ANDREAS PFINGSTEN and ANDREAS WAGENER

BARGAINING SOLUTIONS AS SOCIAL COMPROMISES

(Received 1 February 2001; revised 30 October 2003; accepted 10 February 2004)

ABSTRACT. A bargaining solution is a social compromise if it is metrically rationalizable, i.e., if it has an optimum (depending on the situation, smallest or largest) distance from some reference point. We explore the workability and the limits of metric rationalization in bargaining theory where compromising is a core issue. We demonstrate that many well-known bargaining solutions are social compromises with respect to reasonable metrics. In the metric approach, bargaining solutions can be grounded in axioms on how society measures differences between utility allocations. Using this approach, we provide an axiomatic characterization for the class of social compromises that are based on p-norms and for the attending bargaining solutions. We further show that bargaining solutions which satisfy Pareto Optimality and Individual Rationality can always be metrically rationalized.

KEY WORDS. bargaining solutions, metric rationalizability, social compromise.

1. INTRODUCTION

In a cooperative bargaining problem, an element out of a set of feasible utility vectors for a group of agents has to be selected. The selected outcome is regarded as the cooperative agreement reached by the agents. In the axiomatic approach, the outcome further is interpreted as the recommendation which an impartial arbitrator who holds certain normative positions would give as to how the bargaining problem should reasonably be solved. In this approach, bargaining solutions are found by combining axioms. Numerous axioms have been proposed in the literature and a number of bargaining solutions has been characterized by the specific combination of axioms which only they do satisfy.

In this paper, we suggest a different approach to the bargaining problem. The central idea is to view solutions to bargaining problems as social compromises in the sense that they minimize the

Theory and Decision **55:** 359–389, 2003.

distance between what the bargaining parties ideally want to have (but typically cannot reach) and what they finally get. Alternatively, we can see the idea of bargaining as leaving behind as far as possible some feasible but undesirable fallback outcome – which also suggests an interpretation in terms of distance. Generally, bargaining solutions as social compromises have an optimal (depending on the nature of the problem, maximal or minimal) distance to some reference point inside or outside the feasible set. Such reference points are common ingredients in bargaining problems, the disagreement and claims points being the most prominent examples.

The concept of social compromises is well-established in several areas of social choice theory. A social compromise generally constitutes a decision rule that selects those elements among the feasible outcomes (which may be single items or subsets in some set of alternatives or societal rankings over the choice set) that come as close as possible to some ideal but not achievable outcome. Closeness is measured by suitably defined metrics or quasi-metrics on the outcome space. If the outcomes of a social choice rule coincide with those of a social compromise, the social choice rule is said to be *metrically rationalizable*. Stehling (1978), Farkas and Nitzan (1979), Baigent (1987a,b), and Bossert and Storcken (1992) apply this idea to Arrovian social welfare functions. Bossert and Stehling (1993) model Bergson–Samuelson social welfare functions as social compromises. Nitzan (1981) and Lerer and Nitzan (1985) investigate into the metric rationalization of social choice correspondences. In a general approach, Campbell and Nitzan (1986) discuss social decision procedures as compromises between desirable, but mutually incompatible criteria for group decision rules. General 'minimal distance' choice functions have recently been discussed in Rubinstein and Zhou (1999).

Transferring the idea of social compromises to bargaining problems, we propose to view bargaining solutions as points at an *optimum* distance from some reference point. With respect to the disagreement point, the idea of compromise has been in the core of the bargaining problem since its first formal treatment in Nash (1950): Each single agent wishes to maximize his gain over the disagreement outcome, and a solution has to reconcile these mutually incompatible aims by leaving the disagreement point behind as far as possible from the group's perspective. An analogous interpretation can be given with regard to exterior, non-feasible reference points (claims point, utopia point): Here an agreement is needed to bring to terms the agents' strive for minimal sacrifices.

So far the compromise idea has only rarely been recognized explicitly in solution concepts discussed in the literature. Exceptions we are aware of are Yu (1973) and Freimer and Yu (1976), who, however, see their metric approaches as an alternative rather than a complement to bargaining solutions *à la* Nash. The idea of coming as close as possible to a perceived but unfeasible ideal figures prominently in multi-criteria optimization problems. This area and bargaining theory share some formal analogies that are exploited by Conley et al. (2000) in a characterization of the Euclidean compromise solution (which minimizes the Euclidean distance to the utopia point); a different characterization of this solution is provided by Voorneveld and van den Nouweland (2001).

In this paper we first establish a general framework for the metric approach in the field of bargaining problems (Sections 2 and 3). Second, we demonstrate that many bargaining solutions which have been proposed in the literature can indeed be interpreted as social compromises and we (uniquely) relate bargaining solutions to social compromises via the axioms the former satisfy (Section 4).

Third and more importantly, we argue that the idea of social compromises might open a new perspective on the bargaining problem: In the metric approach, differences between utility vectors are compared with each other in terms of proximity and distance. Obviously, distances can be measured in various ways, some of which seem plausible and normatively appealing while others do not. Imposing reasonable axiomatic properties on the way in which differences between utility vectors are made metrically comparable limits the range of (and, ideally, characterizes uniquely) the distance measures on which social compromises can be based. From this, attending bargaining solutions can be derived which are then normatively grounded in axioms on how society ought to measure differences between utility allocations rather than, as in the standard approach, in axioms related to properties of the bargaining problems themselves. In Section 5 we follow this new route and axiomatically characterize social compromises based on distance measurement with so-called p-norms. This gives rise to a whole continuum of attending bargaining solutions which includes as special cases the utilitarian and the Euclidean solution.

Fourth, given that many well-known bargaining solutions can be viewed as social compromises we proceed by exploring general requirements for the metric rationalizability of bargaining solutions (Section 6). This helps to clarify the applicability of the compromise idea in the context of bargaining problems. In the case of a fixed set of feasible outcomes, we identify the standard properties of Pareto Optimality and Individual Rationality as sufficient, but not indispensable conditions for metric rationalizability. However, the metrics upon that such rationalizations are based may, generally, be rather odd and lack intuitive appeal. Imposing more structure on the bargaining solution will, however, narrow the range of admissible metrics. We discuss this by relating the notion of social compromises to the more general idea of rationalizing bargaining outcomes by preference orderings. Section 7 concludes this paper.

2. PRELIMINARIES

We first have to introduce some notation and definitions. Let $\mathbb{R}$ ($\mathbb{R}_+, \mathbb{R}_{++}$, resp.) denote the sets of all (non-negative, positive, resp.) real numbers. 0^n is the origin of $\mathbb{R}^n$, and 1^n is the vector consisting of n ones. Vector inequalities in $\mathbb{R}^n$ are denoted by $\geqslant$, $>$, and $\gg$, i.e. for $x, y \in \mathbb{R}^n$ we write $x \geqslant y$, $x > y$, and $x \gg y$ if, respectively, $x_i \geqslant y_i$ for all $i = 1, \ldots, n$, $x \geqslant y$ and $x \neq y$, and $x_i > y_i$ for all i. For $x \in \mathbb{R}^n$ we write $x = (x_i, x_{-i})$ to distinguish between the ith component of x and the other ones. A set $S \subset \mathbb{R}^n$ is called *strictly comprehensive* iff $x \in S$ and $x > y$ imply that $y \in S$ and that there exists a $z \in S$ such that $z \gg y$. Given a set S, we denote its interior (i.e., the collection of its inner points) by $\operatorname{int} S$ and its power set by $\mathcal{P}(S)$.

Let Γ^n be the set of all subsets of $\mathbb{R}^n$ that are convex, closed, strictly comprehensive and bounded from above by a hyperplane orthogonal to some strictly positive n-dimensional vector. For $S \in \Gamma^n$, denote by

$$\mathrm{PO}(S) := \{x \in S \mid \nexists y \in S \text{ with } y > x\}$$

the set of *Pareto optimal* points in S.[1]

We describe an n-person cooperative bargaining problem with reference point r by a pair (S, r) where $S \in \Gamma^n$ is the set of feasible utility vectors for the n agents and where the reference point $r \in \mathbb{R}^n$ serves as a benchmark utility vector relative to which the agents or an arbitrator evaluate possible solutions of the bargaining problem.

Let Σ be the class of all such bargaining problems. It can be divided into three disjoint subclasses Σ_R, Σ_T and Σ_U, depending on the location of the reference point:

- If the reference point belongs to the interior of the feasible set ($r \in \operatorname{int} S$), we call it *realistic*. It is an achievable but not necessarily desirable outcome of the bargaining game (e.g., the disagreement point). We collect bargaining problems with realistic reference points in the class Σ_R.
- If the reference point is weakly Pareto optimal ($r \in \mathrm{PO}(S)$), we say that the bargaining problem is *trivial*. We denote the class of trivial problems by Σ_T.
- If the reference point lies outside the feasible set ($r \notin S$) we call it *utopic*. Examples include the claims and the bliss point. We denote the corresponding class of bargaining problems by Σ_U.

