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Evolutionary dynamics in the voting game

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Abstract. Voter participation is immense but theoretically doubtful because there exists cost of voting and the probability of casting the deciding ballot is low. Game theoretic models (Palfrey and Rosenthal, 1985) confirm this paradox of voting. Individual optimization in the voting game is problematic with respect to the rationality and information requirements of traditional game theory. Therefore in this paper a non-optimizing but learning individual is considered. By individual learning the adjustment processes and equilibria of voter turnout are determined. Voters are able to learn to participate and substantial voter turnout is possible.

1. Introduction

Because the main assumption in public choice is methodological individualism and voting is the main political instrument of an individual, explaining voter participation of the individual voter is one of the basic challenges for social scientists. If the decision is between two alternatives, parties or candidates by simple plurality, the rational voter hypothesis (Downs, 1957) states voter participation whenever the expected benefits from voting exceed the cost of voting. Because the probability of casting the deciding ballot is very low no voting is expected by the theory. But in reality there is substantial voter turnout and this contradiction is called the "voting paradox" (Mueller, 1987). The assumption that the voting decision of the analyzed voter has no influence on the probability that this vote is decisive is also made by Tullock (1967), Goodin and Roberts (1975) and Ferejohn and Fiorina (1974). Tullock (1967) considers voters with a taste for voting and Gooding and Roberts (1975) describe ethical voting. The potential voters of Ferejohn and Fiorina (1974) use the minimax-regret decision rule, which has insuperable shortcomings (Mueller, 1989).

Each voter's strategy to vote or not to vote depends on his expectations of the other voters strategies. If each voter decides not to vote because his vote has no chance to be decisive, no one would vote and any one voter could

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determine the election by voting. The expected probability of zero that the vote is decisive is not appropriate to the ex post probability of one.

Therefore Palfrey and Rosenthal (1983, 1985) and Ledyard (1984) introduce rational expectations and thus the endogenous calculation of the decisions to vote and the probability of decisiveness. The suitable game theoretic solution concept is the Nash equilibrium. But game theory does not solve the voting paradox either, as long as traditional solutions like Nash equilibrium are used. Among other things[1] the problems are the bounded rationality and the incomplete information of voters. In our setting we replace the rational individual by an adaptive individual in a survival approach (Selten, 1991; Magee, 1993). No voter knows a priori optimal strategies but the voters learn voting strategies which are of high economic fitness.

The paper is organized as follows. The next part is a brief summary of the Palfrey and Rosenthal (1983) voting game. In Section 3 the survival approach of learning is introduced and the basic elements of evolutionary game theory are reviewed. Section 4 shows voter turnout in the evolutionary voting game. The paper ends with a short summary and an outlook on further research. All proofs of this paper are located in the Appendix.

2. The voting game

Let $i \in \{1,...,\bar{N}\}$ be an element of the index set of players who can vote among alternative A_1 and A_2. They can either vote for A_1, vote for A_2 or they can stay at home and let the others vote. The additive utility function of voter i is U_i. The participation costs for voter i are $\tilde{c}_i$. For all voters with $U_i(A_1) > U_i(A_2)$ the strategy vote for A_1 dominates the strategy vote for A_2. Let $T_1 = \{1,...,M\}$ be the index set of voters with $U_i(A_1) > U_i(A_2)$, $T_2 = \{M+1,...,M+N\}$ the index set of voters who prefer A_2 and $T_3 = \{M+N+1,...,\bar{N}\}$ the set of indifferent voters. If voting costs are non-negative the strategy "not participate" dominates for voters in T_3; so they are not considered in this investigation.

Therefore only two pure strategies $s_i \in \{1,0\}$, participate or not, are considered. Participate means voting for A_j for all voters $i \in T_j$. The mixed strategy is p, i.e. to participate with probability p.

Standardization leads to $U_i(A_j) = 1$ and $U_i(A_{k \neq j}) = 0 \; \forall_i \in T_j$. Let $c_i = U_i(\tilde{c}_i)$ be the opportunity costs of voting. We consider the folowing voting rule: A simple plurality decides the ballot and a tie is broken randomly with a probability 1/2. Thus the revenue rule W_i for a voter in T_i is

$$W_i(s_1, ..., s_{M+N}) = \begin{cases} 0 & \text{for } \Sigma_{k \in T_i} s_k < \Sigma_{k \in T_j} s_k \\ 1 & \text{for } \Sigma_{k \in T_i} s_k > \Sigma_{k \in T_j} s_k \\ 1/2 & \text{for } \Sigma_{k \in T_i} s_k = \Sigma_{k \in T_j} s_k \end{cases}, j \neq i.$$

