Eigenvalues and Eigenvectors

with graphical representations

Reference: "Statistics and Data Analysis in Geology, 2nd ed., J.C. Davis

1. Eigenvalue Problem

1) Definition

2) Solving method

Definition: Eigenvalue Problem

Let's consider the following matrix form,

The vector of unknowns (the x_i's)

some constant (λ)

$$[A]\cdot[X]=\lambda[X]$$

The matrix of coefficients (the a_{ij}'s)

the unknown vector itself

Our concern is to find values of λ that satisfy this relationship.

Above relation also can be written as…

$$([A]-\lambda[I])\cdot[X]=[0]$$

where [I] is an identity matrix.

$\longrightarrow$ Eigenvalue Problem!

Solving Method: Eigenvalue Problems (2×2 case) - 1

$$\lambda[I] = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$([A] - \lambda[I]) \cdot [X] = [0]$$

$$(a_{11} - \lambda)x_1 + a_{12}x_2 = 0$$

$$a_{21}x_1 + (a_{22} - \lambda)x_2 = 0$$

Let's assume that there are solutions to these equations other than the trivial case where all the unknown x'_s=0.

Solving Method: Eigenvalue Problems (2×2 case) - 2

$$\underline{([A]-\lambda[I])}\cdot\underset{\neq 0}{[X]}=[0]$$

the determinant must be zero.

$$|A-\lambda I|=0=\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

$$(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}a_{21}=0$$

$$(a_{11}a_{22})-(a_{21}a_{12})-(a_{11}\lambda)-(a_{22}\lambda)+\lambda^2=0$$

$$\lambda^2+\alpha_1\lambda+\alpha_2=0$$

For a quadratic equation, $ax^2+bx+c=0$ **, the general solution is** $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

2. Numerical Examples for Eigenvalue Problem

Numerical Examples of Eigenvalue Problems (2×2 case)

Numerical Example 1

$$[A] = \begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix}$$

Let's get the eigenvalues of matrix [A]

$$\begin{vmatrix} 3-\lambda & 6 \\ 6 & 3-\lambda \end{vmatrix} = 0$$

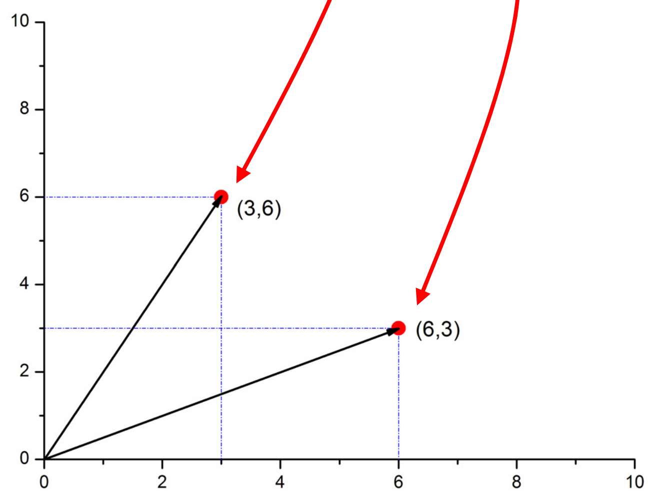
$$(3-\lambda)^2 - 6\times 6 = 0$$

$$9 - 6\lambda + \lambda^2 - 36 = 0$$

$$\lambda^2 - 6\lambda - 27 = 0$$

$$(\lambda - 9)(\lambda + 3) = 0$$

$$\boxed{\lambda_1 = 9,\ \lambda_2 = -3}$$



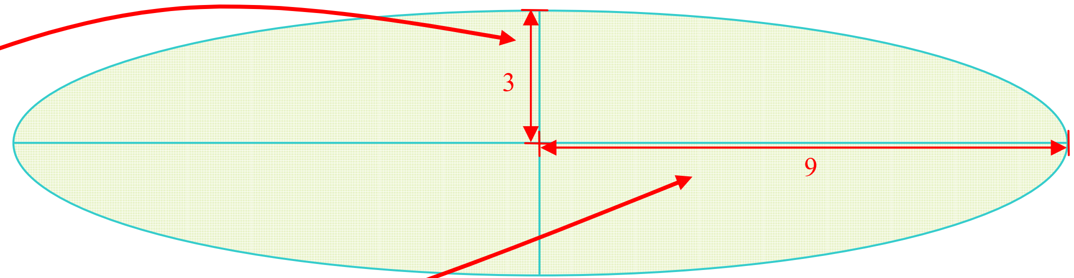
Two eigenvalues from
2 variables

The eigenvalues represent the magnitudes, or lengths, of the major and minor axes of an ellipse.
See Next Page!

