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ORIGINAL PAPER

Pricing and upper price bounds of relax certificates

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Abstract Relax certificates are written on multiple underlying stocks. Their payoff depends on a barrier condition and is thus path-dependent. As long as none of the underlying assets crosses a lower barrier, the investor receives the payoff of a coupon bond. Otherwise, there is a cash settlement at maturity which depends on the lowest stock return. Thus, the products consist of a knock-out coupon bond and a knock-in claim on the minimum of the stock prices. In a Black-Scholes model setup, the price of the knock-out part can be given in closed (or semi-closed) form in the case of one or two underlyings only. With the exception of the trivial case of one underlying, the price of the knock-in minimum claim always has to be calculated numerically. Hence, we derive semi-closed form upper price bounds. These bounds are the lowest upper price bounds which can be calculated without the use of numerical methods. In addition, the bounds are especially tight for the vast majority

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of relax certificates which are traded at a discount relative to the corresponding coupon bond. This is also illustrated with market data.

Keywords Certificates · Barrier option · Price bounds

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1 Introduction

Before the financial market crisis, traded retail certificates were characterized by increasingly complex structures, e.g. products written on several instead of one underlying. Amongst them are so-called relax certificates which can be interpreted as a generalized version of bonus certificates.[1] Although, after the crisis, the market volume of traded bonus certificates decreased from over 20 to 6%, relax certificates are still actively traded, and an increasing number of newly created certificates on several underlyings are launched under different names and product types.[2]

This paper analyzes relax certificates as an example for complex certificates on several underlyings. Normally, relax certificates[3] are written on three stocks belonging to a similar market segment, like blue chips or primary products but they are also traded on indices. Their payoff depends on whether and when any of the underlyings touches a lower barrier. As long as the barrier is not reached, the "bonus payments" of the certificates correspond to those of a coupon bond where the coupon payments well exceed the current level of interest rates.[4] However, if the lower barrier is breached, all future payments from the bond component are cancelled. Instead, the investor receives the minimum of the prices of the underlyings at maturity. Relax certificates thus combine a knock-out component (the bond) and a knock-in component (the minimum claim).[5] For the time to maturity, a typical choice is three years and three months with reference dates every 13 months or a maturity of about one year with a single reference date at maturity.

Relax certificates are advertised as follows: the bonus payments are appealing even in sideways and slightly bearish markets. The risk of losing the bonus payments is low since this event is triggered by a significant loss in one of the underlying stocks. However, relax certificates are less attractive in highly bullish and highly bearish markets. In the first case, the investor would be better off with a direct

[1] Bonus certificates pay the maximum of the underlying value and a fixed payoff if the underlying never reaches a lower boundary until maturity. If the barrier is crossed, however, the investor instead receives the underlying.

[2] Cf. monthly reports of the Börse Stuttgart (EUWAX) and the monthly statistics of the Deutsche Derivate Verband (DDI).

[3] Similar products are also called Top-10-Anleihe, Easy Relax Express, Easy Relax Bonus, Multi-Capped Bonus or Aktienrelax. Furthermore, there are also relax certificates which bear some features of express certificates.

[4] Some examples for contracts which are currently traded will be given in Sect. 5.

[5] In the literature this minimum claim is also known as "cheapest-to-deliver", i.e. an option on the worse of n assets, see Wilkens et al. (2001).

investment in the stocks because with relax certificates, she foregoes the participation in increasing stock prices.[6] In the second case, she has to participate in the (highest) losses at the stock market, which certainly contradicts the label “relax”.

In the following paper, we provide a detailed analysis of relax certificates. Our main findings are as follows: In order to understand the structure of relax certificates it is helpful to decompose them into a knock-out coupon bond and a knock-in minimum claim. The contracts are usually designed in a way that allows to offer relax certificates cheaper than the associated coupon bond. We call these relax certificates attractive and show that conditions for attractiveness can be summarized as follows: The bonus payments exceed a lower bound and/or the barrier level is below an upper bound. In this case, a trivial upper price bound is given by the corresponding coupon bond. This price bound can be tightened by subtracting the price of a put option on the minimum of the underlying assets with a strike price equal to the barrier.

In addition, we show that further price bounds can be determined by considering subsets of the underlyings. In the extreme case, we reduce the number of underlyings to one, so that the upper price bound can be calculated in closed-form in a Black-Scholes setup. This price bound is decreasing in the volatility of the underlying, and the lowest upper bound is thus given by using the stock with the highest volatility as underlying. Since the basic idea of multiple underlyings is obviously contradicted by taking only one underlying into account, we also include more underlyings. In particular, we show that tight but still tractable price bounds result from considering all subsets consisting of two underlyings.

In order to test the practical relevance of our theoretical results, we analyze relax certificates written on two or three underlyings which are currently traded at the market. For typical contract specifications, the price of the relax certificates is up to 10% lower than the price of the corresponding coupon bond. The risk that at least one of the underlying stocks hits the lower barrier can thus not be neglected and is highly economically significant. We also compare the market prices to the upper price bounds which are based on two underlyings only. The comparison confirms that the prices of relax certificates are well above the upper price bounds and consequently overpriced.[7] Additionally, the upper price bounds yield a tight approximation of the fair price.

Our paper is related to the literature on the pricing of structured products, i.e. products that combine stocks or bonds with positions in derivatives. Burth et al. (2001) and Grünbichler and Wohlwend (2005) analyze the Swiss market and find that these products are overpriced both in the primary and in the secondary market. Wilkens et al. (2003) report similar findings for the German market. They find evidence for a so-called “life-cycle hypothesis”: the overpricing is largest at and shortly after issuance of the products (when issuers mainly sell these products) and decreases over time (when issuers also start to buy back the products). Stoimenov

[6] There are also certificates where the investor can participate in the development of the underlying assets if the terminal value of the worst performing stock is larger than the face value of the coupon bond.

[7] The price bounds are calculated in a Black-Scholes model. For attractive relax certificates, however, the price bounds would be even lower if one takes the possibility of (downward) jumps or default risk of the issuer into account.

and Wilkens (2005) furthermore find that the overpricing is the larger the more complex the product is. Muck (2006, 2007), Mahayni and Suchanecki (2006) and Wilkens and Stoimenov (2007) analyze the pricing of turbo certificates, i.e. barrier options, for the German market. Wallmeier and Diethelm (2008) and Lindauer and Seiz (2008) examine (multi-) barrier reverse convertibles which are traded in Switzerland and resemble the German relax certificates.

Recall that relax certificates can be interpreted as a knock-out coupon bond and a knock-in minimum claim. Closed-form solutions for standard barrier options are given by Rubinstein and Reiner (1991), Rich (1994) and Haug (1998). More exotic barrier options are, for example, considered in Kunitomo and Ikeda (1992) (two-sided barriers) and Heynen and Kat (1994) (outside barriers). For multi-asset barrier options, we refer to Wong and Kwok (2003) and Kwok et al. (1998). Closed-form solutions for pricing options on the minimum or maximum of two risky assets were first derived by Stulz (1982). An extension to more than two risky assets can be found in Johnson (1987).

The probability that at least one underlying reaches the barrier is important for the pricing and risk management of relax certificates. In the simple case of one underlying asset, the distribution of the first hitting time is well-known in a Black-Scholes setup, cf., for example, Merton (1973). It can be calculated using the reflection principle as shown in Karatzas and Shreve (1999) or Harrison (1985). For two underlyings, a semi-closed form solution is given in He et al. (1998) and Zhou (2001) where the distribution function is approximated by using an infinite Bessel function. We rely on these results in the following. The first hitting time distribution of more than two underlyings, however, cannot be given in closed-form for a general correlation structure.

The remainder of the paper is organized as follows. In Sect. 2, the payoff structure of relax certificates is defined and analyzed. In addition, we derive conditions on the contract parameters for which the certificates are attractive. This allows us to derive model-independent upper price bounds in Sect. 3. In Sect. 4, we assume a Black-Scholes model and give the (exact) prices as well as (model-dependent) upper price bounds. In particular, we give a tight upper price bound in semi-closed form and discuss the dependence of the prices and price bounds on the characteristics of the underlyings. A comparison to market prices can be found in Sect. 5. Section 6 concludes.

2 Product specification

In general, a relax certificate is written on n underlying stocks, where n is equal to 2 or 3 for currently traded relax certificates. Let $S_t^{(j)}$ be the price of stock j at time t. For ease of exposition, we set the initial value of all stocks equal to one, i.e. $S_0^{(j)} = 1\ (j = 1, \ldots, n)$.[8] The continuously compounded risk-free rate is denoted by r and it is assumed to be constant and positive.

[8] This is in line with currently traded relax certificates where the minimum option is written on the *return* of the underlying stocks from time 0 to time t_N.

The payoff of the relax certificate depends on whether at least one of the stocks has hit its lower barrier $m \quad (m < 1)$, i.e. has lost the fraction $1 - m$ of its value. Usually, m is chosen to be quite low, e.g. $m = 0.5$, so that this event constitutes a significant loss in this stock. The first hitting time of stock $j\,(j = 1, \ldots, n)$ with respect to the barrier level m is indicated by $\tau_{m,j}$. The first hitting time of the portfolio of all underlying stocks is denoted $\tau_m^{(n)}$, i.e.

$$\tau_{m,j} := \inf\left\{t \geq 0, S_t^{(j)} \leq m\right\}, \quad \text{and} \quad \tau_m^{(n)} := \min\{\tau_{m,1}, \ldots, \tau_{m,n}\}. \tag{1}$$

If none of the underlyings reaches the level m, $\tau_m^{(n)}$ is set to $\tau_m^{(n)} = \infty$.

