Stability loss of zero balance state of nonlinear boundary-value problem with deviate in boundary condition

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Boundary-value problem

$$\dot{u} = u'' + \gamma u - u^3, \tag{1}$$

$$u'(0,t) = 0, \quad u'(1,t) = \alpha\, u(x_0, t), \tag{2}$$

$$t \geqslant 0, \quad x \in [0,1], \quad \alpha, \gamma \in \mathbb{R}, \quad x_0 \in [0,1).$$

Eigenvalue problem

$$u(x,t) = e^{\lambda t} v(x).$$

$$v'' + (\gamma - \lambda) v = 0, \tag{3}$$

$$v'(0) = 0, \qquad v'(1) = \alpha\, v(x_0). \tag{4}$$

$$v(x) = c \operatorname{ch} \mu x, \quad c \in \mathbb{R}, \quad \mu = \sqrt{-\gamma + \lambda}.$$

Critical values of α

- $\lambda = 0: \ \mu = \sqrt{-\gamma}$,

$$\alpha_u = \frac{\sqrt{-\gamma} \operatorname{sh} \sqrt{-\gamma}}{\operatorname{ch} \sqrt{-\gamma} x_0}.$$

- $\lambda = \pm i\omega: \ \mu = \sqrt{-\gamma + i\omega}$,

$$\alpha_c = \frac{\sqrt{-\gamma + i\omega} \operatorname{sh} \sqrt{-\gamma + i\omega}}{\operatorname{ch} \sqrt{-\gamma + i\omega} x_0}.$$

Modeling of linearized boundary value problem

$$\dot{u}_j = n^2 (u_{j+1} - 2u_j + u_{j-1}) + \gamma u_j, \quad j = \overline{1, n}, \tag{5}$$

$$u_0 = u_1, \quad u_{n+1} = u_n + \frac{\alpha}{n} u_k, \qquad k \in [1, n]. \tag{6}$$

Normal form

$$u = \sqrt{\varepsilon} u_0 + \varepsilon u_1 + \varepsilon^{\frac{3}{2}} u_2 + O(\varepsilon^2), \tag{7}$$

$$\varepsilon = |\alpha - \alpha_{cr}|,$$

$$\varepsilon \ll 1, \quad s = \varepsilon t.$$

Sequence of boundary value problems

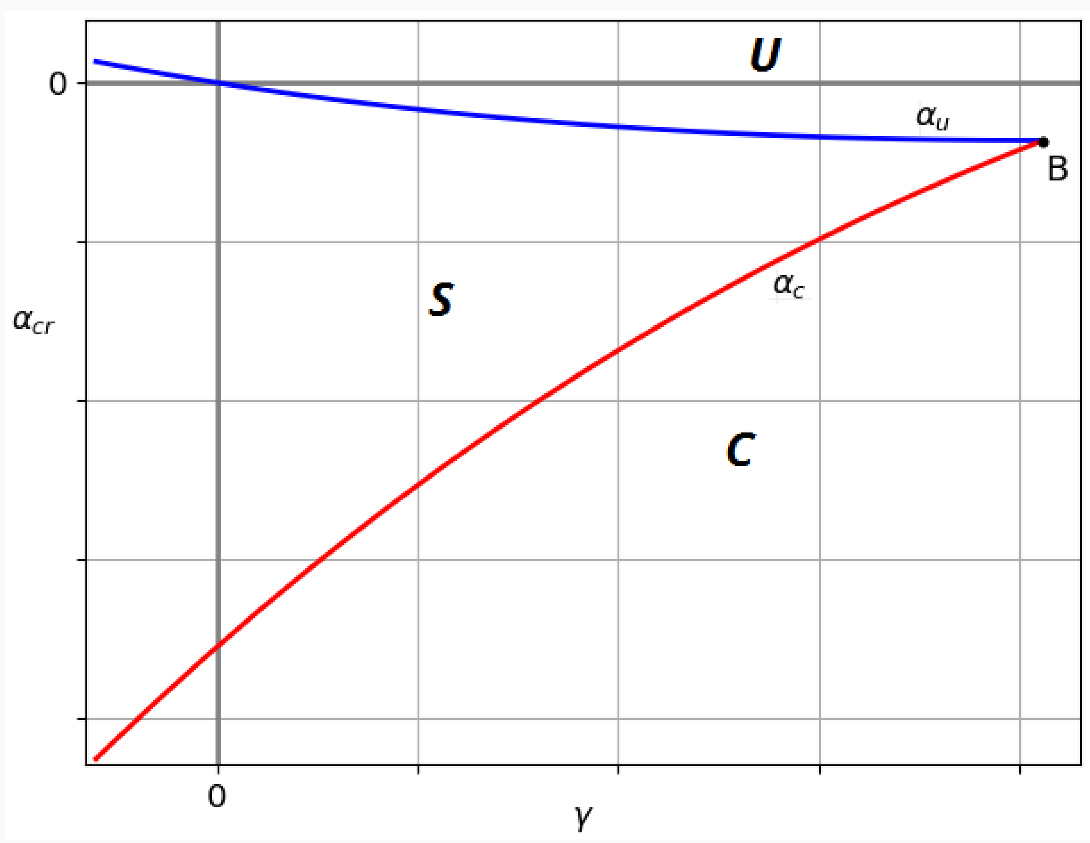
$$u_0 = u_0'' + \gamma u_0, \tag{8}$$

$$\dot{u}_2 + \frac{\partial u_0}{\partial s} = u_2'' + \gamma u_2 - u_0^3, \tag{9}$$

