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Consequences of the sharpness of supercritical Bernoulli Percolation

Semester project

Master of Science (M. Sc.)

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Abstract

In this semester project, we prove properties of the supercritical phase in Bernoulli bond percolation on the Euclidean lattice. We take the theorem by Grimmett and Marstrand as a starting point, which states that in the supercritical phase, there is percolation on slabs. We first use this result to prove the exponential decay of the radius of finite clusters and then the existence and uniqueness of large clusters locally around a vertex. We then use the second result and a static renormalization to prove the positivity of the surface tension and the exponential decay of finite cluster sizes in the supercritical phase.

Contents

1 Bernoulli Percolation on the Euclidean lattice

Percolation theory was introduced to model the flow of a fluid or gas through a porous material. It is one of the simplest models that exhibit a phase transition and is, therefore, of great interest to statistical physics. Bond percolation studies the probabilistic properties of a given graph, or network if one randomly assigns edges to be either open or closed. The open edges are those that can be traversed when travelling from one vertex to another. Another perspective is that one deletes all the closed edges and obtains a random subgraph induced by the open edges. A few examples of applications of percolation theory include the Ising model for ferromagnetism, the spread of disease, and the stability of communication networks.

In this project, we consider Bernoulli bond percolation on the Euclidean lattice. The given graph is the Euclidean (or cubic) lattice $\mathbb{L}^d$ in d dimensions. The vertices are d-tuples of integers $\mathbb{Z}^d$ and are connected by an edge, if they differ exactly by 1 in one coordinate. Each edge is considered independently of the others and undergoes a Bernoulli trial, which gives 1 with probability p and 0 with probability $1-p$. If the outcome of a Bernoulli trial is 1, we define the edge to be open (or keep it) and if the outcome is 0, we define the edge to be closed (or delete it). Percolation theory then studies how such a graph will typically look. See Figure 1 for an illustration of bond percolation on $\mathbb{L}^2$.

In this section, we first introduce the necessary mathematical notation and then state the main results in Section 1.3.

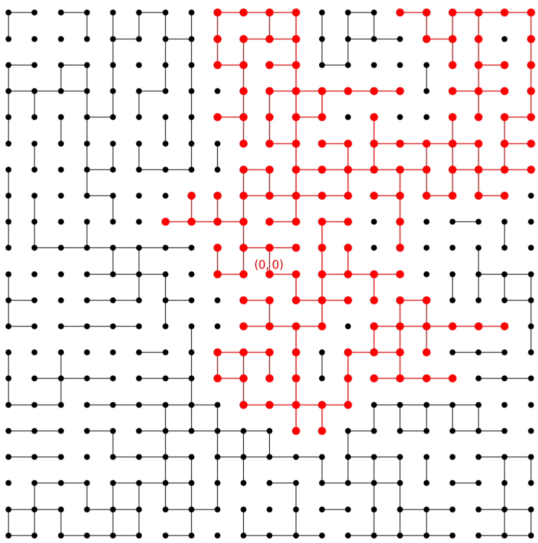

Figure 1: Random configuration on $\mathbb{L}^2$ with $p = 0.47$, where all the closed edges are deleted. The cluster of the origin is coloured in red.

1.1 Model

The Euclidean lattice is defined as the graph $\mathbb{L}^d = (V, E)$ with $V := \mathbb{Z}^d$ and $E := \{\{x, y\} : x, y \in V, d_1(x, y) = 1\}$, where $d_1(x, y) = \sum_{i=1}^{d} |y_i - x_i|$ is the l_1-metric.
The **p-Bernoulli bond percolation** on $\mathbb{L}^d$ is defined by fixing the edge probability $p \in [0, 1]$. The probability space we use is $(\Omega, \mathcal{F}, \mathbb{P}_p)$. The sample space is $\Omega := \{0, 1\}^E$, where $\omega = (\omega(e) : e \in E) \in \Omega$ is called a configuration, and $\omega(e) = 1$ if e is open and $\omega(e) = 0$ if e is closed. The σ-algebra $\mathcal{F}$ is the product σ-algebra. The measure is $\mathbb{P}_p = \prod_{e\in E} \mu_e$, where μ_e is the Bernoulli measure on $\{0, 1\}$, i.e., $\mu(\omega(e) = 1) = p$ and $1 - \mu(\omega(e) = 0) = p$. Note that $\mathbb{P}_p$ is characterized by the number of open edges in any finite set of edges. That is, $\forall n \in \mathbb{N}, \forall e_1, \dots, e_n \in E, \forall s_1, \dots, s_n \in \{0, 1\}$, the law is given by $\mathbb{P}_p(\omega(e_1) = s_1, \dots, \omega(e_n) = s_n) = p^{|s|} \cdot (1 - p)^{n-|s|}$, where $|s| = \sum_{i=1}^{n} s_i$ is the number of open edges.
Next, we define some graph-theoretic notation. We will denote the undirected edges by $xy := \{x, y\}$ and the box of radius n around $x \in \mathbb{Z}^d$ by $\Lambda_n(x) := \{-n, \dots, n\}^d + x$. For the **box around the origin**, we write $\Lambda_n := \Lambda_n(0)$. For a subset $S \subset \mathbb{Z}^d$, let $\partial S := \{x \in S : \exists y \in \mathbb{Z}^d \setminus S, xy \in E\}$ be the **vertex boundary** of S. Furthermore, we denote by $\partial^{\text{bot}}\Lambda_n := \{-n, \dots, n\}^{d-1} \times \{-n\}$ and $\partial^{\text{top}}\Lambda_n := \{-n, \dots, n\}^{d-1} \times \{n\}$ the **bottom and top faces** of the box around the origin.
We call a sequence of distinct vertices $\gamma = (x_0, x_1, x_2, \dots, x_n)$ a path of length $n = |\gamma|$ if $x_i x_{i+1} \in E$ for all $0 \le i \le n - 1$. If the sequence of vertices is pairwise distinct, except for $x_0 = x_n$, we call it a cycle. Moreover, we call γ an **open path or cycle** if all its edges are open. We write $x \leftrightarrow y$ for the event that there is an open path between $x, y \in V$. Similarly, for two subsets $A, B \subset V$, we write $A \leftrightarrow B$ if there is an open path between any two vertices $x \in A$ and $y \in B$, and $A \not\leftrightarrow B$ if there is no open path between any $x \in A$ and $y \in B$. If there is a path between A and B using only vertices in a subset $S \subset \mathbb{Z}^d$, we write $A \overset{S}{\leftrightarrow} B$, and $A \overset{S}{\not\leftrightarrow} B$ if there is no path between A and B using only vertices in S.
A configuration ω induces a subgraph $G_\omega \subset \mathbb{L}^d$ by deleting all the closed edges. For $x \in V$, the (open) cluster of x is the connected component in G_ω containing x and is denoted by $C(x)$. For short, let $C := C(0)$ be the **cluster of the origin** 0. By $\left|C(x)\right|$, we denote the number of vertices in $C(x)$. Note that as the measure is invariant under translation, $C(x)$ has the same distribution as C.

An important question is whether a given vertex belongs to an infinite open cluster. If x is contained in an infinite open cluster, we write $x \leftrightarrow \infty$ or $\left|C(x)\right| = \infty$. The **percolation probability** is defined as $\theta(p) := \mathbb{P}_p(0 \leftrightarrow \infty)$. It can be proven that $\theta(p)$ is a nondecreasing function in p. The **critical probability** is therefore well defined by

$$p_c := p_c(\mathbb{L}^d) := \inf\{p \in [0, 1] : \theta(p) > 0\} \tag{1}$$

The natural question that arises is whether the system exhibits a *phase transition*, i.e., $0 < p_c < 1$. Intuitively, a phase transition means that the system looks globally different in the different phases $p < p_c$ (subcritical) and $p > p_c$ (supercritical). From Kolmogorov's 0-1 law, it follows that in the supercritical phase there is almost surely an infinite open cluster. In the subcritical phase, there is almost surely no infinite open cluster. If the dimension is at least two, there is a phase transition. This is proven by a so-called entropy versus energy argument.

Theorem 1.1.1. *If $d \geq 2$ then $0 < p_c(\mathbb{L}^d) < 1$*

Proof. We first prove $p_c(\mathbb{L}^d) > 0$ by showing that $\theta(p) = 0$ for all $p < 1/(2d)$. Let $n \geq 1$ and observe that the number of paths of length n from the origin is less than $(2d)^n$. This holds because at each step of the path there are at most $2d$ possible directions. Therefore, we obtain an upper bound for the percolation probability

$$\begin{aligned}\theta(p) \leq \mathbb{P}_p(\bigcup_{\substack{\gamma \text{ path from } 0\\ |\gamma|=n}} \{\gamma \text{ is open}\}) &\leq \sum_{\substack{\gamma \text{ path from } 0\\ |\gamma|=n}} \mathbb{P}_p(\gamma \text{ is open}) \\ &\leq p^n \cdot \big|\{\gamma \text{ path from } 0, |\gamma| = n\}\big| \leq p^n(2d)^n \xrightarrow{n\to\infty} 0 \quad (2)\end{aligned}$$

where we used in Equation 2 that $\mathbb{P}_p(\gamma \text{ is open}) \leq p^n$, since we have to open at most the n edges of γ.
Next we prove that $p_c(\mathbb{L}^d) < 1$. Note that since we can embed $\mathbb{L}^2$ in $\mathbb{L}^d$ for $d \geq 2$, we have that $p_c(\mathbb{L}^d)$ is nonincreasing in d because an infinite open cluster in $\mathbb{Z}^2$ is also an infinite open cluster in $\mathbb{Z}^d$. It is therefore sufficient to prove $p_c(\mathbb{L}^2) < 1$. We construct the dual lattice $(\mathbb{L}^2)^* = (V^*, E^*)$ of $\mathbb{L}^2$ by shifting $\mathbb{L}^2$ by $(1/2, 1/2)$, where we see the graph naturally embedded in $\mathbb{R}^2$. Observe that any edge $e \in E$ is crossed orthogonally by exactly one edge $e^* \in E^*$. For a configuration ω on $\mathbb{L}^2$ define its dual configuration ω^* on $(\mathbb{L}^2)^*$ by $\omega^*(e^*) := 1 - \omega(e)$ for all $e^* \in E^*$.
This mapping is a bijection between ω, having law $\mathbb{P}_p$ and ω^* having law $\mathbb{P}_{1-p}$. The important observation is that the origin is not in an infinite open cluster in ω if and only if there is an open cycle surrounding the origin in ω^*. Such a cycle γ must be at least of length $|\gamma| = 4$. We use this to calculate the probability that the origin is in a finite cluster. In Equation 3, we sum over all cycles of length at least $|\gamma| =: n \geq 4$. Then, we fix a starting point $0^* \in V^*$ of the cycle and sum over all translations by at most $\lfloor n/2 \rfloor$. If we shift the cycle by more than $\lfloor n/2 \rfloor$, it will not contain the origin anymore. We therefore get an upper bound of the event that 0 is not in an infinite open cluster $\{0 \nleftrightarrow \infty\}$ by summing over the cycles γ in the dual lattice, weighted by their probability of being open. For a cycle of length $|\gamma| = n$ the probability is $\mathbb{P}_{1-p}(\gamma \text{ open}) \leq (1-p)^n$. For $p > 15/16$ the

