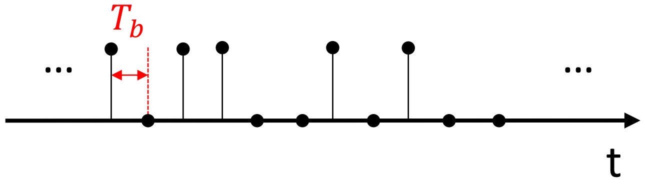
Digital communications

How can we transmit a sequence of bits?

- What happens if we want to transmit a *sequence of bits* in the place of an analog signal?

$$d_k = \cdots 1,0,1,1,0,0,1,0,1,0,1,0,0, \ldots$$



$$\sum_i d_k \delta(t - kT_b)$$

- A train of delta occupies infinite bandwidth, before transmission the bits need to be passed through a low pass filter.
- For reasons already discussed, any signal transmitted in the air needs to be translated in frequency.

How can we transmit a sequence of bits?

- Each bit of the sequence can be modelled as an equiprobable random variable

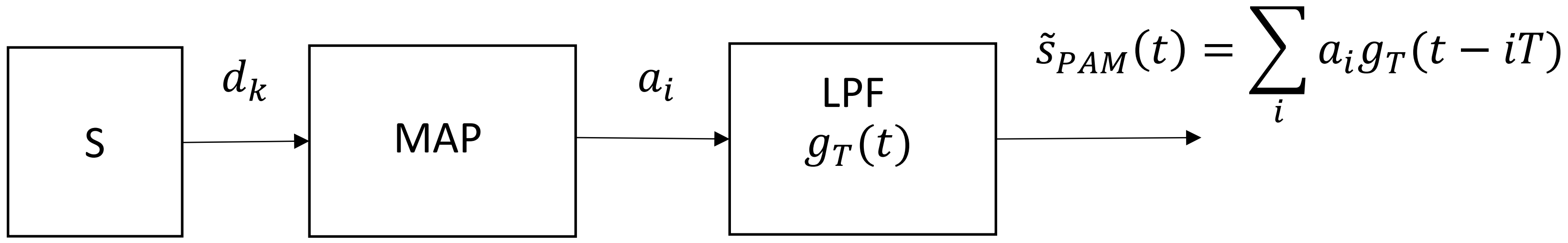
$$P\{d_k = 0\} = P\{d_k = 1\} = \frac{1}{2}$$

so that $E\{d_k\} = \frac{1}{2}$.

- In general, to save energy it is better to transmit 0-mean information.
- Bits d_k are mapped to 0-mean *information symbols*: $a_i = 2d_i - 1.$
- One information symbol can be used to map more than just one bit.

Pulse amplitude modulation

- Pulse amplitude modulation (PAM) is the modulation obtained by
  1. Mapping the bits d_k to the information symbols a_i
  2. Filtering the symbols with a low pass filter with impulse response $g_T(t)$



- Since the mapper can map a sequence of m bits on just one information symbol, the bit duration T_b and the symbol duration T may be different.

Pulse amplitude modulation

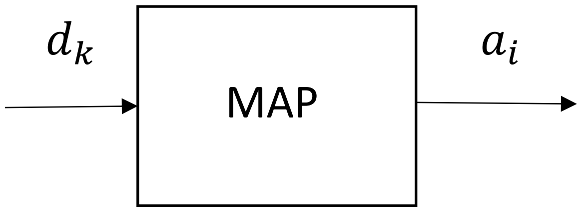
- The signal $\tilde{s}_{PAM}(t)$ is a *real* baseband signal that can be modulated at any frequency f_c

$$s_{PAM}(t) = \sum_i a_i g_T(t - iT)\cos(2\pi f_c t)$$

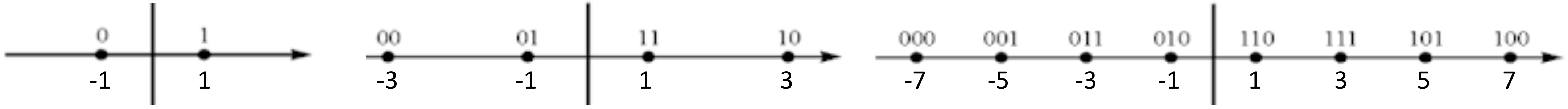
- The PAM signal is equivalent to an analog DSB where the modulating (and complex envelope) signal $m(t)$ is

