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Tractable hedging with additional hedge instruments

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Abstract In an uncertain volatility model where only the stock and the money market account are traded, the upper price bound of a European claim is given by the solution of a Black-Scholes-Barenblatt equation. If an additional hedge instrument is available, the price bound can be tightened. This is also true if the set of admissible strategies is restricted to tractable strategies, which are defined as sums of Black-Scholes strategies. We study the structure of both strategies, the general strategies and the tractable strategies, when an additional convex instrument is available. For a call and a bullish vertical spread, we give closed-form solutions for the optimal tractable hedge when the additional instrument is a call option. We show that the position in the additional convex claim as well as the reduction in the price bounds allow to capture the amount of convexity risk a claim is exposed to.

Keywords Stochastic volatility · Robust hedging · Tractable hedging · Uncertain volatility model · Additional hedge instrument · Coherent risk measure

JEL Classification G13

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1 Introduction

In a complete market, any contingent claim can be replicated by a self-financing portfolio strategy consisting of basis assets, and can thus also be hedged perfectly. In practice, however, markets are not complete. One reason are stochastic volatility and stochastic jumps,[1] where in the latter case infinitely many traded options would be needed to complete the market. Another problem are trading restrictions and the implementation of trading strategies in discrete instead of continuous time.[2] Recently, a lot of attention is also drawn to the topic of model risk, that is to the problem that the true data-generating process is not known.[3] Model risk poses severe problems when it comes to hedging, since in most cases the investor has to know the true model to determine either a perfect hedge within a complete market model or a hedge which satisfies some optimality criteria (like minimal variance of the hedging error, e.g.) in an incomplete market.

A setup that explicitly takes model risk into account is the uncertain volatility model (UVM) of Avellaneda et al. (1995). The stock price is assumed to follow a diffusion process with an unknown (but bounded) volatility. The class of possible stochastic processes includes, among others, the case of a constant volatility as in Black–Scholes, a time-dependent volatility, or stochastic volatility models with bounded volatility. For a given claim which is not affine in the price of the underlying, there does not exist a hedging strategy using the stock and the money market account only which is perfect with respect to all these processes simultaneously.[4] Instead, we consider superhedging strategies, i.e. self-financing strategies for which the terminal payoff is in all models at least as large as the payoff of the claim. The initial capital needed for such a superhedging strategy is an upper price bound for the claim, and the initial capital needed for the cheapest superhedge is the lowest upper price bound. It can be shown that the cheapest superhedging strategy is equal to a replicating strategy in one of the possible models, the so-called worst-case model. The solution of this problem is firstly given in Avellaneda et al. (1995) and Lyons (1995), and we call the corresponding strategies ALP-hedges in the following. The lowest upper price bound solves a Black–Scholes–Barenblatt equation, and the number of stocks is given by the delta of this lowest upper price bound. In general, the solution has to be determined

[1] Models with stochastic volatility are discussed by Hull and White (1987) and Heston (1993), while Merton (1976) considers a jump-diffusion models. Bakshi et al. (1997), Duffie et al. (2000), Eraker et al. (2003) and Broadie et al. (2007), among others, analyze models with stochastic volatility and jumps.

[2] Discretely adjusted option hedges are analyzed in Boyle and Emanuel (1980) and Bertsimas et al. (2000). Option replication in discrete time and the implication of transaction costs is studied by Leland (1985), Bensaid et al. (1992), Boyle and Vorst (1992), Avellaneda and Parás (1994), Grannan and Swindle (1996) and Toft (1996). Portfolio strategies in discrete time are e.g. analyzed in Rogers (2001) or Branger et al. (2008).

[3] The impact of model risk on hedging is e.g. studied in Avellaneda et al. (1995), Lyons (1995), Bergman et al. (1996), El Karoui et al. (1998), Hobson (1998), Dudenhausen et al. (1998), Bossy et al. (2000), Mahayni (2003) and Bossy et al. (2006).

[4] Note that we do not allow for trading variance swaps. Carr and Sun (2007) show that a European-style payoff for a path-independent claim can be replicated by dynamic trading in futures and variance swaps. For details concerning hedging strategies including variance contracts, we refer to their paper and the literature mentioned herein.

numerically. More recently, the problem of superhedging in an uncertain volatility model is also considered in Vanden (2006) and Branger and Mahayni (2006). Vanden (2006) provides an exact solution for a class of European payoffs, including digital options and option spreads. In contrast to this, Branger and Mahayni (2006) introduce an additional constraint to the optimization problem. They restrict the set of admissible strategies to the sum of Black/Scholes-type strategies, so-called tractable strategies, which are a simple and parsimonious choice if the true model is not known.

The concept of superhedging is often criticized as too expensive. Intuitively, it is clear that a hedge which needs to be effective for a whole set of models simultaneously may afford a very high initial investment. This problem can be mitigated by the introduction of an additional hedge instrument.[5] The expensive superhedge then has to be implemented for the remaining payoff only.

The use of an additional instrument in the hedge portfolio will reduce the initial capital. In an uncertain volatility model, the basic intuition can best be explained by considering an additional instrument that is convex. The upper price bound of a claim when no additional instrument is used is given by the worst case model which is implied by the claim to be hedged. In particular, convex payoffs are hedged (respectively priced) at the upper volatility and concave payoffs at the lower volatility bound. For payoffs which are piecewise convex and concave, the volatility in the worst case model switches between the upper and the lower volatility depending on the sign of the worst case gamma. By trading in the additional convex claim, the investor buys (sells) its convexity at the market price instead of at the upper (lower) volatility bound, which allows him to reduce the initial capital needed. A similar argument holds for an additional claim which is neither convex nor concave, where the investor trades its curvature at the market price instead of the price implied within the worst-case model.

We first give conditions on the optimal position in the additional instrument and on the remaining payoff. We assume that the additional claim is convex, e.g. a (highly liquid) ATM-option. Second, we analyze the structure of the optimal hedge. In particular, we are interested in whether the investor uses a long or a short position in the additional instrument, that is whether he cares more about the convexity risk or the concavity risk of the claim to be hedged. We are also interested in the reduction in initial capital he can achieve. Third, we compare our results to the case of ALP-strategies. We want to know for which hedge the investor profits more from the availability of the additional instrument, and for which hedge the optimal position in this instrument is larger.

The price bounds allow to assess the amount of convexity and concavity risk a claim is exposed to. In the UVM, convexity risk is basically priced at the upper volatility bound, while concavity risk is priced at the lower volatility bound. The sensitivity of the price bounds with respect to lower (upper) volatility bound thus gives the impact of changes in concavity (convexity) risk while keeping the other risk constant. Secondly, the position in an additional convex instrument like an ATM-call allows to asses

[5] There is a wide strand of literature concerning the information contained in observed (option) prices. E.g., Ritchken (1985) uses the additional information to restrict the set of feasible stochastic discount factors which then gives a limited range of option prices. Without postulating completeness, we also refer to Ryan (2000) and, more recently, Franke et al. (2006).

whether the claim to be hedged has a higher exposure to convexity or concavity risk. Intuitively, the investor will take a long position in the call if the convexity risk of the claim to be hedged is more important, and a short position if the concavity risk is more important. A third measure for convexity and concavity risk is given by the reduction in the price bound when an additional instrument is available.

Beyond the general characterization of the optimal ALP-hedge and tractable hedge with an additional instrument, we consider two benchmark cases in detail. In the first case, we consider a long position in a call option, i.e. a claim which is only exposed to convexity risk. As expected, the position in the additional ATM-option is the larger the more similar the two claims are in terms of their strikes and the larger the upper volatility bound, that is the price of hedging convexity risk. The reduction in the initial capital due to the use of the additional instrument is larger for the ALP-hedge than for the tractable hedge. Furthermore, the investor takes a larger position in the additional instrument in case of the ALP-hedge than in case of tractable hedging. The intuition for both these findings is that the original payoff is convex, while the remaining payoff which has to be hedged after buying the call option is neither convex nor concave. And while the tractable hedge and the ALP-hedge coincide for convex claims, the tractable hedge is more expensive than the ALP-hedge for mixed payoffs. This makes the investor more reluctant when trading the additional claim.

The second benchmark case is given by a bullish vertical spread, i.e. by the simplest choice of a payoff which is neither uniformly convex nor concave. Consequently, the investor will either take a long or a short position in the additional call option, depending on whether the exposure of the bullish vertical spread to convexity risk or concavity risk is larger. Again, the position is the larger the more similar the additional call is to the claim to be hedged. However, the absolute size of the position may be larger with the ALP-hedge or with the tractable hedge, depending on the current stock price. The same holds true for the reduction in the initial capital. In particular, we show that a tractable hedge with an additional claim can be cheaper than the ALP-hedge without this additional claim.

The paper is organized as follows. Section 2 recalls some well known results about the price bounds in an uncertain volatility model with and without the restriction to tractable strategies. The corresponding price bounds are identified as coherent risk measures, and we show that the UV-model allows for a different pricing of convexity and concavity risk. In Section 3, we allow for the use of an additional instrument and analyze the structure of the optimal tractable robust hedge. Section 4 discusses the two benchmark cases given by a call option and a bullish vertical spread. Section 5 concludes.

2 Price bounds in an uncertain volatility model

Along the lines of Avellaneda et al. (1995) we consider an *uncertain volatility model*. There are two traded assets, a risky asset X and a zero bond B with maturity in T. All prices are already expressed in units of this zero bond, which implies $B \equiv 1$. The asset price X follows a diffusion process with a diffusion coefficient σ which is bounded above by $\sigma_{\max}$ and bounded below by $\sigma_{\min}$, i.e.

$$dX_t = X_t(\mu_t dt + \sigma_t dW_t) \tag{1}$$

where μ_t and σ_t are non-anticipative functions such that $\sigma_{\min} \le \sigma_t \le \sigma_{\max}$, and W is a Brownian Motion under the real world probability measure P. In particular, we assume that the martingale part of X is a stochastic integral with respect to a one-dimensional Brownian motion W, i.e.

$$d(X_t)^M = \sigma_t X_t dW_t,$$

where $(X_t)^M$ denotes the martingale part of the Doob–Meyer decomposition of the process X. Apart from integrability conditions, there are no restrictions on σ.[6] Since the volatility is unknown and may well be stochastic, the investor cannot implement a perfect hedge for non-affine claims, and the market is incomplete. Instead, we will consider super- (and sub-) hedges in the following, i.e. self-financing strategies which dominate (are domatinated by) the claim to be hedged with probability one in all models with bounded volatility given by Eq. 1. In the following, these superhedging strategies are called *robust hedging strategies*.

