Algoritmi di Crittografia Corso di Laurea Magistrale in Informatica

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Algoritmi di Crittografia

- Randomness
 - Basic probability
 - Random (and Pseudo-Random) Number Generators

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Randomness and cryptography

- Modern crypto applications would be simply not possible without the availability of some source of randomness
- For instance, randomness is required to generate:
 - asymmetric keys in RSA-based protocols;
 - symmetric keys in SSH sessions;
 - stored salt data to be hashed with passwords;
 - initialization vectors to a block cipher's mode of operation.
- As we shall see, the ideal source of randomness for these (and other crypto) applications produces sequences of bits that are: (1) independent, (2) uniformly distributed, (3) always ready to hand
- Not as easy as it might sound...
- But let's start with some basic probability



Probability (informal)

- The probability of an event is a measure of "how likely is the occurrence of that event"
- Such measures are normalized values in the interval $[0,1] \subseteq \mathbb{R}$
- Over discrete set of events, 0 means "impossible" while 1 means "certain"
- Over the reals (e.g., when dealing with time) things are little more complicated
- In computer science applications (and hence in Cryptography) the sets of events are always discrete
- But what exactly is the probability of an event? How can we assign such numbers to the events that might occur?
- Not always easy an answer



Probability (informal)

- Start by considering an experiment involving randomness (e.g., tossing a coin, rolling a dice, ...)
- Each possible outcome (e.g., the face which the coin ends up showing) is an elementary event
- Each elementary event is given (here is where we can make mistakes ...) a probability, i.e., a real number in [0, 1]
- The constraint is that the sum (finite or infinite) of the probabilities to all the events must be 1
- An event is then any subset of the set of all the elementary events
- Example, "the dice ends up showing a face with an even number of dots" is an event composed of 3 elementary events
- The probability of an event is the sum of the probabilities of the corresponding elementary events



Probability (decently formal)

- To reason about probability we need the following ingredients
- We are given a set Ω , finite or countably infinite, called the *sample* space
- We consider a function (a *measure*) over 2^{Ω} with values in [0, 1]; call them p (for probability...)
- p is defined for any $A \subseteq \Omega$ (i.e., $A \in 2^{\Omega}$)
- In particular, p is defined for any $\omega \subseteq \Omega$ such that $|\omega| = 1$ (ω is an elementary event)
- For any $A \subseteq \Omega$, it holds that

$$p(A) = \sum_{\omega \in A} p(\omega)$$

• If A is empty, p(A) = 0 while, if $A = \Omega$, p(A) = 1



Example

- We consider the experiment of tossing a coin and denote with head and tail the corresponing elementary events
- $\Omega = \{head, tail\}$
- Assuming the two outcomes are equally likely, and that it is impossible for the coin to land on the edge, we can set:

$$p(head) = p(tail) = \frac{1}{2}$$



Probability distributions

- The "form" of the probability function is known as the (probability) distribution
- When all the elementary events have the same probability (which implies that the sample space is finite) we speak of *uniform* distribution
- The distribution of the coin tossing example of the previous slide is uniform
- Uniform distributions are the most important in cryptographic applications, but many different (and important) distributions exist
- We will see one in the next two slides



Poisson distribution

- Suppose you (as a web server admin) are interested in the following question "How many http requests will arrive in the next t seconds"?
- Clearly, even if we may sensibly assume that the number of requests will be a finite quantity, we cannot place an à priori bound
- Hence, we want to assign a probability to all the events: no request (in t seconds), just one request, exactly two requests, and so on
- To do this, we need a model, which can be more or less accurate
- We assume to know the $\it rate$ with which http requests arrive, a quantity that we denote by λ



Poisson distribution

Let us define:

$$p(k \text{ requests in } t \text{ secs}) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

• Since $p(k \text{ requests in } t \text{ secs}) \ge 0$ and

$$\sum_{k=0}^{\infty} p(k \text{ requests in } t \text{ secs}) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} = 1$$

it follows that *p* is a valid probability function

 Under reasonable assumptions, it is also a good choice, meaning that it produces accurate estimates of the process under analysis