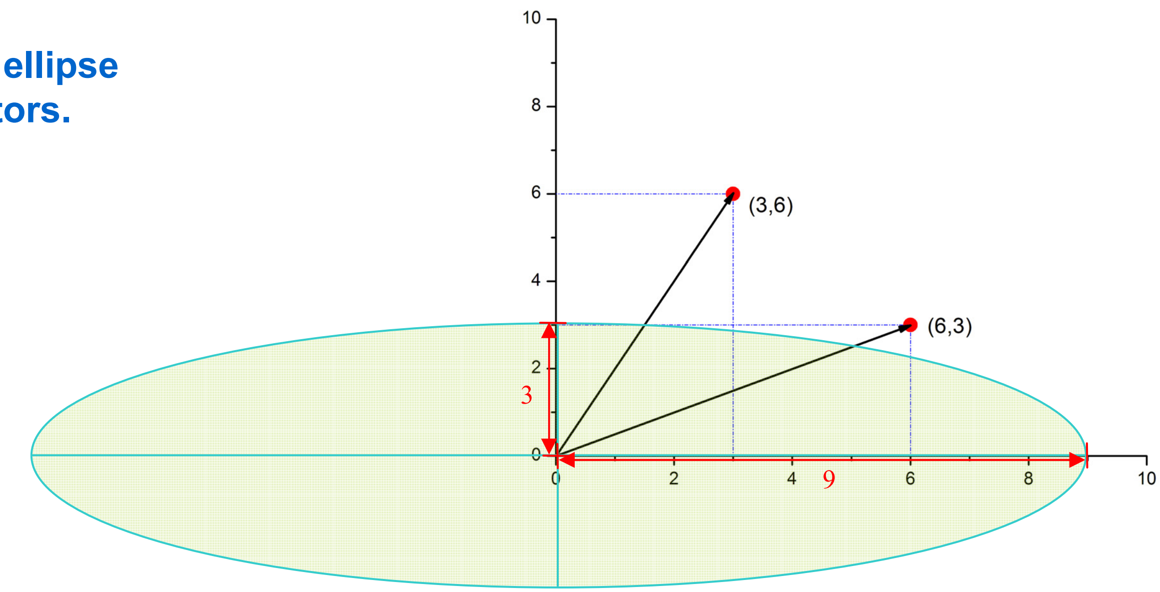
Numerical Example 1: Continued

The eigenvalues represent the magnitudes, or lengths, of the major and minor axes of an ellipse.