The relax certificate can be decomposed into two parts, a knock-out (RO) and a knock-in (RI) component. Its total payoff at maturity t_N is $\mathrm{RC}_{t_N}^{(n)} = \mathrm{RO}_{t_N}^{(n)} + \mathrm{RI}_{t_N}^{(n)}$, where we assume that payments before maturity are accumulated at the risk-free rate r. The set of all payment dates is denoted by $\underline{\underline{T}} = \{t_1, \ldots, t_N\}$, the current point in time is $t_0 = 0 < t_1$. If the barrier is not hit until $t_i \in \underline{\underline{T}}$ $(i = 1, \ldots, N)$, the investor receives a bonus payment which is given by δ times the nominal value and which resembles a coupon payment. At maturity t_N, she also receives the nominal value of the certificate which we normalize to one. This part of the payoff can be interpreted as a knock-out component $\mathrm{RO}_{t_N}^{(n)}$

$$\mathrm{RO}_{t_N}^{(n)} = \sum_{i=1}^{N} \delta e^{r(t_N - t_i)} 1_{\{\tau_m^{(n)} > t_i\}} + 1_{\{\tau_m^{(n)} > t_N\}} \tag{2}$$

where $1_{\{\}}$ denotes the indicator function. If the barrier is hit before t_N, the investor forgoes all future bonus payments as well as the repayment of the nominal value. Instead, she obtains the minimum of the n underlying stocks at the maturity date t_N. The payoff from this European knock-in component $\mathrm{RI}_{t_N}^{(n)}$ maturing at time t_N is given by

$$\mathrm{RI}_{t_N}^{(n)} = \min\left\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\right\} 1_{\{\tau_m^{(n)} \leq t_N\}}. \tag{3}$$

We summarize the payoff from the relax certificate in the following definition:

Definition 1 (Relax certificate) The compounded payoff of a relax certificate with nominal value 1, bonus payments δ, lower boundary m, payment dates $\underline{\underline{T}} = \{t_1, \ldots, t_N\}$, and n underlying stocks $S^{(1)}, \ldots, S^{(n)}$ is

$$\mathrm{RC}_{t_N}^{(n)} = \sum_{i=1}^{N} \delta e^{r(t_N - t_i)} 1_{\{\tau_m^{(n)} > t_i\}} + 1_{\{\tau_m^{(n)} > t_N\}} + \min\left\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\right\} 1_{\{\tau_m^{(n)} \leq t_N\}}.$$

Note that we ignore any default risk of the issuer, which reduces the price of the certificate as compared to the prices without default risk. A detailed analysis of this issue is e.g. provided in Baule et al. (2008).

2.1 Attractive relax certificates

Relax certificates are advertised via rather high bonus payments and a price below the price of the corresponding coupon bond. We call these relax certificates *attractive*:

Definition 2 (Attractive relax certificate) A relax certificate is called attractive iff

$$\mathrm{RC}_0^{(n)} < \sum_{i=1}^{N} \delta e^{-rt_i} + e^{-rt_N}. \tag{4}$$

The *discount* as compared to the price of a coupon bond is achieved by the knock-out feature of the bond component. However, note that in case of a knock-out, the payoff is not replaced by zero but by the minimum of the stock prices at the maturity date. For the relax certificate to be attractive, the investor has to switch from a "higher" to a "lower" value in this case, i.e. the foregone future bond payments must be worth more than the minimum claim. A condition to ensure that this is indeed the case is given in the following lemma:

Lemma 1 (Attractive relax certificate: sufficient conditions) *A sufficient condition on the bonus payments δ and the lower barrier m to ensure that the relax certificate is attractive is given by*

$$m \leq \min_{\{j=0,\ldots,n-1\}} \delta \sum_{i:t_i > t_j} e^{-r(t_i - t_j)} + e^{-r(t_N - t_j)}. \tag{5}$$

In particular, a sufficient condition for Eq. (5) *to hold is given by*

$$m \leq \frac{(1+\delta)e^{-rt_N}}{1 + e^{-rt_N}}. \tag{6}$$

Proof If the barrier is not hit, the payoff of the relax certificate is equal to that of a coupon bond. If the barrier is hit at time τ, the investor foregoes the future payments from this bond and receives a minimum claim instead. The terminal payoff of this claim is bounded from above by any of the stock prices, which implies that its value is also bounded from above by any of the stock prices. The smallest stock price at the hitting time τ is m, so that the value of the minimum claim at τ cannot exceed m.[9] Condition (5) ensures that this upper price bound on the minimum claim is smaller than the value of the coupon bond immediately after a coupon payment. In between the coupon dates, the price of the coupon bond increases and thus also exceeds m. Thus, the investor suffers a loss when the payments from the coupon bond are cancelled at τ and replaced by the minimum claim, so that the price of the relax certificate is indeed lower than the price of the coupon bond.

To prove the second part, note that

$$\min_{\{j=0,\ldots,n-1\}} \delta \sum_{i:t_i > t_j} e^{-r(t_i - t_j)} + e^{-r(t_N - t_j)} \geq \delta e^{-rt_N} + e^{-rt_N} \geq \frac{(1+\delta)e^{-rt_N}}{1 + e^{-rt_N}}.$$

If Condition (6) holds, then m is smaller than the right hand side, which implies Condition (5). □

[9] In the case of gap risk due to jump or liquidity risk the lowest stock price can be lower than m.

3 Risk-neutral pricing and upper price bounds

In the following, we assume an arbitrage free market, i.e. the existence of a risk-neutral (pricing) measure Q. We do not restrict the type of model here. For the specific examples in Sect. 4, we will rely on a Black-Scholes-model.

For ease of exposition, we ignore any dividend payments of the stocks. Basically, dividends would reduce both the prices of attractive relax certificates and their price bounds. To get the intuition, note that dividends reduce the prices of the stocks and thus increase the probability that the lower barrier is hit, in which case the investor goes from a "high" to a "low" payoff for an attractive relax certificate. Since dividend payments also reduce the value of the minimum claim, the price of the relax certificate will decrease.

Let $\mathrm{RC}_{t_0}^{(n)}$ denote the price at t_0 of a relax certificate which is written on n underlying assets $S^{(1)}, \ldots, S^{(n)}$. Pricing by no arbitrage immediately gives:

Proposition 1 (Price of a relax certificate) *The price at time t_0 $(t_0 = 0 < t_1)$ of a relax certificate with bonus payments δ, lower barrier m, payment dates $\underline{\underline{T}} = \{t_1, \ldots, t_N\}$ and n underlying assets is given by $RC_{t_0}^{(n)} = RO_{t_0}^{(n)} + RI_{t_0}^{(n)}$. The prices of the components are*

$$RO_{t_0}^{(n)} = \delta \sum_{i=1}^{N} e^{-rt_i}\, Q\Big(\tau_m^{(n)} > t_i\Big) + e^{-rt_N}\, Q\Big(\tau_m^{(n)} > t_N\Big), \tag{7}$$

$$RI_{t_0}^{(n)} = E_Q\left[\int_{t_0}^{t_N} e^{-ru} C_u^{\mathrm{Min},n} dN_u\right] \tag{8}$$

where $N_t := 1_{\{\tau_m^{(n)} \le t\}}$ and $C_t^{\mathrm{Min},n} := E_Q\Big[e^{-r(t_N - t)} \min\Big\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\Big\} \mid \mathcal{F}_t\Big]$.

Proof Pricing by no arbitrage immediately gives

$$\begin{aligned}\mathrm{RC}_{t_0}^{(n)} &= \delta \sum_{i=1}^{N} e^{-rt_i}\, Q\Big(\tau_m^{(n)} > t_i\Big) + e^{-rt_N}\, Q\Big(\tau_m^{(n)} > t_N\Big)\\ &\quad + E_Q\Big[e^{-rt_N} \min\Big\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\Big\} 1_{\{\tau_m^{(n)} \le t_N\}}\Big].\end{aligned}$$

Using the definition of N_t and iterated expectations yields

$$\begin{aligned}&E_Q\Big[e^{-rt_N} \min\Big\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\Big\} N_{t_N}\Big]\\ &= E_Q\left[\int_{t_0}^{t_N} e^{-rt_N} \min\Big\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\Big\} dN_u\right]\\ &= E_Q\left[\int_{t_0}^{t_N} e^{-ru} E_Q\Big[e^{-r(t_N-u)} \min\Big\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\Big\} | \mathcal{F}_u\Big] dN_u\right].\end{aligned}$$

With the definition of $C_t^{\mathrm{Min},n}$, the pricing formula follows. □

The price of the knock-out bond component in Eq. (7) depends on the distribution of the first hitting time $\tau_m^{(n)}$. i.e. the first time when one of the stocks hits the barrier. In the model of Black-Scholes, the first hitting time distribution for one stock is well known and can be derived using the reflection principle as in Karatzas and Shreve (1999) or Harrison (1985). For $n = 2$, Zhou (2001) derives a semi-closed form solution for the first hitting time by approximating the distribution function using an infinite Bessel function. For the special cases of uncorrelated stock prices or perfectly positively correlated stock prices, the distribution of the first hitting time for $n \geq 2$ follows from the one-dimensional case. In general, however, even semi-closed form solutions do not exist for $n \geq 3$, and the price of the knock-out component has to be calculated numerically.

The price (8) of the knock-in minimum claim depends on the joint distribution of the first hitting time and all stock prices at this first hitting time. Here, a closed form solution exists in the model of Black-Scholes and for $n = 1$. For $n \geq 2$, however, an analytical pricing formula no longer exists in general.

Even in the case of a simple Black-Scholes-type model setup, the prices of relax certificates thus have to be determined numerically in general. Possible methods are binomial or trinomial lattices—see e.g. Hull and White (1993)—or finite difference schemes—see e.g. Dewynne and Wilmott (1994)—which become rather time-consuming for more than one underlying. In this case, a Monte-Carlo simulation is usually preferred.[10]

3.1 Upper price bounds based on coupon bonds

The price of an attractive relax certificate is by definition lower than the price of the corresponding coupon bond. From this derives the first model independent price bound and a trivial superhedge tightened by selling a put option on the minimum of the stock prices with strike price m.

