7th lesson

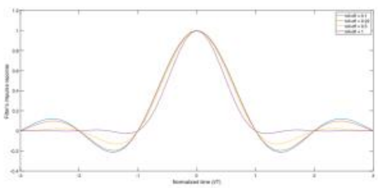
Monday, 19 October 2020 09:51

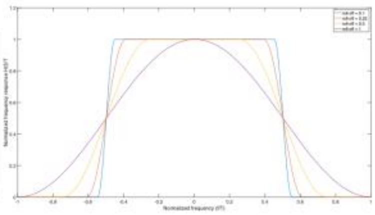
Raised cosine filters

Raised cosine filters satisfy the Nyquist criterion: the occupied bandwidth is

$$B_{RC} = \frac{1+\alpha}{T}$$

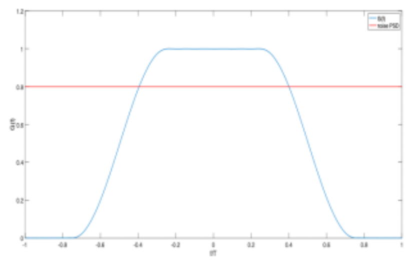
The roll-off factor α is a design parameter, RC with $\alpha = 0$ is a rect and it is the *minimum bandwidth filter*.





Receive filter design: matched filter

- Neglecting, for the moment, the effect of the channel, the other major impairment at the receiver is the presence of Gaussian noise.
- The receiver should be designed to minimize the negative effect of Gaussian noise.
- The choice of $g_R(t) = g_T(-t)$, or, being the filter real, $G_R(f) = G_T^*(f)$ in the frequency domain, *maximizes* the signal-to-noise ratio at the receiver.
- The receive filter is said to be *matched* to the transmit filter.



Root raised cosine filters

- Root raised cosine filters are filters whose frequency response is the square root of a raised cosine, i.e., $H_{RRC}(f,\alpha) = \sqrt{H_{RC}(f,\alpha)}$.
- If $G_R(f) = G_T(f) = H_{RRC}(f,\alpha)$, the transmit and receive filter pair satisfies the two independent *optimality* conditions:
  1. The cascade of $g_T(t)$ and $g_R(t)$ obeys the *Nyquist criterion*:
     $$G_R(f)G_T(f) = \left(H_{RRC}(f,\alpha)\right)^2 = H_{RC}(f,\alpha).$$
     $H_{RC}(f)$ always satisfies the Nyquist criterion
  2. The receive filter is matched to the transmit filter.
     Since it is $G_T(f) \in \Re \rightarrow G_R(f) = G_T(f) = G_T^*(f)$, the transmit and receive filter are *matched*.

Bandwidth of a PAM signal

- If $G_T(f) = H_{RRC}(f)$ and the ymbols are zero-mean, it is

$$S_s(f) = \frac{1}{T}S_a(f)|G_T(f)|^2 = \frac{A}{T}H_{RC}(f,\alpha).$$

- The bandwidth occupied by the PAM *complex envelope* is

$$B_{PAM}^{(BB)} = \frac{1+\alpha}{2T} = \frac{1+\alpha}{2}\frac{1}{\log_2 M\, T_b} = \frac{1+\alpha}{2}\frac{R_b}{\log_2 M}$$

- The bandwidth occupied by the corresponding *passband signal* is

$$B_{PAM}^{(PB)} = 2B_{PAM}^{(BB)} = \frac{1+\alpha}{T} = (1+\alpha)\frac{R_b}{\log_2 M}$$

!

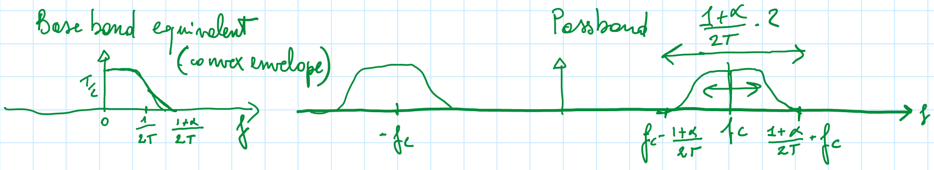
$S_a(f) \rightleftharpoons R_e(m) = A\,\delta(m) \Rightarrow S_e(f) = A$

$A = E\{a_i^2\}$ mean square value of the symbols

Se $\alpha = 1$, $M = 32$ e $R_B = 50$ Mb/s, banda?