$$m(t) = \sum_i a_i g_T(t - iT)$$

PAM: symbol mapping



- The mapper block associates a sequence of m bits to a symbol.
- The symbol constellation contains $M = 2^m$ bits, $m = \log_2 M$.
- If the source generates bits with a rate $R_b = \frac{1}{T_b}$, the mapper outputs symbols with a rate m times slower, i.e. $R = \frac{R_b}{m} = \frac{R_b}{\log_2 M}$ or, in terms of bit and symbol timing, $T = T_b \log_2 M$
- Usually bit-to-symbol mapping is performed so that $E\{a_i\} = 0$.

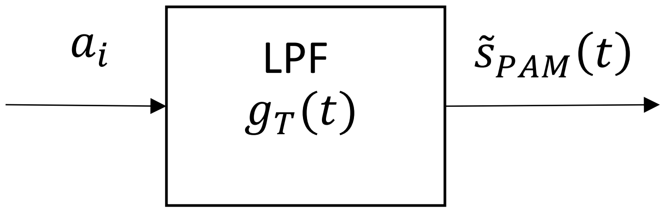


$M = 2$
$m = 1$

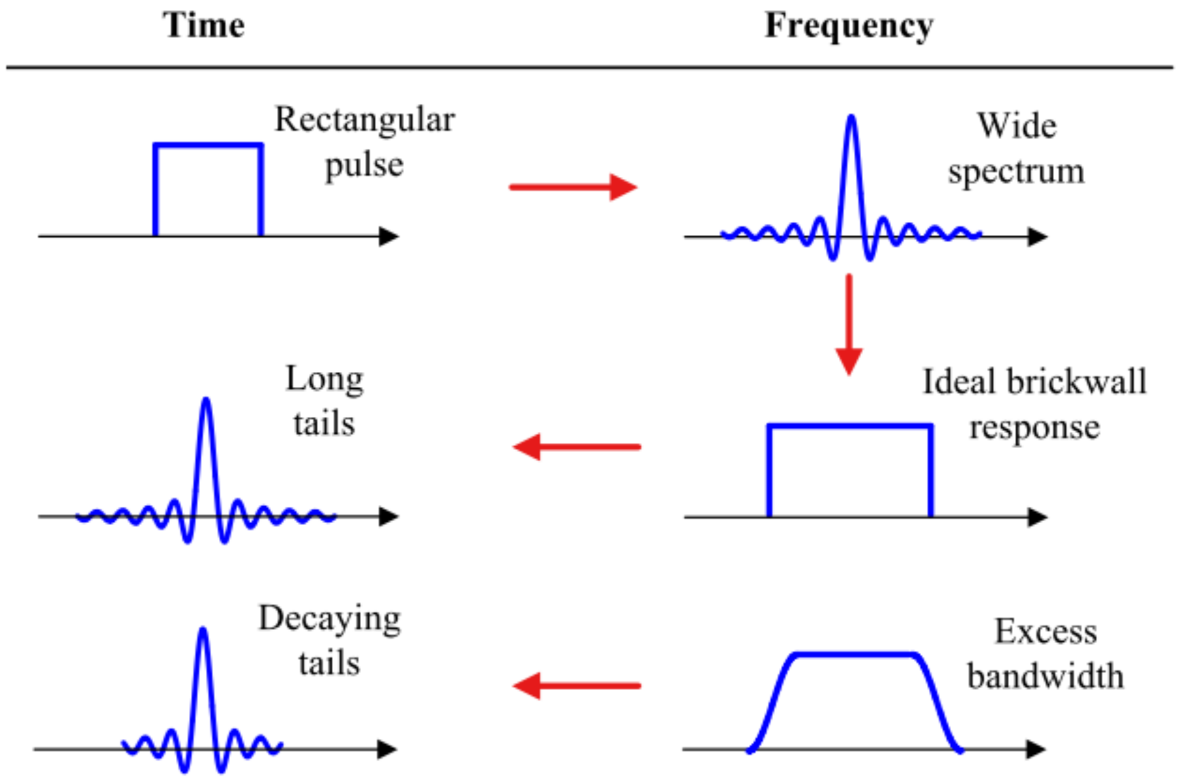
$M = 4$
$m = 2$

$M = 8$
$m = 3$

PAM: pulse shaping



- Intuitively, the choice of the impulse response of the low-pass pulse shaping filter determines the bandwidth of the PAM signal.
- If the pulse shape has duration longer than 1 symbol time T, the spectrum is more compact but the energy of one symbols is spread over several intervals.