Proposition 2 (Semi-Static Superhedge) *Assume that δ and m satisfy Eq. (6). Then, the following semi-static strategy is a superhedge for the relax certificate: At $t_0 = 0$, buy the corresponding coupon bond (with coupon payments δ and payment dates $\underline{\underline{T}}$) and sell a put option on the minimum of the stocks with maturity date t_N and strike m. If $\tau_m^{(n)} < t_N$,* liquidate the portfolio at $\tau_m^{(n)}$ *and use the proceeds to buy the cheapest underlying asset.*

Proof First, consider the case where one of the stocks hits the barrier. At the hitting time $\tau_m^{(n)} < t_N$, the value of the hedge portfolio is $CB_{\tau_m^{(n)}} - P_{\tau_m^{(n)}}^{\mathrm{Min},n}$, where CB and $P^{\mathrm{Min},n}$ denote the value of the coupon bond and the price of the put option on the minimum of n stocks, respectively. The value of the bond is at least as large as the discounted value of the payment at the maturity date, i.e. it is at least as large as $e^{-r(t_N - \tau_m^{(n)})}(1 + \delta)$. The payoff of the put on the minimum stock price at t_N is

[10] Notice that the barrier feature causes some problems for the Monte Carlo simulation, see Boyle et al. (1997).

$$P_{t_N}^{\mathrm{Min},n} = \max\left\{m - \min\{{}_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\}, 0\right\}$$

and is bounded from above by m. At $\tau_m^{(n)}$, it thus holds that $P_{\tau_m^{(n)}}^{\mathrm{Min},n} \leq e^{-r(t_N - \tau_m^{(n)})} m$. This gives

$$\begin{aligned} CB_{\tau_m^{(n)}} - P_{\tau_m^{(n)}}^{\mathrm{Min}} &\geq e^{-r(t_N - \tau_m^{(n)})}(1+\delta) - e^{-r(t_N - \tau_m^{(n)})} m \\ &= e^{-r(t_N - \tau_m^{(n)})}(1 + \delta - m). \end{aligned}$$

With condition (6), it follows that

$$CB_{\tau_m^{(n)}} - P_{\tau_m^{(n)}}^{\mathrm{Min}} \geq m.$$

Therefore, the value of the hedge portfolio at time $\tau_m^{(n)}$ is large enough to buy the cheapest asset, which is worth at most m at $\tau_m^{(n)}$. Obviously, this asset superhedges the minimum claim, which also holds true if it pays dividends.

In case none of the stocks hits the barrier, the put on the minimum expires worthless, and the bond component of the relax certificate is not knocked out. Thus, the payoffs of the hedge portfolio and of the relax certificate both coincide with the payoffs from the coupon bond and are thus equal. □

Corollary 1 (Upper bound on $\mathrm{RC}_{t_0}^{(n)}$) *Assume that δ and m satisfy Eq.* (6). *Then, an upper price bound for the relax certificate is given by*

$$RC_{t_0}^{(n)} \leq \sum_{i=1}^{N} \delta e^{-rt_i} + e^{-rt_N} - P_{t_0}^{\mathrm{Min},n}. \tag{9}$$

Proof The proof follows immediately from Proposition 2. □

The semi–static superhedge in Proposition 2 can be simplified by considering only a subset of underlyings, as will be shown in Sect. 3.2. The consideration of only one underlying leads to a semi-static hedge where only one plain-vanilla put option instead of the more exotic put on the minimum is needed. The optimal choice which gives the lowest initial capital is then the most expensive put.

An issuer who sells the relax certificate as a substitute for selling a coupon bond might follow yet another hedging strategy. As long as the barrier is not hit, she might just refrain from hedging at all. If the barrier is hit, however, she is no longer short a coupon bond but a minimum option. Then, she can hedge by taking a long position in the worst performing stock that is the stock which has first hit the barrier. This implies paying back the bond before maturity at a rather low level m.

3.2 Upper price bounds based on "Smaller" relax certificates

The next proposition shows that the price of an attractive relax certificate is decreasing in the number of underlyings. Considering a smaller number of underlyings thus gives an upper price bound.

Proposition 3 (Upper price bound: relax certificates on a subset of underlyings only) *Let $S = \left(S^{(1)}, \ldots, S^{(n)}\right)$ denote a set of underlyings. In addition, let $RC_{t_0}\left(\hat{S}\right)$ denote the price of a relax certificate with bonus payments δ, lower barrier m, payment dates $\underline{\underline{T}}$ and underlyings $\hat{S}$ where $\hat{S} \subseteq S$. If condition (5) on the bonus payments δ and the barrier m holds, then*

$$RC_{t_0}(S) \leq RC_{t_0}\left(\hat{S}\right) \quad \textit{for all}\, \hat{S} \subseteq S. \tag{10}$$

In particular, it holds

$$RC_{t_0}(S) \leq \min_{k,l \in \{1,\ldots,n\}} RC_{t_0}\left(S^{(k)}, S^{(l)}\right) \leq \min_{i \in \{1,\ldots,n\}} RC_{t_0}\left(S^{(i)}\right). \tag{11}$$

Proof The "big" certificate on S is knocked out no later than the "small" one on $\hat{S}$. Depending on whether and when the two certificates are knocked out, there are three cases. First, if both certificates survive until maturity, their payments coincide. Second, if both are knocked out at the same point in time, the minimum claim resulting from the 'big' certificate is written on more underlyings and thus dominated by the minimum claim resulting from the "small" certificate. Third, if the "big" certificate is knocked out while the "small" one still survives, the minimum claim resulting from the "big" certificate is again dominated by the minimum claim on the smaller set of underlyings, which is by condition (5) dominated by the value of the attractive "small" certificate. In all three cases, the value of the "small" certificate is thus at least as high as the value of the "big" certificate. This proves the first part of the proposition. The second part then follows as a special case. □

The above result is model-independent. Given the distribution of the first hitting time—for which there is a closed form solution in the model of Black-Scholes for $n = 1$ and a semi-closed form solution for $n = 2$—the price of the knock-out component (7) can be calculated in closed form. For the price of the knock-in component (8) which depends on the joint distribution of the first-hitting time and the stock prices at the first hitting time, we now give an upper bound which depends on the distribution of the first hitting time only

Proposition 4 (Upper price bound for knock-in part) *For $n \geq 2$, an upper price bound on the knock-in component is given by*

$$RI_{t_0}^{(n)} \leq m \int_{t_0}^{t_N} e^{-ru} f_{\tau_m^{(n)}}^{Q}(u)du. \tag{12}$$

where $f_{\tau_m^{(n)}}^{Q}$ denotes the density of the first hitting time $\tau_m^{(n)}$. In particular, this immediately implies that

$$RI_{t_0}^{(n)} \leq mQ(\tau_m^{(n)} \leq t_N). \tag{13}$$

Proof Using the law of iterated expectations gives

$$\begin{aligned}
\mathrm{RI}_{t_0}^{(n)} &= E_Q\left[e^{-rt_N}\min\left\{S_{t_N}^{(1)},\ldots,S_{t_N}^{(n)}\right\}1_{\{\tau_m^{(n)}\le t_N\}}\right]\\
&= E_Q\left[E_Q\left[e^{-rt_N}\min\left\{S_{t_N}^{(1)},\ldots,S_{t_N}^{(n)}\right\} \mid \mathcal{F}_{\tau_m^{(n)}}\right]1_{\{\tau_m^{(n)}\le t_N\}}\right]\\
&= E_Q\left[E_Q\left[\min\left\{\hat{S}_{t_N}^{(1)},\ldots,\hat{S}_{t_N}^{(n)}\right\} \mid \mathcal{F}_{\tau_m^{(n)}}\right]1_{\{\tau_m^{(n)}\le t_N\}}\right]
\end{aligned}$$

where $\hat{S}_t := e^{-rt}S_t$. $\hat{S}$ is a Q-martingale, so that $\min\{\hat{S}^{(1)},\ldots,\hat{S}^{(n)}\}$ is a Q-supermartingale. Together with the Optional Sampling Theorem it follows

$$E_Q\left[\min\left\{\hat{S}_{t_N}^{(1)},\ldots,\hat{S}_{t_N}^{(n)}\right\} \mid \mathcal{F}_{\tau_m^{(n)}}\right] \le \min\left\{\hat{S}_{\tau_m^{(n)}}^{(1)},\ldots,\hat{S}_{\tau_m^{(n)}}^{(n)}\right\} \le me^{-r\tau_m^{(n)}}.$$

This implies

$$\mathrm{RI}_{t_0}^{(n)} \le m\int_{t_0}^{t_N} e^{-ru}f_{\tau_m^{(n)}}^{Q}(u)du.$$

The second bound then follows. □

As a consequence we can state the following theorem.