Obviously, $\Sigma_R \cup \Sigma_T \cup \Sigma_U = \Sigma$. As shortcuts, we write $\Sigma_{RT} := \Sigma_R \cup \Sigma_T$ and $\Sigma_{UT} := \Sigma_U \cup \Sigma_T$. A *solution* on a subclass $D \subset \Sigma$ of bargaining problems is a function $f : D \to \mathbb{R}^n$ such that $f(S, r) \in S$ for all $(S, r) \in D$. We only consider single-valued solutions.

The usual description of an n-person bargaining problem has the following two, possibly three ingredients: a set $S \subset \mathbb{R}^n$ of feasible utility allocations, a disagreement point $d \in S$, and sometimes a point $c \in \mathbb{R}^n$ called the claims point (introduced by Chun and Thomson, 1992). Disagreement point, claims point and some other points have found several interpretations in the literature (see Gerber, 1998), the common feature being that they offer possible origins against which the gains or losses from bargaining can be measured. Reference points are a natural generalization of this concept.

Moving from the standard representation of bargaining games as triples (S, d, c) to a representation with only two primitives (S, r) involves some loss in descriptive richness and functional flexibility for bargaining solutions. However, quite a number of approaches to bargaining problems that start out from triples (S, d, c) employ a reference function $r: \Gamma^n \times \mathbb{R}^n \times \mathbb{R}^n$ to construct a point $r(S, d, c) \in \mathbb{R}^n$ and then only use this reference point and the bargaining set to calculate the solution as $f(S, r(S, d, c))$.[2] I.e., using only a single reference point as a summary for what the agents or the arbitrator regard as a benchmark for bargaining is fairly widespread and accepted. A drawback to this method is that it fails to cover some prominent bargaining solutions which cannot be phrased in terms of reference functions. Examples include the Kalai and Smorodinsky (1975) solution or the proportional and the extended claim-egalitarian solutions suggested (Chun and Thomson, 1992, Bossert, 1993) for bargaining problems with claims.

Our approach is a partial one in so far as we take the reference point as given. A full-fledged bargaining model should include a theory on how the reference point is chosen. This is, however, largely irrelevant for our purposes;[3] we therefore disregard this issue.

3. SOCIAL COMPROMISE AND METRIC RATIONALIZABILITY

In the axiomatic approach to bargaining problems, solutions are characterized by combining axioms (i.e., normative properties which reasonable outcomes should possess in the eye of an impartial arbitrator). We propose a different, metric or distance-based view, namely to regard bargaining solutions as feasible utility vectors that come closest to some ideal (but non-feasible) utility vector or lie farthest from some undesirable (but attainable) utility vector. We call a utility vector a *social compromise* if it has an optimum distance from some reference outcome. To get a meaningful notion of distance, we employ metrics and quasi-metrics as measurement devices.

A function $\delta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is called a *metric* on $\mathbb{R}^n$ iff for all $x, y, z \in \mathbb{R}^n$: (i) $\delta(x, y) = 0 \iff x = y$; (ii) $\delta(x, y) = \delta(y, x)$; and

(iii) $\delta(x,y)+\delta(y,z)\geqslant\delta(x,z)$. If δ satisfies all properties except for the triangular inequality (iii), it is called a *quasi-metric*.

DEFINITION 1. Let δ be a quasi-metric on $\mathbb{R}^n$.

(a) Let $D\subseteq\Sigma_{UT}$ be a domain of bargaining problems with non-realistic reference points. The correspondence $K_\delta : D\to\mathcal{P}(\mathbb{R}^n)$ with

$$K_\delta(S,r)=\{y\in S|\delta(y,r)\leqslant\delta(x,r)\quad\forall x\in S\}$$

is called a *social compromise* with respect to the quasi-metric δ.

(b) Let $D\subseteq\Sigma_{RT}$ be a domain of bargaining problems with non-utopic reference points. The correspondence $K_\delta : D\to\mathcal{P}(\mathbb{R}^n)$ with

$$K_\delta(S,r)=\{y\in S|\delta(y,r)\geqslant\delta(x,r)\quad\forall x\in S\}$$

is called a *social compromise* with respect to the quasi-metric δ.

For bargaining problems with utopic reference point, a social compromise has, among all feasible points, the shortest distance to the reference point. This conveys that agents strive to approach r as closely as possible. On the contrary, realistic reference point are allocations from which the agents want to escape as far as possible, requiring that the distance between the final outcome and the reference point be maximized.

The following definition is inspired by Lerer and Nitzan (1985) and Campbell and Nitzan (1986) and establishes a link between social compromises and bargaining solutions:[4]

DEFINITION 2. Let $D\subseteq\Sigma_{UT}$ or $D\subseteq\Sigma_{RT}$ be a given class of bargaining problems and let δ and f be a quasi-metric on $\mathbb{R}^n$ and a bargaining solution on D, respectively. Then δ is a *metric rationalization* of f on D iff for all $(S,r)\in D$:

$$f(S,r)=K_\delta(S,r).$$

By definition, only single-valued social compromises can be full rationalizations of bargaining solutions, which are functions by assumption.

4. SOME EXAMPLES FOR METRIC RATIONALIZABILITY

To illustrate the idea of metric rationalization, we now demonstrate that a number of prominent bargaining solutions can be understood as social compromises in a meaningful way and that social compromises can be related to bargaining solutions by the axioms the latter satisfy.

4.1. *Utopic reference points and the maximum norm*

We first analyse bargaining problems with utopic reference points: $D=\Sigma_{UT}$. We will employ the following metric (the *maximum norm*):[5]

$$\delta_\infty(x,y)=\max_i |x_i-y_i|.$$

Consider the following examples:

EXAMPLE 1. The claim-egalitarian solution as a social compromise. Denote by c the claims point of a bargaining problem. Bossert (1993) introduced the *Claim-Egalitarian* solution $\mathrm{CE}(S,c)$ as the point $x\in\mathrm{PO}(S)$ such that $c_i-x_i=c_j-x_j$ for all $i,j\in\{1,\dots,n\}$. This requires an equal absolute sharing of losses, measured against the claims point. Choosing c as a reference point, the solution can be metrically rationalized by the maximum norm:

$$\mathrm{CE}(S,c)=K_{\delta_\infty}(S,c).$$

EXAMPLE 2. The equal-loss solution as a social compromise. Chun (1988) introduced the *Equal–Loss* solution $\mathrm{EL}(S,d)$ as the point $x\in\mathrm{PO}(S)$ such that $b_i-x_i=b_j-x_j$ for all $i,j\in\{1,\dots,n\}$. Here $b=(b_1,\dots,b_n)$ denotes the bliss point, given by $b_i=b_i(S,d)=\max\{x_i|x\in S \text{ and } x\geqslant d\}$. Using the bliss point as a reference point, the equal–loss solution can be interpreted as a social compromise with respect to the maximum norm:

$$\mathrm{EL}(S,d)=K_{\delta_\infty}(S,b).$$

EXAMPLE 3. The contested garment-principle as a social compromise. In a bankruptcy problem an estate of value $\bar{s}>0$ has to be divided among several creditors whose claims in total exceed $\bar{s}$.

This constitutes a bargaining problem with claims and transferable utilities. A widely discussed solution in this area is the *Contested Garment* principle $\mathrm{CG}(S,c)$; see e.g., Dagan (1996). It is defined for two-person problems only and uses $\bar{c}(S,c)$ with $\bar{c}_i := \min\{c_i, \bar{s}\}$ as its reference point. We again invoke the maximum norm to interpret it as a social compromise. For $n=2$ and if $\mathrm{PO}(S)=\{(x_1,x_2)|x_1+x_2=\bar{s}\}$, then

$$\mathrm{CG}(S,c)=K_{\delta_\infty}(S,\bar{c}).$$

This brief series of examples generalizes to the following observation: If a bargaining solution f on domain $D\subseteq\Sigma_{UT}$ demands of all agents an equal concession from the reference point r, i.e., if $r_i - f_i(S,r) = r_j - f_j(S,r)$ for all $i,j\in\{1,\dots,n\}$, then it can be metrically rationalized by the maximum norm:

$$f(S,r)=K_{\delta_\infty}(S,r).$$

Now consider the following list of axioms that are frequently imposed on bargaining solutions:

[PO] Pareto Optimality: $f(S,r)\in\mathrm{PO}(S)$ *for all* $(S,r)\in D$.

[SYM] Symmetry: *For all* $(S,r)\in D\subset\Sigma$*, if for all permutations* π *of* $\{1,\dots,n\}$, $(S,r)=(\pi(S),\pi(r))$*, then* $f_i(S,r)=f_j(S,r)$ *for all* $i,j\in\{1,\dots,n\}$.