- Random variable represent a very useful way of dealing with sample spaces
- Consider the experiment of rolling a couple of dice
- What are the elementary events here?
- Clearly, a dice rolls and (when it ends rolling) the important fact is the number of dots in its up-facing side
- So, to describe one possible outcome, we should rigorously (but quite tediously) write something like:

The first dice ended up rolling with the up-facing side showing 3 dots and the other dice ended up rolling with the up-facing side showing 2 dots

- Why not to say simply: one (dice) is 3 and the is 2?
- Because 3 and 2 are numbers, not events
- Here is where random variables come up



- A random variable maps events to a numeric space S
- In S it is easier to make quantitative reasonings
- Random variables also make it possible to describe events in an easier way
- Lex X be the random variable that maps the 6 possible outcomes (elementary events) of the experiment of rolling a dice
- X = 2 is then a (very convenient) way to denote the elementary event: "The dice ended up rolling with the up-facing side showing 2 dots"
- Similarly, $x \le 3$ denotes the event: "The dice ended up rolling with the up-facing side showing 1, 2, or 3 dots"



• If X is a random variable and $x \in S$, then the probability that X has the value s, denoted p[X = s], is

$$p[X = s] = \sum_{\omega \in \Omega | X(\omega) = s} p(\omega)$$

i.e., is the sum of the probabilities of all the elementary events that maps to \boldsymbol{s}

• Let X be a random variable that maps the outcomes of rolling two dice to $S = \{2, 3, ..., 12\}$ in the obvious way, namely:

$$X(\bullet, \bullet) = 2$$

$$X(\bullet, \bullet) = 3$$

$$X(\square,\square) = 12$$



But then for instance:

$$p[X=4]=p(\bullet,\bullet)+p(\bullet,\bullet)+p(\bullet,\bullet)$$

 We can also define the value of X using a richer mathematical armory:

Randomness

$$p[X \leq 3] = p(\bullet, \bullet) + p(\bullet, \bullet) + p(\bullet, \bullet)$$

or

$$\rho[X > 7 \text{ and } X \le 10] = \rho(\blacksquare, \blacksquare) + \rho(\blacksquare) + \rho(\blacksquare)$$

Independence

- This is the last notion we recall explicitly, given its importance in cryptography
- Consider two events, E₁ and E₂, not necessarily defined over the same sample space
- We say that E₁ and E₂ are independent events if the occurrence of E₁ does not affect the probability of E₂ occurring, and vice versa
- If we toss a coin and roll a dice, the probability that the coin lands heads is ¹/₂ (assuming the coin is not biased), independently of the outcome of the dice experiment

Independent random variables

- Consider rolling two six-face dice and assume that the dice are not biased, so that we can fairly assume that all the possible outcomes have probability $\frac{1}{6}$.
- Let X and Y be random variables that refers to the first and second dice experiment, respectively
- Then, X and Y being independent amounts to the following property:

$$p(X = x \text{ and } Y = y) = p(X = x) \cdot p(Y = y) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

where clearly $x, y \in \{1, 2, 3, 4, 5, 6\}$

If X and Y are independent we may also write, e.g.:

$$p(X \le 4 \text{ and } Y > 2) = p(X \le 4) \cdot p(Y > 2) = \frac{4}{6} \cdot \frac{3}{6} = \frac{1}{3}$$



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A first simple example

- For cryptographic applications, it is crucial to have sources of randomness, i.e., logical or physical devices that can produce (measurable) random events in some space
- What ultimately matters is to map the available randomness to sequences of independently generated bits (or, simply, independent bits)
- Suppose we need a randomly chosen binary sequence of length n
 (say, to generate a session key as part of some symmetric protocol)
- All the possible 2ⁿ sequences must have the same chance to be chosen
- Assume also, for simplicity, that n = 4

A first simple example

- Suppose to this end we use a Random Bit Generator, i.e., a piece of software that, each time is called, returns 1 with probability p and 0 with probability 1 – p independently of past history
- In other words, the bits are independently generated
- Our question is: what is the probability of generating (say) 1011 as key?
- Well, this is easy: since the bits are independent, the probability is $p(1-p)pp = p^3(1-p)$
- In general, a key with k ones and 4 k zeroes will show up with probability:

$$\binom{4}{k}p^k(1-p)^{4-k}$$



A first simple example

- Now we ask a different question, with a clear cryptographyc flavor: how hard would be for \mathcal{E} to guess our session key?
- Suppose p = 1 and (clearly) 1 p = 0. In this case only one key is possible, namely 1111 which requires no guess at all...
- Suppose now that p=0.9. All the 16 4-bit long sequences have their chance now. But the sequence 1111 is still the best guesss for \mathcal{E} , who will be right approximately 66% of the times
- So, what is the value of p that cause \mathcal{E} having a hard time to guess correctly?