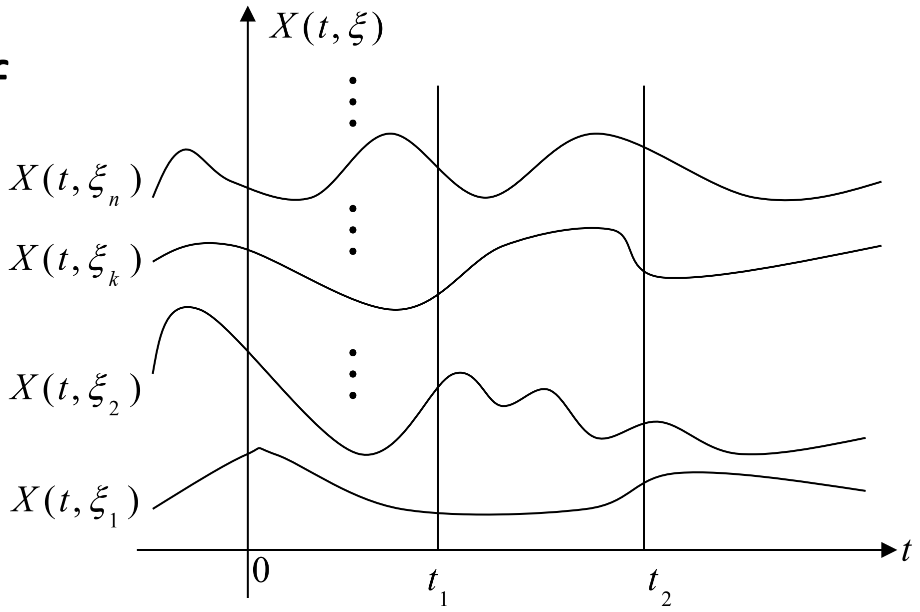


Stochastic processes

- *Deterministic process.* A deterministic process is represented by an explicit mathematical relation.
- *Stochastic process*. A stochastic process is the result of a large number of separate causes, described in probabilistic terms and by properties which are averages.

Stochastic processes

- Let ξ denote the random outcome of an experiment. To every such outcome suppose a waveform $X(t, \xi)$ is assigned. The collection of such waveforms form a *stochastic process*.
- For a fixed ξ (the set of all experimental outcomes), $X(t, \xi)$ is a specific time function.
- For fixed $t = t_0$, $X(t_0, \xi)$ is a random variable.
- The ensemble of all such realizations over time represents the stochastic process $X(t)$.



Categories of stochastic processes

- *Parameter space*: set T of indices $t \in T$.
- *State space*: set S of values $X(t) \in S$.
- Categories:
  - Based on the parameter space:
    - Discrete-time processes: parameter space discrete,
    - Continuous-time processes: parameter space continuous.
  - Based on the state space:
    - Discrete-state processes: state space discrete,
    - Continuous-state processes: state space continuous.

Distribution and probability density function

- If $X(t)$ is a stochastic process, then for fixed $t = t_0$, $X(t_0)$ represents a *random variable*.
- The *distribution function* is given by
$$F_X(x, t_0) = \Pr\{X(t_0) < x\}$$
$F_X(x, t_0)$ depends on the value of t. For different values of t, we obtain a different random variable.
- Further, the first-order *probability density function* of the process $X(t)$ is
$$f_X(x, t_0) = \frac{d}{dx} F_X(x, t_0)$$

Joint distributions

- For $t = t_1$ and $t = t_2$, $X(t)$ represents two different random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$, respectively. Their joint distribution is given by

$$F_X(x_1, x_2, t_1, t_2) = \Pr\{X(t_1) < x_1, X(t_2) < x_2\}$$

  and

$$f_X(x_1, x_2, t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2, t_1, t_2)$$

  represents the second-order density function of the process $X(t)$.
- Similarly, $f_X(x_1, \ldots, x_n, t_1, \ldots, t_n)$ represents the n-th order density function of the process $X(t)$.

