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AUTO-OSCILLATION IN ONE-DIMENSIONAL TWO-PHASE SYSTEM WITH THERMAL FEEDBACK

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1. INTRODUCTION

Nowadays synergetics, especially in the part dealing with theory of nonlinear oscillations and waves, is the rapidly developing field of knowledge. Its subject includes the variety of physical, chemical and biological systems and processes which are referred to as dynamic systems (DS). Among them there is a certain class of DS (control and stabilization systems) grouped on the basis of external feedback. Unlike feedback in DS of natural origin, usually consisting in local interaction of degrees of freedom, the external feedback is nonlinear and nonlocal self-action of one of the thermodynamic degrees of freedom. With a certain value of its depth the steady state of DS loses stability, accompanied by spacio-temporal structure development in the distributed elements of feedback loop.

One such system with the feedback circuit closed through the temperature field of some medium (Fig. 1) was considered in literature [1]. It was shown that the loss of stability results in auto-oscillation in DS and excitation of temperature waves in dielectric medium. The feature of the system is that due to the external nonlinearity the spatio-temporal structure is formed in media that are near to thermal equilibrium. The latter circumstances permits to use the linear equation of heat conduction

$$\dot{T} = aT''(x, t) \tag{1}$$

with zero boundary condition

$$T(0, t) = 0 \tag{2}$$

as the fundamental mathematical model, which is closed by the feedback equation

$$T'|_{x=\delta} = \frac{K^2}{\lambda SR}[u_0 - \alpha T(x_0, t)]^2 \sigma[u_0 - \alpha T(x_0, t)] \tag{3}$$

where $T(x, t)$ is the deviation of medium temperature from that of thermostat, a is thermal diffusivity, K is gain factor, λ is thermal conductivity, S and R are heater surface area and resistance respectively, u_0 is reference voltage, α is thermocouple e.m.f., x_0 is thermocouple coordinate, σ is the Heaviside step function cutting the positive branch of feedback function.

This model has been thoroughly investigated in literature [1, 2]. In particular, the asymptotic periodic solution of the boundary value problem (1)–(3) and expressions for DS critical parameters calculation were obtained in [1]. The posterior nonlinear analysis, carried out ibidem, had revealed the auto-oscillation excitation mode to be soft,

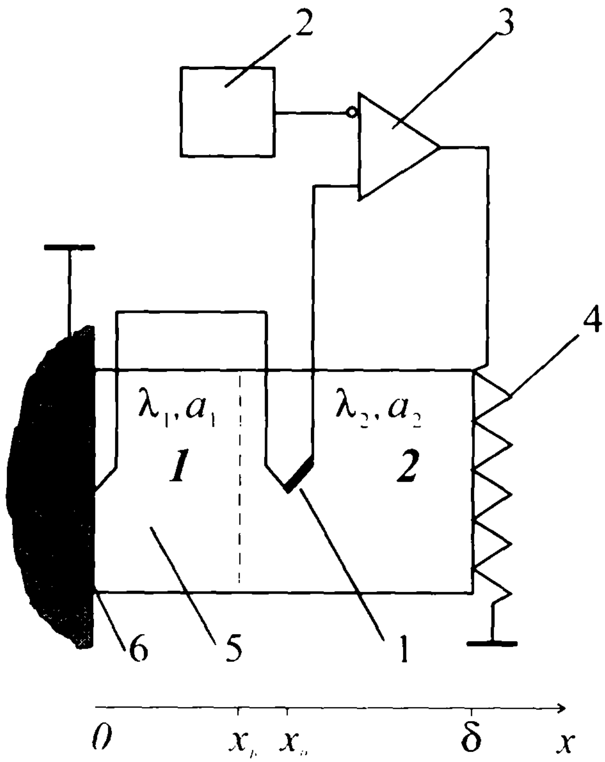

Fig. 1. A system for temperature stabilization consisting of thermocouple 1, reference voltage calibrator 2, controller 3, heater 4, medium 5 and thermostat 6. The thermocouple voltage defined by temperature difference in $x = x_0$ and $x = 0$ as $\alpha T(x_0)$ is fed into the controller inverting input where it is compared with the reference voltage u_0. The output voltage $K[u_0 - \alpha T(x_0, t)] \times \sigma[u_0 - \alpha T(x_0, t)]$ is supplied to the heater generating the heat flux $q = (K^2/SR)[u_0 - \alpha T(x_0, t)]^2 \times \sigma[u_0 - \alpha T(x_0, t)]$ closing the feedback circuit. The interface plane which coordinate x_b is determined by condition $T(x_b, t) = T_c$ divides the medium into phases 1 and 2.

i.e. temperature oscillation amplitude $V(x, \varepsilon)$ was tending to zero with $\varepsilon \to 0$, where ε was system supercriticity. This fact had confirmed the applicability of asymptotic methods and the results of linear analysis.

The problem on investigation of auto-oscillation in two-phase systems arises when the range of temperature control includes the phase transition points. The problems on determination of temperature field and interface position through phase transition in a substance are conventionally called "Stefan problems". The present one, formulated as the investigation of small oscillations of interface about its steady position with auto-oscillation of a one-dimensional two-phase system, only partly belongs to this class. Besides the external nonlinearity of Stefan condition it contains one more nonlinear boundary condition [feedback equation (3)] which, to an even greater degree, influences the DS behavior. One of the conventional methods to solve the Stefan problem is the method of small parameter, consisting in the expansion of solution sought for in power series of Stefan number Sf. It is necessary to stress that the small parameter method we use below bears little resemblance to the one mentioned above and is the application of the algorithmic part of the Andronov-Hopf bifurcation theorem [3].