If alternative A_i wins, the revenue is 1 for a voter in T_i and 0 for a voter in T_j, $j \neq i$. If the election ends in a tie, the revenue for all voters is 1/2. Therefore the payoff functions π_i are

$$\pi_i = W_1 - s_i c_i \ \forall i \in \ T_1 \text{ and } \pi_i = 1 - W_1 - s_i c_i \ \forall i \in \ T_2.$$

Palfrey and Rosenthal (1983) consider the special case in which all voters i have the same cost of voting $c_i = c$. They prove the following theorem:

Theorem 2.1. (Palfrey and Rosenthal, 1983)

1. If $1 > c > 1/2$, the Nash equilibrium is $s_i = 0 \ \forall i \in T_1 \cup T_2$.
2. If $0 < c < 1/2$ and $M = N$, the Nash equilibrium is $s_i = 1 \ \forall i \in T_1 \cup T_2$.
3. If $M \neq N$ and mixed strategies $q \in (0,1) \ \forall i \in T_1$ and $r \in (0,1) \ \forall i \in T_2$, a necessary and sufficient condition for a so-called quasi-symmetric Nash equilibrium is

$$2c = \sum_{k=0}^{\min(M-1,N)} \binom{N}{k} \binom{M-1}{k} q^k (1-q)^{(M-1-k)} r^k (1-r)^{N-k}$$
$$+ \sum_{k=0}^{\min(M-1,N-1)} \binom{N}{k+1} \binom{M-1}{k} q^k (1-q)^{(M-1-k)} r^{(k+1)} (1-r)^{N-1-k}$$

and

$$2c = \sum_{k=0}^{\min(M,N-1)} \binom{M}{k} \binom{N-1}{k} q^k (1-q)^{(M-k)} r^k (1-r)^{N-1-k}$$
$$+ \sum_{k=0}^{\min(M-1,N-1)} \binom{M}{k+1} \binom{N-1}{k} q^k (1-q)^{(M-1-k)} r^k (1-r)^{N-1-k}.$$

If voting participation costs more than 1/2, in the Nash equilibrium no one votes. One vote can change the ballot merely from a defeat to a tie or from a tie to a win. Therefore the utility of an additional vote is at most 1/2 which is smaller than the cost of voting and no voting is dominant.

If the sizes of both groups are equal and voting costs are less than 1/2, the unique pure Nash equilibrium is everybody voting. Whenever the group sizes are different but large and c suitable, there are two (quasi-symmetric) Nash equilibria, one with essentially no one voting and one with essentially everyone voting. In a quasi-symmetric Nash equilibrium every voter in the same team applies the same (mixed) strategy. Furthermore there are other Nash equilibria of the following type: All voters in one group are divided into two subgroups, one in which voters definitely abstain and one in which voters definitely vote.

To summarize, in the voting game with complete information and identical voting costs for all voters there are numerous Nash equilibria. Except for the case of equal group sizes all Nash equilibria are mixed, i.e. there are voters who decide to participate by a random number or a coin toss by example.

In a second step Palfrey and Rosenthal (1985) drop out the assumption of identical voting costs and consider that the voting cost of every individual on a given team T_i is drawn from a common distribution for that team. Every potential voter knows these distributions, his voting costs and all "relevant" parameters. The result is that there exists a Nash equilibrium in pure strategies, which is described by a pair of critical costs $(c^*{}_1, c^*{}_2)$. Every voter in team T_i whose cost of voting is less than $c^*{}_i$ will participate and every voter in team T_i whose cost of voting is greater than $c^*{}_i$ will not vote. Furthermore, as N and M simultaneously increases, $c^*{}_i$ and thus voter turnout approaches zero. Voters with positive voting costs will abstain.

3. Adaptive learning and evolutionary game theory

The problem for the rational voter to vote or not to vote is not yet solved. If he tries to maximize his expected utility and supposes a probability, that his vote is decisive, how to calculate it? If he knows a Nash equilibrium, do the other voters know it too, and what should he do, if they don't? If there are multiple equilibria, which is selected? Furthermore, the quasi-symmetric Nash equilibria are not strict, i.e. an individual player receives the same payoffs regardless of whether he plays the Nash or an arbitrary other strategy. Why shall he play Nash? The political science is controversial in this respect and there is no agreement, which decision rule or strategy is correct.

Furthermore, bounded rationality prevents the voter to vote as assumed in the theories. The voting situation is complicated and some or maybe all voters cannot calculate best responses to their environment (Herzberg and Wilson, 1988). Hence, we introduce the learning individual. The voter does not solve decision problems, but the voter learns how to operate. Behavioral psychology (Schwartz and Lacey, 1982) described the learning process in a complex situation like the voting game as follows: Actions followed by pleasure increase in frequency, i.e. they are learned, actions followed by punishment decrease in frequency. Therefore, people learn which strategies are good by observing what has worked well for them in the past and what has worked well for other people. Imitation is an important part of this learning process because successful behavior tends to be imitated. Furthermore, successful strategies are taught (Selten, 1991; Kandori, Mailath and Rob, 1993).

