frac{1 - \cos(2x)}{2}, \quad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

Also, the identity $\sin x \cos x = \frac{1}{2}\sin(2x)$ might be useful. The goal is to reduce the powers until integrable terms remain (possibly requiring further odd/even analysis on terms like $\cos^k(2x)$).

Example 11.2 (Even Powers Case)**.** Evaluate $\int \sin^2 x\,dx$. *Solution:* Use $\sin^2 x = \frac{1-\cos(2x)}{2}$.

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\int (1-\cos(2x))\,dx \\ &= \frac{1}{2}\left(\int 1\,dx - \int \cos(2x)\,dx\right) \\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C\end{aligned}$$

(For $\int \cos(2x)dx$, let $u = 2x, du = 2dx$).

Example 11.3 (Even Powers Case)**.** Evaluate $\int \sin^2 x \cos^2 x\,dx$. *Solution:* Method 1: Use half-angle identities for both.

$$\begin{aligned}\int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)dx &= \frac{1}{4}\int (1-\cos^2(2x))\,dx \\ &= \frac{1}{4}\int \sin^2(2x)\,dx\end{aligned}$$

Now use the half-angle identity again, this time for $\sin^2(2x)$: $\sin^2(A) = \frac{1-\cos(2A)}{2}$. Let $A = 2x$, so $2A = 4x$.

$$\begin{aligned}&= \frac{1}{4}\int \frac{1-\cos(4x)}{2}\,dx = \frac{1}{8}\int (1-\cos(4x))\,dx \\ &= \frac{1}{8}\left(x - \frac{1}{4}\sin(4x)\right) + C = \frac{1}{8}x - \frac{1}{32}\sin(4x) + C\end{aligned}$$

Method 2: Use $\sin x \cos x = \frac{1}{2}\sin(2x)$.

$$\int (\sin x \cos x)^2\,dx = \int \left(\frac{1}{2}\sin(2x)\right)^2 dx = \int \frac{1}{4}\sin^2(2x)\,dx$$

This leads to the same integral as in Method 1.

11.2 Integrals of $\tan^m x \sec^n x$

Case 1: Power of secant n is an even positive integer ($n = 2k, k \geq 1$). Save one $\sec^2 x$ factor ($d(\tan x) = \sec^2 x dx$). Convert the remaining $\sec^{n-2} x = \sec^{2k-2} x = (\sec^2 x)^{k-1}$ to tangents using $\sec^2 x = 1 + \tan^2 x$. The integral becomes $\int(\text{polynomial in } \tan x)\sec^2 x\,dx$. Substitute $u = \tan x$.

Example 11.4 (Even Secant Power)**.** Evaluate $\int \tan^{1/3} x \sec^4 x\,dx$. *Solution:* Power of secant ($n = 4$) is even. Save $\sec^2 x$. Convert $\sec^2 x$.

$$\begin{aligned}\int \tan^{1/3} x \sec^4 x\,dx &= \int \tan^{1/3} x \sec^2 x(\sec^2 x\,dx) \\ &= \int \tan^{1/3} x(1+\tan^2 x)(\sec^2 x\,dx)\end{aligned}$$

Let $u = \tan x$, $du = \sec^2 x\,dx$.

$$\begin{aligned}&= \int u^{1/3}(1+u^2)\,du = \int (u^{1/3} + u^{1/3+2})\,du = \int (u^{1/3} + u^{7/3})\,du \\ &= \frac{u^{4/3}}{4/3} + \frac{u^{10/3}}{10/3} + C = \frac{3}{4}u^{4/3} + \frac{3}{10}u^{10/3} + C\end{aligned}$$

Substitute back $u = \tan x$:

$$= \frac{3}{4}\tan^{4/3} x + \frac{3}{10}\tan^{10/3} x + C$$

Case 2: Power of tangent m is an odd positive integer ($m = 2k+1, k \geq 0$). Save one $\sec x \tan x$ factor ($d(\sec x) = \sec x \tan x dx$). Convert the remaining $\tan^{m-1} x = \tan^{2k} x = (\tan^2 x)^k$ to secants using $\tan^2 x = \sec^2 x - 1$. The integral becomes $\int(\text{polynomial in } \sec x)(\sec x \tan x\, dx)$. Substitute $u = \sec x$.

Example 11.5 (Odd Tangent Power)**.** Evaluate $\int \tan^3 x \sec^4 x\, dx$. *Solution:* Power of tangent ($m = 3$) is odd. Save $\sec x \tan x$. Convert $\tan^2 x$.

$$\int \tan^3 x \sec^4 x\, dx = \int \tan^2 x \sec^3 x(\sec x \tan x\, dx)$$

$$= \int (\sec^2 x - 1) \sec^3 x(\sec x \tan x\, dx)$$

Let $u = \sec x$, $du = \sec x \tan x\, dx$.

$$= \int (u^2 - 1)u^3\, du = \int (u^5 - u^3)\, du$$

$$= \frac{u^6}{6} - \frac{u^4}{4} + C$$

Substitute back $u = \sec x$:

$$= \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C$$

(Note: This integral also fits Case 1 (even secant power). Applying that method yields $\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C$. These two results differ by a constant, which can be verified using identities.)

Case 3: Power of tangent m is even and power of secant n is odd. This case is more complex. Often involves converting tangents to secants using $\tan^2 x = \sec^2 x - 1$, resulting in powers of $\sec x$ only. Integration by parts might be needed to evaluate integrals of powers of $\sec x$. Standard reduction formulas exist for $\int \sec^n x dx$. Special cases: $\int \tan x dx = \ln|\sec x| + C$. $\int \sec x dx = \ln|\sec x + \tan x| + C$. $\int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$ (requires integration by parts).

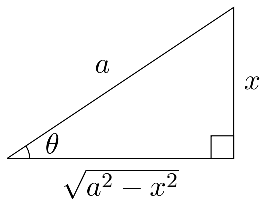
12 Trigonometric Substitution

This technique is useful for integrating functions containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ (where $a > 0$). The strategy is to make a substitution involving a trigonometric function that simplifies the radical expression using Pythagorean identities.

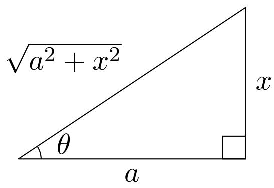
12.1 The Three Main Cases

1. **Integrands involving $\sqrt{a^2 - x^2}$:**

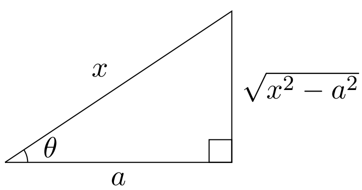
   - Substitute $x = a \sin\theta$, with $-\pi/2 \leq \theta \leq \pi/2$.
   - Then $dx = a \cos\theta\, d\theta$.
   - The radical becomes: $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta}$.
   - Since $-\pi/2 \leq \theta \leq \pi/2$, $\cos\theta \geq 0$, so $\sqrt{a^2 \cos^2\theta} = a \cos\theta$.
   - Reference Triangle: Based on $\sin\theta = x/a$ (Opposite/Hypotenuse). Adjacent side is $\sqrt{a^2 - x^2}$.



2. **Integrands involving $\sqrt{a^2 + x^2}$:**
    - Substitute $x = a \tan \theta$, with $-\pi/2 < \theta < \pi/2$.
    - Then $dx = a \sec^2 \theta \, d\theta$.
    - The radical becomes: $\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2 \theta} = \sqrt{a^2(1 + \tan^2 \theta)} = \sqrt{a^2 \sec^2 \theta}$.
    - Since $-\pi/2 < \theta < \pi/2$, $\sec \theta > 0$, so $\sqrt{a^2 \sec^2 \theta} = a \sec \theta$.
    - Reference Triangle: Based on $\tan \theta = x/a$ (Opposite/Adjacent). Hypotenuse is $\sqrt{a^2 + x^2}$.



3. **Integrands involving $\sqrt{x^2 - a^2}$:**
    - Substitute $x = a \sec \theta$, with $0 \leq \theta < \pi/2$ or $\pi \leq \theta < 3\pi/2$ (or often restricted to $0 \leq \theta < \pi/2$ or $\pi/2 < \theta \leq \pi$ to ensure $\tan \theta$ is well-defined).
    - Then $dx = a \sec \theta \tan \theta \, d\theta$.
    - The radical becomes: $\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = \sqrt{a^2(\sec^2 \theta - 1)} = \sqrt{a^2 \tan^2 \theta}$.
    - $\sqrt{a^2 \tan^2 \theta} = a|\tan \theta|$. The absolute value is important. If we restrict θ to $0 \leq \theta < \pi/2$ (for $x \geq a$) then $\tan \theta \geq 0$, so $\sqrt{x^2 - a^2} = a \tan \theta$. If $\pi/2 < \theta \leq \pi$ (for $x \leq -a$), then $\tan \theta \leq 0$, so $\sqrt{x^2 - a^2} = -a \tan \theta$. Careful handling of the interval for θ is needed, especially for definite integrals.
    - Reference Triangle: Based on $\sec \theta = x/a$ (Hypotenuse/Adjacent). Opposite side is $\sqrt{x^2 - a^2}$.



After performing the integration in terms of θ, use the reference triangle to substitute back in terms of x.