2. PROBLEM STATEMENT

Let us suppose that in DS displayed in Fig. 1 thermostat temperature $\mathfrak{Z}$ is such, that temperature interval $T(\delta) - T(0)$ includes the point of phase transition $T_c = T_T - \mathfrak{Z}$ where T_T is an absolute temperature of phase transition. The interface, which coordinate x_b is determined from condition $T(x_b) = T_c$ divides the medium into two parts labelled in Fig. 1 by *1* and *2*. As the medium state is close to equilibrium it is natural to assume that thermal conductivities λ_1, λ_2 and thermal diffusivities a_1, a_2 of both phases do not depend on temperature, but suffer the break in x_b. As the temperature dependence of kinetic coefficients is not the analytic one, the boundary value problem (1)-(3) is written separately for each particular phase

$$(1) \quad \dot{T}_1(x, t) = a_1 T_1''(x, t), \quad T_1(0, t) = 0, \tag{4}$$

$$(2) \quad \dot{T}_2(x, t) = a_2 T_2''(x, t), \tag{5}$$

while their solutions should be sewn together in x_b by means of continuity

$$T_1(x_b, t) = T_2(x_b, t) = T_c \tag{6}$$

and Stefan

$$\lambda_2 T_2'(x, t)|_{x=x_b} = \lambda_1 T_1'(x, t)|_{x=x_b} - \rho_1 Q \dot{x}_b \tag{7}$$

conditions, where Q is specific heat of phase transition, ρ_1 is the phase *1* density. To close the system (4)-(7) the feedback equation (3) is rewritten for two possible cases of free boundary position:

(1) the interface is situated between heater and thermocouple

$$T_2'(x, t)|_{x=\delta} = \frac{K^2\alpha^2}{\lambda_2 SR}[\beta - T_1(x_0, t)]^2\sigma[\beta - T_1(x_0, t)], \qquad x_b \in]x_0, \delta[; \tag{8}$$

(2) the interface is situated between thermocouple and thermostat

$$T_2'(x, t)|_{x=\delta} = \frac{K^2\alpha^2}{\lambda_2 SR}[\beta - T_2(x_0, t)]^2\sigma[\beta - T_2(x_0, t)], \qquad x_b \in]0, x_0[, \tag{9}$$

where $\beta = u_0/\alpha$, x_b is the interface coordinate. Introducing dimensionless independent $\xi = x/\delta$ and dependent $\Theta_1(\xi, t) = T_1(\xi, t)/\beta$, $\Theta_2(\xi, t) = T_2(\xi, t)/\beta$ variables and parameters $\eta = x_0/\delta$, $\Theta_c = T_c/\beta$, and taking into account the relation $a_1 = \lambda_1/c_1\rho_1$, we obtain

$$\begin{aligned}
\dot{\Theta}_1(\xi, t) &= \frac{a_1}{\delta^2}\Theta_1''(\xi, t), \\
\dot{\Theta}_2(\xi, t) &= \frac{a_2}{\delta^2}\Theta_2''(\xi, t), \\
\Theta_1(0, t) &= 0, \\
\Theta_1(\xi_b, t) &= \Theta_2(\xi_b, t) = \Theta_c, \\
\frac{\lambda_2}{\lambda_1}\Theta_2'(\xi, t)|_{\xi=\xi_b} &= \Theta_1'(\xi, t)|_{\xi=\xi_b} - \frac{\delta^2 Q}{a_1 c_1 \beta}\dot{\xi}_b, \\
\Theta_2'(\xi, t)|_{\xi=1} &= \kappa[1 - \Theta_1(\eta, t)]^2\sigma[1 - \Theta_1(\eta, t)], \qquad \xi_b \in]\eta, 1[\\
\Theta_2'(\xi, t)|_{\xi=1} &= \kappa[1 - \Theta_2(\eta, t)]^2\sigma[1 - \Theta_2(\eta, t)], \qquad \xi_b \in]0, \eta[,
\end{aligned} \tag{10}$$

where $\kappa = K^2\alpha u_0\delta/\lambda_2 SR$, c_1 is phase *1* specific heat.

Suppose that with the certain (critical) value of $\kappa = \kappa_c$ the DS loses stability. Then it is quite expedient to introduce the new parameter

$$\varepsilon = \frac{\kappa - \kappa_c}{\kappa_c}, \tag{11}$$

characterizing the relative deviation of DS from its critical state, which we assume to be small $\varepsilon \ll 1$ and which shall be referred to as the supercriticity of the system. Let us express κ as $\kappa = (1 + \varepsilon)\kappa_c$ and expend ε in power series of auxiliary parameter ζ, which makes sense of the first harmonic amplitude

$$\varepsilon = b_2\zeta^2 + b_4\zeta^4 + \cdots. \tag{12}$$

Assuming that

$$\begin{gathered} t = (1 + c)\tau, \qquad c = c_2\zeta^2 + c_4\zeta^4 \cdots, \\ \Theta_1(\xi, t) = T_{10}(\xi) + \zeta T_{11}(\xi, \tau) + \zeta^2 T_{12}(\xi, \tau) + \cdots \\ \Theta_2(\xi, t) = T_{20}(\xi) + \zeta T_{21}(\xi, \tau) + \zeta^2 T_{22}(\xi, \tau) + \cdots \\ \xi_b(\tau) = \bar{\xi}_b + \zeta\xi_1(\tau) + \zeta^2\xi_2(\tau) + \cdots \end{gathered} \tag{13}$$

substituting (12), (13) in (10) and grouping the terms of the identical ζ powers, we obtain the recurrent sequence of linear inhomogeneous boundary problems. The details of this procedure as well as the sequence proper will be considered at the end of the paper. Now we shall concentrate on the first two problems in the sequence, the form of which is obvious

$$\begin{aligned} T_{10}''(\xi) &= 0, \\ T_{20}''(\xi) &= 0, \\ T_{10}(0) &= 0, \\ T_{10}(\bar{\xi}_b) &= T_{20}(\bar{\xi}_b) = \Theta_c, \\ \lambda_1 T_{10}'(\xi)|_{\xi = \bar{\xi}_b} &= \lambda_2 T_{20}'(\xi)|_{\xi = \bar{\xi}_b} \\ T_{20}'(\xi)|_{\xi = 1} &= -\frac{A_{c1}}{2\eta}[1 - T_{10}(\eta)]\sigma[1 - T_{10}(\eta)], \\ A_{c1} &= -2\eta\kappa_{c1}[1 - T_{10}(\eta)], \qquad \xi_b \in]\eta, 1[, \\ T_{20}'(\xi)|_{\xi = 1} &= -\frac{A_{c2}}{2\eta}[1 - T_{20}(\eta)]\sigma[1 - T_{20}(\eta)], \\ A_{c2} &= -2\eta\kappa_{c2}[1 - T_{20}(\eta)], \qquad \xi_b \in]0, \eta[; \end{aligned} \tag{14}$$

