Channel as a tapped delay line

- When a signal with symbol time T propagates through the channel $h(t)$, the channel impulse response $h(t) = \sum_{m=0}^{N_c-1} \alpha_m e^{j\phi_m} \delta(t - \tau_m)$ can be resampled at intervals multiple di T and the *equivalent channel impulse* response is

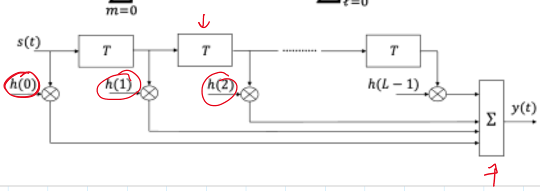
$$h_{eq}(t) = \sum_{\ell=0}^{L-1} h(\ell)\delta(t - \ell T)$$

- Even if L might be different from N_c, the channel characteristics do not change.

OFDM signal model (1)

- The complex envelope of the signal received through the multipath channel is

$$y(t) = \sum_{m=0}^{N_c-1} \underbrace{\alpha_m e^{j\phi_m}}_{\text{PATH}} s(t - \tau_m) = \sum_{\ell=0}^{L-1} \underbrace{h(\ell)}_{\text{TAP}} s(t - \ell T)$$



OFDM signal model (2)

Let's consider a block $\boldsymbol{s} = [s(0), s(1), \ldots, s(N-1)]$ of N samples. After passing through the channel, the received samples are

$$\begin{aligned} y(k) &= \sum_{\ell=0}^{L-1} h(\ell) s(k - \ell) \\ &= h(0)s(k) + \cdots + h(L-1)s(k - L + 1) \end{aligned}$$

Since the elements of $\boldsymbol{s}$ are not defined for negative indices, the values of the samples $s(-1), s(-2), \ldots, s(L-1)$ is 0. Accordingly, the received signal is

$$\begin{aligned} y(0) &= h(0)s(0) \\ y(1) &= h(0)s(1) + h(1)s(0) \\ &\vdots \\ y(N-1) &= h(0)s(N-1) + h(1)s(N-2) + \cdots + h(L-1)s(N-L) \end{aligned}$$

OFDM signal model (3): matrix notation

TOEPLITZ
NOT
CIRCULANT

- In matrix notation the block of received samples $\mathbf{y}$ can be represented as

$$\mathbf{y} = \mathcal{H}\mathbf{s}$$

$N \times N$

$\mathcal{H}$

$$N \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ h(1) & h(0) & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & h(1) & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ h(L-1) & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & h(L-1) & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & h(L-1) & \ddots & \ddots & h(1) & h(0) \end{bmatrix} \begin{bmatrix} s(0) \\ s(1) \\ \vdots \\ \vdots \\ s(N-1) \end{bmatrix} N$$

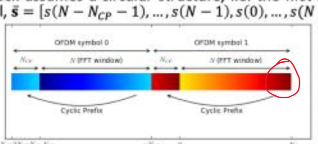
The elements along any diagonal of the $N \times N$ matrix $\mathcal{H}$ are all equal and $\mathcal{H}$ is called a *Toeplitz* matrix.

$y(1) = h(1) \cdot s_0 + h(0)s(1)$ N^2 L different values

L≈L

OFDM signal model (4): cyclic extension

- By copying the last $N_{CP} > L$ samples of $\boldsymbol{s}$ and adding them at the beginning of the block, the block assumes a *circular* structure, i.e. the first N_{CP} and last N_{CP} samples are equal, $\bar{\mathbf{s}} = [s(N - N_{CP} - 1), \ldots, s(N-1), s(0), \ldots, s(N-1)]$.



CP
↳ Cyclic
Prefix

- Keeping the same indexing, the samples with negative indexes take the values $\bar{s}(-1) = s(N-1), \bar{s}(-2) = s(N-2), \ldots, \bar{s}(-L+1) = s(N-L+1)$

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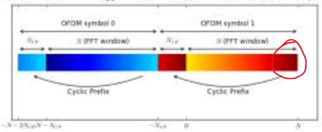

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$length(s) = N$; $length(\bar{s}) = N + N_{CP}$

$s(0), s(1), \dots \dots s(N-1)$ $\quad s$

$s(-N_{CP}+1), s(-N_{CP}+2) \dots s(-2), s(-1), s(0), s(1) \quad s(N-1)$ $\quad \bar{s}$