[TRANS] Translation Invariance: *For all* $(S,r)\in D\subset\Sigma$*, and for all functions* $\lambda:\mathbb{R}^n\to\mathbb{R}^n$ *with* $\lambda(x)=x+b$ *where* $b\in\mathbb{R}^n$*:*

$$f(\lambda(S),\lambda(r))=\lambda(f(S,r)),$$

where $\lambda(S)$ *is the image of* S *under* λ.

[MON] Monotonicity: *For all* $(S,r),(S',r')\in D\subset\Sigma$*, if* $r'=r$ *and* $S'\subset S$*, then* $f(S,r)\geqslant f(S',r')$.

Together, these axioms give rise to

RESULT 1. A bargaining solution $f(S,r)$ with domain Σ_{UT} satisfies [PO], [SYM], [TRANS], and [MON] if and only it is metrically rationalized by δ_∞.

Proof.

(a) It is easy to check that $f(S,r)=K_{\delta_\infty}(S,r)$ satisfies [PO], [SYM], [TRANS], and [MON].

(b) Let f satisfy [PO], [SYM], [TRANS], and [MON]. The following arguments are borrowed from the standard characterization of egalitarian-type solutions due to Kalai (1977) and Bossert (1993).
By [TRANS] choose $r=1^n$. If (S,r) is symmetric, then [PO] and [SYM] require that $f(S,r)$ is the (maximal) egalitarian element in $PO(S)$ which equals $K_{\delta_\infty}(S,1^n)$. If (S,r) is not symmetric, consider the symmetric set $S'=\bigcap_\pi \pi(S)$ where the intersection is taken over all permutations π of $\{1,\ldots,n\}$. [PO] and [SYM] imply that $f(S',1^n)=K_{\delta_\infty}(S',1^n)=K_{\delta_\infty}(S,1^n)$. As $S'\subset S$, [MON] requires $f(S,1^n)\geqslant f(S',1^n)=K_{\delta_\infty}(S,1^n)$. As S is strictly comprehensive and due to [PO], $f(S,1^n)=K_{\delta_\infty}(S,1^n)$. □

Note that Result 1 does *not* provide a unique characterization of the maximum norm. E.g., the metric $2\cdot\delta_\infty$ also metrically rationalizes all bargaining solutions that satisfy the list of axioms in Result 1. Metric rationalizability is an ordinal feature: If a quasi-metric rationalizes a bargaining solution, so will every strictly monotonic transformation that keeps intact the defining properties of quasi-metrics.

4.2. *Realistic reference points and the minimum function*

Let us turn to bargaining problems with realistic reference points: $D=\Sigma_{RT}$. A social compromise on that domain is a feasible utility vector that maximizes the distance to the interior reference point. A general problem is that on strictly comprehensive sets the distance to the reference point can, for many metrics (e.g., for the Euclidean distance), be increased beyond all limits by moving ‘south-west’-bound. Hence, a social compromise fails to exist. An acceptable remedy is to require that only utility vectors which weakly dominate the reference point can qualify as social compromises:

DEFINITION 3. Let $D \subseteq \Sigma_{RT}$ and let δ be a quasi-metric on $\mathbb{R}^n$. The correspondence $K_\delta^{IR} : D \to \mathcal{P}(\mathbb{R}^n)$ with

$$K_\delta^{IR}(S,r) = \{y \in S | y \geqslant r \text{ and } \delta(y,r) \geqslant \delta(x,r) \quad \forall x \in S \text{ with } x \geqslant r\}$$

is called an *individually rational social compromise* with respect to the quasi-metric δ.

Definition 3 is inspired by the axiom of *Individual Rationality* which, with respect to the disagreement vector, is commonly used in axiomatic bargaining theory and which precludes that an individual ends up with a lower utility than in the benchmark situation:[6]

[IR] Individual Rationality with respect to the reference point: *If $r \in S$, then $f(S,r) \geqslant r$.*

Obviously, a bargaining solution on Σ_{RT} which is an individually rational social compromise satisfies [IR].

Similar to bargaining solutions on Σ_{UT} which require equal concessions by all agents from a utopic reference point, a variety of bargaining solutions on Σ_{RT} allocate equal utility gains to all agents, starting from the interior reference point. Geometric analogy to the observations in the previous section might suggest to use the minimum function,

$$\delta_0(x,y) = \min_i |x_i - y_i|,$$

as a vehicle to metrically rationalize such solutions. However, δ_0 is not a quasi-metric (since $x_i = y_i$ for some i already implies $\delta_0(x,y) = 0$). That problem can be remedied quite easily (yet not very elegantly). For $x, y \in \mathbb{R}^n$ define the quasi-metric

$$\bar{\delta}_0(x,y) := \begin{cases} 0 & \text{if } \quad x = y \\ 1 + \delta_0(x,y) & \text{else.} \end{cases} \tag{1}$$

EXAMPLE 4. The egalitarian solution as a social compromise. The *Egalitarian* solution $E(S,d)$, initially proposed by Kalai (1977), selects the point on $\text{PO}(S)$ which awards equal absolute utility gains to all players, starting from the disagreement point. The modified minimum function provides a metric rationalization for this solution:

$$E(S,d)=K^{IR}_{\bar{\delta}_0}(S,d).$$

This observation generalizes as follows: If a bargaining solution f with domain $D\subseteq\Sigma_{RT}$ assigns equal gains over the reference point to all agents, i.e., if $f_i(S,r)-r_i=f_j(S,r)-r_j$ for all $i,j\in\{1,\dots,n\}$, then it is an individually rational social compromise generated by the modified minimum function:

$$f(S,r)=K^{IR}_{\bar{\delta}_0}(S,r).$$

The analogue to Result 1 is

RESULT 2. A bargaining solution $f(S,r)$ with domain Σ_{RT} satisfies [PO], [SYM], [TRANS], and [MON] if and only it is an individually rational social compromise w.r.t. $\bar{\delta}_0$.

Proof. Part (a) in the proof of Result 1 holds without modification. For part (b), start with choosing $r=0^n$ by [TRANS]. If (S,r) is symmetric, [PO] and [SYM] require that $f(S,0^n)$ is the (maximal) egalitarian element in PO(S). This is equal to $K^{IR}_{\bar{\delta}_0}(S,0^n)$. Now proceed as in the proof of Result 1. □

5. AN AXIOMATIC APPROACH TO DISTANCE MEASUREMENT

5.1. *General remarks*

In Results 1 and 2 we imposed axioms on the bargaining solution to limit the range of (quasi-)metrics that generate ‘reasonable’ rationalizations. In their compromise approaches, Conley et al. (2000) and Voorneveld and van den Nouweland (2001) proceed similarly: They demand certain desiderata for the solution and then search for the metrics (in both papers the Euclidean metric δ_2) that support these properties as compromises. Here we suggest to proceed along a different route: Rather than analysing what the distance measure must look like when we want it to produce well-behaved solutions, we ask for the implications which ‘reasonable’ methods of measuring distances between utility allocations have for the outcome of the bargaining problem. I.e., we start from plausible properties imposed on metrics, and then search for the implied social compromises. In the social choice context, a similar procedure has been applied by Lerner and Nitzan (1985).

We use (quasi-)metrics to measure the magnitude of changes that a group of agents jointly experiences when shifting from one utility vector to another. Axioms imposed on (quasi-)metrics therefore reflect normative views on how distances between utility vectors ought to be measured and compared. Such axioms do not only restrict the technical measurement devices but in the first place determine which aspects should count and how they should be weighed when society assesses the scale of its moving from one utility allocation to another.

The axiomatic approach to distance measurement allows us to view social compromises as being rooted in normative positions on how society should proceed when assessing differences between utility allocations. This opens an alternative perspective on bargaining problems that markedly differs from the standard axiomatic approach to bargaining solutions where properties of the bargaining outcome are derived from structural elements of the bargaining situations. As we will see below, from the new perspective well-known bargaining solutions might appear in a different light and formerly unknown solutions might newly emerge.

5.2. *An axiomatic characterization of p-norms*

To exemplify and substantiate our general remarks, we will now provide a characterization result for the class of measurement devices commonly labelled p-norms: For $p > 0$ and $x, y \in \mathbb{R}^n$ define

$$\delta_p(x, y) := \left[\sum_{i=1}^{n} |x_i - y_i|^p \right]^{1/p}. \tag{2}$$

For $p \geqslant 1$, δ_p is a metric, otherwise a quasi-metric. For $p = 2$ we get the Euclidean distance. For $p \to \infty$ we obtain the *maximum norm* $\delta_\infty(x, y) = \max_i |x_i - y_i|$ that we used in Section 4.1, and for $p \to 0$ we approach the *minimum function* $\delta_0(x, y) = \min_i |x_i - y_i|$ that we have already encountered in Section 4.2.

