Oscillating loss of stability of trivial solution for boundary-value problem with linear deviate in boundary condition

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postgraduate students

Nonlinear boundary-value problem

$$u' = \beta \ddot{u} - \gamma u(t-\tau) + F(u), \tag{1}$$

$$\begin{aligned} u'(0,t) &= 0, \\ u'(1,t) &= \alpha\, u(x_0, t-\tau). \end{aligned} \tag{2}$$

$$\alpha \in \mathbb{R}, \quad \beta, \gamma > 0, \quad \tau \geqslant 0, \quad x_0 \in [0,1].$$

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Kaschenko S.A. About bifurcations with small disturbances in logistic equation with delay // Modelling and Analysis of Information Systems, v.24(2), p. 168–185 (2017).

Nonlinear boundary-value problem without delay

$$u' = \beta \ddot{u} - \gamma u + F(u), \tag{3}$$

$$\begin{gathered} u' \mid_{x=0} = 0, \\ u' \mid_{x=1} = \alpha\, u \mid_{x=x_0} . \end{gathered} \tag{4}$$

$$\alpha \in \mathbb{R}, \quad \beta, \gamma > 0, \quad x_0 \in [0, 1].$$

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Substitutions

$$t_1 = d \times t,$$

$$u(x,t) = w(x) \exp\left(\lambda - \frac{\gamma}{d} t\right).$$

Simplified boundary-value problem

$$w'' - \lambda w = 0, \tag{5}$$

$$\begin{aligned} w'(0) &= 0, \\ w'(1) &= \alpha\, w(x_0). \end{aligned} \tag{6}$$

Boundary conditions

$$w(x) = c\,\mathrm{ch}(\mu\, x),$$

$$\mu^2 = \lambda.$$

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$$\mu^2 = \lambda.$$

$$x = 1:$$

$$\mu\,\mathrm{sh}\,\mu = \alpha\,\mathrm{ch}(\mu x_0).$$

System of equations

$$\lambda \in \mathbb{C} : \quad \mu = \tau + i\omega.$$

$$\begin{cases} f(\tau, \omega) = 0 \\ g(\tau, \omega) - \alpha = 0. \end{cases} \tag{7}$$

Oscillating loss of stability of zero balance state

Theorem

There exists $\alpha = \alpha_{cr}$, for that $\mathrm{Re}(\lambda_*) = \dfrac{\gamma}{\beta}$ and for the rest eigenvalues of problem (3), (4) $\mathrm{Re}(\lambda) < \dfrac{\gamma}{\beta}$

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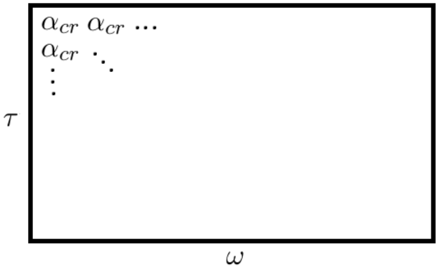
Ivanovsky L.I., Kaschenko S.A. Stability loss of the trivial solution of boundary-value problem with linear deviate in boundary conditions // International Scientific Conference "New trends in nonlinear dynamics". Abstracts, p. 32-33 (2017).

Numerical research

python™

OpenMP®

Algorithm



$$\omega \to \tau_* \to \alpha_*$$

$$\mathrm{Re}(\lambda_*) = \frac{\gamma}{\beta} = \tau_*^2 - \omega^2$$

Algorithm

$$\alpha = \alpha_{cr} : \quad |\alpha_* - \alpha| < \varepsilon$$

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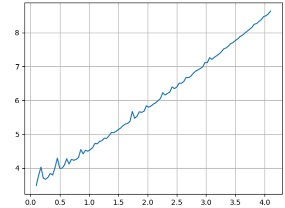
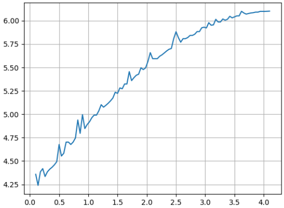
$$\mathrm{Re}(\lambda) < \mathrm{Re}(\lambda_*) + \varepsilon$$

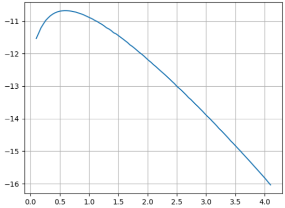
Results



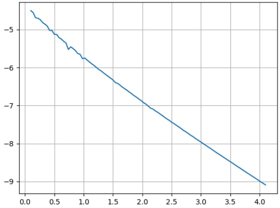
$x_0 = 0.0$



$x_0 = 0.15$



$x_0 = 0.45$



$x_0 = 0.6$

$\beta = 1.0$

Nonlinear boundary-value problem with delay

$$u' = \beta\ddot{u} - \gamma u(t-\xi) + F(u), \tag{8}$$

$$\begin{gathered} u' \,|_{x=0} = 0, \\ u' \,|_{x=1} = \alpha\, u(1, t-\tau). \end{gathered} \tag{9}$$

$$\alpha \in \mathbb{R}, \quad \beta, \gamma > 0, \quad \tau, \xi > 0.$$

Linear problem

$$u' = \beta \ddot{u} - \gamma u(t - \xi) \tag{10}$$

$$u(x,t) = X(x)\exp(\lambda t) \tag{11}$$

$$(8,9) \quad \rightarrow \quad \begin{cases} \ddot{X} - \frac{1}{\beta}\left(\lambda + \gamma \exp(-\lambda\xi)\right) X = 0 \\ \dot{X}|_{x=0} = 0 \\ \dot{X} - \alpha \exp\left(-\lambda\tau\right) X|_{x=1} = 0 \end{cases} \tag{12}$$