probability that the origin is not in an infinite open cluster is strictly smaller than 1, as

$$\mathbb{P}_p(0 \nleftrightarrow \infty) = \mathbb{P}_{1-p}(\exists \gamma \text{ open cycle around } 0) \leq \sum_{n\geq 4} \sum_{\substack{x \in V^* \\ d(x,0)\leq \lfloor n/2 \rfloor}} \sum_{\substack{\gamma \text{ cycle} \\ x\in\gamma \\ |\gamma|=n}} \mathbb{P}_{1-p}(\gamma \text{ open})$$

$$\leq \sum_{n\geq 4} (\frac{n}{2})^2 \cdot (1-p)^n \cdot \big|\{\gamma \text{ cycle from } 0^*, |\gamma| = n\}\big| \leq \frac{1}{4}\sum_{n\geq 0} n^2(1-p)^n 4^n = \frac{(1-p)(1+4(1-p))}{(1-4(1-p))^3} \tag{3}$$

is an upper bound. In the last equality of Equation 3 we use the geometric series to calculate the value. We conclude that $1-\theta(p) < 1$ and thus $\theta(p) > 0$. □

Remark 1.1.2. In the previous proof, the choice of $p > 15/16$ can be improved. For example, taking $p > 3/4$ is enough if one considers the event that a large box is not connected to infinity $\{\Lambda_k \nleftrightarrow \infty\}$. In dimension $d = 2$ the value of the critical probability was proven in [Kes80] to be $p_c(\mathbb{L}^2) = \frac{1}{2}$.

We can subdivide the percolation into the subcritical ($p < p_c$), critical ($p = p_c$) and supercritical ($p > p_c$) phase. The next question might be to ask what the phases typically look like. A possible starting point in the subcritical phase is the radius of open clusters, i.e. the event $\{0 \leftrightarrow \partial\Lambda_n\}$. The proof of Theorem 1.1.1 above actually implies that the probability of this event decays exponentially for p close to 0. The following theorem, proven independently in [AB87] and [Men86], says that this is true up to the critical point. This result is known as subcritical sharpness and has geat importance in percolation theory.

Theorem 1.1.3. *For every $p < p_c$, there is a constant $c > 0$ such that*

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \leq e^{-cn}.$$

In the supercritical phase, the probability of this event does not decay, since the origin belongs to an infinite cluster with positive probability. One therefore considers the radius of finite clusters, i.e. the event $\{0 \leftrightarrow \partial\Lambda_n, 0 \nleftrightarrow \infty\}$. The study of the subcritical phase tends to be easier, since the events are typically increasing and we therefore have a bigger toolbox available. In this project, we focus on the supercritical phase, with the goal of understanding its geometry.

1.2 Supercritical phase

It is proven in [BK89] that the **infinite open cluster is unique** in the supercritical phase. In two dimensions, the Euclidean lattice is (planar) dual to itself, as used in the proof of Theorem 1.1.1. This allows one to study the supercritical percolation for $d = 2$ by considering the subcritical percolation on the dual lattice. There is no such duality in higher dimensions. The reader is referred to [Gri99, p. 16] for a detailed description of the duality. Therefore, from now on for the rest of the project, we will **only consider the dimensions** $d \geq 3$.

For dimensions $d \geq 3$, other methods are needed to study the supercritical phase. Historically, the first approach was to consider percolation on slabs. For $L \geq 0$, define the **slab** of thickness L

$$S_L := \mathbb{Z}^2 \times \{0, 1, ..., L\}^{d-2} \subset \mathbb{Z}^d.$$

A slab can be imagined as a thickened version of $\mathbb{Z}^2$, consisting of a finite number of copies of $\mathbb{Z}^2$. This allows one to apply tools from percolation in two dimensions.

For such a slab, we define the **critical slab probability** analogous to the percolation probability, but restricted to the slab.

$$p_c(S_L) := \inf\{p \geq 0 : \mathbb{P}_p(0 \overset{S_L}{\longleftrightarrow} \infty) > 0\}.$$

This means that for $p > p_c(S_L)$ there is almost surely an infinite open cluster, using only vertices in S_L. Since $S_L \subset S_{L+1} \subset \mathbb{Z}^d$, we have $p_c(S_L) \geq p_c(S_{L+1}) \geq p_c$ for all $L \geq 0$. Therefore, the decreasing limit exists and we define as follows.

$$p_c^{slab} := \lim_{L \to \infty} p_c(S_L)$$

We say that there is **percolation on slabs of thickness L**, if $p > p_c(S_L)$. Multiple properties of the supercritical phase can be proven by assuming that $p > p_c^{slab}$. For example, all the statements in the following sections can be proven, based on the assumption $p > p_c^{slab}$. Grimmett and Mardstrand proved that $p_c^{slab} = p_c$ in [GM90], thus showing that all these properties hold up to the critical point. (This is the reason, why this result is sometimes refered to as supercritical sharpness.)

Theorem 1.2.1.

$$p_c^{slab} = p_c$$

Another approach to analyze the supercritical phase is using a technique called coarse graining or static renormalization. This technique is based on a finite size criterion which is called **local uniqueness**.

Definition 1.2.2. *For* $k \geq 1$ *define the event*

$$Unique(k) := \{\Lambda_k \leftrightarrow \partial\Lambda_{7k}\} \cap \{\textit{There exists at most one cluster } C, \textit{ s.t. } \Lambda_{3k} \overset{C \cap \Lambda_{6k}}{\longleftrightarrow} \partial\Lambda_{6k}\}.$$

If $Unique(k)$ occurs, there is a crossing cluster intersecting Λ_k and $\partial\Lambda_{7k}$, and any cluster in Λ_{6k} crossing $\Lambda_{6k} \setminus \Lambda_{3k}$ is connected to it. This local criterion gives the existence and uniqueness of a large cluster locally around a given vertex. Moreover, if the event occurs around two adjacent boxes of radius k, the big clusters are connected to each other. This property will be crucial for using the static renormalization in Section 2.5.
In analogy to the subcritical phase, one could also refer to the exponential decay of the radius of finite clusters as supercritical sharpness. However, there is not really a standard convention in the supercritical phase. One could take any of the following results in the supercritical phase and refer to it as supercritical sharpness.

1.3 Main results

In this section, we collect the main results that will be proven in this project. We show that the Grimmett-Marstrand Theorem 1.2.1 implies local uniqueness in the supercritical phase, i.e. Theorem 1.3.1. By this we mean that for $p > p_c$ we can make the probability of $Unique(k)$ arbitrarily close to 1 by choosing $k \geq 1$ large enough.

Theorem 1.3.1. *For all $p > p_c, \delta > 0$, there is $k \geq 1$ such that*

$$\mathbb{P}_p(Unique(k)) \geq 1 - \delta.$$

We also show the exponential decay of the radius of finite clusters in the supercritical phase.

Theorem 1.3.2. *For any $p > p_c$, there exists a constant $c > 0$, such that*

$$\forall n \geq 1 : \quad \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n, 0 \nleftrightarrow \infty) \leq e^{-cn}.$$

We then use the local uniqueness property (Theorem 1.3.1), combined with a static renormalization to prove the following two theorems.
We show that for $p > p_c$ the size of finite clusters decays exponentially with surface order.

Theorem 1.3.3. *For $p > p_c$, there exist constants $c > 0$, such that*

$$\forall n \geq 1 : \mathbb{P}_p(n \leq |C| < \infty) \leq e^{-cn^{(d-1)/d}}.$$

Additionally, we show that the probability of disconnection between two opposing faces of the box Λ_L decays exponentially with surface order L^{d-1}.

Theorem 1.3.4. *For all $p > p_c$, there exists a constant $c > 0$ such that*

$$P_p(\partial^{bot}\Lambda_L \overset{\Lambda_L}{\nleftrightarrow} \partial^{top}\Lambda_L) \leq e^{-cL^{d-1}} \qquad \forall L \geq 1.$$

2 Consequences of supercritical sharpness

In this section, we prove properties of the supercritical phase. We take the Theorem 1.2.1 by Grimmett and Marstrand as a starting point, which states that in the supercritcical phase, there is percolation on slabs. We first use this result to prove the exponential decay of the radius of finite clusters in Section 2.1. We will then prove that we can make the probability of $Unique(k)$ arbitrary close to 1 in Section 2.2. This event, depending on a local neighbourhood around a vertex, will be called local uniqueness. We then use local uniqueness and a static renormalization to prove the positivity of the surface tension and the exponential decay of finite cluster sizes in the supercritical phase. In order to do so, we first collect a toolbox of geometric and probabilistic lemmas in Section 2.3. We first prove the positivity of surface tension and the exponential decay of finite cluster sizes by a combinatorial argument for a sufficiently large $p_0 < 1$ in Section 2.4. Afterwards, we extend these two results to the whole supercritical phase $p_c < p \leq 1$ by using renormalization and local uniqueness in Section 2.5.