NEGATIVE INDICES

$s(N-2)$ $s(N-1)$

OFDM signal model (4): cyclic extension

- After the cyclic extension, the received signal becomes

$$y(k) = \sum_{\ell=0}^{L-1} h(\ell)\bar{s}(k-\ell)$$

CIRCULAR CONVOLUTION

$$y(0) = h(0)\bar{s}(0) + h(1)\bar{s}(-1) + \dots + h(L-1)\bar{s}(-L+1)$$

$$y(0) = h(0)s(0) + h(1)s(N-1) + \dots + h(L-1)s(N-L+1)$$

$$y(1) = h(0)\bar{s}(1) + h(1)\bar{s}(0) + \dots + h(L-1)\bar{s}(-L+2)$$

$$y(1) = h(0)s(1) + h(1)s(0) + \dots + h(L-1)s(N-L+2)$$

$$y(0) = \sum_{\ell=0}^{L-1} h(\ell)\,\bar{s}(0-\ell) = h(0)\bar{s}(0) + h(1)\bar{s}(-1) + \dots + h(L-1)\bar{s}(-L+1)$$
$$= h(0)s(0) + h(1)s(N-1) + \dots + h(L-1)s(N-L+1)$$

before it was $y(0) = h(0)s(0)$
Now it is $y(0) = h(0)s(0) + h(1)s(N-1) + \dots + h(L-1)s(N-L+1)$

OFDM signal model (5): matrix notation

- In matrix notation, the N-dimensional received vector $\mathbf{y}$ can be represented as

$$\mathbf{y} = \bar{\mathcal{H}}\mathbf{s}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & \ddots & \ddots & \ddots & h(3) & h(2) & h(1) \\ h(1) & h(0) & \ddots & \ddots & \ddots & \ddots & h(3) & h(2) \\ \vdots & h(1) & \ddots & \ddots & \ddots & \ddots & \ddots & h(3) \\ h(L-1) & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & h(L-1) & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & h(0) & 0 \\ 0 & \ddots & 0 & h(L-1) & \ddots & \ddots & h(1) & h(0) \end{bmatrix} \begin{bmatrix} s(0) \\ s(1) \\ \vdots \\ \vdots \\ \vdots \\ s(N-1) \end{bmatrix}$$

- The N columns of the $N \times N$ matrix $\bar{\mathcal{H}}$ are obtained by a cyclic shift one of each other and the matrix is called *circulant.*
- There is a loss of *power* and *spectral* efficiency: since a vector of length $N + N_{CP}$ samples is transmitted for a length-N data vector

Discrete convolution → circular convolution

OFDM signal model (6)

- The interesting property of circulant matrices is that they can be diagonalized as

$$\bar{\mathcal{H}} = \mathbf{F}^H\mathbf{H}\mathbf{F}$$

where $\mathbf{F}$ is the normalized Fourier transform matrix, i.e.

$$[\mathbf{F}]_{k,n} = \frac{1}{\sqrt{N}} e^{-\frac{j2\pi kn}{N}}$$

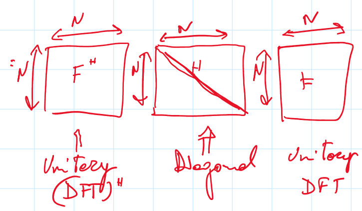
and $\mathbf{H}$ is a diagonal matrix where the n-th element along the diagonal is

$$[\mathbf{H}]_{n,n} = H(n) = \sum_{\ell=0}^{L-1} h(\ell) e^{-\frac{j2\pi \ell n}{N}}$$

- The matrix $\mathbf{F}$ is unitary, i.e., $\mathbf{F}^H\mathbf{F} = \mathbf{F}\mathbf{F}^H = \mathbf{I}_N$.