Independence

- For an *independent* stochastic process, the random variables obtained by sampling the process at any n times $t_1, \ldots, t_n$ are independent random variables for any n.
- Accordingly, the distribution is

$$F_X(x_1, \ldots, x_n, t_1, \ldots, t_n) = \Pr\{X(t_1) < x_1\} \cdots \Pr\{X(t_n) < x_n\}$$
$$= F_X(x_1, t_1) \cdots F_X(x_n, t_n)$$

and the probability density function is

$$f_X(x_1, \ldots, x_n, t_1, \ldots, t_n) = f_X(x_1, t_1) \cdots f_X(x_n, t_n)$$

Mean and autocorrelation

- **Mean** of a stochastic process**:**

$$\mu_X(t_0) = E\{X(t_0)\} = \int_{-\infty}^{+\infty} x f_X(x, t_0) dx$$

is the mean value of the process $X(t)$ at time t_0. In general, the mean of a process depends on the time index t.

- **Autocorrelation function** of a process:

$$R_{XX}(t_1, t_2) = E\{X(t_1)X^*(t_2)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1 x_2^* f_X(x_1, x_2, t_1, t_2) dx_1 dx_2$$

and it represents the interrelationship between the random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ obtained by sampling the process $X(t)$ at times t_1 and t_2.

Stationarity

- A stationary process exhibits statistical properties that are invariant to shift in the time index.
- *First-order stationarity* implies that the statistical properties of $X(t_0)$ and $X(t_0 + c)$ are the same for any *c*.
$$f_X(x, t_0) = f_X(x)$$
  - The *mean* is a constant and does not depend on t
- *Second-order stationarity* implies that the statistical properties of the pairs $\{X(t_1), X(t_2)\}$ and $\{X(t_1 + c), X(t_2 + c)\}$ are the same for any c.
$$f_X(x_1, x_2, t_1, t_2) = f_X(x_1, x_2, t_2 - t_1)$$
  - The autocorrelation depends only on the difference of the time indices.

Wide sense stationarity

- The basic conditions for the first and second order stationarity are usually difficult to verify.
- In that case, we can use a looser definition of stationarity. A process $X(t)$ is said to be *wide-sense stationary* (WSS) if the two following conditions hold:
  1) $E\{X(t)\} = \mu_X$
  2) $E\{X(t_1), X(t_2)\} = R_{XX}(t_2 - t_1)$
- For a wide-sense stationary process, the mean is a constant and the autocorrelation function depends only on the difference between the time indices.

Power spectral density

- Wiener-Kintchine theorem. For stationary processes, the *power spectral density* (PSD) describes how the power of the signal is distributed over frequency

$$S_{XX}(f) = \mathcal{F}\{R_{XX}(\tau)\} = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{j2\pi f\tau} d\tau$$

- The signal power of $X(t)$ can be computed as

$$P_X = \int_{-\infty}^{+\infty} S_{XX}(f) df$$

PAM: power spectral density

- A PAM signal is modelled as a *stochastic process* because the symbols a_i are samples of a discrete-time discrete-state stochastic process.
- The bandwidth occupied by a stochastic process is measured by its *power spectral density* (Fourier transform of its autocorrelation function).
- The PSD of the PAM signal $\tilde{s}(t)$ is

$$S_{\tilde{s}}(f) = \frac{1}{T} S_a(f) |G_T(f)|^2$$

  where $S_a(f)$ is the PSD of a_i and $G_T(f)$ is the frequency response of the transmit filter $g_T(t)$.

  From now on, we omit the tilde for ease of notation.

