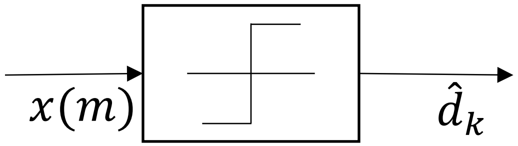
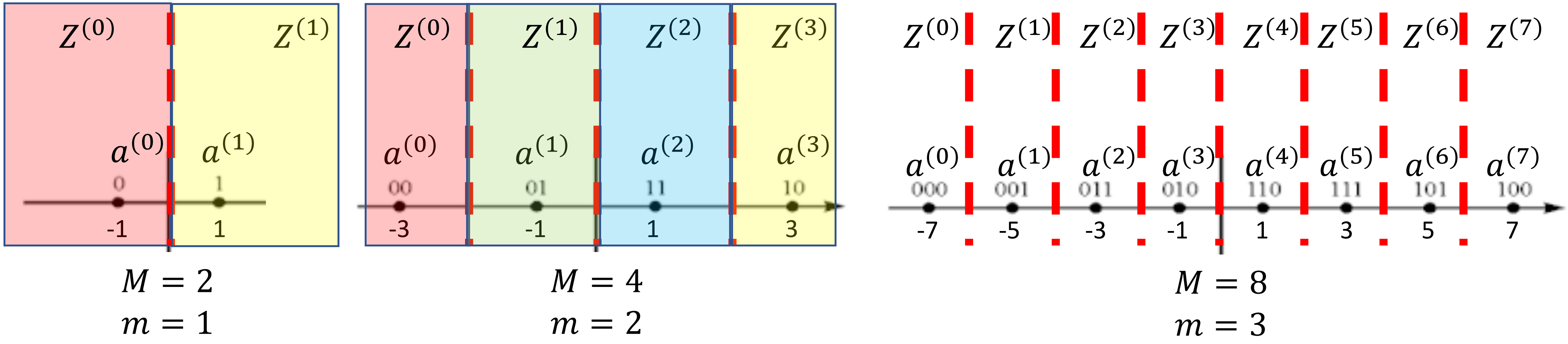
Decision strategy



- Adopting the maximum likelihood criterion, we can partition the signal space in *zone of decisions*, where zone $Z^{(i)}$ is the set of points that are closer to the symbol $a^{(i)}$ than to any other symbol

$$Z^{(i)} = \{x | d(x, a^{(i)}) < d(x, a^{(j)}), j \neq i, j = 1, \dots, M\}$$



The decision threshold are in the midpoints of the segment connecting any two adjacent symbols. For example, for $M = 4$ the thresholds are in $-2,0$ and 2.

PAM error probability

- Even if the maximum likelihood decision strategy is optimal, the receiver still make errors due to the presence of noise.
- The error probability is averaged over all the symbol of the constellation

$$P_e = \lim_{N^{(s)} \to \infty} \frac{N_e^{(s)}}{N^{(s)}} = \frac{1}{M} \sum_{i=0}^{M-1} P\left(e \middle| a^{(i)}\right)$$

