Dynamics of diffused connected systems of differential equations with internal connection

Leonid Ivanovsky

P.G. Demidov Yaroslavl State University

Diffused connected system of differential equations

$$\dot{u}_j = N^2(u_{j+1} - 2u_j + u_{j-1}) + \gamma u_j, \qquad j = \overline{1, N}, \tag{1}$$

$$u_0 = u_1, \quad u_{N+1} = u_N + \frac{\alpha}{N} u_k + \frac{\beta}{N} u_k^3, \qquad 1 \leq k < N, \tag{2}$$

$$u_j = u_j(t), \quad t \geqslant 0, \quad \alpha, \beta, \gamma \in \mathbb{R}.$$

Linearized system of differential equations

$$\dot{u}_j = N^2(u_{j+1} - 2u_j + u_{j-1}) + \gamma u_j, \qquad j = \overline{1, N}, \tag{3}$$

$$u_0 = u_1, \quad u_{N+1} = u_N + \frac{\alpha}{N} u_k, \qquad 1 \leqslant k < N, \tag{4}$$

Eigenvalue problem

$$u_j = e^{\lambda t}\,\mathrm{ch}\,\delta x_j,$$

$$x_j = -\frac{1}{2N} + \frac{j}{N}.$$

Eigenvalue problem

$$u_j = e^{\lambda t} \operatorname{ch} \delta x_j,$$

$$x_j = -\frac{1}{2N} + \frac{j}{N}.$$

- $j \leqslant N - 1:$

$$\delta = 2N \operatorname{arsh} \frac{\sqrt{-\gamma + \lambda}}{2N}. \tag{5}$$

Eigenvalue problem

$$u_j = e^{\lambda t} \operatorname{ch} \delta x_j,$$

$$x_j = -\frac{1}{2N} + \frac{j}{N}.$$

- $j \leqslant N - 1$:

$$\delta = 2N \operatorname{arsh} \frac{\sqrt{-\gamma + \lambda}}{2N}. \tag{5}$$

- $j = N$:

$$\alpha = \frac{\sqrt{-\gamma + \lambda} \operatorname{sh} \delta}{\operatorname{ch} \delta x_k}. \tag{6}$$

Stability loss of zero solution

- $\lambda = 0:$

$$\alpha_u = \frac{\sqrt{-\gamma}\ \operatorname{sh}\delta_u}{\operatorname{ch}\delta_u x_k}, \qquad \delta_u = 2N \operatorname{arsh}\frac{\sqrt{-\gamma}}{2N}.$$

Stability loss of zero solution

- $\lambda = 0:$

$$\alpha_u = \frac{\sqrt{-\gamma}\,\operatorname{sh}\delta_u}{\operatorname{ch}\delta_u x_k}, \qquad \delta_u = 2N \operatorname{arsh}\frac{\sqrt{-\gamma}}{2N}.$$

- $\lambda = \pm i\omega:$

$$\alpha_c = \frac{\sqrt{-\gamma + i\omega}\,\operatorname{sh}\delta_c}{\operatorname{ch}\delta_c x_k}, \qquad \delta_c = 2N \operatorname{arsh}\frac{\sqrt{-\gamma + i\omega}}{2N}.$$

Stability loss of zero solution

- $\lambda = 0:$

$$\alpha_u = \frac{\sqrt{-\gamma}\ \operatorname{sh}\delta_u}{\operatorname{ch}\delta_u x_k}, \qquad \delta_u = 2N \operatorname{arsh}\frac{\sqrt{-\gamma}}{2N}.$$

- $\lambda = \pm i\omega:$

$$\alpha_c = \frac{\sqrt{-\gamma + i\omega}\ \operatorname{sh}\delta_c}{\operatorname{ch}\delta_c x_k}, \qquad \delta_c = 2N \operatorname{arsh}\frac{\sqrt{-\gamma + i\omega}}{2N}.$$