We deal with European path-independent claims with maturity $T > 0$ and characterize a claim by its payoff-function $h : \mathbb{R} \to \mathbb{R}_+$. The upper price bound for a claim h is the lowest initial capital needed for a self-financing strategy which dominates h, i.e. the initial capital for the cheapest robust strategy. Note that a robust strategy for h dominates the payoff h and thus actually provides a hedge for a short-position in the claim with payoff-function h.

In the unconstrained case, there is no restriction on the set of admissible strategies.[7] The lowest upper price bound u^{ALP} of a payoff-function h is given by[8]

$$u^{\text{ALP}}(t, x; h) := \text{Sup}_{Q \in Q^*} E_t^Q[h(X_T)] \tag{2}$$

where Q^* denotes the class of all probability measures on the set of paths $\{X_t, 0 \le t \le T\}$, such that for some bounded σ_t it holds that

$$dX_t = X_t \sigma_t dW_t^*.$$

In Avellaneda et al. (1995) and Lyons (1995) it is shown that this lowest upper price bound is the solution of a Black–Scholes–Barenblatt (BSB) equation.[9] In particular, for all $x \ge 0$ it holds

6 The assumption that the asset price is only driven by one Brownian motion (or by the first component of a d-dimensional Brownian motion) does not restrict the analysis but simplifies the notation. Furthermore, we assume that there exists at least one equivalent martingale measure which guarantees that the financial market model is arbitrage-free.

7 The set of admissible strategies is given by all predictable processes $\phi = \left(\phi^X, \phi^B\right)$.

8 The duality between the problem of the cheapest superhedge and the lowest upper price bound is for example given in El Karoui and Quenez (1995).

9 The existence, uniqueness and smoothness for this equation is analyzed in detail in Vargiolu (2001).

$$\frac{\partial u^{\mathrm{ALP}}}{\partial t}(t,x;h)+\frac{1}{2}\sigma^2(t,x;h)x^2\frac{\partial^2 u^{\mathrm{ALP}}}{\partial x^2}(t,x;h)=0$$
$$u^{\mathrm{ALP}}(T,x;h)=h(x)$$

where

$$\sigma(t,x;h)=\sigma_{\max}1_{\left\{\frac{\partial^2 u^{\mathrm{ALP}}}{\partial x^2}(t,x;h)\geq 0\right\}}+\sigma_{\min}1_{\left\{\frac{\partial^2 u^{\mathrm{ALP}}}{\partial x^2}(t,x;h)<0\right\}}.$$

The corresponding cheapest robust hedge is given by a portfolio consisting of ϕ_t^X assets X and ϕ_t^B bonds where

$$\phi_t^X=\frac{\partial u^{\mathrm{ALP}}}{\partial x}(t,X_t;h),\quad \phi_t^B=u^{\mathrm{ALP}}(t,X_t;h)-\frac{\partial u^{\mathrm{ALP}}}{\partial x}(t,X_t;h)X_t, \tag{3}$$

cf. Avellaneda et al. (1995). In general, the price bound $u^{\mathrm{ALP}}(t,x;h)$ and the corresponding volatility $\sigma(t,x;h)$ have to be determined simultaneously. The solution is given by a Hamilton–Jacobi–Bellman equation and is linked to a stochastic control problem. If h is convex (concave), the problem simplifies considerably. In this case, $\sigma(t,x;h)=\sigma_{\max}$ ($\sigma(t,x;h)=\sigma_{\min}$), and the BSB partial differential equation reduces to a Black–Scholes (BS) equation. Convex (concave) claims are thus hedged and priced at the upper (lower) volatility bound. In addition, closed-form-solutions are available for digital options and vertical spreads, cf. Vanden (2006).

We now restrict the set of admissible strategies to tractable strategies along the lines of Branger and Mahayni (2006). A *tractable robust hedge* for h is represented by $(\overline{h},\underline{h})$ where $\overline{h}$ is convex and $\underline{h}$ is concave and where

$$\overline{h}(y)+\underline{h}(y)\geq h(y)\quad\forall y\geq 0.$$

The tractable robust hedge itself is given by the sum of a BS-hedge for $\overline{h}$ at the upper volatility bound and a BS-hedge for $\underline{h}$ at the lower volatility bound, cf. Definition 3.1 of Branger and Mahayni (2006). It is easy to see that this strategy is indeed a superhedge for h.

The price bound corresponding to the *cheapest tractable robust hedge* is denoted by u^{Trac}, and the optimal choice of the two payoffs is denoted by $\left(\bar{h}^*,\underline{h}^*\right)$. In the special case where the claim to be hedged is convex (concave), the ALP-hedge is given by a Black–Scholes hedge at the upper (lower) volatility bound and is thus tractable. It then holds that

$$u^{\mathrm{Trac}}\left(t,x;\bar{h}\right)=u^{\mathrm{ALP}}\left(t,x;\bar{h}\right),\quad u^{\mathrm{Trac}}\left(t,x;\underline{h}\right)=u^{\mathrm{ALP}}\left(t,x;\underline{h}\right).$$

For the general case, it immediately follows that the initial investment which is needed to achieve a tractable robust hedge is at least as high as the lowest upper price bound, i.e.

$$u^{\mathrm{Trac}}(t,x;h) = u\left(t,x;\bar{h}^*\right) + u\left(t,x;\underline{h}^*\right) \\ \geq u(t,x;\bar{h}) + u(t,x;\underline{h}) \geq u^{\mathrm{ALP}}(t,x;h). \tag{4}$$

To motivate the use of tractable strategies, note that the true data-generating process is not known. In such a situation, the use of BS-strategies is certainly the most simple choice. However, these strategies provide a superhedge in the UVM only for convex or concave claims, but not for mixed payoffs. This problem is solved by the use of tractable strategies, without foregoing the simplicity of BS-strategies.

Branger and Mahayni (2006) determine the optimal tractable robust hedge. They show that it follows from the optimal decomposition of an optimal dominating payoff. The optimal decomposition of a payoff is given in their Proposition 3.3, which basically says that the two component payoffs must not include any additional curvature. Proposition 3.6 characterizes the optimal dominating payoff. In particular, they show that it might be optimal to dominate the original payoff in case of tractable hedging. The basic intuition for this result is that the increase in initial capital due to dominating the payoff may be more than offset by the reduced overall curvature of the dominating payoff. For a more detailed explanation and some examples, we refer the reader to Branger and Mahayni (2006).

In the following, we state some basic and helpful properties of the price bounds which hold both for the ALP-hedge and for the tractable hedge.[10] Note that $-u^w(t,x;-h)$ gives the highest lower price bound for the claim h.

Lemma 2.1 (Properties of price bounds) *For $w \in \{Trac, ALP\}$, the lowest upper price bounds $u^w(t,x;f_1)$ and $u^w(t,x;f_2)$ at time $t \in [0,T[$ and state x as a function of the payoffs f_1 and f_2 at maturity T satisfy the following conditions:*

(i) *Monotonicity:* $f_1(y) \geq f_2(y)\ \forall y \geq 0. \Rightarrow u^w(t,x;f_1) \geq u^w(t,x;f_2)\ \forall x \geq 0.$
(ii) *Subadditivity:* $u^w(t,x;f_1) + u^w(t,x;f_2) \geq u^w(t,x;f_1+f_2)\ \forall x \geq 0$. *If f_1 and f_2 are convex or if f_1 and f_2 are concave, equality holds.*
(iii) $f_2(y) = a + by\ \forall y \geq 0$
$\Rightarrow u^w(t,x;f_1) + u^w(t,x;f_2) = u^w(t,x;f_1+f_2)\ \forall x \geq 0.$
(iv) $u^w(t,x;f_2) \geq -u^w(t,x;-f_2)\ \forall x \geq 0.$

Proof The proof of the case $w = \mathrm{ALP}$ is given in Branger and Mahayni (2006). The case $w = \mathrm{Trac}$ can be proved along the same lines. □

It is worth mentioning that the above lemma implies that for both $w \in \{\mathrm{Trac}, \mathrm{ALP}\}$, the upper price bound u^w defines a coherent risk measure as defined by Artzner et al. (1999). In particular, the price bounds satisfy the axioms of translation invariance, subadditivity, positive homogeneity and monotonicity.

Lemma 2.2 (Monotonicity of price bounds in volatility bounds) *For $w \in \{Trac, ALP\}$, the lowest upper price bound $u^w(t,x;h,\sigma_{\min},\sigma_{\max})$ at time $t \in [0,T[$ and in state x for the payoff h at maturity T is monotonically increasing in the upper volatility bound $\sigma_{\max}$ and monotonically decreasing in the lower volatility bound $\sigma_{\min}$.*

[10] See also Lyons and Smith (1999).

Proof To prove the monotonicity in the upper volatility bound, let $\mathcal{Q}_i^*$ be the set of equivalent martingale measures in the UVM with lower volatility bound $\sigma_{\min}$ and upper volatility bound $\sigma_{\max} = \sigma_i$. $\sigma_{\min} \leq \sigma_1 \leq \sigma_2$ implies $\mathcal{Q}_1^* \subseteq \mathcal{Q}_2^*$. The case $w =$ ALP is then obvious, since

$$\sup_{Q \in \mathcal{Q}_1^*} \mathbb{E}^Q[h(X_T)|\mathcal{F}_t] \leq \sup_{Q \in \mathcal{Q}_2^*} \mathbb{E}^Q[h(X_T)|\mathcal{F}_t].$$

Furthermore, it holds that

$$\begin{aligned} &Inf_{\overline{h},\underline{h}}\left[\sup_{Q \in \mathcal{Q}_1^*} \mathbb{E}^Q[\overline{h}(X_T)|\mathcal{F}_t] + \sup_{Q \in \mathcal{Q}_1^*} \mathbb{E}^Q[\underline{h}(X_T)|\mathcal{F}_t]\right] \\ &\leq Inf_{\overline{h},\underline{h}}\left[\sup_{Q \in \mathcal{Q}_2^*} \mathbb{E}^Q[\overline{h}(X_T)|\mathcal{F}_t] + \sup_{Q \in \mathcal{Q}_2^*} \mathbb{E}^Q[\underline{h}(X_T)|\mathcal{F}_t]\right]. \end{aligned}$$

which proves the case $w =$ Trac. The proof for the monotonicity in the lower volatility bound is analogous. □

The upper volatility bound gives the price for hedging convexity risk, and the larger it is, the more expensive convexity risk becomes. Analogously, the price of hedging concavity risk is decreasing in the lower volatility bound. The UVM allows to increase the price of hedging convexity while keeping the price of concavity risk constant. The impact of the upper volatility bound on the price can then be interpreted as one measure of the amount of convexity risk a claim is exposed to. This is also illustrated in the left part of Fig. 1, which gives the price of a bullish vertical spread as a function of the stock price. The impact of the upper volatility bound on the price is largest for low stock prices, which is in line with the intuition that the convexity risk of the claim is also largest in these situations. In the BS-model, on the other hand, the two volatilities coincide, and a larger price for hedging convexity risk automatically implies that hedging concavity risk becomes cheaper. The right part of Fig. 1 shows that the price of the bullish vertical spread then increases for low stock prices (when convexity risk dominates) and decreases for high stock prices (when concavity risk dominates).