$$\boxed{\lambda_1 = 9, \lambda_2 = -3}$$

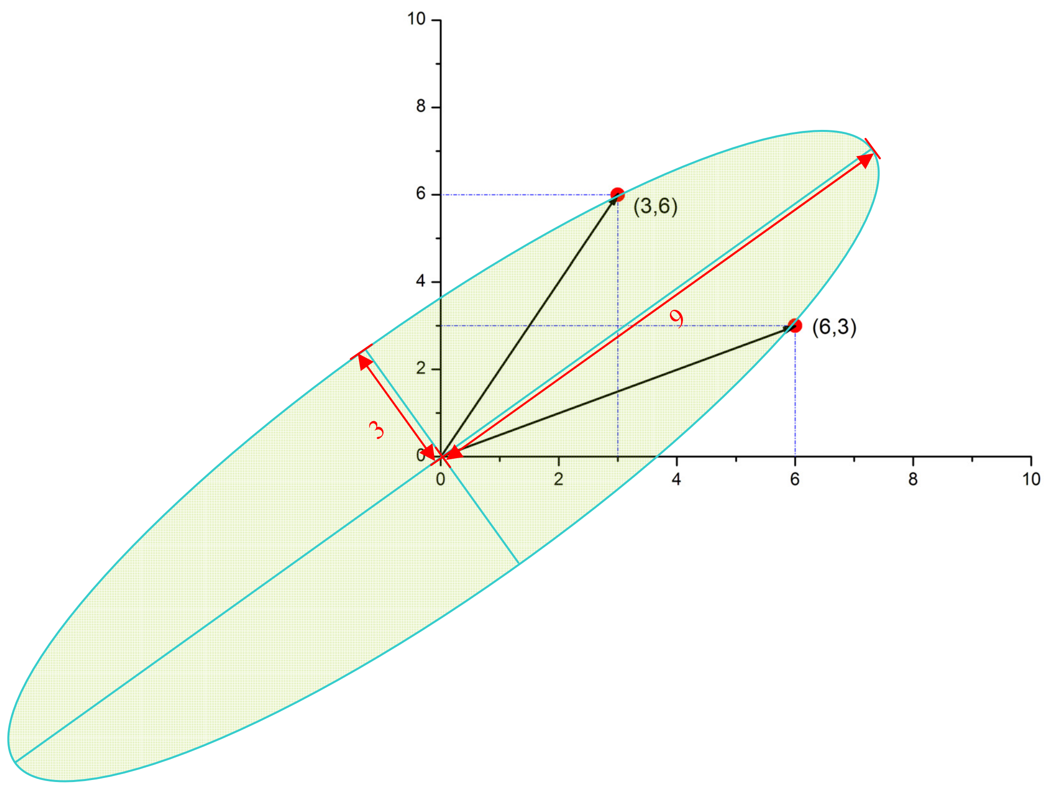


Let's overlap the center of ellipse onto center of the two vectors.



Numerical Example 1: Continued

Then rotate the ellipse to its envelope can be on the two points.



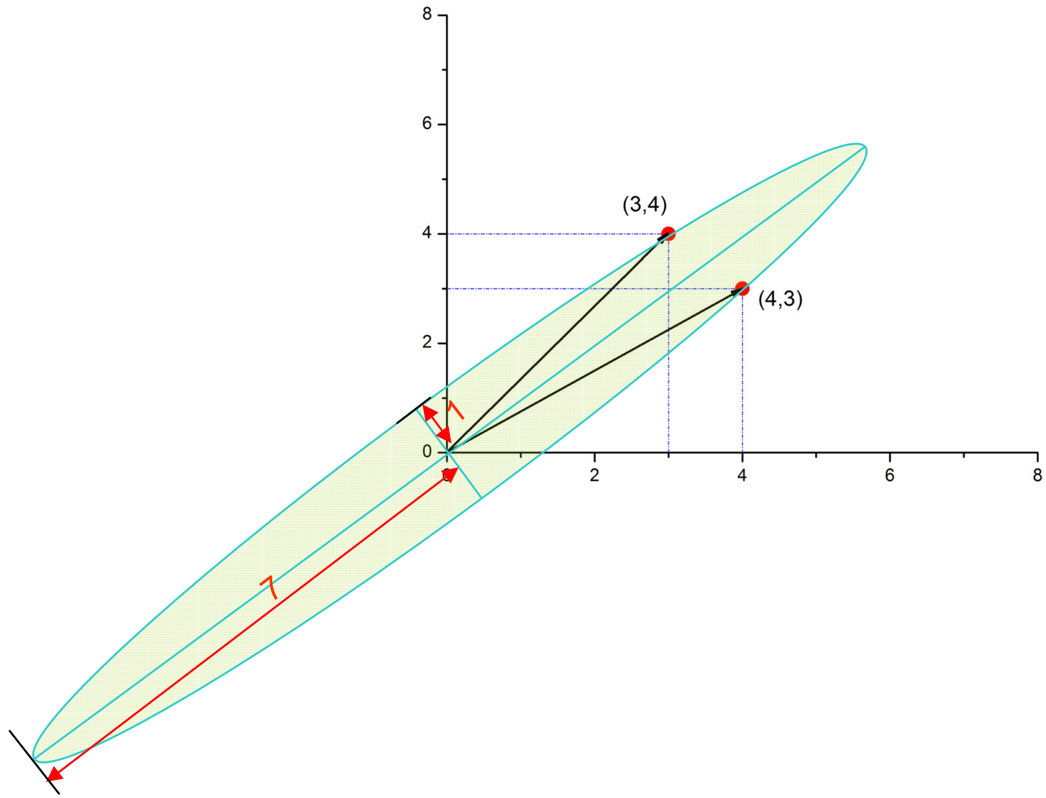
Numerical Example 2.