$$
\begin{aligned}
\dot{T}_{11} &= \frac{a_2}{\delta^2} T''_{11}, \\
\dot{T}_{21} &= \frac{a_2}{\delta^2} T''_{21}, \\
T_{11}(0, \tau) &= 0, \\
T'_{10}(\bar{\xi}_b)\xi_1(\tau) + T_{11}(\bar{\xi}_b, \tau) &= 0, \\
T'_{20}(\bar{\xi}_b)\xi_1(\tau) + T_{21}(\bar{\xi}_b, \tau) &= 0, \\
\frac{\lambda_2}{\lambda_1} T'_{21}(\xi, \tau)|_{\xi=\xi_b} &= T'_{11}(\xi, \tau)|_{\xi=\xi_b} - \frac{\delta^2 Q}{a_1 c_1 \beta}\dot{\xi}_1(\tau), \\
T'_{21}(\xi, \tau)|_{\xi=1} &= \frac{A_{c1}}{\eta} T_{11}(\eta, \tau), \qquad \xi_b \in]\eta, 1[, \\
T'_{21}(\xi, \tau)|_{\xi=1} &= \frac{A_{c2}}{\eta} T_{21}(\eta, \tau), \qquad \xi_b \in]0, \eta[,
\end{aligned}
\tag{15}
$$

where A has a significance of generalized gain factor while A_c is its critical value. Subscripts attached to variable parameters indicate the mutual disposition of thermocouple and interface: 1—thermocouple is in phase *1*, 2—thermocouple is in phase *2*.

3. STEADY STATE

Let us consider the first problem of sequence (14), (15). Its solutions with respect to zero boundary condition is

$$
\begin{aligned}
T_{10} &= C\xi, \\
T_{20} &= C_3\xi + C_4.
\end{aligned}
$$

Expressing constants C_3, C_4 through C from continuity and Stefan conditions, we obtain

$$
\begin{aligned}
T_{10} &= C\xi, \\
T_{20} &= C\chi(\xi),
\end{aligned}
\tag{16}
$$

where $\chi(\xi) = (\lambda_1/\lambda_2)(\xi - \bar{\xi}_b) + \bar{\xi}_b$. The stationary temperature gradients C_n ($n = 1, 2$ indicates the interface position toward η) satisfying feedback equations (14), are

$$
\begin{aligned}
C_1 &= \frac{A_{c1}}{A_{c1} - 2(\lambda_1/\lambda_2)} \frac{1}{\eta}, \qquad \bar{\xi}_b \in]\eta, 1[, \\
C_2 &= \frac{A_{c2}}{A_{c2} - 2(\lambda_1/\lambda_2)[\eta/\chi(\eta)]} \frac{1}{\chi(\eta)}, \qquad \bar{\xi}_b \in]0, \eta[.
\end{aligned}
\tag{17}
$$

The expression to determine A_{cn} will be deduced below from the feedback condition. Getting ahead and assuming A_{cn} to be a known value, we obtain the expressions for the stationary temperature distribution

$$\left.\begin{aligned} T_{10} &= C_1\xi = \frac{A_{c1}}{A_{c1} - 2(\lambda_1/\lambda_2)}\frac{\xi}{\eta} \\ T_{20} &= C_1\chi(\xi) = \frac{A_{c1}}{A_{c1} - 2(\lambda_1/\lambda_2)}\frac{\chi(\xi)}{\eta} \end{aligned}\right\} \bar{\xi}_b \in]\eta, 1[, \tag{18}$$

$$\left.\begin{aligned} T_{10} &= C_2\xi = \frac{A_{2c}}{A_{c2} - 2(\lambda_1/\lambda_2)[\eta/\chi(\eta)]}\frac{\xi}{\chi(\eta)} \\ T_{20} &= C_2\chi(\xi) = \frac{A_{c2}}{A_{c2} - 2(\lambda_1/\lambda_2)[\eta/\chi(\eta)]}\frac{\chi(\xi)}{\chi(\eta)} \end{aligned}\right\} \bar{\xi}_b \in]0, \eta[. \tag{19}$$

These solutions admit the simple verification: with $\lambda_1 = \lambda_2$ they transform into the following expression

$$T_{10} = T_{20} = \frac{A_c}{A_c - 2}\frac{\xi}{\eta}, \qquad \bar{\xi}_b \in]0, 1[,$$

which coincides with that obtained in [1] for the case of the homogeneous system.

Both solutions (18) and (19) are valid only with critical values of generalized gain factor A_{cn}. To extend them into precritical domain we should omit the subscript c and replace A_n by its relation to system parameters. The latter is found from the feedback conditions, preliminary transformed into

$$T_{20}'(\xi)|_{\xi=1} = \frac{1}{2\eta(D-1)}[1 - T_{10}(\eta)]^2\sigma[1 - T_{10}(\eta)], \qquad \bar{\xi}_b \in]\eta, 1[, \tag{20}$$

$$T_{20}'(\xi)|_{\xi=1} = \frac{1}{2\eta(D-1)}[1 - T_{20}(\eta)]^2\sigma[1 - T_{20}(\eta)], \qquad \bar{\xi}_b \in]0, \eta[, \tag{21}$$

where $D - 1 = \lambda_2 SR/2K^2\alpha u_0 x_0$. Substitution of (16) in (20), (21) gives two quadratic equations

$$C_1^2 - \frac{2}{\eta}\left[1 + \frac{\lambda_1}{\lambda_2}(D-1)\right]C_1 + \frac{1}{\eta^2} = 0, \qquad \bar{\xi}_b \in]\eta, 1[,$$

$$C_2^2 - \frac{2}{\chi(\eta)}\left[1 + \frac{\eta}{\chi(\eta)}\frac{\lambda_1}{\lambda_2}(D-1)\right]C_2 + \frac{1}{\chi^2(\eta)} = 0, \qquad \bar{\xi}_b \in]0, \eta[.$$