- $\lambda = 0: \quad u_0 = z(s) \operatorname{ch} \mu x.$
- $\lambda = \pm i\omega: \quad u_0 = z(s) e^{i\omega t} \operatorname{ch} \mu x + \overline{z(s)} e^{-i\omega t} \overline{\operatorname{ch} \mu x}.$

$$z' = \phi_0 z + d_0 z |z|^2. \tag{10}$$

Areas of stability for zero solution



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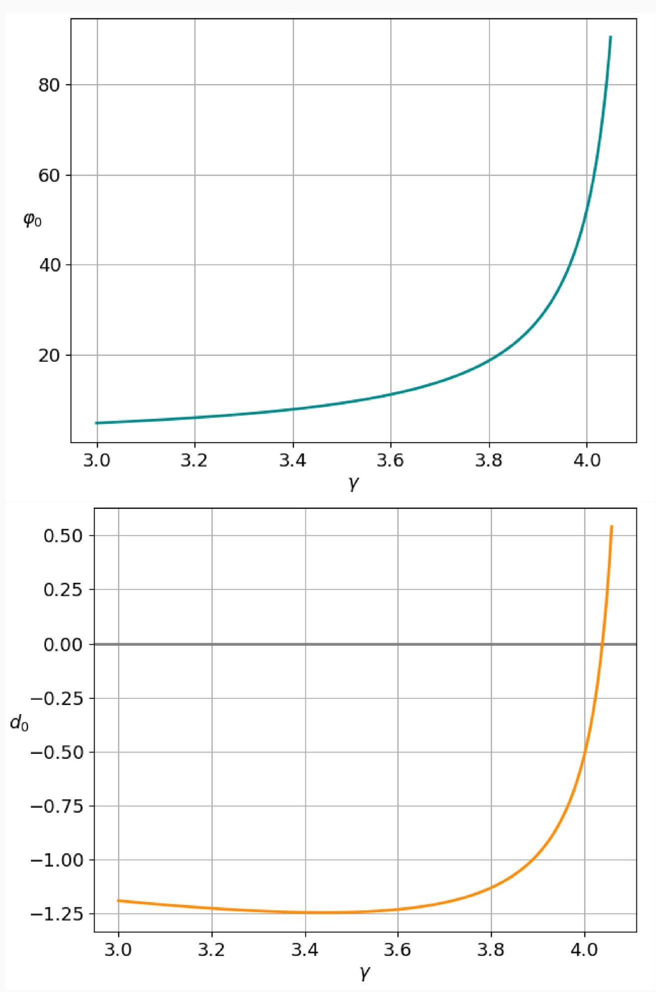
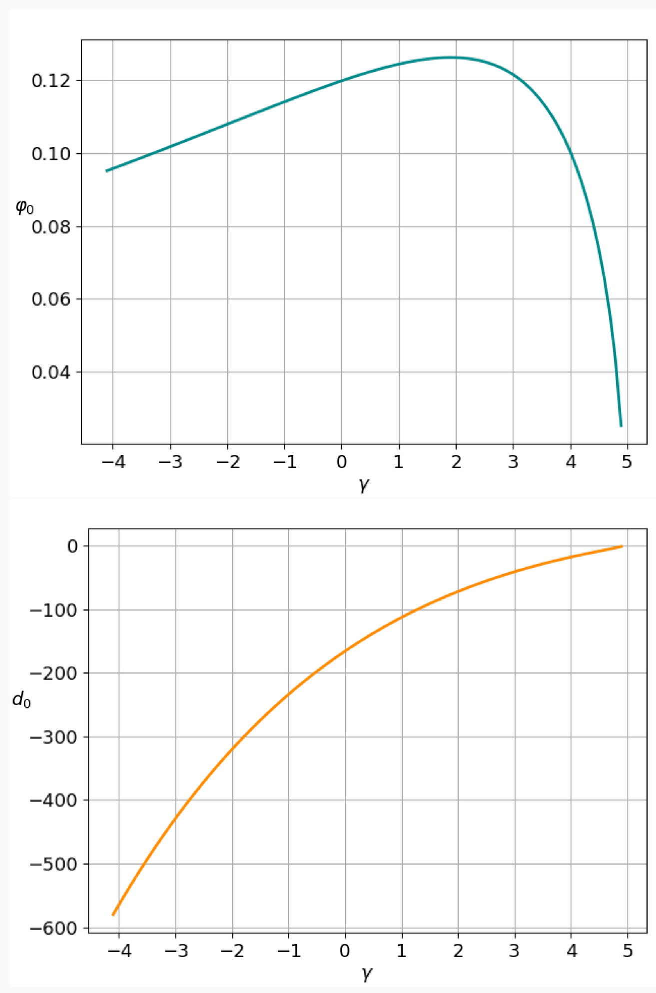
Divergent loss of zero balance state $(\lambda = 0)$



$x_0 = 0.0$

Oscillating loss of zero balance state $(\lambda = \pm i\omega)$



$x_0 = 0.33$