$$N = 50.$$

Extreme case

$$N \to \infty : \qquad \delta \to \sqrt{-\gamma + \lambda}.$$

Extreme case

$$N \to \infty : \qquad \delta \to \sqrt{-\gamma + \lambda}.$$

$$\dot{u} = u'' + \gamma u, \tag{7}$$

$$u'(0,t) = 0, \qquad u'(1,t) = \alpha\, u(x_0,t) + \beta u^3(x_0,t), \tag{8}$$

$$x \in [0,1], \quad x_0 \in [0,1).$$

Extreme case

$$N \to \infty : \qquad \delta \to \sqrt{-\gamma + \lambda}.$$

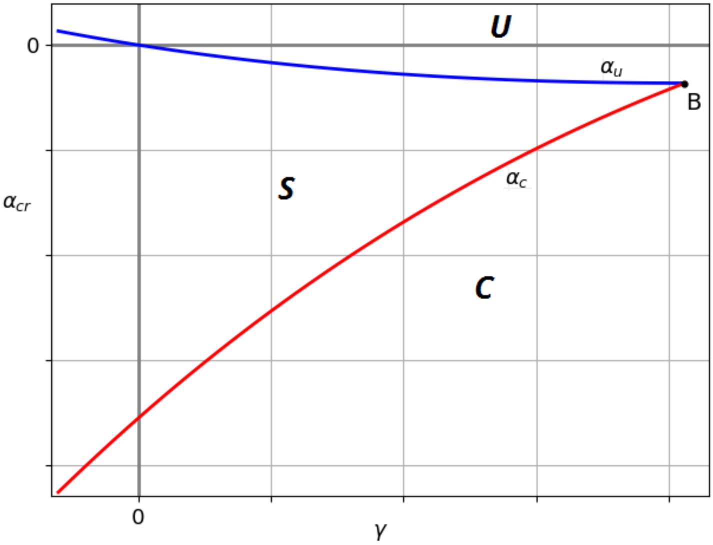
$$\dot{u} = u'' + \gamma u, \tag{7}$$

$$u'(0,t) = 0, \qquad u'(1,t) = \alpha\, u(x_0,t) + \beta u^3(x_0,t), \tag{8}$$

$$x \in [0,1], \quad x_0 \in [0,1).$$

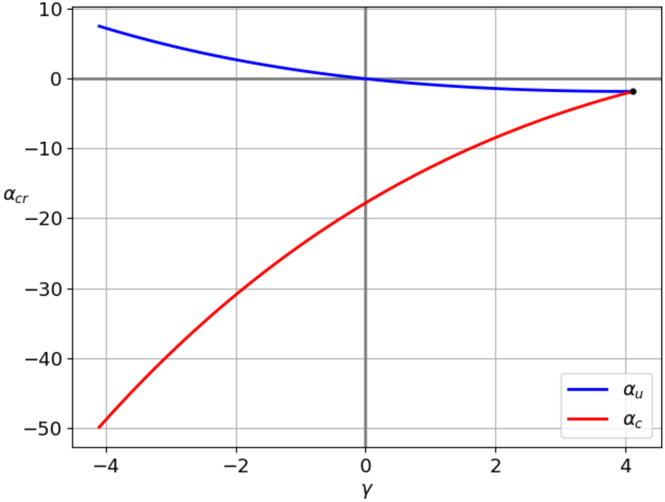
$$\alpha = \frac{\sqrt{-\gamma+\lambda}\,\mathrm{sh}\,\sqrt{-\gamma+\lambda}}{\mathrm{ch}\,\sqrt{-\gamma+\lambda}\,x_0}. \tag{9}$$

Visualization of critical dependencies $\alpha_{cr}(\gamma)$, $1 \leqslant k \leqslant 17$