Empirical evidence speaks well for that the voter does not know the payoffs for the other voters, the size of the preference groups T_i and the voting cost structure. There are no common probability distributions on these variables. The voter only can compare his payoff to the average payoff for the other voters and we will take this ratio as proxy for the social position of the voter. The voters experience is his social position.

Sometimes a voter changes his strategy, just by accident or because he tries to find another better strategy by trial and error. The new strategy is deemed more successful than the old one and the new strategy is learned if the voter's social position is increased. Therefore just one trial shows whether the new strategy or the status quo strategy is right to apply further. In this view frequency of voting is unimportant for learning, just as a child, for example, is not able to calculate which temperature is too hot for his hand, but touching a hot hearth only once is enough to learn that this temperature is too hot for his hand. Therefore the individual learns within one election.

This individual learning behavior is the motor of the adjustment process and describes the main dynamics in the voting game. Starting with an arbitrary situation the learning dynamics show the further development of voter participation. If every voter tries new strategies with positive probabilities, the evolutionary learning process does not end until all voters play optimal strategies. The identification of equilibria is possible and although there is no rational individual, the learning process optimizes the evolutionary fitness.

Both the individual learning process and the identification of equilibria are valid for non repeated voting as, e.g., U.S. presidential elections. Statements about the direction of the voter turnout trajectory and the stability of the equilibria are possible though there are many voters. But because many voters may need much time for gradually learning or may learn at the same time, our investigation will not answer the question if it is possible to reach the equilibrium is reasonable time, but solve the problem if and which equilibria exist and how voter participation evolves. The important difference of this learning approach used in evolutionary game theory in contrast to the Nash equilibrium in conventional game theory is as follows: In a Nash equilibrium all strategies are best responses as long as the strategies of the other players remain unchanged. In an evolutionary equilibrium (van Damme, 1987) a strategy is a relative best response. Whenever a voter changes his strategy both his and the opponents payoffs change. Suppose that there is the costly, i.e. payoff decreasing, opportunity for a voter to do the opponents harm. If the decline of opponent's payoffs is larger than the decline of the voter's payoff, his situation with regard to the opponent is improved. A relative best response leads to the individually best situation of relative payoffs considering the altered payoffs for the opponents. Because absolute success may be relative lack of success, the conceptions are different.

This view of economic evolution by individually learning is equivalent to the conception of population dynamics in biological game theory (van Damme, 1987; Friedman, 1991). The learning process in human behavior fulfills the task of the selection process in nature. The equalization of economic (social) evolution and population dynamics is not straight forward (Hirshleifer, 1977; Selten, 1991), but the replicator dynamics used in biology is often (and as well here) seen as a model of learning with bounded rationality (Dekel and Scotchmer, 1992; Kuan and White, 1994).

For a formal investigation of the learning approach to the voting game, we look at a situation with $M + N$ voters with $M + N$ strategy sets which consist of pure strategies $s_i \in \{1,0\}$ and mixed strategies $q_i \in (0,1)$. We compare this situation to the situation in which only one voter, the deviant voter, changes his strategy. We define the following notation:

q_i:	strategy of voter $i \in T_1 \cup T_2$.
$\pi_i =$	$\pi_i(q_1, ..., q_{M+N})$: expected payoff for a voter $i \in T_1 \cup T_2$ with strategy q_i.
$\pi^A =$	$\pi^A(q_1, ..., q_{M+N})$: average expected payoff for a non deviant voter.
${}_iE =$	${}_iE(p)={}_iE(q_1, ..., q_{j-1}, p, q_{j+1}, ..., q_{M+N})$: expected revenue for a voter in T_i with strategy p.
$({}_i\pi^A),{}_i\pi$:	(average) expected payoff for a voter in T_i.
${}_i\pi^D =$	${}_i\pi^D(p)={}_i\pi^D(q_1, ..., q_{j-1}, p, q_{j+1}, ..., q_{M+N})$: expected payoff for a deviant $j \in T_i$ with strategy $p \in [0, 1]$.

Standard concepts in evolutionary game theory as evolutionary stable strategies ESS (Smith and Price, 1973) describes the long-run effects of selection for more successful strategies. But in those concepts, it is assumed that invidiuals are randomly matched, in pairs, to play a two-person game. Because the voting game considered here is not a two but a $M + N$ person game the adequate solution concept could be the symmetric evolutionary equilibrium SEE (Schaffer, 1989; Crawford, 1990). SEE is based on a playing the field approach, i.e. simultaneous play of more (here: $M + N$) than two identical players.