3 Additional hedge instrument

The claim to be hedged is given by its payoff-function h. Besides the stock X and the bond B, the investor can now also use an additional claim g for hedging. Throughout the following, we assume that g is convex and nonnegative, and we exclude the trivial case of an affine g.[11] The (market) price M of g satisfies for all $t \in [0, T]$ the no arbitrage condition $M_t(g) \in]-u(t, X_t; -g), u(t, X_t; g)[$. We exclude the bounds, since in this case we could infer from the market price of g that the true volatility is

[11] Notice that a concave claim is simply given by a short position in a convex claim. We do not include mixed claims.

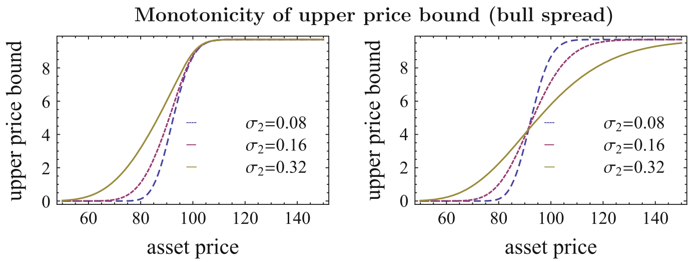

Fig. 1 The figure shows the price of a bullish vertical spread as a function of the stock price in the UVM and in the BS-model. The strike prices are $K_1 = 90$, $K_2 = 100$, i.e. $h(x) = [x - 90]^+ - [x - 100]^+$, the maturity is $T = 0.5$, the risk-free rate is $r = 0.06$. The left graph shows the lowest upper price bound in the UVM with a lower volatility bound of $\sigma_{\min} = \sigma_1 = 0.08$ and a varying upper volatility bound σ_2. The right graph shows the price in the BS-model for varying volatilities σ_2

equal to either the upper or the lower volatility bound. Then, we would no longer be in the uncertain volatility model.

If the investor takes a long position in the additional claim g, he buys convexity risk at the market price instead of hedging it at the upper volatility bound. Intuitively, this will be particularly attractive for him when the convexity risk of the claim is large and when hedging convexity risk is expensive. Analogously, if he takes a short position in the additional claim, he buys concavity risk at the market price. This will be the more beneficial the more the concavity risk of the claim dominates and the more expensive it is to hedge concavity risk.

3.1 Hedge with additional hedge instrument

Along the lines of Branger and Mahayni (2006) we define a tractable robust hedge with an additional hedge instrument as follows:

Definition 3.1 (*Tractable Robust Hedge with Additional Hedge Instrument*) A tractable robust hedge with additional instrument g for h is represented by a triple $(\overline{h}, \underline{h}, \phi)$ where $\overline{h}$ is a convex payoff, $\underline{h}$ is a concave payoff, and where

$$\overline{h}(x) + \underline{h}(x) + \phi g(x) \geq h(x) \quad \forall x \geq 0.$$

The tractable robust hedge is given by ϕ static positions in g, a BS-hedge for $\overline{h}$ at the upper volatility bound, and a BS-hedge for $\underline{h}$ at the lower volatility bound.

We are interested in the optimal position in the additional claim, and in the reduction in the initial capital due to the availability of the additional instrument g. The optimal robust hedge and the optimal tractable robust hedge with an additional instrument are defined in[12]

[12] For the unconstrained case $w = ALP$, we also refer to Avellaneda and Parás (1996).

Definition 3.2 (*Optimal Robust Hedge with Additional Hedge Instrument*) For $w \in \{\text{Trac}, \text{ALP}\}$, the optimal position in g for hedging h is denoted by ϕ_w^* and solves the minimization problem

$$\phi_w^* := \operatorname{argmin}_\phi \left\{ \phi M_t(g) + u^w\left(t, X_t; h - \phi g\right) \right\}.$$

The lowest upper price bounds u^w with additional hedging instrument are

$$u^w(t, X_t; h|g) = \operatorname{Inf}_\phi \left\{ \phi M_t(g) + u^w\left(t, X_t; h - \phi g\right) \right\}.$$

For $w = \text{Trac}$, $\left(\bar{h}^*_{\text{Trac}}, \underline{h}^*_{\text{Trac}}\right)$ represents the optimal tractable robust hedge for the modified payoff $h - \phi^*_{\text{Trac}} g$, and the optimal tractable robust hedge with additional instrument for the payoff h is represented by $\left(\bar{h}^*_{\text{Trac}}, \underline{h}^*_{\text{Trac}}, \phi^*_{\text{Trac}}\right)$.

We are interested in the positions which give the cheapest tractable hedge, i.e. we want to characterize the triple $\left(\bar{h}^*_{\text{Trac}}, \underline{h}^*_{\text{Trac}}, \phi^*_{\text{Trac}}\right)$ which is defined in Definition 3.2. To simplify the notation, we only write $(\bar{h}^*, \underline{h}^*, \phi^*)$ instead of $\left(\bar{h}^*_{\text{Trac}}, \underline{h}^*_{\text{Trac}}, \phi^*_{\text{Trac}}\right)$ in the following.

Before we give a characterization of the optimal position in g, it is convenient to formulate the analog of Lemmas 2.1 and 2.2 in the case of an additional instruments.

Lemma 3.3 (Properties of price bounds with additional instrument) *For $w \in \{\text{Trac}, \text{ALP}\}$, the lowest upper price bounds with additional instrument $u^w(t, x; f_1|g)$ and $u^w(t, x; f_2|g)$ at time $t \in [0, T[$ and state x as a function of the payoffs f_1 and f_2 at maturity T satisfy the following conditions:*

(i) *Monotonicity:* $f_1(y) \geq f_2(y)\ \forall y \geq 0 \Rightarrow u^w(t, x; f_1|g) \geq u^w(t, x; f_2|g)$ $\forall x \geq 0$
(ii) *Subadditivity:* $u^w(t, x; f_1|g) + u^w(t, x; f_2|g) \geq u^w(t, x; f_1 + f_2|g)\ \forall x \geq 0$
(iii) $f_2(y) = a + by\ \forall y \geq 0$
$\Rightarrow u^w(t, x; f_1|g) + u^w(t, x; f_2|g) = u^w(t, x; f_1 + f_2|g)\ \forall x \geq 0$
(iv) $u^w(t, x; f_2|g) \geq -u^w(t, x; -f_2|g)\ \forall x \geq 0.$

Proof The proof is analogous to the one of Lemma 2.1. □

The lowest upper price bound can thus again be interpreted as a coherent risk measure. In addition, the monotonicity with respect to the volatility bounds is also true in the case of an additional instrument, i.e.

Lemma 3.4 (Monotonicity of price bounds with additional instrument in volatility bounds) *For $w \in \{\text{Trac}, \text{ALP}\}$, the lowest upper price bound $u^w(t, x; h, \sigma_{\min}, \sigma_{\max}|g)$ at time $t \in [0, T[$ and in state x for the payoff h at maturity T is monotonically increasing in the upper volatility bound $\sigma_{\max}$ and monotonically decreasing in the lower volatility bound $\sigma_{\min}$.*

Proof Lemma 2.2 states that for $\sigma_{\min} \leq \sigma_1 \leq \sigma_2$

$$u^w(t, x; h, \sigma_{\min}, \sigma_1) \leq u^w(t, x; h, \sigma_{\min}, \sigma_2).$$

It then holds

$$
\begin{aligned}
u^w(t,x;h,\sigma_{\min},\sigma_2|g) &= \mathrm{Inf}_\phi \left\{\phi M_t(g) + u^w\left(t,x;h-\phi g,\sigma_{\min},\sigma_2\right)\right\} \\
&\geq \mathrm{Inf}_\phi \left\{\phi M_t(g) + u^w\left(t,x;h-\phi g,\sigma_{\min},\sigma_1\right)\right\} \\
&= u^w(t,x;h,\sigma_{\min},\sigma_1|g).
\end{aligned}
$$

The proof for the monotonicity in the lower volatility bound is analogous. □

3.2 Optimal position in the additional instrument

First, we derive some properties of the optimal position in the additional instrument. To get the intuition, note that a positive ϕ implies that the hedger buys convexity at a market price which is lower than the price implied by the upper volatility bound. If $\phi < 0$, he sells convexity at the market price, which is cheaper than selling it at the lower volatility bound.

The optimal position in the additional claim will depend on the price of this claim. The next lemma shows that, in line with intuition, the position is the larger the cheaper the claim is:

Lemma 3.5 (Monotonicity of optimal static position in market price) *For $w \in \{Trac, ALP\}$, the optimal position ϕ^*_w in the additional instrument g is monotonically decreasing in its market price $M_t(g)$.*

Proof For $i = 1, 2$, let $M_{t,i}(g)$ be the market price of the claim g and let $\phi^*_{w,i}$ be the optimal static position in the instrument g. From the optimality of $\phi^*_{w,i}$, we first get

$$
\begin{aligned}
\phi^*_{w,1} M_{t,1}(g) + u^w\left(t,x;h-\phi^*_{w,1}g\right) &\leq \phi^*_{w,2} M_{t,1}(g) + u^w\left(t,x;h-\phi^*_{w,2}g\right) \\
\phi^*_{w,2} M_{t,2}(g) + u^w\left(t,x;h-\phi^*_{w,2}g\right) &\leq \phi^*_{w,1} M_{t,2}(g) + u^w\left(t,x;h-\phi^*_{w,1}g\right).
\end{aligned}
$$

Combining these equations gives

$$
\begin{aligned}
&\left(\phi^*_{w,1} - \phi^*_{w,2}\right) M_{t,1}(g) + u^w\left(t,x;h-\phi^*_{w,1}g\right) \\
&\quad \leq \left(\phi^*_{w,1} - \phi^*_{w,2}\right) M_{t,2}(g) + u^w\left(t,x;h-\phi^*_{w,1}g\right)
\end{aligned}
$$

which implies

$$
\left(\phi^*_{w,1} - \phi^*_{w,2}\right)\left(M_{t,1}(g) - M_{t,2}(g)\right) \leq 0.
$$

$M_{t,1}(g) < M_{t,2}(g)$ thus implies $\phi^*_{w,1} \geq \phi^*_{w,2}$, so that the optimal static position in the claim is decreasing in its market price. □

Furthermore, if the investor faces a situation where he can either hedge a convex claim at the upper volatility bound or buy it at the market price, he will of course choose the cheaper alternative and buy it at the market price. This allows us to derive upper and lower bounds on the optimal position in the additional claim:

Proposition 3.6 (Bounds for optimal static position) *Let* $l := \sup\{\phi | h - \phi g$ *is convex*$\}$ *and* $u := \inf\{\phi | h - \phi g$ *is concave*$\}$. *For* $w \in \{Trac, ALP\}$ *it holds that* $\phi_w^* \in [l, u]$. *In particular, for h convex (concave) it holds that* $\phi_w^* \geq 0$ *(*$\phi_w^* \leq 0$*).*

Proof The proof is given in Appendix A.1. □

To get the intuition, consider e.g. the lower boundary l and start with a position of l in the additional claim. Since the modified payoff $h - lg$ is convex, the investor might want to buy additional convexity at the market price (choosing $\phi > l$). However, it is never optimal to sell additional convexity (choosing $\phi < l$) and make the modified claim even more convex. In the latter case, the investor would actually sell convexity at the market price and (re-)buy it at the higher price implied by the upper volatility bound, which just increases the initial capital needed.

