Enhanced Exact Algorithms for Discrete Bilevel Linear Problems

Seminar: Selected Topics in Bilevel Optimization

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Problem

$$
\begin{aligned}
\text{(DBLP)} \quad & \min_{x,y} F(x,y) = c_1^T x + c_2^T y \\
& \quad \text{s.t.} \quad Cx + Dy \le e \\
& \quad\quad\quad\quad x \in \mathbb{Z}_+^n \\
& \quad\quad\quad\quad y \in \arg\min_{y} f(y) = d^T y \\
& \quad\quad\quad\quad \text{s.t.} \qquad Ax + By \le b \\
& \quad\quad\quad\quad\quad\quad\quad\quad y \in \mathbb{Z}_+^m
\end{aligned}
$$

Assumptions

- No upper level constraints
- Optimistic approach

Notation

- *Constrained Region*

$$S = \{(x,y) \mid x \in \mathbb{Z}_+^m, y \in \mathbb{Z}_+^m, Cx + Dy \leq e, Ax + By \leq b\}$$

- *Followers Feasible Set*

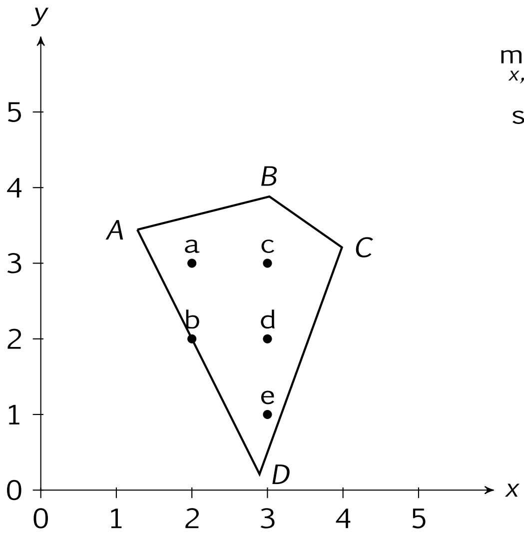
$$\Omega_y(x) = \{y \mid y \in \mathbb{Z}_+^m, By \leq b - Ax\}.$$

- *Reaction Set*

$$R_y(x) = \arg\min_y\{f(y) \text{ s.t. } y \in \Omega_y(x).$$

- *Inducible Region*

$$IR = \{(x,y) \mid x \in \mathbb{Z}_+^n, Cx + Dy \leq e, y \in R_y(x)\}$$

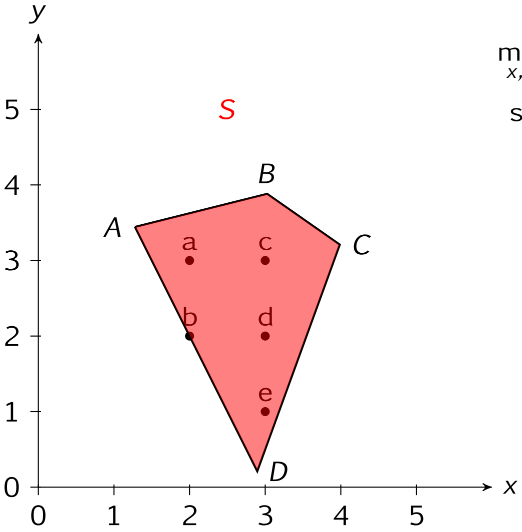


$$\min_{x,y} F(x,y) = 2x + 7y$$

$$\text{s.t.} \quad x \in \mathbb{Z}_+$$

$$y \in \arg\min_{y} f(y) = -y$$

$$\begin{aligned} \text{s.t.} \quad & 2x - 8y \geq -25 \\ & 7x + 10y \leq 60 \\ & 2x + y \geq 6 \\ & 11x - 4y \leq 31 \\ & y \in \mathbb{Z}_+ \end{aligned}$$