Definition 3.1. An M and N-tupel of strategies $(q,...,q)$ is a SEE_q, whenever

$$\pi_j^D(p) < \pi^A(q,...,q,p,q,...,q) \forall p \neq q \text{ and } \forall j, 1 \leq j \leq M+N.$$[2]

That is, in a population of $M + N$ players and of one deviant player, the SEE players do strictly better than the deviant player, no matter what the deviant player's strategy. This is an equilibrium in the following sense: If all individuals adopt the SEE strategy, then no player can reach higher (relative) payoffs by applying a deviant strategy. Therefore, there is no incentive to learn another (deviant) strategy and definition 3.1. characterizes an equilibrium. Whenever the mutant receives the same relative payoff $\pi^D - \pi^A$ as in the SEE, he may change his strategy respectively or retains his new one. Because such a situation is not an equilibrium, we require $<$ in the definition instead of $\leq$.

This is a reasonable definition only under the assumption of identical players or at least under the assumption that the non deviating mutant receives the same payoffs as the average. But the players of the voting game are not identical. To apply the survival approach to the voting game, we have to extend the SEE definition to an evolutionary voting equilibrium (EVE). The problem is as follows: There are two groups T_1 and T_2 of possibly different size. Without precise calculation we cannot say if a very small group has got a chance to beat the average. In such a situation the definition of equilibrium must be generalized. If a deviant player can increase his relative success by changing his strategy, he gets a positive incentive to learn the new strategy and to apply it in future. This does not depend on whether the relative success is positive or negative, because only the change in relative success is important.

We do not look at the relative success of the deviant to the players from the own group because preferences, costs and the election are anonymous, and therefore, no voter can realize his fitness in relation to his own group. Therefore, we take the payoff in relation to the total voting population (without deviant) as a proxy for the relative income, i.e. the social position or fitness of the voter. We say, that a voter is able to learn a new strategy if this strategy increases his social position.[3]

Definition 3.2. A voter i with strategy $q \in [0,1]$ is able to learn strategy $p \in [0,1]$ if

$$\begin{aligned} &\pi_i^D(p) - \pi^A(s_1,...,s_{i-1},p,s_{i+1},...,s_{M+N}) \\ \geq\ &\pi_i^D(q) - \pi^A(s_1,...,s_{i-1},q,s_{i+1},...,s_{M+N}). \end{aligned}$$

An EVE is reached whenever no other strategy can increase the relative payoff and thus no other strategy can be learned:

Definition 3.3. An $M + N$-tupel of strategies $(s_1^{\mathrm{EVE}},...,s_{M+N}^{\mathrm{EVE}})$ is an EVE of the voting game, whenever there is no deviant strategy p with a higher relative payoff than the equilibrium strategy:

$$\pi^D(p) - \pi^A\left(s_1^{\mathrm{EVE}}, ..., s_{j-1}^{\mathrm{EVE}}, p, s_{j+1}^{\mathrm{EVE}}, ..., s_{M+N}^{\mathrm{EVE}}\right)$$
$$< \pi^D\left(s_j^{\mathrm{EVE}}\right) - \pi^A\left(s_1^{\mathrm{EVE}}, ..., s_{j-1}^{\mathrm{EVE}}, s_j^{\mathrm{EVE}}, s_{j+1}^{\mathrm{EVE}}, ..., s_{M+N}^{\mathrm{EVE}}\right)$$
$$\forall p \neq s_j^{\mathrm{EVE}} \text{ and } \forall 1 \leq j \leq M + N.$$

This definition includes the SEE whenever the non deviating mutant receives the same payoff as the average, i.e. $\pi^D(s_j^{\mathrm{EVE}}) = \pi^A$. The SEE and the EVE definitions are analogous to the conception for the non-cooperative Nash solution in Shubik's (Shubik and Levitan, 1980) zero-sum "beat-the-average" game (Schaffer, 1988).

4. The evolutionary voting game

4.1. *EVE in pure strategies*

In this subsection we look at the voting game with pure strategies. We assume voters with only two strategies: the active voter with strategy s = 1, who participates in the election and the passive voter with strategy s = 0, who does not participate. A voter i with strategy $s \in \{0;1\}$ is able to learn strategy $1 - \pi_i^D(1-s) - \pi^A(s_1,...,s_{i-1}, 1-s,s_{i+1},...,s_{M+N}) \geq \pi^D{}_i(s) - \pi^A(s_1,...,s_{i-1},s,s_{i+1},...,s_{M+N})$ holds. A discussion of all possible cases leads to:

Lemma 4.1. Let M_a be the number of active voters in T_1 and N_a the number of active voters in T_2 and let $M_a \geq N_a$.