We now present a number of axioms which one might regard as plausible properties of distance measurement (in addition to employing a quasi-metric). We label axioms imposed on metrics by small letters to distinguish them from axioms imposed on bargaining solutions.

[trans] Translation Invariance: *For all* $x, y, z \in \mathbb{R}^n$, $\delta(x+z, y+z) = \delta(x, y)$.

[lhom] Linear Homogeneity: *For all* $x, y \in \mathbb{R}^n$ *and* $\lambda > 0$, $\delta(\lambda \cdot x, \lambda \cdot y) = \lambda \cdot \delta(x, y)$.

[add] Additive Decomposability: *There exist continuous functions* $\Psi\colon \mathbb{R} \to \mathbb{R}_+$, $\xi_i\colon \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, \ldots, n)$ *such that:* $\delta(x, y) = \Psi(\sum_{i=1}^n \xi_i(x_i, y_i))$ *for all* $x, y \in \mathbb{R}^n$.

[smon] Strict Monotonicity: *For all* $x, y, z \in \mathbb{R}^n$ *with* $x > y \geqslant z$, $\delta(x, z) > \delta(y, z)$.

The interpretation of these axioms is straightforward: Axiom [trans] represents the idea that distances between utility vectors are independent of the chosen origin. It is the metric analogue of axiom [TRANS] on bargaining solutions.[7]

Axiom [lhom] implies that changes in the units of all utilities are reflected one-to-one in the distance between utility vectors, when calibrated in new units.

Axiom [add] conveys that the distance between utility vectors depends on the sum of distances between the individual components in that vector. Hence, the social distance between utility vectors depends on the sum of individual gains or regrets. Lerer and Nitzan (1985, p. 194) use a similar axiom in their characterization of social choice functions. In particular, [add] excludes interdependencies between agents: Agent i's gains or regrets do not affect the valuation of gains or regrets experienced by another agent j.

Axiom [smon] represents the natural idea that utility vectors with larger components (in absolute terms) are farther away. The axiom transfers the basic understanding of 'more is better' that is reflected by the ordering '$\geqslant$' on the real numbers to the comparison of utility allocations.

Using this list of axioms we arrive at the following characterization which is a modified version of a result by Gehrig and Hellwig (1982) and Gehrig (1984, Theorem 3.3).[8]

RESULT 3. A quasi-metric δ satisfies [trans], [lhom], [add], and [smon] if and only if there exist $\eta, b_i, p > 0$ such that for all x, $y \in \mathbb{R}^n$,

$$\delta(x,y) = \eta \cdot \left[\sum_{i=1}^{n} b_i |x_i - y_i|^p\right]^{1/p}. \tag{3}$$

Proof. Obviously, all functions of type (3) satisfy all axioms (to confirm [add] set $\Psi(z) = \eta z^{1/p}$ and $\xi_i(x_i, y_i) = b_i|x_i - y_i|^p$). We will now prove that the axioms (together with the fact that δ is a quasi-metric) imply properties (A1) through (A4) in Gehrig and Hellwig (1982). Since axioms [trans] and [lhom] are identical with properties (A1) and (A2) in Gehrig and Hellwig (1982) we only have to verify (A3) and (A4).[9]

From [add] and [trans] with $z = -y$ we obtain that Ψ must be of the form:

$$\Psi\left(\sum_{i=1}^{n} \xi_i(x_i, y_i)\right) = \Psi\left(\sum_{i=1}^{n} h_i(x_i - y_i)\right) \tag{4}$$

with $h_i(w) := \xi_i(w, 0)$. To show that (A3) is satisfied, we explore the monotonicity properties of Ψ and the h_i. Define $H_0 := \sum_i h_i(0)$. Since δ is a metric with $\delta(x,y) = 0$ if and only if $x = y$, we must have $\Psi(H_0) = 0$. Now suppose that, starting from $x = y$, we change x_1 to $x_1 \neq y_1$, i.e., we have $w_1 \neq 0$. Then $h_1(w_1) \neq h_1(0)$ must hold.

(a) Assume that $h_1(w_1) > h_1(0)$. If we now vary x_2 such that $w_2 = x_2 - y_2 \neq 0$, we must also have $h_2(w_2) > h_2(0)$ because otherwise we could choose, by continuity of the h_i, w_1 and w_2 such that $h_1(w_1) + h_2(w_2) + \sum_{j \geqslant 3} h_j(0) = H_0$, which would contradict $\delta(x,y) = 0$ *only if* $x = y$. Consequently, all h_i must have their minimum at 0. Denoting the range of $\sum_i h_i(w_i)$ in that case by M_+ we, thus, have $M_+ \subseteq [H_0, \infty]$ and $H_0 \in M_+$.
(b) By a similar token, if $h_1(w_1) < h_1(0)$, all h_i must reach their maximum at 0. The range M_- of $\sum_i h_i(w_i)$ must therefore belong to $[-\infty, H_0]$ and must include H_0.

Together with [smon] this leaves us with two cases: In case (a), all h_i are U-shaped around zero, Ψ is increasing to the right of H_0, and is defined at least on M_+. In case (b), all h_i are inversely U-shaped, Ψ decreases to the left of H_0, and is defined at least on M_-.

Suppose that case (b) holds. Then define, for all i, $\hat{h}_i(w_i) := -h_i(w_i)$. Let $\hat{H}_0 := \sum_i \hat{h}_i(0) = -H_0$. Further define a non-negative function $\hat{\Psi}$ such that $\hat{\Psi}(-z) = \Psi(z)$ for all $z \in M_-$. Note that $\hat{\Psi}$ is at least defined to the right of $\hat{H}_0$. By definition, all hat-functions possess monotonicity properties that are opposite to their counterparts without hats. Clearly, for all $x, y \in \mathbb{R}^n$,

$$\delta(x,y) = \Psi\left(\sum_i h_i(x_i - y_i)\right) = \hat{\Psi}\left(\sum_i \hat{h}_i(x_i - y_i)\right).$$

Hence, we can substitute $\hat{\Psi}$ and $\hat{h}_i$ for Ψ and h_i without affecting distance measurement. Without loss of generality, we can therefore choose Ψ to be a strictly increasing function. Then, all $h_i(w)$ must be strictly increasing [strictly decreasing] on $(0, \infty)$ [on $(-\infty, 0)$]. This, however, is what Gehrig and Hellwig (1982) require in property (A3) beyond our axiom [add].

Let us now turn to (A4) in Gehrig and Hellwig (1982). Due to the symmetry of a metric, $\delta(x,y) = \delta(y,x)$ and the strict monotonicity of Ψ (see previous step), we have $\sum_{i=1}^n \xi_i(x_i, y_i) = \sum_{i=1}^n h_i(y_i - x_i)$ which can only hold for all $x, y \in \mathbb{R}^n$ if

$$h_i(w) = h_i(-w) \tag{5}$$

for all $w \in \mathbb{R}$ and $i = 1, \ldots, n$. To see (5), choose $x, y \in \mathbb{R}^n$ such that $x_i - y_i = w_0 \neq 0$ and $x_j = y_j$ for all $j \neq i$. By symmetry of δ, this implies

$$\Psi\left(h_i(w_0) + \sum_{j \neq i} h_j(0)\right) = \Psi\left(h_i(-w_0) + \sum_{j \neq i} h_j(0)\right).$$

Strict monotonicity of the function Ψ therefore requires $h_i(w_0) = h_i(-w_0)$. As i and w_0 were chosen arbitrarily, (5) follows and, hence, (A4) in Gehrig and Hellwig (1982) is satisfied.

Gehrig and Hellwig (1982, p. 234) show that (A1) through (A4) imply (3). □

To arrive at social compromises with respect to p-norms (2) we add to the axioms in Result 3 a standard anonymity property:

[neutr] Neutrality: *For all permutations* π *of* $\{1,\ldots,n\}$, $\delta(\pi(x),\pi(y))=\delta(x,y)$.

By this axiom, which is also used, e.g., in Lerer and Nitzan (1985), the b_i in (3) have to be identical for all i. Hence,

RESULT 4. A metric δ satisfies [trans], [lhom], [add], [smon], and [neutr] if and only if it is proportional to a p-norm δ_p with $0<p<\infty$ (cf. (2)).