Example 12.1 (Case $\sqrt{a^2 - x^2}$)**.** Evaluate $\int \frac{x^2}{\sqrt{4 - x^2}} \, dx$. *Solution:* Recognize the form $\sqrt{a^2 - x^2}$ with $a = 2$. Substitute $x = 2 \sin \theta$. Then $dx = 2 \cos \theta \, d\theta$. $\sqrt{4 - x^2} = \sqrt{4 - 4 \sin^2 \theta} = \sqrt{4 \cos^2 \theta} = 2 \cos \theta$ (since $-\pi/2 \leq \theta \leq \pi/2$). The integral becomes:

$$\int \frac{(2 \sin \theta)^2}{2 \cos \theta} (2 \cos \theta \, d\theta) = \int 4 \sin^2 \theta \, d\theta$$

Use the half-angle identity $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$:

$$= \int 4\left(\frac{1-\cos(2\theta)}{2}\right) d\theta = 2\int (1-\cos(2\theta))\, d\theta$$

$$= 2\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C = 2\theta - \sin(2\theta) + C$$

Use the double-angle identity $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$= 2\theta - 2\sin\theta\cos\theta + C$$

Now substitute back using the reference triangle or the substitution relations: From $x = 2\sin\theta$, we have $\sin\theta = x/2$. Then $\theta = \arcsin(x/2)$. From the triangle (Hyp=2, Opp=x), Adjacent $= \sqrt{4-x^2}$. So $\cos\theta = \frac{\sqrt{4-x^2}}{2}$. Substitute these into the result:

$$= 2\arcsin\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$$

$$= 2\arcsin\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C$$

Example 12.2 (Case $\sqrt{a^2+x^2}$ after Completing the Square)**.** Evaluate $\int \frac{1}{(x^2+2x+5)^{3/2}}\, dx$. *Solution:* First complete the square in the denominator: $x^2+2x+5 = (x^2+2x+1)+4 = (x+1)^2+4$. The integral is $\int \frac{1}{((x+1)^2+4)^{3/2}}\, dx$. This involves $\sqrt{u^2+a^2}$ where $u = x+1$ and $a = 2$. Let $u = a\tan\theta \implies x+1 = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$. The term $(x+1)^2+4 = (2\tan\theta)^2+4 = 4\tan^2\theta + 4 = 4(\tan^2\theta+1) = 4\sec^2\theta$. So, $((x+1)^2+4)^{3/2} = (4\sec^2\theta)^{3/2} = (\sqrt{4\sec^2\theta})^3 = (2\sec\theta)^3 = 8\sec^3\theta$ (assuming $\sec\theta > 0$). Substitute into the integral:

$$\int \frac{1}{8\sec^3\theta}(2\sec^2\theta\, d\theta) = \frac{2}{8}\int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta = \frac{1}{4}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{4}\int \cos\theta\, d\theta = \frac{1}{4}\sin\theta + C$$

Now substitute back. From $x+1 = 2\tan\theta$, $\tan\theta = (x+1)/2$. Reference triangle: Opp $= x+1$, Adj $= 2$. Hyp $= \sqrt{(x+1)^2+2^2} = \sqrt{x^2+2x+5}$. $\sin\theta = \frac{\text{Opp}}{\text{Hyp}} = \frac{x+1}{\sqrt{x^2+2x+5}}$. Result:

$$= \frac{1}{4}\frac{x+1}{\sqrt{x^2+2x+5}} + C$$

13 Integration by Partial Fractions

This technique is used to integrate **rational functions**, which are ratios of polynomials, $f(x) = \frac{P(x)}{Q(x)}$.

13.1 Preprocessing: Proper Fractions

Partial fraction decomposition only applies directly to **proper rational functions**, where the degree of the numerator $P(x)$ is strictly less than the degree of the denominator $Q(x)$.

If $\deg(P) \geq \deg(Q)$ (an improper fraction), perform **polynomial long division** first:

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $S(x)$ is the quotient polynomial and $\frac{R(x)}{Q(x)}$ is the remainder fraction, which is now proper ($\deg(R) < \deg(Q)$). Then integrate $S(x)$ (which is easy) and apply partial fractions to the remainder $\frac{R(x)}{Q(x)}$.

14 Improper Integrals

Standard definite integrals $\int_a^b f(x)\,dx$ require two conditions:

1. The interval of integration $[a, b]$ is finite (bounded).
2. The integrand $f(x)$ is bounded on $[a, b]$ (i.e., $|f(x)| \leq M$ for some constant M). If f is continuous on $[a, b]$, this condition is automatically satisfied.

Integrals where one or both of these conditions are not met are called **improper integrals**. There are two main types.

14.1 Type 1: Improper Integrals over Unbounded Intervals

These integrals involve limits of integration extending to infinity.

Definition 14.1 (Type 1 Improper Integrals)**.** *1. If f is continuous on $[a, \infty)$, the improper integral $\int_a^\infty f(x)\,dx$ is defined as:*

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

2. If f is continuous on $(-\infty, b]$, the improper integral $\int_{-\infty}^b f(x)\,dx$ is defined as:

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

3. If f is continuous on $(-\infty, \infty)$, the improper integral $\int_{-\infty}^\infty f(x)\,dx$ is defined by splitting it at an arbitrary point c (often $c = 0$):

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

$$= \left(\lim_{a\to-\infty} \int_a^c f(x)\,dx\right) + \left(\lim_{b\to\infty} \int_c^b f(x)\,dx\right)$$

*In cases (1) and (2), if the limit exists and is finite, the improper integral is said to **converge**, and its value is the limit. If the limit does not exist or is infinite, the integral **diverges**. In case (3), the integral $\int_{-\infty}^\infty f(x)\,dx$ converges only if *both* improper integrals on the right side converge independently. If either one diverges, then $\int_{-\infty}^\infty f(x)\,dx$ diverges.*

Example 14.1 (Convergent Type 1 Integral)**.** Evaluate $\int_2^\infty \frac{1}{x^3}\,dx$. *Solution:* This is improper because the upper limit is infinite.

$$\int_2^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_2^b x^{-3}\,dx$$

First, evaluate the definite integral:

$$\int_2^b x^{-3}\,dx = \left[\frac{x^{-2}}{-2}\right]_2^b = \left[-\frac{1}{2x^2}\right]_2^b = \left(-\frac{1}{2b^2}\right) - \left(-\frac{1}{2(2)^2}\right) = -\frac{1}{2b^2} + \frac{1}{8}$$

Now, take the limit as $b \to \infty$:

$$\lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{8}\right) = 0 + \frac{1}{8} = \frac{1}{8}$$

Since the limit is finite, the integral converges to 1/8.

Example 14.2 (Divergent Type 1 Integral)**.** Evaluate $\int_1^\infty x^2\,dx$. *Solution:*

$$\int_1^\infty x^2\,dx = \lim_{b\to\infty}\int_1^b x^2\,dx$$

$$\int_1^b x^2\,dx = \left[\frac{x^3}{3}\right]_1^b = \frac{b^3}{3} - \frac{1^3}{3} = \frac{b^3}{3} - \frac{1}{3}$$

$$\lim_{b\to\infty}\left(\frac{b^3}{3} - \frac{1}{3}\right) = \infty$$

Since the limit is infinite, the integral diverges.

Example 14.3 (Integral over $(-\infty, \infty)$)**.** Evaluate $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$. *Solution:* Split the integral at $c = 0$:

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$

Evaluate the first part:

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx$$

Recall $\int \frac{1}{1+x^2}dx = \tan^{-1} x$.

$$= \lim_{a\to-\infty}[\tan^{-1} x]_a^0 = \lim_{a\to-\infty}(\tan^{-1} 0 - \tan^{-1} a)$$

$$= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

(Since $\lim_{a\to-\infty}\tan^{-1} a = -\pi/2$). This part converges.

Evaluate the second part:

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty}[\tan^{-1} x]_0^b = \lim_{b\to\infty}(\tan^{-1} b - \tan^{-1} 0)$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

(Since $\lim_{b\to\infty}\tan^{-1} b = \pi/2$). This part also converges.

Since both parts converge, the original integral converges, and its value is the sum:

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Theorem 14.1 (p-Integral Test for $\int_a^\infty \frac{1}{x^p}dx$)**.** *For $a > 0$, the improper integral $\int_a^\infty \frac{1}{x^p}\,dx$:*

- *Converges if $p > 1$.*
- *Diverges if $p \le 1$.*

Proof. If $p = 1$: $\int_a^\infty \frac{1}{x}dx = \lim_{b\to\infty}[\ln|x|]_a^b = \lim_{b\to\infty}(\ln b - \ln a) = \infty$. Diverges. If $p \ne 1$: $\int_a^\infty x^{-p}dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_a^b = \lim_{b\to\infty}\frac{1}{1-p}(b^{1-p} - a^{1-p})$. The limit depends on the behavior of b^{1-p} as $b\to\infty$. If $p > 1$, then $1 - p < 0$. Let $1 - p = -q$ where $q > 0$. Then $\lim_{b\to\infty} b^{1-p} = \lim_{b\to\infty} b^{-q} = \lim_{b\to\infty}\frac{1}{b^q} = 0$. The integral converges to $\frac{1}{1-p}(-a^{1-p}) = \frac{a^{1-p}}{p-1}$. If $p < 1$, then $1 - p > 0$. Then $\lim_{b\to\infty} b^{1-p} = \infty$. The integral diverges. □

14.2 Type 2: Improper Integrals with Infinite Discontinuities

These integrals involve an integrand $f(x)$ that becomes unbounded (approaches $\pm\infty$) at one or more points within the interval of integration $[a, b]$.

