Self-excited oscillations in a parabolic system with nonlinear external feedback

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Temperature is one of the most difficult thermodynamic parameters to control and at the same time is one of the most important. Patent difficulties of a general sort are encountered in regulating the temperature of a particular object, whether an industrial machine or an integrated microcircuit. One problem is that when the feedback ratio is increased, as is necessary in order to improve the precision of active temperature control, the temperature stabilization system can become unstable. We are therefore faced with the problem of determining the limit of stability of the temperature stabilizing and regulating system. In this article we investigate Hopf bifurcation in the class of proportional temperature control systems.

1. Regardless of the structure or function of such systems, they can always be conditionally partitioned into a nonlinear external feedback circuit with lumped parameters and a distributed controlled object (Fig. 1). A mathematical model of such a system can be constructed in the form of a thermal conduction boundary-value problem specified in dimensionless variables with a nonlocal boundary condition:[1]

$$\dot{T}(x,t)=T''(x,t), \quad T(0,t)=0, \tag{1}$$

$$T'(x,t)|_{x=1}=f[1-T(x_0,t)]\sigma[1-T(x_0,t)], \tag{2}$$

where $x_0\in[0,1]$, f is a certain smooth function of the temperature, and σ is the Heaviside unit step function, which cuts off the positive feedback branch from f.

The special case of a quadratic nonlinearity of the form $f=[1-T(0,5;t)]^2$ has already been investigated.[1] Since f can contain cubic terms, which influence the Hopf bifurcation regime (bifurcation of limit cycles), it is necessary to investigate the more general case (2) of nonlinear feedback.

2. We consider steady-state, periodic solutions of the boundary-value problem (1), (2). Its steady-state solution has the form $\overline{T}(x)=Cx$, where C is the root of the nonlinear equation

$$C=f(1-Cx_0). \tag{3}$$

The function f is positive concave over the entire domain of definition for nearly all proportional temperature control systems:

$$\left.\begin{aligned} f[1-T(x_0,t)]&>0\\ \partial f/\partial C\equiv A&<0\\ \partial^2 f/\partial T^2&>0\end{aligned}\right\}\ \forall T\in[0,1],$$

such that $f'(0)=0$. By virtue of these conditions Eq. (3) has a unique root, for which $A<0$ and the corresponding steady-state solution is stable.

3. Let us assume that the steady-state solution $\overline{T}=Cx$ acquires self-excited oscillations $T(x,t)=\overline{T}(x)+u(x,t)$ such that $|u(x_0,t)|\ll|\overline{T}(x_0)|$. The function f can then be expanded in a Taylor series in powers of the unsteady temperature $u(x_0,t)$:

$$f[1-\overline{T}(x_0)-u(x_0,t)] = C+\frac{A}{x_0}u(x_0,t)+f''\frac{u^2(x_0,t)}{2!}+f'''\frac{u^3(x_0,t)}{3!}\ldots+, \tag{4}$$

where $f^{(n)}$ is the nth derivative at $T=\overline{T}(x_0)$.

4. Next we consider the boundary-volume problem linearized in the steady state

$$\dot{u}(x,t)=u''(x,t),$$

$$u(0,t)=0,$$

$$u'(x,t)|_{x=1}=\frac{A}{x_0}u(x_0,t). \tag{5}$$

Its periodic solution is the function

$$u(x,t)=\xi\,\sinh(\sqrt{i\omega x})\exp(i\omega t) \tag{6}$$

and its complex conjugate. The condition imposed by feedback on the complex amplitude of the temperature oscillations at the points x_0 can be used to determine the critical value $A_c=A(\omega_c)$:

$$A_c=\frac{\mathrm{Re}\sqrt{i\omega_c}x_0\,\cosh\,\sqrt{i\omega_c}}{\sinh\,\sqrt{i\omega_c}x_0}, \tag{7}$$

where ω_c is that root of the equation

$$\frac{\cosh\sqrt{\omega/2}\,\cos\,\sqrt{\omega/2}-\sinh\,\sqrt{\omega/2}\,\sin\,\sqrt{\omega/2}}{\sinh(x_0\sqrt{\omega/2})\cos(x_0\sqrt{\omega/2})} = \frac{\cosh\sqrt{\omega/2}\,\cos\,\sqrt{\omega/2}+\sinh\,\sqrt{\omega/2}\,\sin\,\sqrt{\omega/2}}{\cosh(x_0\sqrt{\omega/2})\sin(x_0\sqrt{\omega/2})}, \tag{8}$$

for which $A(\omega_c)>A(\omega_n)$ for any n.