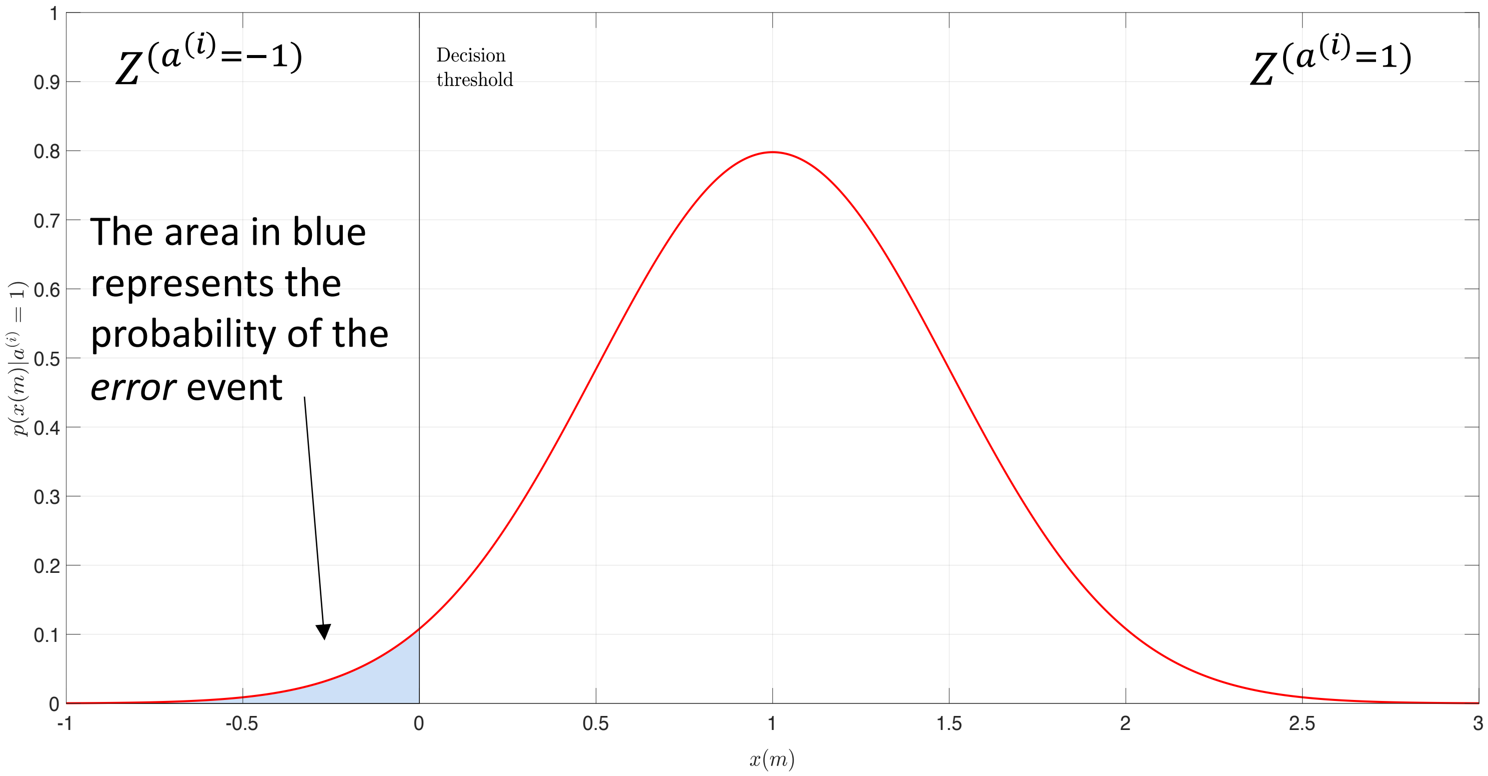
  where $N_e^{(s)}$ is the number of symbol errors and $N^{(s)}$ is the number of transmitted symbols.

- The probability of error $P\left(e \middle| a^{(i)}\right)$ is the probability that, having transmitted $a^{(i)}$, the decision variable $x(m)$ does not fall in the decision region $Z^{(i)}$.

