Prescriptive Analytics: Optimisation

0.1 Optimisation in R

Following[1] customer demands, FitBite, a chain of Dutch walk-up fast food restaurants of the automatiek type, is preparing a new Hamburger that is lower on calories, fat and sodium by mixing beef with chicken. This new Hamburger should weight at least 125 grams and have at most 350 calories, 15 grams of fat, and 360 milligrams of sodium. Each gram of beef used has 2.5 calories, 0.2 gram of fat, and 3.5 milligrams of sodium. Each gram of chicken has 1.8 calories, 0.1 gram of fat, and 2.5 milligrams of sodium. What is the mix that maximises beef content while meeting all requirements?

1. Formulate a mathematical programming model in the standard form (i.e., with inequality and equality constraints as necessary). In particular, specify the decision variables, the objective function, and the constraints.

2. Verify that this is a linear program, and derive the vector $\vec{C}$ of objective function coefficients, and for the constraints the coefficient matrix A on the left-hand-side as well as the $\vec{B}$-vector on the right-hand-side.

$$\max_{x_1 \in R} 1 \cdot x_1 + 0 \cdot x_2$$

subject to

$$\begin{aligned}
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_1 + x_2 & \geq 125 \\
2.5x_1 + 1.8x_2 & \leq 350 \\
0.2x_1 + 0.1x_2 & \leq 15 \\
3.5x_1 + 2.5x_2 & \leq 360
\end{aligned}$$

3. Solve this problem using the function lp() of the *lpSolve*-library in R. Check whether the optimisation succeeded. Report the optimal (according to the model) combination of beef and chicken. Which constraints are active (binding) at this optimum?

```
library("lpSolve")
f.dir<-"max"
f.obj<-c(1,0)
f.con<-matrix(c(1,1,2.5,1.8,0.2,0.1,3.5,2.5),ncol=2,byrow=TRUE)
f.condir<-c(">=","<=","<=","<=")
f.rhs<-c(125,350,15,360)
obj<-lp(f.dir,f.obj,f.con,f.condir,f.rhs)
print(obj)
if (obj$status==0){
  print(paste("Success. Optimum at ",toString(obj$solution),
    " with objective function value ",obj$objval))
}else{
  print("error: optimisation was not successful")
}
# Check which of the constraints are active at optimum:
print(cbind(f.con%*%obj$solution, f.condir, f.rhs))
# constraints 1 (weight) and 3 (fat)
```

The optimal mix contains 25 grams of beef and 100 grams of chicken. Constraints 1 (weight) and 3 (fat) are active.

4. What changes if the weight requirement is increased from 125 to 200 grams?

[1]Exercise inspired by [Winston, 1997, Exercise 2-4].