Surprise and ... information

- Let's analyze things from another perspective
- Assume \mathcal{E} knows the value of p and consider again the state of affairs where p=1.
- In this case releasing to $\mathcal E$ the "information" that the key is 1111 actually amounts to giving her no-info at all
- Suppose now that p=0.9. This time releasing the same info as above would indeed carry some information, however one that definitely would not knock Eve out
- It then seems that the question raised in the previous slide can be rephrased as follows: what is the value of p that implies maximum gain of information from the outcome?

Entropy

- Entropy is a well-known concept in thermodynamics, which is however not relevant to us
- In Information theory, the entropy of a random event is a measure of the average information that one acquires when the outcome is revealed
- ... and this is exactly what we need
- If the event has k possible outcomes, and p_1, p_2, \ldots, p_k denote their probabilities, then the entropy is defined as

$$\sum_{i=1}^{k} p_{i} \log_{2} \frac{1}{p_{i}} = -\sum_{i=1}^{k} p_{i} \log_{2} p_{i}$$

• The values we are looking for are then those p_i that maximize the above quantity, under the constraints: $p_i \ge 0$ and $\sum_{i=1}^k p_i = 1$

Entropy

- If k = 2, hence $p_1 = p$ and $p_2 = 1 p$, it is an easy excercise to prove that entropy is maximum for $p = \frac{1}{2}$
- The result carries over the general case, namely $p_i = \frac{1}{k}$, for i = 1, ..., k, are the values that implies maximum entropy
- For our toy example, it $p = \frac{1}{2}$, then all the keys have the same probability of $\frac{1}{16}$ and the entropy is

$$16\left(-\frac{1}{16}\log_2\frac{1}{16}\right) = -\log_2\frac{1}{16} = \log_216 = 4$$

- In other words, in case of $p = \frac{1}{2}$ we learn four bits of information and this is supposed to be maximum information
- Indeed, it makes sense: how would it be possible to get more than four bits of information when the key is exactly four bit long?

Entropy

- Any other (i.e., non-uniform) distribution gives lower entropy
- The same happens if the bits are not independent
- For instance, suppose the probability of the next bit being 1 is 1/4;
 then the entropy E associated to the next two bits is:

$$E = -\text{prob}(00) \log \frac{1}{\text{prob}(00)} - \text{prob}(01) \log \frac{1}{\text{prob}(01)} -$$

$$\text{prob}(10) \log \frac{1}{\text{prob}(10)} - \text{prob}(11) \log \frac{1}{\text{prob}(11)}$$

$$= \left(\frac{3}{4}\right)^2 \log \left(\frac{4}{3}\right)^2 + 2\frac{3}{4^2} \log \frac{4^2}{3} + \left(\frac{1}{4}\right)^2 \log 4^2$$

$$\approx 1.62$$

Random Number Generators (RNG)

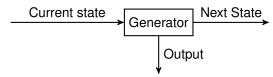
- The ideal source of randomness would then be a Random Bit Generator which always returns 0 or 1 with uniform probability (i.e., independently of past bits)
- In general, in the crypto literature we speak of Random Number Generators, rather than of random bit generators, irrespectively of the actual device being oriented to bits or numbers
- Aside for terminology, to generate truly random bits is all but easy in practice
- For non-crypto applications that embody a probabilistic model of some sort, pseudo-randomness is almost always adequate, and pseudo-randomness is much easier to obtain
- For crypto application pseudo-randomness is almost unavoidably the origin of disasters

The problem in a nutshell

- Having random bits ready at hand (and at will) when they are needed is problematic
- True randomness can be extracted by the physical environment (including quantum mechanical phenomena)
- There might be availability problems (we need physical devices attached to the computer)
- Also, it might not be easy to transform the environmental randomness in a perfectly uniform and independent sequence of bits
- Gathering randomness from the computer itself (e.g., from disk activity, keyboard or mouse inputs) is clearly easier but might not be completely secure
- In all cases, the rate at which the random bits can be produced is not sufficient in many application settings (e.g., think of Amazon web servers...)