Theorem 1 (Upper price bound for $n \ge 2$) *For $n \ge 2$, an upper price bound on the relax certificate on the underlyings $S = (S^{(1)},\ldots,S^{(n)})$ is given by*

$$\begin{aligned}
\min_{k,l\in\{1,..,n\}}\Bigg\{ & mQ\big(\min\{\tau_{m,k},\tau_{m,l}\}\le t_N\big) + \delta\sum_{i=1}^{N} e^{-rt_i}Q\big(\min\{\tau_{m,k},\tau_{m,l}\} > t_i\big)\\
& + e^{-rt_N}Q\big(\min\{\tau_{m,k},\tau_{m,l}\} > t_N\big)\Bigg\}.
\end{aligned} \tag{14}$$

Proof According to Proposition 3, it holds that

$$\begin{aligned}
\mathrm{RC}_{t_0}^{(n)} &\le \min_{k,l\in\{1,..,n\}}\left\{\mathrm{RC}_{t_0}(S^{(k)},S^{(l)})\right\}\\
&= \min_{k,l\in\{1,..,n\}}\left\{\mathrm{RO}_{t_0}(S^{(k)},S^{(l)}) + \mathrm{RI}_{t_0}(S^{(k)},S^{(l)})\right\}
\end{aligned}$$

The value of the knock-out component follows from Proposition 1, while Proposition 4 gives an upper bound on the value of the knock-in minimum claim. Putting the results together gives Eq. (14). □

The upper price bound for a relax certificate on n underlyings is based on the prices of all relax certificates on two underlyings only. We can also use relax certificates on three underlyings. However, it is not possible to determine the hitting time probabilities for $n \ge 3$ in (semi-)closed form. Therefore the tightest bounds for $n = 3$ which are not based on numerical approximations are achieved by using:

Lemma 2 (Semi closed-form bounds on survival probabilities for $n = 3$) *The* *probability* $Q\left(\tau_m^{(3)} \le t\right)$ *can be bounded from below and above as follows:*

$$\underline{Q}\left(\tau_m^{(3)} \le t\right) \le Q\left(\tau_m^{(3)} \le t\right) \le \overline{Q}\left(\tau_m^{(3)} \le t\right)$$

where

$$\begin{aligned}\overline{Q}\left(\tau_m^{(3)} \le t\right) &= \min\left\{Q(\min\left\{\tau_{m,1}, \tau_{m,2}\right\} \le t) + Q(\tau_{m,3} \le t), Q(\min\left\{\tau_{m,1}, \tau_{m,3}\right\} \le t)\right.\\ &\left.+Q(\tau_{m,2} \le t), Q(\min\left\{\tau_{m,2}, \tau_{m,3}\right\} \le t) + Q(\tau_{m,1} \le t)\right\}\\ \underline{Q}\left(\tau_m^{(3)} \le t\right) &= \max\left\{Q(\min\left\{\tau_{m,1}, \tau_{m,2}\right\} \le t),\right.\\ &\left.Q(\min\left\{\tau_{m,1}, \tau_{m,3}\right\} \le t), Q(\min\left\{\tau_{m,2}, \tau_{m,3}\right\} \le t)\right\}.\end{aligned}$$

Proof It holds that

$$Q\left(\tau_m^{(3)} \le t\right) = Q\left(\min\left\{\tau_{m,1}, \tau_{m,2}, \tau_{m,3}\right\} \le t\right)$$

Notice that

$$\left\{\min\left\{\tau_{m,1}, \tau_{m,2}, \tau_{m,3}\right\} \le t\right\} = \left\{\min\left\{\tau_{m,1}, \tau_{m,2}\right\} \le t\right\} \cup \left\{\tau_{m,3} \le t\right\}$$

Using

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B)$$

immediately gives the upper bound. The lower bound follows from $P(A \cup B) \ge P(A)$. □

To derive an upper price bound on a relax certificate, $Q\left(\tau_m^{(3)} > t\right)$ is replaced by $\left(1 - \underline{Q}\left(\tau_m^{(3)} \le t\right)\right)$ while $Q\left(\tau_m^{(3)} \le t\right)$ is replaced by $\overline{Q}\left(\tau_m^{(3)} \le t\right)$. It is straightforward to show that the resulting upper price bound is higher than the one given in Theorem 1.

4 Numerical examples

4.1 Risk-neutral measure

For the specific examples, we rely on a Black-Scholes-type model setup with no dividends. Each stock price $S_t^{(j)}$ satisfies the stochastic differential equation

$$dS_t^{(j)} = \mu_j S_t^{(j)} dt + \sigma_j S_t^{(j)} dW_t^{(j)}, \tag{15}$$

where $W^{(j)}$ is a standard Brownian motion under the real world measure P. The Wiener processes are in general correlated, i.e. for $i \ne j$ it holds that $d\langle W^{(i)}, W^{(j)}\rangle_t = \rho_{ij}\, dt$ where we assume constant correlations. Equation (15) implies that the dynamics of the stock prices under the risk neutral measure Q are

$$dS_t^{(j)} = rS_t^{(j)} dt + \sigma_j S_t^{(j)} dW_t^{Q,(j)} \tag{16}$$

where $W^{Q,(j)}$ is a standard Brownian motion under Q.

4.2 Prices of relax certificates

For the model of Black-Scholes and in the case of one underlying, the first hitting time distribution is well known and was derived using the reflection principle as in Karatzas and Shreve (1999) or Harrison (1985).Using these formulas for the first hitting time, the price of the relax certificate can be calculated in closed-form:

Proposition 5 (Price of a relax certificate on one underlying) *For $n = 1$, the price $RC_{t_0}^{(1)}$ can be given in closed-form. The survival probability $Q\left(\tau_m^{(1)} \geq t\right)$ needed in Eq. (7) to price the knock-out component is:*

$$Q(\tau_m^{(1)} \geq t) = N\left(\frac{-ln\frac{m}{S_0} + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) + e^{2\frac{r-\frac{1}{2}\sigma^2}{\sigma^2} ln\frac{m}{S_0}} N\left(\frac{ln\frac{m}{S_0} + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right)$$

where N denotes the cumulative distribution function of the standard normal distribution. The minimum claim in Eq. (8) reduces to the underlying itself, and the price of the knock-in component is

$$RI_{t_0}^{(1)} = m \int_{t_0}^{t_N} e^{-ru} f_{\tau_m^{(1)}}^{Q}(u)du \tag{17}$$

where the density $f_{\tau_m^{(1)}}^{Q}$ of the first hitting time $\tau_m^{(1)}$ is given in Proposition 7 or Corollary 2 respectively of Appendix A.

Proof The expression for the density $f_{\tau_m^{(1)}}^{Q}$ is based on well known results which, for the sake of completeness, are given in Appendix A. For the knock-in component, first note that

$$C_t^{\text{Min},1} = E_Q\left[e^{-r(t_N-t)} \min\left\{S_{t_N}^{(1)}\right\} \mid \mathcal{F}_t\right] = E_Q\left[e^{-r(t_N-t)} S_{t_N}^{(1)} \mid \mathcal{F}_t\right] = S_t^{(1)}.$$

In addition, we know that at the hitting time $\tau_m^{(1)} = u$ it holds that $S_u = m$. This gives

$$E\left[\int_{t_0}^{t_N} e^{-ru} C_u^{\text{Min},1} dN_u\right] = mE\left[\int_{t_0}^{t_N} e^{-ru} dN_u\right] = m \int_{t_0}^{t_N} e^{-ru} f_{\tau_m^{(1)}}^{Q}(u)du.$$

□

For two or more underlyings, we rely on Theorem 1. The distribution of the first hitting time is known in semi-closed form for $n = 2$, where we rely on He et al. (1998) and Zhou (2001). This allows us to calculate the price of the knock-out component and an upper bound for the knock-in component. The resulting upper bound for the price of a relax certificate on two underlyings is also an upper price bound for relax certificates on more than two underlyings.

The distribution of the first hitting time is given in the next proposition.

Proposition 6 (Distribution of first hitting time for $n = 2$) *The distribution of the first hitting time min* $\{\tau_{m,k}, \tau_{m,l}\}$ *is given by*

$$Q\left(\min\{\tau_{m,k}, \tau_{m,l}\} > t\right) = \frac{2}{\alpha t} e^{a_k \ln\left(\frac{S_0^{(k)}}{m}\right) + a_l \ln\left(\frac{S_0^{(l)}}{m}\right) + bt} \times \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) e^{-\frac{r_0^2}{2t}} \int_0^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta) d\theta.$$

The parameters and the function g_n *are defined in Corollary 3 in Appendix B.*

Proof The survival probability $Q(\min\{\tau_{m,k}, \tau_{m,l}\} > t)$ follows from the results of He et al. (1998) and Zhou (2001). Details are given in Appendix B. □

The upper price bound in Theorem 1 results from looking at all subsets with two underlyings. If the relax certificate itself is written on two underlyings only, the knock-out component can be priced exactly, and only the knock-in part is approximated from above. Since for most realistic parameter values the "main part" of the product is explained by the knock-out part, the price bound is especially tight in this case which is illustrated by the following simulation study.

We illustrate our results in a Black-Scholes economy with short rate $r = 0.05$, volatilities $\sigma_1 = \sigma_2 = \sigma = 0.4$, and correlations between all assets set to $\rho = 0.25$. The prices are calculated using a Monte-Carlo simulation with 10.000 simulation runs and a step size of 100 steps per day.[11]

Our base contract is an attractive relax certificate written on two underlyings $S^{(1)}$ and $S^{(2)}$ with initial values $S_0^{(1)} = S_0^{(2)} = 1$. The time to maturity is 3 years, intermediate payment dates are $t_1 = 1$ and $t_2 = 2$ (years), the bonus payment is $\delta = 0.11$ and the barrier is $m = 0.5$. The probability that none of the stocks hits the barrier until time 3 is 71.11%. The price of the knock-out component is 0.8624, and the price of the knock-in minimum claim is 0.1276. The knock-out component thus represents a large part of the price of the relax certificate.

The upper bound for the knock-in minimum component follows from Proposition 4 and is equal to 0.1445. This gives an upper price bound of 1.0069 for the relax certificate, which exceeds the true price by less than 2%.

In a first step, we analyze the impact of volatility and correlation. Figure 1 gives the price, the survival probability, and the prices of the knock-out and the knock-in component as a function of volatility and for various correlations ρ. In line with intuition, the survival probability is decreasing in volatility and increasing in correlation. The same holds true for the price of the knock-out coupon bond, which is equal to the discounted sum of survival probabilities as can be seen in Eq. (7). The price of the knock-in minimum component is more involved. A larger volatility or a lower correlation increases the probability that the barrier is hit and that the payoff from the knock-in component is positive. This leads to a larger price of the knock-in

[11] To control the accuracy of the approximation, the simulation results for the survival probabilities and the prices of the knock-out component are compared to the exact closed-form solutions. These closed-form solutions, which are valid for one and two underlyings in the Black-Scholes model only, also allow for a quick calculation of the upper price bound.