U is unitary $\quad y = Ux \quad \|y\|^2 = y^H y = \underbrace{x^H U^H}_{y^H} \underbrace{U x}_{y} = x^H x = \|x\|^2$ $\quad (U^H U = I)$

$\bar{\mathcal{H}} =$



OFDM signal model (7)

- If we define $\mathbf{Y}=\mathbf{F}\mathbf{y}$, $\mathbf{S}=\mathbf{F}\mathbf{s}$, the FFT of $\mathbf{y}$ yields
$$\mathbf{Y}=\mathbf{F}\mathbf{y}=\mathbf{F}\bar{\mathcal{H}}\mathbf{s}=\mathbf{F}\mathbf{F}^H\mathbf{H}\mathbf{F}\mathbf{s}=\mathbf{H}\mathbf{S}$$
- Since $\mathbf{H}$ is diagonal, the signal received on subcarrier n depends *exclusively* on the signal transmitted on subcarrier n.
- There is no ISI in the frequency domain!!!
$$Y(n)=H(n)S(n)$$

$\bar{\mathcal{H}}=\underline{F^H H F}$

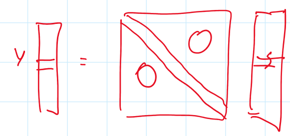
$Y=Fy \quad S=Fs$

$Y=Fy=F\bar{\mathcal{H}}s=\cancel{F F^H}HFs \Rightarrow Y=HS$ (I)

$S=Fs$

$F^H S=F^H F s=s \Rightarrow s=F^H S$

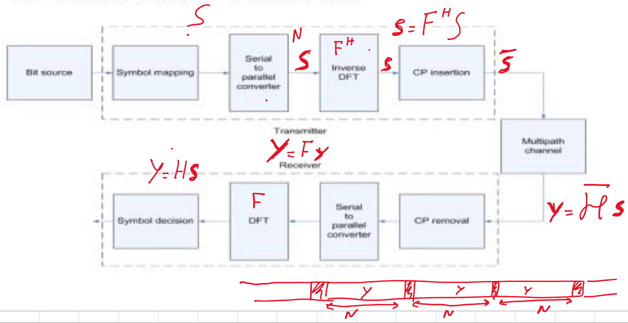
$Y=HS$



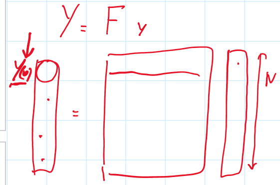
$Y(n)=H(n)S(n) \qquad n=0,\dots,N-1 \qquad Y=HS$

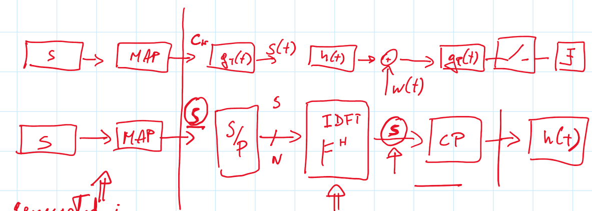
OFDM baseband transceiver

(N) symbols



$Y=Fy$





generated in the frequency domain

OFDM baseband transceiver

1. In the serial-to-parallel block, a block of N consecutive data symbols are collected in the vector $\mathbf{S}=[S(0),S(1),\dots,S(N-1)]$.
2. The IDFT block converts $\mathbf{S}$ into a 'time-domain' vector
$$\mathbf{s}=\mathbf{F}^H\mathbf{S}$$
3. A N_{CP}–long cyclic prefix is inserted to create the new time-domain vector of length $N+N_{CP}$
$$\bar{\mathbf{s}}=[s(N-N_{CP}-1),\dots,s(N-1),s(0),\dots,s(N-1)]$$

OFDM baseband transceiver

4. The signal propagates through the wireless channel with impulse response $\mathbf{h}=[h(0),h(1),\dots,h(L-1)]$
$$y(k)=\sum_{\ell=0}^{L-1}h(\ell)\bar{s}(k-\ell)$$
circular convolution
5. At the receiver the samples corresponding to the CP, which do not carry any information, are discarded and the remaining samples are frequency converted $\mathbf{Y}=\mathbf{F}\mathbf{y}$, yielding
$$\mathbf{Y}=\mathbf{F}\bar{\mathcal{H}}\mathbf{s}=\mathbf{F}(\mathbf{F}^H\mathbf{H}\mathbf{F})\mathbf{s}$$

H is a complex number

$H=|H|e^{j\phi_H}$

OFDM on multipath channel

- The overall signal bandwidth is B.
- The sampling duration is $T = 1/B$.
- The OFDM block duration is $T_{OFDM} = T(N + N_{CP})$
- The bandwidth for each subcarrier is $\Delta f = B/N$
- By accurately choosing N we have

$$T < \sigma_\tau \ll T_{OFDM}, B > B_c \gg \Delta f$$

- On each subcarrier the channel is *flat*!!