From two roots of each equation should be selected only those, with which the Heavyside function argument is positive. In the case $\bar{\xi}_b \in]\eta, 1[$ temperature gradients in both phases are constants, i.e. they do not depend on interface coordinate

$$C_1 = \frac{1}{\eta}(D_1 - \sqrt{D_1^2 - 1}), \tag{22}$$

where $D_1 = 1 + (\lambda_1/\lambda_2)(D-1) = 1 + \lambda_1 SR/2K^2\alpha u_0 x_0$. When $\bar{\xi}_b \in]0, \eta[$, temperature gradient of both phases depends on $\bar{\xi}_b$

$$C_2 = \frac{1}{\chi(\eta)}[D_2(\bar{\xi}_b) - \sqrt{D_2^2(\bar{\xi}_b) - 1}] \tag{23}$$

through parameters

$$\chi(\eta) = \frac{\lambda_1}{\lambda_2}(\eta - \bar{\xi}_b) + \bar{\xi}_b,$$

$$D_2(\bar{\xi}_b) = 1 + \frac{\lambda_1 SR}{2\alpha u_0 x_0 K^2}\frac{\eta}{(\lambda_1/\lambda_2)(\eta - \bar{\xi}_b) + \bar{\xi}_b}.$$

Comparison of expression (17) to (22), (23) yields the relation of A_n to other system parameters

$$A_1 = \frac{\lambda_1}{\lambda_2}\left[1 - \sqrt{\frac{D_1 + 1}{D_1 - 1}}\right]$$

$$A_2 = \frac{\lambda_1}{\lambda_2}\frac{\eta}{\chi(\eta)}\left[1 - \sqrt{\frac{D_2(\bar{\xi}_b) + 1}{D_2(\bar{\xi}_b) - 1}}\right].$$

Verifying the last expressions by assumption $\lambda_1 = \lambda_2$ we immediately obtain the result $D_1 = D_2 = D$ and $A_1 = A_2 = A = 1 - \sqrt{(D+1)/(D-1)}$ consistent with that of [1].

Substitution of (22), (23) in (16) yields the stationary temperature distribution in medium

$$\left.\begin{aligned} T_{10} &= (D_1 - \sqrt{D_1^2 - 1})\frac{\xi}{\eta} \\ T_{20} &= (D_1 - \sqrt{D_1^2 - 1})\frac{\chi(\xi)}{\eta} \end{aligned}\right\} \bar{\xi}_b \in]\eta, 1[, \tag{24}$$

$$\left.\begin{aligned} T_{10} &= [D_2(\bar{\xi}_b) - \sqrt{D_2^2(\bar{\xi}_b) - 1}]\frac{\xi}{\chi(\eta)} \\ T_{20} &= [D_2(\bar{\xi}_b) - \sqrt{D_2^2(\bar{\xi}_b) - 1}]\frac{\chi(\xi)}{\chi(\eta)} \end{aligned}\right\} \bar{\xi}_b \in]0, \eta[. \tag{25}$$

Expressions (24), (25) enable one to solve the inverse problem, i.e. to find $C_{1,2}$ when system parameters and interface coordinate are known values, and then to find the thermostat temperature satisfying condition $\Theta_{1,2}(\bar{\xi}_b, t) = C_{1,2}\bar{\xi}_b = \Theta_c$. More often it is necessary to determine the interface position for fixed thermostat temperature and system parameters. This is attained by substitution of $\bar{\xi}_b = \Theta_c/C_1$ and $\bar{\xi}_b = \Theta_c/C_2$ in (16) which in the case $\bar{\xi}_b \in]\eta, 1[$ brings directly to the required expression. With $\bar{\xi}_b \in]0, \eta[$ the resultant expression

$$T_{20} = \frac{\lambda_1}{\lambda_2}C_2\xi + \frac{\Delta\lambda}{\lambda_2}\Theta_c, \qquad \Delta\lambda = \lambda_2 - \lambda_1, \tag{26}$$

should be substituted into the feedback condition

$$C_2 = \frac{\lambda_2}{\lambda_1}\frac{1-(\Delta\lambda/\lambda_2)\Theta_c}{\eta}[D_2(\Theta_c) - \sqrt{D_2^2(\Theta_c) - 1}],$$

where

$$D_2(\Theta_c) = 1 + \frac{\lambda_2 SR}{2\alpha u_0 x_0 K^2}\frac{1}{1-(\Delta\lambda/\lambda_2)\Theta_c}.$$

The final form of temperature field expressions with prescribed value of Θ_c, is the following

$$\left.\begin{aligned} T_{10} &= (D_1 - \sqrt{D_1^2 - 1})\frac{\xi}{\eta} \\ T_{20} &= \frac{\lambda_1}{\lambda_2}(D_1 - \sqrt{D_1^2 - 1})\frac{\xi}{\eta} + \frac{\Delta\lambda}{\lambda_2}\Theta_c \end{aligned}\right\} \bar{\xi}_b \in]\eta, 1[, \tag{27}$$

$$\left.\begin{aligned} T_{10} &= \frac{\lambda_2}{\lambda_1}\left[1 - \frac{\Delta\lambda}{\lambda_2}\Theta_c\right][D_2(\Theta_c) - \sqrt{D_2^2(\Theta_c) - 1}]\frac{\xi}{\eta} \\ T_{20} &= \left[1 - \frac{\Delta\lambda}{\lambda_2}\Theta_c\right][D_2(\Theta_c) - \sqrt{D_2^2(\Theta_c) - 1}]\frac{\xi}{\eta} + \frac{\Delta\lambda}{\lambda_2}\Theta_c \end{aligned}\right\} \bar{\xi}_b \in]0, \eta[. \tag{28}$$

