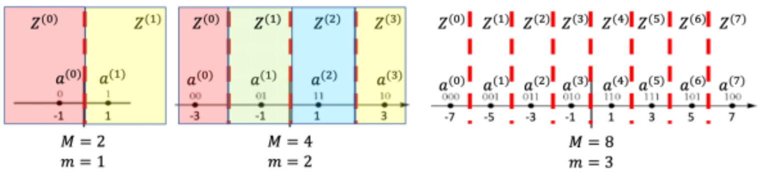
8th lesson

Monday, 19 October 2020 17:22

Decision strategy

- Adopting the maximum likelihood criterion, we can partition the signal space in *zone of decisions*, where zone $Z^{(i)}$ is the set of points that are closer to the symbol $a^{(i)}$ than to any other symbol

$$Z^{(i)} = \{x | d(x, a^{(i)}) < d(x, a^{(j)}), j \neq i, j = 1, \dots, M\}$$



The decision threshold are in the midpoints of the segment connecting any two adjacent symbols. For example, for $M = 4$ the thresholds are in -2,0 and 2.

$x(m) = a^{(i)} + n(m)$

$x(m) \in \mathcal{N}(a^{(i)}, \sigma^2)$

PAM error probability

- Even if the maximum likelihood decision strategy is optimal, the receiver still make errors due to the presence of noise.
- The error probability is averaged over all the symbol of the constellation

$$P_e = \lim_{N^{(s)} \to \infty} \frac{N_e^{(s)}}{N^{(s)}} = \frac{1}{M} \sum_{i=0}^{M-1} P(e|a^{(i)})$$

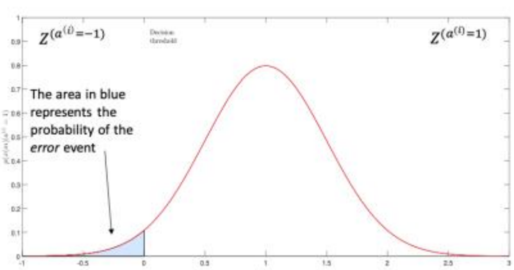
where $N_e^{(s)}$ is the number of symbol errors and $N^{(s)}$ is the number of transmitted symbols.

- The probability of error $P(e|a^{(i)})$ is the probability that, having transmitted $a^{(i)}$, the decision variable $x(m)$ does not fall in the decision region $Z^{(i)}$.

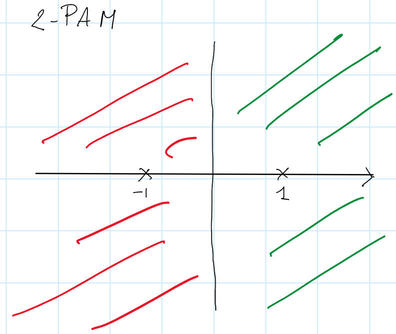
PAM error probability

- To compute $P(e|a^{(i)})$ we assume that the transmitted symbol is $a_m = a^{(i)}$, so that it is $x(m) = a^{(i)} + n(m)$ and the probability of error is

$$P(e|a^{(i)}) = Pr\{x(m) \notin Z^{(i)} | a_m = a^{(i)}\}$$



2-PAM



if $x(m) > 0$

$\Rightarrow \hat{a}_m = 1$

else

$\hat{a}_m = -1$

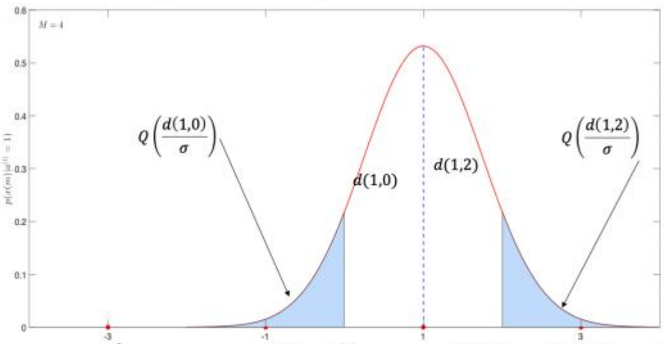
$x(m) < 0$

$P(e|a^{(1)}) \Rightarrow P\{x(m) < 0 \mid a_m = 1\}$

$Pr\{x(m) < 0 | a_m = 1\} = \int_{-\infty}^{0} p(x(m)|a_m = 1)\, dx(m) \Rightarrow$ We can use Q function $Q(x)$

$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-t^2/2}\, dt$

PAM error probability



$$P(e|a^{(i)}=1)=\int_{-\infty}^{0}p(x|a^{(i)}=1)dx+\int_{2}^{+\infty}p(x|a^{(i)}=1)dx=Q\left(\frac{d(1,0)}{\sigma}\right)+Q\left(\frac{d(1,2)}{\sigma}\right)=2Q\left(\frac{1}{\sigma}\right)$$