1. The passive voter of the winning team will not be able to learn the part of an active voter.
2. The active voter of the loosing party will be able to learn the part of a passive voter.
3. If $M_a > N_a$+1 holds, the active voter of the winning team will be able to learn the part of a passive, and the passive voter of the loosing team will not be able to learn the part of an active voter.
4. If $M_a = N_a$+1 holds, the active voter of the winning team will be able to learn the part of a passive voter, only if $c_i \geq N/(M+N-1)$ holds, and

the passive voter of the loosing team will be able to learn the part of an active voter, only if $c_i \leq M/(M+N-1)$ holds.

5. If $M_a = N_a$ holds, the active voter $i \in T_2$ will be able to learn the part of the passive voter, only if $c_i \geq M/(M+N-1)$ holds, and the passive voter $i \in T_1$ will be able to learn the part of an active voter, only if $c_i \leq N/(M+N-1)$ holds.

The interpretation of the cost condition for a voter i in Team T_j is as follows: The cost of voting c_i is compared with the ratio of voters in the opposing team to the whole voting population (without voter i). Because there is no variation in the voting costs of the non deviant voters, nothing but the revenues and c_i count. But the voters in the team T_j receive the same revenues as the voter i. Therefore the cost conditions are independent of these revenues and dependent on the revenues for the opposing team, which are at most one for all members of the opposing team. The dynamic movement of the voter turnout system depends on the cost conditions referred to in lemma 4.1. Therefore Figure 1 shows directional arrows with an index for the cost conditions that hold for a deviant, if the movement described by the arrow is possible. No index indicates that the movement is possible in general. Because every voter is a potential deviant and voting costs are arbitrary a lot of combinations of directional arrows and resulting movements of voter turnout exist. The characteristics of the adjustment process are as follows. The individual learning is unique. If an active voter is not able to learn the part of a passive voter, a passive voter from the same team and with the same cost of voting is able to learn the part of an active voter and vice versa. But the uniqueness is not valid for the aggregate level, i.e. for the whole voter population. On one hand, there are different teams, and voters in the teams may act differently. On the other hand, voters in the same team with different cost of voting may act differently. As a rule the evolution of voter turnout is ambiguous.

Whenever the election result is a two or more votes lead the evolutionary trend is to abstain and thus the evolution tends to the "voting paradox", i.e. no one votes. If the decision is a tie or a one vote's lead there are two trends: High voting costs speak well for increasing voter participation and vice versa. That is, voters are able to learn participation and this is the trend against the "voting paradox". The "voting paradox" is not an EVE if voting costs are less 1/2.[4] Whenever the costs of voting are low, an increasing voter turnout trajectory is possible (see trajectory 1 in Figure 2). Furthermore, cycles as the trajectory 2 are possible. Remarkable are the following rules: If the cost of voting is appropriate an active winner may increase his social position by staying away and a passive loser may decrease his social position by voting though their votes are decisive.

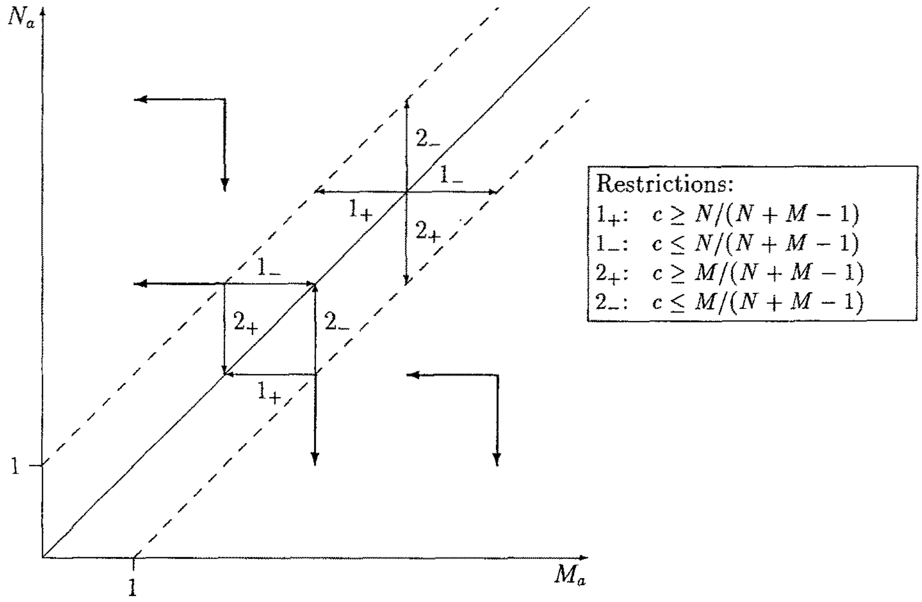

Figure 1. Possible movements in the voter turnout.