If the two points are closer than the case of numerical example 1,

$$[A] = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

Then the eigenvalues are

$$\lambda_1 = 7, \lambda_2 = -1$$

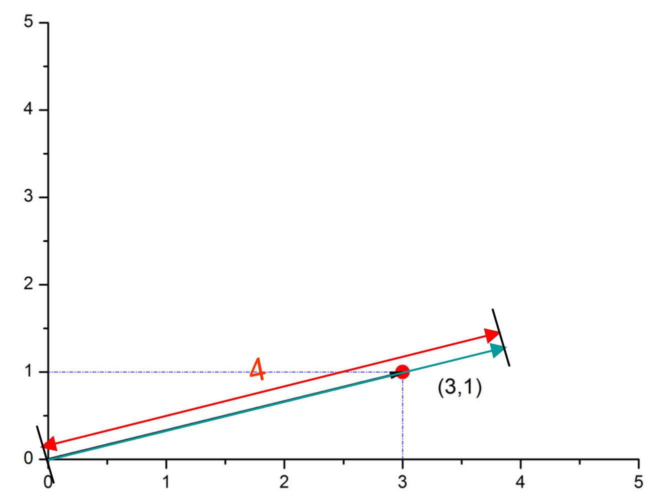


The sum of the eigenvalues($=\lambda_1 + \lambda_2 = 7-1=6$) is always equal to the sum of the diagonal elements(3+3), i.e. trace of the original matrix.

Numerical Example 3.

If the two points are identical,

$$[A]=\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$$



Then the eigenvalues are

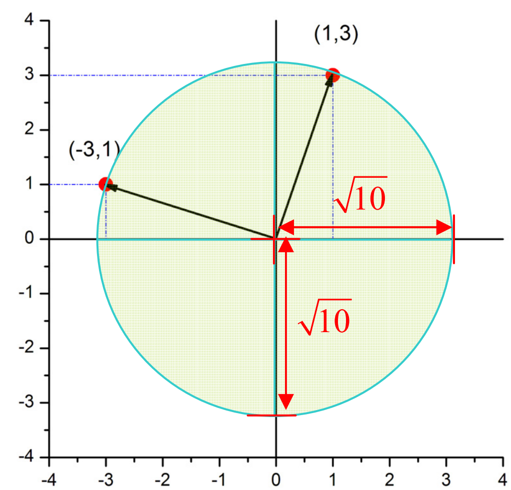
$$\lambda_1 = 4, \lambda_2 = 0$$

The sum of the eigenvalues is always equal to the sum of the diagonal elements, i.e. trace of the original matrix.

Numerical Example 4.

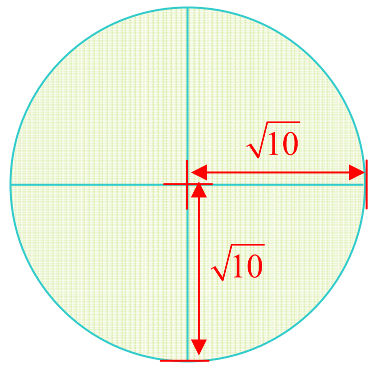
If the two points are perpendicular to one another,

$$[A] = \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$$



Then the eigenvalues are

$$\lambda_1 = \sqrt{10}, \lambda_2 = -\sqrt{10}$$



The sum of the eigenvalues is always equal to the sum of the diagonal elements, i.e. trace of the original matrix.