Definition 14.2 (Type 2 Improper Integrals). *1. If f is continuous on $[a, b)$ and discontinuous at b (e.g., $\lim_{x\to b^-} |f(x)| = \infty$), then:*

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

2. If f is continuous on $(a, b]$ and discontinuous at a (e.g., $\lim_{x\to a^+} |f(x)| = \infty$), then:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

3. If f has a discontinuity at c within the open interval (a, b), and is continuous elsewhere on $[a, b]$, then split the integral at c:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \left(\lim_{t\to c^-} \int_a^t f(x)\,dx\right) + \left(\lim_{t\to c^+} \int_t^b f(x)\,dx\right)$$

*In cases (1) and (2), if the limit exists and is finite, the integral **converges**. Otherwise, it **diverges**. In case (3), the integral $\int_a^b f(x)\,dx$ converges only if *both* improper integrals on the right side converge independently. If either one diverges, the original integral diverges.*

Example 14.4 (Discontinuity at Lower Limit). Evaluate $\int_0^4 \frac{1}{\sqrt{x}}\,dx$. *Solution:* The integrand $f(x) = 1/\sqrt{x} = x^{-1/2}$ is discontinuous (unbounded) at the lower limit $x = 0$.

$$\int_0^4 x^{-1/2}\,dx = \lim_{t\to 0^+} \int_t^4 x^{-1/2}\,dx$$

Evaluate the definite integral:

$$\int_t^4 x^{-1/2}\,dx = \left[\frac{x^{1/2}}{1/2}\right]_t^4 = [2\sqrt{x}]_t^4 = 2\sqrt{4} - 2\sqrt{t} = 4 - 2\sqrt{t}$$

Now take the limit:

$$\lim_{t\to 0^+} (4 - 2\sqrt{t}) = 4 - 2\sqrt{0} = 4$$

Since the limit is finite, the integral converges to 4.

Example 14.5 (Discontinuity at Upper Limit). Evaluate $\int_0^2 \frac{1}{\sqrt{2-x}}\,dx$. *Solution:* The integrand $f(x) = (2-x)^{-1/2}$ is discontinuous (unbounded) at the upper limit $x = 2$.

$$\int_0^2 (2-x)^{-1/2}\,dx = \lim_{t\to 2^-} \int_0^t (2-x)^{-1/2}\,dx$$

Evaluate the definite integral. Let $u = 2 - x$, $du = -dx$. When $x = 0, u = 2$. When $x = t, u = 2 - t$.

$$\int_{x=0}^{x=t} (2-x)^{-1/2}\,dx = \int_{u=2}^{u=2-t} u^{-1/2}(-du) = -\int_2^{2-t} u^{-1/2}\,du$$

$$= -\left[\frac{u^{1/2}}{1/2}\right]_2^{2-t} = -[2\sqrt{u}]_2^{2-t} = -(2\sqrt{2-t} - 2\sqrt{2}) = 2\sqrt{2} - 2\sqrt{2-t}$$

Now take the limit:

$$\lim_{t\to 2^-} (2\sqrt{2} - 2\sqrt{2-t}) = 2\sqrt{2} - 2\sqrt{2-2} = 2\sqrt{2} - 0 = 2\sqrt{2}$$

The integral converges to $2\sqrt{2}$.

Example 14.6 (Discontinuity within the Interval)**.** Evaluate $\int_{-1}^{1} \frac{1}{x^{5/3}}\, dx$. *Solution:* The integrand $f(x) = x^{-5/3}$ has an infinite discontinuity at $x = 0$, which is within $(-1, 1)$. We must split the integral at $x = 0$:

$$\int_{-1}^{1} x^{-5/3}\, dx = \int_{-1}^{0} x^{-5/3}\, dx + \int_{0}^{1} x^{-5/3}\, dx$$

Evaluate the first part:

$$\int_{-1}^{0} x^{-5/3}\, dx = \lim_{t\to 0^-} \int_{-1}^{t} x^{-5/3}\, dx$$

$$\int_{-1}^{t} x^{-5/3}\, dx = \left[\frac{x^{-5/3+1}}{-5/3+1}\right]_{-1}^{t} = \left[\frac{x^{-2/3}}{-2/3}\right]_{-1}^{t} = -\frac{3}{2}[x^{-2/3}]_{-1}^{t}$$

$$= -\frac{3}{2}\left(t^{-2/3} - (-1)^{-2/3}\right) = -\frac{3}{2}\left(\frac{1}{t^{2/3}} - \frac{1}{((-1)^2)^{1/3}}\right) = -\frac{3}{2}\left(\frac{1}{t^{2/3}} - 1\right)$$

Now take the limit:

$$\lim_{t\to 0^-} \left(-\frac{3}{2}\left(\frac{1}{t^{2/3}} - 1\right)\right)$$

As $t \to 0^-$, $t^2 \to 0^+$, so $t^{2/3} = (t^2)^{1/3} \to 0^+$. Thus, $1/t^{2/3} \to +\infty$. The limit is $-\frac{3}{2}(\infty - 1) = -\infty$. Since the first part diverges, the original integral $\int_{-1}^{1} x^{-5/3}\, dx$ diverges. There is no need to evaluate the second part.

Theorem 14.2 (p-Integral Test for $\int_0^a \frac{1}{x^p} dx$)**.** *For $a > 0$, the improper integral $\int_0^a \frac{1}{x^p}\, dx$:*

- *Converges if $p < 1$.*
- *Diverges if $p \geq 1$.*

Proof. Similar to the Type 1 p-integral test, evaluating $\lim_{t\to 0^+} \int_t^a x^{-p} dx$. The limit involves t^{1-p}. If $p < 1$, then $1 - p > 0$, so $\lim_{t\to 0^+} t^{1-p} = 0$. Converges. If $p > 1$, then $1 - p < 0$. Let $1 - p = -q$ where $q > 0$. $\lim_{t\to 0^+} t^{1-p} = \lim_{t\to 0^+} t^{-q} = \lim_{t\to 0^+} \frac{1}{t^q} = \infty$. Diverges. If $p = 1$, $\int_t^a \frac{1}{x} dx = [\ln|x|]_t^a = \ln a - \ln t$. $\lim_{t\to 0^+}(\ln a - \ln t) = \infty$. Diverges. □

Comparing the two p-tests: $\int_a^\infty \frac{1}{x^p} dx$ converges for $p > 1$, while $\int_0^a \frac{1}{x^p} dx$ converges for $p < 1$.

Exercise 14.1. Evaluate the improper integrals or show divergence.

1. $\int_{-\infty}^{3} e^{2x}\, dx$
2. $\int_1^\infty \frac{\ln x}{x}\, dx$ (Use substitution $u = \ln x$)
3. $\int_e^\infty \frac{1}{x(\ln x)^3}\, dx$ (Use substitution $u = \ln x$)
4. $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^{3/2}}\, dx$ (Check if odd function?)

5. $\int_{-1}^{\infty} \frac{1}{x^2+2x+2}\,dx$ (Complete square, use arctan)

6. $\int_0^{\infty} e^{-x}\sin x\,dx$ (Use integration by parts twice, then limit)

7. $\int_0^{\pi} \frac{\sin x}{1+\cos x}\,dx$ (Discontinuity at $x=\pi$)

Applications of Integration

Having defined the definite integral primarily in terms of net signed area and developed methods for evaluating it, we now explore its applications, starting with calculating the geometric area of more complex regions.

15 Area Between Curves

15.1 Area Between $y = f(x)$ and the x-axis (Total Area)

Recall that the definite integral $\int_a^b f(x)\,dx$ gives the *net signed area*. If we want the total geometric area enclosed between the graph $y = f(x)$ and the x-axis over $[a, b]$, treating areas below the axis as positive, we must integrate the absolute value of the function.

Definition 15.1 (Total Area)**.** *Suppose the function $y = f(x)$ is continuous on $[a, b]$. The **total area** A bounded by its graph $y = f(x)$ and the x-axis on $[a, b]$ is given by:*

$$A = \int_a^b |f(x)|\,dx$$

To evaluate this, we typically find the intervals where $f(x) \ge 0$ and where $f(x) < 0$ within $[a, b]$. Then we split the integral accordingly:

$$\int_a^b |f(x)|\,dx = \int_{\text{intervals where } f\ge 0} f(x)\,dx + \int_{\text{intervals where } f<0} (-f(x))\,dx$$

Example 15.1 (Total Area: $y = x^2 - 4$)**.** Find the total area bounded by the graph $y = x^2 - 4$ and the x-axis on the interval $[-1, 3]$. *Solution:* First, determine where $f(x) = x^2 - 4$ is positive or negative. The roots are $x^2 - 4 = 0 \implies x = \pm 2$. The parabola opens upwards. * On $[-1, 2)$, $x^2 - 4 < 0$, so $|f(x)| = -(x^2 - 4) = 4 - x^2$. * On $[2, 3]$, $x^2 - 4 \ge 0$, so $|f(x)| = x^2 - 4$.

Split the integral at $x = 2$:

$$A = \int_{-1}^{3} |x^2 - 4|\,dx = \int_{-1}^{2} -(x^2 - 4)\,dx + \int_2^3 (x^2 - 4)\,dx$$

$$A = \int_{-1}^{2} (4 - x^2)\,dx + \int_2^3 (x^2 - 4)\,dx$$

Evaluate each integral:

$$\begin{aligned}
\int_{-1}^{2} (4 - x^2)\,dx &= \left[4x - \frac{x^3}{3}\right]_{-1}^{2} \\
&= \left(4(2) - \frac{2^3}{3}\right) - \left(4(-1) - \frac{(-1)^3}{3}\right) \\
&= \left(8 - \frac{8}{3}\right) - \left(-4 - (-\frac{1}{3})\right) \\
&= \frac{16}{3} - \left(-\frac{11}{3}\right) = \frac{27}{3} = 9
\end{aligned}$$

$$
\begin{aligned}
\int_2^3 (x^2 - 4)\,dx &= \left[\frac{x^3}{3} - 4x\right]_2^3 \\
&= \left(\frac{3^3}{3} - 4(3)\right) - \left(\frac{2^3}{3} - 4(2)\right) \\
&= (9 - 12) - \left(\frac{8}{3} - 8\right) \\
&= -3 - \left(-\frac{16}{3}\right) = -3 + \frac{16}{3} = \frac{-9+16}{3} = \frac{7}{3}
\end{aligned}
$$

Total Area $A = 9 + \frac{7}{3} = \frac{27+7}{3} = \frac{34}{3}$. and [2, 3]]

15.2 Area Between Two Curves $y = f(x)$ and $y = g(x)$

To find the area of the region bounded between the graphs of two continuous functions $y = f(x)$ and $y = g(x)$ over an interval $[a, b]$, we integrate the absolute difference between the functions.