Recall that we assume that the additional hedging instrument is given by a convex claim, i.e. an option. A long position ϕ^* implies that the investor wants to buy convexity at the market price and thus cares more about hedging the convexity risk at the upper volatility bound than he cares about hedging the concavity risk at the lower volatility bound. ϕ^* can thus be interpreted as a measure for the convexity risk of the claim. If h is a mixed payoff which is neither convex nor concave, then the interesting question is of course whether the optimal solution is given by buying or selling convexity, that is whether the convexity or the concavity risk of the claim dominates. We will come back to this question when we study some benchmark payoffs. Second, it is interesting to see whether the sign of the optimal position in g depends on whether we consider a tractable hedge or an ALP-hedge. From Proposition 3.6, we know that if the original claim is convex, then both hedges imply a long position (or no position) in the additional claim, and if the original claim is concave, they both imply a short position (or no position). Again, things are more complicated for mixed payoffs.

3.3 Optimal tractable hedge of the remaining payoff with additional instrument

Up to now, we have considered the optimal position in the additional instrument. Now we turn to the remaining payoff $h - \phi g$ which still has to be hedged at the upper and lower volatility bound.

Proposition 3.7 (Structure of Optimal Tractable Robust Hedge) *For the cheapest tractable robust hedge* $\left(\bar{h}^*, \underline{h}^*, \phi^*\right)$ *with additional instrument g, it holds that the cheapest tractable robust hedge* $\left(\bar{h}^*, \underline{h}^*\right)$ *of the modified payoff* $h - \phi^* g$ *must not include a long or a short position in g, i.e. there does not exist any* $\alpha > 0$ *such that* $\bar{h}^* - \alpha g$ *is convex or* $\underline{h}^* + \alpha g$ *is concave.*

Proof The proof is given in Appendix A.2. □

The intuition is similar as for Proposition 3.6. Consider the payoff $\bar{h}^*$. The investor hedges this payoff at the upper volatility bound. If there is an $\alpha > 0$ such that $\bar{h}^* - \alpha g$ is convex, then he can buy the convexity of α positions in the additional instrument at the market price, and the payoff he has to hedge at the (more expensive) upper volatility bound is reduced to $\bar{h}^* - \alpha g$. Put differently, hedging at the upper volatility bound can only be optimal if buying g does not provide an easy alternative. A similar argument of course holds for the concave payoff $\underline{h}^*$.

4 Benchmark payoffs

To illustrate the use of an additional hedge instrument, we consider the price bounds and optimal positions with respect to two benchmark payoffs. First, we deal with the simplest case where both the claim to be hedged and the additional instrument are call options. It turns out that the optimal tractable hedge can be given in closed form and is interpretable by a simple convexity argument. Besides, this benchmark case gives an easy motivation for more general scenarios. As a second benchmark payoff, we consider a bullish vertical spread and thus deal with a claim that is neither convex nor concave.

4.1 Hedging call options

The payoff function of a plain vanilla call option with maturity T and strike K is denoted by c_K, i.e. we have

$$c_K(x) := (x - K)^+ .$$

We consider the cases $h = c_K$ and $h = -c_K$, and we will show that there are indeed distinctive differences between superhedging the payoff of a call long and a call short. The additional claim is a call option, too, i.e. $g = c_{K_M}$. We assume that the maturity of all options which are considered is T, but that the strikes are different, i.e. $K \neq K_M$.

Before we give the solution for the cheapest tractable robust hedge, we define some special strike prices which will turn out to be useful later on:

Lemma 4.1 *Depending on the upper and lower volatility bound and the strike and market price of the additional instrument, we can define the following six special strike prices:*

(a) *The equation*

$$\frac{\partial u(t, x; c_b)}{\partial b} = \frac{u(t, x; c_b) - M_t(c_{K_M})}{b - K_M} \tag{5}$$

has two solutions $b = b^l$ and $b = b^u$ with $b^l < K_M < b^u$. In the limiting case $\sigma_{\max} \to \infty$, these solutions converge to $\lim_{\sigma_{\max}\to\infty} b^l = 0, \quad \lim_{\sigma_{\max}\to\infty} b^u = \infty$.

(b) *For $i = l, u$, the equation*

$$\frac{-u(t, x; -c_{\tilde{b}}) - M_t(c_{K_M})}{\tilde{b} - K_M} = \frac{u(t, x; c_{b^i}) - M_t(c_{K_M})}{b^i - K_M}. \tag{6}$$

has two solutions $\tilde{b} = b^{il}$ and $\tilde{b} = b^{iu}$ with $b^{il} < K_M < b^{iu}$.

Proof The proof is given in Appendix B.1. □

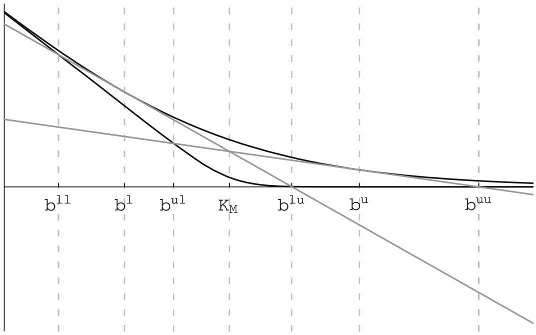

Fig. 2 The figure illustrates the definition of the strikes in Lemma 4.1. The tangents to the upper price bound at the strikes b^l and b^u go through the market price of the additional claim. The intersections of these tangents with the lower price bound then define the strikes b^{ll}, b^{lu}, b^{ul} and b^{uu}

In Lemma 4.1, the upper and lower price bounds $u(.; c_b)$ and $-u(.; -c_b)$ of a European call are considered as a function of the strike price b. The solution of Eq. (5) is a strike such that the tangent line to the upper price bound goes through the price of the traded call with strike K_M, cf. Fig. 2. According to Lemma 4.1, Eq. (5) has two solutions b^l and b^u with $b^l < K_M < b^u$. Furthermore, each tangent line intersects the lower price bound twice, and this gives the strikes $b^{ll} < K_M < b^{lu}$ (for the tangent at $b = b^l$) and $b^{ul} < K_M < b^{uu}$ (for the tangent at $b = b^u$) which solve Eq. (6).

In the following, we represent the cheapest tractable hedge along the lines of Definition 3.2, i.e. by its optimal static position ϕ^* in the additional claim g and the tuple $\left(\bar{h}^*, \underline{h}^*\right)$ which gives the optimal tractable robust hedge for the remaining payoff $h - \phi^* g$. In particular, the position in the underlying is given by the delta hedge for $\bar{h}^*$ at the upper volatility bound and $\underline{h}^*$ at the lower volatility bound.

Proposition 4.2 (Cheapest Tractable Robust Hedge with AI—Call Long) *The cheapest tractable hedge of a call with strike K is given by*

$$\begin{aligned}
\phi^* &= \frac{K - b^l}{K_M - b^l} 1_{\{b^l < K < K_M\}} + \frac{K - b^u}{K_M - b^u} 1_{\{K_M < K < b^u\}} \\
\bar{h}^* &= \left(1 - \phi^*\right) c_{b^l} 1_{\{b^l < K < K_M\}} + \left(1 - \phi^*\right) c_{b^u} 1_{\{K_M < K < b^u\}} + c_K 1_{\{K \notin [b^l, b^u]\}} \\
\underline{h}^* &= 0
\end{aligned}$$

where b^l and b^u are defined as in Lemma 4.1. It holds that $\phi^* \in [0, 1]$.

Proof The proof is given in Appendix B.2. □

The upper price bound for a tractable hedge when an additional call is available is illustrated in the left panel of Fig. 3. If the strike of the claim to be hedged is between b^l and b^u, then the new price bound is indeed lower than the old one. Following Proposition 4.2, the call with strike K is dominated by a portfolio of two calls with a lower and a higher strike price. If all three calls were priced and hedged at the upper volatility bound, then the initial capital needed for the dominating payoff would of course be larger. However, the call with strike K_M can be bought at a lower market price instead of being hedged at the upper volatility bound, which reduces the initial capital needed for the convex combination. It then depends on the tradeoff between these two effects

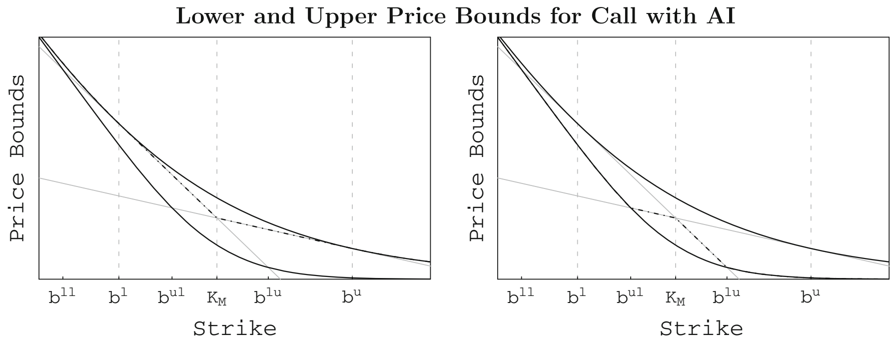

Fig. 3 The figures show the upper and lower price bound (*solid black lines*) for a call as a function of the strike, as well as the tangents (*solid grey lines*) to the upper price bound that pass through the price of the traded call. If an additional call with strike K_M is available, then the price bounds can be tightened. The left graph shows the new upper price bound (*dash-dotted line*), which is lower if $b^l < K < b^u$. The right graph shows the new lower price bound (*dash-dotted line*), which is larger if $b^{ul} < K < b^{lu}$

whether the additional call can be used to lower the upper price bound. If K is between b^l and b^u, the option to be hedged is still similar enough to the additional option to profit from its lower price of hedging convexity risk.