Quasi analytic solution

Let

$$g = \frac{1}{\beta}\left(\lambda + \gamma \exp(-\lambda\xi)\right), \tag{13}$$

so

$$(??, ??) \quad \rightarrow \quad \begin{cases} g^{1/2}\,\mathrm{sh}\,g^{1/2} - \alpha \exp\left(-\lambda\tau\right) \mathrm{ch}\,g^{1/2} = 0 & \text{if } \tau \neq 0 \\ g^{1/2}\,\mathrm{sh}\,g^{1/2} - \alpha\; 1 \;\mathrm{ch}\,g^{1/2} = 0 & \text{if } \tau = 0 \end{cases} \tag{14}$$

Quasi analytic solution

Let's solve (??)

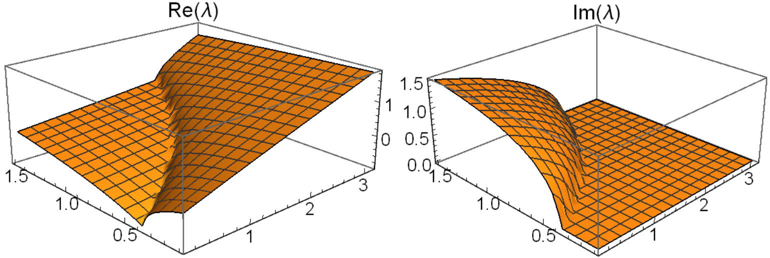
$$\lambda = \frac{g\beta\xi - \mathrm{W}(k, -\exp(-g\beta\xi)\gamma\xi)}{\xi}, \tag{15}$$

where g can be found as a numerical solution of (??)

Roots of (??) can be found numerically for different α

$$\alpha = 0.5 \quad g \to 0.59552;$$
$$\alpha = 1.0 \quad g \to 1.43923;$$
$$\alpha = 1.5 \quad g \to 2.63030;$$

$\alpha = 0.5 \quad g \to 0.59552$



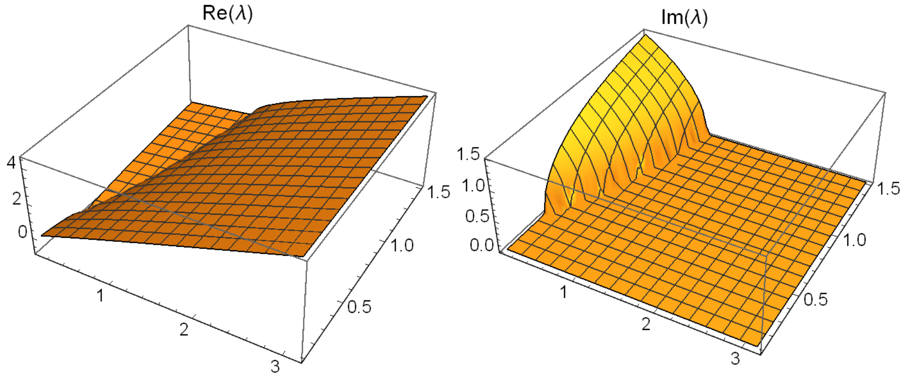
$k = 0, \tau = 0, \xi = 1$

$$\beta_{crit} \to 0.93612 \quad \gamma_{crit} \to 1.23716$$

$$Im(\lambda_{crit}) \to 1.10347$$

$$Re(\lambda_{crit}) << 10^{-6}$$

$\alpha = 1.0 \quad g \to 1.43923$



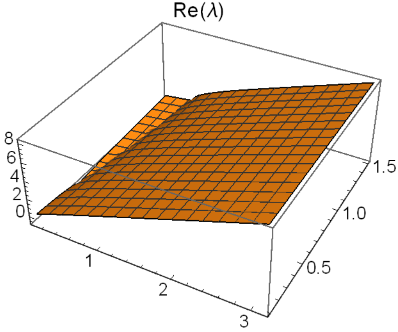
$k = 0, \tau = 0, \xi = 1$

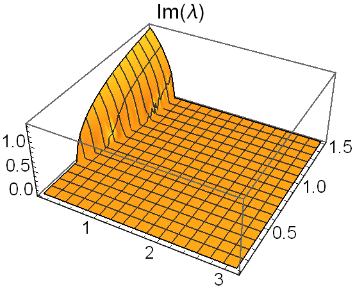
$$\beta_{crit} \to 0.33334 \quad \gamma_{crit} \to 1.28224$$

$$Im(\lambda_{crit}) \to 1.18855$$

$$Re(\lambda_{crit}) << 10^{-6}$$

$\alpha = 1.5 \quad g \to 2.63030$





$k = 0, \tau = 0, \xi = 1$

$$\beta_{crit} \to 0.18224 \quad \gamma_{crit} \to 1.28323$$

$$Im(\lambda_{crit}) \to 1.19032$$

$$Re(\lambda_{crit}) << 10^{-6}$$

LambertW

1. $f(W) = W \exp(W)$
2. $\mathrm{W}(-\frac{1}{e}) = -1$
3. $\mathrm{W}(e) = 1$
4. $\frac{\mathrm{W}(z)}{dz} = \frac{1}{1+\exp(\mathrm{W}(z))}$