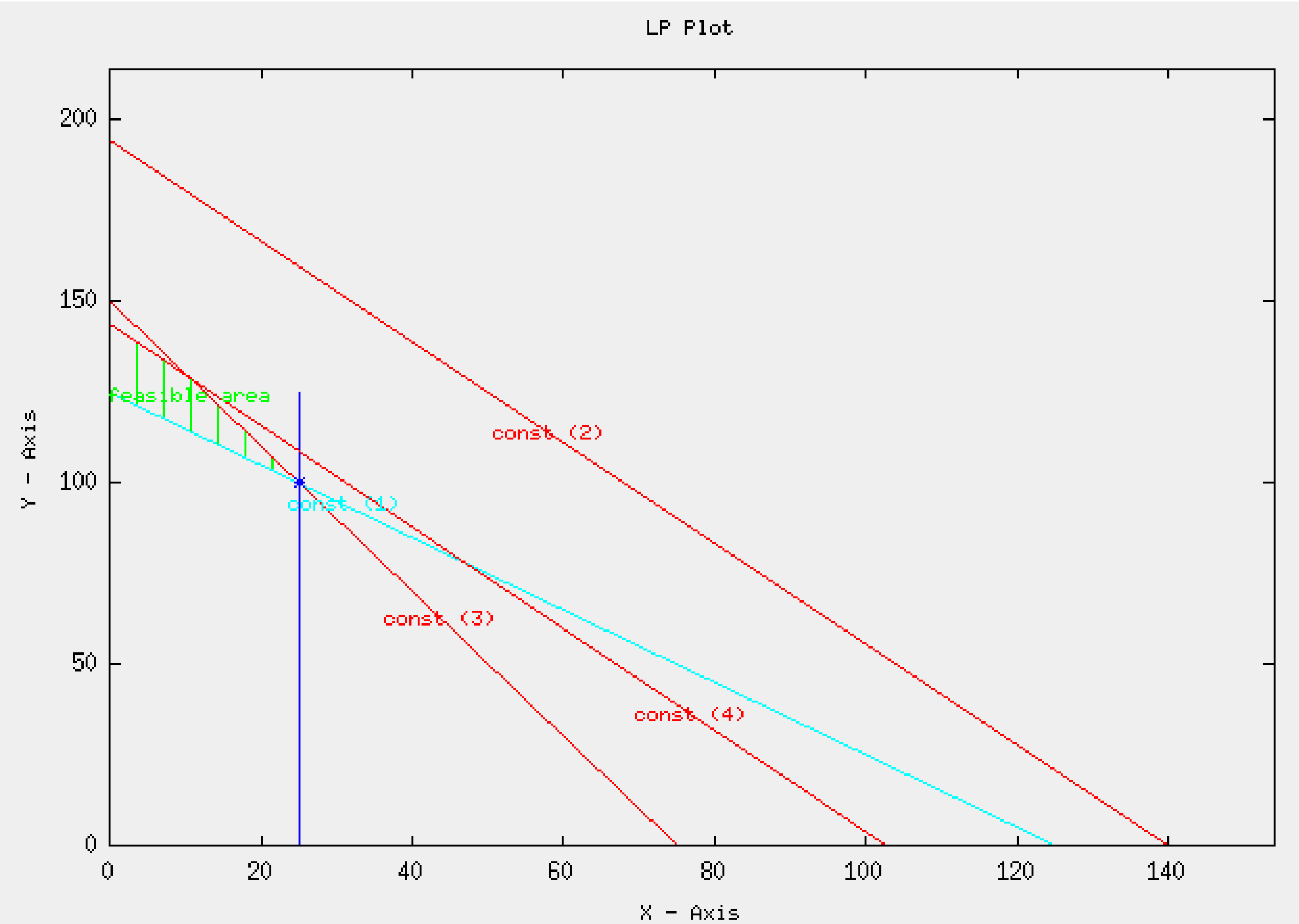

Figure 1: Plot of the FitBite LP

```
f.rhs<-c(200,350,15,360)
obj<-lp(f.dir,f.obj,f.con,f.condir,f.rhs)
print(obj)
```

The model becomes infeasible.

0.2 Optimisation in R

Dutch Farms operates 10 hectars of greenhouses in the Westland region of the Netherlands. This area can be used to plant either tomatoes or bell peppers (paprikas). Based on the previous' years market prices, Dutch Farms estimates a profit of 450,000 Euro per hectar of planted tomatoes, or 200,000 Euro per hektar of planted bell peppers. As precaution against pests, insects, and market price changes, Dutch Farms wants to diversify its production. Thus, it will not use more than 70% of its total greenhouse capacity on any single vegtable. Furthermore, being a sustainable food producer, Dutch Frams has dediced to limit its use of irrigation water to a maximum of 70 units water per season. However, tomatoes require 10 units of water per hectar, while bell peppers require only 7 units. Dutch Farms is consulting you to help them developing a planting plan that maximises their profit while also meeting their sustainability objective.

1. Formulate a mathematical programming model in the standard form (i.e., with inequality and equality constraints as necessary). In particular, specify the decision variables, the objective function, and the constraints.

2. Verify that this is a linear program, and derive the vector $\vec{C}$ of objective function coefficients, and for the constraints the coefficient matrix A on the left-hand-side as well as the $\vec{B}$-vector on the right-hand-side.

$$\max_{x_1,x_2\in R^2} 450\cdot x_1 + 200\cdot x_2$$

$$\begin{aligned} &\text{subject to} \\ x_1 & \quad \geq 0 \\ x_2 & \quad \geq 0 \\ x_1 + x_2 & \quad \leq 10 \\ 10x_1 + 7x_2 & \quad \leq 70 \\ x_1 & \quad \leq 7 \\ x_2 & \quad \leq 7 \end{aligned}$$