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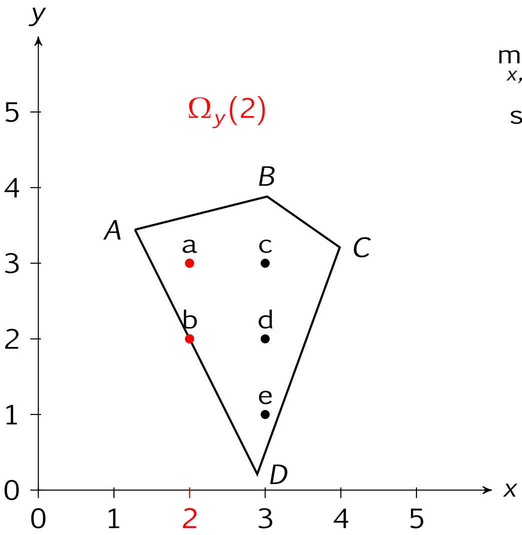

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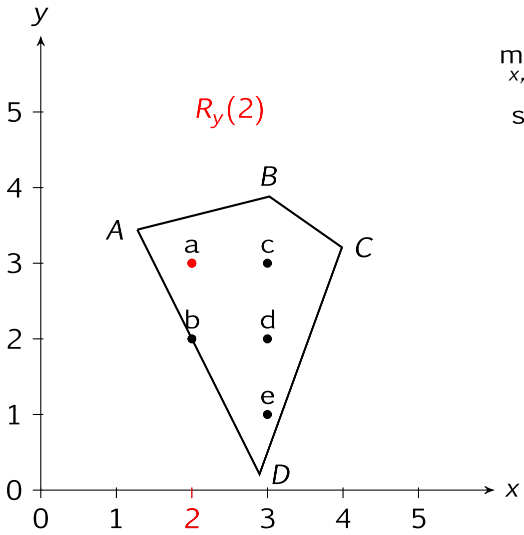

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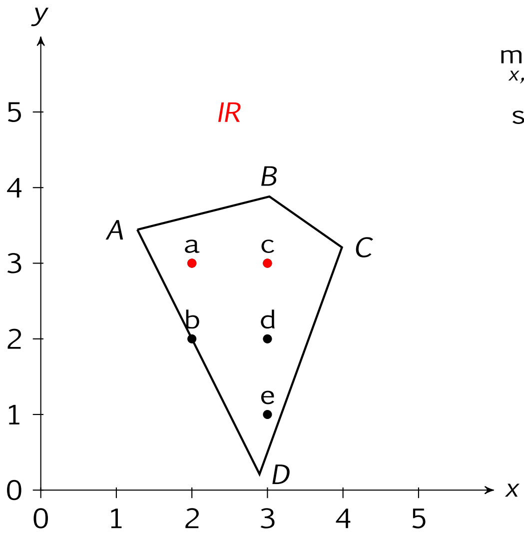

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Single Level Linear Problem

- The Single Level Linear Problem is

$$\text{(SLP)} \quad \min_{x,y} F(x,y) = c_1^T x + c_2^T y$$
$$\text{s.t.} \quad (x,y) \in S.$$

- This is a relaxation of DBLP
- Dropping integrality and maintaining bilevel structure is not a relaxation

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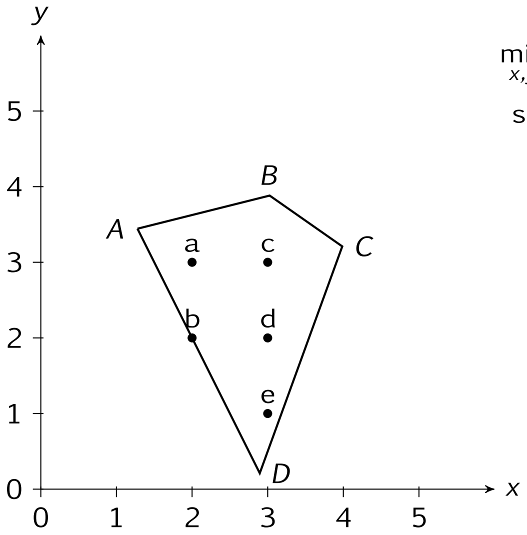
Cutting Plane Method