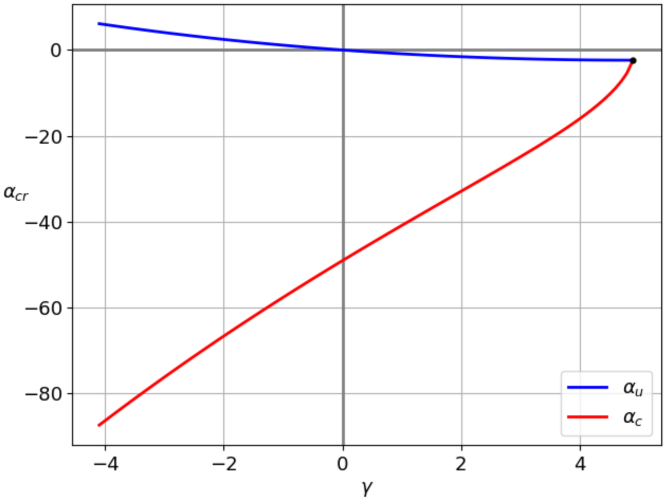
$$B = (\gamma_*, \alpha_*)$$

Plot $\alpha_{cr}(\gamma),\ k = 1$



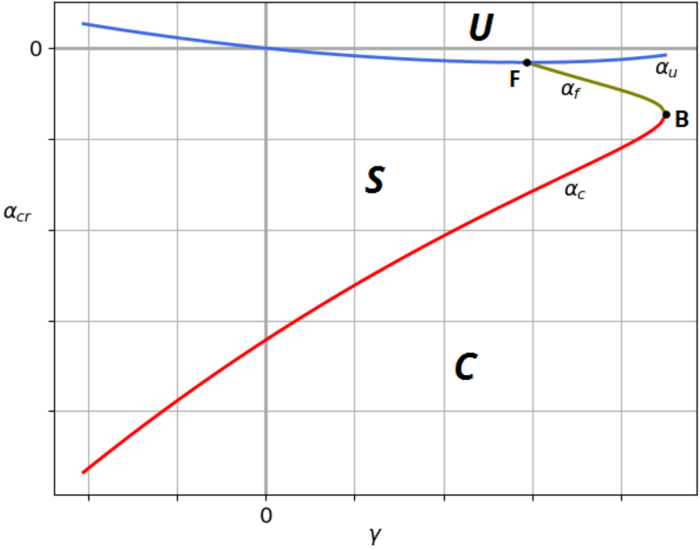
$$\gamma_* \approx 4.116$$

Plot $\alpha_{cr}(\gamma),\ k = 17$



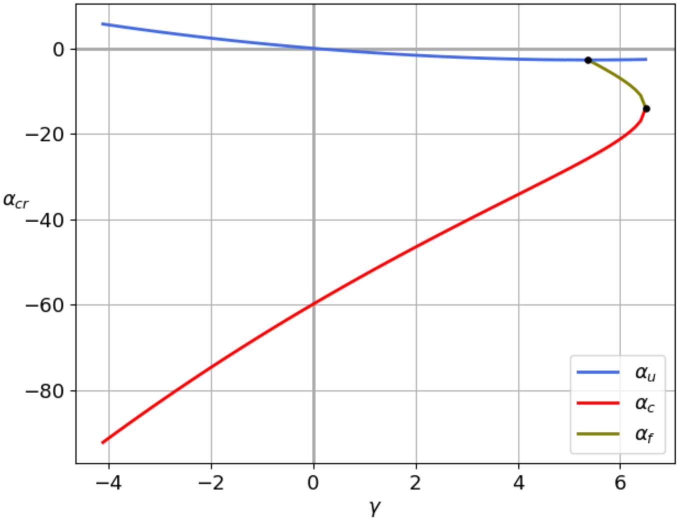
$$\gamma_* \approx 4.896$$

Visualization of critical dependencies $\alpha_{cr}(\gamma)$, $18 \leqslant k \leqslant 23$



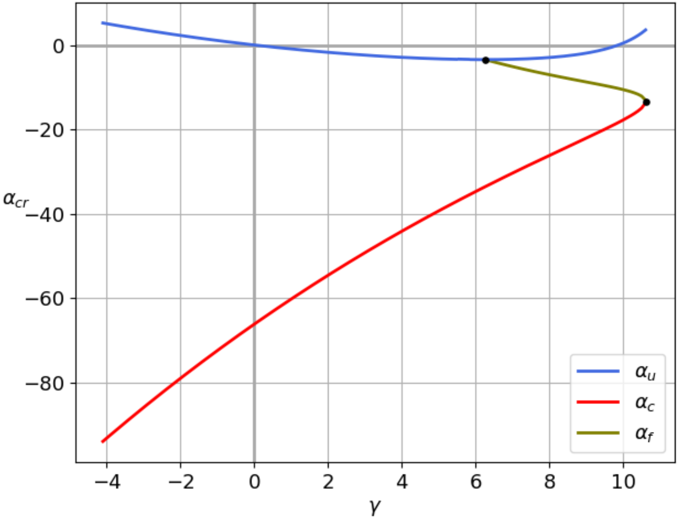
$$F = (\overline{\gamma}, \overline{\alpha})$$

Plot $\alpha_{cr}(\gamma),\ k = 20$



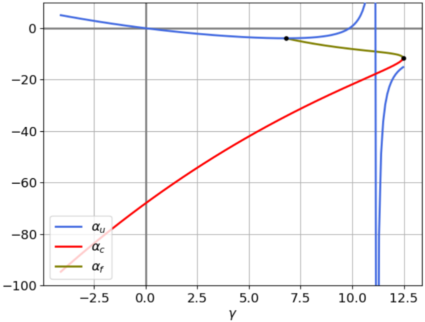
$$\overline{\gamma} \approx 5.375,\ \gamma_* \approx 6.497$$