3. Solve this problem using the function lp() of the *lpSolve*-library in R. Check whether the optimisation succeeded, report the optimal values of the decision variables and the profit at this optimum.

```
library("lpSolve")
f.dir<-"max"
f.obj<-c(450,200)
f.con<-matrix(c(1,1,10,7,1,0,0,1),ncol=2,byrow=TRUE)
f.condir<-c("<=","<=","<=","<=")
f.rhs<-c(10,70,7,7)
obj<-lp(f.dir,f.obj,f.con,f.condir,f.rhs)
print(obj)
if (obj$status==0){
  print(paste("Success. Optimum at ",toString(obj$solution),
    " with objective function value ",obj$objval))
}else{
  print("error: optimisation was not successful")
}
# Check which of the constraints are active at optimum:
print(cbind(f.con%*%obj$solution, f.condir, f.rhs))
# constraints 2 (water) and 3 (diversity of tomato production)
```

The best production plan is to produce tomatoes on 7 hectars, and nothing on the remaining 3. This yields a profit of 3,150,000 Euros.

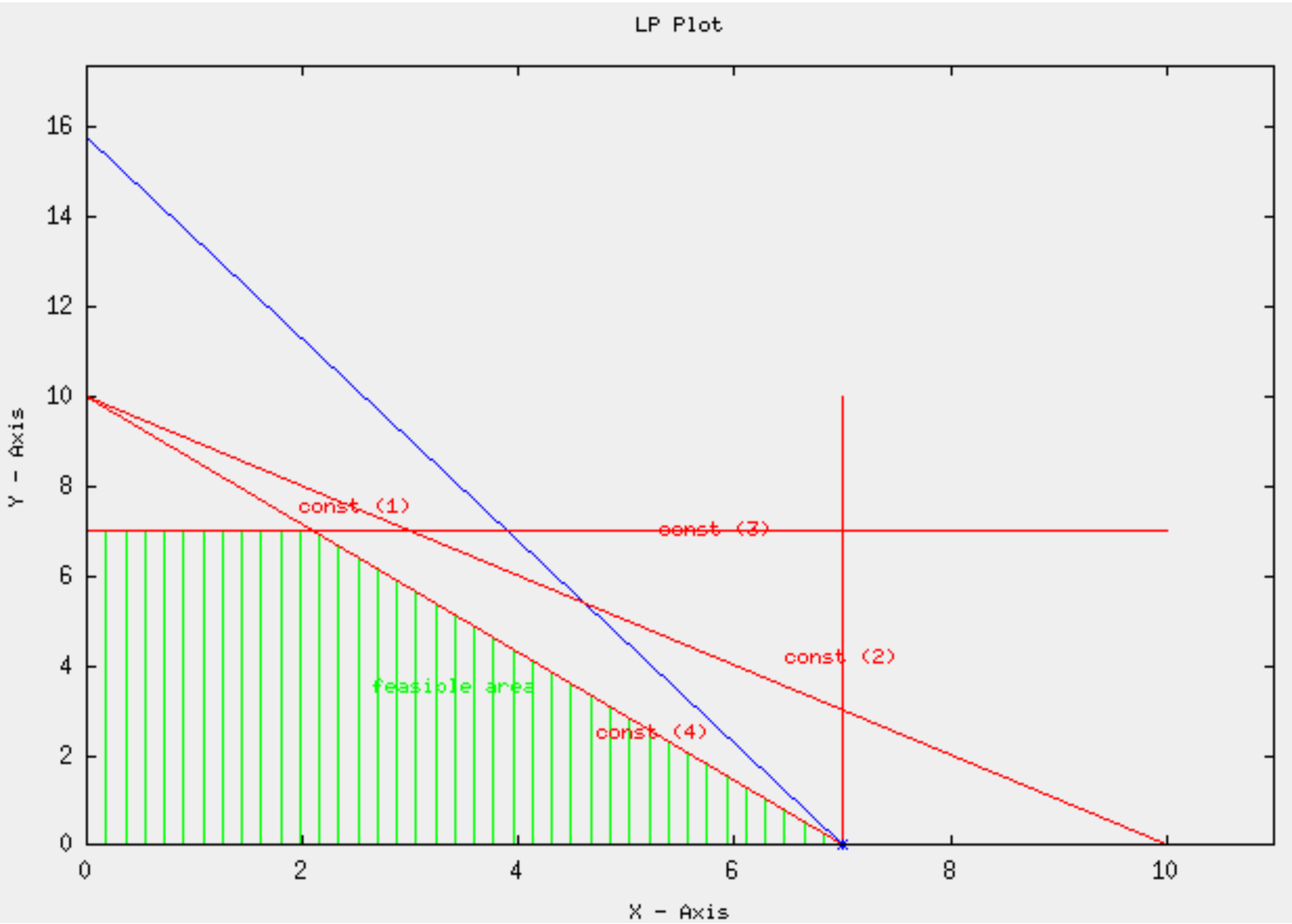

Figure 2: Plot of the Dutch Farms LP

4. Are both products planted? No, bell peppers are not planted at all.
5. Which constraints are active (binding) at this optimum? Constraints 2 (water) and 3 (max. amount of tomatoes).
6. How changes the solution, if 1 hectar less / more can be planted?

```
f.rhs<-c(09,70,7,7)
f.rhs<-c(11,70,7,7)
obj<-lp(f.dir,f.obj,f.con,f.condir,f.rhs)
if (obj$status==0){
  print(paste("Success. Optimum at ",toString(obj$solution),
    " with objective function value ",obj$objval))
  print(cbind(f.con%*%obj$solution, f.condir, f.rhs))
}else{
  print("error: optimisation was not successful")
}
```

This does not affect the production plan, as constraint 1 is not active. That is, we will still plant 7 hectars of tomatoes.

7. How changes the solution, if 1 unit of water less / more can be used?

```
f.rhs<-c(10,69,7,7)
f.rhs<-c(11,71,7,7)
obj<-lp(f.dir,f.obj,f.con,f.condir,f.rhs)
if (obj$status==0){
  print(paste("Success. Optimum at ",toString(obj$solution),
    " with objective function value ",obj$objval))
  print(cbind(f.con%*%obj$solution, f.condir, f.rhs))
}else{
  print("error: optimisation was not successful")
}
```

The water constraint is active. Increasing the available water will result in additional planting of bell peppers, decreasing it will reduce the amout of planted tomatoes.