3. Eigenvector Problem

Eigenvector Problems with Numerical Example 1

We already calculated eigenvalues in example 1

$$[A] = \begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix}$$ with $$\lambda_1 = 9,\ \lambda_2 = -3$$

For $\lambda_1 = 9$

$$\left.\begin{aligned}
&\begin{bmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\begin{bmatrix} 3-9 & 6 \\ 6 & 3-9 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix} \\
&\begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}\right\} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvector

Eigenvector of the largest eigenvalue represents a slope of the major axis of ellipse

Eigenvector Problems with Numerical Example 1 (continued)

For $\lambda_2 = -3$

$$\begin{bmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-(-3) & 6 \\ 6 & 3-(-3) \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

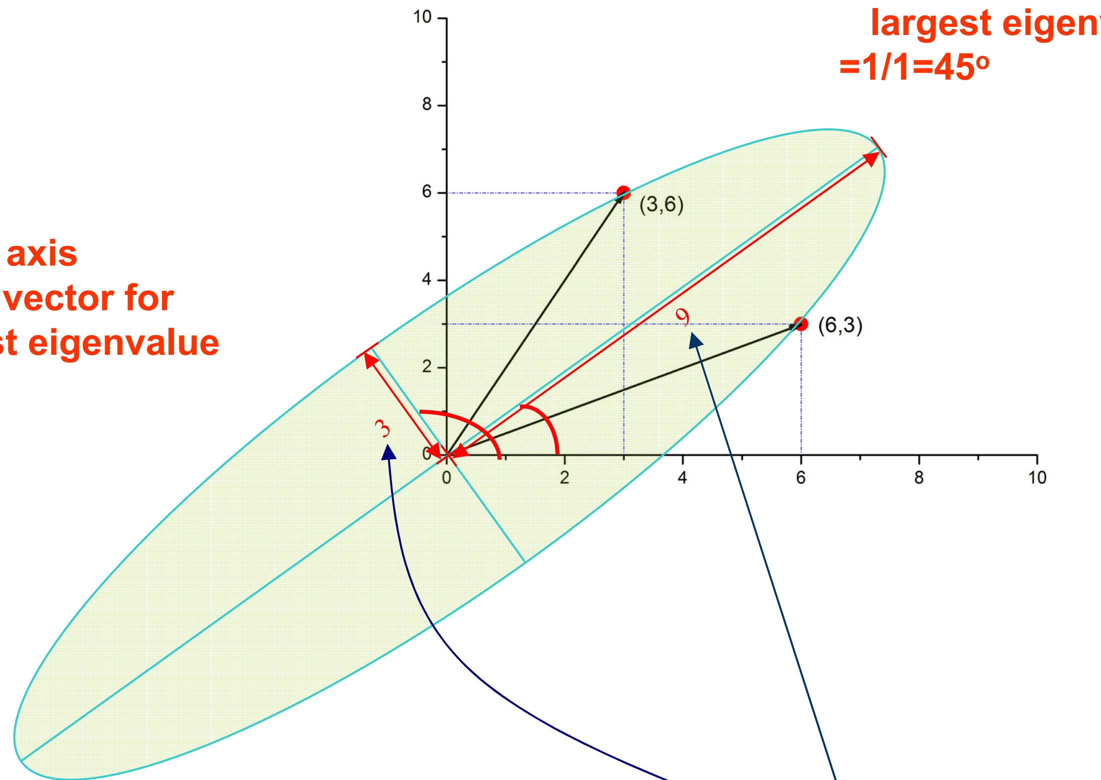
Eigenvector

The eigenvector of second largest eigenvalue represents a slope of the minor axis of ellipse

Eigenvector Problems with Numerical Example 1 (continued)

Slope of major axis
=Ratio of eigenvector for the largest eigenvalue
$=1/1=45^{o}$

Slope of minor axis
=Ratio of eigenvector for second largest eigenvalue
$=1/(-1)=135^{o}$



Eigenvalues

For symmetric matrices, their eigenvectors always are at right angles to each other: ORTHOGONAL!!!