5. For $A>A_c$ the null equilibrium state of the boundary-value problems (1), (2) is locally exponentially stable, and for $A<A_c$ it loses stability by oscillating. Let

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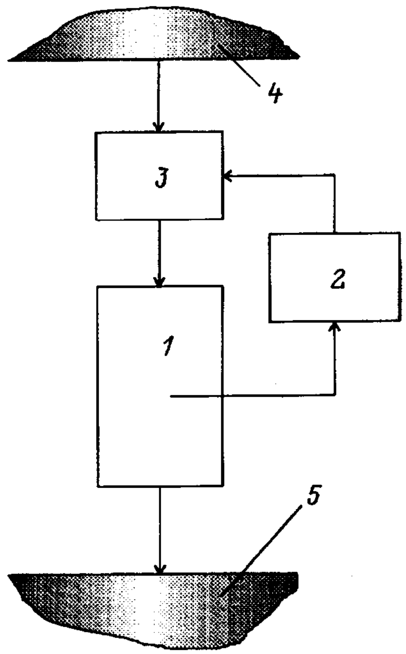

FIG. 1. Diagram of a temperature stabilization system. *1* — Distributed controlled object; *2* — feedback loop; *3* — controller; *4* — energy source; *5* — thermostat.

$$A = A_c(1+\varepsilon), \quad 0 < \varepsilon \ll 1, \tag{9}$$

so that under the constraints (9) the Andronov–Hopf bifurcation theorem[2] is valid for proving the existence and stability of self-excited oscillations for the nonlinear boundary-value problem (1), (2).

We introduce the notation

$$\Theta(x,t) = \frac{f''}{2} u(x,t); \quad \gamma = \frac{f'''}{(f'')^2} \tag{10}$$

and transform the boundary-value problem as follows:

$$\dot{\Theta}(x,t) = \Theta''(x,t), \quad \Theta(0,t) = 0,$$

$$\Theta'(x,t)|_{x=1} = \frac{A_c(1+\varepsilon)}{x_0}\Theta(x_0,t) + \Theta^2(x_0,t) + \frac{4}{3}\gamma\Theta^3(x_0,t). \tag{11}$$

Setting

$$t = (1+c)\tau; \quad c = c_2\xi^2 + c_4\xi^4 + \ldots;$$

$$\varepsilon = b_2\xi^2 + b_4\xi^4 + \ldots,$$

$$\Theta(x,\tau) = \xi\Theta_1(x,\tau) + \xi^2\Theta_2(x,\tau) + \ldots, \tag{12}$$

we obtain a recursive sequence of nonlinear, inhomogeneous, boundary-value problems for the determination of $\Theta_1, \Theta_2, \ldots$:

$$\dot{\Theta}_1(x,\tau) = \Theta_1''(x,\tau), \quad \Theta_1(0,\tau) = 0, \tag{13}$$

$$\Theta_1'(x,\tau)|_{x=1} = \frac{A_c}{x_0}\Theta_1(x_0,\tau),$$

$$\dot{\Theta}_2(x,\tau) = \Theta_2''(x,\tau), \quad \Theta_2(0,\tau) = 0,$$

$$\Theta_2'(x,\tau)|_{x=1} = \frac{A_c}{x_0}\Theta_2(x_0,\tau) + \Theta_1^2(x_0,\tau); \tag{14}$$

$$\dot{\Theta}_3(x,\tau) = \Theta_3''(x,\tau) + c_2\Theta_1''(x,\tau), \quad \Theta_3(0,\tau) = 0,$$

$$\Theta_3'(x,\tau)|_{x=1} = \frac{A_c}{x_0}[\Theta_3(x_0,\tau) + b_2\Theta_1(x_0,\tau)] + 2\Theta_1(x_0,\tau)\Theta_2(x_0,\tau) + \frac{4}{3}\gamma\Theta_1^3; \ \ldots, \tag{15}$$