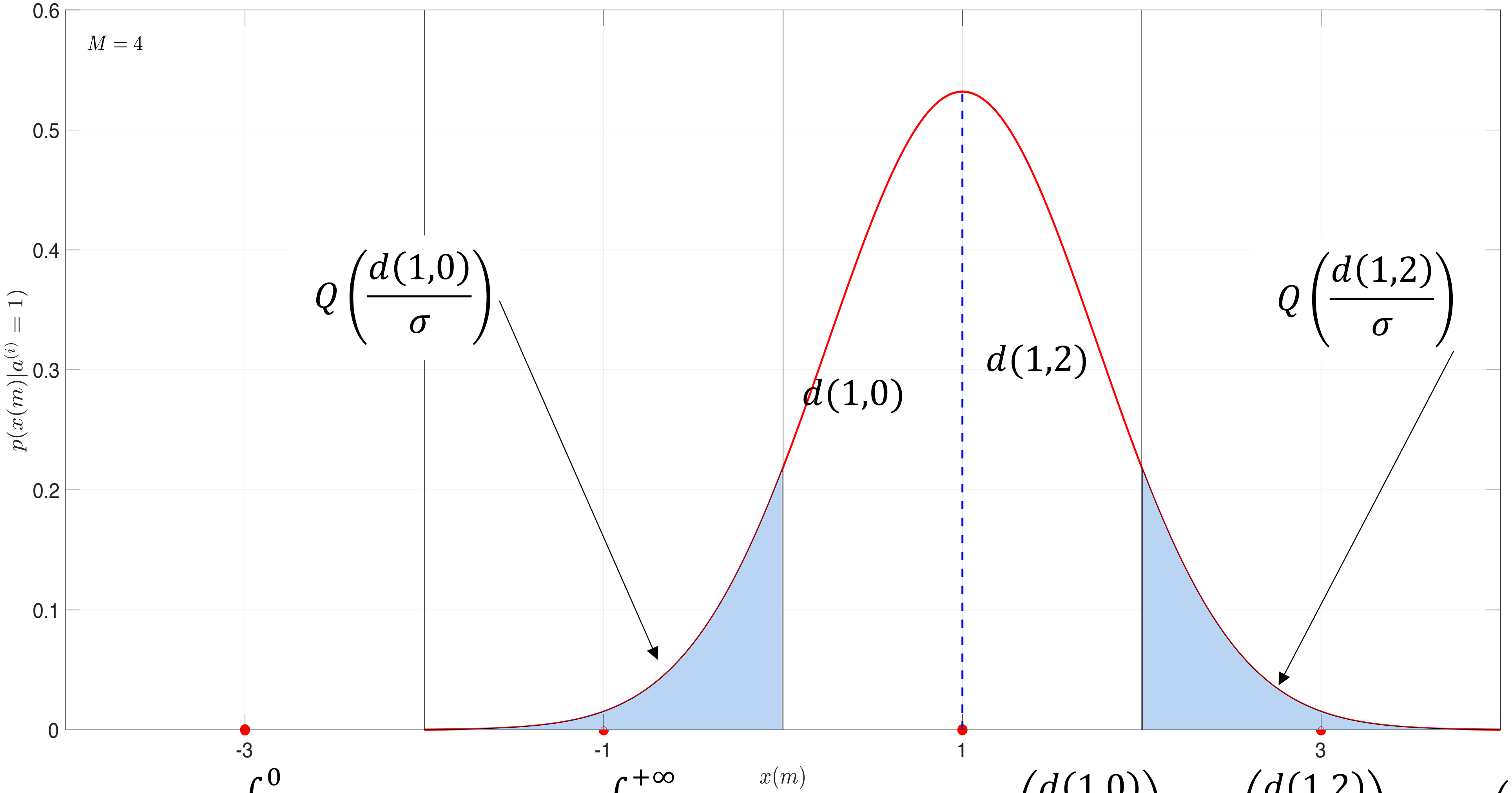
PAM error probability

- To compute $P\left(e \middle| a^{(i)}\right)$ we assume that the transmitted symbol is $a_m = a^{(i)}$, so that it is $x(m) = a^{(i)} + n(m)$ and the probability of error is

$$P\left(e \middle| a^{(i)}\right) = Pr\{x(m) \notin Z^{(i)} | a_m = a^{(i)}\}$$



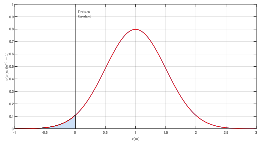
PAM error probability

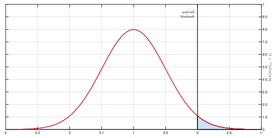


$$P\left(e \middle| a^{(i)} = 1\right) = \int_{-\infty}^{0} p\left(x \middle| a^{(i)} = 1\right) dx + \int_{2}^{+\infty} p\left(x \middle| a^{(i)} = 1\right) dx = Q\left(\frac{d(1,0)}{\sigma}\right) + Q\left(\frac{d(1,2)}{\sigma}\right) = 2Q\left(\frac{1}{\sigma}\right)$$

PAM error probability: Q-function

- The Q-function computes the integral of the *tail* of a Gaussian distribution.
- The probability that $x \in \mathcal{N}(m, \sigma^2)$ is smaller than t_1 or larger than t_2 are the integral of Gaussian tails and they are computed as





$$\left.\begin{aligned} \int_{-\infty}^{t_1} pdf(x)dx &= Q\left(\frac{m - t_1}{\sigma}\right) \\ \int_{t_2}^{+\infty} pdf(x)dx &= Q\left(\frac{t_2 - m}{\sigma}\right) \end{aligned}\right\} = Q\left(\frac{d(t_i, m)}{\sigma}\right), i = 1,2$$

- In our case, m is the symbol $a^{(i)}$ and t_1 or t_2 are the detection thresholds.
- The main properties of the Q-function are

$$Q(-\infty) = 1, Q(\infty) = 0, Q(0) = 0.5, Q(-x) = 1 - Q(x).$$

PAM error probability

- 2-PAM

$$P_e^{(2-PAM)} = \frac{1}{2}\left(Q\left(\frac{d(-1,0)}{\sigma}\right) + Q\left(\frac{d(1,0)}{\sigma}\right)\right) = Q\left(\frac{1}{\sigma}\right)$$

