Algorithms and their Applications CS2004 (2020-2021)

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3.1 Mathematical Foundation

NOTICES

Laboratory sessions

- ❑ Last week laboratory went well – I think!
- ❑ More helpers in the lab ☺
- ❑ Online + in-person session today
- ❑ Booking system is working!
- ❑ Feel free to use discussion boards on Teams

CodeRunner

- ❑ CodeRunner is used for Task #1 and Task #2
- ❑ CodeRunner worksheet is released!
- ❑ CodeRunner mock test will be released today
- ❑ CodeRunner can only be accessed using VPN from home

Previously on CS2004…

- We looked at how to compare a number of algorithms
- The core topic of counting **Primitive Operations**
- We discussed why experimental studies could not always be used as a comparison
- We looked at Pseudo-Code

Mathematics in this module. Why?

- Mathematics is a descriptive language
- It can be used to precisely describe how numerical items relate to each other
- It can be used to model the real world
- Carl Gauss referred to it as "the Queen of the Sciences"
- Why do we need mathematics in this module and the other modules?

What We are Going to Cover

- How to interpret simple equations and implement them in Java
- The basics:
  - Variables
  - Sets
  - Equations
  - Functions
  - Subscripts
  - Summation
  - Products
  - And a few other topics...

What We are NOT Going to Cover

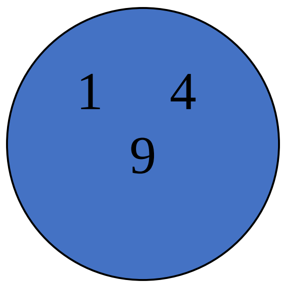
- Differentiation
- Integration
- Solving Equations
- Writing mathematical proofs
- Statistics
  - But we will use simple summary statistics in this module…

Variables

- A variable is a symbol used to represent a mathematical construct
  - E.g. numbers, sets, lists , vectors, matrices, …
  - Often lower case letters or Greek letters are used
  - E.g. $x, y, z, \Omega, \alpha, \beta, \ldots$
- Variables in Mathematics are treated the same as variables within a programming language
- They can be thought of as a box containing a value that can be read from or written to

Sets – Bassics 1

- ❑ A set is a collection of objects called elements



$= \{1,4,9\} = A$

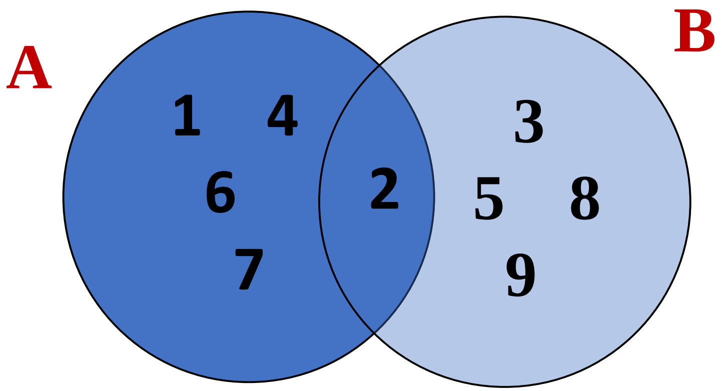
- ❑ Sets can be finite or infinite
- ❑ A set has **no order**
- ❑ A set only contains one copy of an item

Sets – Basics 2

- Some well known sets:
  - $\boldsymbol{R}$: Real numbers
  - $\boldsymbol{Z}$: Integers
  - $\boldsymbol{N}$: Natural numbers (integers ≥ 0)
  - The alphabet
- $a \in A$: item *a* is a member of set A
  - $\notin$: not a member
- $|A|$ is the *cardinality* of A, i.e. How many items in A
- The empty set: $\phi = \{\}$

Set Operators – Part 1

- A = {1,2,4,6,7}
- B = {2,3,5,8,9}

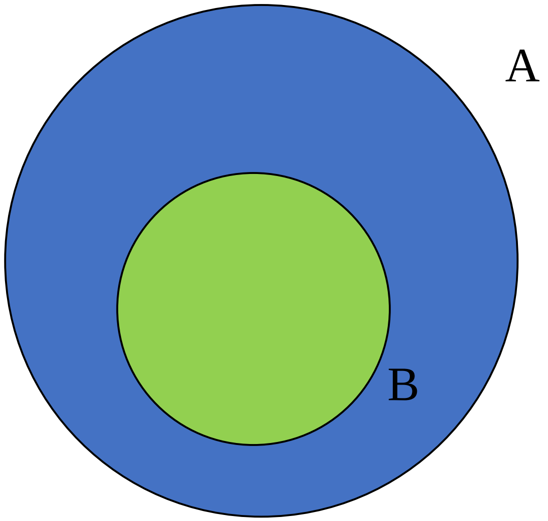


- Intersection:
  - $A \cap B = \{2\}$
- Union:
  - $A \cup B = \{1,2,3,4,5,6,7,8,9\}$

