Digital communications

Quadrature modulations (QAM)

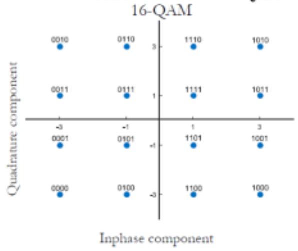
Quadrature modulations

- In analog modulations, QAM is obtained by transmitting two orthogonal DSB signals $m_I(t), m_Q(t)$ and the complex envelope is
$$\tilde{s}_{QAM}(t) = m_I(t) + jm_Q(t)$$
- Quadrature PAM is obtained exactly in the same manner by transmitting two PAM signals in quadrature $m_I(t) = \sum_i a_i g_T(t - iT)$ and $m_Q(t) = \sum_i b_i g_T(t - iT)$, with a_i, b_i PAM symbols.
- The QAM signal is
$$s_{QAM}(t) = \sum_i (a_i + jb_i) g_T(t - iT) = \sum_i c_i g_T(t - iT)$$
and the QAM complex symbols take the form $c_i = a_i + jb_i$.

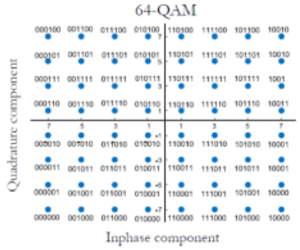
QAM symbols

- Because QAM is the combination of two orthogonal PAM, the values of $M_{QAM} = M_{PAM}^2$ are squared powers of 2, i.e. m is always even.
  - If the two PAMs have $M_{PAM} = 4$ symbols than the QAM has $M_{QAM} = 16$ symbols, if $M_{PAM} = 8$ than $M_{QAM} = 64$.

16-QAM



64-QAM



Energy of a QAM symbol

- In the computation of power and energy the only difference between PAM and QAM is in the mean square value of the symbols.
- Keeping in mind that th in-phase and quadrature symbols are indipendent and zero-mean, it is
$$A = E\{c_i c_i^*\} = E\{a_i^2\} + E\{b_i^2\} = 2\frac{M_{PAM}^2 - 1}{3} = 2\frac{M_{QAM} - 1}{3}$$
- Compared to PAM, QAM constellation is much more compact and requires less energy per symbol.
  - $A^{(4-PAM)} = \frac{16-1}{3} = 5;$ $\quad A^{(4-QAM)} = 2\frac{4-1}{3} = 2.$
  - $A^{(16-PAM)} = \frac{256-1}{3} = 85;$ $\quad A^{(16-QAM)} = 2\frac{16-1}{3} = 10.$
- The energy per symbol is
$$E_s = \frac{A}{2} = \frac{M_{QAM} - 1}{3}$$

$|A|^2 = A \cdot A^*$ $\quad A \in \mathbb{R} \Rightarrow |A|^2 = A^2$

Mean square value $E\{c_i c_i^*\} = E\{|c_i|^2\}$

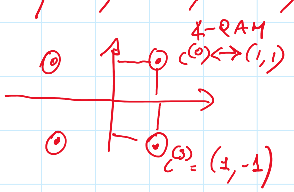
a_m and b_m are indipendent

$c_m = a_m + j b_m$ $\quad E\{c_m c_m^*\} = E\{\underbrace{(a_m + jb_m)(a_m - jb_m)}\} = E\{a_m^2 - (jb_m)^2\} = E\{a_m^2 + b_m^2\} = E\{a_m^2\} + E\{b_m^2\}$

4-QAM is obtained as two 2-PAM in quadrature

$E\{|c_m|^2\} = E\{a_m^2\} + E\{b_m^2\} = 1 + 1 = 2$

4-QAM $= \{1+j,\ 1-j,\ -1+j,\ -1-j\}$

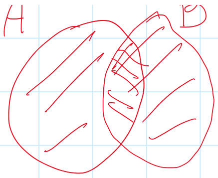


$1 + 1 = 2$

QAM error probability

- The *complex* decision variable is
$$x(m) = c_m + n(m) = (a_m + jb_m) + \left(n_I(m) + jn_Q(m)\right)$$
$$= a_m + n_I(m) + j\left(b_m + n_Q(m)\right)$$
- The in-phase and quadrature noise components $n_I(m)$ and $n_Q(m)$ are independent.
- Error events depend on noise. If the noise is independent also the error events on the two components are independent.
- The probability of error can be computed as
$$Pr\{error\} = Pr\{\{error\ on\ the\ I\ channel\} \cup \{error\ on\ the\ Q\ channel\}\}$$
$$\leq Pr\{error\ on\ the\ I\ channel\} + Pr\{error\ on\ the\ Q\ channel\}$$
- The error probability can be *approximated* as the sum of the probability of making an error on the in-phase symbol a_m and probability of making an error on the quadrature symbol b_m.