In conclusion let us clarify the conditions of states (27), (28) realization, specifically whether both states are possible with the same values of DS parameters. In other words, is this system bistable or not? The condition of bistability (see Fig. 2) is formulated as follows

$$C_2\eta > \Theta_c > C_1\eta.$$

The first inequality $C_2\eta > \Theta_c$ easily transforms in

$$\frac{\lambda_2}{\lambda_1}\left[1 - \frac{\Delta\lambda}{\lambda_2}\Theta_c\right][D_2(\Theta_c) - \sqrt{D_2^2(\Theta_c) - 1}] > \Theta_c.$$

Taking into account that

$$D_2(\Theta_c) = 1 + \frac{D-1}{1-(\Delta\lambda/\lambda_2)\Theta_c},$$

after necessary computations we obtain

$$\frac{1}{D-1} > 2\frac{\lambda_1}{\lambda_2}\frac{\Theta_c}{(1-\Theta_c)^2}.$$

The last inequality

$$C_1\eta < \Theta_c \quad \text{or} \quad D_1 - \sqrt{D_1^2 - 1} < \Theta_c$$

after substitution of $D_1 = 1 + (\lambda_1/\lambda_2)(D-1)$ yields

$$\frac{1}{D-1} < 2\frac{\lambda_1}{\lambda_2}\frac{\Theta_c}{(1-\Theta_c)^2}.$$

Consequently conditions $C_2\eta > \Theta_c$ and $C_1\eta < \Theta_c$ are incompatible and with prescribed D the system has the single stable state.

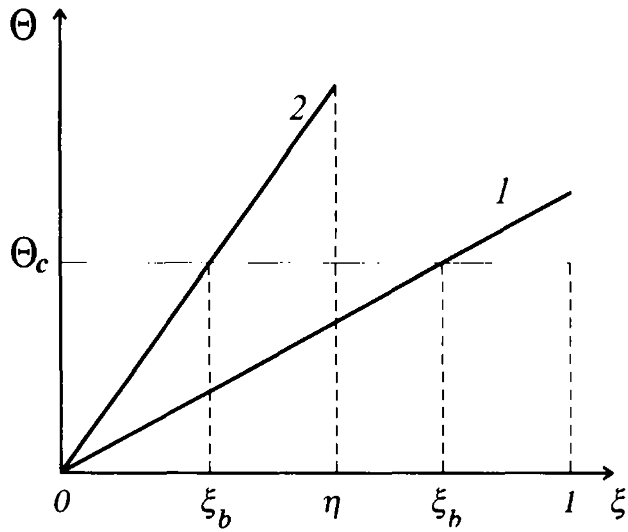

Fig. 2. Illustration of bistability condition $C_2\eta > \Theta_c > C_1\eta$. Lines 1, 2 are the graphs of stationary temperatures $T_{10}(\xi) = C_1\xi$ and $T_{l0}(\xi) = C_2\xi$, respectively. The system is bistable if both states are valid with the same values of variable parameters.

4. LINEAR ANALYSIS

To solve the problem (15) we make use of the Fourier method and try solutions in the form

$$\begin{aligned} T_{11}(\xi, \tau) &= V(\xi)\exp(\mathrm{i}\omega\tau), \\ T_{21}(\xi, \tau) &= W(\xi)\exp(\mathrm{i}\omega\tau). \end{aligned} \tag{29}$$

The spatial parts of solutions (29) are the eigenfunctions of the operator

$$\begin{aligned} V''(\xi) &= \frac{\mathrm{i}\omega}{a_1}\delta^2 V(\xi), \\ W''(\xi) &= \frac{\mathrm{i}\omega}{a_2}\delta^2 W(\xi), \\ V(0) &= 0, \\ C(\bar{\xi}_b)\xi_1(\tau) + V(\bar{\xi}_b)\exp(\mathrm{i}\omega\tau) &= 0, \\ \frac{\lambda_1}{\lambda_2}C(\bar{\xi}_b)\xi_1(\tau) + W(\bar{\xi}_b)\exp(\mathrm{i}\omega\tau) &= 0, \\ \frac{\lambda_2}{\lambda_1}W'(\xi)|_{\xi=\xi_b} &= V'(\xi)|_{\xi=\xi_b} - \frac{\delta^2 Q}{a_1 c_1 \beta}\dot{\xi}_1(\tau)\exp(-\mathrm{i}\omega\tau), \\ W'(\xi)|_{\xi=1} &= \frac{A_{c1}}{\eta}V(\eta), \qquad \xi_b \in\,]\eta, 1[, \\ W'(\xi)|_{\xi=1} &= \frac{A_{c2}}{\eta}W(\eta), \qquad \xi_b \in\,]0, \eta[, \end{aligned} \tag{30}$$

where $C(\bar{\xi}_b) = C_1 = (D_1 - \sqrt{D_1^2 - 1})/\eta$ with $\bar{\xi}_b \in]\eta, 1[$ and $C(\bar{\xi}_b) = C_2 = (\lambda_2/\lambda_1) \times [1 - (\Delta\lambda/\lambda_2)\Theta_c][D_2(\Theta_c) - \sqrt{D_2^2(\Theta_c) - 1}]/\eta$ with $\bar{\xi}_b \in]0, \eta[$ the stationary temperature gradient in phase *1*. The oscillating component of interface coordinate

$$\xi_1 = -\frac{V(\bar{\xi}_b)}{C(\bar{\xi}_b)}\exp(\mathrm{i}\omega\tau)$$
$$\xi_1 = -\frac{\lambda_2}{\lambda_1}\frac{W(\bar{\xi}_b)}{C(\bar{\xi}_b)}\exp(\mathrm{i}\omega\tau) \tag{31}$$

and its velocity

$$\dot{\xi}_1 = -\mathrm{i}\omega\frac{V(\bar{\xi}_b)}{C(\bar{\xi}_b)}\exp(\mathrm{i}\omega\tau)$$
$$\dot{\xi}_1 = -\mathrm{i}\omega\frac{\lambda_2}{\lambda_1}\frac{W(\bar{\xi}_b)}{C(\bar{\xi}_b)}\exp(\mathrm{i}\omega\tau) \tag{32}$$