$B^{(BB)} = \frac{1+1}{2}\cdot\frac{50\cdot 10^6}{\log_2 32} = 10$ MHz

$B^{(PB)} = 2B^{(BB)} = 20$ MHz



Power of a PAM signal

- The mean power of the *complex envelope* of a PAM signal with zero-mean symbols and root raised cosine filtering is

$$P_{\tilde{s}} = \int_{-\infty}^{+\infty} S_{\tilde{s}}(f)df = \frac{A}{T}\int_{-\infty}^{+\infty}|G_T(f)|^2 df = \frac{A}{T}\int_{-\infty}^{+\infty}H_{RC}(f,\alpha)df$$

since it is $\int_{-\infty}^{+\infty} H_{RC}(f,\alpha)df = h_{RC}(t)|_{t=0} = 1$,

$$P_{\tilde{s}} = \frac{A}{T}$$

- The power of the corresponding *passband signal* is

$$P_s = \frac{1}{2}P_{\tilde{s}} = \frac{A}{2T}$$

$\int_{-\infty}^{+\infty} H_{RC}(f,\alpha)\,df = \int_{-\infty}^{+\infty} H_{RC}(f)e^{j2\pi ft}\,df\Big|_{t=0} = h_{RC}(0) = 1$

Energy of a PAM symbol

- The mean square value of the symbols for a PAM constellation is

$$A = E\{a_i^2\} = \frac{M^2-1}{3}$$

- The energy per symbol is computed as the power multiplied by the symbol duration

$$E_s = P_s T = \frac{A}{2T}T = \boxed{\frac{M^2-1}{6}}$$

2-PAM; $M=2$

$A = E\{a_i\} = \frac{1}{2}(1^2) + \frac{1}{2}(-1)^2 = 1 \Rightarrow \frac{2^2-1}{3} = \frac{4-1}{3} = 1$

4-PAM

$A = \quad a_i = \begin{cases}-3\\-1\\1\\3\end{cases} \quad A = \frac{1}{4}(-3)^2 + \frac{1}{4}(-1)^2 + \frac{1}{4}(1)^2 + \frac{1}{4}(3)^2 = 5 \quad \frac{4^2-1}{3} = 5$

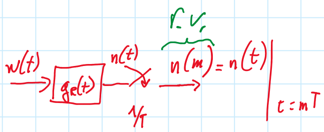
M-PAM

$A = \frac{M^2-1}{3}$

Additive white Gaussian noise

- The noise $w(t)$ is the zero-mean white Gaussian process with PSD $S_w(f) = \frac{N_0}{2}$.
- The complex envelope of the noise is $\tilde{w}(t) = w_I(t) + jw_Q(t)$ with PSD $S_{\tilde{w}}(f) = 2N_0$.
- The noise $n(t) = n_I(t) + jn_Q(t) = g_R(t) \otimes \tilde{w}(t)$ is a zero-mean Gaussian complex stochastic process and its PSD is

$$S_n(f) = S_w(f)|G_R(f)|^2 = 2N_0|G_R(f)|^2.$$



Additive white Gaussian noise

- The sample $n(m) = n(t)|_{t=mT} = n_I(m) + jn_Q(m)$ is a zero-mean Gaussian complex random variable and its variance is

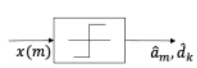
$$\sigma_n^2 = E\{|n(m)|^2\} = \int_{-\infty}^{+\infty} S_n(f)df = 2N_0 \int_{-\infty}^{+\infty} |G_R(f)|^2 df$$

- If the receive filter is a RRC, then it is $\int_{-\infty}^{+\infty} |G_R(f)|^2 df = 1$ and

$$\sigma_n^2 = 2N_0$$

- The in-phase and quadrature components $n_I(m), n_Q(m)$ are independent and the variance of each component is $\sigma^2 = \sigma_{n_I}^2 = \sigma_{n_Q}^2 = N_0$.

Decision strategy



- Under the hypothesis of RRC filtering at the transmit and at the receiver, the decision variable is

$$x(m) = a_m + n(m)$$

- The optimal decision strategy is the one that chooses the symbol maximize the probability conditioned on having received $x(m)$.