$P(A \cup B) = \underbrace{P(A) + P(B)} - P(A \cap B)$



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) \leq P(A) + P(B)$$

$$P(A \cup B) = P(A) + P(B) \iff A \cap B = \emptyset$$

4-QAM error probability

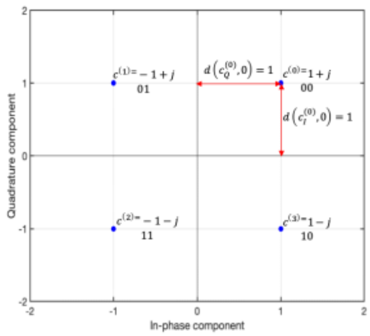
- 4-QAM is obtained as the composition of two 2-PAM in quadrature.
- The symbol error probability is

$$P_e^{(4-QAM)} = \frac{1}{4}\sum_{i=0}^{3} P(e|c^{(i)}) = P(e|c^{(0)})$$

$$P(e|c^{(0)}) \approx Q\left(\frac{d\left(c_I^{(0)},0\right)}{\sigma_{n_I}}\right) + Q\left(\frac{d\left(c_Q^{(0)},0\right)}{\sigma_{n_Q}}\right)$$

$$= 2Q\left(\frac{1}{\sigma}\right)$$

$$P_e^{(4-QAM)} \approx 2P_e^{(2-PAM)}$$



$$P_e^{(4-QAM)} = \frac{1}{4}\sum_{i=0}^{3} P(e|c^{(i)}) = \frac{1}{\cancel{4}} \cancel{4} P(e|c^{(0)})$$

$$\sigma_{n_I} = \sigma_{n_Q} = \sigma$$

$$P(e|c^{(0)}) \simeq Q\left(\frac{1}{\sigma_{n_I}}\right) + Q\left(\frac{1}{\sigma_{n_Q}}\right) = 2Q\left(\frac{1}{\sigma}\right)$$

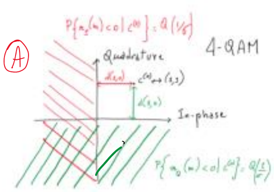
$$P_e^{(4-QAM)} \simeq 2Q\left(\frac{1}{\sigma}\right)$$

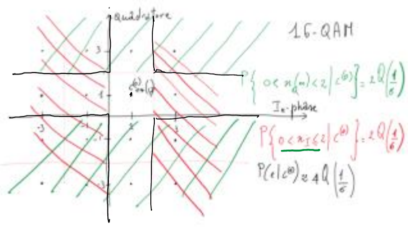
M-QAM error probability

- M-QAM is obtained as the composition of two PAM in quadrature, each with $\sqrt{M}$ symbols.
- The symbol error probability can always be approximated as

$$P_e^{(M-QAM)} \approx 2P_e^{(\sqrt{M}-PAM)}$$

A





1) $Pr\{A \cap B\}$ in this case is difficult to computed
2) $Pr\{A \cap B\} \ll Pr\{A\} = Pr\{B\}$ so it can be neglected

QAM symbol error probability

- 4-QAM: $M = 4, E_s = \frac{4-1}{3} = 1 \Longrightarrow E_s = 1$, and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{E_s}{N_0}}$

$$P_e^{(4-QAM)} \approx 2Q\left(\sqrt{\frac{E_s}{N_0}}\right);$$

16-QAM: $M = 16, E_s = \frac{16-1}{3} = 5 \Longrightarrow \frac{1}{5}E_s = 1$, and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{E_s}{5N_0}}$

$$P_e^{(16-QAM)} \approx 2\frac{3}{2}\left(\frac{1}{\sigma}\right) = 3Q\left(\sqrt{\frac{E_s}{5N_0}}\right).$$

M-QAM bit error probability

- The total number of M-QAM transmitted bits is the sum of the number of bits transmitted on the in-phase and quadrature channels.
- Because each channel is independent, the bit error probability per channel is independent.
- Accordingly, $P_e^{(M-QAM),b}$ can be *exactly* computed as the sum of the bit error probability on the in-phase channel and the quadrature channel, divided by two.

$$P_e^{(M-QAM),b} = \frac{1}{2}2P_e^{(\sqrt{M}-PAM),b} = P_e^{(\sqrt{M}-PAM),b}$$

16-QAM $\quad m = \log_2 16 = 4$ bits x symbol

$\underbrace{2 \cdot 2 \text{ bits}}_{4\text{-PAM}}$ x symbol

QAM bit error probability

- 4-QAM:

$$P_e^{(4-QAM),b} = P_e^{(2-PAM),b} \approx Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- 16-QAM

$$P_e^{(16-QAM),b} = P_e^{(4-PAM),b} \approx \frac{3}{4}Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$