PAM: power spectral density

- $S_a(f)$ is computed as the Fourier transform of the autocorrelation function $R_a(m)$ of the stationary, discrete, independent process a_i.

$$R_a(m) = E\{a_i a_{i+m}\} = \begin{cases} E\{a_i^2\} = A & m = 0 \\ (E\{a_i\})^2 & m \neq 0 \end{cases}$$

when symbols are zero-mean, it is

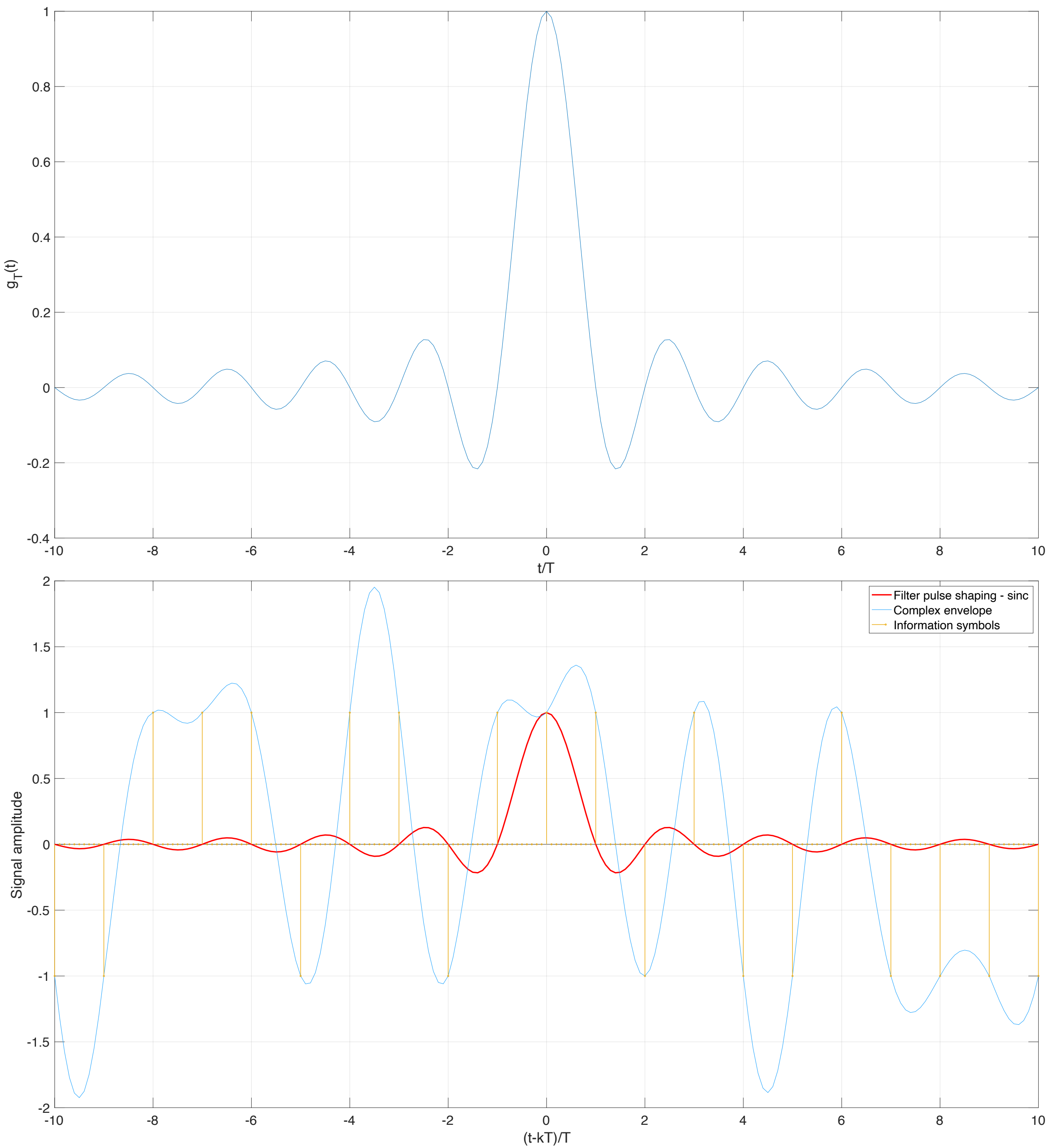
$$R_a(m) = A\delta(m)$$

and

$$S_a(f) = A$$

PAM: pulse shaping

- The most compact spectrum is obtained when $G_T(f) = \mathrm{rect}(fT)$, which in the time domain corresponds to a *sinc*.
- The pulse shape of a *sinc* spans an interval of several symbols.
- One single symbol ‘mixes’ its information with several adjacent symbols.
- This type of interference is denominated *inter-symbol interference* (ISI).