In the following sections, we will repeatedly need some formalism of exploring the cluster $C(v)$ of a fixed vertex $v \in V$ in a percolation configuration. Define the **edge boundary** of a subset $S \subset \mathbb{Z}^d$ by $\Delta S := \{xy \in E : x \in S, y \notin S\}$. An **exploration of the cluster** $C(v)$ is an increasing sequence $(C_i)_{i\geq 1}$ of subgraphs of $\mathbb{L}^d$. Start with the subgraph $C_1 := (\{v\}, \emptyset)$. We inductively construct C_{m+1}. Order all the edges E in some fixed and deterministic way with $\mathbb{N}$. Assume that C_m is constructed. We define C_{m+1} by adding to C_m the earliest open edge that lies in ΔC_m, i.e. $C_{m+1} := C_m \cup \{e_k\}$ with $k := \inf\{i \in \mathbb{N} : e_i \in \Delta C_m, e_i \text{ is open}\}$. If there are no more open edges in the edge boundary, we define $C_{m+1} := C_m$. The result is an increasing sequence of subgraphs with $\lim_{i\to\infty} C_i = C(v)$, if $C(v)$ is finite.[1]

2.1 Exponential decay of truncated connection probabilities

We prove that the probability of the radius of finite clusters, i.e. the event $\{0 \leftrightarrow \partial\Lambda_n, 0 \not\leftrightarrow \infty\}$, decays exponentially. The idea of the proof is that if the cluster has a large radius and is not infinite, it traverses a large number of disjoint regions. If chosen thick enough, each of these regions contains almost surely an infinite open cluster, by the Grimmett Marstrand theorem. Therefore, the probability that the cluster traverses the region without intersecting the infinite cluster can be bound independently and away from 1. Let us proof this formally.

Proof of Theorem 1.3.2. First let C be the cluster of the origin. For $i \in \{1, 2, ...\}$ define the regions

$$R_k(i) := \{x \in \mathbb{Z}^d : (i-1)k \leq x_1 < ik\}$$

[1] C.f. [Gri99, pp. 205–206]

Since $p > p_c$, there is percolation on slabs and we can find $k \geq 1$ large enough, such that $\theta(p,k) := \mathbb{P}_p(0 \xleftrightarrow{R_k(1)} \infty) > 0$ (Using the Grimmett-Marstrand Theorem 1.2.1 and $S_k \subset R_{k+1}(1)$).

Assume that $n = k \cdot r$ for some $r \in \mathbb{N}$. Define the hyperplane $H_j := \{x \in \mathbb{Z}^d : x_1 = j\}$. Also note that for any $v \in R_k(i)$ we have that $\mathbb{P}_p(v \xleftrightarrow{R_k(i)} \infty) = \theta(p,k)$ by the translation invariance of the measure. The boundary of the box $\partial\Lambda_n$ can be covered by $2d$ hyperplanes H_n. Thus, by symmetry, we get the upper bound

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n, 0 \not\leftrightarrow \infty) \leq 2d \cdot \mathbb{P}_p(0 \leftrightarrow H_n, 0 \not\leftrightarrow \infty) \tag{4}$$

and we will continue to bound the RHS. Next, let $(C_m)_{m\geq 1}$ be an exploration of the cluster of the origin C and let

$$T := \sup\{i \in \{1, ..., r\} : C \cap R_k(i) \neq \emptyset\}$$

be the index of the maximal region that is reached by C. Use the exploration to define a (random) sequence of first vertices $(v_i)_{1\leq i \leq T}$ in each region $R_k(i)$, i.e. the unique vertex in $C_j \cap R_k(i)$ with $j := \inf\{j \in \mathbb{N} : C_j \cap R_k(i) \neq \emptyset\}$. By definition, we have a sequence $v_1, v_2, ..., v_T$ with $v_1 = 0$ and of random length T. See Figure 2.1 as an illustration of the previously described steps with $k = 5$.

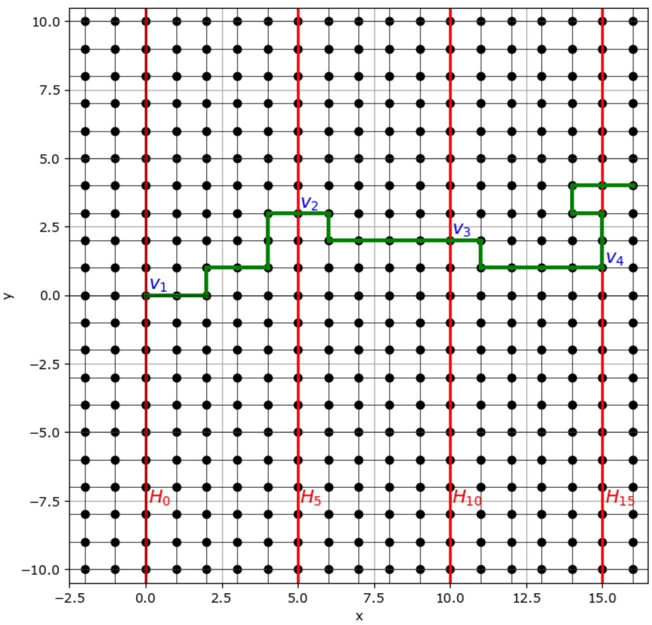

Figure 2: Example of an open path in C that connects $0 \leftrightarrow H_{15}$ and the corresponding sequence of first vertices $(v_i)_{i\geq 1}$ in each region $R_5(i)$

Define the event

$$A_i := \{T \geq i\} \cap \Big(\bigcap_{j=1}^{i} \{v_j \overset{R_k(j)}{\not\longleftrightarrow} \infty\} \Big).$$

Note that $\{0 \leftrightarrow H_n, 0 \not\leftrightarrow \infty\}$ implies that A_r occurs and thus $\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n, 0 \not\leftrightarrow \infty) \leq 2d \cdot \mathbb{P}_p(A_r)$. Since $A_i \subset A_{i-1}$, we can calculate the probability using the conditional probability

$$\mathbb{P}_p(A_r) = \mathbb{P}_p(A_1) \cdot \Pi_{i=2}^{r} \mathbb{P}_p(A_i \mid A_{i-1}) \tag{5}$$

Next, we calculate the conditional probabilities $\mathbb{P}_p(A_i \mid A_{i-1})$. Here, it is important to observe that the event $\{v \overset{R_k(i)}{\not\longleftrightarrow} \infty\}$ is independent of A_{i-1}. Also, $T \geq i$ is independent of $\{v \overset{R_k(i)}{\not\longleftrightarrow} \infty\}$. Therefore for a $v \in H_{(i-1)k}$ we get

$$\mathbb{P}_p(v \overset{R_k(i)}{\not\longleftrightarrow} \infty \mid v_i = v, T \geq i, A_{i-1}) \leq (1 - \theta(p,k)).$$

We now take the union over all vertices of $v \in H_{(i-1)k}$ that can be v_i

$$\begin{aligned} \mathbb{P}_p(A_i \mid A_{i-1}) &\leq \sum_{v \in H_{(i-1)k}} \mathbb{P}_p(v \overset{R_k(i)}{\not\longleftrightarrow} \infty \mid v_i = v, T \geq i, A_{i-1}) \cdot \mathbb{P}_p(v_i = v, T \geq i \mid A_{i-1}) \\ &\leq (1 - \theta(p,k)) \sum_{v \in H_{(i-1)k}} \mathbb{P}_p(v_i = v \mid T \geq i, A_{i-1}) \leq (1 - \theta(p,k)). \end{aligned} \tag{6}$$

Combining this result with (5) and (4), we get

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n, 0 \not\leftrightarrow \infty) \leq 2d \cdot (1 - \theta(p,k))^{n/k} \leq e^{-cn}. \tag{7}$$

The last inequality follows by choosing $c := -1/k \cdot \log(1 - \theta(p,k)) > 0$.

□

2.2 Percolation on Slabs implies local uniqueness

In this section, we will show that for $p > p_c$ we can make the probability of $Unique(k)$ arbitrarily close to 1 by choosing k large enough.

Definition 2.2.1. *Let $U(n, 2n)$ be the event that the number of clusters in Λ_{2n}, which intersect both Λ_n and $\partial\Lambda_{2n}$, is either zero or one.*

Recall the Definition 1.2.2 and rewrite $Unique(k) = \{\Lambda_k \leftrightarrow \partial\Lambda_{7k}\} \cap U(3k, 6k)$. For $x \in \mathbb{Z}^d$, $U_k(x)$ is the event that $Unique(k)$ occurs centered around x.
Furthermore, we will need an important inequality from percolation theory, called FKG-inequality. We first define the notion of increasing events. One can equip the space of

configurations Ω with a partial ordering by defining

$$\omega \leq \omega' :\Leftrightarrow \forall e \in E : \omega(e) \leq \omega'(e).$$

An event $A \in \mathcal{F}$ is defined to be **increasing** if

$$\omega \leq \omega', \omega \in A \Rightarrow \omega' \in A$$

holds. Intuitively speaking, an event is increasing, if adding edges can only help satisfying the property. The following Theorem 2.2.2 is called the **FKG-inequality**. We omit the proof here and refer the reader to [Gri99, pp. 32 – 36].

Theorem 2.2.2. *Let $A, B \in \mathcal{F}$ be two increasing events. Then the following inequality holds*

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Our goal is to prove Theorem 1.3.1, i.e. that for $p > p_c$ the probability of $Unique(k)$ can be made arbitrarily large by choosing $k \in \mathbb{N}$ large enough.
We first prove that for $\eta > 0$, we can find $n_0 \geq 1$, such that $\mathbb{P}_p(U(n, 2n)) \geq 1 - \eta$ for all $n \geq n_0$. The idea is similar to the proof of the last Theorem 1.3.2. To illustrate the event $U(n, 2n)$ we show in Figure 12 an example of a random configuration that has two disjoint crossing clusters of $\Lambda_{20} \setminus \Lambda_{10}$ and hence is in $U(10, 20)^C$.
Define $T_m(L) := \{-m, ..., m\}^{d-1} \times \{0, ..., L\}$. To show that the probability of $U(n, 2n)$

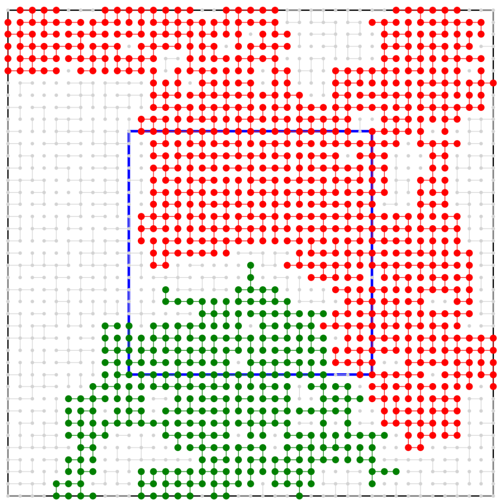

Figure 3: Random configuration that has two disjoint crossing clusters (in green and red) of $\Lambda_{20} \setminus \Lambda_{10}$, i.e. that satisfies $U(10, 20)^C$

decays exponentially, we partition the uniqueness zone $\Lambda_{2n} \setminus \Lambda_n$ into multiple layers of fixed size. We then cover the layers by 2d slices, isomorphic to $T_m(L)$. We will show that in each of these layers, the probability of disconnection of two crossing clusters can be bounded away from 1, since there is percolation on slabs. See Figure 4 for an illustration of partitioning the uniqueness zone into two layers of thickness $L = 10$ and covering each layer by $2d = 4$ overlapping slices.