Definition 15.2 (Area Between Two Graphs)**.** *Let f and g be continuous functions on $[a, b]$. The **total area** A bounded between their graphs $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ is given by:*

$$A = \int_a^b |f(x) - g(x)|\,dx$$

Remark 15.1 (Upper minus Lower)**.** Geometrically, $|f(x) - g(x)|$ represents the vertical distance between the two curves at x. If we can determine which function is consistently above the other on subintervals, we can simplify the integral. If $f(x) \geq g(x)$ on $[c, d]$, the area between them on that interval is $\int_c^d (f(x) - g(x))\,dx$. If $g(x) \geq f(x)$ on $[c, d]$, the area between them is $\int_c^d (g(x) - f(x))\,dx$. In general, on any interval, the integrand can be thought of as:

$$A = \int_a^b (\text{Upper function} - \text{Lower function})\,dx$$

To apply this, we usually need to: 1. Find the points of intersection by solving $f(x) = g(x)$. 2. Determine which function is upper and which is lower on the interval(s) between intersection points (or the given interval $[a, b]$). This often involves testing a point within each interval. 3. Set up and evaluate the integral(s).

Example 15.2 (Area Between Parabola and Line)**.** Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$. *Solution:* The problem implies finding the area between the intersection points. 1. Find intersections: Solve $2x^2 + 10 = 4x + 16$. $2x^2 - 4x - 6 = 0$ $x^2 - 2x - 3 = 0$ $(x - 3)(x + 1) = 0$ Intersection points occur at $x = -1$ and $x = 3$. The interval is $[-1, 3]$. 2. Determine Upper/Lower: Choose a test point in $(-1, 3)$, e.g., $x = 0$. At $x = 0$, $y = 2x^2 + 10 \implies y = 10$. At $x = 0$, $y = 4x + 16 \implies y = 16$. Since $16 > 10$, the line $y = 4x + 16$ is the upper function on $(-1, 3)$. 3. Set up and Evaluate Integral:

$$A = \int_{-1}^3 (\text{Upper} - \text{Lower})\,dx = \int_{-1}^3 ((4x + 16) - (2x^2 + 10))\,dx$$

$$A = \int_{-1}^3 (-2x^2 + 4x + 6)\,dx$$

$$A = \left[-\frac{2x^3}{3} + \frac{4x^2}{2} + 6x\right]_{-1}^3 = \left[-\frac{2}{3}x^3 + 2x^2 + 6x\right]_{-1}^3$$

Evaluate at limits: At $x = 3$: $-\frac{2}{3}(27) + 2(9) + 6(3) = -18 + 18 + 18 = 18$. At $x = -1$: $-\frac{2}{3}(-1) + 2(1) + 6(-1) = \frac{2}{3} + 2 - 6 = \frac{2}{3} - 4 = \frac{2-12}{3} = -\frac{10}{3}$.

$$A = (18) - \left(-\frac{10}{3}\right) = 18 + \frac{10}{3} = \frac{54+10}{3} = \frac{64}{3}$$

The area is 64/3.

Example 15.3 (Area Over a Specified Interval)**.** Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$, and $x = 5$. *Solution:* The curves intersect at $x = -1$ and $x = 3$. The specified interval $[-2, 5]$ includes these intersection points. We need to consider which function is upper/lower on each subinterval: $[-2, -1]$, $[-1, 3]$, and $[3, 5]$. * On $[-1, 3]$: $4x+16$ is upper. * Check interval $[-2, -1]$: Test $x = -1.5$. $y = 2(-1.5)^2 + 10 = 2(2.25) + 10 = 4.5 + 10 = 14.5$. $y = 4(-1.5) + 16 = -6 + 16 = 10$. Parabola $y = 2x^2 + 10$ is upper on $[-2, -1]$. * Check interval $[3, 5]$: Test $x = 4$. $y = 2(4)^2 + 10 = 32 + 10 = 42$. $y = 4(4) + 16 = 16 + 16 = 32$. Parabola $y = 2x^2 + 10$ is upper on $[3, 5]$.

The integral must be split:

$$A = \int_{-2}^{5} |(4x+16) - (2x^2+10)|\, dx$$

$$A = \int_{-2}^{-1} [(2x^2+10) - (4x+16)]\, dx + \int_{-1}^{3} [(4x+16) - (2x^2+10)]\, dx + \int_{3}^{5} [(2x^2+10) - (4x+16)]\, dx$$

$$A = \int_{-2}^{-1} (2x^2 - 4x - 6)\, dx + \int_{-1}^{3} (-2x^2 + 4x + 6)\, dx + \int_{3}^{5} (2x^2 - 4x - 6)\, dx$$

We already calculated $\int_{-1}^{3}(-2x^2 + 4x + 6)\, dx = 64/3$. Evaluate the other two:

$$\begin{aligned}\int_{-2}^{-1} (2x^2 - 4x - 6)\, dx &= \left[\frac{2x^3}{3} - 2x^2 - 6x\right]_{-2}^{-1} \\ &= (\frac{-2}{3} - 2 + 6) - (\frac{-16}{3} - 8 + 12) \\ &= (\frac{-2}{3} + 4) - (\frac{-16}{3} + 4) = \frac{10}{3} - (-\frac{4}{3}) = \frac{14}{3}\end{aligned}$$

$$\begin{aligned}\int_{3}^{5} (2x^2 - 4x - 6)\, dx &= \left[\frac{2x^3}{3} - 2x^2 - 6x\right]_{3}^{5} \\ &= (\frac{2(125)}{3} - 2(25) - 6(5)) - (\frac{2(27)}{3} - 2(9) - 6(3)) \\ &= (\frac{250}{3} - 50 - 30) - (18 - 18 - 18) \\ &= (\frac{250}{3} - 80) - (-18) = \frac{250-240}{3} + 18 = \frac{10}{3} + 18 = \frac{10+54}{3} = \frac{64}{3}\end{aligned}$$

Total Area $A = \frac{14}{3} + \frac{64}{3} + \frac{64}{3} = \frac{142}{3}$.

15.3 Integrating with Respect to y

Sometimes it is easier to find the area between curves by integrating with respect to y. This is particularly useful when the boundaries are given as functions of y, i.e., $x = f(y)$ and $x = g(y)$.

If a region is bounded by $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where $f(y) \geq g(y)$ for $y \in [c, d]$, the area is:

$$A = \int_{c}^{d} (f(y) - g(y))\, dy$$

In general, integrate the absolute difference, or more intuitively:

$$A = \int_c^d (\text{Right function} - \text{Left function})\, dy$$

Here, "Right" and "Left" refer to the functions when viewed as x in terms of y. [Image showing area integrated wrt y, right-left curves]

Example 15.4 (Integrating wrt y)**.** Determine the area of the region bounded by $2x = y^2 - 6$ and $y = x - 1$. *Solution:* First, express both curves as x in terms of y. Curve 1: $x = \frac{1}{2}y^2 - 3$. (Parabola opening right) Curve 2: $x = y + 1$. (Line) Find intersection points by setting x-values equal:

$$\frac{1}{2}y^2 - 3 = y + 1$$
$$y^2 - 6 = 2y + 2$$
$$y^2 - 2y - 8 = 0$$
$$(y - 4)(y + 2) = 0$$

Intersections occur at $y = -2$ and $y = 4$. These will be our limits of integration $c = -2, d = 4$. Determine Right/Left curves on $[-2, 4]$. Test $y = 0$. Curve 1 ($x = \frac{1}{2}y^2 - 3$): At $y = 0$, $x = -3$. Curve 2 ($x = y + 1$): At $y = 0$, $x = 1$. Since $1 > -3$, the line $x = y + 1$ is the Right function, and the parabola $x = \frac{1}{2}y^2 - 3$ is the Left function on this interval. Set up and evaluate the integral:

$$A = \int_{-2}^{4} (\text{Right} - \text{Left})\, dy = \int_{-2}^{4} \left((y + 1) - (\frac{1}{2}y^2 - 3) \right) dy$$
$$A = \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4 \right) dy$$
$$A = \left[-\frac{1}{2}\frac{y^3}{3} + \frac{y^2}{2} + 4y \right]_{-2}^{4} = \left[-\frac{y^3}{6} + \frac{y^2}{2} + 4y \right]_{-2}^{4}$$

Evaluate at limits: At $y = 4$: $-\frac{64}{6} + \frac{16}{2} + 16 = -\frac{32}{3} + 8 + 16 = -\frac{32}{3} + 24 = \frac{-32+72}{3} = \frac{40}{3}$. At $y = -2$: $-\frac{-8}{6} + \frac{4}{2} + 4(-2) = \frac{4}{3} + 2 - 8 = \frac{4}{3} - 6 = \frac{4-18}{3} = -\frac{14}{3}$.

$$A = \left(\frac{40}{3}\right) - \left(-\frac{14}{3}\right) = \frac{40 + 14}{3} = \frac{54}{3} = 18$$

The area is 18.

Exercise 15.1. Determine the area of the region bounded by $x = -y^2 + 10$ and $x = (y - 2)^2$. *Hint:* Integrate with respect to y. Find intersections by setting $-y^2 + 10 = (y - 2)^2$. Determine which curve is Right and which is Left.

Part II

Multivariable Calculus: Functions of Several Variables

We now extend the concepts of functions, limits, and derivatives to functions involving more than one independent variable.