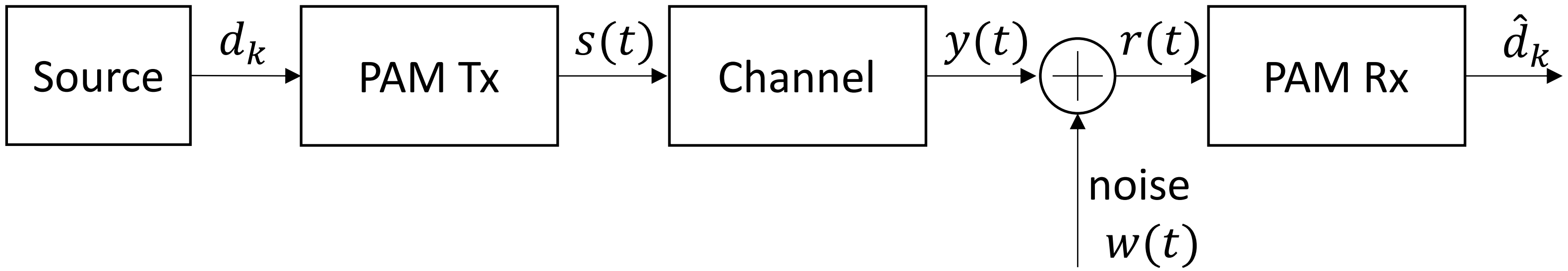


PAM: occupied bandwidth

- Because of the espression of the PSD, the bandwidth occupied by the PAM signal depends on $G_T(f)$, the frequency response of the transmit filter.
- There is a trade-off to make:
  - compact spectrum → large amount of interference in the time domain (Extreme choice: a *rect* in the frequency domain and a *sinc* in time).
  - wide spectrum → most of the symbol energy is contained within one symbol interval (Extreme choice: a *rect* in the time domain and a *sinc* in frequency).