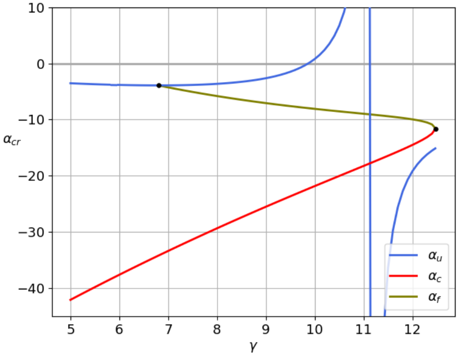
Plot $\alpha_{cr}(\gamma),\ k = 23$



$$\overline{\gamma} \approx 6.258,\ \gamma_* \approx 10.608$$

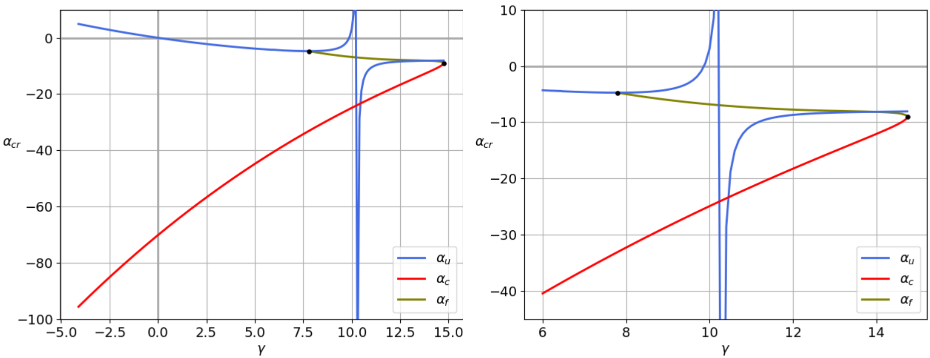
Plot $\alpha_{cr}(\gamma),\ k = 24$





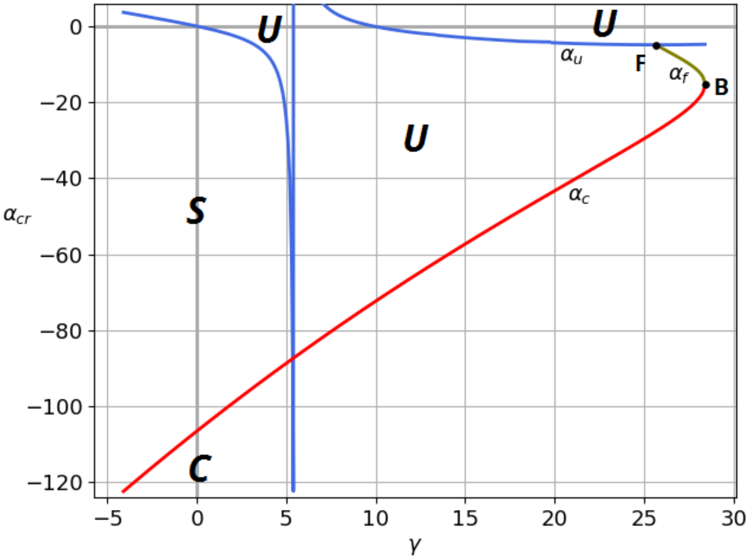
$$\overline{\gamma} \approx 6.798,\ \gamma_* \approx 12.467$$