Functions of Several Variables and Partial Derivatives

16 Functions of Several Variables

Definition 16.1 (Function of Two Variables)**.** *A **function of two variables**, $z = f(x, y)$, maps each ordered pair (x, y) in a subset $\mathcal{D}$ of the real plane $\mathbb{R}^2$ to a unique real number z.*

- *The set $\mathcal{D}$ is called the **domain** of the function. It represents the set of all possible input pairs (x, y).*
- *The **range** of f is the set of all possible output values z. That is, Range $= \{z \in \mathbb{R} \mid z = f(x, y)$ for some $(x, y) \in \mathcal{D}\}$.*

*We call x and y the **independent variables** and z the **dependent variable**. [Image illustrating domain in xy-plane mapping to range on z-axis]*

Remark 16.1 (Visualization)**.** The graph of a function $z = f(x, y)$ is typically a surface in three-dimensional space $\mathbb{R}^3$. Each point on the surface has coordinates (x, y, z) where $z = f(x, y)$. [Image examples: $z = \sqrt{9 - x^2 - y^2}$ (hemisphere), $z = x^2 + y^2$ (paraboloid)]

Example 16.1 (Finding Domains)**.** Find the domain of each function:

(a) $f(x, y) = 3x + 5y + 2$. This is a linear function (its graph is a plane). It is defined for all real numbers x and y. Domain $\mathcal{D} = \mathbb{R}^2$.

(b) $g(x, y) = \sqrt{9 - x^2 - y^2}$. For the square root to be defined, the radicand must be non-negative: $9 - x^2 - y^2 \geq 0$. This inequality can be rewritten as $x^2 + y^2 \leq 9$. This represents all points (x, y) on or inside a circle centered at the origin with radius 3. Domain $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}$.

Definition 16.2 (Function of Three or More Variables)**.** *The concept extends naturally. A function of n variables, $w = f(x_1, x_2, \ldots, x_n)$, maps each n-tuple $(x_1, \ldots, x_n)$ in a domain $\mathcal{D} \subseteq \mathbb{R}^n$ to a unique real number w.*

17 Partial Derivatives

For a function of a single variable, $y = f(x)$, the derivative $\frac{dy}{dx}$ represents the instantaneous rate of change of y with respect to x. For a function of two variables, $z = f(x, y)$, we can consider the rate of change of z with respect to one variable while holding the other variable constant. This leads to the concept of partial derivatives.

Definition 17.1 (Partial Derivatives)**.** *Let $z = f(x, y)$ be a function of two variables.*

1. *The **partial derivative of f with respect to x** (treating y as a constant) is denoted by $\frac{\partial f}{\partial x}$ or $f_x(x, y)$ and is defined as:*

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

provided the limit exists.

2. *The **partial derivative of f with respect to y** (treating x as a constant) is denoted by $\frac{\partial f}{\partial y}$ or $f_y(x, y)$ and is defined as:*

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

provided the limit exists.

Other notations include $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, D_x f, D_y f$. *The partial derivatives evaluated at a specific point* (a, b) *are denoted by* $\frac{\partial f}{\partial x}\Big|_{(a,b)}$, $f_x(a, b)$, $\frac{\partial f}{\partial y}\Big|_{(a,b)}$, $f_y(a, b)$.

Remark 17.1 (Calculating Partial Derivatives)**.** To calculate $\frac{\partial f}{\partial x}$, treat y as a constant and apply the usual rules of single-variable differentiation with respect to x. To calculate $\frac{\partial f}{\partial y}$, treat x as a constant and apply the usual rules of single-variable differentiation with respect to y. This works because fixing one variable reduces the function temporarily to a function of a single variable. For example, let $g(x) = f(x, y)$ where y is fixed. Then $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial f}{\partial x}$. Similarly for the partial derivative with respect to y.

Example 17.1 (Partial Derivative using Limit Definition)**.** Find $f_x(x, y)$ for $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$ using the limit definition. *Solution:* We need $f(x+h, y)$: $f(x+h, y) = (x+h)^2-3(x+h)y+2y^2-4(x+h)+5y-12 = (x^2+2xh+h^2)-(3xy+3hy)+2y^2-(4x+4h)+5y-12$ Now apply the limit definition for f_x:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12) - (x^2 - 3xy + 2y^2 - 4x + 5y - 12)}{h}$$

Cancel terms:

$$= \lim_{h\to 0} \frac{2xh + h^2 - 3hy - 4h}{h}$$

Factor out h from the numerator (since $h \neq 0$ in the limit):

$$= \lim_{h\to 0} \frac{h(2x + h - 3y - 4)}{h}$$

$$= \lim_{h\to 0} (2x + h - 3y - 4)$$

Evaluate the limit by substituting $h = 0$:

$$= 2x + 0 - 3y - 4 = 2x - 3y - 4$$

So, $f_x(x, y) = 2x - 3y - 4$.

Example 17.2 (Partial Derivatives using Rules)**.** Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$. *Solution:* Using the rule: treat the other variable as constant. To find $\frac{\partial f}{\partial x}$, treat y as a constant:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial x}(2y^2) - \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial x}(5y) - \frac{\partial}{\partial x}(12)$$

$$= 2x - (3y)\frac{\partial}{\partial x}(x) + 0 - 4(1) + 0 - 0$$

$$= 2x - 3y(1) - 4 = 2x - 3y - 4$$

(This matches the result from the limit definition).

To find $\frac{\partial f}{\partial y}$, treat x as a constant:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial y}(3xy) + \frac{\partial}{\partial y}(2y^2) - \frac{\partial}{\partial y}(4x) + \frac{\partial}{\partial y}(5y) - \frac{\partial}{\partial y}(12)$$

$$= 0 - (3x)\frac{\partial}{\partial y}(y) + 2(2y) - 0 + 5(1) - 0$$

$$= -3x(1) + 4y + 5 = -3x + 4y + 5$$

Example 17.3 (Partial Derivatives with Chain Rule). Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = \sin(x^2y - 2x + 4)$. *Solution:* Let $u = x^2y - 2x + 4$. Then $f = \sin u$. Use the Chain Rule. To find $\frac{\partial f}{\partial x}$, treat y as constant:

$$\frac{\partial f}{\partial x} = \frac{d(\sin u)}{du} \cdot \frac{\partial u}{\partial x} = (\cos u) \cdot \frac{\partial}{\partial x}(x^2y - 2x + 4)$$
$$= \cos(x^2y - 2x + 4) \cdot (2xy - 2 + 0) = (2xy - 2)\cos(x^2y - 2x + 4)$$

To find $\frac{\partial f}{\partial y}$, treat x as constant:

$$\frac{\partial f}{\partial y} = \frac{d(\sin u)}{du} \cdot \frac{\partial u}{\partial y} = (\cos u) \cdot \frac{\partial}{\partial y}(x^2y - 2x + 4)$$
$$= \cos(x^2y - 2x + 4) \cdot (x^2(1) - 0 + 0) = x^2\cos(x^2y - 2x + 4)$$

Example 17.4 (Evaluating Partial Derivatives). Let $f(x, y) = x^3 + x^2y^3 - 2y^2$. Find $f_x(2, 1)$ and $f_y(2, 1)$. *Solution:* First find the partial derivatives as functions of x and y. $f_x(x, y) = \frac{\partial}{\partial x}(x^3 + x^2y^3 - 2y^2) = 3x^2 + 2xy^3 - 0 = 3x^2 + 2xy^3$. $f_y(x, y) = \frac{\partial}{\partial y}(x^3 + x^2y^3 - 2y^2) = 0 + x^2(3y^2) - 4y = 3x^2y^2 - 4y$. Now evaluate at $(x, y) = (2, 1)$: $f_x(2, 1) = 3(2)^2 + 2(2)(1)^3 = 3(4) + 4 = 12 + 4 = 16$. $f_y(2, 1) = 3(2)^2(1)^2 - 4(1) = 3(4)(1) - 4 = 12 - 4 = 8$.

Example 17.5 (Partial Derivatives with Product Rule). Let $f(x, y) = x^4e^{xy}$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. *Solution:* To find $\frac{\partial f}{\partial x}$, treat y as constant. Use the Product Rule for $(x^4)(e^{xy})$.

$$\begin{aligned}\frac{\partial f}{\partial x} &= x^4\frac{\partial}{\partial x}(e^{xy}) + e^{xy}\frac{\partial}{\partial x}(x^4) \\ &= x^4(e^{xy} \cdot \frac{\partial}{\partial x}(xy)) + e^{xy}(4x^3) \\ &= x^4(e^{xy} \cdot y) + 4x^3e^{xy} \\ &= (x^4y + 4x^3)e^{xy} = x^3(xy + 4)e^{xy}\end{aligned}$$

To find $\frac{\partial f}{\partial y}$, treat x as constant. x^4 is just a constant factor.

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^4e^{xy}) = x^4\frac{\partial}{\partial y}(e^{xy}) \\ &= x^4(e^{xy} \cdot \frac{\partial}{\partial y}(xy)) \\ &= x^4(e^{xy} \cdot x) \\ &= x^5e^{xy}\end{aligned}$$

17.1 Functions of More Than Two Variables

The concept of partial derivatives extends directly to functions of three or more variables. For a function $w = f(x, y, z)$, we can find the partial derivative with respect to each independent variable by treating the other independent variables as constants.

Definition 17.2 (Partial Derivatives for Three Variables). *Let $w = f(x, y, z)$ be a function of three variables.*

1. *The **partial derivative of f with respect to x** is:*

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x + h, y, z) - f(x, y, z)}{h}$$

Notations: $\frac{\partial f}{\partial x}, f_x, \frac{\partial w}{\partial x}, D_xf$.

2. *The **partial derivative of f with respect to y** is:*

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$$

Notations: $\frac{\partial f}{\partial y}, f_y, \frac{\partial w}{\partial y}, D_y f$.

3. *The **partial derivative of f with respect to z** is:*

$$\frac{\partial f}{\partial z} = \lim_{h \to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$$

Notations: $\frac{\partial f}{\partial z}, f_z, \frac{\partial w}{\partial z}, D_z f$.

In each case, the limit must exist. The calculation follows the rule: treat all variables except the one being differentiated with respect to as constants.