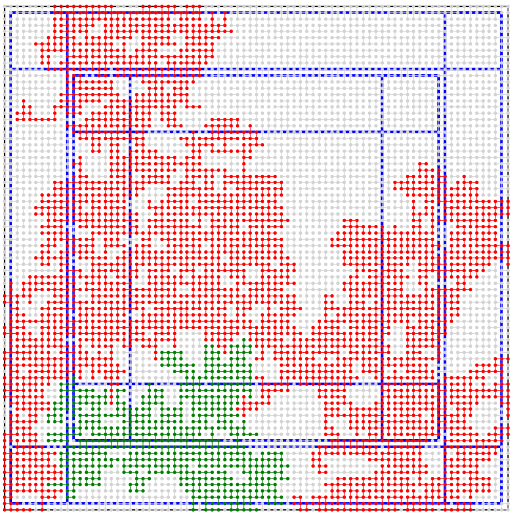

Figure 4: Illustration of partitioning the uniqueness zone $\Lambda_{40} \setminus \Lambda_{20}$ into two layers of thickness 10 in two dimensions. Each layer is covered by 4 overlapping slices of thickness $L = 10$ (coloured in blue). The two largest clusters are coloured in red and green. ($p = 0.501$)

We first show that we can find a uniform lower bound for the connection probability in a slice $T_m(L)$ for sufficiently large $L \geq 1$ and $p > p_c$. This follows from Theorem 1.2.1.

Lemma 2.2.3. *For $p > p_c$, there is an integer $L \geq 1$ and a constant $\delta = \delta(p, L) > 0$ such that $\mathbb{P}_p(x \xleftrightarrow{T_m(L)} y) \geq \delta$ for all $x, y \in T_m(L)$ and $m \geq L$.*

Proof. By Theorem 1.2.1 there is a L, such that $p_c(S_L) < p$. We will make use of a so-called sprinkling argument. Therefore, let p' be such that $p_c(S_L) < p' < p$. Write for short $\theta_{p'} := \mathbb{P}_{p'}(0 \xleftrightarrow{S_L} \infty)$ and note that $\theta_{p'} > 0$. We will show that there is $\delta_1 > 0$ such that $\mathbb{P}_p(0 \xleftrightarrow{T_m(L)} z) \geq \delta_1$ for all $z \in T_m(L)$. With FKG-inequality, this will imply $\mathbb{P}_p(x \xleftrightarrow{T_m(L)} y) \geq \delta_1^2$ for all $x, y \in T_m(L)$.
First assume that z has only one nonzero coordinate, say the first $z = (z_1, 0, 0, ..) \in T_m(L)$. Let z_1 be positive without loss of generality. Define two sets $H_l(z_1) := \{0\} \times \{0, ..., z_1\} \times \{0, ..., L\}^{d-2}$ and $H_r(z_1) := \{z_1\} \times \{0, ..., z_1\} \times \{0, ..., L\}^{d-2}$. Note that the box in the slab $B := \{-z_1, ..., z_1\}^2 \times \{0, ..., L\}^{d-2} \subset S_L$ can be covered by eight hyperplanes isomorphic

to $H_r(z_1)$. The origin is connected to infinity with probability $\theta_{p'}$ which implies that one of these hyperplanes is connected to the origin. Hence we get

$$\mathbb{P}_{p'}(0 \overset{B}{\longleftrightarrow} H_r(z_1)) \geq \frac{1}{8}\theta_{p'}. \tag{8}$$

By the same arguments and invariance under translation we get $\mathbb{P}_{p'}(z \overset{B}{\longleftrightarrow} H_l(z_1)) \geq \frac{1}{8}\theta_{p'}$. Since both are increasing events, it is possible to apply FKG-inequality to bound the probability that both occur.

$$\mathbb{P}_{p'}(A) = \mathbb{P}_{p'}\Big(0 \overset{B}{\longleftrightarrow} H_r(z_1), z \overset{B}{\longleftrightarrow} H_l(z_1)\Big) \geq \Big(\frac{\theta_{p'}}{8}\Big)^2 \tag{9}$$

See Figure 5 for an illustration of the event $A := \{0 \overset{B}{\longleftrightarrow} H_r(z_1), z \overset{B}{\longleftrightarrow} H_l(z_1)\}$, projected to the first two coordinates. It is important to notice that conditional on this event, we have to open at most the $L(d-2)$ edges below the (almost) intersection point, such that the origin is connected to z. The following bound is obtained by a sprinkling technique

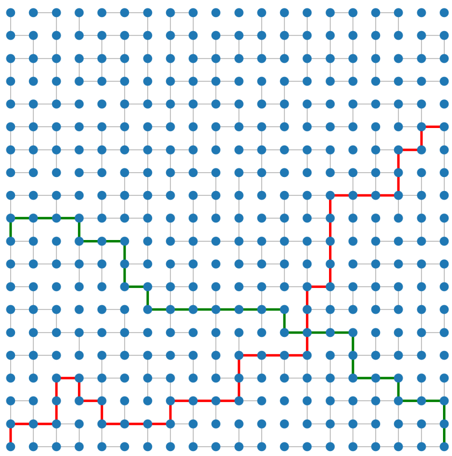

Figure 5: Illustration of the event $\{0 \overset{S_n(0)}{\longleftrightarrow} H_r(20), z \overset{S_n(0)}{\longleftrightarrow} H_l(20)\}$

and is adapted from the **sprinkling lemma** by [Aiz+83]. The idea is to give the finite set of possibly closed edged a second chance of being open by *sprinkling* edges on top with a small probability.

Definition 2.2.4. *Let $\epsilon := p - p'$ and ω' be a p'-Bernoulli configuration. Furthermore, let ω'' be an independent $\epsilon/(1-p')$-Bernoulli configuration.*

Observe that we obtain a p-Bernoulli configuration ω by taking the union (of open edges)

$$\omega := \omega' \cup \omega'', \qquad \forall e \in E : \omega(e) := \max\{\omega'(e), \omega''(e)\}.$$

The configuration ω has density p because for an edge $e \in E$ we can decompose the event of the edge being open into disjoint events $\{\omega(e) = 1\} = \{\omega'(e) = 1\} \cup \{\omega'(e) = 0, \omega''(e) = 1\}$. Hence, the density of ω is

$$p' + (1-p')\frac{\epsilon}{1-p'} = p.$$

Now observe that if the event A occurs in ω', there is a (random) set of at most $L(d-2)$ edges, which if all are open in ω'' would produce the event $\{0 \overset{B}{\leftrightarrow} z\}$ in ω. Therefore, we get by the independence of the two configurations

$$\mathbb{P}_p(0 \overset{B}{\leftrightarrow} z) \geq \big(\frac{\epsilon}{1-p'}\big)^{L(d-2)} \cdot \mathbb{P}_{p'}(A).$$

Observe that the box B is contained in $T_m(L)$, as $m \geq L$ by assumption. Together with (9), this gives the following uniform lower bound for the connection probability

$$\mathbb{P}_p(0 \xleftrightarrow{T_m(L)} z) \geq \big(\frac{\epsilon}{1-p'}\big)^{L(d-2)} \cdot \big(\frac{\theta_{p'}}{8}\big)^2. \tag{10}$$

Next take an arbitrary $z \in T_m(L)$. We can construct a sequence of at most d vertices, such that every vertex differs from the previous one by at most one coordinate, such that the last one is z. Hence, we can bound the connection probability between 0 and z by using FKG-inequality

$$\mathbb{P}_p(0 \xleftrightarrow{T_m(L)} z) \geq \Big(\big(\frac{\epsilon}{1-p'}\big)^{L(d-2)} \cdot \big(\frac{\theta_{p'}}{8}\big)^2\Big)^d =: \delta_1 > 0. \tag{11}$$

The claim of the lemma follows with $\delta := \delta_1^2 > 0$. $\square$

Next, we show that the probability of more than one disjoint crossing cluster of $\Lambda_{2n} \setminus \Lambda_n$ can be made arbitrarily small by choosing n large enough.

Lemma 2.2.5. *For $p > p_c, \eta > 0, \exists n_0 \geq 1$, such that:*

$$\mathbb{P}_p(U(n, 2n))) \geq 1 - \eta \qquad \forall n \geq n_0$$

Proof. We start by considering two vertices $x, y \in \Lambda_n$ and bounding the probability that they are contained in disjoint crossing clusters. Define for $x, y \in \Lambda_n$ the event

$$E_n(x, y) := \{x, y \leftrightarrow \partial\Lambda_{2n}, x \overset{\Lambda_{2n}}{\not\leftrightarrow} y\}.$$

We show that for $p > p_c$ there is a constant $c > 0$, such that

$$\mathbb{P}_p(E_n(x,y)) \le e^{-c\cdot n} \qquad \forall x, y \in \Lambda_n, \forall n \ge 1. \tag{12}$$

To prove this, we successively peel layers of thickness $M := L + 1$ from the uniqueness zone $\Lambda_{2n} \setminus \Lambda_n$. Using Lemma 2.2.3, choose for $\delta > 0$ $L \ge 1$ large enough, such that $\mathbb{P}_p(x \xleftrightarrow{T_n(L)} y) \ge \delta > 0$ for all $x, y \in T_n(L)$ and $n \ge L$.
Let $x, y \in \Lambda_n$ be arbitrary. For $i \in \mathbb{N}$ define the event

$$A_i := \{x, y \leftrightarrow \partial\Lambda_{n+iM}, x \overset{\Lambda_{n+iM-1}}{\not\longleftrightarrow} y\}.$$