the first of which coincides with problem (5). The solution of problem (14) has the form

$$\Theta_2 = 2\frac{|V_1(x_0)|^2}{1-A_c}x + V_2(x)e^{i2\omega\tau} + V_2^*(x)e^{-i2\omega\tau}, \tag{16}$$

where

$$V_2(x) = \frac{V_1^2(x_0)}{\sqrt{i2\omega}\cosh\sqrt{i2\omega} - (A_c/x_0)\sinh\sqrt{i2\omega}x_0} \times \sinh\sqrt{i2\omega}x, \tag{17}$$

and $V_1(x)$ is the solution of the Sturm-Liouville problem for the boundary-value problem (5).

From the solvability condition

$$\frac{c_2}{2}\left\{ i\omega\frac{V_1(1)}{V_1(x_0)} - \frac{A_c}{x_0}\left[x_0\frac{V_1'(1)}{V_1(x_0)} - 1\right]\right\} + \frac{A_c}{x_0}b_2 + |V_1(x_0)|^2\left[\frac{2}{z} + 4\left(\gamma + \frac{x_0}{1-A_c}\right)\right] = 0 \tag{18}$$

for the third problem in the sequence we find the Lyapunov coefficients

$$c_2 = 4\frac{x_0}{A_c}|V_1(x_0)|^2\frac{z_2}{p_2|z|^2},$$

$$b_2 = -\frac{x_0}{A_c}|V_1(x_0)|^2\left[\frac{2}{|z|^2}\left(z_1 + \frac{p_1}{p_2}z_2\right) + 4\frac{x_0}{1-A_c}\right] - 4\frac{x_0}{A_c}|V_1(x_0)|^2\gamma, \tag{19}$$

where

$$z = z_1 + iz_2 = \frac{\sqrt{i2\omega}\cosh\sqrt{i2\omega}}{\sinh\sqrt{i2\omega}x_0} - \frac{A_c}{x_0},$$

$$p_1 + ip_2 = 1 + \frac{x_0}{|V_1(x_0)|^2}\left[\frac{i\omega}{A_c}V_1(1)V_1^*(x_0) - V_1'(1)V_1^*(x_0)\right]. \tag{20}$$

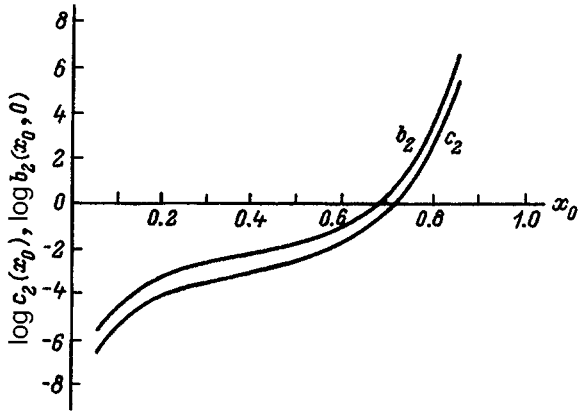

FIG. 2. Lyapunov coefficients $c_2(x_0)$ and $b_2(x_0,0)$ versus x_0 for a system with quadratic nonlinearity ($\gamma=0$).

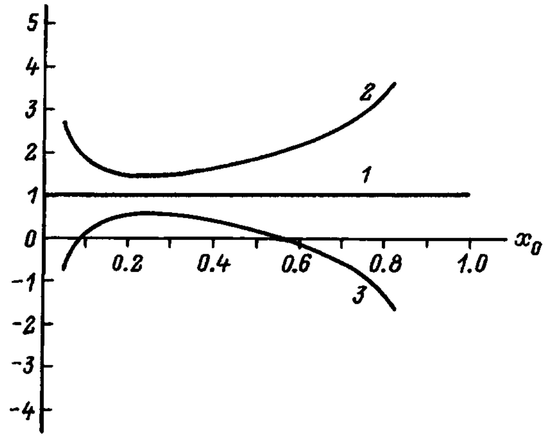

FIG. 3. Lyapunov coefficient $b_2(x_0,\gamma)$ normalized to $b_2(x_0,0)$ versus x_0. *1* — $\gamma=0$; *2* — $\gamma=0.01$; *3* — $\gamma=-0.01$.