- 4-PAM

$$P_e^{(4-PAM)} = \frac{1}{4}\left(Q\left(\frac{d(-3,-2)}{\sigma}\right) + Q\left(\frac{d(-1,-2)}{\sigma}\right) + Q\left(\frac{d(-1,0)}{\sigma}\right)\right.$$
$$\left. Q\left(\frac{d(1,0)}{\sigma}\right) + Q\left(\frac{d(1,2)}{\sigma}\right) + Q\left(\frac{d(3,2)}{\sigma}\right)\right) = \frac{3}{2}Q\left(\frac{1}{\sigma}\right)$$

PAM symbol error probability

- It is often useful to express the P_e in terms of E_s/N_0.
  - 2-PAM: $E_s = \frac{2^2-1}{6} = \frac{1}{2} \Longrightarrow 2E_s = 1$ and $\sigma^2 = N_0$, and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{2E_s}{N_0}}$.

$$P_e^{(2-PAM)} = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

  - 4-PAM: $E_s = \frac{4^2-1}{6} = \frac{5}{2} \Longrightarrow \frac{2}{5}E_s = 1$, and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{2E_s}{5N_0}}$.

$$P_e^{(4-PAM)} = \frac{3}{2}Q\left(\sqrt{\frac{2E_s}{5N_0}}\right)$$

PAM bit error probability

- To have a fair comparison, the modulation performance are expressed in terms of *bit error probability* $P_e^{(b)}$ as function of E_b/N_0.
- The energy E_b per bit is computed as the energy per symbol divided by the number of bits per symbol

$$E_b = \frac{E_s}{\log_2 M}$$

- Although one symbol carries $\log_2 M$ bits, it is reasonable to assume that in a well-designed system (*Gray mapping* and medium-high SNR) *a symbol error causes only one-bit errors*.
- If $N^{(b)}$ and $N_e^{(b)}$ are the number of transmitted bits and the number of bit errors, the bit error probability is computed as

$$P_e^{(b)} = \lim_{N^{(b)}\to\infty} \frac{N_e^{(b)}}{N^{(b)}} \approx \lim_{N^{(s)}\to\infty} \frac{N_e^{(s)}}{\log_2 M\, N^{(s)}} = \frac{1}{\log_2 M} \lim_{N^{(s)}\to\infty} \frac{N_e^{(s)}}{N^{(s)}} = \frac{1}{\log_2 M} P_e.$$