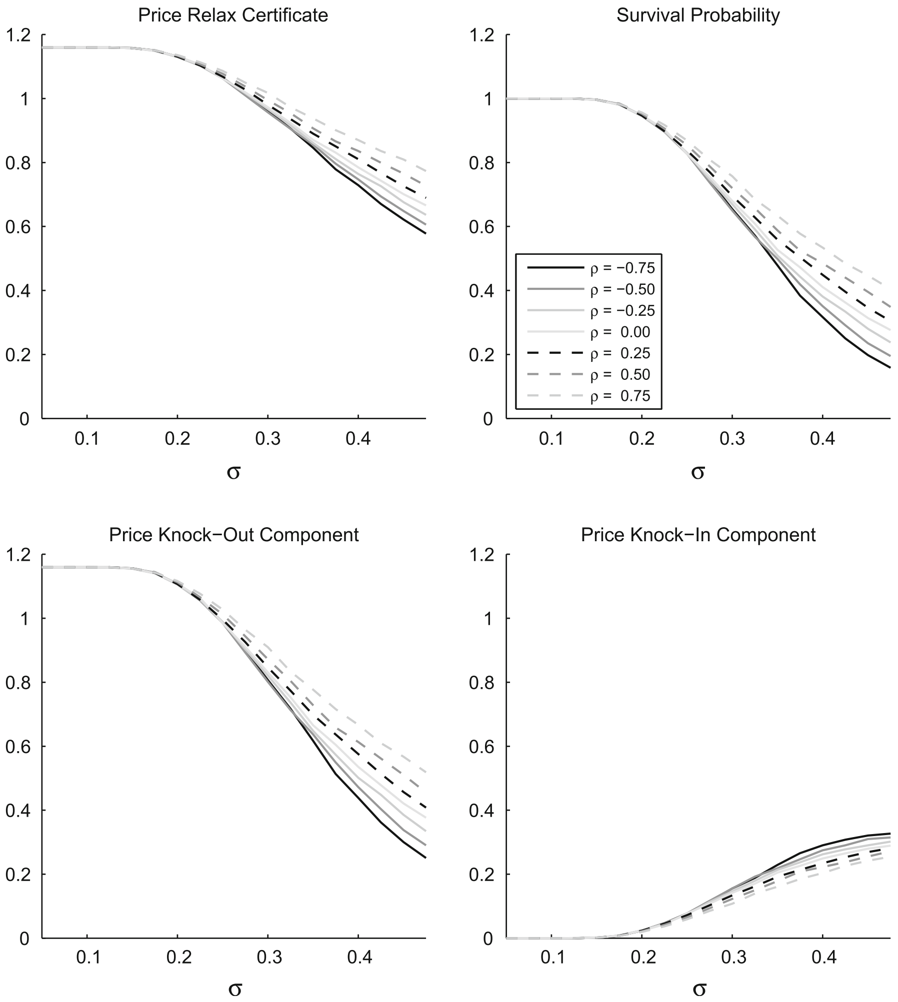

Fig. 1 Relax certificate on two underlyings: impact of volatility and correlation. The figure gives the price, the survival probability, the price of the knock-out component and the price of the knock-in component of a relax certificate as a function of the volatility of the two stocks for varying correlations. The parameters are $m = 0.5$, $\delta = 0.11$, $\underline{\underline{T}} = \{1, 2, 3\}$, $S_0^{(1)} = S_0^{(2)} = 1$ and $r = 0.05$

component in Eq. (13). At the same time, a high volatility or a low correlation increases the probability for at least one very low terminal stock price, which reduces the price of the knock-in component. For our parameters, the first effect dominates, and the price of the knock-in component is increasing in volatility and decreasing in correlation.

The results also show that the knock-out part represents a large part of the price of the relax certificate for nearly all correlations and volatilities. Its price contribution ranges from nearly 100% for a volatility of 0.1 and all correlations to at least 50% for all positive correlations and all volatilities. For negative correlations and very high volatilities ($\sigma \geq 0.4$) the knock-in part dominates

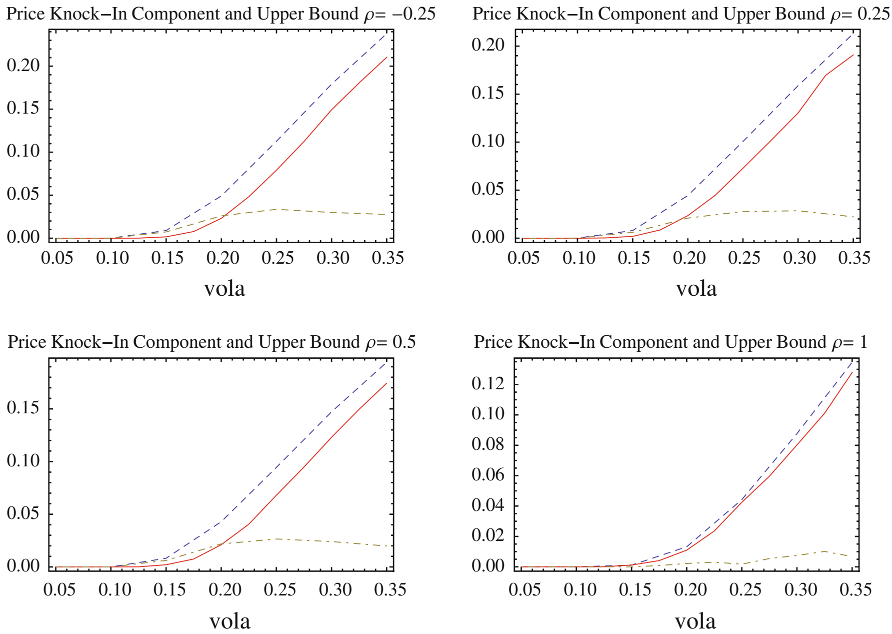

Fig. 2 Exact price vs. upper bound of knock-in part. The figure compares the exact price of the knock-in component (*solid line*) with the upper price bound derived in Proposition 4 (*dashed line*). The *dash-dotted line* shows the difference between the upper price bound and the exact price

because of the low survival probabilities. The difference between the upper price bound and the exact price of the knock-in part is illustrated in Fig. 2. This difference is increasing in volatility and decreasing in correlation. The overestimation of the true price by the upper price bounds given in Proposition 4 and Theorem 1 respectively is rather small. It ranges from basically zero (for $\sigma \leq 0.1$ and all correlations) to 2.8% (for $\sigma = 0.35$ and $\rho = -0.25$) of the price of the relax certificate.

Secondly, we illustrate the impact of the number of underlyings, where we know from Proposition 3 that the price is decreasing in the number of underlyings. Figure 3 shows the survival probability and the prices of the relax certificate, the knock-out component and the knock-in component as a function of volatility for 1, 2, 3, 4, 5, and 10 stocks. In line with intuition, both the survival probability and the price of the knock-out component are the smaller the larger the number of underlyings. The price of the knock-in component is again more involved. While a larger number of underlyings increases the probability for a positive payoff from this component, it also increases the risk that at least one stock price at maturity is very low. For a small volatility, the first effect dominates, and the price of the knock-in component increases in the number of underlyings. For a large volatility, however, the price drops when the number of underlyings increases to ten. Furthermore, for ten underlyings the price of the knock-in component is no longer an increasing function of volatility, but decreases for large volatilities.

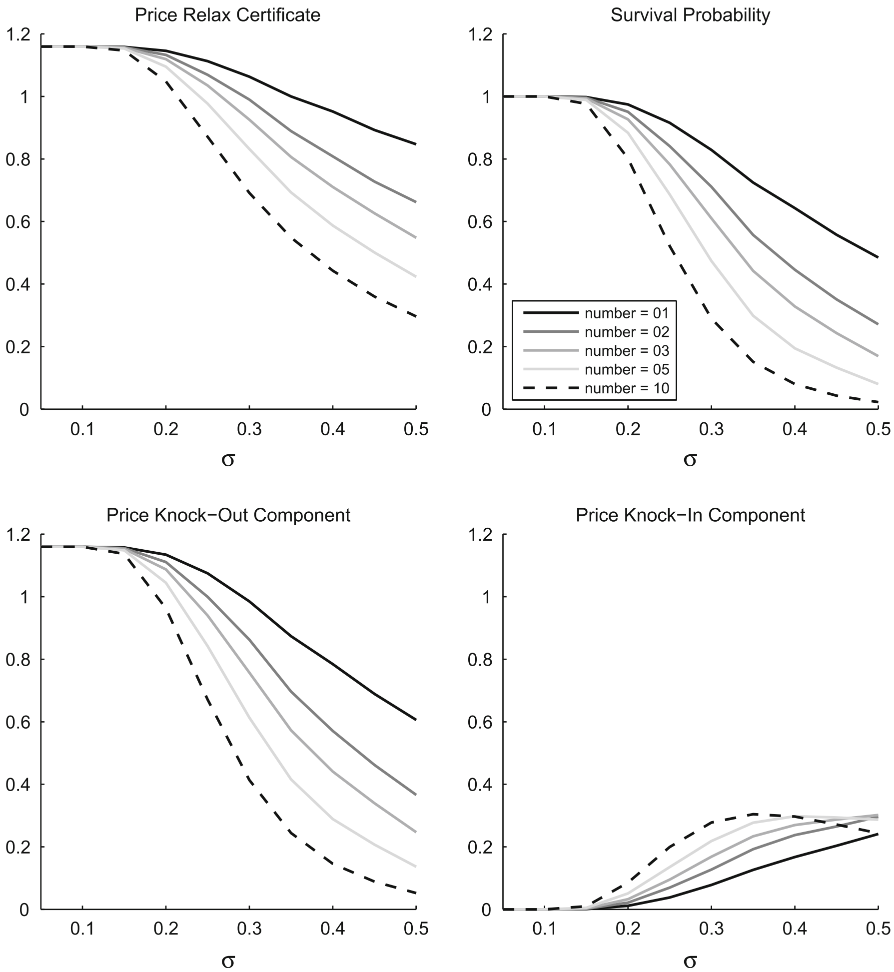

Fig. 3 Relax certificate on several underlyings: impact of volatility and number of underlyings. The figure gives the price, the survival probability, the price of the knock-out component and the price of the knock-in component of a relax certificate as a function of the volatility of the stocks for 1, 2, 3, 4, 5, and 10 uncerlyings. The parameters are $m = 0.5$, $\delta = 0.11$, $\underline{\underline{T}} = \{1, 2, 3\}$, $S_0^{(i)} = 1$, $\rho = 0.25$, and $r = 0.05$