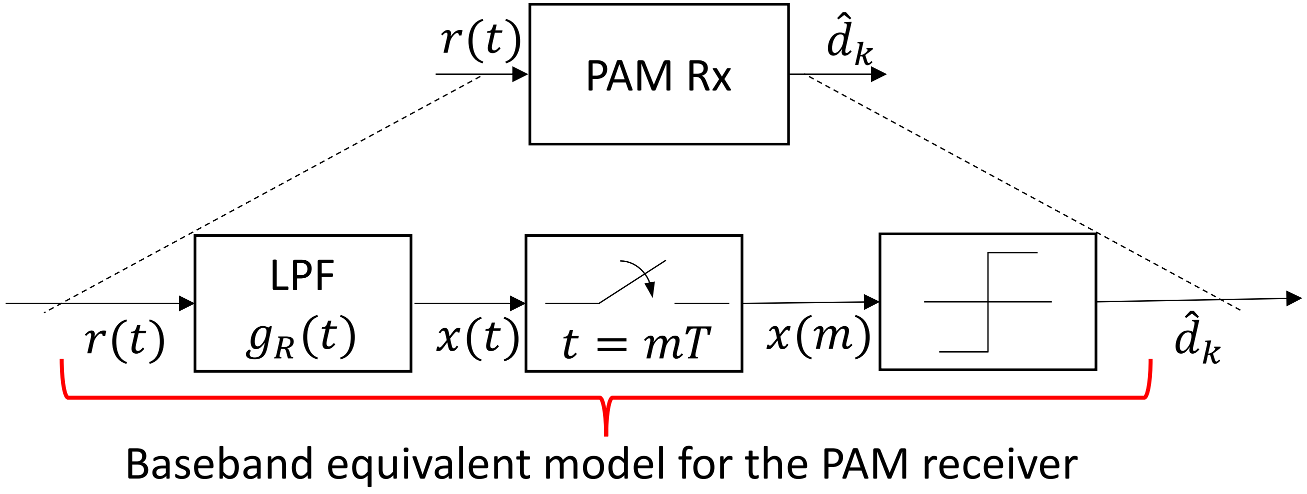
PAM: receiver architecture

- PAM system block diagram



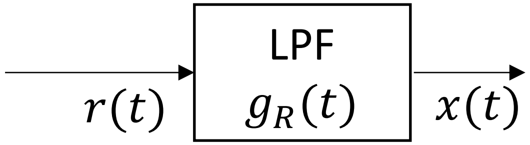
  - The propagation channel is in general modelled as a LTI filter with impuls response $h(t)$. When the channel is ideal, it is $h(t) = \delta(t)$.
  - The noise term is a white, zero-mean, Gaussian stationary process with PSD $S_w(f) = N_0/2$ ($S_w(f) = 2N_0$ for its complex envelope).
  - The receiver's task is to reconstruct the sequence of transmitted bits from the received signal $r(t)$.

PAM: receiver architecture



- The PAM receiver performs the inverse operation of the transmitter: extract the transmitted bits from the analog received signal $r(t)$.
    1. Filters the interference and spurious components from the received signal;
    2. Samples the filtered signal once per symbol time T;
    3. Recovers the transmitted bits from the signal samples.

PAM: Receive filter



- The received baseband equivalent signal has the form

$$r(t) = s(t) \otimes h(t) + w(t)$$

- The filter output is

$$x(t) = r(t) \otimes g_R(t) = \sum_i a_i g(t - iT) + n(t)$$

where $g(t) = g_T(t) \otimes h(t) \otimes g_R(t)$ is the convolution of the impulse response of the channel, the transmit and the receiver filter, $n(t)$ is the filtered (and colored!) noise.

PAM receive filter

$$r(t) \longrightarrow \boxed{\begin{array}{c}\text{LPF}\\ g_R(t)\end{array}} \longrightarrow x(t)$$

- One of the tasks of the receive filter $g_R(t)$ is to remove the intersymbol interference affecting the received samples.
- The samples of the received signal take this form:

$$x(m) = x(t)\Big|_{t=mT} = \sum_i a_i g(mT - iT) + n(mT)$$

$$= a_m\, g(0) + \underbrace{\sum_{\ell,\ell\neq 0} a_{m-\ell}\, g(\ell T)}_{\text{ISI}} + n(mT)$$

- Neglecting the noise term, the condition on $g(\ell T) = g(t)|_{t=\ell T}$ to have zero ISI is

$$g(\ell T) = \begin{cases} 1 & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$

- Under these conditions (Nyquist criterion), the received sample $x(m)$ is

$$x(m) = a_m + n(mT)$$

Nyquist criterion in the frequency domain

- The frequency response of the cascade of the channel, the transmit and the receive filter is $G(f)$, the Fourier transform of $g(t)$.
- Since sampling in time determines *periodicity* in the frequency domain, $\mathcal{F}\{g(\ell T)\}$, the Fourier transform of $g(\ell T)$, $g(t)$ sampled every T seconds, is

$$\mathcal{F}\{g(\ell T)\} = \sum_{\ell} g(\ell T) e^{-j2\pi f \ell T} = \frac{1}{T} \sum_{k} G\left(f - \frac{k}{T}\right)$$

Nyquist criterion in the frequency domain

- On the other hand, if the sampled response $g(\ell T)$ satisfies the Nyquist criterion, then it is a Kronecker delta, i.e. $g(\ell T) = \delta(\ell)$.
- The Fourier transform of $\delta(\ell)$ is $\mathcal{F}\{\delta(\ell)\} = 1$.
- Accordingly, it is