For social compromises, the previous results have the following interpretation: If one agrees that distances between utility allocations should be socially evaluated by quasi-metrics that satisfy [trans], [lhom], [add], [smon], and [neutr], then the social compromise must be based on a p-norm with $0<p<\infty$.[10]

5.3. *Applications*

The (quasi)-metrics emerging from Results 3 and 4 have found some attention in the literature on compromise and bargaining solutions. In the context of multi-objective programming, some properties of compromises with respect to p-norms for $p\geqslant 1$ are discussed, e.g., by Yu (1973) and Freimer and Yu (1976). Neither of these studies attempts a full characterization or discusses applications in bargaining problems. Moreover, two of the infinitely many p-norms figure quite prominently in the (bargaining) literature:

EXAMPLE 5. Utilitarian solutions $(p=1)$. For bargaining problems in Σ_{UT} the (*weighted*) *utilitarian solution* minimizes the (weighted) sum of individual concessions from the utopic reference point while it maximizes the (weighted) sum of individual gains over the reference point on the domain Σ_{RT}. By definition, unweighted utilitarian solutions are rationalized by the metric

$$\delta_1(x,y)=\sum_{i=1}^{n}|x_i-y_i|;$$

the weighted extensions are straightforward. Uniqueness of utilitarian solutions can only be ensured for bargaining problems with *strictly* convex feasible sets. In terms of a bargaining solution on Σ_{RT}, the utilitarian solution is characterized by Thomson (1981) while an interpretation and discussion in terms of a compromise on the domain Σ_{UT} is provided by Yu (1973).

EXAMPLE 6. Euclidean solutions ($p=2$). For $p=2$, we arrive at the (weighted) Euclidean compromise solution. In terms of axioms imposed on the outcome, this solution has been characterized by Conley et al. (2000) and Voorneveld and van den Nouweland (2001).

Social compromises with respect to p-norms do not only emerge from distance measures that satisfy the axioms listed in Results 3 and 4; they also possess attractive axiomatic properties in terms of bargaining solutions. In addition to the axioms introduced in Section 4, consider the following properties:

[CONT] Continuity: *For all* $(S,r)\in\Sigma_{RT}$, *for all sequences* $\{(S^k,r^k)\}_{k\in\mathbb{N}}$ *with* $(S^k,r^k)\in\Sigma_{RT}$*: If* $\lim_{k\to\infty}S^k=S$ *in the Hausdorff topology and* $\lim_{r\to\infty}r^k=r$*, then* $\lim_{k\to\infty}f(S^k,r^k)=f(S,r)$.

[IIA] Independence of Irrelevant Alternatives: *For all* $(S,r),(S',r)\in\Sigma_{RT}$*, if* $S\subset S'$ *and* $f(S',r)\in S$*, then* $f(S,r)=f(S',r)$.

The next proposition enumerates some features of bargaining solutions which are social compromises with respect to p-norms:

RESULT 5. If a bargaining solution on Σ_{UT} is metrically rationalized by a p-norm with $1\leqslant p<\infty$, then it satisfies [TRANS], [IIA], [WPO], [SYM], and [CONT].

If a bargaining solution on Σ_{RT} is an individually rational social compromise with respect to a p-norm with $0<p\leqslant 1$, then it satisfies [TRANS], [IIA], [WPO], [SYM], and [CONT].

Proof. [TRANS] is obvious given that p-norms satisfy [trans]. For Σ_{UT}, [IIA], [WPO], [SYM], and [CONT] are (in a different context) proved in Yu and Leitmann (1974). The results for Σ_{RT} can be established in a similar fashion. □

6. APPLICABILITY OF THE CONCEPT

6.1. *Some general observations*

Can every bargaining solution be interpreted as a social compromise? If no, which properties must a bargaining solution possess to allow for a metric rationalization? Unfortunately, only partial answers to these questions are available to date. In a first attempt to identify sufficient conditions for the existence of metric rationalizations of bargaining solutions, we fix the set S of feasible outcomes and allow only the reference point to vary.

In the Appendix we prove

RESULT 6. Fix $S=\bar{S}\in\Gamma^n$. Let $f(\bar{S},r)$ be a bargaining solution on a domain D such that either $(\bar{S},r)\in\Sigma_{UT}$ or $(\bar{S},r)\in\Sigma_{RT}$ for all $(\bar{S},r)\in D$. If f satisfies [PO] and [IR], then there exists a metric δ such that $f(\bar{S},r)=K_\delta(\bar{S},r)$ for all r.

On a fixed set of feasible utility vectors, any bargaining solution that exhibits individual rationality [IR] and Pareto optimality [PO] can, thus, be viewed as a social compromise with respect to at least one metric. A formula for this metric (more precisely, one formula for each of the domains Σ_{UT} and Σ_{RT}) is provided in the Appendix. Our proof is inspired by Lerer's and Nitzan's (1985) work on social compromises in the context of social choice correspondences (which map preference profiles into subsets of alternatives). Lerer and Nitzan (1985) show that social choice correspondences are metrically rationalizable on the set of preference profiles if they respect the unanimity condition. Their proof is constructive and presents a special metric that always does the job and that serves as a model for the metrics in our proof. Unfortunately, our metrics also share their flaws with Lerer's and Nitzan's metric: They can only be used when the set of feasible outcomes is fixed (also see Nitzan, 1989 on this point). Furthermore, they are induced by the utility vectors chosen by the bargaining solution and cannot be applied without knowledge of that solution. To paraphrase the words of Lerer and Nitzan (1985, p. 194), the metrics are based on what the bargaining solution does rather than on what the bargaining problem looks like. While this certainly is not very attractive, the previous sections have already shown that more reasonable metrics can often be found that can be used for the purpose of rationalizing bargaining solutions.

In Lerer and Nitzan (1985)'s framework, the unanimity condition is sufficient and necessary to allow for a metric rationalization of social choice correspondences. In the bargaining framework here, [IR] and [PO] are only sufficient but not indispensable for metric rationalizability; the converse of Result 6 does not hold.

RESULT 7. Metrically rationalizable bargaining solutions might violate [IR] or [PO].

Proof. Let $n=2$. We restrict attention to the subclass of bargaining problems in Σ_{RT} that contains at least one strictly positive utility vector: $S \cap \mathbb{R}^2_{++} \neq \emptyset$. Consider the *constant bargaining solution* that chooses the origin as the outcome, regardless of where the reference point is located:

$$f_0(S,r)=0^2. \tag{6}$$

Clearly, this bargaining solution violates both [IR] (e.g., if $r \gg 0^2$) and [PO] (if $0^2 \notin \mathrm{WPO}(S)$). Now consider the following distance measure:

$$\tilde{\delta}(x,y)=\begin{cases} 0 & \text{if } x=y, \\ 1 & \text{if } x\neq y \;\text{ and }\; (x=0^2 \;\text{ or }\; y=0^2), \\ \alpha & \text{if } x_i \cdot y_i<0 \text{ for some } i, \\ \hat{\delta}(x,y) & \text{else} \end{cases} \tag{7}$$

where $1/2<\alpha<1$ and we define

$$\hat{\delta}(x,y)=\begin{cases} \dfrac{\delta_2(x,y)}{\delta_2(x,0^2)+\delta_2(y,0^2)} & \text{if } x\neq 0^2 \text{ or } y\neq 0^2 \\ 0 & \text{if } x=y=0^2. \end{cases} \tag{8}$$

Recall that δ_2 denotes the Euclidean metric. In the Appendix we show that $\hat{\delta}$ is a metric (on the entire $\mathbb{R}^2$ and therefore also on $\mathbb{R}^2_+$). It is then straightforward to verify that also $\tilde{\delta}$ is a metric.[11]

A peculiar feature of $\hat{\delta}$ is that it is confined to the interval $[0,1]$; this is an implication of the Euclidean metric satisfying the triangular inequality. Moreover, note that $\hat{\hat{\delta}}(x,y)$ assumes its maximum with value 1 on $\mathbb{R}^2_+$ if exactly one of x and y equals 0^2.[12] This implies, however, that $\tilde{\tilde{\delta}}$ (which makes use of $\hat{\hat{\delta}}$) indeed metrically rationalizes the trivial bargaining solution (6) for all $r \neq 0^2$: $f_0(S,r) = K_{\tilde{\tilde{\delta}}}(S,r)$. □

Result 7 highlights an important asymmetry in the interpretations of social compromises and bargaining solutions. For the latter we have a basic understanding of 'more is better', given by the ordering '$\geqslant$' on the utility space $\mathbb{R}^n$. For the former we only have an understanding of proximity and distance reflected in the definition of a social compromise, but – as the metric $\hat{\hat{\delta}}$ in the proof exemplifies – no relation to any sort of monotonicity. This may become clearer when we include the requirement that metrics satisfy the axiom of strict monotonicity [smon] introduced above. For Σ_{RT} we then immediately obtain that bargaining solutions that are individually rational social compromises for strictly monotonic metrics always satisfy [PO] (and, by definition, also [IR]):

RESULT 8. Let $K^{IR}_{\delta}(S,r)$ be an individually rational social compromise on $D \subseteq \Sigma_{RT}$ with a metric δ that satisfies [smon]. If $f(S,r) = K^{IR}_{\delta}(S,r)$ for all $(S,r) \in D$, then f satisfies [PO].

