6th lesson

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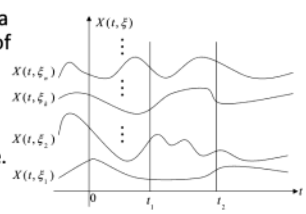
Stochastic processes

- *Deterministic process.* A deterministic process is represented by an explicit mathematical relation.
- *Stochastic process*. A stochastic process is the result of a large number of separate causes, described in probabilistic terms and by properties which are averages.

Stochastic processes

- Let ξ denote the random outcome of an experiment. To every such outcome suppose a waveform $X(t,\xi)$ is assigned. The collection of such waveforms form a *stochastic process*.
- For a fixed ξ (the set of all experimental outcomes), $X(t,\xi)$ is a specific time function.
- For fixed $t = t_0$, $X(t_0,\xi)$ is a random variable.
- The ensemble of all such realizations over time represents the stochastic process $X(t)$.



Categories of stochastic processes

- *Parameter space*: set T of indices $t \in T$.
- *State space*: set S of values $X(t) \in S$.
- Categories:
  - Based on the parameter space:
    - Discrete-time processes: parameter space discrete,
    - Continuous-time processes: parameter space continuous.
  - Based on the state space:
    - Discrete-state processes: state space discrete,
    - Continuous-state processes: state space continuous.

Distribution and probability density function

- If $X(t)$ is a stochastic process, then for fixed $t = t_0$, $X(t_0)$ represents a *random variable*.
- The *distribution function* is given by
$$F_X(x, t_0) = \Pr\{X(t_0) < x\}$$
$F_X(x, t_0)$ depends on the value of t. For different values of t, we obtain a different random variable.
- Further, the first-order *probability density function* of the process $X(t)$ is
$$f_X(x, t_0) = \frac{d}{dx} F_X(x, t_0)$$

Joint distributions

- For $t = t_1$ and $t = t_2$, $X(t)$ represents two different random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$, respectively. Their joint distribution is given by
$$F_X(x_1, x_2, t_1, t_2) = \Pr\{X(t_1) < x_1, X(t_2) < x_2\}$$
and
$$f_X(x_1, x_2, t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2, t_1, t_2)$$
represents the second-order density function of the process $X(t)$.
- Similarly, $f_X(x_1, \dots, x_n, t_1, \dots, t_n)$ represents the n-th order density function of the process $X(t)$.

Independence

- For an *independent* stochastic process, the random variables obtained by sampling the process at any n times $t_1, \dots, t_n$ are independent random variables for any n.
- Accordingly, the distribution is
$$F_X(x_1, \dots, x_n, t_1, \dots, t_n) = \Pr\{X(t_1) < x_1\} \cdots \Pr\{X(t_n) < x_n\}$$
$$= F_X(x_1, t_1) \cdots F_X(x_n, t_n)$$
and the probability density function is
$$f_X(x_1, \dots, x_n, t_1, \dots, t_n) = f_X(x_1, t_1) \cdots f_X(x_n, t_n)$$

Mean and autocorrelation

- **Mean** of a stochastic process:
$$\mu(t_0) = E\{X(t_0)\} = \int_{-\infty}^{+\infty} x f_X(x, t_0) dx$$
is the mean value of the process $X(t)$ at time t_0. In general, the mean of a process depends on the time index t.
- **Autocorrelation function** of a process:
$$R_{XX}(t_1, t_2) = E\{X(t_1)X^*(t_2)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1 x_2^* f_X(x_1, x_2, t_1, t_2) dx_1 dx_2$$
and it represents the interrelationship between the random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ obtained by sampling the process $X(t)$ at times t_1 and t_2.

Stationarity

- A stationary process exhibits statistical properties that are invariant to shift in the time index.
- *First-order stationarity* implies that the statistical properties of $X(t_0)$ and $X(t_0 + c)$ are the same for any *c*.

$$f_X(x, t_0) = f_X(x)$$

  - The *mean* is a constant and does not depend on *t*
- *Second-order stationarity* implies that the statistical properties of the pairs $\{X(t_1), X(t_2)\}$ and $\{X(t_1 + c), X(t_2 + c)\}$ are the same for any c.

$$f_X(x_1, x_2, t_1, t_2) = f_X(x_1, x_2, t_2 - t_1)$$

  - The autocorrelation depends only on the difference of the time indices.

Wide sense stationarity

- The basic conditions for the first and second order stationarity are usually difficult to verify.
- In that case, we can use a looser definition of stationarity. A process *X*(*t*) is said to be *wide-sense stationary* (WSS) if the two following conditions hold:
  1) $E\{X(t)\} = \mu$
  2) $E\{X(t_1), X(t_2)\} = R_{XX}(t_2 - t_1)$
- For a wide-sense stationary process, the mean is a constant and the autocorrelation function depends only on the difference between the time indices.