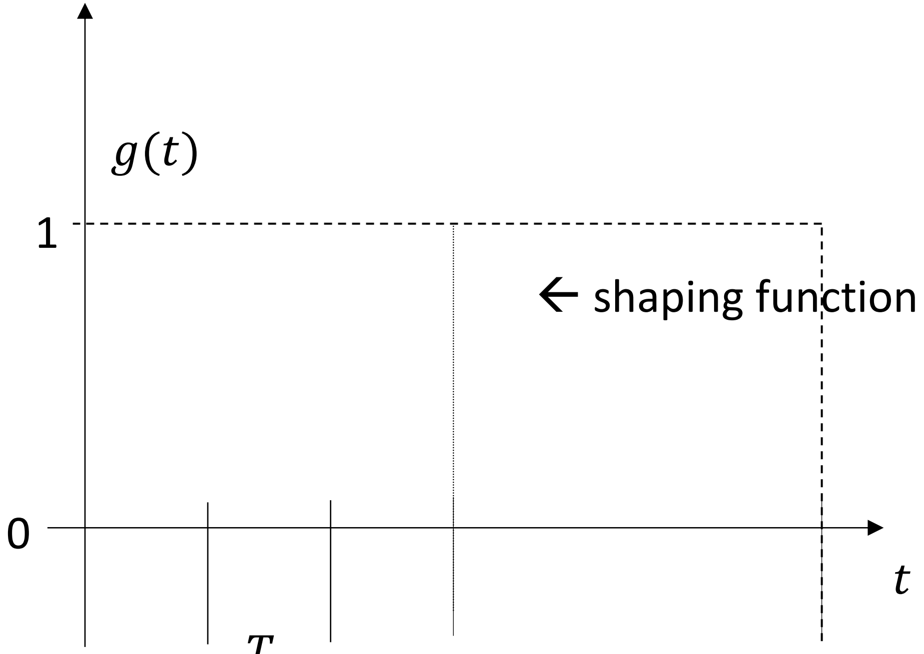
$$\mathcal{F}\{g(\ell T)\} = \frac{1}{T}\sum\nolimits_{k} G\left(f - \frac{k}{T}\right) = \mathcal{F}\{\delta(\ell)\} = 1$$

- From which we can extrapolate the Nyquist criterion for zero ISI in the frequency domain

$$\sum\nolimits_{k} G\left(f - \frac{k}{T}\right) = T$$

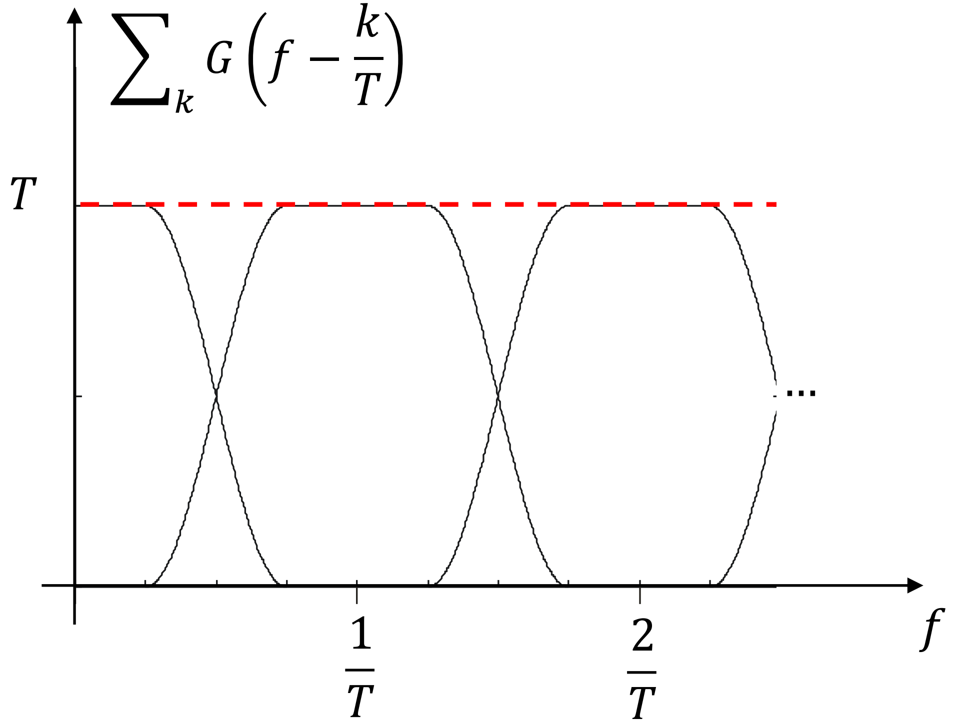
Nyquist criterion

Time domain



Equally spaced zeros

Frequency domain



Raised cosine filters

Raised cosine filters satisfy the Nyquist criterion: the occupied bandwidth is

$$B_{RC} = \frac{1+\alpha}{T}$$

The roll-off factor α is a design parameter, RC with $\alpha = 0$ is a rect and it is the *minimum bandwidth filter*.