If the price of the traded call increases, the strikes b^l and b^u move closer to the strike K_M. Then, as also shown in Lemma 3.5, the optimal position in the additional call decreases. The reduction in the initial capital decreases, too, as can also be seen from Fig. 3. If the upper volatility bound increases, the strikes b^l and b^u move away from the strike K_M. The optimal position in the additional option thus becomes larger. This is in line with the intuition that the investor will profit more from buying convexity at the market price if hedging convexity at the upper volatility bound is becoming more expensive. It is also interesting to look at the limiting hedge when the upper volatility bound goes to infinity or when the strike of the additional option converges towards the strike of the option to be hedged.

Corollary 4.3 (Limits for the Cheapest Tractable Hedge with AI—Call Long)

(i) *In an uncertain volatility model where $\sigma_{\max} \to \infty$, the optimal tractable robust hedge for c_K converges to a static portfolio:*
  (a) *for $K_M > K$: portfolio of the stock $(1 - \phi^*)$ and the additional call with $\phi^* = \frac{K}{K_M}$*
  (b) *for $K_M < K$: position in the additional call with $\phi^* = 1$*

(ii) *The optimal static position is the higher the closer the additional strike is to the strike of the option to be hedged, i.e.* $\lim_{K_M \to K} \phi^* = 1$.

Proof The above corollary is an immediate consequence of Lemma 4.1 and Proposition 4.2. □

The above corollary is also true in the unconstrained case which is considered in Avellaneda and Parás (1996). For the limiting cases $\sigma_{\max} \to \infty$ and $K_M \to K$, we thus have $\phi^*_{\text{Trac}} = \phi^*_{\text{ALP}}$. In general, the optimal static position in case of tractable

Table 1 The table compares the tractable hedge and the ALP-hedge with and without an additional instrument

$\frac{K}{100}$	0.85	0.90	0.95	1.00	1.05	1.10	1.15
$u^{\text{ALP}}(t,x;c_K)$	19.7623	16.2269	13.1063	10.4343	8.1739	6.3223	4.82708
$u_x^{\text{ALP}}(t,x;c_K)$	0.8324	0.7616	0.6818	0.5971	0.5120	0.4303	0.3550
$u^{\text{ALP}}(t,x;c_K\|c_{K_M})$	18.6884	14.3800	10.1787	6.0773	4.9385	3.9769	3.1640
$u_x^{\text{ALP}}(t,x;c_K\|c_{K_M})$	0.5936	0.4122	0.2092	0.0000	−0.0776	−0.0902	−0.0784
ϕ^*_{ALP}	0.4150	0.6038	0.8074	1.0000	0.9444	0.8172	0.6727
$u^{\text{Trac}}(t,x;c_K)$	19.7623	16.2269	13.1063	10.4343	8.1739	6.3223	4.82708
$u_x^{\text{Trac}}(t,x;c_K)$	0.8324	0.7616	0.6818	0.5971	0.5120	0.4303	0.3550
$u^{\text{Trac}}(t,x;c_K\|c_{K_M})$	19.1799	14.8124	10.4449	6.0773	5.3963	4.7153	4.03431
$u_x^{\text{Trac}}(t,x;c_K\|c_{K_M})$	0.5639	0.3759	0.1880	0.0000	0.0349	0.0698	0.1047
ϕ^*_{Trac}	0.3964	0.5976	0.7988	1.0000	0.8228	0.6455	0.4683

The claim to be hedged is a call option with moneyness $K/100$, the additional instrument is an ATM-call. Both options have a time to maturity of half a year. The current stock price is $x = 100$, the risk-free rate is $r = 0.06$, and the volatility bounds are $\sigma_{\min} = 0.08$ and $\sigma_{\max} = 0.32$. The additional ATM-call has an implied volatility of 0.16

hedging differs from the one in the unconstrained case. This and the above results are illustrated by a numerical example.

Along the lines of Avellaneda and Parás (1996), we consider the problem of hedging OTC options with different strikes using ATM options. For simplicity, we set $K_M = x = 100$. We use the same parameters as ALP. The volatility band is defined by $\sigma_{\min} = 0.08$ and $\sigma_{\max} = 0.32$. The risk-free interest rate is set to $r = 0.06$, and time to maturity is equal to half a year. Finally, the implied volatility of the ATM option available for hedging is $\sigma_{imp} = 0.16$. The call options to be hedged have moneyness (defined as strike price over stock price) ranging from 0.85 to 1.15.

Table 1 shows the results both for the ALP-hedge and the tractable hedge with and without the additional instrument. Without the additional instrument, both the restricted and the unrestricted hedge coincide, and the optimal strategy is just to hedge the call at the upper volatility bound. If the additional ATM-call is available, the initial capital is lower in the unrestricted ALP-case than for the restricted tractable hedge. To get the intuition, note that the remaining payoff is mixed, which implies that the hedge of this remaining payoff is more expensive for the tractable strategy than for the ALP-strategy, as already argued in Eq. (4).

For all strike prices, the investor takes a long position in the additional instrument both for the tractable hedge and for the ALP-hedge. Given that he wants to hedge an already convex payoff, this result is intuitive and also follows from Proposition 3.6. The position in the additional instrument is larger in case of the ALP-hedge. Intuitively, this can again be explained by noting that the remaining payoff is neither convex nor concave. This makes the tractable hedge more expensive than the ALP-hedge. The investor is thus more reluctant to buy the option and add concavity to the remaining payoff.

In the next proposition, we give the results for superhedging a call short:

Proposition 4.4 (Cheapest Tractable Robust Hedge with AI—Call Short) *The cheapest tractable hedge of a call short with strike K is given by*

$$\begin{aligned}
\phi^* &= -\frac{K-b^u}{K_M-b^u} 1_{\{b^{ul}<K<K_M\}} - \frac{K-b^l}{K_M-b^l} 1_{\{K_M<K<b^{lu}\}} \\
\bar{h}^* &= -\phi^* \left(\frac{K-K_M}{K-b^u} c_{b^u} \; 1_{\{b^{ul}<K<K_M\}} + \frac{K-K_M}{K-b^l} c_{b^l} \; 1_{\{K_M<K<b^{lu}\}} \right) \\
\underline{h}^* &= -c_K \; 1_{\{K \notin [b^{ul}, b^{lu}]\}}
\end{aligned}$$

where b^l, b^{lu}, b^u, and b^{ul} are defined in Lemma 4.1. In particular, $\phi^* \in [-1, 0]$.

Proof The proof is given in Appendix B.3. □

The result is illustrated in the right panel of Fig. 3 which shows the lower price bound if an additional instrument is available. The payoff of the call short is dominated by the payoff of a short position in a call with strike K_M and a long position in a call with strike b^i. If the short position would be priced at the lower volatility bound and the long position at the upper volatility bound, then the resulting lower price bound would indeed be smaller than the old one. However, the short position in the traded call is traded at the market price, which increases the lower price bound. And again, if $b^{ul} < K < b^{lu}$, then the weight of the traded call in the dominating portfolio is large enough for the investor to profit from the additional call.

4.2 Hedge of bullish vertical spread

In the next example, we consider hedging a bullish vertical spread with strikes $K_1 < K_2$, i.e. $h(x) = (x - K_1)^+ - (x - K_2)^+$. While the call was an example for a convex payoff, the bullish vertical spread is neither convex nor concave. This leads to two interesting questions. First, the tractable robust hedge and the ALP-hedge do not coincide even in the base case without an additional traded instrument, and the tractable robust hedge is more expensive than the ALP-hedge, i.e. $u^{\text{ALP}}(t, x; h) < u^{\text{Trac}}(t, x; h)$. It is then no longer obvious whether the additional instrument will reduce the initial capital by more in the unconstrained case or in the constrained case. Second, the optimal static position in the additional claim can be positive as well as negative. Then, the question is under which conditions the investor will buy convexity at the market price, and under which conditions he will sell it.

For the case $K_M < K_1 < K_2$, the representation of the optimal tractable hedge with an additional call is given in the following proposition. The remaining cases $K_1 < K_M < K_2$ and $K_1 < K_2 < K_M$ can be found in Appendix B.5 and B.6.

Proposition 4.5 (Tractable Robust Hedge of Bullish Call Spread with Additional Instrument) *The cheapest tractable robust hedge with additional instrument c_{K_M} for a bullish vertical spread with payoff-function $h(x) = (x - K_1)^+ - (x - K_2)^+$ is for $K_M < K_1 < K_2$ given by*

$$\begin{aligned}
\phi^* &= -\frac{K_2 - K_1}{K_M - b^l} 1_{\{K_2 < b^{lu}\}} + \frac{K_2 - K_1}{K_2 - K_M} 1_{\{b^{lu} < K_2 < b^{uu}\}} + \frac{b^u - K_1}{b^u - K_M} 1_{\{K_1 < b^u\}} 1_{\{b^{uu} < K_2\}} \\
\bar{h}^* &= \frac{K_2 - K_1}{K_M - b^l} c_{b^l} 1_{\{K_2 < b^{lu}\}} \\
&\quad + \left(\frac{K_1 - K_M}{b^u - K_M} c_{b^u} 1_{\{K_1 < b^u\}} + \frac{K_2 - K_1}{K_2 - a^*} c_{a^*} 1_{\{b^u < K_1\}} \right) 1_{\{b^{uu} < K_2\}} \\
\underline{h}^* &= -\frac{K_2 - K_1}{K_2 - K_M} c_{K_2} 1_{\{b^{lu} < K_2 < b^{uu}\}} \\
&\quad + \left(-c_{K_2} 1_{\{K_1 < b^u\}} - \frac{K_2 - K_1}{K_2 - a^*} c_{K_2} 1_{\{b^u < K_1\}} \right) 1_{\{b^{uu} < K_2\}}
\end{aligned}$$

where $a^* = \min\{a^{cand}, K_1\}$ *and* a^{cand} *solves the equation*

$$\frac{\partial u(t, X_t; c_a)}{\partial a} = \frac{-u(t, X_t; -c_{K_2}) - u(t, x; c_a)}{K_2 - a}.$$

b^l *(*b^{ll}, b^{lu}*) and* b^u *(*b^{ul}, b^{uu}*) are defined as in Lemma 4.1. In particular,* $\phi^* \in (-\infty, 1]$.