Assume that $n = r \cdot M$ for some $r \in \mathbb{N}$ (else, we take $r := \inf\{j \in \mathbb{N} : j \cdot M \ge n\}$). We observe that $E_n(x,y) \subset A_r \subset A_{r-1} \subset .... \subset A_0$, which implies the following inequality

$$\mathbb{P}_p(E_n(x,y)) \le \prod_{i=1}^{r} \mathbb{P}_p(A_i \mid A_{i-1}). \tag{13}$$

Next, we claim that the conditional probability can be bound uniformly away from 1, i.e. that $P_p(A_i \mid A_{i-1}) \le 1 - \delta^d$ for some $\delta > 0$. To do so, we fix an exploration of the two clusters $C(x)$ and $C(y)$. We stop each of these explorations as soon as each of them touches a vertex in $\partial\Lambda_{n+(i-1)M}$. Conditional on A_{i-1}, there are two disjoint vertices $u, v \in \partial\Lambda_{n+(i-1)M}$, such that $x \xleftrightarrow{\Lambda_{n+(i-1)M}} u$ and $y \xleftrightarrow{\Lambda_{n+(i-1)M}} v$ and both paths use only edges in $\Lambda_{n+(i-1)M-1} \cup \Delta(\Lambda_{n+(i-1)M-1})$.
Define the **layer** $D_i := \Lambda_{n+iM-1} \setminus \Lambda_{n+(i-1)M-1}$. Conditional on A_{i-1}, we know that A_i can only occur if the event $\{u \overset{D_i}{\not\longleftrightarrow} v\}$ occurs. Note that this event only depends on edges in D_i. We show that $\mathbb{P}_p(u \xleftrightarrow{D_i} v) \ge \delta^d$ for all $u, v \in D_i$. To do so, observe that the maximum distance that u and v can have in D_i is when they lie in two opposing edges. With $r_i := n + iM - 1$, we can cover D_i by $2d$ slices isomorphic to $T_{r_i}(L)$, as illustrated in Figure 4. It is important that they always overlap at the corners. Next, we can take a shortest deterministic path γ, connecting u and v in D_i. The path intersects at most d different connected slices. Take a deterministic sequence of vertices $u = v_1, v_2, .., v_d = v$ from γ, such that each of the v_i for $2 \le i \le d - 1$ lies in a region, where at least two slices overlap. Since we have chosen L large enough, such that $\mathbb{P}_p(x \xleftrightarrow{T_{r_i}(L)} y) \ge \delta$ for all $x, y \in T_{r_i}(L)$, we can lower bound the probability that u and v are connected by using FKG-inequality

$$\mathbb{P}_p(v_1 \xleftrightarrow{D_i} v_2 \xleftrightarrow{D_i} ... \xleftrightarrow{D_i} v_d) \ge \prod_{i=1}^{d} \mathbb{P}_p(v_i \xleftrightarrow{D_i} v_{i+1}) \ge \delta^d. \tag{14}$$

This implies the bound on $E_n(x,y)$ with $c_1 := -\frac{1}{L+1}\log(1-\delta^d) > 0$

$$\mathbb{P}_p(E_n(x,y)) \le \prod_{i=1}^{r} \mathbb{P}_p(A_i \mid A_{i-1}) \le (1-\delta^d)^r = (1-\delta^d)^{\frac{n}{L+1}} = e^{-c_1 n}. \tag{15}$$

Using (15), and considering the complement of $U(n,2n)$, we obtain

$$\mathbb{P}_p(U(n,2n)) \ge 1 - \sum_{x,y\in\Lambda_n} E_n(x,y) \ge 1 - e^{-c_1 n}(\sum_{x,y\in\Lambda_n} 1) \ge 1 - e^{-c_1 n}\cdot \binom{(2n+1)^d}{2}. \tag{16}$$

In (16), the binomial coefficient grows polynomially and is therefore dominated by the exponential. Choose $n_0 \ge 1$ such that $e^{-c_1 n}\cdot\binom{(2n+1)^d}{2} \le \eta$.

□

It is important to note that $U(n,2n)$ does not state that there is a crossing cluster from Λ_n to $\partial\Lambda_{2n}$, but only that if it exists, it is also unique.

Next we will prove that by choosing a sufficiently large $k \ge 1$, we can make the probability that there is a crossing cluster of $\Lambda_{7k}\setminus\Lambda_k$ arbitrarily close to 1. This is a consequence of the existence of an infinite open cluster. Clearly, $\mathbb{P}_p(\Lambda_k \leftrightarrow \infty) \le \mathbb{P}_p(\Lambda_k \xleftrightarrow{\Lambda_{7k}} \partial\Lambda_{7k})$. The probability of $\{\Lambda_k \leftrightarrow \infty\}$ is increasing in k and thus we can rewrite the limit in the following way.

$$\lim_{k\to\infty} P_p(\Lambda_k \leftrightarrow \infty) = \mathbb{P}_p(\exists k \ge 1 : \Lambda_k \leftrightarrow \infty)$$

Now the RHS is equal to 1 because there is almost surely an infinite open cluster for $p > p_c$. Hence, for $\delta > 0$ there is a $k_2 \ge 1$, such that $\mathbb{P}_p(\Lambda_k \xleftrightarrow{\Lambda_{7k}} \partial\Lambda_{7k}) \ge 1-\delta$ for all $k \ge k_2$. Now Theorem 1.3.1 is an easy consequence.

Proof of Theorem 1.3.1. As $Unique(k) = \{\Lambda_k \xleftrightarrow{\Lambda_{7k}} \partial\Lambda_{7k}\} \cap U(3k,6k)$, we use the complement to get

$$\mathbb{P}_p(Unique(k)) \ge 1 - \mathbb{P}_p(\{\Lambda_k \overset{\Lambda_{7k}}{\not\leftrightarrow} \partial\Lambda_{7k}\}) - \mathbb{P}_p(U(3k,6k)^C) \tag{17}$$

Using Lemma 2.2.5, take k_1 large enough, such that $\mathbb{P}_p(U(3k,6k)^C) \le \delta/2$ for all $k \ge k_1$. Take k_2 large enough, such that $\mathbb{P}_p(\Lambda_k \overset{\Lambda_{7k}}{\not\leftrightarrow} \partial\Lambda_{7k}) \le \delta/2$ for all $k \ge k_2$. Then, we have with $k := \max\{k_1,k_2\}$ that $\mathbb{P}_p(Unique(k)) \ge 1-\delta$. □

The important property of local uniqueness is that if the events occurs around two adjacent boxes of size k in $\mathbb{Z}^d$, there are crossing clusters of $\Lambda_{7k}\setminus\Lambda_k$ locally around each of them and these are connected to each other by the uniqueness event $U(3k,6k)$. This will be made more explicit in Section 2.5, where we use it for a coarse graining technique, called static renormalization.

2.3 Toolbox

In this section, we collect some lemmas that we will need for proving the positivity of the surface tension and the exponential decay of the volume of finite clusters. Define the **external boundary** of a subset $S \subset \mathbb{Z}^d$ by

$$\Delta_{ext} S := \{xy \in \Delta S : y \overset{V \setminus S}{\longleftrightarrow} \infty\}.$$

Intuitively speaking, the external boundary of S is the edge boundary without the edge boundary of bubbles inside of S. The following lemma states that the size of the **external boundary is of surface order**.

Lemma 2.3.1. *For $d \geq 2$ there exists $c > 0$, such that for all finite $A \subset \mathbb{Z}^d$:*

$$|\Delta_{ext} A| \geq c|A|^{(d-1)/d}$$

Proof. We start by defining the **projection operators** to the coordinate planes. Define for $i \in \{1, ..., d\}$ the projection operators to the coordinate planes $P_i : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$ by dropping the i-th coordinate:

$$P_i(x_1, ..., x_d) := (x_1, ..., x_{i-1}, x_{i+1}, ..., x_d)$$

Any vertex $v \in P_i(A)$ corresponds to a line in the e_i direction, intersecting A. We can order the intersected vertices by their i-th coordinate. Since A is finite, there is a first and a last vertex we denote by v_f and v_l, respectively. Match to each vertex $v \in P_i(A)$ the two edges $\{v_f - e_i, v_f\}$ and $\{v_l, v_l + e_i\}$. By summing over all $i \in \{1, ..., d\}$ and $v \in P_i(A)$, we get exactly $\Delta_{ext} A$ and thus $|\Delta_{ext} A| = 2\sum_{i=1}^d \big|P_i(A)\big|$.
By applying the arithmetic mean-geometric mean inequality, this gives us:

$$\Pi_{i=1}^d \big|P_i(A)\big| \leq \Big(\frac{1}{d}\sum_{i=1}^d \big|P_i(A)\big|\Big)^d = \Big(\frac{|\Delta_{ext} A|}{2d}\Big)^d \tag{18}$$

Furthermore, by applying Loomis-Whitney inequality, we have that for all finite $A \subset \mathbb{Z}^d$ holds $|A| \leq (\Pi_{i=1}^d \big|P_i(A)\big|)^{1/(d-1)}$. See [LP17, 201 ff.] for a nice proof of Loomis-Whitney inequality. Thus the claim of the lemma follows with $c := (2d)$. ☐

A crucial ingredient for being able to bound the number of configurations is a notion of connectedness for subsets of edges. Note that the external boundary of a connected and finite subset is generally not connected in the line graph of the Euclidean lattice for $d \geq 3$. We therefore introduce a slightly larger graph, in which the external boundary of a finite and connected subset and a separating surface will be proven to be connected in the following lemmas.