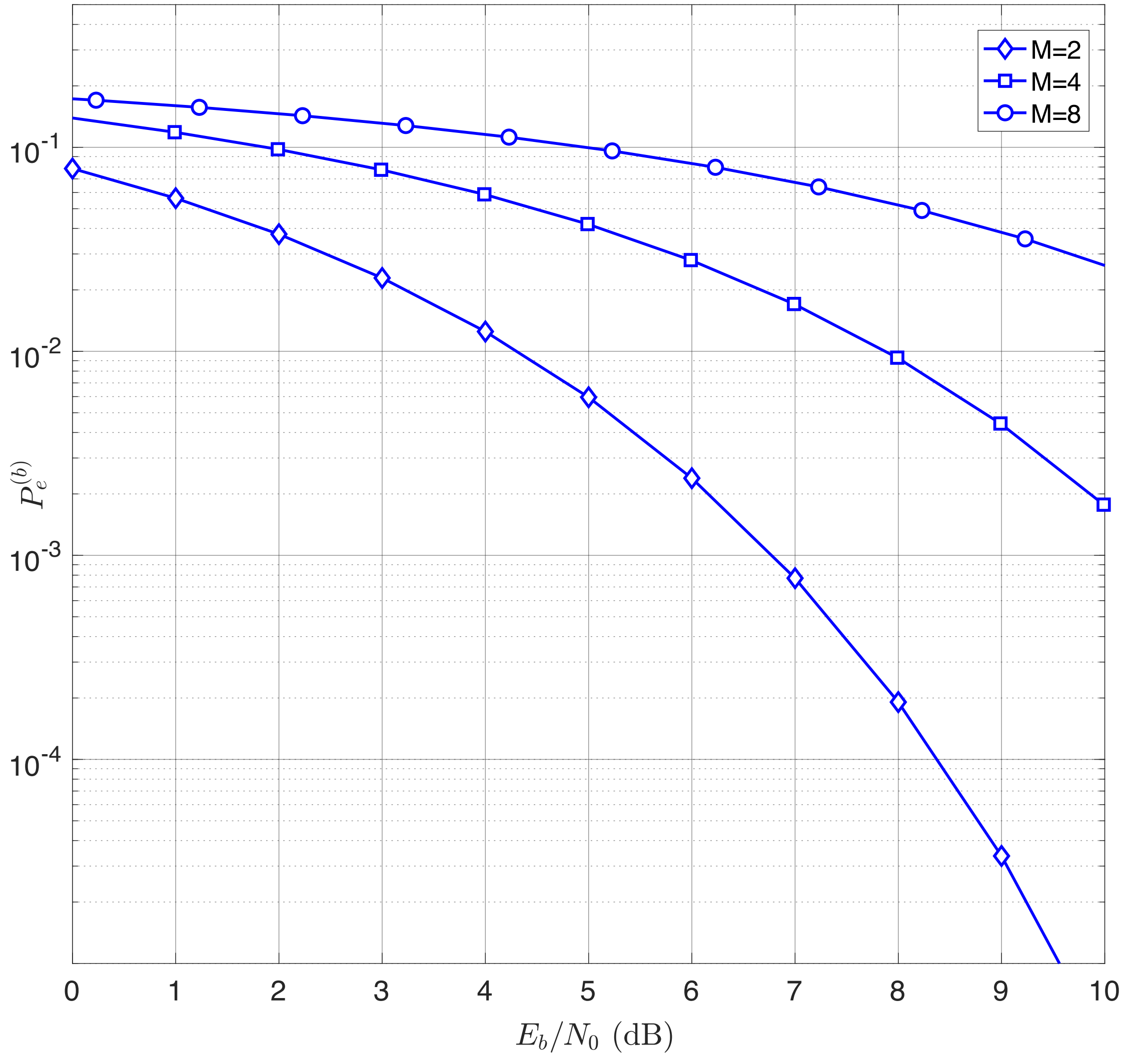
PAM bit error probability

- 2-PAM: $M = 2, m = 1$ bit per symbol $\Longrightarrow P_e^{(b)} = P_e, E_b = E_s$

$$P_e^{(2-PAM),b} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- 4-PAM: $M = 4, m = 2$ bit per symbol $\Longrightarrow P_e^{(b)} = \frac{1}{2}P_e, E_b = \frac{1}{2}E_s$

$$P_e^{(4-PAM),b} = \frac{3}{4}Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$



Digital communications

Quadrature modulations (QAM)

Quadrature modulations

- In analog modulations, QAM is obtained by transmitting two orthogonal DSB signals $m_I(t), m_Q(t)$ and the complex envelope is

$$\tilde{s}_{QAM}(t) = m_I(t) + jm_Q(t)$$

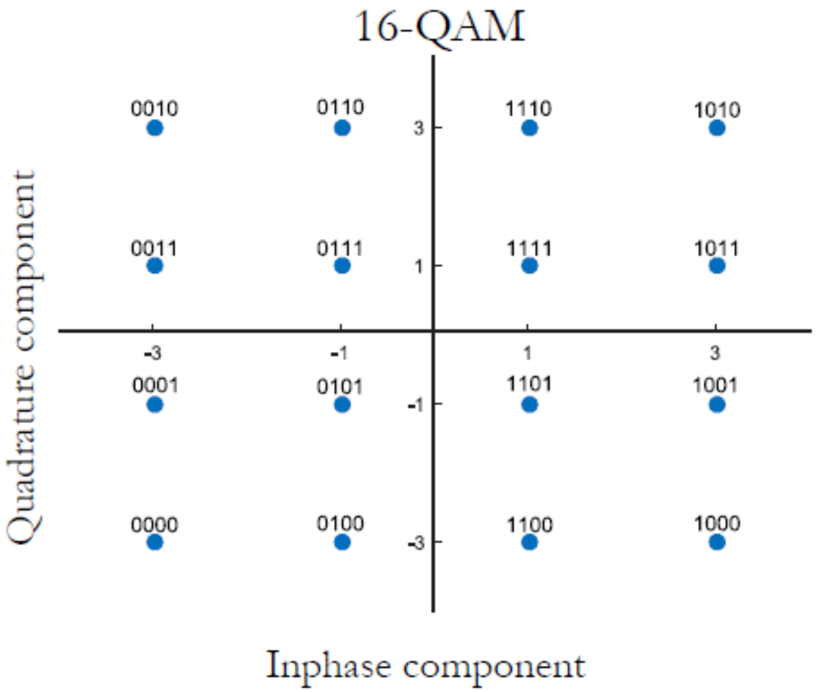
- Quadrature PAM is obtained exactly in the same manner by transmitting two PAM signals in quadrature $m_I(t) = \sum_i a_i g_T(t - iT)$ and $m_Q(t) = \sum_i b_i g_T(t - iT)$, with a_i, b_i PAM symbols.
- The QAM signal is