Power spectral density

- Wiener-Kintchine theorem. For stationary processes, the *power spectral density* (PSD) describes how the power of the signal is distributed over frequency

$$S_{XX}(f) = \mathcal{F}\{R_{XX}(\tau)\} = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{j2\pi f\tau} d\tau$$

- The signal power of $X(t)$ can be computed as

$$P_X = \int_{-\infty}^{+\infty} S_{XX}(f) df$$

PAM: power spectral density

- A PAM signal is modelled as a *stochastic process* because the symbols a_i are samples of a discrete-time random process.
- The bandwidth occupied by a stochastic process is measured by its *power spectral density* (Fourier transform of its autocorrelation function).
- The PSD of the PAM signal $\tilde{s}(t)$ is

$$S_{\tilde{s}}(f) = \frac{1}{T} S_a(f) |G_T(f)|^2$$

  where $S_a(f)$ is the PSD of a_i and $G_T(f)$ is the frequency response of the transmit filter $g_T(t)$.

  From now on, we omit the tilde for ease of notation.

PAM: receiver architecture

- PAM system block diagram

  - The propagation channel is in general modelled as a LTI filter with impuls response $h(t)$. When the channel is ideal, it is $h(t) = \delta(t)$.
  - The noise term is a white, zero-mean, Gaussian stationary process with PSD $S_w(f) = N_0/2$ ($S_w(f) = 2N_0$ for its complex envelope).
  - The receiver's task is to reconstruct the sequence of transmitted bits from the received signal $r(t)$.

Linear Time Invariant
↳ completely described by its impulse response

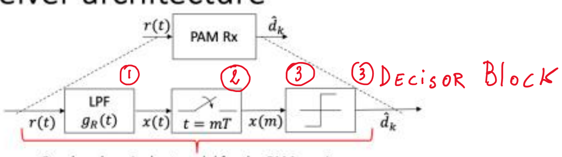
$S_w(f) = \frac{N_0}{2}$

$\tilde{w}(t) = w_I(t) + j w(t)$

$S_{\tilde{w}}(f) = \underline{\underline{2N_0}}$

$S_{w_I}(f) = S_{w_Q}(f) = N_0$

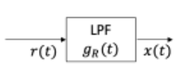
PAM: receiver architecture



Baseband equivalent model for the PAM receiver

- The PAM receiver performs the inverse operation of the transmitter: extract the transmitted bits from the analog received signal $r(t)$.
  1. Filters the interference and spurious components from the received signal;
  2. Samples the filtered signal once per symbol time T;
  3. Recovers the transmitted bits from the signal samples.

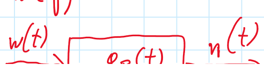
PAM: Receive filter



- The received baseband equivalent signal has the form
$$r(t) = s(t) \otimes h(t) + w(t)$$
- The filter output is
$$x(t) = r(t) \otimes g_R(t) = \sum_i a_i g(t - iT) + n(t)$$

where $g(t) = g_T(t) \otimes h(t) \otimes g_R(t)$ is the convolution of the impulse response of the channel, the transmit and the receiver filter, $n(t)$ is the filtered (and colored!) noise.

$S_w(f) = 2N_0$ ← White



$n(t)$ not white anymore

$S_w(f) = 2N_0 \leftarrow$ White

$w(t) \longrightarrow \boxed{g_R(t)} \xrightarrow{n(t)}$ $n(t)$ not white anymore

$S_n(f) = S_w(f)\,|G_R(f)|^2 = 2N_0 |G_R(f)|^2$ PSD depends on the freq. response $|G_R(f)|^2$

$$\underbrace{\longrightarrow \boxed{g_T(t)} \longrightarrow \boxed{h(t)} \longrightarrow \boxed{g_R(t)}}_{h(t)} \longrightarrow$$

PAM receive filter

$$\xrightarrow{r(t)} \boxed{\text{LPF } g_R(t)} \xrightarrow{x(t)} \underset{t=mT}{\times} \longrightarrow x(m)$$

- One of the tasks of the receive filter $g_R(t)$ is to remove the intersymbol interference affecting the received samples.
- The samples of the received signal take this form:

$$x(m) = x(t)\big|_{t=mT} = \sum_i a_i g(mT - iT) + n(mT)$$

$$= a_m\, g(0) + \sum_{\ell, \ell \neq 0} \underline{a_{m-\ell}}\, g(\ell T) + n(mT)$$

ISI $= \emptyset$

- Neglecting the noise term, the condition on $g(\ell T) = g(t)|_{t=\ell T}$ to have zero ISI is

$$g(\ell T) = \begin{cases} 1 & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$

- Under these conditions (Nyquist criterion), the received sample $x(m)$ is

$$x(m) = a_m + n(mT)$$

if $h(t) = \delta(t) \Rightarrow g(t) = g_T(t) \otimes \delta(t) \otimes g_R(t) = g_T(t) \otimes g_R(t)$

$x(t) = \sum_i a_i\, g(t - iT)$

$\underline{x(m)} = x(t)\big|_{t=mT} = \sum_i a_i\, g(mT - iT) = \sum_i a_i\, g((\underline{m-i})T)$