Example 17.6 (Partial Derivatives of a Three-Variable Function)**.** Let $f(x, y, z) = e^{xy} \ln(z)$. Find f_x, f_y, f_z, and evaluate $f_z(1, 0, 1)$. (Assume $z > 0$). *Solution:* To find f_x, treat y and z as constants. $\ln(z)$ is a constant factor.

$$f_x(x, y, z) = \frac{\partial}{\partial x}(e^{xy} \ln(z)) = \ln(z) \frac{\partial}{\partial x}(e^{xy})$$

Using the Chain Rule for e^{xy} with respect to x:

$$= \ln(z) \cdot (e^{xy} \cdot \frac{\partial}{\partial x}(xy)) = \ln(z) \cdot e^{xy} \cdot y = ye^{xy} \ln(z)$$

To find f_y, treat x and z as constants. $\ln(z)$ is a constant factor.

$$f_y(x, y, z) = \frac{\partial}{\partial y}(e^{xy} \ln(z)) = \ln(z) \frac{\partial}{\partial y}(e^{xy})$$

Using the Chain Rule for e^{xy} with respect to y:

$$= \ln(z) \cdot (e^{xy} \cdot \frac{\partial}{\partial y}(xy)) = \ln(z) \cdot e^{xy} \cdot x = xe^{xy} \ln(z)$$

To find f_z, treat x and y as constants. e^{xy} is a constant factor.

$$f_z(x, y, z) = \frac{\partial}{\partial z}(e^{xy} \ln(z)) = e^{xy} \frac{\partial}{\partial z}(\ln(z))$$

$$= e^{xy} \cdot \frac{1}{z} = \frac{e^{xy}}{z}$$

To evaluate $f_z(1, 0, 1)$, substitute $x = 1, y = 0, z = 1$ into the expression for f_z:

$$f_z(1, 0, 1) = \frac{e^{(1)(0)}}{1} = \frac{e^0}{1} = \frac{1}{1} = 1$$

Exercise 17.1. Find f_x, f_y, f_z for $f(x, y, z) = \sin(x^2 y - z) + \cos(x^2 - yz)$. *Solution Outline:* $f_x = \cos(x^2 y - z) \cdot (2xy) - \sin(x^2 - yz) \cdot (2x)$ $f_y = \cos(x^2 y - z) \cdot (x^2) - \sin(x^2 - yz) \cdot (-z)$ $f_z = \cos(x^2 y - z) \cdot (-1) - \sin(x^2 - yz) \cdot (-y)$

18 Higher-Order Partial Derivatives

Since the first partial derivatives f_x and f_y of a function $f(x, y)$ are themselves functions of x and y, they can be differentiated again to obtain **second-order partial derivatives**.

Definition 18.1 (Second-Order Partial Derivatives)**.** *Let $z = f(x, y)$. The four second-order partial derivatives are:*

1. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$ *(Differentiate wrt x twice)*

2. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$ *(Differentiate wrt x, then wrt y)*

3. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ *(Differentiate wrt y, then wrt x)*

4. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$ *(Differentiate wrt y twice)*

*The derivatives f_{xy} and f_{yx} are called **mixed partial derivatives**.*

Example 18.1 (Calculating Second Partial Derivatives)**.** Calculate all four second partial derivatives for $f(x, y) = xe^{-3y} + \sin(2x - 5y)$. *Solution:* First, find the first partial derivatives:

$$f_x = \frac{\partial}{\partial x}(xe^{-3y} + \sin(2x - 5y)) = (1)e^{-3y} + \cos(2x - 5y) \cdot 2 = e^{-3y} + 2\cos(2x - 5y)$$

$$f_y = \frac{\partial}{\partial y}(xe^{-3y} + \sin(2x - 5y)) = x(e^{-3y} \cdot (-3)) + \cos(2x - 5y) \cdot (-5) = -3xe^{-3y} - 5\cos(2x - 5y)$$

Now, find the second derivatives:

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(e^{-3y} + 2\cos(2x - 5y)) \\ &= 0 + 2(-\sin(2x - 5y) \cdot 2) = -4\sin(2x - 5y) \end{aligned}$$

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(e^{-3y} + 2\cos(2x - 5y)) \\ &= e^{-3y} \cdot (-3) + 2(-\sin(2x - 5y) \cdot (-5)) \\ &= -3e^{-3y} + 10\sin(2x - 5y) \end{aligned}$$

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(-3xe^{-3y} - 5\cos(2x - 5y)) \\ &= -3(1)e^{-3y} - 5(-\sin(2x - 5y) \cdot 2) \\ &= -3e^{-3y} + 10\sin(2x - 5y) \end{aligned}$$

$$\begin{aligned} f_{yy} &= \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(-3xe^{-3y} - 5\cos(2x - 5y)) \\ &= -3x(e^{-3y} \cdot (-3)) - 5(-\sin(2x - 5y) \cdot (-5)) \\ &= 9xe^{-3y} - 25\sin(2x - 5y) \end{aligned}$$

18.1 Clairaut's Theorem: Equality of Mixed Partials

In the previous example, we observed that $f_{xy} = f_{yx}$. This is not a coincidence and holds under certain conditions.

Theorem 18.1 (Clairaut's Theorem (Equality of Mixed Partial Derivatives))**.** *Suppose $f(x, y)$ is defined on an open disk D containing the point (a, b). If the mixed partial derivatives f_{xy} and f_{yx} are both continuous throughout D, then*

$$f_{xy}(a, b) = f_{yx}(a, b)$$

If f_{xy} and f_{yx} are continuous on the entire domain where f is defined, then $f_{xy} = f_{yx}$ everywhere.

Remark 18.1. Clairaut's Theorem simplifies calculations, as we only need to compute one of the mixed partials if the continuity conditions are met (which they are for most functions encountered, such as polynomials, rational functions away from zero denominators, compositions of trig/exp/log functions, etc.). It also extends to higher-order derivatives (e.g., $f_{xxy} = f_{xyx} = f_{yxx}$ if the relevant derivatives are continuous) and functions of more variables.

Example 18.2 (Higher-Order Mixed Partials)**.** Let $f(x, y) = y^2e^x + y$. Find $\frac{\partial^3 f}{\partial y^2 \partial x}$. *Solution:* This notation means we differentiate first wrt x, then twice wrt y: f_{xyy}. $f_x = \frac{\partial}{\partial x}(y^2e^x + y) = y^2e^x$. $f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(y^2e^x) = 2ye^x$. $f_{xyy} = \frac{\partial}{\partial y}(f_{xy}) = \frac{\partial}{\partial y}(2ye^x) = 2e^x$.

Alternatively, due to Clairaut's Theorem (assuming continuity), we could calculate f_{yxy} or f_{yyx}. Let's try f_{yyx}: $f_y = \frac{\partial}{\partial y}(y^2e^x + y) = 2ye^x + 1$. $f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(2ye^x + 1) = 2e^x$. $f_{yyx} = \frac{\partial}{\partial x}(f_{yy}) = \frac{\partial}{\partial x}(2e^x) = 2e^x$. The results match.

Exercise 18.1 (Verifying a Solution to a Partial Differential Equation)**.** A **Partial Differential Equation (PDE)** is an equation involving partial derivatives of an unknown function. Show that the function $u(x, y, t) = 5\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$ is a solution to the two-dimensional **wave equation**:

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

where $c^2 = 4$ in this case (so $c = 2$). *Solution Outline:* Calculate the required second partial derivatives:

- $u_t = 5\sin(3\pi x)\sin(4\pi y)[-\sin(10\pi t)\cdot 10\pi]$ $u_{tt} = 5\sin(3\pi x)\sin(4\pi y)[-\cos(10\pi t)\cdot(10\pi)^2] = -500\pi^2 u(x, y, t)/\cos(10\pi t)$ (using original u for brevity) Better: $u_{tt} = -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$

- $u_x = 5[\cos(3\pi x)\cdot 3\pi]\sin(4\pi y)\cos(10\pi t)$ $u_{xx} = 5[-\sin(3\pi x)\cdot(3\pi)^2]\sin(4\pi y)\cos(10\pi t) = -45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$.

- $u_y = 5\sin(3\pi x)[\cos(4\pi y)\cdot 4\pi]\cos(10\pi t)$ $u_{yy} = 5\sin(3\pi x)[-\sin(4\pi y)\cdot(4\pi)^2]\cos(10\pi t) = -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$.

Now check the equation $u_{tt} = 4(u_{xx} + u_{yy})$: Right side: $4(u_{xx} + u_{yy}) = 4(-45\pi^2\sin\ldots\cos\cdots - 80\pi^2\sin\ldots\cos\ldots) = 4((-45-80)\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) = 4(-125\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t))$ $= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$. This equals the expression for u_{tt}. Therefore, the function $u(x, y, t)$ satisfies the wave equation.

The Chain Rule for Functions of Several Variables

The Chain Rule extends to functions of several variables, allowing us to differentiate composite functions where intermediate variables depend on other variables. There are several forms depending on the number of intermediate and independent variables.

19 Chain Rule with One Independent Variable

This case applies when $z = f(x, y)$, but both x and y are themselves functions of a single variable t, i.e., $x = x(t)$ and $y = y(t)$. Then z can be considered a function of t, $z(t) = f(x(t), y(t))$. We want to find $\frac{dz}{dt}$.

Theorem 19.1 (Chain Rule for One Independent Variable)**.** *Suppose $x = x(t)$ and $y = y(t)$ are differentiable functions of t, and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t, and its derivative is given by:*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Here, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are evaluated at $(x(t), y(t))$, while $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are evaluated at t.

Remark 19.1 (Interpretation)**.** The total rate of change of z with respect to t ($\frac{dz}{dt}$) is the sum of contributions from the change in t through each intermediate variable (x and y). The contribution through x is the rate of change of z with respect to x ($\frac{\partial z}{\partial x}$) multiplied by the rate of change of x with respect to t ($\frac{dx}{dt}$). Similarly for the contribution through y.