are obtained from the continuity condition. Substituting (32) for $\dot{\xi}_1(\tau)$ in Stefan condition of operator (30), eliminating ξ_1 from continuity condition and introducing notations $\nu_1^* = (1 + \mathrm{i})\nu_1 = (1 + \mathrm{i})k_1\delta$, $\nu_2^* = (1 + \mathrm{i})\nu_2 = (1 + \mathrm{i})k_2\delta$ where $k_1 = \sqrt{\omega/2a_1}$, $k_2 = \sqrt{\omega/2a_2}$ we arrive at the nonlocal Sturm–Liouville problem

$$\begin{aligned}
V''(\xi) &= \nu_1^{*2}V(\xi),\\
W''(\xi) &= \nu_2^{*2}W(\xi),\\
V(0) &= 0,\\
\lambda_1 V(\bar{\xi}_b) &= \lambda_2 W(\bar{\xi}_b),\\
\frac{\lambda_2}{\lambda_1}W'(\xi)|_{\xi=\bar{\xi}_b} &= V'(\xi)|_{\xi=\bar{\xi}_b} + 2\mathrm{i}\frac{\omega\delta^2}{2a_1}\frac{Q}{\beta c_1}\frac{V(\bar{\xi}_b)}{C(\bar{\xi}_b)},\\
W'(\xi)|_{\xi=1} &= \frac{A_{c1}}{\eta}V(\eta), \qquad \bar{\xi}_b \in]\eta, 1[,\\
W'(\xi)|_{\xi=1} &= \frac{A_{c2}}{\eta}W(\eta), \qquad \bar{\xi}_b \in]0, \eta[,
\end{aligned} \tag{33}$$

If temperature gradient in phase *1* is expressed through normalized critical temperature $\Theta_c = T_c/\beta$ and interface coordinate $\bar{\xi}_b$ as $C(\bar{\xi}_b) = \Theta_c/\bar{\xi}_b$ the Stefan condition, taking into account $2\mathrm{i}\omega\delta^2/2a_1 = \nu_1^{*2}$, can be transformed into

$$\frac{\lambda_2}{\lambda_1}W'(\xi)|_{\xi=\xi_b} = V'(\xi)|_{\xi=\xi_b} + \nu_1^{*2}\frac{\bar{\xi}_b}{Sf}V(\bar{\xi}_b),$$

where $Sf = c_1T_c/Q$ is the Stefan number.

The solutions of problem (33), satisfying zero boundary condition, are as follows

$$V(\xi) = 2C_5\sinh(\nu_1^*\xi),$$
$$W(\xi) = C_3\exp(\nu_2^*\xi) + C_4\exp(-\nu_2^*\xi).$$

Parameters k_1 and k_2 make sense of the wavenumbers of temperature waves propagating in both phases, while $\nu_1^* \xi = (1 + \mathrm{i})k_1 x$ and $\nu_2^* \xi = (1 + \mathrm{i})k_2 x$ are complex phases of temperature oscillations in x. Unknown constants C_3, C_4 are determined from continuity and Stefan conditions

$$\begin{aligned} C_3 \exp(\nu_2^* \bar{\xi}_b) + C_4 \exp(-\nu_2^* \bar{\xi}_b) &= \frac{\lambda_1}{\lambda_2} 2C_5 \sinh(\nu_1^* \bar{\xi}_b) \\ C_3 \exp(\nu_2^* \bar{\xi}_b) - C_4 \exp(-\nu_2^* \bar{\xi}_b) &= \frac{\lambda_1}{\lambda_2} 2C_5 \left[\frac{\nu_1^*}{\nu_2^*} \cosh(\nu_1^* \bar{\xi}_b) + \frac{\nu_1^{*2}}{\nu_2^*} \frac{\bar{\xi}_b}{Sf} \sinh(\nu_1^* \bar{\xi}_b) \right]. \end{aligned} \tag{34}$$

Solving system (34)

$$\begin{aligned} C_3 &= C_5 \frac{\lambda_1}{\lambda_2} \left[\left(1 + \frac{\nu_1^{*2}}{\nu_2^*} \frac{\bar{\xi}_b}{Sf}\right) \sinh(\nu_1^* \bar{\xi}_b) + \frac{\nu_1^*}{\nu_2^*} \cosh(\nu_1^* \bar{\xi}_b) \right] \exp(-\nu_2^* \bar{\xi}_b), \\ C_4 &= C_5 \frac{\lambda_1}{\lambda_2} \left[\left(1 - \frac{\nu_1^{*2}}{\nu_2^*} \frac{\bar{\xi}_b}{Sf}\right) \sinh(\nu_1^* \bar{\xi}_b) - \frac{\nu_1^*}{\nu_2^*} \cosh(\nu_1^* \bar{\xi}_b) \right] \exp(\nu_2^* \bar{\xi}_b) \end{aligned} \tag{35}$$

and substituting (35) in feedback conditions we obtain two equations for determination of temperature wave complex amplitude