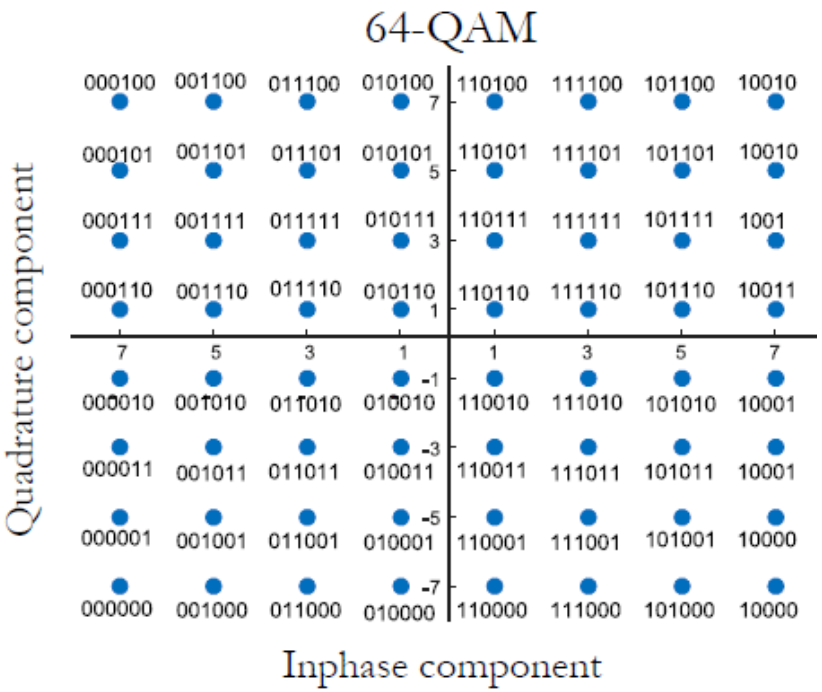
$$s_{QAM}(t) = \sum_i (a_i + jb_i) g_T(t - iT) = \sum_i c_i g_T(t - iT)$$

and the QAM complex symbols take the form $c_i = a_i + jb_i$.

QAM symbols

- Because QAM is the combination of two orthogonal PAM, the values of $M_{QAM} = M_{PAM}^2$ are squared powers of 2, i.e. m is always even.
  - If the two PAMs have $M_{PAM} = 4$ symbols than the QAM has $M_{QAM} = 16$ symbols, if $M_{PAM} = 8$ than $M_{QAM} = 64$.





Energy of a QAM symbol

- In the computation of power and energy the only difference between PAM and QAM is in the mean square value othe symbols.
- Keeping in mind that th in-phase and quadrature symbols are indipendent and zero-mean, it is

$$A = E\{c_i c_i^*\} = E\{a_i^2\} + E\{b_i^2\} = 2\,\frac{M_{PAM}^2 - 1}{3} = 2\,\frac{M_{QAM} - 1}{3}$$

- The energy per symbol is

$$E_s = \frac{A}{2} = \frac{M_{QAM} - 1}{3}$$

- QAM constellation is much more more compact and requires less energy per symbol compared to PAM.
  - $A^{(4-PAM)} = \frac{16-1}{3} = 5;$ $A^{(4-QAM)} = 2\,\frac{4-1}{3} = 2$
  - $A^{(16-PAM)} = \frac{256-1}{3} = 85;$ $A^{(16-QAM)} = 2\,\frac{16-1}{3} = 10.$

QAM error probability

- The *complex* decision variable is

$$x(m) = c_m + n(m) = (a_m + jb_m) + \left(n_I(m) + jn_Q(m)\right)$$
$$= a_m + n_I(m) \ + j\left(b_m + n_Q(m)\right)$$

- The in-phase and quadrature noise components $n_I(m)$ and $n_Q(m)$ are independent.
- Error events depends on noise. If the noise is independent also the error events on the two components are independent.
- The error probability can be *approximated* as the sum of the probability of making an error on the in-phase symbol a_m and probability of making an error on the quadrature symbol b_m.