Definition 2.3.2. *For the Euclidean lattice* $\mathbb{L}^d = (V, E)$ *let* $G_E := (V_E, E_E)$ *with* $V_E := E$ *and* $E_E := \{\{e, f\} \subset \mathcal{P}(E) : d_1(e, f) \leq 1\}$*. Short, we say that a subset of edges* $A \subset E$ *is* G_E*-connected, if it is connected in* G_E*.*

For showing that the external boundary of a finite and connected subset is G_E-connected, we use a result by Timar [Tim11] and apply it in Lemma 2.3.4 and 2.3.5 to our case. First, we intruduce the necessary notation.
The **cycle space** $Cycle(G)$ of a graph G is the set of all even-degree subgraphs (each vertex has an even degree).
A subset $\mathcal{C} \subset Cycle(G)$ is said to generate the cycle space of G, if any cycle in the cycle space can be obtained from elements in $\mathcal{C}$ by composing them with the symmetric difference as set operation. For example, it can be shown that $Cycle(\mathbb{L}^d)$ is generated by the set of basic 4-cycles. That is the set of cycles that surround some 2-face of a unit cube in $\mathbb{L}^d$. The cycle space can also be described as a vector space $(\{0, 1\}^E, +)$ over the two-element field $\mathbb{F}_2$. A subgraph $O \in Cycle(G)$ is described (analogous to a percolation configuration) by a sequence $O = (\omega(e))_{e \in E}$ with $\omega(e) = 1$ if and only if $e \in O$. The addition in this vector space corresponds to the operation of symmetric difference for sets.
Furthermore, let $Ends(G)$ be the **set of ends** in G. An end is an equivalence class of infinite paths, where two are equivalent if they can be connected by infinitely many pairwise disjoint paths. Intuitively, an end stands for a direction in which the graph extends to infinity. E.g., if a graph F is finite, we have $Ends(F) = \emptyset$. For the Euclidean lattice holds $\left|Ends(\mathbb{L}^d)\right| = 1$.
A **separating set of edges** between $x, y \in G \cup Ends(G)$ is a subset of $E(G)$ that every path between x and y intersects. We say that a separating set of edges is minimal if when removing any edge, it is not a separating set anymore. Now we can state the key result by Timar [Tim11] for proving connectedness in G_E.

Lemma 2.3.3. *Let* G *be some graph and* Π *a minimal separating set of edges between two points* $x, y \in G \cup Ends(G)$*. Let* $\mathcal{C}$ *be a set of cycles that generate the cycle space of* G*. Then, for any partition* (Π_1, Π_2) *of* Π*, there is some cycle* $O \in \mathcal{C}$ *that intersects both* Π_1 *and* Π_2*.*

Proof. The proof is taken from [Tim11, p. 3].
If x (or y) is an end, define x' (or y') to be a vertex, such that there is a path between x and x' (y and y') in $G \setminus \Pi$. Else let $x' := x$ ($y' := y$).
Let P_1 be a path between x' and y' that does not intersect Π_2 and let P_2 be a path between

x' and y' that does not intersect Π_1. This is possible because Π is minimal.
As $P_1 + P_2 \in Cyle(G)$, there is a subset $A \subset \mathcal{C}$, such that

$$P_1 + P_2 = \sum_{C \in A} C.$$

Let $A_1 := \{C \in A : C \cap \Pi_1 \neq \emptyset\}$ and $A_2 := A \setminus A_1$. Hence, the cycles in A_2 do not intersect Π_1. Rewrite the previous equation.

$$P_1 + \sum_{C \in A_1} C = P_2 + \sum_{C \in A_2} C \tag{19}$$

The RHS of Equation 19 does not intersect Π_1 and so it has to intersect Π_2. The path P_2 contains an odd number of elements from Π_2 because it crosses the separating set an odd number of times. All the cycles in A_2 contain an even number of elements from Π_2. Thus the total number of elements from Π_2 contained in the RHS is odd. The path P_1 does not intersect Π_2 by definition. Therefore, A_1 has to contain a cycle that intersects A_2 an odd number of times. Therefore, there is a $O \in \mathcal{C}$ such that $O \cap \Pi_2 \neq \emptyset$ (and $O \cap \Pi_1 \neq \emptyset$ by definition of A_1).

□

Next, we apply the previous lemma to show that the external boundary of a finite and connected subset is G_E-connected.

Lemma 2.3.4. *Let $S \subset \mathbb{Z}^d$ be finite and connected. Then the external boundary $\Delta_{ext} S$ is G_E-connected.*

Proof. The proof follows from Lemma 2.3.3.
Let $S \subset \mathbb{Z}^d$ be an arbitrary finite and connected subset. Let $\Pi := \Delta_{ext} S$ and $x := Ends(\mathbb{L}^d \setminus S)$ (using that $\left|Ends(\mathbb{L}^d \setminus S)\right| = 1$, since S is finite). Then Π is a minimal separating set of edges in $\mathbb{L}^d$ between S and x. Let $\mathcal{C}$ be the set of basic 4-cylces in $\mathbb{L}^d$.
Let $\Pi = \Pi_1 \cup \Pi_2$ be an arbitrary partition. By Lemma 2.3.3 there is a cycle $O \in \mathcal{C}$ such that $O \cap \Pi_1 \neq \emptyset$ and $O \cap \Pi_2 \neq \emptyset$. Take edges $e_1 \in O \cap \Pi_1$ and $e_2 \in O \cap \Pi_2$ from each of these intersections. Since O is a basic 4-cycle, the distance is $d_1(e_1, e_2) \leq 1$ and hence e_1 and e_2 are adjacent in G_E. Therefore, Π is connected in G_E because the partition of Π was arbitrary. □

The next lemma will be used for proving the exponential decay of the disconnection probability of two opposing faces of the box Λ_n. The intuition is that if two opposing faces are not connected, there is a separating set of edges that is closed. For an illustration of the disconnection event see Figure 6.

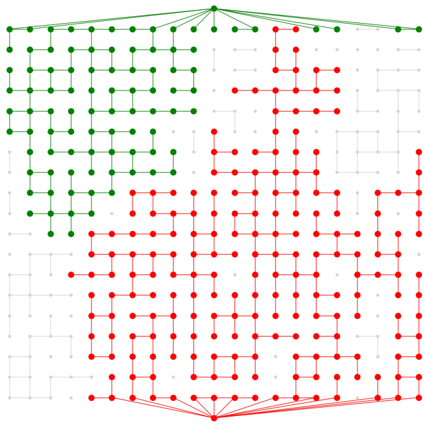

Figure 6: Random configuration that satisfies $\{\partial^{bot}\Lambda_{10} \overset{\Lambda_{10}}{\not\leftrightarrow} \partial^{top}\Lambda_{10}\}$ in two dimensions. The cluster of v_t is coloured in green and the cluster of v_b in red.

Lemma 2.3.5. *Let $L \geq 1$ and $\Pi \subset E(\Lambda_L)$ be a minimal set of edges that separates $\partial^{bot}\Lambda_L$ and $\partial^{top}\Lambda_L$. Then Π is G_E-connected.*

Proof. In order to prove this, we again use Lemma 2.3.3 by Timar. Consider the box Λ_L and define an auxiliary graph H, which is obtained from $\mathbb{L}^d[\Lambda_L]$ by contracting all the edges that are strictly contained in $\partial^{top}\Lambda_L$ or $\partial^{bot}\Lambda_L$. Denote the two contracted vertices by v_t and v_b respectively.
The cycle space of $\mathbb{L}^d$ is generated by the set of basic 4-cycles denoted by $\mathcal{C}$. If we take the subgraph, induced by the box $\mathbb{L}^d[\Lambda_L] \subset \mathbb{L}^d$, its cycle space $\mathcal{C}_{\Lambda_L}$ is generated by all the four cycles that are strictly contained in the box. Let $\mathcal{C}_H$ be the set of cycles that are obtained from $\mathcal{C}_{\Lambda_L}$ by the edge contraction. The set $\mathcal{C}_H$ generates the cylce space of H.
Now take a minimal separating set of edges in H between v_b and v_t. Let $\Pi = \Pi_1 \cup \Pi_2$ be an arbitrary partition. From Lemma 2.3.3 follows that there is a cycle $O \in \mathcal{C}_H$ such that $O \cap \Pi_1 \neq \emptyset$ and $O \cap \Pi_2 \neq \emptyset$. Take edges $e_1 \in O \cap \Pi_1$ and $e_2 \in O \cap \Pi_2$ from each of these intersections. Since O is either a basic 4-cycle or a 3-cycle, the distance is $d_1(e_1, e_2) \leq 1$ and hence e_1 and e_2 are adjacent in G_E. Therefore, Π is connected in G_E. $\square$

The next lemma will be used to bound the number of G_E-connected edge sets with one fixed edge.

Lemma 2.3.6. *Let $G = (V, E)$ be the d-dimensional Euclidean lattice. There exists a constant $C > 0$, such that $\forall e \in E$:*

$$\left|\{A \subset E : e \in A, |A| = n, A \text{ is } G_E - \text{connected}\}\right| \leq C^n$$

Proof. First recall the definition of the edge graph $G_E = (V_E, E_E)$. We can therefore rewrite the set.

$$\{A \subset E : e \in A, |A| = n, G_E - connected\} = \{A \subset V_E : e \in A, |A| = n, \text{ connected}\} =: \mathcal{A}_n$$

The problem is therefore to bound the number of lattice animals of size n $\mathcal{A}_n$ in G_E. We will continue by showing that for every $A \in \mathcal{A}_n$ there exists closed walk (sequence of vertices that are not necessarily disjoint) γ starting and ending at e, visiting each vertex of A and having length $|\gamma| = 2(n-1)$. Then, we can bound the number of lattice animals by counting the number of walks.
Let $A \in \mathcal{A}_n$ and $e \in A$ be arbitrary. Since A is connected, it contains a spanning tree $T \subset A$. Either by induction, one can prove that there is a walk γ starting and ending at e and having length $|\gamma| = 2(n-1)$. (Or explore the tree in depth first order starting at e.) This proves that there is a surjective function

$$f : \{\gamma \text{ closed walk in } \mathbb{Z}^d : |\gamma| = 2(n-1), e \in \gamma, \text{visiting n vertices}\} \to \mathcal{A}_n.$$

Since both are finite sets, we can upper bound $|\mathcal{A}_n| \leq \left|f^{-1}(\mathcal{A}_n)\right|$. Furthermore, we can upper bound the number of these walks of length $2(n-1)$ by eliminating the restrictions

$$\left|\{\gamma \text{ closed walk in } \mathbb{L}^d : |\gamma| = 2(n-1), \text{ starting at e, visiting n vertices}\}\right| \leq \\ \left|\{\gamma \text{ walk in } \mathbb{L}^d : |\gamma| = 2(n-1), \text{ starting at e}\}\right|.$$

Note that every vertex $e \in G_E$ has the same degree and denote it by $deg(e) := deg_{G_E}(e)$. To bound the number of walks of length $2(n-1)$ from e, simply observe that at each step of the walk, one has exactly $deg(e)$ possible directions to go. Therefore, we get $\left|\{\gamma \text{ walk in } \mathbb{L}^d : |\gamma| = 2(n-1), \text{ starting at e}\}\right| \leq deg(e)^{2(n-1)}$.
It remains to calculate $deg(e)$ in G_E. Instead of calculating it explicitly, we give a constant upper bound. Recall that $f \in V_E$ is adjacent to e if $d_1(e, f) \leq 1$. The number of vertices $v \in V$ with $d_1(e, v) \leq 1$ is $4 \cdot 3^{d-1}$. Each of the vertices is adjacent to $2d$ edges in G. Therefore, the degree is at most $deg(e) \leq 4 \cdot 3^{d-1} \cdot 2d$.
Combining these calculations, the claim of the lemma follows with $c := (8d \cdot 3^{d-1})^2$. □

The next lemma is important for the renormalization in Section 2.5. The problem is that in the new lattice the edges are not independent. They are only independent of each other if they are far enough apart from each other. Let $Y = \{Y_e, e \in E(\mathbb{L}^d)\}$ be a family of random variables. Y is called n**-dependent** if any two sub families $\{Y_e, e \in A\}$ and $\{Y_e, e \in A'\}$ with $A, A' \subset E$ are independent, whenever $d_1(e, f) > n$ for all $e \in A$ and

$f \in A'$.
For a set $A \subset E$ of independent edges with $|A| = m$, we have $\mathbb{P}_p(A \text{ is closed}) \leq (1-p)^m$. We need a similar bound for a n-dependent percolation.