Properties of distance $d(x,y)$

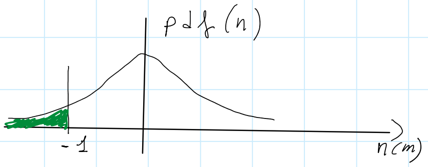
$d(x,y)=d(y,x)$

$d(x,y)\geq 0$

$d(x,y)=0 \Leftrightarrow x=y$

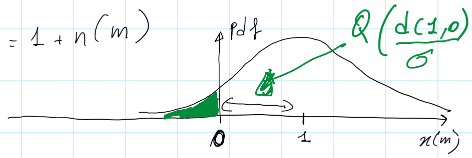
$d(x,z)\leq d(x,y)+d(y,z)$

$x(m)\in\mathcal{N}(1,\sigma^2)$



$x(m)=1+n(m)$

$\Longrightarrow$



PAM error probability: Q-function

- The Q-function computes the integral of the *tail* of a Gaussian distribution.
- The probability that $x\in\mathcal{N}(m,\sigma^2)$ is smaller than t_1 or larger than t_2 are the integral of Gaussian tails and they are computed as

$$\left.\begin{array}{l}\int_{-\infty}^{t_1}pdf(x)dx=Q\left(\frac{m-t_1}{\sigma}\right)\\ \int_{t_2}^{+\infty}pdf(x)dx=Q\left(\frac{t_2-m}{\sigma}\right)\end{array}\right\}=Q\left(\frac{d(t_i,m)}{\sigma}\right), i=1,2$$

- In our case, m is the symbol $a^{(i)}$ and t_1 or t_2 are the detection thresholds.
- The main properties of the Q-function are

$$Q(-\infty)=1, Q(\infty)=0, Q(0)=0.5, Q(-x)=1-Q(x).$$

PAM error probability

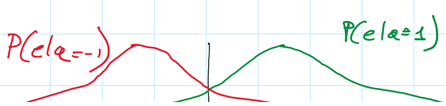
- 2-PAM

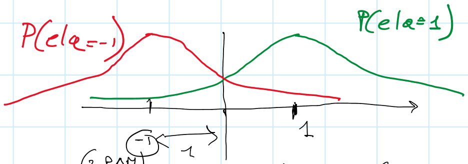
$$P_e^{(2-PAM)}=\frac{1}{2}\left(Q\left(\frac{d(-1,0)}{\sigma}\right)+Q\left(\frac{d(1,0)}{\sigma}\right)\right)=Q\left(\frac{1}{\sigma}\right)$$

- 4-PAM

$$P_e^{(4-PAM)}=\frac{1}{4}\left(Q\left(\frac{d(-3,-2)}{\sigma}\right)+Q\left(\frac{d(-1,-2)}{\sigma}\right)+Q\left(\frac{d(-1,0)}{\sigma}\right)\right.$$
$$\left.Q\left(\frac{d(1,0)}{\sigma}\right)+Q\left(\frac{d(1,2)}{\sigma}\right)+Q\left(\frac{d(3,2)}{\sigma}\right)\right)=\frac{3}{2}Q\left(\frac{1}{\sigma}\right)$$

2-PAM

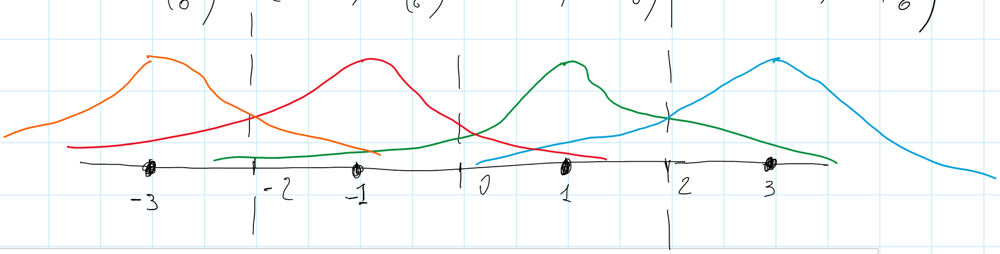




$$P_e^{(2-PAM)} = \frac{1}{2} P(e \mid a=1) + \frac{1}{2} P(e \mid a=-1)$$
$$= \frac{1}{2} Q\left(\frac{1}{\sigma}\right) + \frac{1}{2} Q\left(\frac{1}{\sigma}\right) = Q\left(\frac{1}{\sigma}\right)$$

4-PAM

$P(e \mid a=-3) = Q\left(\frac{1}{\sigma}\right)$ $P(e \mid a=-1) = 2Q\left(\frac{1}{\sigma}\right)$ $P(e \mid a=1) = 2Q\left(\frac{1}{\sigma}\right)$ $P(e \mid a=3) = Q\left(\frac{1}{\sigma}\right)$