- Solve (SLP). Get an integer solution $(\bar{x}, \bar{y})$.
- If $(\bar{x}, \bar{y})$ is bilevel infeasible, add valid inequality to eliminate bilevel-infeasible solutions at $\bar{x}$.
- Valid inequality is non linear but can be reformulated as bilevel and linear
- Cut turns (SLP) into bilevel problem with continuous follower variables.
- This can again be transformed into a single-level problem

Example Cutting Plane Method



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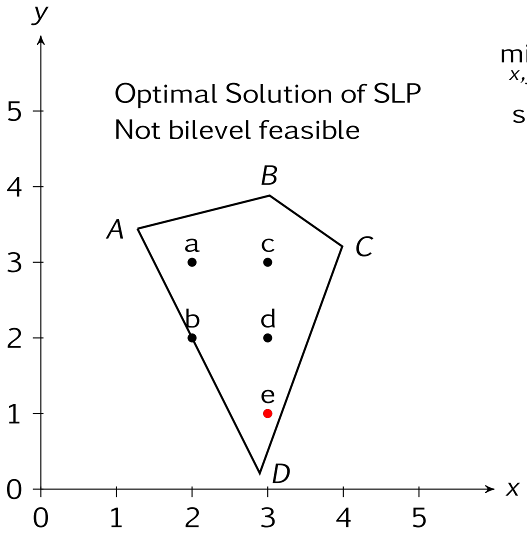

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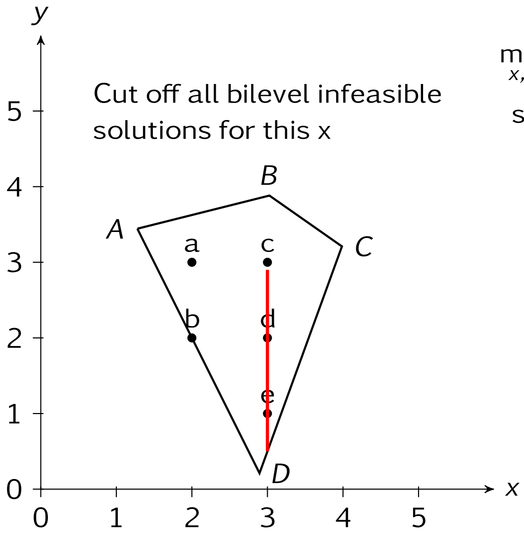

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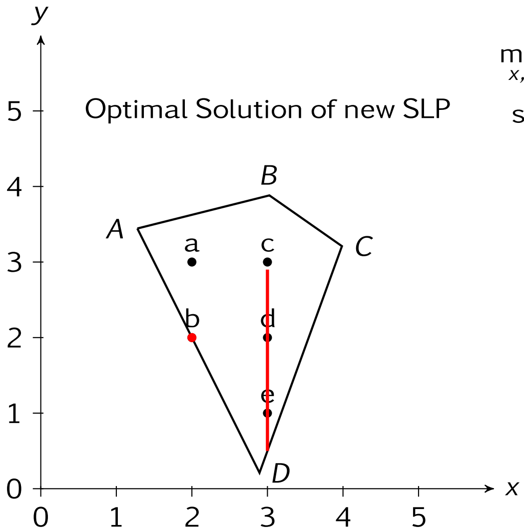

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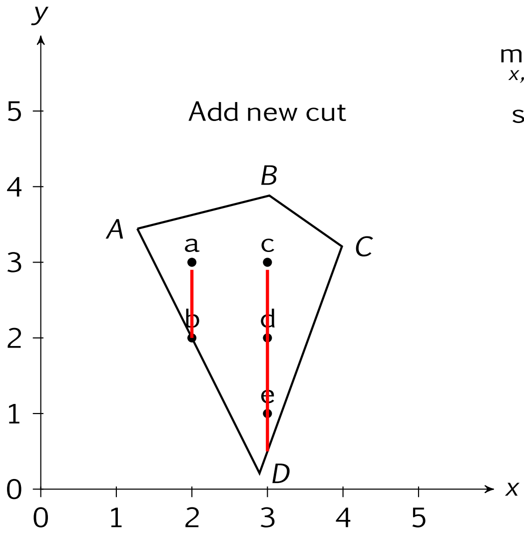