Plot $\alpha_{cr}(\gamma),\ k = 25$

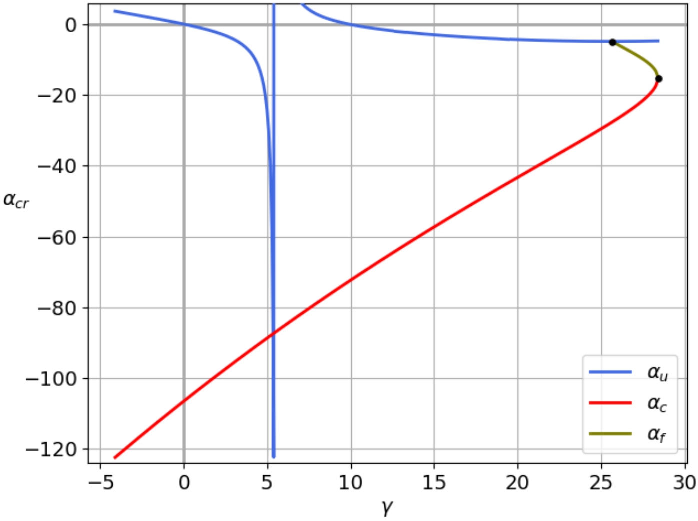


$$\overline{\gamma} \approx 7.794,\ \gamma_* \approx 14.738$$

Visualization of critical dependencies $\alpha_{cr}(\gamma)$, $26 \leqslant k \leqslant 50$



Plot $\alpha_{cr}(\gamma),\ k = 33$



$$\overline{\gamma} \approx 25.682,\ \gamma_* \approx 28.407$$

Local analysis

$$u_j = \sqrt{\varepsilon} u_{j,0} + \varepsilon u_{j,1} + \varepsilon^{\frac{3}{2}} u_{j,2} + O(\varepsilon^2), \qquad j = \overline{1, N} \tag{10}$$

$$u_j = u_j(s), \quad s = \varepsilon t,$$

$$\varepsilon = |\alpha - \alpha_{cr}|, \quad \varepsilon \ll 1.$$

Divergent case of stability loss

- $\lambda = 0: \quad \varepsilon = \alpha - \alpha_u,$

$$\dot{u}_{j,0} = N^2(u_{j+1,0} - 2u_{j,0} + u_{j-1,0}) + \gamma u_{j,0}, \tag{11}$$

$$u_{0,0} = u_{1,0}, \quad u_{N+1,0} = u_{N,0} + \frac{\alpha}{N} u_{k,0}, \tag{12}$$

$$u_{j,0} = \rho(s)\,\mathrm{ch}\,\delta_u x_j,$$

$$\delta_u = 2N\,\mathrm{arsh}\,\frac{\sqrt{-\gamma}}{2N}, \quad x_j = -\frac{1}{2N} + \frac{j}{N}.$$

Divergent case of stability loss

$$\dot{u}_{j,2} + \frac{\partial u_{j,0}}{\partial s} = N^2(u_{j+1,2} - 2u_{j,2} + u_{j-1,2}) + \gamma u_{j,2}, \tag{13}$$

$$u_{0,2} = u_{1,2}, \quad u_{N+1,2} = u_{N,2} + \frac{\alpha}{N} u_{k,0} + \frac{\beta}{N} u_{k,0}, \qquad 1 \leq k < N, \tag{14}$$

$$u_{j,2} = \operatorname{ch} \delta_u x_j.$$

Divergent case of stability loss

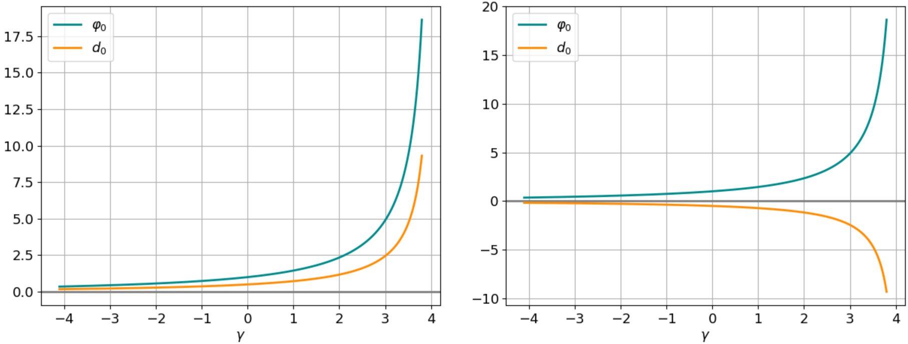
$$\rho' = \phi_0 \rho + d_0 \rho^3, \tag{15}$$

$$\phi_0 = Q \operatorname{ch} \delta_u x_k, \qquad d_0 = \beta Q \operatorname{ch}^3 \delta_u x_k,$$

$$Q = \frac{2\delta_u}{\delta_u \operatorname{ch} \delta_u + \operatorname{sh} \delta_u - \alpha_u x_k \operatorname{sh} \delta_u x_k},$$

$$\alpha_u = \frac{\sqrt{-\gamma}\ \operatorname{sh} \delta_u}{\operatorname{ch} \delta_u x_k}.$$