4-QAM error probability

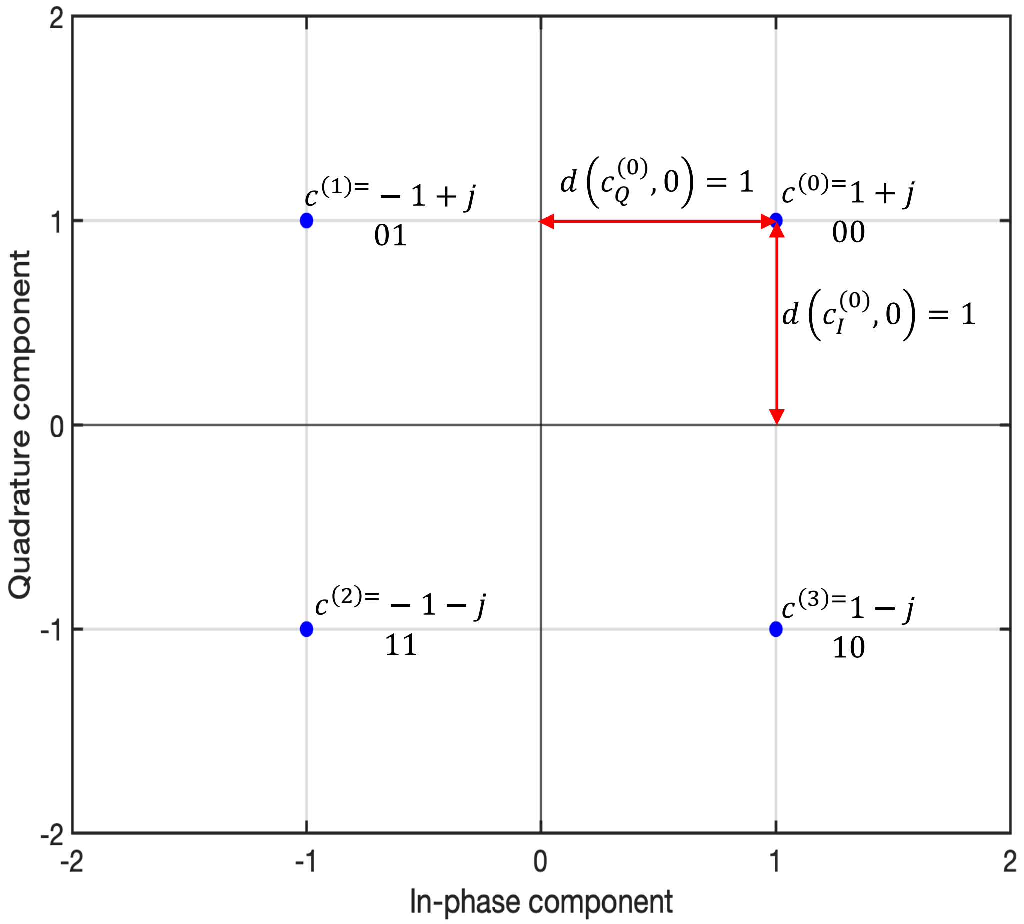
- 4-QAM is obtained as the composition of two 2-PAM in quadrature.
- The symbol error probability is

$$P_e^{(4-QAM)} = \frac{1}{4}\sum_{i=0}^{3} P\left(e|c^{(i)}\right) = P\left(e|c^{(0)}\right)$$

$$P\left(e|c^{(0)}\right) \approx Q\left(\frac{d\left(c_I^{(0)},0\right)}{\sigma_{n_I}}\right) + Q\left(\frac{d\left(c_Q^{(0)},0\right)}{\sigma_{n_Q}}\right)$$

$$= 2Q\left(\frac{1}{\sigma}\right)$$

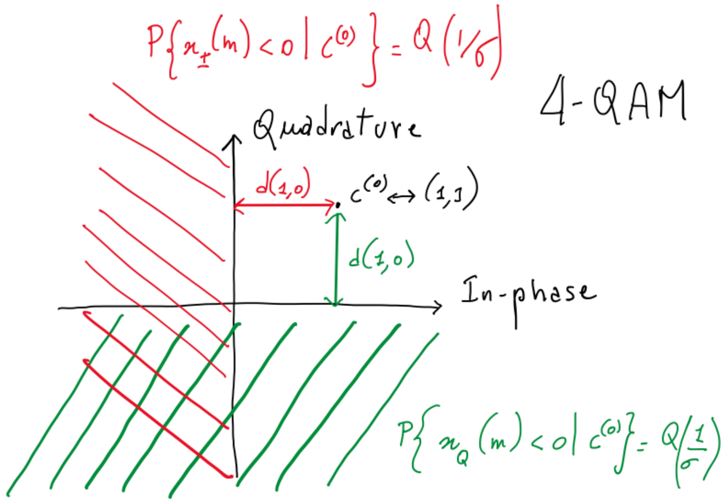
$$P_e^{(4-QAM)} \approx 2P_e^{(2-PAM)}$$