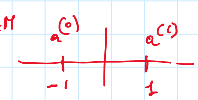
$$\hat{a}_m = \arg \max_{a^{(i)} \in \mathcal{A}} p\left(a^{(i)} | x(m)\right)$$

- It can be shown that in case of equiprobable symbols it is

$$p\left(x(m)|a^{(i)}\right) \approx p\left(a^{(i)}|x(m)\right)$$

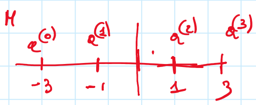
so that the symbol $a^{(i)}$ that maximizes $p\left(x(m)|a^{(i)}\right)$ maximizes also $p\left(a^{(i)}|x(m)\right)$, *maximum likelihood decision.*

2-PAM



$\mathcal{A} = \{-1, 1\}$

4-PAM



$\mathcal{A} = \{-3, -1, 1, 3\}$

$a^{(i)}$ i-th element of of the symbol constellation $\mathcal{A}$

a_m m-th transmitted symbol (random variable)

$x(m) = a_m + n(m) \Rightarrow x(m)$ is a complex sample

$x(m) = x_I(m) + j\,x_Q(m) = a_m + n_I(m) + j\,n_Q(m)$

$\hookrightarrow x_I(m) = a_m + n_I(m) \Rightarrow x(m) = x_I(m)\,;\quad n(m) = n_I(m)$

PAM decision strategy

- Since PAM signal is *real*, we consider only the in-phase component of the received signal, i.e. $x(m) = x_I(m)$ and $n(m) = n_I(m)$.
- Because of the conditioning, the symbol value $a^{(i)}$ is fixed and $x(m) \in \mathcal{N}\left(a^{(i)}, N_0\right)$ is a Gaussian random variable with mean $a^{(i)}$ and variance $\sigma^2 = \sigma_{n_I}^2 = N_0$.
- The probability density function is

$$p\left(x(m)|a^{(i)}\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(x(m)-a^{(i)}\right)^2}{2\sigma^2}}$$

so that the symbol $a^{(i)}$ that maximizes $p\left(x(m)|a^{(i)}\right)$ is the one that minimizes the distance between the symbol and the received sample

$$\hat{a}_m = \arg \min_{a^{(i)} \in \mathcal{A}} \left|x(m) - a^{(i)}\right|$$

Decision strategy

- Because of the conditioning, the symbol value $a^{(i)}$ is fixed and $x(m) \in \mathcal{N}\left(a^{(i)}, N_0\right)$ is a Gaussian random variable with mean $a^{(i)}$ and variance $\sigma^2 = N_0$.
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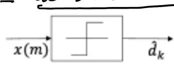
<u>Received sample</u> = <u>decision variable</u>

Decision strategy



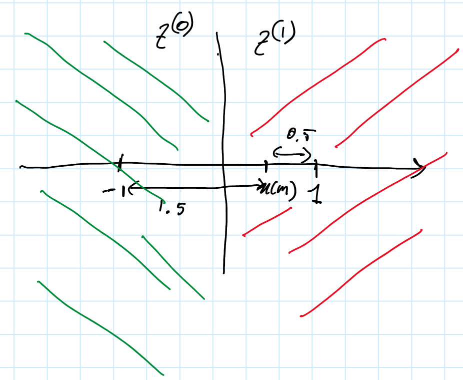
- Example. Consider a 2-PAM ($M = 2$) and assume that $x(m) = 0.5$. The two symbols are $a^{(0)} = -1$ and $a^{(1)} = 1$, i.e. $\mathcal{A} = \{-1,1\}$.
- The decision block will compute the two distances:

  $d\left(x(m), a^{(0)}\right) = |0.5 - (-1)| = 1.5$

  $d\left(x(m), a^{(1)}\right) = |0.5 - 1| = 0.5$

  and will decide for the one that minimizes the distance, so that

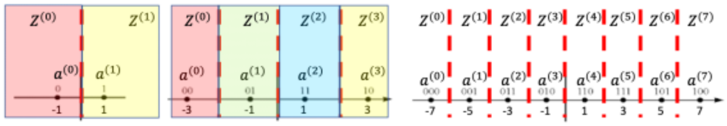
$$\hat{a}_m = a^{(1)} = 1.$$



Decision strategy

- Adopting the maximum likelihood criterion, we can partition the signal space in *zone of decisions*, where zone $Z^{(i)}$ is the set of points that are closer to the symbol $a^{(i)}$ than to any other symbol

$$Z^{(i)} = \{x | d(x, a^{(i)}) < d(x, a^{(j)}), j \neq i, j = 1, \dots, M\}$$



$M = 2$, $m = 1$ | $M = 4$, $m = 2$ | $M = 8$, $m = 3$

The decision threshold are in the midpoints of the segment connecting any two adjacent symbols. For example, for $M = 4$ the thresholds are in $-2, 0$ and 2.