Figure 4 shows the impact of the time to maturity. In line with intuition, the survival probability is the lower the longer the time to maturity. For low volatilities, the decrease in the survival probability is rather small. For a bonus rate of 11% which well exceeds the risk-free rate of 5%, the price of the knock-out bond component increases in the time to maturity. In contrast, for high volatilities the decrease in the survival probability dominates, and the price of the knock-out component decreases in the time to maturity. Since the knock-out component accounts for a large part of the overall price of the relax certificate, we see the same dependence on time to maturity for the overall price. The price of the knock-in component again depends on the probability of being knocked in, which increases in

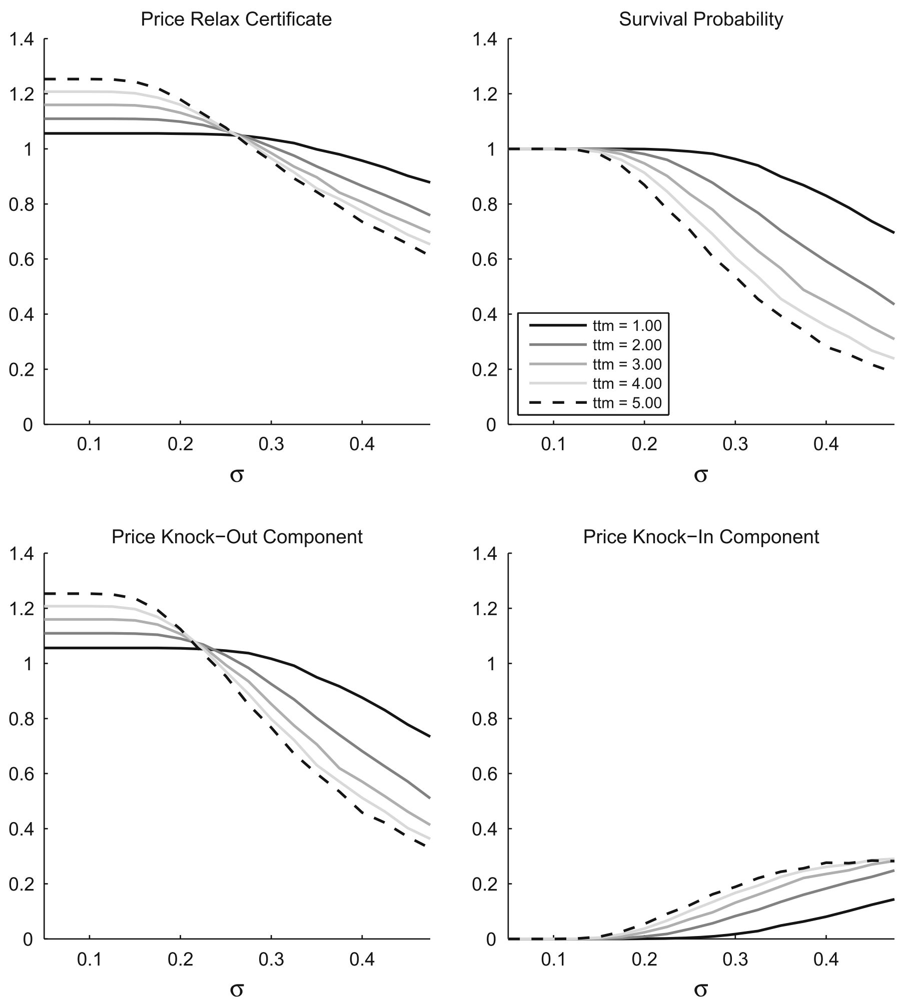

Fig. 4 Relax certificate on two underlyings: impact of volatility and time to maturity. The figure gives the price, the survival probability, the price of the knock-out component and the price of the knock-in component of a relax certificate as a function of the volatility of the two stocks for varying times to maturity. The parameters are $m = 0.5$, $\delta = 0.11$, $S_0^{(1)} = S_0^{(2)} = 1$, $\rho = 0.25$, and $r = 0.05$

the time to maturity, and on the payoff of the minimum claim, which decreases in the time to maturity. For the volatilities and times to maturity in our example, the first effect dominates, and the price of the knock-in component increases in the time to maturity.

Finally, we look at the impact of the barrier level m and the bonus payments δ, which is shown in Fig. 5. The survival probability and the price of the knock-in component are both independent of the bonus payments δ. The survival probability is decreasing in m, while the price of the knock-in component increases in m. The price of the knock-out component, on the other hand, decreases in m. It furthermore increases in the level of the bonus payments.

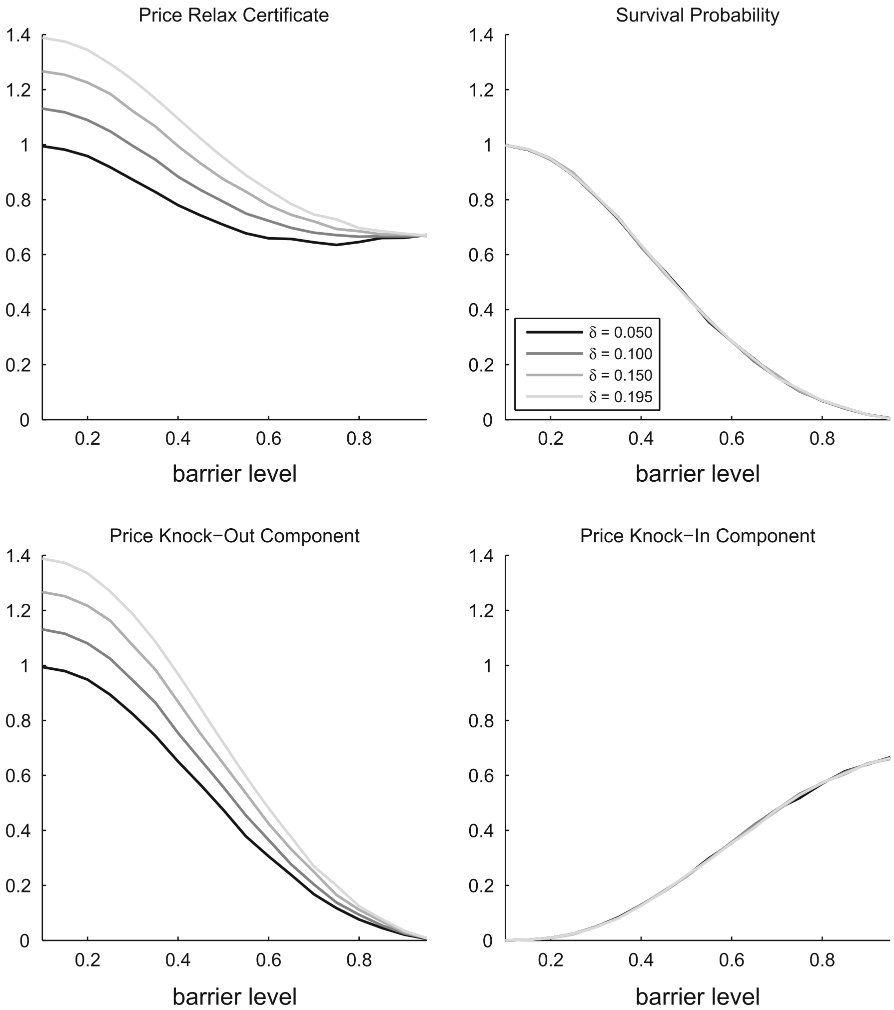

Fig. 5 Relax certificate on two underlyings: impact of barrier level and bonus payments. The figure gives the price, the survival probability, the price of the knock-out component and the price of the knock-in component of a relax certificate as a function of the barrier level m for varying bonus payments δ. The parameters are $\underline{\underline{T}} = \{1, 2, 3\}$, $S_0^{(1)} = S_0^{(2)} = 1$, $\sigma = 0.4$, $\rho = 0.25$, and $r = 0.05$

5 Market comparison

5.1 Contract specifications

We now analyze some relax certificates issued in 2008 and 2009 and compare their issue prices to our price bounds. Table 1 gives the contract specifications of six typical certificates. All barriers are set to a rather low value (50 or 60%), so that at least one of the underlying stocks has to loose a high fraction of its initial value for

Table 1 Summary of traded product specifications and interest rates

	n	δ	m	Underlyings	IV	Payment dates	Interest rate
C1	2	0.11	0.5	Daimler AG	0.33	14 months	0.0464
Commerz-bank				Siemens AG	0.35		
C2	3	0.16	0.5	Daimler AG	0.33	17 months,	0.0464
HSBC				Siemens AG	0.35	4 days	
				EON	0.27		
C3	3	0.10	0.5	Allianz	0.32	13 months	0.0464
HVB				BASF	0.25	26 months	0.0458
				Deutsche Post	0.30	39 months	0.0455
C4	3	0.11	0.5	EON	0.23	39 months	0.0455
HVB				Siemens AG	0.30		
				Tui	0.60		
C5	3	0.35	0.5	S&P 500	0.26	44 months,	0.0455
Societé				DJ Euro STOXX	0.25	4 days	
Géneral				NIKKEI 225	0.21		
C6	3	0.18	0.6	S&P 500	0.26	18 months,	0.0464
WestLB				DJ Euro STOXX	0.25	15 days	
				NIKKEI 225	0.21		

For each certificate C1–C6, the table gives the issuer, the number of underlyings n, the bonus payments δ and the lower barrier m. It also gives the underlyings and the implied volatilities of at-the-money options on these underlyings with a time to maturity equal to the time to maturity of the certificates. The last two columns contain the payment dates (only C3 has intermediate payments) and the risk-free rate for the corresponding investment horizons.

the coupon bond to be replaced by the minimum claim. Furthermore, the bonus payments are large enough for all certificates to be attractive in the sense of Definition 2.