$\boxed{K = m - i} \Rightarrow i = m - K$

$= \sum_K a_{m-K}\, g(KT) = a_m\, g(0) + \sum_{K \neq 0} a_{m-K}\, g(KT)$

Nyquist criterion in the frequency domain

- The frequency response of the cascade of the channel, the transmit and the receive filter is $G(f)$, the Fourier transform of $g(t)$.
- Since sampling in time determines *periodicity* in the frequency domain, $\mathcal{F}\{g(\ell T)\}$, the Fourier transform of $g(\ell T)$, $g(t)$ sampled every T seconds, is

$$\mathcal{F}\{g(\ell T)\} = \sum_\ell g(\ell T) e^{-j2\pi f \ell T} = \frac{1}{T} \sum_k G\left(f - \frac{k}{T}\right)$$

$x(m) = a(m)\,g(0) + \underbrace{\sum_K a_{m-K}\, g(KT)}_{ISI} + n(mT)$

$g(0) = 1$; $g(KT) = 0$, when $K \neq 0$

$\hookrightarrow x(m) = a(m) + n(mT)$

$g(t) \rightleftharpoons G(f)$

$g(mT) \rightleftharpoons \frac{1}{T} \sum_\ell G\left(f - \frac{\ell}{T}\right)$

$g(\ell T) = \begin{cases} 1 & \ell = \emptyset \\ 0 & \ell \neq 0 \end{cases} \Rightarrow g(\ell T) = \delta(\ell)$

Nyquist criterion in the frequency domain

$$g(\ell T) = \begin{cases} 1 & \ell = 0 \\ 0 & \ell \neq 0 \end{cases} \Rightarrow g(\ell T) = \delta(\ell)$$

Nyquist criterion in the frequency domain

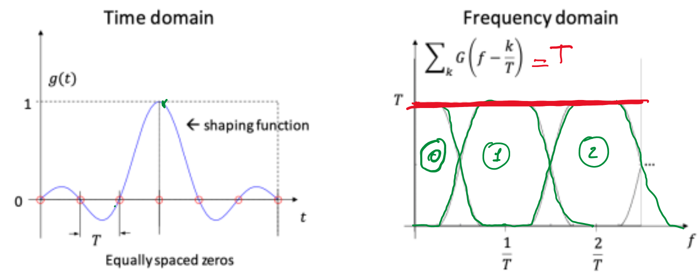
- On the other hand, if the sampled response $g(\ell T)$ satisfies the Nyquist criterion, then it is a Kronecker delta, i.e. $g(\ell T) = \delta(\ell)$.
- The Fourier transform of $\delta(\ell)$ is $\mathcal{F}\{\delta(\ell)\} = 1$.
- Accordingly, it is

$$\mathcal{F}\{g(\ell T)\} = \frac{1}{T}\sum_k G\left(f - \frac{k}{T}\right) = \mathcal{F}\{\delta(\ell)\} = 1$$

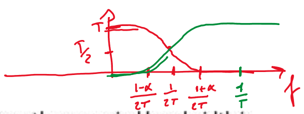
- From which we can extrapolate the Nyquist criterion for zero ISI in the frequency domain

$$\sum_k G\left(f - \frac{k}{T}\right) = T$$

Nyquist criterion



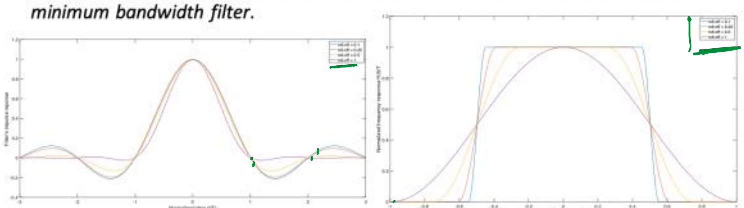
Raised cosine filters



Raised cosine filters satisfy the Nyquist criterion: the occupied bandwidth is

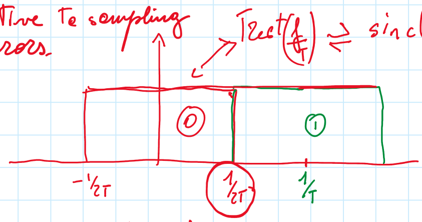
$$B_{RC} = \frac{1+\alpha}{T}$$

The roll-off factor α is a design parameter, RC with $\alpha = 0$ is a rect and it is the *minimum bandwidth filter*.



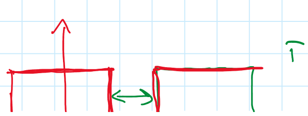
Minimum bandwidth filter not practical
- Can not be truncated
- Very sensitive to sampling time errors.

$\text{Rect}\left(\frac{f}{T}\right) \rightleftharpoons \text{sinc}(tT)$

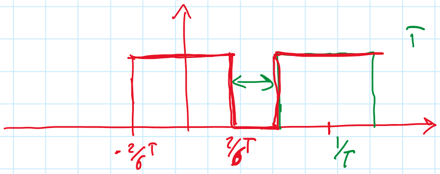


Satisfy Nyquist criterion

Minimum bandwidth $\Rightarrow \alpha = 0$ $\quad B = 2 \cdot \frac{1}{2T} = \frac{1}{T}$



Does NOT satisfy Nyquist criterion



Does NOT satisfy
Nyquist criterion