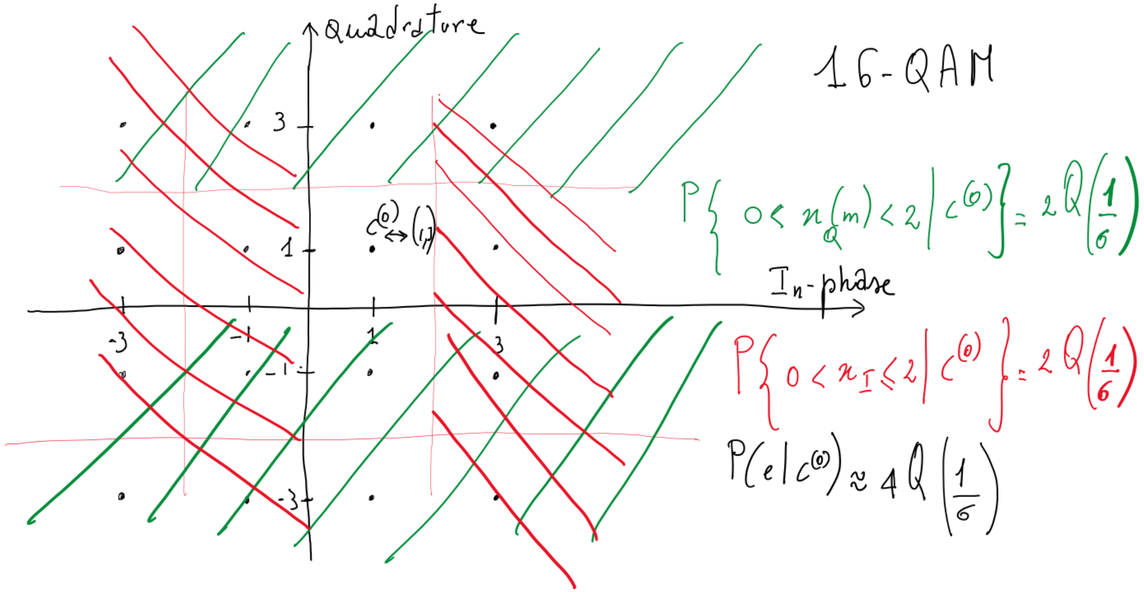


M-QAM error probability

- M-QAM is obtained as the composition of two PAM in quadrature, each with $\sqrt{M}$ symbols.
- The symbol error probability can always be approximated as

$$P_e^{(M-QAM)} \approx 2P_e^{(\sqrt{M}-PAM)}$$





QAM symbol error probability

- 4-QAM: $E_s = \frac{4-1}{3} = 1 \Longrightarrow E_s = 1, M = 4, m = 2$ and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{E_s}{N_0}}$.

$$P_e^{(4-QAM)} \approx 2Q\left(\sqrt{\frac{E_s}{N_0}}\right);$$

16-QAM: $E_s = \frac{16-1}{3} = 5 \Longrightarrow \frac{1}{5}E_s = 1, M = 16, m = 4$ and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{E_s}{5N_0}}$.

$P_e^{(16-QAM)} \approx 2\frac{3}{2}\left(\sqrt{\frac{1}{\sigma}}\right) = 3Q\left(\sqrt{\frac{E_s}{5N_0}}\right); P_e^{(16-QAM),b} \approx \frac{1}{4}3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) = \frac{3}{4}Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$

M-QAM error probability

- The total number of M-QAM transmitted bits is the sum of the number of bits transmitted on the in-phase and quadrature channels.
- Because each channel is independent, the bit error probability per channel is independent.
- Accordingly, $P_e^{(M-QAM),b}$ can be *exactly* computed as the sum of the bit error probability on the in-phase channel and the quadrature channel, divided by two.

$$P_e^{(M-QAM),b} = \frac{1}{2} 2 P_e^{\left(\sqrt{M}-PAM\right),b} = P_e^{\left(\sqrt{M}-PAM\right),b}$$

QAM bit error probability

- 4-QAM:

$$P_e^{(4-QAM),b} = P_e^{(2-PAM),b} \approx Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- 16-QAM

$$P_e^{(16-QAM),b} = P_e^{(4-PAM),b} \approx \frac{3}{4} Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$