The table also gives the underlyings as well as the implied volatilities of at-the-money options on these underlyings with a time to maturity equal to the maturity of the certificate. The time to maturity of all contracts is below four years, and only one of the six certificates has intermediate payment dates.

Finally, the second column in Table 2 gives the issue price of the certificates. All certificates are issued one Euro above par.

5.2 Survival probabilities and price bounds

For an attractive relax certificate, the price of the corresponding coupon bond is a trivial upper price bound. The interest rates are inferred from the corresponding zero coupon bonds (swaps) via bootstrapping and are given in the last column of Table 1. The resulting prices of the coupon bonds are given in Table 2. For all certificates, the issue price is significantly lower than the price of the corresponding coupon bond. The risk that at least one of the stocks looses more than 50 respectively 40%

Table 2 Price bounds for traded certificates

C	n	Issue price incl. load	Corresp. coupon bond	Survival probability		Upper price bound		
				One underlying (%)	Two underlyings (%)	Knock-out comp.	Knock-in comp.	Price
C1	2	101.00	104.25	96.88	93.60	97.32	3.20	100.52
C2	3	101.00	108.68	94.97	89.90	96.11	5.25	101.36
C3	3	101.00	113.60	80.76	65.35	75.19	17.32	92.51
C4	3	101.00	116.32	81.77	65.35	80.03	17.32	97.35
C5	3	101.00	114.46	87.804	70.62	80.65	14.69	95.34
C6	3	1,001.00	1,101.66	91.57	80.92	891.10	114.00	1,005.10

For each certificate C1–C6, the table gives the number of underlyings, the issue price, the price of the corresponding coupon bond, the survival probabilities based on one and two underlyings, and the upper price bounds based on two underlyings. The calculations are based on a volatility of $\sigma = 0.3$ for the stocks and $\sigma = 0.25$ for the indices. The correlation is $\rho = 0.3$

of the initial value reduces the value of the relax certificate by 4 to 11%. This risk should thus not be neglected.

To assess the risk inherent in the relax certificate, we look at the (risk-neutral) probability that the barrier will not be hit. This survival probability is bounded from above by the survival probability for one or two underlyings. In the calculation, we set $\rho_{k,l} = 0.3$ and $\sigma_k = \sigma_l = 0.3$ for C1–C4 and $\sigma_k = \sigma_l = 0.25$ for the remaining certificates.[12] The results show that adding a second underlying significantly increases the risk that the bond will be knocked out. They also confirm that the risk of a knock-out is rather high.

In the next step, we consider the upper price bounds that result from Theorem 1. They are also given in Table 2. For all certificates, the upper price bound of the knock-out coupon bond largely exceeds the upper bound on the value of the knock-in minimum claim. Furthermore, the resulting upper price bound is below the issue price for all but two certificates. If we account for dividend payments of the stocks, the upper price bound would even decrease further. The same holds true if we take credit risk into account.[13]

For C1 and C2, we also calculate the upper price bounds using the implied volatilities of the underlyings and a correlation which ranges from -1 to 1. For $n = 1$, the upper price bound follows from Proposition 5, while we rely on Theorem 1 for $n = 2$. For C1, we find that the issue price exceeds the lowest upper price bound for all correlation levels. For C2, the issue price is below the upper bound only if the correlation is larger than 0.85.

There are two possible conclusions. First, relax certificates may be overpriced in the market. This is in line with the empirical results of Wallmeier and

[12] For all certificates, the implied volatilities of at least two underlying stocks as given in Table 1 are above 30%, so that a volatility of $\sigma = 0.3$ yields an upper bound for the survival probability. The same holds for the indices setting $\sigma = 0.25$.

[13] In 2009, CDS spreads of Commerzbank e.g. increased to more than 100 basis points. In a very rough approximation, this would reduce the prices of our certificates by around 1%. We thank an anonymous referee for pointing out this example.

Diethelm (2008) for the Swiss certificate market. Furthermore, the mispricing is the higher the higher the bonus payments (and thus the higher the discount due to the knock-out feature of the bond). We conjecture that the investors do not correctly estimate the risk associated with the barrier feature, but overweight the sure coupon.

Second, the model of Black-Scholes may not be the appropriate choice. If we include (on average downward) jumps as in Merton (1976), however, the knock-out probability increases. The resulting price bounds are lower than in the model of Black-Scholes such that the overpricing is even higher under a more realistic model setup. The same holds true if we account for default risk of the issuer, which again reduces the upper price bound calculated in a model. Dividend payments of the underlying, which we have not taken into account, have a similar effect and also reduce the upper price bound. Finally, our price bounds are based on two underlyings only, and they would be lower if we accounted for the larger number of underlyings. We thus conclude that it is hard to find a model-based motivation for the large prices of relax certificates at the market and that there is strong evidence that these contracts are indeed overpriced.

6 Conclusion

Relax certificates can be decomposed into a knock-out coupon bond and a knock-in minimum claim on the underlying stocks. The contracts are designed such that relax certificates can be offered at a discount compared to the associated coupon bond. Formally, this gives a condition on admissible (or *attractive*) contract parameters in terms of the barrier and the bonus payments.

The knock-out/knock-in event takes place when the worst-performing of the underlying stocks hits a lower barrier. Our analysis shows that the probability of a knock-out cannot be neglected and induces a significant price discount of the relax certificate as compared to the corresponding coupon bond. The risk is the larger the higher the volatility of the underlyings, the lower their correlation, the larger the number of stocks the certificate is written on, and the longer the time to maturity.

In general, numerical methods are needed to price relax certificates, and even in the Black-Scholes model closed form solutions only exist for one underlying. However, closed-form or semi-closed form solutions are available for upper price bounds. A trivial upper price bound is given by the corresponding coupon bond. Furthermore, the price of a relax certificate on several underlyings is bounded from above by the price of the (cheapest) relax certificate on a subset of underlyings. We show that two underlyings allow to achieve meaningful and tractable price bounds. The most likely candidates to give this lowest upper price bound are the relax certificates on the most risky assets and/or the assets with the lowest correlation between the underlyings.

Finally, we test the practical relevance of our theoretical results by comparing the price bounds to market prices. The upper price bounds are calculated based on the implied volatilities of call options on the respective underlyings. It turns out that relax certificates which are currently traded are significantly overpriced. This result is true for nearly all correlation scenarios.

Appendix A: First hitting time: one-dimensional case

To derive the distribution of the first hitting time in the one-dimensional case, we use results given in He et al. (1998).They consider the probability density and distribution function of the maximum or minimum of a one-dimensional Brownian motion with drift. Along the lines of He et al. (1998), we define

$$\underline{X}_t := \min_{0 \le s \le t} X_s \qquad \overline{X}_t := \max_{0 \le s \le t} X_s$$

where $X_t = \alpha t + \sigma W_t,\ \ t \ge 0$ and α, σ are constants. W is a Brownian motion defined on some probability space.

Proposition 7 *Let $G(x, t;\alpha)$ and $g(y, x, t;\alpha_1)$ be defined as*

$$G(x, t; \alpha) := N\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right) - e^{\frac{2\alpha x}{\sigma^2}} N\left(\frac{-x - \alpha t}{\sigma\sqrt{t}}\right),$$
$$g(y, x, t; \alpha_1) := \frac{1}{\sigma\sqrt{t}} N'\left(\frac{x - \alpha_1 t}{\sigma\sqrt{t}}\right)\left(1 - e^{-\frac{4x^2 - 4x4y}{2\sigma^2 t}}\right)$$

where N denotes the cumulative distribution function of the standard normal distribution and $N'(z)$ the density of the standard normal distribution. For $x \ge 0$, it holds

$$P(\overline{X}_t \le x) = G(x, t; \alpha), \qquad f^P_{\tau_m^{(1)}} = g(y, x, t; \alpha_1)dy.$$

For $x < 0$, it holds

$$P(\underline{X}_t \ge x) = G(-x, t; -\alpha), \qquad P(X_1(t) \in dy, \underline{X}_1(t) \ge x) = g(-y, -x, t; -\alpha_1)dy.$$

Proof C.f. Theorem 1 of He et al. (1998). and the proof given here. $\square$

Corollary 2 *Let*

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

where μ, σ ($\sigma > 0$) are constants. W is a Brownian motion defined on some probability space. For the first hitting time $\tau_m := \inf\{t \ge 0 | S_t \le m\}$, $(m < S_0)$ it holds that

$$P(\tau_m \le t) = N\left(\frac{\ln\frac{m}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) + e^{2\frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2} \ln\frac{m}{S_0}} N\left(\frac{\ln\frac{m}{S_0} + \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right),$$

$$f^P_{\tau_m^{(1)}} = \frac{-\ln\frac{m}{S_0}}{\sqrt{2\pi\sigma^2 t^3}} e^{-\frac{1}{2}\frac{\left(\ln\frac{m}{S_0} - (\mu - \frac{1}{2}\sigma^2)t\right)^2}{\sigma^2 t}} dt.$$