$$\begin{aligned} &\frac{\lambda_1}{\lambda_2} \Bigg[\nu_1^* \cosh(\nu_1^* \bar{\xi}_b) \cosh[\nu_2^*(1 - \bar{\xi}_b)] + \nu_2^* \sinh(\nu_1^* \bar{\xi}_b) \sinh[\nu_2^*(1 - \bar{\xi}_b)] \\ &\qquad + \nu_1^{*2} \frac{\bar{\xi}_b}{Sf} \sinh(\nu_1^* \bar{\xi}_b) \cosh[\nu_2^*(1 - \bar{\xi}_b)] \Bigg] = \frac{A_{c1}}{\eta} \sinh(\nu_1^* \eta), \qquad \bar{\xi}_b \in]\eta, 1[, \\ &\nu_1^* \cosh(\nu_1^* \bar{\xi}_b) \cosh[\nu_2^*(1 - \bar{\xi}_b)] + \nu_2^* \sinh(\nu_1^* \bar{\xi}_b) \sinh[\nu_2^*(1 - \bar{\xi}_b)] \\ &\qquad + \nu_1^{*2} \frac{\bar{\xi}_b}{Sf} \sinh(\nu_1^* \bar{\xi}_b) \cosh[\nu_2^*(1 - \bar{\xi}_b)] \\ &= \frac{A_{c2}}{\nu_2^*} \Bigg[\frac{\nu_1^*}{\nu_2^*} \cosh(\nu_1^* \bar{\xi}_b) \sinh[\nu_2^*(\eta - \bar{\xi}_b)] + \sinh(\nu_1^* \bar{\xi}_b) \cosh[\nu_2^*(\eta - \bar{\xi}_b)] \\ &\qquad + \frac{\nu_1^{*2}}{\nu_2^*} \frac{\bar{\xi}_b}{Sf} \sinh(\nu_1^* \bar{\xi}_b) \sinh[\nu_2^*(\eta - \bar{\xi}_b)] \Bigg], \qquad \bar{\xi}_b \in]0, \eta[. \end{aligned} \tag{36}$$

Transforming the products of hyperbolic functions in (36), introducing notations $\nu_1 + \nu_2 = (\nu_1^* + \nu_2^*)/(1 + \mathrm{i}) = \nu$, $\nu_1 - \nu_2 = (\nu_1^* - \nu_2^*)/(1 + \mathrm{i}) = \Delta\nu$, $(\nu_1^* - \nu_2^*)\bar{\xi}_b + \nu_2^* = (1 + \mathrm{i})\varphi_1$, $(\nu_1^* + \nu_2^*)\bar{\xi}_b - \nu_2^* = (1 + \mathrm{i})\psi_1$, $(\nu_1^* - \nu_2^*)\bar{\xi}_b + \nu_2^* \eta = (1 + \mathrm{i})\varphi_2$, $(\nu_1^* + \nu_2^*)\bar{\xi}_b - \nu_2^* \eta = (1 + \mathrm{i})\psi_2$ and separating in the resultant equation real and imaginary terms we get

$$\begin{aligned} &\nu(\cosh\varphi_1 \cos\varphi_1 - \sinh\varphi_1 \sin\varphi_1) + \Delta\nu(\cosh\psi_1 \cos\psi_1 - \sinh\psi_1 \sin\psi_1) \\ &\qquad - 2\nu_1^2 \frac{\bar{\xi}_b}{Sf} (\cosh\varphi_1 \sin\varphi_1 + \cosh\psi_1 \sin\psi_1) = 2 \frac{A_{c1}}{\eta} \frac{\lambda_2}{\lambda_1} \sinh\nu_1 \eta \cos\nu_1 \eta, \\ &\nu(\cosh\varphi_1 \cos\varphi_1 + \sinh\varphi_1 \sin\varphi_1) + \Delta\nu(\cosh\psi_1 \cos\psi_1 + \sinh\psi_1 \sin\psi_1) \\ &\qquad + 2\nu_1^2 \frac{\bar{\xi}_b}{Sf} (\sinh\varphi_1 \cos\varphi_1 + \sinh\psi_1 \cos\psi_1) = \frac{2A_{c1}}{\eta} \frac{\lambda_2}{\lambda_1} \cosh\nu_1 \eta \sin\nu_1 \eta \end{aligned}$$

with $\bar{\xi}_b \in]\eta, 1[$,

$$
\begin{aligned}
&\nu(\cosh\varphi_1\cos\varphi_1 - \sinh\varphi_1\sin\varphi_1) + \Delta\nu(\cosh\psi_1\cos\psi_1 - \sinh\psi_1\sin\psi_1)\\
&\quad - 2\nu_1^2\frac{\bar{\xi}_b}{Sf}(\cosh\varphi_1\sin\varphi_1 + \cosh\psi_1\sin\psi_1)\\
&\quad = \frac{A_{c2}}{\eta\nu_2}\Bigg[\nu\sinh\varphi_2\cos\varphi_2 - \Delta\nu\sinh\psi_2\cos\psi_2 + \nu_1^2\frac{\bar{\xi}_b}{Sf}\\
&\quad \times(\cosh\varphi_2\cos\varphi_2 - \sinh\varphi_2\sin\varphi_2 - \cosh\psi_2\cos\psi_2 + \sinh\psi_2\sin\psi_2)\Bigg],\\
&\nu(\cosh\varphi_1\cos\varphi_1 + \sinh\varphi_1\sin\varphi_1) + \Delta\nu(\cosh\psi_1\cos\psi_1 + \sinh\psi_1\sin\psi_1)\\
&\quad + 2\nu_1^2\frac{\bar{\xi}_b}{Sf}(\sinh\varphi_1\cos\varphi_1 + \sinh\psi_1\cos\psi_1)\\
&\quad = \frac{A_{c2}}{\eta\nu_2}\Bigg[\nu\cosh\varphi_2\sin\varphi_2 - \Delta\nu\cosh\psi_2\sin\psi_2 + \nu_1^2\frac{\bar{\xi}_b}{Sf}\\
&\quad \times(\cosh\varphi_2\cos\varphi_2 + \sinh\varphi_2\sin\varphi_2 - \cosh\psi_2\cos\psi_2 - \sinh\psi_2\sin\psi_2)\Bigg]
\end{aligned}
\tag{37}
$$

with $\bar{\xi}_b \in]0, \eta[$.