Remark 19.2 (Tree Diagram)**.** A tree diagram helps visualize the dependencies and remember the formula.

- Start with the final dependent variable (z).
- Draw branches to the intermediate variables it depends on (x, y). Label these branches with the corresponding partial derivatives ($\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$).
- From each intermediate variable, draw branches to the independent variable(s) it depends on (t). Label these branches with the ordinary derivatives ($\frac{dx}{dt}, \frac{dy}{dt}$).
- To find $\frac{dz}{dt}$, multiply the derivatives along each path from z to t and sum the results for all paths.

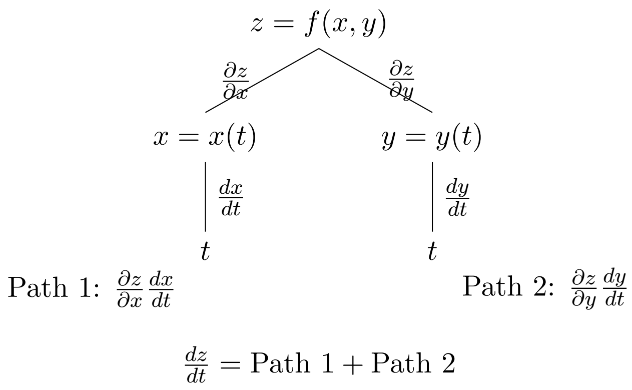


Figure 8: Tree diagram for the Chain Rule with one independent variable t.

Example 19.1 (Applying Chain Rule: $z(t)$)**.** Find $\frac{dz}{dt}$ if $z = f(x, y) = 4x^2 + 3y^2$, where $x = \sin t$ and $y = \cos t$. *Solution:* Method 1: Using the Chain Rule formula. First, find the required derivatives:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(4x^2 + 3y^2) = 8x$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(4x^2 + 3y^2) = 6y$$

$$\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

Apply the formula:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (8x)(\cos t) + (6y)(-\sin t)$$

Substitute $x = \sin t$ and $y = \cos t$ to express the result solely in terms of t:

$$= (8\sin t)(\cos t) + (6\cos t)(-\sin t)$$

$$= 8\sin t\cos t - 6\sin t\cos t = 2\sin t\cos t$$

Using the identity $\sin(2t) = 2\sin t\cos t$, we get $\frac{dz}{dt} = \sin(2t)$.

Method 2: Substitute first, then differentiate. Substitute $x(t)$ and $y(t)$ into $z = f(x, y)$:

$$z(t) = 4(\sin t)^2 + 3(\cos t)^2 = 4\sin^2 t + 3\cos^2 t$$

Now differentiate $z(t)$ directly with respect to t using the chain rule for single-variable functions:

$$\frac{dz}{dt} = \frac{d}{dt}(4\sin^2 t) + \frac{d}{dt}(3\cos^2 t)$$

$$= 4(2\sin t \cdot \frac{d}{dt}(\sin t)) + 3(2\cos t \cdot \frac{d}{dt}(\cos t))$$

$$= 8\sin t(\cos t) + 6\cos t(-\sin t)$$

$$= 8\sin t\cos t - 6\sin t\cos t = 2\sin t\cos t = \sin(2t)$$

Both methods yield the same result. The Chain Rule (Method 1) is often advantageous when the substitution is complex or when we only need the derivative at a specific point without needing the explicit form of $z(t)$.

Example 19.2. Find $\frac{dz}{dt}$ if $z = f(x, y) = \sqrt{x^2 - y^2}$, where $x = e^{2t}$ and $y = e^t$. *Solution:* Using the Chain Rule.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}((x^2 - y^2)^{1/2}) = \frac{1}{2}(x^2 - y^2)^{-1/2} \cdot (2x) = \frac{x}{\sqrt{x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}((x^2 - y^2)^{1/2}) = \frac{1}{2}(x^2 - y^2)^{-1/2} \cdot (-2y) = \frac{-y}{\sqrt{x^2 - y^2}}$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^{2t}) = 2e^{2t}$$

$$\frac{dy}{dt} = \frac{d}{dt}(e^t) = e^t$$

Apply the formula:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \left(\frac{x}{\sqrt{x^2 - y^2}}\right)(2e^{2t}) + \left(\frac{-y}{\sqrt{x^2 - y^2}}\right)(e^t)$$

Substitute $x = e^{2t}$ and $y = e^t$:

$$= \left(\frac{e^{2t}}{\sqrt{(e^{2t})^2 - (e^t)^2}} \right) (2e^{2t}) - \left(\frac{e^t}{\sqrt{e^{4t} - e^{2t}}} \right) (e^t)$$

$$= \frac{2e^{4t}}{\sqrt{e^{4t} - e^{2t}}} - \frac{e^{2t}}{\sqrt{e^{4t} - e^{2t}}}$$

$$= \frac{2e^{4t} - e^{2t}}{\sqrt{e^{2t}(e^{2t} - 1)}} = \frac{e^{2t}(2e^{2t} - 1)}{e^t\sqrt{e^{2t} - 1}}$$

$$= \frac{e^t(2e^{2t} - 1)}{\sqrt{e^{2t} - 1}}$$

(Check by substitution: $z(t) = \sqrt{(e^{2t})^2 - (e^t)^2} = \sqrt{e^{4t} - e^{2t}} = \sqrt{e^{2t}(e^{2t} - 1)} = e^t\sqrt{e^{2t} - 1}$. Differentiate using product rule and chain rule... result should match).

20 Chain Rule with Two Independent Variables

This case applies when $z = f(x, y)$, and both x and y are functions of two other variables, say u and v. So $x = g(u, v)$ and $y = h(u, v)$. Then z can be considered a function of u and v: $z(u, v) = f(g(u, v), h(u, v))$. We now want to find the partial derivatives $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Theorem 20.1 (Chain Rule for Two Independent Variables)**.** *Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v, and its partial derivatives are given by:*

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

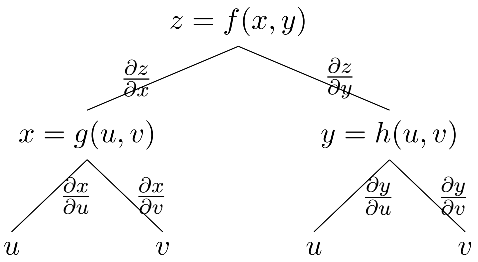
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Here, $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are evaluated at $(x(u, v), y(u, v))$, and the partial derivatives of x and y are evaluated at (u, v).

Remark 20.1 (Interpretation)**.** To find the rate of change of z with respect to u (holding v constant), we sum the contributions through each intermediate variable (x and y). The contribution through x is (rate of change z wrt x) $\times$ (rate of change x wrt u). Similarly for y. The same logic applies for $\frac{\partial z}{\partial v}$.

Remark 20.2 (Tree Diagram)**.** The tree diagram extends naturally.

- Start with z. Branches to x, y (labeled $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$).
- From x, draw branches to u, v (labeled $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}$).
- From y, draw branches to u, v (labeled $\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$).
- To find $\frac{\partial z}{\partial u}$, multiply along all paths from z to u and sum the results.
- To find $\frac{\partial z}{\partial v}$, multiply along all paths from z to v and sum the results.



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Figure 9: Tree diagram for the Chain Rule with two independent variables u, v.

Example 20.1 (Applying Chain Rule: $z(u, v)$)**.** Calculate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $z = f(x, y) = 3x^2y + y^2$, where $x = 3u + 2v$ and $y = 4u - v$. *Solution:* First, find the required partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(3x^2y + y^2) = 6xy$$
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(3x^2y + y^2) = 3x^2 + 2y$$
$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(3u + 2v) = 3$$
$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(3u + 2v) = 2$$
$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(4u - v) = 4$$
$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(4u - v) = -1$$

Apply the formulas:

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\\ &= (6xy)(3) + (3x^2 + 2y)(4)\\ &= 18xy + 12x^2 + 8y\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\\ &= (6xy)(2) + (3x^2 + 2y)(-1)\\ &= 12xy - 3x^2 - 2y\end{aligned}$$

Optionally, we can express the results entirely in terms of u and v by substituting $x = 3u + 2v$ and $y = 4u - v$, but the question doesn't explicitly require it.

21 Generalized Chain Rule

The Chain Rule can be generalized to any number of intermediate and independent variables.

Theorem 21.1 (Generalized Chain Rule)**.** *Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m intermediate variables $x_1, \ldots, x_m$. Let each x_i be a differentiable function of n independent variables $t_1, t_2, \ldots, t_n$, i.e., $x_i = x_i(t_1, t_2, \ldots, t_n)$ for $i = 1, \ldots, m$. Then w is a differentiable function of $t_1, \ldots, t_n$, and the partial derivative of w with respect to any independent variable t_j (for $j = 1, \ldots, n$) is given by:*

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

$$\frac{\partial w}{\partial t_j} = \sum_{i=1}^{m} \frac{\partial w}{\partial x_i}\frac{\partial x_i}{\partial t_j}$$

Remark 21.1 (Tree Diagram for Generalized Case)**.** The tree diagram extends:

- Root: w.
- Level 1: Branches to intermediate variables $x_1, \ldots, x_m$ (labeled $\frac{\partial w}{\partial x_i}$).
- Level 2: From each x_i, branches to independent variables $t_1, \ldots, t_n$ (labeled $\frac{\partial x_i}{\partial t_j}$).
- To find $\frac{\partial w}{\partial t_j}$, sum the products of derivatives along all paths from w to t_j.

Example 21.1 (Tree Diagram and Formulas for $w(t, u, v)$)**.** Let $w = f(x, y, z)$, where $x = x(t, u, v)$, $y = y(t, u, v)$, and $z = z(t, u, v)$. Create a tree diagram and write out the formulas for $\frac{\partial w}{\partial t}$, $\frac{\partial w}{\partial u}$, and $\frac{\partial w}{\partial v}$.