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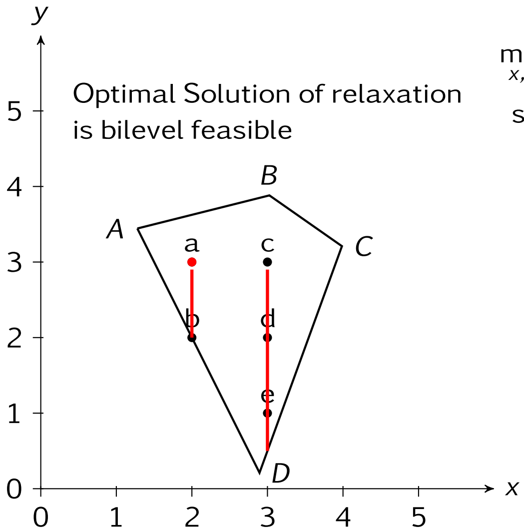

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The valid inequality

- Assume solution of SLP is $(\bar{x}, \bar{y})$ bilevel infeasible.
- There exists $(\bar{x}, \hat{y})$ (bilevel feasible) with $f(\hat{y}) < f(\bar{y})$.
- Valid inequality is

$$f(y) \leq f(\hat{y}) + L\|x - \bar{x}\|_\infty. \tag{1}$$

Reformulation I

The inequality ??? can be reformulated as

$$f(y) \leq f(\hat{y}) + L\hat{z}$$

where $\hat{z}$ is the optimal solution of

$$
\begin{aligned}
(P_{\hat{z}}) \quad & \min_{z} z \\
\text{s.t.} \quad & z \geq x_i - \bar{x}_i \quad i = 1, \ldots, n \\
& z \geq \bar{x}_i - x_i \quad i = 1, \ldots, n
\end{aligned}
$$

Reformulation II

Adding this to SLP results in a bilevel continuous Problem

$$
\begin{aligned}
\text{(SLPC)} \quad & \min_{x,y} F(x,y) = c_1^T x + c_2^T y \\
& \text{s.t.} \quad (x,y) \in S. \\
& f(y) \leq f(\hat{y}) + L\hat{z} \\
& \bar{z} \in \arg\min_{z} z \\
& \text{s.t.} \qquad z \geq x_i - \bar{x}_i \quad i = 1,\dots,n \\
& \qquad\qquad z \geq \bar{x}_i - x_i \quad i = 1,\dots,n \\
& \qquad\qquad z \in \mathbb{R}
\end{aligned}
$$

This can again be reformulated as a single level problem.

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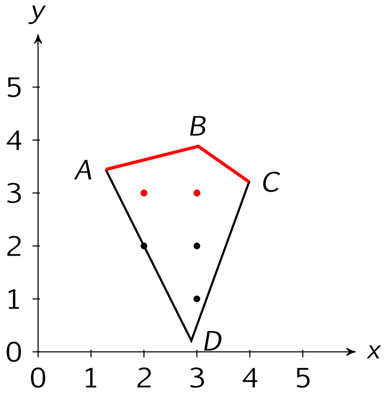
Branch and Cut Method

- Divide S into smaller constrained regions S', S''.
- Solve both subproblems using Branch and Cut.
- Optimal solution is likely to be in S'.
- S'' might be discarded based on relaxation and upper bound of S'.

Motivation

Proposition

The inducible region of a bilevel linear problem can be written equivalently as a piecewise linear inequality constraint comprised of supporting hyperplanes of S.