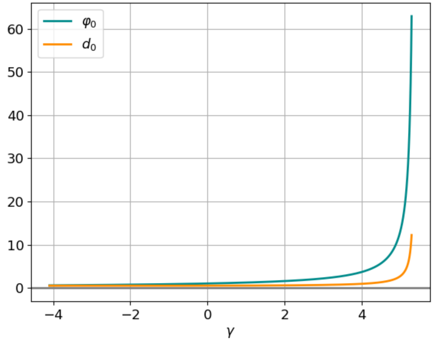
Plot $\phi_0(\gamma)$ and $d_0(\gamma)$, $k = 1$



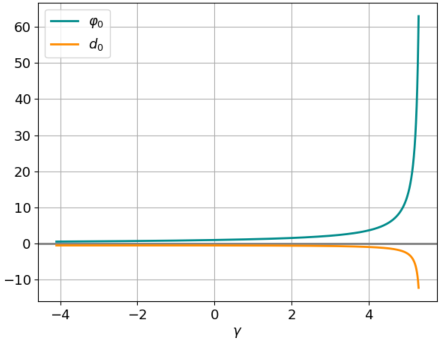
a) $\beta = 0.5$ b) $\beta = -0.5$

$$\gamma_* \approx 4.116$$

Plot $\phi_0(\gamma)$ and $d_0(\gamma)$, $k = 20$



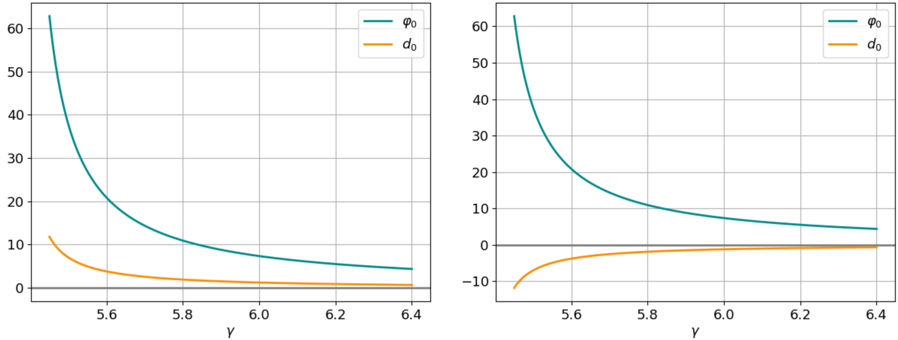
a) $\beta = 0.5$



b) $\beta = -0.5$

$$\overline{\gamma} \approx 5.375$$

Plot $\phi_0(\gamma)$ and $d_0(\gamma)$, $k = 20$



a) $\beta = 0.5$ b) $\beta = -0.5$

$$\overline{\gamma} \approx 5.375, \; \gamma_* \approx 6.497$$

Oscillating case of stability loss

- $\lambda = \pm i\omega : \quad \varepsilon = \alpha_c - \alpha,$

$$\dot{u}_{j,0} = N^2(u_{j+1,0} - 2u_{j,0} + u_{j-1,0}) + \gamma u_{j,0}, \tag{16}$$

$$u_{0,0} = u_{1,0}, \quad u_{N+1,0} = u_{N,0} + \frac{\alpha}{N} u_{k,0}, \tag{17}$$

$$u_{j,0} = z(s)e^{i\omega t} \operatorname{ch} \delta_c x_j + \overline{z(s)} e^{-i\omega t} \overline{\operatorname{ch} \delta_c x_j},$$

$$\delta_c = 2N \operatorname{arsh} \frac{\sqrt{-\gamma + i\omega}}{2N}, \quad x_j = -\frac{1}{2N} + \frac{j}{N}.$$