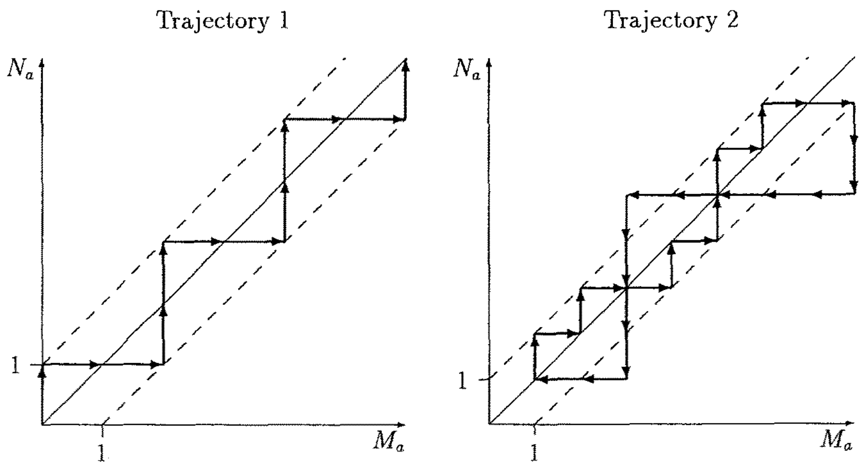

Figure 2. Possible trajectories.

An interesting phenomenon is shown in Figure 3. If group sizes are different and for all voting costs c_i, $i \in T_i \cup T_2$ the inequation $M/(M+N-1) < c_i < N/(M+N-1)$ holds (M is the group size of the smaller group), then the voter turnout will end in the long run with one active voter for the smaller group and no active voter for the larger group. If, for example, group sizes are 10 and 20,

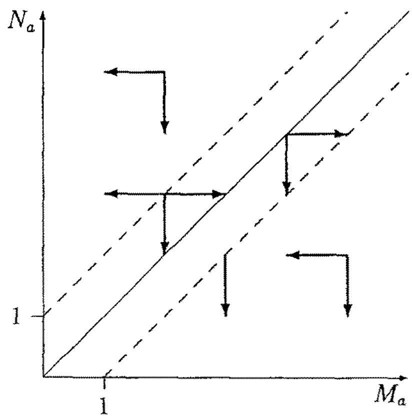

Figure 3. Different group sizes and voting costs near 1/2.

this phenomenon can be seen with voting costs $1/3 < c_i < 2/3$ for all voters i. Starting with an arbitrary turnout, we end up in the evolutionary equilibrium with one active voter in the smaller group and an ever losing larger group with no one voting. This result corresponds to Olson's (1965) results, that the behavior of small groups is completely different to the behavior of large one's because there is a systematic tendency for exploitation of the great by the small. A voter from the smaller team may take the cost of voting because the sum of external revenue in favour of the members of his own team is relatively low and the result is an increased social position whereas a voter from the larger team produces a high amount of external revenues and thus a decreased social position.

4.2. *EVE and Nash equilibrium*

In this subsection the Nash equilibrium in the game with identical cost of voting $c_i = c$ for all voters i is compared with the evolutionary voting equilibrium. It is shown that the evolutionary approach may lead to different results.

Lemma 4.2. If $1 > c > 1/2$, the symmetric Nash equilibrium with no one voting is not an EVE if $\max\{N/(M+N-1), M/(M+N-1)\} > c$.

In case the group sizes are different and the costs are not too high, there is an incentive for voters in the smaller group to go to the election. Although the net return is negative because of voting costs $c_i > 1/2$ and although they are relatively unsuccessful in comparison to their own team, they are relatively successful in comparison to the other large group. Therefore the social position increases.

Lemma 4.3. If $0 < c < 1/2$ and $M = N$ the Nash equilibrium with everyone voting is an EVE.

If the size of both groups are equal and voting costs are less than 1/2, everyone voting is not only a Nash equilibrium but also an EVE. The non-rational individual as well can learn to participate in the election. But two deviants appearing simultaneously in each of the T_i can break up the equilibrium. If both deviant players do not participate they do not only receive higher payoffs, but also increase their status. Therefore the EVE with everyone voting endures only one deviant, i.e. the degree of stability is less than 2.

Lemma 4.4. No quasi-symmetric Nash equilibrium (q,r), $0 < q,r < 1$, is an EVE.

The learning process of voters never leads to a quasi-symmetric Nash equilibrium, i.e. if there is no rational individual there is no quasi-symmetric Nash equilibrium. The proof of lemma 4.4. shows: In a quasi-symmetric Nash equilibrium deviants of the smaller groups tend to learn to participate, deviants of the larger groups tend to learn to stay away. There is exploitation of the great by the small (Olson, 1965).