Solution: Tree Diagram:

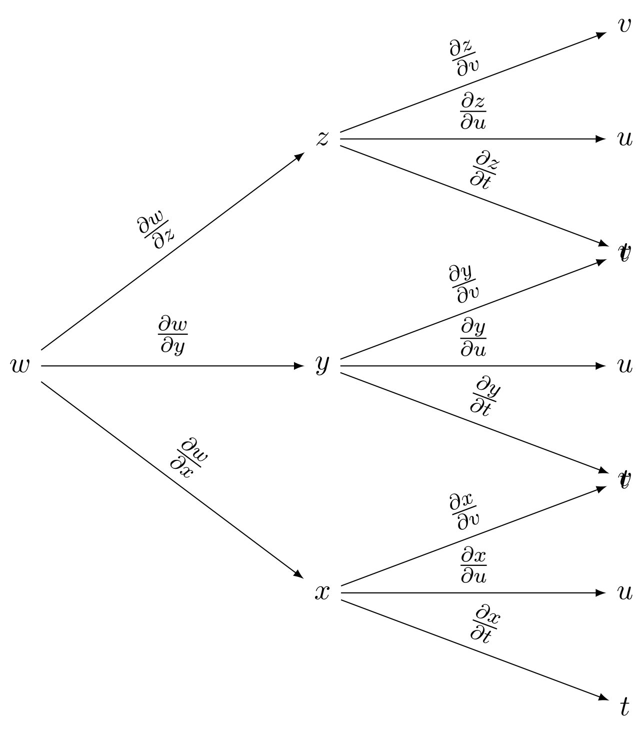


Figure 10: Tree diagram for $w = f(x(t, u, v), y(t, u, v), z(t, u, v))$.

Formulas (summing products along paths to each independent variable):

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

Example 21.2 (Applying Generalized Chain Rule)**.** Let $w = f(x, y, z) = 3x^2 - 2xy + 4z^2$. Let $x = e^u \sin v$, $y = e^u \cos v$, $z = e^u$. Find $\frac{\partial w}{\partial u}$. *Solution:* We need the partial derivatives of w wrt x, y, z and the partial derivatives of x, y, z wrt u.

$$\frac{\partial w}{\partial x} = 6x - 2y$$

$$\frac{\partial w}{\partial y} = -2x$$

$$\frac{\partial w}{\partial z} = 8z$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(e^u \sin v) = e^u \sin v \quad (\sin v \text{ is constant wrt } u)$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(e^u \cos v) = e^u \cos v \quad (\cos v \text{ is constant wrt } u)$$

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}(e^u) = e^u$$

Apply the formula for $\frac{\partial w}{\partial u}$:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= (6x - 2y)(e^u \sin v) + (-2x)(e^u \cos v) + (8z)(e^u)$$

Substitute x, y, z in terms of u, v:

$$= (6e^u \sin v - 2e^u \cos v)(e^u \sin v) + (-2e^u \sin v)(e^u \cos v) + (8e^u)(e^u)$$

$$= e^{2u}(6 \sin v - 2 \cos v)(\sin v) - 2e^{2u} \sin v \cos v + 8e^{2u}$$

$$= e^{2u}[(6 \sin^2 v - 2 \sin v \cos v) - 2 \sin v \cos v + 8]$$

$$= e^{2u}[6 \sin^2 v - 4 \sin v \cos v + 8]$$

(Using $\sin(2v) = 2 \sin v \cos v$, this is $e^{2u}[6 \sin^2 v - 2 \sin(2v) + 8]$).

Exercise 21.1. For the functions in the previous example, find $\frac{\partial w}{\partial v}$.

Implicit Differentiation Revisited

The concept of implicitly defined functions extends to multiple variables, and partial derivatives can be found using implicit differentiation.

22 Implicit Differentiation with Two Variables

Consider an equation $F(x, y) = 0$ that defines y implicitly as a function of x, i.e., $y = g(x)$ such that $F(x, g(x)) = 0$. We previously found $\frac{dy}{dx}$ by differentiating the entire equation with respect to x. We can derive a formula using partial derivatives of F.

Let $z = F(x, y)$. Since y is a function of x, we can think of z as a function of x only: $z = F(x, y(x))$. According to the equation $F(x, y) = 0$, z must be constantly zero. Therefore, its derivative with respect to x must be zero: $\frac{dz}{dx} = 0$. We can compute $\frac{dz}{dx}$ using the Chain Rule (Theorem 19.1), treating x as the independent variable t, and x, y as intermediate variables where $x = x$ and $y = y(x)$:

$$\frac{dz}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

Since $\frac{dx}{dx} = 1$ and $\frac{dz}{dx} = 0$, we have:

$$0 = \frac{\partial F}{\partial x}(1) + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$, provided $\frac{\partial F}{\partial y} \neq 0$:

Theorem 22.1 (Implicit Differentiation Formula: Two Variables)**.** *If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then*

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

provided $F_y = \frac{\partial F}{\partial y} \neq 0$.

Example 22.1 (Using the Implicit Differentiation Formula)**.** Find $\frac{dy}{dx}$ if $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$. *Solution:* Let $F(x, y) = 3x^2 - 2xy + y^2 + 4x - 6y - 11$. Find the partial derivatives:

$$F_x = \frac{\partial F}{\partial x} = 6x - 2y + 0 + 4 - 0 - 0 = 6x - 2y + 4$$

$$F_y = \frac{\partial F}{\partial y} = 0 - 2x + 2y + 0 - 6 - 0 = -2x + 2y - 6$$

Apply the formula:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{6x - 2y + 4}{-2x + 2y - 6} = \frac{6x - 2y + 4}{2x - 2y + 6} = \frac{3x - y + 2}{x - y + 3}$$

(Assuming $F_y \neq 0$).

Find the slope of the tangent line at $(2, 1)$. Check if $(2, 1)$ is on the curve: $3(2^2) - 2(2)(1) + 1^2 + 4(2) - 6(1) - 11 = 12 - 4 + 1 + 8 - 6 - 11 = 0$. Yes. Evaluate $\frac{dy}{dx}$ at $(2, 1)$:

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{3(2) - 1 + 2}{2 - 1 + 3} = \frac{6 - 1 + 2}{4} = \frac{7}{4}$$

The slope is $7/4$. Equation: $y - 1 = \frac{7}{4}(x - 2)$.

23 Implicit Differentiation with Three Variables

Consider an equation $F(x, y, z) = 0$ that defines z implicitly as a function of two independent variables x and y, i.e., $z = g(x, y)$ such that $F(x, y, g(x, y)) = 0$. We can find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using a similar approach.

Let $w = F(x, y, z)$. Since $F(x, y, z) = 0$, w is constantly zero. Therefore, its partial derivatives with respect to the independent variables x and y must be zero: $\frac{\partial w}{\partial x} = 0$ and $\frac{\partial w}{\partial y} = 0$.

To find $\frac{\partial w}{\partial x}$, we treat x as the independent variable and y as a constant. The variable z depends on x. We use the Chain Rule (Generalized Form, Theorem 21.1) where $w = F(x, y, z)$, $x = x$, $y = y$(constant wrt x), $z = z(x, y)$.

$$\frac{\partial w}{\partial x} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

Since y is treated as constant when finding $\frac{\partial w}{\partial x}$, $\frac{\partial y}{\partial x} = 0$. Also $\frac{\partial x}{\partial x} = 1$. And we know $\frac{\partial w}{\partial x} = 0$.

$$0 = \frac{\partial F}{\partial x}(1) + \frac{\partial F}{\partial y}(0) + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

$$0 = F_x + F_z\frac{\partial z}{\partial x}$$

Solving for $\frac{\partial z}{\partial x}$, provided $F_z \neq 0$.

Similarly, to find $\frac{\partial w}{\partial y}$, we treat y as the independent variable and x as a constant. z depends on y.

$$\frac{\partial w}{\partial y} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}$$

Since x is treated as constant, $\frac{\partial x}{\partial y} = 0$. Also $\frac{\partial y}{\partial y} = 1$. And $\frac{\partial w}{\partial y} = 0$.

$$0 = \frac{\partial F}{\partial x}(0) + \frac{\partial F}{\partial y}(1) + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}$$

$$0 = F_y + F_z\frac{\partial z}{\partial y}$$

Solving for $\frac{\partial z}{\partial y}$, provided $F_z \neq 0$.

Theorem 23.1 (Implicit Differentiation Formula: Three Variables)**.** *If the equation* $F(x, y, z) = 0$ *defines* z *implicitly as a differentiable function of* x *and* y*, then*

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

provided $F_z = \frac{\partial F}{\partial z} \neq 0$.

Example 23.1 (Implicit Partial Derivatives)**.** Suppose $x^2e^{yz} + z\sin(y) = 0$. Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. *Solution:* Let $F(x, y, z) = x^2e^{yz} + z\sin(y)$. Find the partial derivatives of F:

$$F_x = \frac{\partial}{\partial x}(x^2e^{yz} + z\sin(y)) = 2xe^{yz} + 0 = 2xe^{yz}$$

$$F_y = \frac{\partial}{\partial y}(x^2e^{yz} + z\sin(y)) = x^2(e^{yz}\cdot z) + z\cos(y) = x^2ze^{yz} + z\cos y$$

$$F_z = \frac{\partial}{\partial z}(x^2e^{yz} + z\sin(y)) = x^2(e^{yz}\cdot y) + (1)\sin(y) = x^2ye^{yz} + \sin y$$

Apply the formulas:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xe^{yz}}{x^2ye^{yz} + \sin y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x^2ze^{yz} + z\cos y}{x^2ye^{yz} + \sin y}$$

These are valid where $F_z \neq 0$.

Exercise 23.1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using Theorem 23.1 for the following equations:

1. $x^3z^2 - 5xy^5z = x^2 + y^3$
2. $x^2\sin(2y - 5z) = 1 + y\cos(6zx)$