Lemma 2.3.7. *Let $A \subset E$ be a subset of edges in a n-dependent bond percolation η with $|A| = m$ and $\forall e \in E : \mathbb{P}_p(\eta(e) = 1) \geq p$. Then there is a constant $c > 0$, such that*

$$\mathbb{P}_p(A \textit{ is closed}) \leq (1-p)^{c \cdot m}.$$

Proof. From A we construct a subset $\tilde{A} \subset A$ that is obtained from A by starting at any edge $e_0 := e$ and deleting all the edges $f \in A$ with $d(f, e_0) \leq n$. Take a new edge $e_1 \in A$, that was not deleted and repeat this procedure. Around each vertex v there are $(2 \cdot n + 1)^d$ vertices with distance less or equal to n. Each vertex is adjoint to at most $2d$ edges. Therefore, we delete at most $c^{-1} := (2 \cdot n + 1)^{(d-1)} \cdot (2 \cdot n + 2) \cdot 2d$ edges in each step (with a lot of double counting). We repeat this procedure until for all $e, f \in \tilde{A}$ we have $d(e, f) > n$. As a result, the states of the edges in $\tilde{A}$ are independent and $\left|\tilde{A}\right| \geq c \cdot m$. Therefore, we can bound the probability by saying that if all the edges in A are closed, this implies that all the edges in $\tilde{A}$ are closed.

$$\mathbb{P}_p(A \text{ is closed}) \leq \mathbb{P}_p(\tilde{A} \text{ is closed}) \leq (1-p)^{c \cdot m}$$

□

2.4 Perturbative Results

From now on, the goal is to prove the positivity of surface tension and the exponential decay of the volume of finite clusters. We will first prove them in a perturbative way, i.e. there is a $p_0 < 1$, such that the desired property holds for all $p \in [p_0, 1]$. Afterwards, we will extend these results to any $p > p_c$ by applying a static renormalization in Section 2.5. The perturbative results are proven similar to Griffiths-Peierls argument by a kind of energy versus entropy estimate. This means bounding the probability of some event and counting the number of configurations that satisfy the event.

2.4.1 Positivity of surface tension

In this section, we will prove that the probability of disconnection between the bottom and the top of a box Λ_L decays exponentially with surface order L^{d-1}. This implies the positivity of the surface tension, which we will not introduce here. For the formal definition and applications of the surface tension, the reader is referred to [Pis96].
Recall the definition of bottom and top of the box by $\partial^{bot}\Lambda_L := \{-L, ..., L\}^{d-1} \times \{-L\}$

$\partial^{top}\Lambda_L := \{-L, ..., L\}^{d-1} \times \{L\}$. Our goal is to upper bound the probability of the event of disconnection between bottom and top $\{\partial^{bot}\Lambda_L \overset{\Lambda_L}{\not\leftrightarrow} \partial^{top}\Lambda_L\}$.
The proof is based on the intuition that a certain *separating surface* of closed edges in the box implies that there is no connection between top and bottom. The number of edges in such a surface will be shown to be of order L^{d-1}, and we can therefore bound the probability that all the edges are closed. To obtain a bound on the connection probability between top and bottom, we then sum over all possible separating surfaces. See Figure 6 for a random configuration that satisfies $\{\partial^{bot}\Lambda_{10} \overset{\Lambda_{10}}{\not\leftrightarrow} \partial^{top}\Lambda_{10}\}$ in two dimensions. This motivates the definition of a separating surface.

Definition 2.4.1. *Let $L \geq 1$. We call a subset of edges $\Pi \subset E$ a* ***separating surface****, if Π is a minimal separating set of edges between $\partial^{bot}\Lambda_L$ and $\partial^{top}\Lambda_L$ in the induced subgraph $\mathbb{L}^d[\Lambda_L]$.*

Observe that $\partial^{bot}\Lambda_L$ is not connected to $\partial^{top}\Lambda_L$ in Λ_L if and only if there is a separating surface Π with all edges closed. We first prove that there is a $p_0 < 1$ large enough, such that the disconnection probability decays for all $p \geq p_0$ and then extend the result to the whole supercritical phase by renormalization in Section 2.5. To be able to apply renormalization, we prove it for a n-dependent bond percolation.

Lemma 2.4.2. *Let $n \in \mathbb{N}$. For a n-dependent bond percolation η on $\mathbb{L}^d$, there is a $p_0 < 1$, such that if the marginals satisfy $\forall e \in E : \mathbb{P}_p(\eta(e) = 1) \geq p_0$ then there exists a constant $c > 0$, such that*

$$\mathbb{P}_p(\partial^{bot}\Lambda_L \overset{\Lambda_L}{\not\leftrightarrow} \partial^{top}\Lambda_L) \leq e^{-cL^{d-1}} \qquad \forall L \geq 1 \tag{20}$$

Proof. We bound the probability of disconnection between top and bottom of the box by summing over the size of possible separating surfaces, weighted by their probability. The sum starts at $(2L+1)^{d-1}$, since any smaller edge set in the box cannot separate top and bottom. Hence, we get as an upper bound

$$\begin{aligned}\mathbb{P}_p(\partial^{bot}\Lambda_L \overset{\Lambda_L}{\not\leftrightarrow} \partial^{top}\Lambda_L) &= \mathbb{P}_p(\exists \text{ separating surface } \Pi \text{ that is closed}) \\ &\leq \sum_{m \geq (2L+1)^{d-1}} \mathbb{P}_p(\Pi \subset E : |\Pi| = m, \Pi \text{ is a separating surface and closed}). \end{aligned}\tag{21}$$

In order to upper bound the number of possible separating surfaces (apply Lemma 2.3.6), we need a separating surface to be G_E-connected. This is proven in Lemma 2.3.5. Thus, for $m \in \mathbb{N}$ and $e \in E$ there exists a constant $c_1 > 0$, such that

$$\big|\{\Pi \subset E : e \in \Pi, |\Pi| = m, \Pi \text{ is } G_E - connected\}\big| \leq c_1^m. \tag{22}$$

Let us therefore consider the probability of a closed separating surface of fixed size m. The bound is obtained by fixing an edge $e = \{u, u + e_d\}$, with $u \in \{0\}^{d-1} \times \{-L, ..., L-1\}$ and summing over all G_E-connected sets Π of size m that contain e. Then we sum over all translations in the e_d direction that are possibly still contained in the box.
From Lemma 2.3.7 we obtain that the probability that such an edge set of size m is closed in a n-dependent percolation can be bound by $\mathbb{P}_p(\Pi \text{ is closed}) \leq (1-p)^{c_2 m}$ with a constant $c_2 > 0$. Combining these bounds on the probability and the set size, we obtain

$$\begin{aligned}
&\mathbb{P}_p(\Pi \subset E : \Pi \text{ closed separating surface}, |\Pi| = m) \\
&\qquad \leq \sum_{\substack{e=\{u,u+e_d\} \\ u \in \{0\}^{d-1} \times \{-L,...,L-1\}}} \sum_{\substack{|\Pi|=m, e \in \Pi \\ \Pi \text{ is } G_E\text{-connected}}} \mathbb{P}_p(\Pi \text{ is closed}) \\
&\qquad \leq 2L(1-p)^{c_2 m} \big|\{\Pi : e \in \Pi, |\Pi| = m, \Pi \text{ is } G_E - connected\}\big| \leq 2L(1-p)^{c_2 m} c_1^m \\
&\qquad = e^{m \log(c_1(1-p)^{c_2}) + \log(2L)} \leq e^{-c_3 m}.
\end{aligned} \tag{23}$$

For the last inequality, we choose p_0, such that $c_3 := -(\log(c_1(1-p_0)^{c_2}) + 1) > 0$. To conclude the proof, we bound the probability inside the sum in Equation (21) by the result of Equation (23) and get

$$\mathbb{P}_p(\partial^{bot}\Lambda_L \overset{\Lambda_L}{\not\leftrightarrow} \partial^{top}\Lambda_L) \leq \sum_{m \geq (2L+1)^{d-1}} e^{-c_3 m} \leq \frac{e^{-c_3(2L+1)^{d-1}}}{1 - e^{-c_3}} \leq e^{-cL^{d-1}}. \tag{24}$$

Where in the last inequality, we choose $c := c_3 \cdot 2^{d-1} > 0$.

□

2.4.2 Exponential decay of the volume of finite clusters

In this section, we establish the result that for $p < 1$ large enough, the probability of clusters of size $n \leq |C| < \infty$ decays exponentially with surface order $n^{(d-1)/d}$.
The proof of the following lemma is similar in style to the previous section. This means bounding the probability of an event and counting the possible configurations in which this event occurs. Here, the intuition is that a finite cluster of size n has an external boundary of order $n^{(d-1)/d}$. This can be used to bound the probability, combined with summing over possible external boundaries. We also state this perturbative result for a n-dependent percolation, such that we can apply renormalization in the following section.