Proof Note that

$$\tau_m := \inf\left\{t \geq 0 | S_t \leq m\right\} = \inf\left\{t \geq 0 \middle| \ln\frac{S_t}{S_0} \leq \ln\frac{m}{S_0}\right\}.$$

Let X_t denote the logarithm of the normalized asset price, i.e. $X_t := \ln\frac{S_t}{S_0} = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t$ and set $\alpha = \mu - \frac{1}{2}\sigma^2$. The stopping time τ_m is related to the first hitting time of a one-dimensional Brownian motion with drift α. With $x := \ln\frac{m}{S_0} < 0$ it follows

$$P(\tau_m \leq t) = P(\underline{X}_t \leq x) = 1 - P(\underline{X}_t \geq x).$$

According to Proposition 7, we have

$$\begin{aligned} 1 - P(\underline{X}_t \geq x) &= 1 - G(-x, t; -\alpha) = 1 - N\left(\frac{-x + \alpha t}{\sigma\sqrt{t}}\right) + e^{\frac{2\alpha x}{\sigma^2}} N\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right) \\ &= N\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right) + e^{\frac{2\alpha x}{\sigma^2}} N\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right). \end{aligned}$$

Inserting α and x gives the distribution function. To derive the density function, define

$$f(t) := N\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right) + e^{\frac{2\alpha x}{\sigma^2}} N\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right).$$

This implies

$$\begin{aligned} f'(t) = N'\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right) &\times \left(\frac{-\alpha\sigma\sqrt{t} - \frac{\sigma}{2\sqrt{t}}(x - \alpha t)}{\sigma^2 t}\right) \\ &+ e^{\frac{2\alpha x}{\sigma^2}} \times N'\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right) \times \left(\frac{\alpha\sigma\sqrt{t} - \frac{\sigma}{2\sqrt{t}}(x + \alpha t)}{\sigma^2 t}\right) \end{aligned}$$

Using $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, we get

$$\begin{aligned} e^{\frac{2\alpha x}{\sigma^2}} N'\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right) &= \frac{1}{\sqrt{2\pi}} e^{\frac{2\alpha x}{\sigma^2} - \frac{1}{2}\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{\sigma^2}\left[2\alpha x - \frac{1}{2t}(x + \alpha t)^2\right]} \\ &= \frac{1}{\sqrt{2\pi}} e - \frac{1}{2}\frac{(x - \alpha t)^2}{\sigma^2 t} = N'\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right). \end{aligned}$$

Inserting this in the above equation for $f'(t)$ gives

$$f'(t) = N'\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right) \times \frac{-\sigma}{2\sqrt{t}\sigma^2 t}(x - \alpha t + x + \alpha t) = \frac{-x}{\sqrt{2\pi\sigma^2 t^3}} e^{-\frac{1}{2}\frac{(x - \alpha t)^2}{\sigma^2 t}}.$$

Using $\alpha = \mu - \frac{1}{2}\sigma^2$ and $x = \ln\frac{m}{S_0}$ gives the result. $\square$

Appendix B: First hitting time: two dimensional case

The distribution of the first hitting time of a two-dimensional arithmetic Brownian motion is given in He et al. (1998) and Zhou (2001):

Proposition 8 *Let* $X_t^{(j)} = \alpha_j t + \sigma_j W_t^{(j)}$ $(j = 1, 2)$, *where* α_j *and* σ_j *are constants.* $W^{(1)}$, $W^{(2)}$ *are two correlated Brownian motions with* $< W^{(1)}, W^{(2)} >_t = \rho\ t$. *Then, the probability that* $X^{(1)}$ and $X^{(2)}$ *will not hit the upper boundaries* $x^{(1)} > 0$ and $x^{(2)} > 0$ *up to time t is given by*

$$Q\left(\overline{X}_t^{(1)} \le x^{(1)}, \overline{X}_t^{(2)} \le x^{(2)}\right) = \frac{2}{\alpha t} e^{a_1 x_1 + a_2 x_2 + bt} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right)$$

$$\times\, e^{-\frac{r_0^2}{2t}} \int_0^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta) d\theta$$

where $\overline{X}_t := \max_{0 \le s \le t} X_s$. *The parameters are defined by*

$$a_1 = \frac{-\alpha_1\sigma_2 + \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2} \qquad a_2 = \frac{-\alpha_2\sigma_1 + \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_2^2\sigma_1}$$

$$d_1 = a_1\sigma_1 + a_2\sigma_2\rho \qquad d_2 = a_2\sigma_2\sqrt{1-\rho^2}$$

and by

$$b = \alpha_1 a_1 + \alpha_2 a_2 + \frac{1}{2}\sigma_1^2 a_1^2 + \frac{1}{2}\sigma_2^2 a_2^2 + \rho\sigma_1\sigma_2 a_1 a_2$$

$$\alpha = \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise} \end{cases}$$

$$\theta_0 = \begin{cases} \tan^{-1}\left(\frac{\frac{x2}{\sigma_2}\sqrt{1-\rho^2}}{\frac{x1}{\sigma_1} - \rho\frac{x2}{\sigma_2}}\right) & \text{if } (.) > 0 \\ \pi + \tan^{-1}\left(\frac{\frac{x2}{\sigma_2}\sqrt{1-\rho^2}}{\frac{x1}{\sigma_1} - \rho\frac{x2}{\sigma_2}}\right) & \text{otherwise} \end{cases}$$

$$r_0 = \frac{x_2}{\sigma_2} / \sin(\theta_0).$$

The function g_n *is defined as*

$$g_n(\theta) = \int_0^{\infty} r e^{-\frac{r^2}{2t}} e^{d_1 r \sin(\theta-\alpha) - d_2 r \cos(\theta-\alpha)} I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{t}\right) dr.$$

$I_v(z)$ *is the modified Bessel function of order* v.

Proof Cf. Proposition 1 of Zhou (2001) and the proof given there. □

Corollary 3 *For two stocks $S^{(k)}$ and $S^{(l)}$ with volatilities σ_k and σ_l and correlation $\rho_{k,l}$, the distribution function of the first hitting time* $\min\{\tau_{m,k}, \tau_{m,l}\}$ *under the risk-neutral measure is given by*

$$Q\big(\min\{\tau_{m,k}, \tau_{m,l}\} \le t\big) = 1 - \frac{2}{\alpha t} e^{a_k \ln\left(\frac{S_0^{(k)}}{m}\right) + a_l \ln\left(\frac{S_0^{(l)}}{m}\right) + bt} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \cdot e^{-\frac{r_0^2}{2t}} \int_0^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta) d\theta$$

where

$$a_k = \frac{\left(r - 0.5\sigma_k^2\right)\sigma_l - \rho_{k,l}\left(r - 0.5\sigma_l^2\right)\sigma_k}{\left(1 - \rho_{k,l}^2\right)\sigma_k^2 \sigma_l} \qquad a_l = \frac{\left(r - 0.5\sigma_l^2\right)\sigma_k - \rho_{k,l}\left(r - 0.5\sigma_k^2\right)\sigma_l}{\left(1 - \rho_{k,l}^2\right)\sigma_l^2 \sigma_k}$$

$$d_k = a_k\sigma_k + a_l\sigma_l\rho_{k,l} \qquad d_l = a_l\sigma_l\sqrt{1 - \rho_{k,l}^2}$$

and

$$b = -\left(r - 0.5\sigma_k^2\right)a_k - \left(r - 0.5\sigma_l^2\right)a_l + \frac{1}{2}\sigma_k^2 a_k^2 + \frac{1}{2}\sigma_l^2 a_l^2 + \rho_{k,l}\sigma_k\sigma_l a_k a_l$$

$$g_n(\theta) = \int_0^{\infty} r e^{-\frac{r^2}{2t}} e^{d_k r \sin(\theta - \alpha) - d_l r \cos(\theta - \alpha)} I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{t}\right) dr.$$

$I_v(z)$ is the modified Bessel function of order v. α, θ_0, and r_0 are given in Proposition 8 in Appendix B for the case where $k = 1$ and $l = 2$.

Proof The stock prices are given by

$$S_t^{(j)} = S_0 e^{\left(r - \frac{1}{2}\sigma_j^2\right)t + \sigma_j W_t^{(j)}} \quad j = k, l.$$

The first hitting time of the lower boundary $m_j < S_0^{(j)}$ by the geometric Brownian motion $S_t^{(j)}$ is

$$\tau_m^{(j)} := \inf\left\{t \ge 0 \middle| S_t^{(j)} \le m_j\right\} = \inf\left\{t \ge 0 \middle| -\ln\frac{S_t^{(j)}}{S_0^{(j)}} \ge \ln\frac{S_0^{(j)}}{m_j}\right\}.$$

With the definition of the arithmetic Brownian motion

$$X_t^{(j)} := -\ln\frac{S_t^{(j)}}{S_0^{(j)}} = -\left(r - \frac{1}{2}\sigma_j^2\right)t - \sigma_j W_t^{(j)},$$

the first hitting time can be rewritten as $\tau_m^{(j)} = \inf\left\{t \ge 0 \middle| X_t^{(j)} \ge \ln\frac{S_0^{(j)}}{m_j}\right\}$. Using the relation $\{\tau_{m,j} > t\} = \left\{\overline{X}_t^{(j)} < \ln\frac{S_0^{(j)}}{m_j}\right\}$ we can conclude

$$Q\left(\min\{\tau_{m,k},\tau_{m,l}\} > t\right) = Q\left(\overline{X}_t^{(k)} < \ln\frac{S_0^{(k)}}{m_k}, \overline{X}_t^{(l)} < \ln\frac{S_0^{(l)}}{m_l}\right).$$

Since both, $\ln\frac{S_0^{(k)}}{m_k} > 0$ and $\ln\frac{S_0^{(l)}}{m_l} > 0$, the result follows from Proposition 8. □

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