8. Assume two additional plants become available:
   - Chilies (no irrigation, 150,000 Euros profit per hectar,
   - Cucumber (9 units water / acre, 400,000 Euros profit per hectar

   How changes the mathematical program? Which plants are planted? Which constraints are active?

$$\max_{x_1,x_2,x_3,x_4 \in R^4} 450 \cdot x_1 + 200 \cdot x_2$$

$$\text{subject to}$$

$$\begin{aligned}
x_1 & & &\geq 0\\
x_2 & & &\geq 0\\
& & x_3 &\geq 0\\
& & x_4 &\geq 0\\
x_1 + x_2 & & +x_3 + x_4 &\leq 10\\
10x_1 + 7x_2 & & +0x_3 + 9x_4 &\leq 70\\
x_1 & & &\leq 7\\
x_2 & & &\leq 7\\
& & x_3 &\leq 7\\
& & x_4 &\leq 7
\end{aligned}$$

```
f.obj<-c(450,200,150,400)
f.con<-matrix(c(1,1,1,1,10,7,0,9,1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1),ncol=4,byrow=TRUE)
f.condir<-c("<=","<=","<=","<=","<=","<=")
f.rhs<-c(10,70,7,7,7,7)
obj<-lp(f.dir,f.obj,f.con,f.condir,f.rhs)
print(obj)
if (obj$status==0){
  print(paste("Success. Optimum at ",toString(obj$solution),
    " with objective function value ",obj$objval))
  print("Checking which constraints are active (binding):")
  print(cbind(f.con%*%obj$solution, f.condir, f.rhs))
}else{
  print("error: optimisation was not successful")
}
```

The best production plan is now to produce 7 hectars with tomatoes and 3 hectars with chilies, with a profit of 3,600,000 Euros. Neither bell pepper nor cucumber are produced. Constraint 1 (area), constraint 2 (water), and constraint 3 (max. tomatoes) are active.

0.3 Which of the following statements are correct?

(X) A constraint that is active at the optimum means that it is restricting the optimal solution. Thus, the optimum will change if this constraint is omitted.

(X) A non-active (or non-binding) constraint means omitting it will not influence the optimal solution.

(X) The feasible area is the set of decision variable value combinations, which satisfy all constraints.

(X) In a linear program, the objective function and the constraints are linear functions.

Appendix

References

[Winston, 1997] Winston, W. L. (1997). *Operations Research: Applications and Algorithms*. Wadsworth Publishing Company, 3rd edition edition.