A general solution

- Combine true RNGs with good Pseudo-Random Number Generators (PRNGs)
- Essentially, the RNG is used to "seed" a cryptographic PRNG (and later to collect other randomness, if and when available)
- A cryptographic hash function or a cipher is then used to return pseudo-random data (the output) and to update the internal state



Simple PRNG operations

Outline

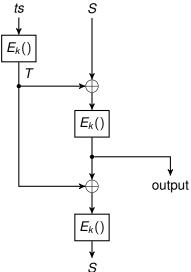
- This "architecture" prevents an attacker from computing the internal state/seed from the output
- This is usually sufficient to prove computational security (rather than unconditional security)
- We shall describe three different (allegedly...) Cryptographically Secure PRNGs (CSPRNGs)
 - 1 The ANSI X9.17 standard, a PRNG which was widely adopted especially in banking applications
 - 2 The Fortuna PRNG, used in Windows® systems
 - The /dev/random and /dev/urandom generators under Linux and other *nix systems

The ANSI X9.17 generator

- Requirements:
 - A secret key k generated at initialization time
 - A Triple-DES symmetric block-cipher
 - A short initial random seed S (64 bits), which is (part of) the internal state
- When pseudorandom bits are required:
 - Compute a timestamp ts of current date and time T at maximum possible resolution (e.g., microseconds)
 - 2 Using the encryption function of 3DES with key k:
 - compute $T = E_k(ts)$
 - compute the output value $O = E_k(T \oplus S)$
 - update the internal state: $S = E_k(T \oplus O)$
 - Return O



ANSI X9.17 block diagram



A Python code for ANSI X9.17

```
from datetime import datetime
from Crypto. Cipher import DES3
from Crypto import Random
from Crypto. Util. strxor import strxor
class ANSIX917:
   def init (self, keylen):
       assert(keylen == 24 \text{ or } keylen == 16)
       key = Random.new().read(keylen)
       self. state = Random.new().read(keylen)
       self.cipher = DES3.new(key, DES3.MODE ECB)
```

A Python code ANSI X9.17 (cont.d)

```
class ANSIX917:
def iter (self):
  return self
def next (self):
  fmt = \text{'}\%y-\text{'}m-\text{'}d, \text{'}H:\text{'}M:\text{'}S.\text{'}f'
  ts = datetime.now().strftime(fmt)
  T = self.cipher.encrypt(ts)
  out = self.cipher.encrypt(strxor(T, self. state)
  self. state = self.cipher.encrypt(strxor(T, out)
  return out
```

In [1]: from ANSIX917 import ANSIX917

In [4]: **next**(PRNG)

Usage

```
In [2]: PRNG = ANSIX917(keylen=16)
In [3]: next(PRNG)
Out[3]: b'\x8c\x98\xbf~\xf3\xae\xabrB\x87\xd8\xe7\x
```

Out[4]: b'10!\xaa:!\x8b\xe2d\xf34\xa3a\x19\xc0*|\x1

Security issues

- At the macroscopic level the goals are obvious: it should be impossible for an attacker to recover both past and future generated bits
- Being able to to protect the past is known as backtracking resistance
- Success in protecting future bits is the property of prediction resistance
- For backtracking resistance the general idea is to use one-way transformations: either a cryptographic hash function, which is one-way, or a cipher, which is not invertible without the key
- For prediction resistance, a good strategy is to regularly update the internal state with fresh random bits, a solution that the ANSI X9.17 PRNG does not implement