Oscillating case of stability loss

$$\dot{u}_{j,2} + \frac{\partial u_{j,0}}{\partial s} = N^2(u_{j+1,2} - 2u_{j,2} + u_{j-1,2}) + \gamma u_{j,2}, \tag{18}$$

$$u_{0,2} = u_{1,2}, \quad u_{N+1,2} = u_{N,2} + \frac{\alpha}{N} u_{k,0} + \frac{\beta}{N} u_{k,0}, \qquad 1 \le k < N, \tag{19}$$

$$u_{j,2} = e^{i\omega t} \operatorname{ch} \delta_c x_j.$$

Oscillating case of stability loss

$$z' = \phi_0 z + d_0 z|z|^2, \tag{20}$$

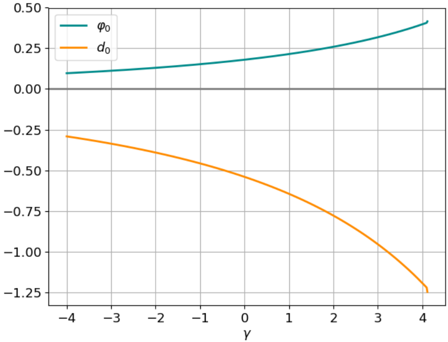
$$\phi_0 = -\mathrm{Re}\left(\frac{2\delta_c \operatorname{ch}\delta_c x_k}{\delta_c \operatorname{ch}\delta_c + \operatorname{sh}\delta_c - \alpha_c x_k \operatorname{sh}\delta_c x_k}\right),$$

$$d_0 = \mathrm{Re}\left(\frac{3\beta\delta_c(\operatorname{ch}\chi x_k + \operatorname{ch}\eta x_k + 2\operatorname{ch}\overline{\delta_c} x_k)}{2(\delta_c \operatorname{ch}\delta_c + \operatorname{sh}\delta_c - \alpha_c x_k \operatorname{sh}\delta_c x_k)}\right),$$

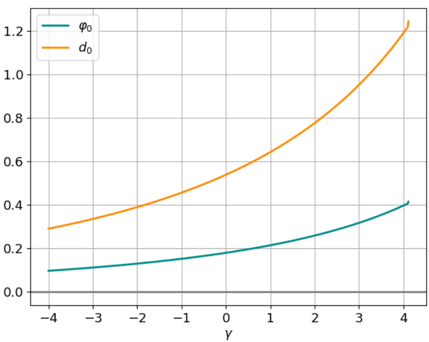
$$\chi = \delta_c + 2\mathrm{Re}\,\delta_c, \quad \eta = \delta_c + 2i\,\mathrm{Im}\,\delta_c,$$

$$\alpha_c = \frac{\sqrt{-\gamma + i\omega}\,\operatorname{sh}\delta_c}{\operatorname{ch}\delta_c x_k}.$$