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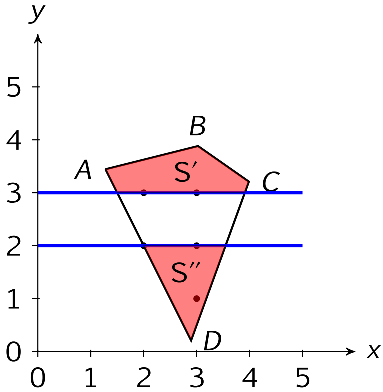

Finding S' and S''

- We solve the max-min problem

$$
\begin{aligned}
(BLP^{max}_{min}) \quad & \max_{x,y} f(y) = d^T y \\
& \text{s.t.} \quad x \in \mathbb{R}^n_+ \\
& \qquad y \in \arg\min_{y} f(y) = d^T y \\
& \qquad \text{s.t.} \quad Ax + By \leq b \\
& \qquad\quad y \in \mathbb{R}^m_+
\end{aligned}
$$

- Let $(\hat{x}, \hat{y})$ be optimal solution of (BLP^{max}_{min})
- Split S into S', S'' by adding $f(y) \leq \lceil f(\hat{y}) \rceil$ and $f(y) \geq \lceil f(\hat{y}) \rceil + 1$ respectively

Branch and Cut for S' and S''

We can solve the two subproblems using any branch and cut or cutting plane method method, i.e.:

- **Step 0** Determine S' and S''.
- **Step 1** Solve the subproblem $(DBLP')$ induced by S'. Use continuous single-level relaxation.
- **Step 2** Branch if solution is not integral, add cut if solution is integral but not bilevel feasible
- **Step 3** Compute LB'' for problem $(DBLP'')$ induced by S''. If this is worse than the UB found before, stop. Otherwise solve $(DBLP'')$.

Proposed Variant

- Use branch and cut to solve $(DBLP')$
  - Use cuts that are computationally less expensive
  - Intuition that S' contains few bilevel-infeasible integral points
- Use previously introduced cutting plane method for $(DBLP'')$
  - Yields strong bounds quickly
  - Possibly prune whole subproblem

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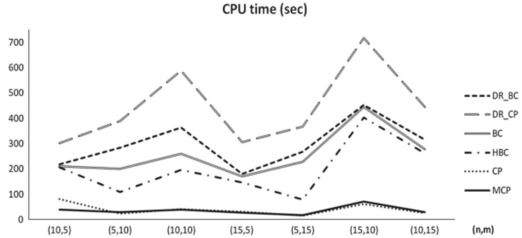
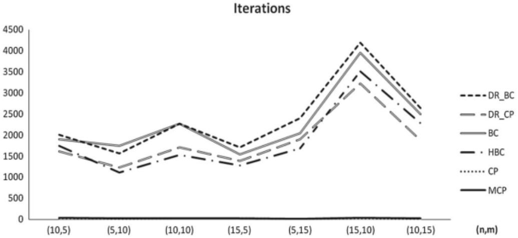
Setup

- Compare with algorithm from DeNegre and Ralphs
- Testsets: Random instances and modified MIPLIB 2010 instances
- All algorithms implemented in C
- PC Pentium Core 2 Duo with a 2 GHz processor and 1 GB RAM
- Solver used is CPLEX 12.3

Random Set

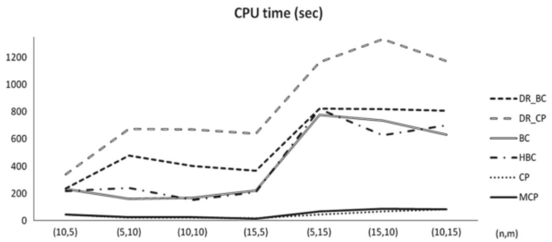


(a) Computational time



(b) Number of iterations

Miplib set



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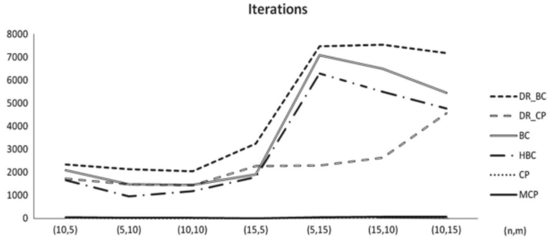

(b) Number of iterations