- More examples:
  - $A = B \cap C$, $A = \{a : a \in B \text{ AND } a \in C\}$
    - A contains only what B and C have in common
  - $A = B \cup C$, $A = \{a : a \in B \text{ OR } a \in C\}$
    - A contains all of B and C

Set Operators – Part 2

- Subset: If A is a set, then $B \subseteq A$ (B is a subset of A) if every element of B is also in A
  - For example, $\boldsymbol{N} \subseteq \boldsymbol{Z}$
- Superset: In the example above [and below] A is a superset of B



Equations – Part 1

- An equation uses mathematical operators to relate one set of variables or numbers to another set of variables or numbers

- For Example:
  - $2+2 = 4$
  - $y = mx + c$
  - $ax^2+bx+c=0$
  - $(x-a)^2+(y-b)^2=r^2$

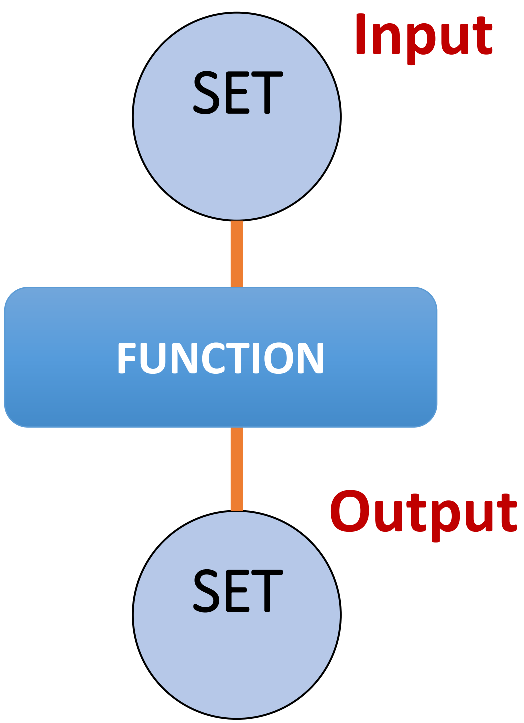
Equations – Part 2

- Often we have to simplify an equation
- This often means adding up similar terms and ordering the powers
- E.g.:
  - $(n-1)(n-2)(n-3)$
  - $=(n^2-2n-n+2)(n-3)$
  - $=(n^2-3n+2)(n-3)$
  - $=n^3-3n^2-3n^2+9n+2n-6$
  - $=n^3-6n^2+11n-6$

- In a previous years exam ~15% of the students did the following:
  - $\boldsymbol{n \times n \times n = 3n}$**!!!**

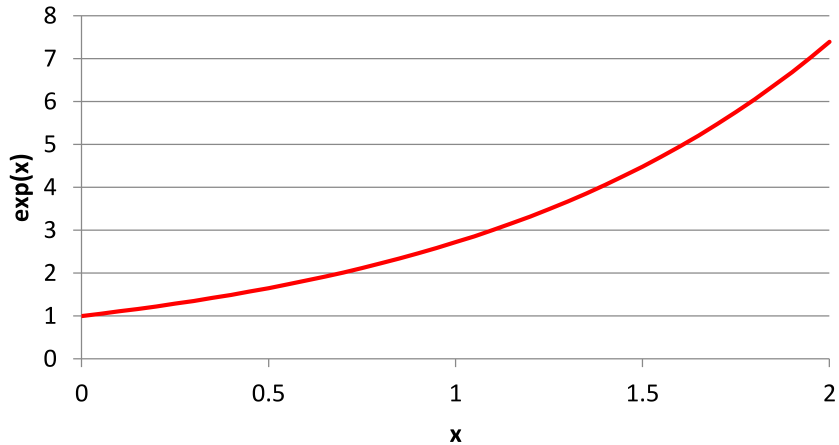
Functions – Part 1

- A function is a relation that uniquely associates members of one set with members of another set
  - E.g.
    - $y = x+1 \rightarrow f(x) = x+1$
- Functions can take parameters,
  - E.g. $f(x,y) = 2x+y$



Functions – Part 2

- There is a special function that we will be using in some of our work
- This is the exponential function
- $f(x) = \exp(x) = e^x$
  - where e = 2.718…