Proof The proof is given in Appendix B.4. □

For the numerical example, we use the same parameters as in Sect. 4.1, i.e. we set $\sigma_{\min} = 0.08$, $\sigma_{\max} = 0.32$, and $r = 0.06$. The inital stock price is $x = 100$, and the additional instrument is an ATM-call with implied volatility $\sigma_{imp} = 0.16$. The time to maturity of the contracts is again equal to half a year. We set $K_2 = K_1 + 10$.

Figure 4 shows the initial capital needed for the ALP-hedge and for the tractable hedge as a function of K_1, where we consider both the case with and without an additional instrument. The results confirm that the tractable hedge is more expensive than the ALP-hedge, which is to be expected for a mixed payoff. For both types of hedges, the initial capital can be reduced when an additional instrument is available. The reduction depends on the relation between the strikes of the spread and the strike of the additional call. If the claim to be hedged and the additional call are too different, that is if the strikes of the spread are too far away from the strike K_M, the reduction in initial capital is zero for the tractable hedge and goes to zero for the ALP-hedge. Furthermore, the reduction in initial capital for the tractable hedge is zero for $K_2 \approx b^{lu}$. In this case, the optimal tractable hedge without the additional instrument cannot be improved upon, as can also be shown analytically. For the other values of K_1 and K_2, the reduction in initial capital can be significant.

A comparison of the reduction in initial capital by the additional instrument shows that there is no ranking between the ALP-hedge and the tractable hedge. Again, it depends on the parameters for which hedge the reduction is larger. The most important result, however, is that the tractable hedge with an additional instrument can be significantly cheaper than the ALP-hedge without additional instrument. In our example, this holds true if the strike of the additional call is smaller than the two strikes of the spread. Then, the disadvantage of the tractable hedge due to the restriction of possible trading strategies can be offset by using an additional hedge instrument.

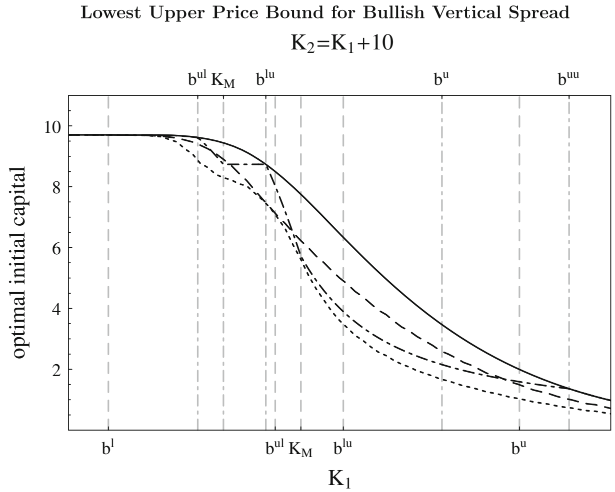

Fig. 4 The figure shows the lowest upper price bound for a spread with strikes K_1 (shown at the *lower axis*) and $K_2 = K_1 + 10$ (shown at the *upper axis*) as a function of K_1. We consider the ALP-hedge (*dashed line*), the ALP-hedge with a traded ATM-call (*dotted line*), the tractable hedge (*solid line*), and the tractable hedge with a traded ATM-call (*dash-dotted line*)

The optimal position in the ATM-call is shown in Fig. 5. The first question is whether the investor takes a long, a short, or no position in the additional call c_{K_M}, that is whether he buys or sells convexity at the market price. Intuitively, the answer should depend on whether the traded call is more similar to the convex part (call with strike K_1 long) or concave part (call with strike K_2 short) of the spread. It should also depend on the current stock price in relation to the strikes of the bullish vertical spread, which governs whether the spread is more similar to a convex or to a concave claim at the moment. Consider the dependence on the strike prices first. As we can see from Proposition 4.5 and the propositions in Appendix B.5 and B.6, the distinctive parameter for the optimal tractable hedge is K_2. For $K_2 > b^{lu} > K_M$ and thus for a large strike price K_2, the investor takes a long position in the additional call, whereas he takes a short position for $K_2 < b^{lu}$. Surprisingly, the sign of the position in the additional call does not depend on K_1. Second, consider the dependence of the optimal position in the additional call on the current stock price. This dependence is indirect in that the optimal tractable hedge depends on the characteristic strikes from Lemma 4.1 which in turn depend on the current stock price. b^{lu} increases in the stock price, and there will be some critical stock price where the investor switches from a long position in the additional call (for stock prices below this level) to a short position (for stock prices above this level). This is in line with intuition: for low stock prices, the spread is more similar to a convex claim (which implies a long position in the additional call), while it is more similar to a concave claim for large stock prices (which implies a short position).

In our example, the sign of the position in the additional call coincides for the tractable hedge and the ALP-hedge. The size of the position in the tractable hedge, however, can (in absolute terms) be both smaller and larger than the position in the

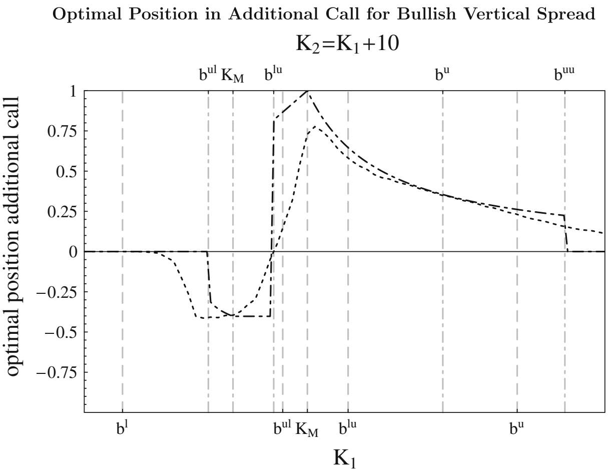

Fig. 5 The figure shows the optimal position in a traded ATM-call for a spread with strikes K_1 (shown at the *lower axis*) and $K_2 = K_1 + 10$ (shown at the *upper axis*) as a function of K_1. We consider the ALP-hedge with a traded ATM-call (*dotted line*) and the tractable hedge with a traded ATM-call (*dash-dotted line*)

ALP-hedge. The position is zero for the tractable hedge if the strike of the additional call is too different from the strikes of the spread, and it goes to zero for the ALP-hedge if the difference between the strike of the call and the strikes of the spread increases. Furthermore, note that the optimal position in the traded call is a continuous function of the strike K_1 for the ALP-hedge, while it jumps for the tractable hedge. At the discontinuity points, which are again defined in terms of K_2, there are two different strategies which are both optimal.

The cases $K_1 = K_M$ and $K_2 = K_M$ are special in that one of the component calls is already traded at the market. Intuitively, one would expect the investor to buy or sell the traded call and to hedge the remaining call with strike K_1 or K_2 at the upper or lower volatility bound. In our example, this intuition gives the right answer for $K_2 = K_M$, but not for $K_1 = K_M$, and it can be seen from Proposition 4.5 and the propositions in Appendix B.5 and B.6 that it holds only under some additional conditions. If $K_1 = K_M$, a long position in the traded call and a hedge of the call with strike K_2 at the lower volatility bound is optimal only if K_2 is large enough. If $K_2 = K_M$, a short position in the traded call and a hedge of the call with strike K_1 at the upper volatility bound is optimal only if K_1 is small enough.

Finally, we compare the optimal hedge for the spread to the sum of the optimal hedges for the component payoffs. If no additional instrument is available, the sum of the ALP-hedges is given by hedging the call long at the upper volatility bound and the call short at the lower volatility bound. It is well known that this strategy is often prohibitively expensive compared to the optimal ALP-hedge for the portfolio. The same holds true if an additional instrument is used. For the tractable hedge, on the other hand, there are situations where the sum of the optimal hedges for the components is optimal for the portfolio, too. Without an additional instrument, this holds true if K_1

is small enough.[13] With an additional instrument, this case may also happen. Assume e.g. $b^l < K_1 < K_M < K_2 < b^{lu}$. Adding up the optimal tractable hedges for the call long from Proposition 4.2 and the call short from Proposition 4.4 results in the optimal tractable hedge for the spread, which is given in the proposition in Appendix B.5. However, there are other parameter scenarios where the sum of the hedges is more expensive than the hedge of the portfolio, and where the investor thus profits from a portfolio effect.

5 Conclusion

In this paper, we analyze the benefits of using an additional traded instrument besides the stock and the money market account for hedging. The model setup is given by an uncertain volatility model with bounded volatility. The investor wants to implement a robust hedge, that is a superhedge for all volatilities within the volatility bounds. We consider both, unrestricted trading strategies and tractable strategies, i.e. strategies which can be written as the sum of Black–Scholes strategies.

We assume that the additional traded instrument is a convex claim, where we think for example of a (highly liquid) ATM-call. By trading this option, the investor can buy (sell) convexity at the market price instead of superhedging it at the upper (lower) volatility bound. If he takes a long position in the option, he is more concerned about the convexity risk than about the concavity risk. The sign of the position in the option thus shows whether the claim has a larger exposure to convexity or to concavity risk. Furthermore, the size of the position and the reduction in the initial capital allow to assess how big this exposure is.

The optimal position in the additional option depends on whether the investor uses an unrestricted trading strategy or is restricted to a tractable strategy. We show that there is no dominance relationship, and that the optimal position can be larger with or without the restriction. If the claim to be hedged is neither convex nor concave, then the same holds true for the reduction in the initial capital, which can again be larger with or without the restriction. Furthermore, we show that the initial capital needed for a tractable hedge with the additional option can be smaller than the initial capital of the unrestricted hedge without this additional option.

Further research could proceed along several lines. First, the uncertain volatility model is one way to capture model risk, and the hedging strategies considered in this paper are robust with respect to this model class. It seems natural to extend the concept of robust hedging to other model classes, too, and to consider the use of additional instruments in these model classes. Second, it would be interesting to analyze so-called semi-static strategies, which are e.g. considered in Carr and Wu (2002), and to combine the criteria of robust hedging and semi-static hedging.

[13] For a detailed argumentation, we refer to Branger and Mahayni (2006).

Appendix

A Proofs ad section 3

A.1. *Proof of Proposition* 3.6 g convex implies $\{\phi | h - \phi g \text{ is convex}\} =]-\infty, l]$ and $\{\phi | h - \phi g \text{ is concave}\} = [u, \infty[$. Thus, it holds that $l \leq u$. To prove $\phi_w^* \in [l, u]$ it is enough to show that $\phi M_t(g) + u^w(t, S_t; h - \phi g)$ is decreasing in ϕ for $\phi \in]-\infty, l]$ and increasing for $\phi \in [u, \infty[$.