Plot $\phi_0(\gamma)$ and $d_0(\gamma)$, $k = 1$



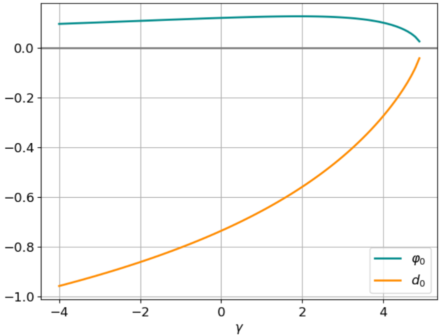
a) $\beta = 1.0$



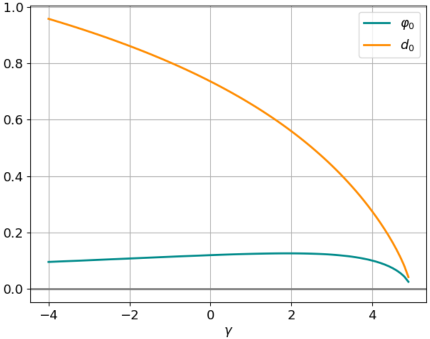
b) $\beta = -1.0$

$$\gamma_* \approx 4.116$$

Plot $\phi_0(\gamma)$ and $d_0(\gamma),\ k = 17$



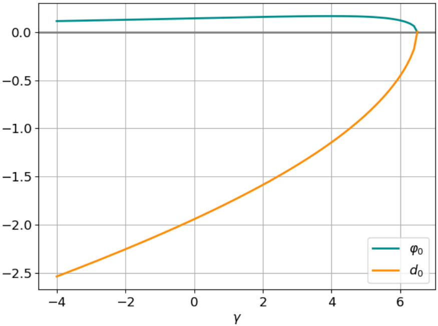
a) $\beta = 1.0$



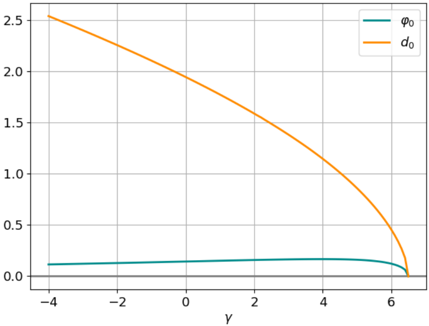
b) $\beta = -1.0$

$$\gamma_* \approx 4.896$$

Plot $\phi_0(\gamma)$ and $d_0(\gamma)$, $\alpha_{cr} = \alpha_c, k = 20$



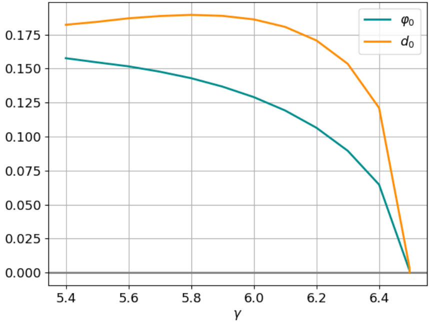
a) $\beta = 1.0$



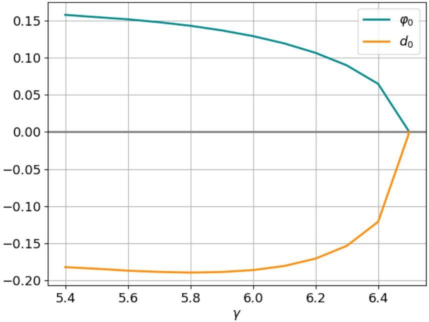
b) $\beta = -1.0$

$$\gamma_* \approx 6.497$$

Plot $\phi_0(\gamma)$ and $d_0(\gamma)$, $\alpha_{cr} = \alpha_f, k = 20$



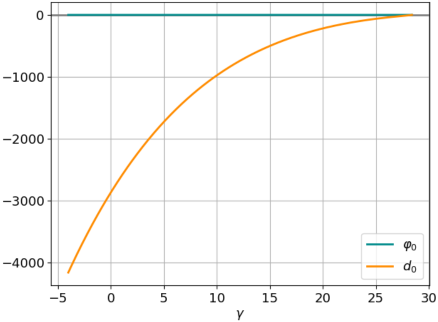
a) $\beta = 1.0$



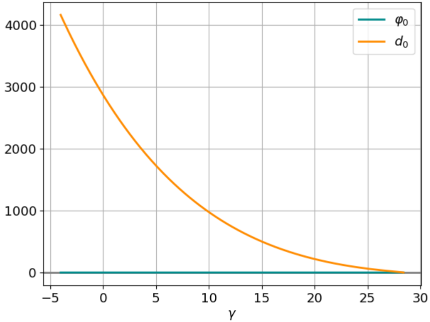
b) $\beta = -1.0$

$$\overline{\gamma} \approx 5.375,\ \gamma_* \approx 6.497$$

Plot $\phi_0(\gamma)$ and $d_0(\gamma)$, $\alpha_{cr} = \alpha_c, k = 33$



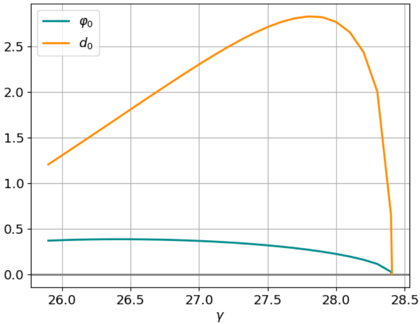
a) $\beta = 1.0$



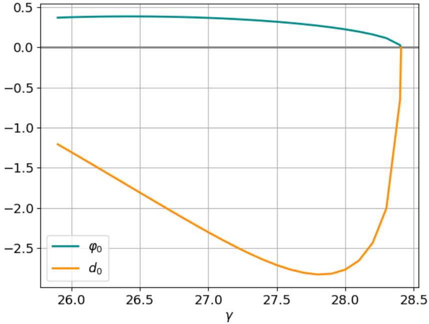
b) $\beta = -1.0$

$$\gamma_* \approx 28.407$$

Plot $\phi_0(\gamma)$ and $d_0(\gamma)$, $\alpha_{cr} = \alpha_f, k = 33$



a) $\beta = 1.0$



b) $\beta = -1.0$

$$\overline{\gamma} \approx 25.682, \ \gamma_* \approx 28.407$$

Dynamics of diffused connected systems of differential equations with internal connection

Leonid Ivanovsky

P.G. Demidov Yaroslavl State University