Lemma 2.4.3. *Let $n \in \mathbb{N}$. For a n-dependent bond percolation η on $\mathbb{L}^d$, there is a $p_0 < 1$, such that if $\forall e \in E : \mathbb{P}_p(\eta(e) = 1) \geq p_0$ there exists a constant $c > 0$, such that*

$$\forall p \geq p_0, \forall n \geq 1 : \mathbb{P}_p(n \leq |C| < \infty) \leq e^{-cn^{(d-1)/d}}.$$

Proof. If C is finite and connected, we use Lemma 2.3.1 to get that there is a constant $c_1 > 0$ such that $|\Delta_{ext}C| \geq c_1|C|^{(d-1)/d}$. Therefore, we can rewrite the probability by summing over external boundaries of fixed size. Since the cluster is finite, its external boundary is also finite and we can upper bound

$$\mathbb{P}_p(n \leq |C| < \infty) \leq \mathbb{P}_p(|\Delta_{ext}C| \geq c_1 n^{(d-1)/d}, |C| < \infty) \leq \sum_{m \geq c_1 n^{(d-1)/d}} \mathbb{P}_p(|\Delta_{ext}C| = m). \tag{25}$$

Let us first bound the probability $\mathbb{P}_p(|\Delta_{ext}C| = m)$ for a fixed m. We sum over all possible external boundaries, fixed at one edge. Then, we sum over all shifts by m. Observe that if we shift the external boundary by more than m, the origin is not contained anymore. Furthermore, the probability $\mathbb{P}_p(\Delta_{ext}C = A)$ with $|A| = m$ can be bound by $(1-p)^{(c_3 \cdot m)}$ by Lemma 2.3.7 with $c_3 > 0$.

$$\begin{aligned}\mathbb{P}_p(|\Delta_{ext}C_0| = m) &\leq \sum_{\substack{e \in E \\ d(e,0) \leq m}} \sum_{\substack{A \subset E \\ |A| = m \\ G_E - connected \\ e \in A}} \mathbb{P}_p(\Delta_{ext}C_0 = A) \\ &\leq \sum_{\substack{e \in E \\ d(e,0) \leq m}} (1-p)^{c_3 m} \big|\{A \subset E : e \in A, |A| = m, A \text{ is } G_E - connected\}\big| \leq (2m+1)^d c_2^m (1-p)^{c_3 m}\end{aligned} \tag{26}$$

In the last step, we used Lemma 2.3.6 to bound the number of possibilities of obtaining such an edge set with a constant $c_2 > 0$.
If we now choose $p_0 < 1$, such that $c_4 := -(\log(c_2(1-p_0)^{c_3}) + 1) > 0$, we get:

$$\mathbb{P}_p(|\Delta_{ext}C_0| = m) \leq e^{-c_4 m}$$

Finally, we obtain the result by summing over all m to infinity and using the value of a geometric sum, where the last inequality holds for $c := c_4 \cdot c_1 > 0$.

$$\sum_{m \geq c_1 n^{(d-1)/d}} \mathbb{P}_p(|\Delta_{ext}| = m) \leq \sum_{m \geq c_1 n^{(d-1)/d}} e^{-c_4 m} \leq \frac{e^{-c_4 c_1 n^{(d-1)/d}}}{1 - e^{-c_4}} \leq e^{-c n^{(d-1)/d}}$$

□

2.5 Renormalization

The next goal is to extend the perturbative results from the previous section to all $p > p_c$. In order to do so, we use a technique, called *static renormalization*.[2] The idea is to group boxes of vertices $\Lambda_k(u) \subset \mathbb{Z}^d$ together to a new vertex in the renormalized lattice $u \in 2k\mathbb{Z}^d$. For an illustration of renormalizing the two-dimensional lattice $\mathbb{L}^2$ with $k = 2$ see Figure 7. The next step of renormalization is to define a percolation on the renormalized lattice.

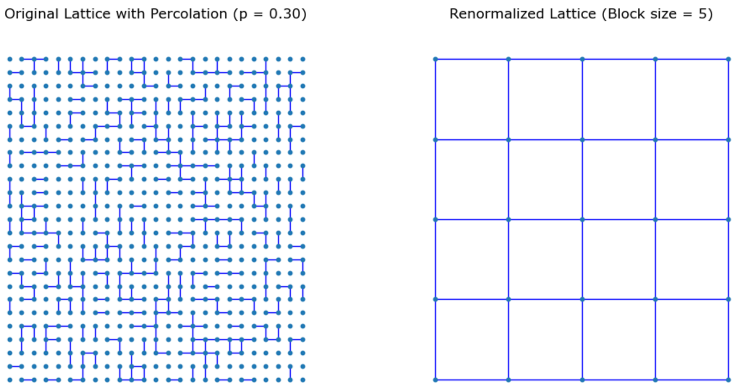

Figure 7: Illustration of renormalizing the lattice in two dimensions by grouping together boxes with 25 vertices.

There are different possibilities to do so. We choose the following approach by using the local uniqueness event, introduced in Section 2.2. We define that an edge between two vertices u, v in the new lattice is open if and only if $Unique(k)$ occurs centered around u and v. The next step of renormalization is to choose k large enough, such that the perturbative results from the previous section apply to the new percolation.

Definition 2.5.1. *For $k \in \mathbb{N}$ we define the **renormalized lattice** as the graph $G_k := (V_k, E_k)$ with $V_k := 2k\mathbb{Z}^d$ and $E_k := \{\{2kx, 2ky\} : x, y \in \mathbb{Z}^d, d_1(x, y) = 1\}$.*

It is important to note that the new renormalized lattice is also isomorphic to $\mathbb{L}^d$. For $u \in V_k$ recall that $U_k(u)$ is the event that $Unique(k)$ occurs centered around u.

Definition 2.5.2. *Define the **renormalized configuration** as a new bond percolation η on G_k by setting $\forall uv \in E_k : \eta(uv) := \mathbf{1}_{U_k(u) \cap U_k(v)}$.*

Recall the definition of n-dependence and observe that η is 15-dependent. By Theorem 1.3.1 we get that for any $p > p_c$ and $\delta > 0$, there is a $k \geq 1$, such that $\mathbb{P}_p(\eta(e) = 1) \geq 1 - 2\delta$ for all $e \in E_k$. An open path in η means that there is a big cluster locally around each of the vertices and these clusters are connected to each other. We now use this renormalized lattice to extend the perturbative results from Section 2.4 to the whole supercritical phase.

[2]For dynamic and static renormalitazion in percolation theory see [Gri99].

2.5.1 Positivity of surface tension

We prove the Theorem 1.3.4 by extending the perturbative Lemma 2.4.2 to the whole supercritical phase.

Proof of Theorem 1.3.4. Let $p_0 < 1$ be from the perturbative Lemma 2.4.2. Using Theorem 1.3.1, choose k large enough, such that $\mathbb{P}_p(\eta(e) = 1) \geq p_0$.
Define the box $\bar{\Lambda}_L$ in the renormalized lattice by $\bar{\Lambda}_L := \{u \in V_k : (\Lambda_{2k}(u) \setminus \Lambda_k(u)) \cap \Lambda_L \neq \emptyset\}$. The faces $\partial^{top}\bar{\Lambda}_L$ and $\partial^{bot}\bar{\Lambda}_L$ are defined to be the vertices in $\bar{\Lambda}_L$ with the maximal and minimal coordinates in e_d direction, respectively.
Observe that the event $\{\partial^{top}\bar{\Lambda}_L k \overset{\bar{\Lambda}_L}{\longleftrightarrow} \partial^{bot}\bar{\Lambda}_L\}$ implies that $\{\partial^{top}\Lambda_L \overset{\Lambda_L}{\longleftrightarrow} \partial^{bot}\Lambda_L\}$ since an open edge in the renormalized lattice implies that the local clusters around two boxes in the original lattice are connected. An open path between bottom and top in the renormalized lattice corresponds to a crossing cluster in the original lattice. It is also important to note that the box $\bar{\Lambda}_L$ is thickened in the e_d direction to ensure that the crossing cluster around any vertex u in the bottom or top of $\bar{\Lambda}_L$ also intersects the bottom or the top of Λ_L.
We use this to bound the probability in the original lattice by the event into the renormalized lattice.

$$\mathbb{P}_p(\partial^{top}\Lambda_L \overset{\Lambda_L}{\not\longleftrightarrow} \partial^{bot}\Lambda_L) \leq \mathbb{P}_p(\partial^{top}\bar{\Lambda}_L \overset{\bar{\Lambda}_L}{\not\longleftrightarrow} \partial^{bot}\bar{\Lambda}_L) \tag{27}$$

By applying Lemma 2.4.2 to the renormalized lattice and η, we get that there is a constant $c > 0$, such that:

$$\mathbb{P}_p(\partial^{top}\bar{\Lambda}_L \overset{\bar{\Lambda}_L}{\not\longleftrightarrow} \partial^{bot}\bar{\Lambda}_L) \leq e^{-cL^{d-1}} \qquad \forall L \geq 1 \tag{28}$$

Combining this result with Equation (27) the statement of the theorem follows. $\square$

2.5.2 Exponential decay of the volume of finite clusters

Next, we prove the Theorem 1.3.3, which extends the perturbative Lemma 2.4.3 to $p > p_c$.

Proof of Theorem 1.3.3. Let $p_0 < 1$ be from the perturbative Lemma 2.4.3. With Theorem 1.3.1, choose k be large enough, such that $\mathbb{P}_p(\eta(e) = 1) \geq p_0$. Note that if an edge in G_k is open, the crossing clusters around u and v are connected to each other, as a crossing cluster of $\Lambda_{7k}(u) \setminus \Lambda_k(u)$ crosses the uniqueness zone of v being $\Lambda_{6k}(v) \setminus \Lambda_{3k}(v)$.
Let C be the cluster of the origin. Define the corresponding cluster in the renormalized lattice by

$$\bar{C} := \{u \in V_k : C \cap \Lambda_k(u) \neq \emptyset\}.$$

We can bound the size of C using $\bar{C}$ by $|C| \leq |\bar{C}||\Lambda_k|$. Therefore, we get

$$\mathbb{P}_p(n \leq |C| < \infty) \leq \mathbb{P}_p(\frac{n}{|\Lambda_k|} \leq |\bar{C}| < \infty) \tag{29}$$

From Lemma 2.4.3 we get that there is a constant $c > 0$ such that

$$\mathbb{P}_p(n/|\Lambda_k| \leq \left|\bar{C}\right| < \infty) \leq e^{-(c/|\Lambda_k|)^{\frac{d-1}{d}} n^{(d-1)/d}}$$

Since k and d are constants, this proves the statement. $\square$

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