PAM symbol error probability

- It is often useful to express the P_e in terms of E_s/N_0.
  - 2-PAM: $E_s = \frac{2^2-1}{6} = \frac{1}{2} \Longrightarrow 2E_s = 1$ and $\sigma^2 = N_0$, and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{2E_s}{N_0}}$.
$$P_e^{(2-PAM)} = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$
  - 4-PAM: $E_s = \frac{4^2-1}{6} = \frac{5}{2} \Longrightarrow \frac{2}{5}E_s = 1$, and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{2E_s}{5N_0}}$.
$$P_e^{(4-PAM)} = \frac{3}{2} Q\left(\sqrt{\frac{2E_s}{5N_0}}\right)$$

PAM bit error probability

- To have a fair comparison, the modulation performance are expressed in terms of *bit error probability* $P_e^{(b)}$ as function of E_b/N_0.
- The energy E_b per bit is computed as the energy per symbol divided by the number of bits per symbol
$$E_b = \frac{E_s}{\log_2 M}$$
- Although one symbol carries $\log_2 M$ bits, it is reasonable to assume that in a well-designed system (*Gray mapping* and medium-high SNR) ***a symbol error causes only one-bit errors.***
- If $N^{(b)}$ and $N_e^{(b)}$ are the number of transmitted bits and the number of bit errors, the bit error probability is computed as
$$P_e^{(b)} = \lim_{N^{(b)}\to\infty} \frac{N_e^{(b)}}{N^{(b)}} \approx \lim_{N^{(s)}\to\infty} \frac{N_e^{(s)}}{\log_2 M\, N^{(s)}} = \frac{1}{\log_2 M} \lim_{N^{(s)}\to\infty} \frac{N_e^{(s)}}{N^{(s)}} = \frac{1}{\log_2 M} P_e.$$

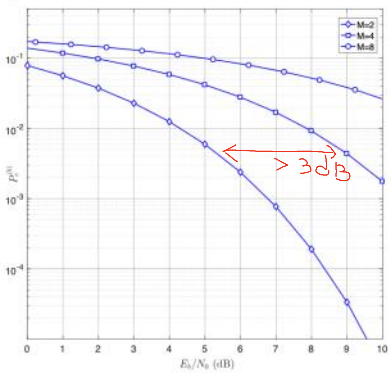
With Gray encoding, adjacent PAM symbols map sequence of bits that are different of only one position
For example a possible Gray

that are different of only one position
For example a possible Gray
encoding strategy for a 4-PAM is

00	01	11	10
-3	-1	1	3

PAM bit error probability

- 2-PAM: $M = 2, m = 1$ bit per symbol $\Longrightarrow P_e^{(b)} = P_e, E_b = E_s$
$$P_e^{(2-PAM),b} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$
- 4-PAM: $M = 4, m = 2$ bit per symbol $\Longrightarrow P_e^{(b)} = \frac{1}{2}P_e, E_b = \frac{1}{2}E_s$
$$P_e^{(4-PAM),b} = \frac{3}{4}Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$



$$P_e^{2-PAM,b} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$P_e^{4-PAM,b} = \cancel{\frac{3}{4}}\,Q\left(\sqrt{\frac{4}{5}\frac{E_b}{N_0}}\right)$$

$$Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \text{ vs. } Q\left(\sqrt{\frac{4}{5}\frac{E_b}{N_0}}\right)$$

$$2\frac{E_b}{N_0} \qquad \underbrace{\frac{4}{5}}_{\sim 1}\cdot\frac{E_b}{N_0} \Longrightarrow \text{guadagno} > 2\times\ (3dB)$$

Digital communications

Quadrature modulations (QAM)

Quadrature modulations

- In analog modulations, QAM is obtained by transmitting two orthogonal DSB signals $m_I(t), m_Q(t)$ and the complex envelope is
$$\tilde{s}_{QAM}(t) = m_I(t) + jm_Q(t)$$
- Quadrature PAM is obtained exactly in the same manner by transmitting two PAM signals in quadrature $m_I(t) = \sum_i a_i g_T(t - iT)$ and $m_Q(t) = \sum_i b_i g_T(t - iT)$, with a_i, b_i PAM symbols.
- The QAM signal is
$$s_{QAM}(t) = \sum_i \underbrace{(a_i + jb_i)} g_T(t - iT) = \sum_i c_i g_T(t - iT)$$
and the QAM complex symbols take the form $c_i = a_i + jb_i$.

$$s_{QAM}(t) = \underbrace{\sum_m \overset{\text{Real}}{a_m}\, g_T(t - mT)}_{\text{In-phase comp.}} + j\underbrace{\sum_m \overset{\text{Real}}{b_m}\, g_T(t - mT)}_{\text{Quadrature comp.}}$$

PAM $\Rightarrow$ symbols are real

QAM $\Rightarrow$ symbols are complex $(c_m = a_m + jb_m)$

QAM symbols

- Because QAM is the combination of two orthogonal PAM, the values of $M_{QAM} = M_{PAM}^2$ are squared powers of 2, i.e. m is always even.
  - If the two PAMs have $M_{PAM} = 4$ symbols than the QAM has $M_{QAM} = 16$ symbols, if $M_{PAM} = 8$ than $M_{QAM} = 64$.

GRAY MAPPED

16-QAM combination of 2 4-PAM