First, assume that $\phi_1 < \phi_2 \leq l$.

$$\begin{aligned}
&\phi_1 M_t(g) + u^w(t, S_t; h - \phi_1 g) - \left[\phi_2 M_t(g) + u^w(t, S_t; h - \phi_2 g)\right] \\
&= (\phi_1 - \phi_2) M_t(g) + u^w(t, S_t; h - \phi_1 g) - u^w(t, S_t; h - \phi_2 g) \\
&= (\phi_1 - \phi_2) M_t(g) + u^w(t, S_t; (\phi_2 - \phi_1) g) \\
&= (\phi_2 - \phi_1) \left[-M_t(g) + u^w(t, S_t; g)\right] \\
&\geq 0
\end{aligned}$$

where we have used that $h - \phi_1 g$, $h - \phi_2 g$ and $(\phi_2 - \phi_1) g$ are convex so that the price bounds are additive, cf. Lemma 2.1. This proves that $\phi M_t(g) + u^w(t, S_t; h - \phi g)$ is indeed decreasing in ϕ on the interval $(-\infty, l]$.

In a similar way, we can show that $\phi M_t(g) + u^w(t, S_t; h - \phi g)$ is increasing in ϕ on the interval $[u, \infty[$. □

A.2. *Proof of Proposition* 3.7 Along the lines of Definition 3.2 we have

$$\begin{aligned}
\left(\bar{h}^*, \underline{h}^*, \phi^*\right) = \operatorname{argmin}_{(\bar{h}, \underline{h}, \phi)} &\left\{\phi M_t(g) + u^{\text{ALP}}\left(t, x; \bar{h}\right) + u^{\text{ALP}}\left(t, x; \underline{h}\right)\right\} \\
\text{s.t. } &\bar{h}(x) + \underline{h}(x) + \phi g(x) \;\geq\; h(x) \text{ for all } x \geq 0.
\end{aligned}$$

Assume that there is a constant $\alpha > 0$ such that $\bar{h}^* - \alpha g$ is convex. Consider the triple $(\bar{h}, \underline{h}, \phi)$ where $\bar{h} = \bar{h}^* - \alpha g$, $\underline{h} = \underline{h}^*$ and $\phi = \phi^* + \alpha$. Obviously this triple also represents a tractable robust hedge. With the no arbitrage condition $M_t(g) \in]-u(t, S_t; -g), u(t, S_t; g)[$, the convexity of $\bar{h}^* - \alpha g$, and the convexity of αg, it follows that

$$\begin{aligned}
&\phi M_t(g) + u^{\text{ALP}}\left(t, x; \bar{h}\right) + u^{\text{ALP}}\left(t, x; \underline{h}\right) \\
&= \left(\phi^* + \alpha\right) M_t(g) + u^{\text{ALP}}\left(t, x; \bar{h}^* - \alpha g\right) + u^{\text{ALP}}\left(t, x; \underline{h}^*\right) \\
&= \phi^* M_t(g) + u^{\text{ALP}}\left(t, x; \bar{h}^*\right) + u^{\text{ALP}}\left(t, x; \underline{h}^*\right) + \alpha\left(M_t(g) - u^{\text{ALP}}(t, x; g)\right) \\
&< \phi^* M_t(g) + u^{\text{ALP}}\left(t, x; \bar{h}^*\right) + u^{\text{ALP}}\left(t, x; \underline{h}^*\right).
\end{aligned}$$

The above contradicts the optimality of the triple $\left(\bar{h}^*, \underline{h}^*, \phi^*\right)$. Analogous arguments prohibit the existence of $\alpha > 0$ such that $\underline{h}^* + \alpha g$ is concave. □

B Proofs ad section 4

B.1. *Proof of Lemma* 4.1 The upper price bound for a European call is given by the BS-price at the upper volatility bound, i.e.

$$
\begin{aligned}
u(t,x;c_b) &= x\mathcal{N}\left(d_1\left(t,x;K;\sigma_{\max}\right)\right) - K\mathcal{N}\left(d_2\left(t,x;K;\sigma_{\max}\right)\right) \\
\text{where } d_1\left(t,x;K;\sigma_{\max}\right) &:= \frac{\ln\left(\frac{x}{K}\right)+\frac{1}{2}\sigma_{\max}^2(T-t)}{\sigma_{\max}\sqrt{T-t}} \\
d_2\left(t,x;K;\sigma_{\max}\right) &:= d_1\left(t,x;K;\sigma_{\max}\right) - \sigma_{\max}\sqrt{T-t}
\end{aligned}
$$

where $\mathcal{N}$ denotes the cumulative distribution function of the normal distribution.

Equation (5) can be rewritten as

$$f(t,x;b) := u(t,x;c_b) + \frac{\partial u(t,x;c_b)}{\partial b}(K_M - b) = M_t(c_{K_M})$$

It can be shown that the function f is monotonically increasing in b for $0 \le b < K_M$ and monotonically decreasing in b for $K_M < b$. Furthermore, it holds that

$$
\begin{aligned}
\lim_{b\to 0} f(t,x;b) &= x - K_M < M_t(c_{K_M}) \\
f(t,x;K_M) &= u(t,x;c_{K_M}) > M_t(c_{K_M}) \\
\lim_{b\to\infty} f(t,x;b) &= 0 < M_t(c_{K_M}).
\end{aligned}
$$

Thus, the equation $f(t,x;b) = M_t(c_{c_{K_M}})$ has two solutions b^l, b^u where $0 < b^l < K_M$ and $b^u > K_M$. Since the upper price bound increases in $\sigma_{\max}$, if furthermore holds that $\lim_{\sigma_{\max}\to\infty} b^l = 0$ and $\lim_{\sigma_{\max}\to\infty} b^u = \infty$.

Equation (6) can be rewritten as

$$f(\tilde{b}) := M_t(c_{K_M}) + \frac{u(t,x;c_{b^i}) - M_t(c_{K_M})}{b^i - K_M}\left(\tilde{b} - K_M\right) = -u(t,x;-c_{\tilde{b}}) \quad (7)$$

where $i = l, u$. The function f is a monotonically decreasing and affine function of $\tilde{b}$, while the lower price bound $-u(t,x;-c_{\tilde{b}})$ is a convex function of $\tilde{b}$. Thus, Eq. (7) has at most two solutions. Since f is the tangent to the convex upper price bound for the call, it holds that

$$f(0) < u(t,x;c_0) = x = -u(t,x;-c_0).$$

Furthermore, we know that

$$
\begin{aligned}
f(K_M) = M_t(c_{K_M}) &> -u(t,x;-c_{K_M}) \\
\lim_{\tilde{b}\to\infty} f\left(t,x;\tilde{b}\right) = -\infty < 0 &= \lim_{\tilde{b}\to\infty} -u\left(t,x;-c_{\tilde{b}}\right).
\end{aligned}
$$

Thus, for each $i = l, u$, Eq. (7) has two solutions b^{il}, b^{iu} where $0 < b^{il} < K_M$ and $b^{iu} > K_M$. □

B.2. *Proof of Proposition* 4.2 From Proposition 3.6, it follows that $\phi \geq 0$. The static position ϕ in the claim $g = c_M$ results in the modified payoff

$$h(x) - \phi g(x) = (x - K)^+ - \phi(x - K_M)^+. \tag{8}$$

This payoff is structurally equal to the payoff of a vertical spread. In the special case $\phi = 1$, we get a bearish (bullish) vertical spread for $K > K_M$ ($K < K_M$).

Branger and Mahayni (2006) show that the cheapest tractable hedge is given by the optimal decomposition of the optimal dominating payoff. Applying Proposition 3.6 of Branger and Mahayni (2006) to the modified payoff (8) shows that the optimal dominating portfolio consists of a short position in a call with strike K_M and a long position in a call with strike b.[14] For $K_M < K$ ($K_M > K$) we have $b > K$ ($b < K$). Let α_i denote the number of calls with strike i ($i = b, K_M$). The portfolio of $\phi + \alpha_{K_M}$ options with strike K_M and α_b calls with strike b has to dominate the original call with strike K, and it must not be possible to reduce the position in the options any further. This gives the following conditions

$$(i) \quad \alpha_{K_M} + \phi + \alpha_b = 1, \qquad (ii) \quad (\alpha_{K_M} + \phi)(b - K_M) = b - K.$$

Furthermore, we must have $\alpha_b \geq 0$. Proposition 3.7 immediately implies $\alpha_{K_M} = 0$, i.e. the hedge of the modified payoff must not include a superhedge of the additional hedging instrument. Summing up, the structure of the optimal hedge is given by

$$\alpha_{K_M} = 0, \qquad \alpha_b = 1 - \phi, \qquad \phi = \frac{K - b}{K_M - b}.$$

It can easily be seen that $\alpha_b \in [0, 1]$ and $\phi \in [0, 1]$. The position α_b in the option with strike b is superhedged dynamically, i.e. a long position in this call is delta-hedged at the upper volatility bound. In contrast, the call with strike K_M is available at the market. The (total) initial investment for the hedging positions is given by

$$\left(1 - \frac{K - b}{K_M - b}\right) u(t, x; c_b) + \frac{K - b}{K_M - b} M_t(c_{K_M}). \tag{9}$$

In the last step, we have to find the optimal b. From the first order condition and Lemma 4.1, we get $b = b^l$ or $b = b^u$. For $K_M < K < b^u$, we thus set $b = b^u$, and for $b^l < K < K_M$, we set $b = b^l$. Otherwise, if $K \notin [b^l, b^u]$, then the optimal choice of b is not viable, i.e. the condition $\phi \in [0, 1]$ is violated. Then, we get the boundary solution $b = K$, and it is optimal just to superhedge the call with strike K at the upper volatility bound. □

[14] Notice that these option positions are superhedged dynamically.

B.3. *Proof of Proposition* 4.4 From Proposition 3.6, it follows that $\phi^* \leq 0$. The modified payoff $\tilde{h}$ is

$$\tilde{h}(x) = h(x) - \phi g(x) = -\phi\,(x - K_M)^+ - (x - K)^+,$$

which is similar to a bullish (bearish) vertical spread for $K_M < K$ ($K_M > K$).