Proof. Suppose the contrary of the assertion. I.e., $f(S,r) = K^{IR}_{\delta}(S,r)$ with strictly monotonic δ and there exists $x \in S$ such that $x > f(S,r)$ for some $(S,r) \in D$. By the strict comprehensiveness of S, this amounts to $x \gg f(S,r)$. As $f = K^{IR}_{\delta}$ we have $f(S,r) \geqslant r$ and thus, due to monotonicity of δ, $\delta(r,x) > \delta(r, f(S,r))$. Hence, δ cannot be a full metric rationalization of f. □

6.2. *Restricting the range of admissable metrics*

Given the fairly abstract results of the previous section, can one be more specific about the type of (quasi-)metrics which qualify as rationalizers for bargaining solutions that are restricted by stronger axioms than merely [IR] and [PO]? To arrive at one positive answer to this question, we take a little detour. We start by observing that the concept of social compromising bears close relationships to the idea of (general) rationalizability of bargaining solutions, as developed in Lensberg (1987), Peters and Wakker (1991) and Bossert

(1994). There, the basic idea is the following: If a bargaining solution represents the outcome suggested by an optimizing impartial arbitrator, then it should be possible to rationalize this suggestion by a preference ordering. I.e., there should exist a class of complete, transitive, and reflexive binary relations $\succeq_r$ on $\mathbb{R}^n$ such that for all $(S,r)\in D$,

$$F(S,r)=\{x\in S|x\succeq_r y \quad \text{for all } y\in S\}. \tag{9}$$

Clearly, if (but not only if) a bargaining solution is a social compromise with respect to a quasi-metric δ, then it is also rationalizable in the sense of (9); for domains $D\subseteq\Sigma_{RT}$ $[D\subseteq\Sigma_{UT}]$ the required orderings can be defined as:

$$x\succeq_r^\delta y:\Longleftrightarrow \delta(x,r)\geqslant[\leqslant]\,\delta(y,r).$$

Let us restrict attention to the domain Σ_{RT} in what follows; the extension to Σ_{UT} is straightforward.

Lensberg (1987, Theorem 1) shows that a bargaining solution $f(S,r)$ satisfies [TRANS], [PO], [CONT], and *multilateral stability* if and only if it has an additively separable numerical representation,

$$H(S):=f(S,0^n)=\arg\max_{x\in S}\sum_{i=1}^{n}h_i(x_i),$$

where all h_i: $\mathbb{R}\to\mathbb{R}$ are strictly increasing and their sum is strictly quasi-concave.

Multilateral Stability is an axiom with roughly the following idea: Consider the outcome for a bargaining problem with $n>2$ agents. Fix the utilities for a subgroup of these agents at the levels in that outcome. Multilateral stability then requires that the solution applied to the bargaining problem that has the remaining individuals as its only agents allocates the same utility levels to these agents as the original solution. A bargaining solution which satisfies [PO], [CONT], and multilateral stability also satisfies [IIA] (see Lensberg, 1987, Lemma 1).

The fact that formula (10) of the bargaining solution can be provided independently of the reference point r is due to [TRANS]. Namely, with [TRANS] a bargaining solution $f(S,r)$ satisfies

$f(S, r) = r + f(S - \{r\}, 0^n)$. Without loss of generality, we can therefore choose the origin as the reference point r.

Bossert (1994, Theorem 2) builds on Lensberg's result for the case $n = 2$. He shows that if a bargaining solution satisfies [TRANS], [PO], [CONT], and [IIA], then it is numerically representable by a strictly monotonic, strictly quasi-concave and upper semi-continuous function. Lensberg's and Bossert's results can be used to restrict – in certain circumstances – the range of 'reasonable' distance measures:

- From Lensberg's result follows for arbitrary n that a social compromise with respect to an additively separable, monotonic metric whose contour sets $\{y | \delta(x, y) \geqslant \bar{\delta}\}$ are strictly convex for all $\bar{\delta} > 0$ and $x \in \mathbb{R}^n$ satisfies [TRANS], [PO], [CONT], [IIA], and multilateral stability. Result 5 confirms this for the quasi-metric p-norms with $p < 1$.
- From Bossert's result we learn that if we want δ to metrically rationalize an individually rational bargaining solution that satisfies [TRANS], [PO], [CONT], and [IIA] in two-person problems, then we can (but need not) confine ourselves to quasi-metrics that are continuous, satisfy (mon), and have strictly convex and closed contour sets. Again this encompasses all p-norms with $p < 1$.

Observe that Lensberg's and Bossert's results exclude the metric p-norms with $p \geqslant 1$; social compromises with respect to these metrics entail problems with respect to single-valuedness or [CONT] on Σ_{RT}. This leaves us with the interesting question whether there exists any metric (i.e., a distance measure that satisfies the triangular inequality) that rationalizes bargaining solutions on Σ_{RT} which possess the axiomatic properties proposed by Lensberg and Bossert. Answering this question might also yield further insights for the intertwined issues of social compromising and numerically representing bargaining solutions.

7. CONCLUSION

This paper introduces the idea of social compromises and metric rationalization into the theory of bargaining. It provides a workable formal framework for these ideas and derives some first results.

From our observations we conclude that the idea of social compromises is worthwhile to be pursued in the bargaining context for at least three reasons:

First, several of the most widely used bargaining solutions have near-at-hand interpretations in lying at maximal proximity or distance from some benchmark point. In these cases the metric approach nicely complements the standard bargaining approach by making explicit the measurement device upon which the interpretation in terms of distance is based.

Second, the metric approach constitutes a framework on its own for solving bargaining problems. Compromises have previously been employed fruitfully to 'invent' new solutions in various other contexts of collective choice and multi-criteria decision making. In the area of bargaining, the notion of social compromises widens the range of possible solution concepts in a consequential and meaningful way.

Third, the metric approach offers a new normative view on the bargaining problem. Social compromising aims at making a possibly large (or, in the utopic setting, possibly small) difference from the reference situation. Axioms imposed on metrics therefore differ conceptually from axioms imposed on bargaining solutions; they refer to the largeness of differences between utility allocations rather than to features of the bargaining situation. We do not argue in favour of or against any specific axiomatic approach. Rather we emphasize that one reason why one might interested in the metric rationalizability of bargaining solutions is that one can then potentially ground the solution in an alternative axiomatic set-up.

As a first attempt our paper is far from comprehensive. It leaves open many questions for future research. In particular, the search for reasonable metrics and for criteria which determine what 'reasonable' means in this context has just begun. We hope, however, that the reader shares our conviction that social compromises are a workable and productive concept in the field of bargaining.

ACKNOWLEDGMENTS

The authors are grateful, with the usual disclaimer, to Thomas Eichner, Hendrik Hakenes, and to conference participants in

Vancouver, Rostock, and Namur for helpful comments. Three anonymous referees provided fruitful criticism, corrections, and suggestions. Andreas Wagener thanks Deutsche Forschungsgemeinschaft (DFG) for financial support.

APPENDIX

Proof of result 6

We hold the feasible set fixed: $S=\bar{S}\in\Gamma^n$. Due to the different domains for f mentioned in the assertion, we proceed in two parts.

(a) Let f be defined for all $(\bar{S},r)$ such that $(\bar{S},r)\in\Sigma_{UT}$ and let f satisfy both [PO] and [IR] on that domain. Define

$$f^+(\bar{S},x):=\begin{cases} f(\bar{S},x) & \text{if} \quad x\notin\bar{S} \\ x & \text{else.}\end{cases}$$

f^+ extends f from $\bar{S}$ to $\mathbb{R}^n$. Since f satisfies [IR], we have $f^+(\bar{S},r)=r=f(\bar{S},r)$ for all $r\in PO(\bar{S})$. Now consider the following metric:

$$\delta^+(x,y)=\begin{cases} 1.75 \text{ if } x\neq y & \text{and} \quad \{x,y\}\cap PO(\bar{S})=\emptyset \\ 1.5 \text{ if } x\neq y & \text{and} \quad \{x,y\}\cap PO(\bar{S})\neq\emptyset \\ & \text{and } f^+(\bar{S},x)\neq f^+(\bar{S},y) \\ 1 \text{ if } x\neq y & \text{and} \quad \{x,y\}\cap PO(\bar{S})\neq\emptyset \\ & \text{and } f^+(\bar{S},x)= f^+(\bar{S},y) \\ 0 \text{ if } x=y. & \end{cases}$$

First assume that $r\in \mathrm{PO}(\bar{S})$. Then $K_\delta(\bar{S},r)=r$ for *all* quasi-metrics δ. Further, by [IR] we also have $f(\bar{S},r)=r$.