Attacks to PRNGs

- We have a direct cryptoanalytic attacks when the attacker is able to distinguish the outputs from truly random bists
- If the attacker, at some time, has access to the internal state of the PRNG (for whatever reason), then s/he will try to exploit this knowledge for as long as possible with state compromise extension attacks
- Finally, cryptoanalytic attacks can take advantage of the attacker being in some control of (possible) PRNG inputs; in this case we speak of input-based attacks
- We now present a simple case study with the hope to shed light upon some of the above points

A known attack to ANSI X9.17 PRNG

- Tecnically, it's a state compromise extension attacks
- However, it also exploits the low entropy in the PRNG inputs (and hence it can also be considered an input-based attack)
- This attack is directed towards discovering the ANSI X9.17 state using the *meet-in-the-middle* attack tecnique
- It works under the assumption that the attacker has somehow learned the key K (due to a flaw in the implementation, a leakage of information, ...)
- The attacker needs also two consecutive outputs from the PRNG

A known attack to ANSI X9.17 PRNG

- The attack works as follows.
- We know that

$$T_i = \mathbf{E}(K, t)$$

 $O_i = \mathbf{E}(K, T_i \oplus S_i)$
 $S_{i+1} = \mathbf{E}(K, T_i \oplus O_i)$

where t is current timestamp while O_i is the i-th output.

• It follows that S_{i+1} can be computed in two different ways, namely:

$$S_{i+1} = \mathbf{E}(K, T_i \oplus O_i) \tag{1}$$

and

$$S_{i+1} = \mathbf{D}(K, O_{i+1}) \oplus T_{i+1}$$
 (2)



A known attack to ANSI X9.17 PRNG

- The point is that we (the alleged attackers...) do not know at which time the outputs and the new states were computed, i.e., which timestamps t were used
- This is, however, a low entropy input to the PRNG
- If we can approximate t, say within a second or few seconds (which is highly reasonable), then we are left with about $\log_2 10^6 \approx 20$ bits of entropy, since we assumed resolution of microseconds
- This is well affordable for any commodity PC so that we can mount the following search:
 - ompute (1) and (2) for different values of t
 - 2 stop when (and if), for some t_1 and t_2 :

$$\mathsf{E}(K,t_1\oplus O_{i+1})=\mathsf{D}(K,O_{i+1})\oplus t_2$$



A Python code for the attack (1)

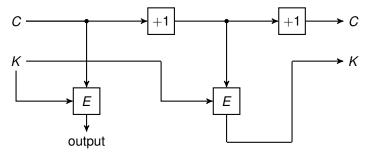
```
def timestamps(T):
    def advance():
       from datetime import timedelta
       nonlocal T
       T += timedelta(microseconds=1)
       return T. strftime (fmt)
    for i in range(1000000): yield advance()
def PSCAttack(ANSIX917gen, key):
   fmt = '\%y - \%m - \%d_{1}\%H: \%M: \%S.\% f'
   cipher = DES3.new(key, DES3.MODE ECB)
   t = datetime.now()
   o1, o2 = next(ANSIX917gen), next(ANSIX917gen)
                       # A list of candidate seeds
   L = []
   ts = timestamps(t) # A generator of timestamps
```

A Python code for the attack (2)

```
from Crypto. Util. strxor import strxor
from bisect import bisect left
for j in range(100000): # Should be sufficient
   t = next(ts)
  T = cipher.encrypt(t)
   s1 = cipher.encrypt(strxor(T,o1))
   L.insert(bisect left(L,s1),s1)
   d = cipher.decrypt(o2)
   s2 = strxor(d,T)
   idx = bisect left(L, s2)
   if idx!=len(L) and L[idx]==s2:
      print("Success'')
  ....break
```

A more sophisticated design: The Fortuna CS-PRNG

- So named after the Roman goddess of chance
- Fortuna's internal state is a pair (K, C), where K is a 256-bit key while C is a 128-bit counter
- The generator works as depicted below:



Output and key update in Fortuna. *E* is the AES (or an AES-like) encryption function.