Subscripts

- A subscript is a natural number that indexes a list of variables.
- For example:
  - Let X be the list (or vector) $[x_1,\ldots,x_n]$, then to access any element we use the notation x_i, where $1 \leq i \leq n$
  - We use the notation $|X|$ to refer to the number of elements in the list X, which is n in this case

$$X = [5,2,-8,3.6,88,2000.003]$$

- Then $x_1=5$, $x_2 = 2$, $x_3 = -8$ etc...
- $|X|=6$

Summation

- Let X be the list $[x_1,\ldots,x_n]$
- To sum all the elements, we would use this notation:

$$s = \sum_{i=1}^{n} x_i$$

Note that we are arbitrarily assigning the result to s.

$$s = x_1 + x_2 + x_3 + \ldots x_n$$

- If we wanted to sum the squares:

$$s = \sum_{i=1}^{n} x_i^2$$

$$s = x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2$$

Products

- Let X be the list $[x_1,\ldots,x_n]$, then if we want multiple together all of the elements, we would use the notation:

$$s = \prod_{i=1}^{n} x_i$$

Note that we are arbitrarily assigning the result to s.

$$s = x_1 x_2 x_3 \ldots x_n$$

- If we wanted to multiply the squares:

$$s = \prod_{i=1}^{n} x_i^2$$

$$s = x_1^2 x_2^2 x_3^2 \ldots x_n^2$$

Factorial

- The notation $n!$ is defined as multiply all of the integers between 1 and n together

$$n! = 1.2.3\ldots n = \prod_{i=1}^{n} i$$

- E.g. 6!
  - = 1.2.3.4.5.6 = 720

Note that, 0! = 1, undefined for $n < 0$

- $n!$ also gives the number of possible arrangements of n items

Permutations

- The number of ways that r ordered items can be picked (**arranged**) from n items
- Defined as:

$$\mathrm{P}_r^n = \frac{n!}{(n-r)!}$$

$$\mathrm{P}_2^4 = \frac{4!}{2!} = \frac{1.2.3.4}{2} = 12$$

Combinations

- The number of ways that r unordered items can be picked (**selected**) from n items
- Defined as:

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{1.2.3.4}{2.2} = 6$$

Permutations and Combinations

- If the order is important then we use permutations otherwise we use combinations
- For example if the key code to a lock is 745 then 574 would not open it!
  - Permutations, there are $P_3^{10} = 720$ ways of choosing a 3 digit key code from the 10 digits
- If we like a fruit salad containing banana, melon and grapes, then the order doesn’t matter!
  - Combinations, there are $C_3^{10} = 120$ ways of choosing three fruits from ten fruits

Logarithms – Part 1

- A logarithm is the power to which a number is raised to get some other number
  - **$\log_{10} 100 = 2$** *because* **$10^2 = 100$**
- There are logarithms using different base units
  - **$\log_2 8 = 3$** *because* **$2^3 = 8$**
- The most common logarithms are base 10 logarithms and natural logarithms
- A base 10 logarithmic equation is usually written in the form:
  - **$\log a = r$**

Logarithms – Part 2

- For example:
  - $\mathrm{Log}_{10}(1000) = 3$
  - $\mathrm{Log}_2(8) = 3$
- Note that:
  - Ln(x) [$\log_e(x)$] is used for log base e, $\ln(e{=}2.718\ldots) = 1$
  - If $y = \exp(x) = e^x$ then $\ln(y) = x$
  - This is known as the **natural logarithm**
- If $y = a^x b^z$ then
  - $\ln(y) = \ln(a^x)+\ln(b^z) = x\ln(a)+z\ln(b)$

- $\mathrm{Log}_b(x) = y$ means that $x = b^y$
- $\mathrm{Log}_b(b) = 1$
  - $b^1 = b$ for all $b \neq 0$
- $\mathrm{Log}_b(1) = 0$
  - $b^0 = 1$ for all $b \neq 0$
- $\mathrm{Log}_b(xy) = \mathrm{Log}_b(x) + \mathrm{Log}_b(y)$
- $\mathrm{Log}_b(x/y) = \mathrm{Log}_b(x) - \mathrm{Log}_b(y)$
- $\mathrm{Log}_b(x^y) = y\mathrm{Log}_b(x)$
- $\mathrm{Log}_b(x) = \mathrm{Log}_a(x) / \mathrm{Log}_a(b)$
- $\mathrm{Log}_b(x)$ where $x \leq 0$ is undefined

Other Topics You May Need

- Probability
  - Generating 1 in n chances
  - Covered in level 1
- Summing series
  - Adding up 1+2+3+4+....+n
  - Covered in level 1
- Summary statistics
  - Mean, median, variance, etc...
  - Covered in level 1

Next Topic

- Lecture
  - We will next look in more detail at Time Complexity, Big(O), T(n), etc…
- Laboratory
  - Mathematical tests
  - CodeRunner worksheet and mock test