4.3. *The EVE of the voting game*

Theorem 4.1. In the voting game with voting costs c_i the $M + N$-tuple of strategies $(q_1,\dots,q_{M+N})$ is an EVE if and only if

$$a) \quad q_i \in \{0,1\} \forall i$$

$$\text{and } b) \quad \frac{d}{dp}\ {}_iE(q_j) < \frac{M+N-1}{2(\delta_{1i}M+\delta_{2i}N)}c_j \ \ \forall j \in T_i \text{ and } q_j = 0$$

$$\frac{d}{dp}\ {}_iE(q_i) > \frac{M+N-1}{2(\delta_{1i}M+\delta_{2i}N)}c_j \ \ \forall j \in T_i \text{ and } q_j = 1.$$

in which δ_{ij} is the Kronecker symbol.[5]

Theorem 4.1. shows that the decision to vote is not a coin toss. No voter uses a mixed strategy and no probability process decides the question whether to vote or not. This result is analogous to the one of Palfrey and Rosenthal (1985), however, we do not establish a symmetric voting behavior. The decision of identical voters with same voting costs can result in different behavior: one votes and the other stays away. Furthermore, Palfrey and Rosenthal (1985) use voters who know their voting costs, a common distribution of voting costs, the actual group sizes and all "relevant" parameters. In reality this information is unknown to the voters and thus unnecessary in our set-up.

All EVE strategies are pure. Therefore only equilibria of the voter turnout dynamics shown in lemma 4.1. could be EVEs of the voting game and the graphical approach is necessary to determine the EVE of a voting game with arbitrary group sizes M and N and arbitrary and different voting costs c_i.

Theorem 4.2. If there exists a voter $i \in T_j$ with $c_i \leq (N\delta_{1j} + M\delta_{2j})/(N+M-1)$, no voting is not an EVE of the voting game.

If the group sizes are different and there is just one voter in the smaller team with cost of voting less than 1/2, the "voting paradox" disappears and voter turnout is unequal to zero. This voter is able to learn to go to the poll.

5. Summary

Voters do not know adequate decision rules for the voting decision, they are of bounded rationality, and information about voting costs and potential quantity of the preference groups is not obtainable. Therefore we replace the rational voter by the learning individual. Actions followed by pleasure are learned, actions followed by punishment are forgotten.

The results are the following: The individual learning process is unique whereas the evolution of voter turnout is ambiguous. Voters with high cost of voting tend to stay away and voters with low costs tend to participate. If voting costs less than half and both voter groups are of the same size, all voters can learn to participate and the unique Nash equilibrium in pure strategies is also an EVE. If the voter groups are of different size, the "voting paradox", i.e. no one voting, is not an EVE of the voting game. In this situation, voters are able to learn participation.

Small groups are in a better position to solve the "free rider" problem. Starting from a quasi-symmetric Nash equilibrium, a voter in the smaller group learns to participate whereas a voter in the larger group learns to stay away. Voters in the relatively small group are able to learn to go to a poll, though the costs of voting are larger than half as much as the utility difference.

In the long run, voters cannot learn to play a mixed strategy. They do or do not participate but they do not toss a coin. The voting situation is so complex that even identical voters, i.e. voters with identical preferences and identical voting costs, can decide differently whether to vote or not to vote.

The final remark is, that interesting issues remain, of course. On the one hand further investigations of the voting behavior in the EVE have to determine the probabilities defined by the sets of permutations in Theorem 4.1. Because this calculation is difficult, progress may be reached with numerical algoritms to solve systems of inequations. On the other hand, further research on simultaneous learning in situations with more than one deviant is necessary.

Notes

1. The shortcomings as multiplicity and weakness of the Nash equilibrium in the voting game are discussed in Chapter 2 and the absence of an adjustment mechanism to reach the equilibrium is well known.
2. The average is a trivial one, because all players are identical:

$$\pi^A(q, ..., q, p, q, ..., q) = \pi_i(q, ..., q, p, q, ..., q) \forall i \neq j.$$

3. Usually the social position is defined as π^D/π^A, but because of $\pi^D - \pi^A \gtreqless 0 \Longleftrightarrow \pi^D/\pi^A \gtreqless 1$, the dynamical consequences are the same.
4. 1^+ and 2^+ are incompatible to $c_i < 1/2\ \forall i$.
5. That is $\delta_{ij} = 1$ if i and $\delta_{ij} = 0$ if $i \neq j$.

Appendix

Detailed proofs are available from the authors.

Proof of lemma 4.1.: The proof is straightforward by direct calculation.