For the case $K_M < K$, we know from Branger and Mahayni (2006) that the optimal dominating payoff is given by a long position in a call with strike $b < K_M$ and a short position in the call with strike K. Using the notation of the proof before and analogous arguments, we have

$$\begin{aligned} (i) \quad & \alpha_b + \phi + \alpha_K = -1 \\ (ii) \quad & \alpha_b(K - b) + \phi(K - K_M) = 0 \end{aligned}$$

and we also know that $\alpha_b \geq 0,\ \alpha_K \leq 0$ where $b \leq K_M$. This immediately gives

$$\begin{aligned} (i) \qquad & \alpha_b = -\phi \frac{K - K_M}{K - b} \\ (ii) \qquad & \alpha_K = -1 - \phi \frac{K_M - b}{K - b}. \end{aligned}$$

Furthermore, it has to hold that

$$(iii) \qquad -\frac{K - b}{K_M - b} \leq \phi \leq 0.$$

In the case of $K_M > K$, analogous arguments give

$$\begin{aligned} (i) \qquad & \alpha_b = -\phi \frac{K - K_M}{K - b} \geq 0 \\ (ii) \qquad & \alpha_K = -1 - \phi \frac{K_M - b}{K - b} \leq 0 \\ (iii) \qquad & -\frac{K - b}{K_M - b} \leq \phi \leq 0 \end{aligned}$$

where $b > K_M$.

The (total) initial investment of the hedging positions is[15]

$$\begin{aligned} & -\alpha_K u(t, x; -c_K) + \alpha_b u(t, x; c_b) + \phi M_t(c_{K_M}) \\ & \quad = \left(1 + \phi \frac{K_M - b}{K - b}\right) u(t, x; -c_K) - \phi \frac{K - K_M}{K - b} u(t, x; c_b) + \phi M_t(c_{K_M}) \end{aligned}$$

[15] Recall again that a long position is dynamically hedged at the upper volatility bound and the short position at the lower volatility bound.

$$
\begin{aligned}
&= u(t,x;-c_K) - \phi\left(\frac{K-K_M}{K-b}u(t,x;c_b)\right.\\
&\quad\left.+\frac{K_M-b}{K-b}[-u(t,x;-c_K)] - M_t(c_{K_M})\right).
\end{aligned}
$$

In case the term in brackets is positive, the optimal choice is $\phi^* = 0$, and the additional hedge instrument is not used. In the opposite case, we get $\phi^* = -\frac{K-b}{K_M-b}$. Then, the (total) initial investment is

$$
\frac{K_M-K}{b-K_M}u(t,x;c_b) - \frac{b-K}{b-K_M}M_t(c_{K_M}),
$$

and we have to find the optimal b in the next step. From the first order condition and Lemma 4.1, we get $b = b^l$ or $b = b^u$. For $K_M < K$, the optimal choice is thus $b = b^l$, and for $K < K_M$, it is $b = b^u$. Finally, it remains to show that

$$
\begin{aligned}
&\frac{K-K_M}{K-b^i}u(t,x;c_{b^i}) + \frac{K_M-b^i}{K-b^i}[-u(t,x;-c_K)] - M_t(c_{K_M}) < 0\\
&\iff K \in]b^{ul}, b^{lu}[.
\end{aligned}
$$

□

B.4. *Proof of Proposition* 4.5 First, consider the structure of the hedge. For a given ϕ, the optimal tractable hedge of the remaining payoff follows from Proposition 3.6 in Branger and Mahayni (2006). Furthermore, Proposition 3.7 in this paper states that this hedge must not include a hedge of the additional call c_{K_m} at the upper or lower volatility bound. The hedge for the remaining payoff is then given by a long position in an option with strike y, where $y < K_M$ for $\phi < 0$ and $y < K_2$ for $\phi \geq 0$, and a short position in the option with strike K_2. For the positions α_y and α_{K_2} in the options, a similar argumentation as in the proof of Proposition 4.2 gives

$$
\begin{aligned}
\alpha_y &= \frac{K_2-K_1}{K_2-y} - \phi\frac{K_2-K_M}{K_2-y}\\
\alpha_{K_2} &= -\frac{K_2-K_1}{K_2-y} - \phi\frac{K_M-K^l}{K_2-y}
\end{aligned}
$$

where y and ϕ have to meet one of the following two conditions

$$
\begin{aligned}
y < K_M \quad &\text{and} \quad \phi \in \left[-\frac{K_2-K_1}{K_M-y}, 0\right]\\
y < K_2 \quad &\text{and} \quad \phi \in \left[\frac{y-K_1}{y-K_M}\, 1_{\{K_1<y<K_2\}}, \frac{K_2-K_1}{K_2-K_M}\right].
\end{aligned}
$$

Second, we have to find the optimal choice of y and ϕ. The initial investment for the hedge is

$$V_t(\phi, y) := \frac{K_2 - K_1}{K_2 - y}\left[u(t, x; c_y) - (-1)u(t, x; -c_{K_2})\right] + \phi f(y)$$

$$\text{where } f(y) := M_t(c_{K_M}) - \frac{K_2 - K_M}{K_2 - y}u(t, x; c_y) - \frac{K_M - y}{K_2 - y}(-1)u(t, x; -c_{K_2}).$$

For fixed y, we denote the optimal choice of ϕ by $\phi(y)$. It is easily seen that $\phi(y)$ is equal to the lower boundary if $f(y) > 0$ and equal to the upper boundary if $f(y) \leq 0$:

$$\phi(y) = \begin{cases} -\frac{K_2-K_1}{K_M-y}1_{\{y<K_M\}} + 0 \cdot 1_{\{K_M<y<K_1\}} + \frac{y-K_1}{y-K_M}1_{\{K_1<y<K_2\}} & f(y) > 0 \\ \frac{K_2-K_1}{K_2-K_M} & f(y) \leq 0. \end{cases}$$

The optimal y then follows from the first order conditions, where we have to distinguish between the cases $f(y) \leq 0$ and $f(y) > 0$ with $y < K_M$, $K_M < y < K_1$, and $K_1 < y < K_2$. Note that a necessary condition for $f(y) > 0$ to hold is $K_2 < b^{lu}$ or $K_2 > b^{uu}$. This gives the following candidate choices for y:

(i) in the interval $[0, K_M]$: b^l
(ii) in the interval $[K_M, K_1]$: $\max\{K_M, a^*\}$
(iii) in the interval $[K_1, K_2]$: $\max\{K_1, \min\{b^u, K_2\}\}\, 1_{\{K_2 \notin (b^{lu}, b^{uu})\}} + K_2 1_{\{K_2 \in [b^{lu}, b^{uu}]\}}$

where we have already dropped some choices that are never optimal. Comparing the initial investments that result from these cases then gives Proposition 4.5. □

B.5 Extension of proposition 4.5 to the case $K_1 < K_M < K_2$

Proposition .1 (Tractable Robust Hedge of Bullish Call Spread with Additional Instrument (ii)) *The cheapest tractable hedge with additional instrument c_{K_M} of a claim with payoff-function $h(x) = (x - K_1)^+ - (x - K_2)^+$ ($K_1 < K_M < K_2$) is given by*

$$\begin{aligned}
\phi^* &= \left[-\frac{K_2 - K_1)}{K_M - b^l} 1_{\{K_1 > b^l\}} - \frac{K_2 - b^l}{K_M - b^l} 1_{\{K_1 < b^l\}}\right] 1_{\{K_2 < b^{lu}\}} \\
&\quad + \frac{K_1 - b^l}{K_M - b^l} 1_{\{K_2 > b^{lu}\}} 1_{\{K_1 > b^l\}} \\
\bar{h}^* &= \left[\frac{K_2 - K_1}{K_M - b^l} c_{b^l} 1_{\{K_1 > b^l\}} + \left(c_{K_1} + \frac{K_2 - K_M}{K_M - b^l} c_{b^l}\right) 1_{\{K_1 < b^l\}}\right] 1_{\{K_2 < b^{lu}\}} \\
&\quad + \left[\frac{K_M - K_1}{K_M - b^l} c_{b^l} 1_{\{K_1 > b^l\}} + c_{K_1} 1_{\{K_1 < b^l\}}\right] 1_{\{K_2 > b^{lu}\}} \\
\underline{h}^* &= -c_{K_2}\ 1_{\{K_2 > b^{lu}\}}
\end{aligned}$$

where b^l (b^{ll}, b^{lu}) and b^u (b^{ll}, b^{lu}) are defined as in Lemma 4.1. In particular, $\phi^* \in [-1, 1]$.

Proof The proof is analogous to the one of Proposition 4.5. □

B.6 Extension of proposition 4.5 to the case $K_1 < K_2 < K_M$

Proposition .2 (Tractable Robust Hedge of Bullish Call Spread with Additional Instrument (iii)) *The cheapest tractable hedge with additional instrument c_{K_M} of a claim with payoff-function $h(x) = (x - K_1)^+ - (x - K_2)^+$ ($K_1 < K_2 < K_M$) is given by*

$$\begin{aligned}
\phi^* &= \left[-\frac{K_2 - K_1}{K_2 - K^*} \frac{b^u - K_2}{b^u - K_M} 1_{\{K^* < K_1\}} - \frac{b^u - K_2}{b^u - K_M} 1_{\{K^* > K_1\}} \right] 1_{\{b^{ul} < K_2\}} \\
\bar{h}^* &= \left[\left(\frac{K_2 - K_1}{K_2 - K^*} c_{K^*} + \frac{K_2 - K_1}{K_2 - K^*} \frac{K_M - K_2}{b^u - K_M} c_{b^u} \right) 1_{\{K^* < K_1\}} \right. \\
&\quad \left. + \left(c_{K_1} + \frac{K_M - K_2}{b^u - K_M} c_{b^u} \right) 1_{\{K^* > K_1\}} \right] 1_{\{b^{ul} < K_2\}} \\
&\quad + \frac{K_2 - K_1}{K_2 - \min\{a^*, K_1\}} c_{\min\{a^*, K_1\}} 1_{\{b^{ul} > K_2\}} \\
\underline{h}^* &= \frac{-(K_2 - K_1)}{K_2 - \min\{a^*, K_1\}} c_{K_2} 1_{\{b^{ul} > K_2\}}
\end{aligned}$$

where K^* solves

$$\frac{\partial u(t, x; c_{K^*})}{\partial K^*} = \frac{u(t, x; c_{K^*}) - \left[\frac{b^u - K_2}{b^u - K_M} M_t(c_{K_M}) + \frac{K_2 - K_M}{b^u - K_M} u(t, x; c_{b^u}) \right]}{K^* - K_2}.$$

a^* is defined as in Proposition 4.5 and b^l (b^{ll}, b^{lu}) and b^u (b^{ll}, b^{lu}) are defined as in Lemma 4.1. In particular, $\phi^* \in [-1, 1]$.

Proof The proof is analogous to the one of Proposition 4.5. □

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