Now assume that $r\notin\bar{S}$. Check that $\delta^+(r,f(\bar{S},r))=1$ by [PO] and [IR]. (Note that we in fact need [IR] here. Otherwise, we cannot ensure that $f(\bar{S},f(\bar{S},r))=f(\bar{S},r)$.) Furthermore, $\delta^+(r,x)=1.5$ for all $x\in\mathrm{PO}(\bar{S})\backslash\{f(\bar{S},r)\}$ (because by [IR], $f(\bar{S},x)=x\neq f(\bar{S},r)$) and $\delta^+(r,x)=1.75$ for all $x\notin\mathrm{PO}(\bar{S})\cup\{r\}$. Hence,

$$1=\delta^+(r,f(\bar{S},r))<\delta^+(r,x)\in\{1.5,1.75\}$$

for all $x\notin PO(\bar{S})\cup\{r,f(\bar{S},r)\}$ and hence $K_{\delta^+}(\bar{S},r)=f(\bar{S},r)$.

(b) Now let f be defined for all $(\bar{S}, r)$ such that $(\bar{S}, r) \in \Sigma_{RT}$ and let f satisfy both [PO] and [IR] on that domain. Define

$$f^{-}(\bar{S}, x) := \begin{cases} f(\bar{S}, x) & \text{if } x \in \bar{S} \\ x & \text{else.} \end{cases}$$

f^{-} extends f from $\bar{S}$ to $\mathbb{R}^n$. Since f satisfies [IR], we have $f^{-}(\bar{S}, r) = r = f(\bar{S}, r)$ for all $r \in PO(\bar{S})$. Now consider the following metric:

$$\delta^{-1}(x, y) = \begin{cases} 0 \text{ if } x = y & \\ 1 \text{ if } x \neq y & \text{and } \{x, y\} \cap \mathrm{PO}(\bar{S}) = \emptyset \\ 1.5 \text{ if } x \neq y & \text{and } \{x, y\} \cap \mathrm{PO}(\bar{S}) \neq \emptyset \\ & \text{and } f^{-1}(\bar{S}, x) \neq f^{-}(\bar{S}, y) \\ 1.75 \text{ if } x \neq y & \text{and } \{x, y\} \cap \mathrm{PO}(\bar{S}) \neq \emptyset \\ & \text{and } f^{-1}(\bar{S}, x) = f^{-}(\bar{S}, y)\,. \end{cases}$$

Check by similar reasoning as above that $1.75 = \delta^{-}(r, f(\bar{S}, r)) > \delta^{-}(r, x)$ for all $x \notin \mathrm{PO}(\bar{S}) \cup \{r, f(\bar{S}, r)\}$. Hence, δ^{-} fully rationalizes f. □

Proof that $\hat{\delta}$ is a metric

LEMMA 1. *The function $\hat{\delta}$ (defined in (8)) is a metric.*

Proof. Obviously, (i) $\hat{\delta}(x, y) \geqslant 0$ for all $x, y \in \mathbb{R}^2$; (ii) $\hat{\delta}(x, y) = 0$ if and only if $x = y$; and (iii) $\hat{\delta}(x, y) = \hat{\delta}(y, x)$. It remains to be shown that $\hat{\delta}$ satisfies the triangular inequality.

Consider arbitrary three points $x, y, z \in \mathbb{R}^2$. The triangular inequality for $\hat{\delta}$ is trivial when one of these points is the origin 0^2 (recall that $\hat{\delta}(x, y)$ is maximal for $y = 0^2$). Therefore we assume that $x, y, z \neq 0^2$. Together with the origin 0^2 these points span up a (possibly degenerate) quadrilateral. To simplify notation, we write $\bar{p} = \delta_2(p, 0^2)$ for the Euclidean distance of vector p to the origin. Further let $\alpha = \delta_2(y, z)$, $\beta = \delta_2(x, z)$, and $\gamma = \delta_2(x, y)$. We then have to show that

$$\begin{aligned}
&\hat{\delta}(x,z) \leqslant \hat{\delta}(x,y) + \hat{\delta}(y,z) \\
&\iff \frac{\beta}{\bar{x}+\bar{z}} \leqslant \frac{\gamma}{\bar{x}+\bar{y}} + \frac{\alpha}{\bar{y}+\bar{z}} \\
&\iff H := \bar{z}^2\gamma + \bar{x}^2\alpha - \bar{y}^2\beta + (\gamma+\alpha-\beta)\cdot[\bar{x}\bar{y}+\bar{y}\bar{z}+\bar{x}\bar{z}] \geqslant 0.
\end{aligned} \tag{10}$$

Assume that $\bar{x} \leqslant \bar{z}$ (the proof for $\bar{z} \leqslant \bar{x}$ follows along analogous lines). Three cases must be distinguished:

(a) $\bar{y} \leqslant \bar{x} \leqslant \bar{z}$. Then

$$\bar{z}^2\gamma + \bar{x}^2\alpha - \bar{y}^2\beta \geqslant \bar{x}^2 \cdot (\gamma+\alpha-\beta) \geqslant 0$$

by the triangular inequality for δ_2 on the triangle $\Delta(x,y,z)$. Thus, $H \geqslant 0$ in (10).

(b) $\bar{x} \leqslant \bar{y} \leqslant \bar{z}$. Calculate

$$\begin{aligned}
H &\geqslant \bar{z}^2\gamma + \bar{x}^2\alpha - \bar{y}^2\beta + (\gamma+\alpha-\beta)\bar{x}\bar{z} \\
&= \beta\cdot[\bar{x}\bar{y}+\bar{y}\bar{z}-\bar{x}\bar{z}-\bar{y}^2] + (\bar{x}+\bar{z})\cdot[\alpha\bar{x}+\gamma\bar{z}-\beta\bar{y}] \\
&= \beta(\bar{y}-\bar{x})(\bar{z}-\bar{y}) + (\bar{x}+\bar{z})\cdot[\alpha\bar{x}+\gamma\bar{z}-\beta\bar{y}] \geqslant 0.
\end{aligned}$$

To see the last inequality, verify that $\alpha\bar{x}+\gamma\bar{z}-\beta\bar{y} \geqslant 0$ is Ptolemy's Inequality for the quadrilateral $\Box(0,x,y,z)$.

(c) $\bar{x} \leqslant \bar{z} \leqslant \bar{y}$. For any vector p consider the vector p' such that $p' = p/\bar{p}^2$ and, thus, $\bar{p}' = 1/\bar{p}$. Verify that for any two points $p, q \neq 0^2$ and their transforms p', q' we have

$$\begin{aligned}
\delta_2(p',q') &= \sqrt{\sum\left(\frac{p_i}{\bar{p}^2} - \frac{q_i}{\bar{q}^2}\right)^2} \\
&= \sqrt{\frac{\sum p_i^2}{\bar{p}^4} - 2\cdot\frac{\sum p_i q_i}{\bar{p}^2\bar{q}^2} + \frac{\sum q_i^2}{\bar{q}^4}} \\
&= \frac{1}{\bar{p}\cdot\bar{q}}\cdot\sqrt{\bar{q}^2 - 2\cdot\sum p_i q_i + \bar{p}^2} \\
&= \frac{1}{\bar{p}\cdot\bar{q}}\cdot\sqrt{\sum(q_i-p_i)^2} = \frac{\delta_2(p,q)}{\bar{p}\cdot\bar{q}}
\end{aligned}$$

(recall that $\bar{p}=\sqrt{\sum p_i^2}$ and $\bar{q}=\sqrt{\sum q_i^2}$). From this we get

$$\hat{\delta}(p',q') = \frac{\delta_2(p',q')}{\bar{p}'+\bar{q}'} = \frac{\delta_2(p,q)}{\bar{p}\bar{q}\left(\frac{1}{\bar{p}}+\frac{1}{\bar{q}}\right)} = \frac{\delta_2(p,q)}{\bar{p}+\bar{q}} = \hat{\delta}(p,q).$$

Therefore,

$$\begin{aligned}\hat{\delta}(x,z) &= \hat{\delta}(x',z') = \hat{\delta}(z',x') \leqslant \hat{\delta}(z',y') + \hat{\delta}(y',x')\\ &= \hat{\delta}(y,z) + \hat{\delta}(x,y)\end{aligned}$$

from case a) (note that $\bar{y}' \leqslant \bar{z}' \leqslant \bar{x}'$). □

NOTES

1. Due to the strict comprehensiveness of S the set PO(S) coincides with the set of *weakly* Pareto optimal points in S.
2. See, e.g., Thomson (1981), Herrero (1998), Conley et al. (1997), or Gerber (1998). The *goal functions*-approach by Klemisch-Ahlert (1996) is a closely related concept.