Substituting Θ_1, Θ_2, and $\xi=\sqrt{\varepsilon/b_2}$ into $u(x,t) = (2/f'')[\xi\Theta_1(x,\tau)+\xi^2\Theta_2(x,\tau)]$, we obtain the periodic solution of problem (1), (2) in the final form

$$u(x,t)=\frac{2}{f''}\Bigg\{\sqrt{\frac{\varepsilon}{b_2}}[V_1(x)e^{i\omega(\varepsilon)t}+V_1^*(x)e^{-i\omega(\varepsilon)t}] + \frac{\varepsilon}{b_2}\frac{|V_1|^2}{1-A_c}x+\frac{\varepsilon}{b_2}[V_2(x)e^{i2\omega(\varepsilon)t} + V_2^*(x)e^{-i2\omega(\varepsilon)t}]\Bigg\},$$

$$\omega(\varepsilon)=\frac{\omega_c}{1+(c_2/b_2)\varepsilon}.^{1)} \qquad (21)$$

The latter relation is valid only when the coefficient b_2 is positive and soft Hopf bifurcation takes place, i.e., when the oscillation amplitude tends to zero in the limit $\varepsilon\to 0$. Graphs of the functions $c_2(x_0)$ and $b_2(x_0,0)$ are shown in Fig. 2. It is evident from Fig. 2 and the second relation (19) that the parameter b_2 is always positive for $f'''>0$, whereas for $f'''<0$ the sign of b_2 can be either positive or negative, depending on γ and x_0 (Fig. 3). This means that both soft and hard Hopf bifurcations are possible. Finally, for $f''=0$ the system is linear, and the oscillation amplitude in the supercritical region is infinite.

The results of our nonlinear analysis are in good agreement with general notions concerning nonlinearity as a factor limiting the amplitude of self-excited oscillations due to the redistribution of energy between the first and higher harmonics. From this point of view, the role of quadratic nonlinearity is purely ''dissipative,'' so that without cubic terms in the expansion (4) self-excited oscillations in a parabolic system always occur in the soft regime (Fig. 2). Cubic nonlinearity is more ''conservative'' since part of the energy returns in the form of oscillations in the fundamental mode. If the coefficient of the cubic nonlinearity is positive, i.e., if A_c and f''' have opposite signs, this additional energy is out-of-phase with the fundamental mode, diminishing the oscillation amplitude. Self-excited oscillations occur in the soft regime in this case as well [Fig. 3, curve *2*: $b_2(x_0,\gamma)>b_2(x_0,0)$ and $\xi(x_0,\gamma)<\xi(x_0,0)$]. But if $f'''<0$, the energy arrives in phase with the first harmonic, and its amplitude increases, $\xi(x_0,\gamma)>\xi(x_0,0)$ (Fig. 3, curve *3*). For a certain relation between the derivatives and the value of x_0 the recovery of energy by cubic nonlinearity exceeds its dissipation within one period as long as the oscillation amplitude remains small. Energy balance sets in when the amplitude of the self-excited oscillations attains a certain (not small) level. This self-excitation regime, which is said to be hard, corresponds to the range of negative values of the parameter b_2 on curve *3* in Fig. 3.

By far the majority of temperature control systems utilize Joule heat sources, for which *f* must be a quadratic function of the temperature. In reality, however, *f* contains several temperature-independent parameters, which introduce cubic terms in (4). For this reason, the behavior of systems with an identical temperature control scheme in the supercritical region can differ significantly. The results of the present study provide a straightforward explanation of this phenomenon.

1)The quantities $\omega(\varepsilon)$ and ω_c are mistakenly switched in Ref. 1.

[1] A. S. Rudy, Int. J. Thermophys. **14**, 159 (1993).
[2] Yu. S. Kolesov and D. I. Shvitra, *Self-Excited Oscillations in Systems with Delay* [in Russian], Mokslas, Vilnius (1979).

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