Proof of lemma 4.2.: A deviant strategy $p > 0$ from T_i receives an expected payoff ${}_1\pi^D = p + (1-p)/2 - pc$, the other voters reveive ${}_1\pi = (1-p)/2$ and ${}_2\pi = (1-p)/2$. Therefore

$$\pi^A = \tfrac{1-p}{2} + p\tfrac{M-1}{M+N-1}, \text{i.e.}\, {}_1\pi^D > \pi^A. \qquad \square$$

Proof of lemma 4.3.: A deviant strategy $p \neq 1$ from T_i receives an expected payoff ${}_i\pi^D = p/2 - pc$, the other voters receive ${}_i\pi = p/2 - c$ and ${}_j\pi = (1-p) + p/2 - c$. Therefore

$$\pi^A = \frac{p}{2} - c + \frac{M}{2M-1}(1-p),$$

i.e. ${}_i\pi^D < \pi^A\ \forall 0 \leq p \leq 1$. Therefore the EVE condition holds. $\square$

Proof of lemma 4.4.: Let the deviant with a strategy p appear in T_1. The revenue for group 1 is

$$\begin{aligned} {}_1\mathrm{E} = \; & pc + \sum_{k=1}^{\min(M-1,N)} \binom{M-1}{k} q^k(1-q)^{M-1-k} \sum_{j=0}^{\min(k-1,N)} \\ & \binom{N}{j} r^j(1-r)^{N-j} + \frac{1}{2} \sum_{k=0}^{\min(M-1,N)} \binom{N}{k}\binom{M-1}{k} \\ & q^k(1-q)^{(M-1-k)} r^k(1-r)^{N-k} \end{aligned}$$

Therefore

$$\frac{d}{dp}\,{}_1E(p) = c \geq 0 \;\forall 0 \leq p \leq 1.$$

Furthermore

$$\pi^A = \frac{1}{M+N-1}((M-N-1)_1E + N - c((M-1)q + Nr)).$$

Therefore

$$\frac{d}{dp}(\pi^D - \pi^A) = -c\frac{M-1-N}{M+N-1} \gtreqless 0 \Longleftrightarrow N \gtreqless M-1.$$

If, for example, $M-1 > N$, then $\frac{d}{dp}(\pi^D - \pi^A) \;<\; 0$. If in this case $p < q$ for the deviant, fitness with strategy p will increase. Therefore the quasi-symmetric Nash equilibrium is not an EVE if $M-1 \neq N$. But if $M-1 = N$ the quasi-symmetric Nash equilibrium is not an EVE whenever a deviant appears in T_2. □

Proof of theorem 4.1.: Let

$$P_n^M = \{(i_1, ..., i_n), i_j \neq i_k \forall j \neq k, i_j \in \{1, ..., M\}\}$$
$$\text{and } P_n^N = \{(i_1, ..., i_n), i_j \neq i_k \forall j \neq k, i_j \in \{M+1, ..., N\}\}.$$

The expected revenue for a voter k in T_1

$$\begin{aligned} &{}_1\mathrm{E}(q_1, q_2, \ldots, q_{M+N}) \\ &= \sum_{k=1}^{M}\left(\sum_{P\in P_k^M}\prod_{l\in P} q_l \prod_{l\notin P}(1-q_l)\right) \\ &\sum_{j=0}^{\min\{k-1,N\}}\left(\sum_{P\in P_j^N}\prod_{l\in P} q_l \prod_{l\notin P}(1-q_l)\right) + \tfrac{1}{2}\sum_{k=1}^{\min\{M,N\}} \\ &\left(\sum_{P\in P_k^M}\prod_{l\in P} q_l \prod_{l\notin P}(1-q_l)\right)\left(\sum_{P\in P_k^N}\prod_{l\in P} q_l \prod_{l\notin P}(1-q_l)\right) \end{aligned}$$

is linear in q_k. Let $k \in T_1$ and $q_k = p$. Since $\pi^D = {}_1E(p) - pc_k$ and

$$\pi^A = \frac{1}{M+N-1}\left((M-1-N)_1E(p) + N - \sum_{\substack{i=1\\ j\neq k}}^{M} q_i c_i - \sum_{i=M+1}^{M+N} q_i c_i\right)$$

holds

$$\frac{d}{dp}(\pi^D - \pi^A) \gtreqless 0 \Longleftrightarrow \frac{d}{dp}\,{}_1E \gtreqless \frac{M+N-1}{2N}c_k$$

i.e. *b*). Because ${}_1E$ is linear in q_i the maximum is in {0,1} or indefinite. Furthermore no maximum in the interior of [0,1] is a strong local one, i.e. *a*). □

Proof of theorem 4.2.: See Figure 1.

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