Fortuna's detailed operations

- Initialization Assign initial values to the counter C and the key K Reseeding Update the internal state (both C and K)
- Block generation Using the AES block cipher, generate a number of 128-bit (16 bytes) blocks sufficient to satisfy the user's request
- Random data generation Generate the number of random data requested by the user
 - Collection Collect real random data from entropy sources, used to reseed the generator

Initialization and reseeding

Initialization is exceedingly simple

```
K = b'0' # i.e. bytes
C = 0
return (K,C) # State of generator
```

- Reseeding mixes the current key with (fresh random) data S
- Fortuna uses double SHA256 hashes (which simply means SHA256 twice)

```
K = SHAd256(K||S) \# // here is string/bytes concat

C = C + 1
```

Block generation

- The input is the number k of required blocks
- State (C, K) is clearly available (e.g., global or instance variables/attributes)
- E is the AES encryption function

```
def GenerateBlocks(k)
  out = b''
  for i in range(k):
    b = E(K,C)
    out = out || b
    C += 1
  return output
```

Block generation is meant as an internal function

Random data generation

- The input is the number *n* of random bytes required
- n must not exceed 2²⁰ (i.e., at most 1MB of data can be generated per-request)
- The reason is that increasing the size would also increase the statistical divergence from true randomness

```
if n>2**20:
    raise some error
B = GenerateBlock(ceil(n/16))[:n]
K = GenerateBlock(2) # Update the key
return B
```

Collection

- This is the most important (and difficult) aspect of a CSPRNG
- Collecting real randomness from the environment requires (if not dedicated hardware, at least) special enablement of various drivers
- From the perspective of the CSPRNG, randomness collection thus means storing and making available the entropy collected from external sources (e.g., the drivers)
- Typically, entropy is gathered from events like mouse movements and keystrokes as well as other various timing sources
- In Fortuna, each source distributes the entropy it collects (evenly) over 32 different pools
- In this way, unless the attacker has full control of all the sources, s/he cannot reconstruct the contents of each pool

Reseeding reviewed

- Reseeding clearly uses data coming from the pools
- Reseeding is done when pool P_0 has enough data
- Reseeding number k uses data from all the pools P_j such that 2^j divides k
- Thus, for instance, pool P_0 is used at each reseeding while P_2 is used at reseedings 4, 8, 12 and so on
- This results in a sort of frequency/amount of entropy tradeoff for the attacker who snatched the internal state
- Suppose s/he does not control source 0: then the system is reseeded too frequently with data s/he does not know
- Suppose now the attacker does not control source 1 (or 2); then the generator is reseeded less frequently with data s/he does not know, but in much larger quantity

PyCrypto implementation of Fortuna

```
from Crypto.Random.Fortuna import FortunaGenerator
from Crypto. Util.py3compat import b
from binascii import b2a hex
fg = FortunaGenerator.AESGenerator()
fg.reseed(b(seedstring))
# Initial key is computed as
# SHAd256(b'\x00\x00...\x00'+b(seedstring))
print(fg.key)
fg.counter.next value() # --> 1
b2a hex(fg.pseudo random data(16))
print(fg.key)
fg.counter.next_value() # --> 4
```

Generating random bytes in Linux

- Under Linux the user may extract high-quality cryptographic random bytes from the special files /dev/random and /dev/urandom
- Both files (actually, a character special files) harvest entropy from the same sources
- The fundamental difference is that requests to one of these (i.e., /dev/random) are blocking, while requests to the other are always immediately honored
- This means that if the generator has not sufficient entropy to satisfy the request, then the process that issued it is put oh hold
- Under some circumstances, this might be preferable

Using /dev/random and /dev/urandom from command shell

 The following command reads 24 bytes of (pseudo)random data from /dev/urandom and prints it to the terminal using base64 encoding (for readability)

```
dd if=/dev/urandom bs=8 count=3 2>/dev/null|base64
(redirection of stderr is required to keep the output clean)
```

 When using /dev/random it may be useful to keep an eye on the available entropy beforehand

```
cat /proc/sys/kernel/random/entropy_avail
```

• ... and then using /dev/random under timeout guard

```
timeout 5s dd if=/dev/random/ bs=....
```

to avoid blocking (while possibly getting less data)



PRNGs in Python

from os import urandom

```
from Crypto.Random.OSRNG import os from binascii import b2a_hex
```

```
urandom is os.urandom # eval to True
r = b2a_hex(urandom(32)) # returns 32 bytes
# of random data
```