   Trivially, all bargaining solutions which only use either the disagreement point (e.g., the Nash solution) or the claims point (e.g., Bossert's (1993) claim-egalitarian solution) can be interpreted in that way. An example for a reference function which uses both the disagreement and the claims point is the *adjusted threat point* $t(S,d,c)$ due to Curiel et al. (1987) (called *natural reference point* by Herrero, 1998). Its components are for $i=1,\ldots,n$ defined by $t_i(S,d,c)=\max\{d_i, \max\{x_i|(x_i,c_{-i})\in S\}\}$.
3. Below we will sometimes use axiomatic properties related to the reference point. We implicitly always assume that these axioms are consistent with the choice of the reference point.
4. Precisely, the displayed equation in Definition 2 should read: $K_\delta(S,r)=\{f(S,r)\}$. Here and in the sequel we identify a singleton set $\{x\}$ with its only element x.
5. The reason for the notation δ_∞ will be become obvious below.
6. Our axiom differs from the one typically used under the name [IR] in literature which requires that agents benefit from an agreement relative to the disagreement point rather than relative to the reference point. We nevertheless stick to the label [IR] and hope that this does not create misinterpretations.
7. Translation invariance is, e.g., implied by the agents' utility functions being cardinal unit comparable or by scale invariance as defined in Nash (1950).
8. The main difference between Result 3 and Gehrig and Hellwig (1982) is that we impose from the outset the restriction that the object to be characterized

is a quasi-metric (not a general function). A different axiomatization of p-norms is provided by Ebert (1984) in the context of measuring income inequality. His approach also deals with a variable number of agents, an issue which we do not wish to touch here.

9. Originally these properties read as:
(A3) *For all* $x, y \in \mathbb{R}^n$, $\delta(x,y) = \Psi(\sum_{i=1}^{n} \xi_i(x_i, y_i))$ *where* $\Psi : \mathbb{R} \to \mathbb{R}$, $\xi_i : \mathbb{R}^2 \to \mathbb{R} (i = 1, \ldots, n)$ *are continuous functions,* Ψ *is strictly monotonic,* $h_i(w) := \xi_i(w, 0)$ *is strictly decreasing on* $(-\infty, 0)$ *and strictly increasing on* $(0, \infty)$.
(A4) *For all i,* $h_i(w_0) = h_i(-w_0)$ *for at least one* $w_0 \neq 0$.
10. Note that Results 3 and 4 do not cover the limit cases of p-norms for $p \to 0$ (minimum function) and $p \to \infty$ (maximum function); neither satisfies the additivity axiom.
11. We choose the metric $\tilde{\delta}$ and not solely $\hat{\delta}$ for the following reason: With $\hat{\delta}$, two points on a ray through, but on different sides of, the origin have a distance of 1. Since $\hat{\delta}(x, 0^2) = 1$ for all $x \neq 0^2$ and since we want the origin to be the unique distance maximizer on $\mathbb{R}^2$, we have to somehow circumvent this case. The problem cannot be fixed by assigning any other value than 1 (β, say) to the distance between two points on a ray through the origin only. Namely, then it would always be possible to find x and y, both located in the same orthant, such that $\hat{\delta}(x, y)$ is larger than $\hat{\delta}(x, -\lambda y) + \delta(-\lambda y, y) = \hat{\delta}(x, -\lambda y) + \beta$ for some $\lambda > 0$; hence the triangular inequality would be violated. Assigning a distance of $\alpha > 0.5$ to all points that differ in the sign of at least one coordinate avoids that problem because it assigns a distance of $2\alpha > 1 > \hat{\delta}(x, y)$ to the detour from x to y via $-\lambda y$.
12. The function $\hat{\delta}$ (and consequently $\tilde{\delta}$) has a singularity at $(0^2, 0^2)$ which does not pose any problem here, however.

REFERENCES

Baigent, N. (1987a), Metric rationalisation of social welfare functions according to the principles of social choice, *Mathematical Social Sciences* 13, 59–65.

Baigent, N. (1987b), Preference proximity and anonymous social choice, *Quarterly Journal of Economics* 102, 161–169.

Bossert, W. (1993), An alternative solution to two-person bargaining problems with claims, *Mathematical Social Sciences* 25, 205–220.

Bossert, W. (1994), Rational choice and two-person bargaining solutions, *Journal of Mathematical Economics* 23, 549–563.

Bossert, W. and Stehling, F. (1993), Social preferences as optimal compromises, in Diewert, W.E. et al. (eds.), *Mathematical Modelling in Economics. Essays in Honour of Wolfgang Eichhorn*. Springer-Verlag, Berlin etc., pp. 56–64.

Bossert, W. and Storcken, T. (1992), Strategy-proofness of social welfare functions: The use of the Kemeny distance between preference orderings, *Social Choice and Welfare* 9, 345–360.

Campbell, D.E. and Nitzan, S.I. (1986), Social compromise and social metrics, *Social Choice and Welfare* 3, 1–16.

Chun, Y. (1988), The equal-loss principle for bargaining problems, *Economics Letters* 26, 103–106.

Chun, Y. and Thomson, W. (1992), Bargaining problems with claims, *Mathematical Social Sciences* 24, 19–33.

Conley, J.P., McLean, R.P. and Wilkie, S. (1997), Reference functions and possibility theorems for cardinal social choice problems, *Social Choice and Welfare* 14, 65–78.

Conley, J.P., McLean, R. and Wilkie, S. (2000), Axiomatic foundations for compromise theory: The duality of bargaining theory and multi-objective programming. Forthcoming, *Games and Economic Behaviour*.

Curiel, I.J., Maschler, M. and Tijs, S.H. (1987), Bankruptcy games, *Zeitschrift für Operations Research* 31, A143–A159.

Dagan, N. (1996), New characterizations of old bankruptcy rules, *Social Choice and Welfare* 13, 51–59.

Ebert, U. (1984), Measures of distance for income distributions, *Journal of Economic Theory* 32, 266–274.

Farkas, D. and Nitzan, S. (1979), The Borda rule and pareto stability. A Comment, *Econometrica* 47, 1305–1306.

Freimer, M. and Yu, P.L. (1976), Some new results on compromise solutions for group decision problems, *Management Science* 22, 688–693.

Gehrig, W. (1984), Functional equations methods applied to economic problems: Some examples, in Aczél, J. (ed.), *Functional Equations: History, Applications and Theory.* D. Reidel, Dordrecht etc., pp. 17–32.

Gehrig, W. and Hellwig, K. (1982), Eine charakterisierung der gewichteten L_r-distanz, *OR Spektrum* 3, 233–237.

Gerber, A. (1998), Reference functions and solutions to bargaining problems with and without claims, IMW Working Paper No. 278, University of Bielefeld. Forthcoming, *Social Choice and Welfare*.

Herrero, C. (1998), Endogenous reference points and the adjusted proportional solution for bargaining problems with claims, *Social Choice and Welfare* 15, 113–119.

Kalai, E. (1977), Proportional solutions to bargaining problems: Interpersonal utility comparisons, *Econometrica* 45, 1623–1630.

Kalai, E. and Smorodinsky, M. (1975), Other solutions to nash's bargaining problem, *Econometrica* 43, 513–518.

Klemisch-Ahlert, M. (1996), *Bargaining in Ethical and Economic Environments*, Springer-Verlag, Berlin etc.

Lensberg, T. (1987), Stability and collective rationality, *Econometrica* 55, 935–961.

Lerer, E. and Nitzan, S. (1985), Some general results on the metric rationalization of social decision rules, *Journal of Economic Theory* 37, 191–201.

Nash, J.F. (1950), The bargaining problem, *Econometrica* 18, 155–162.

Nitzan, S. (1981), Some measures of closeness to unanimity and their implications, *Theory and Decision* 13, 129–138.

Nitzan, S. (1989), More on the preservation of preference proximity and anonymous social choice, *Quarterly Journal of Economics* 104, 187–190.

Peters, H. and Wakker, P. (1991), Independence of irrelevant alternatives and revealed group preference, *Econometrica* 59, 1787–1801.

Rubinstein, A. and Zhou, L. (1999), Choice problems with a 'reference' point, *Mathematical Social Sciences* 37, 205–209.

Stehling, F. (1978), Indices of preference inequality and the construction of social preference relations, in Eichhorn, W. et al. (eds.), *Theory and Applications of Economic Indices*. Physica, Würzburg.

Thomson, W. (1981), A class of solutions to bargaining problems, *Journal of Economic Theory* 25, 431–441.

Voorneveld, M. and van den Nouweland, A. (2001), *An Axiomatization of the Euclidean Compromise Solution*, Mimeo, University of Oregon, Eugene.

Yu, P.L. (1973), A class of solutions for group decision problems, *Management Science* 19, 936–946.

Yu, P.L. and Leitmann, G. (1974), Compromise solutions, domination structures, and Salukvadze's solution, *Journal of Optimization Theory and Applications* 13, 362–378.

Address for correspondence: Andreas Pfingsten, Westfälische Wilhelms-Universität Münster, Institut für Kreditwesen, Universitätsstr, 14-16, 48143 Münster, Germany.

E-mail: 21anpf@wiwi.uni-muenster.de; Tel./Fax: +49-251-832 2881/2882.

Andreas Wagener, Department of Economics, University of Vienna, Hohenstaufengasse 9, 1010 Vienna, Austria.

E-mail: andreas.wagener@univie.ac.at; Tel. +43-1-4277 37423.