Equations (37) express the conditions the temperature wave complex amplitude should obey. The phase condition can be deduced from (37) by elimination of A in each pair of equations

$$
\frac{\begin{aligned}&\nu(\cosh\varphi_1\cos\varphi_1 - \sinh\varphi_1\sin\varphi_1) + \Delta\nu(\cosh\psi_1\cos\psi_1 - \sinh\psi_1\sin\psi_1)\\ &\quad - 2\nu_1^2\frac{\bar{\xi}_b}{Sf}(\cosh\varphi_1\sin\varphi_1 + \cosh\psi_1\sin\psi_1)\end{aligned}}{\begin{aligned}&\nu(\cosh\varphi_1\cos\varphi_1 + \sinh\varphi_1\sin\varphi_1) + \Delta\nu(\cosh\psi_1\cos\psi_1 + \sinh\psi_1\sin\psi_1)\\ &\quad + 2\nu_1^2\frac{\bar{\xi}_b}{Sf}(\sinh\varphi_1\cos\varphi_1 + \sinh\psi_1\cos\psi_1)\end{aligned}} = \tanh\eta\nu_1\cot\eta\nu_1,
\tag{38}
$$

with $\bar{\xi}_b \in]\eta, 1[$ and

$$
\begin{aligned}
&\frac{\begin{aligned}&\nu(\cosh\varphi_1\cos\varphi_1 - \sinh\varphi_1\sin\varphi_1) + \Delta\nu(\cosh\psi_1\cos\psi_1 - \sinh\psi_1\sin\psi_1)\\ &\quad - 2\nu_1^2\frac{\bar{\xi}_b}{Sf}(\cosh\varphi_1\sin\varphi_1 + \cosh\psi_1\sin\psi_1)\end{aligned}}{\begin{aligned}&\nu(\cosh\varphi_1\cos\varphi_1 + \sinh\varphi_1\sin\varphi_1) + \Delta\nu(\cosh\psi_1\cos\psi_1 + \sinh\psi_1\sin\psi_1)\\ &\quad + 2\nu_1^2\frac{\bar{\xi}_b}{Sf}(\sinh\varphi_1\cos\varphi_1 + \sinh\psi_1\cos\psi_1)\end{aligned}}\\
&= \frac{\begin{aligned}&\nu\sinh\varphi_2\cos\varphi_2 - \Delta\nu\sinh\psi_2\cos\psi_2 + \nu_1^2\frac{\bar{\xi}_b}{Sf}\\ &\quad \times(\cosh\varphi_2\cos\varphi_2 - \sinh\varphi_2\sin\varphi_2 - \cosh\psi_2\cos\psi_2 + \sinh\psi_2\sin\psi_2)\end{aligned}}{\begin{aligned}&\nu\cosh\varphi_2\sin\varphi_2 - \Delta\nu\cosh\psi_2\sin\psi_2 + \nu_1^2\frac{\bar{\xi}_b}{Sf}\\ &\quad \times(\cosh\varphi_2\cos\varphi_2 + \sinh\varphi_2\sin\varphi_2 - \cosh\psi_2\cos\psi_2 - \sinh\psi_2\sin\psi_2)\end{aligned}},
\end{aligned}
\tag{39}
$$

with $\bar{\xi}_b \in]0, \eta[$.

A certain complexity of equations (38), (39), if compared with the analogous expressions obtained in [1], proceeds from additional shift of the temperature wave phase by interface oscillation. Both conditions select the frequencies with which temperature oscillation in $\xi = \eta$ is out of phase by π from that of the heat flux in $\xi = 1$, while the rest π falls on the feedback signal inversion by the controller. The total phase delay of feedback signal in medium is formed by temperature phase delay from power oscillations in $\xi = 1$, phase difference of temperature oscillation in $\xi = 1$ and $\xi = \bar{\xi}_b + |\zeta\xi_1(\tau)|$, the phase shift produced through interface motion in the span $[\bar{\xi}_b - |\zeta\xi_1(\tau)|, \bar{\xi}_b + |\zeta\xi_1(\tau)|]$ and finally the difference of temperature phases in $\bar{\xi}_b - |\zeta\xi_1(\tau)|$ and $\xi = \eta$. Taking into account that in each phase temperature wave is a superposition of two waves running in opposite directions,† equations (38), (39) do not seem to be too complicated.

The question which naturally arises on (38), (39) deduction is their roots existence. The statement that there is at least one root with which $A < 0$ (i.e. providing that the feedback is the reverse one) for the general case is hardly provable. But for a particular case it can be easily verified by substitution of some real system parameters to equations (38), (39). This procedure reveals quite verisimilar and explicable dynamic system behavior with $\bar{\xi}_b$ variation. Namely the substitution of equation (38) roots in the first of conditions (37) yields the A_{c1} values which are too high to be achievable for real systems, hence with $\bar{\xi}_b \in]\eta, 1[$ the system remains stable. This result is more evident for the equivalent thermal scheme of two two-phase system, where the interface should be replaced by reactive thermal resistance bypassing the segment between the interface and thermostat which is a ground equivalent. Consequently, interface displacement behind the thermocouple must raise the system quality factor and lower A critical value. Indeed, equations (37)–(39) investigation reveals that with $\bar{\xi}_b = \eta$ parameter A_c suffers the break and with $\bar{\xi}_b < \eta$ its value is less than the corresponding parameter of the homogeneous system completely composed of phase 2.

The indirect evidence of (38), (39) validity is their asymptotic behavior with $a_1 \to a_2$, $\lambda_1 \to \lambda_2$, $Q \to 0$ when they convert into phase condition

$$\frac{\cosh \nu_2 \cos \nu_2 - \sinh \nu_2 \sin \nu_2}{\sinh \eta\nu_2 \cos \eta\nu_2} = \frac{\cosh \nu_2 \cos \nu_2 + \sinh \nu_2 \sin \nu_2}{\cosh \eta\nu_2 \sin \eta\nu_2}, \tag{40}$$

obtained in literature [1] for the problem which is the particular case of the present one. The second-order phase transition ($1/Sf = 0$) as well as auto-oscillations in a two-layered system ($1/Sf = 0$, $\bar{\xi}_b = \text{const}$) are also particular cases of the problem under consideration.

Phase conditions (38), (39) close the chain of equations for temperature field and system parameters calculation thus completing the solution of the problem in linear approximation. Indeed, provided that parameters $\bar{\xi}_b$, η, Sf, a_1, a_2 are known values, equations (38), (39) and relation

$$\nu_n = \sqrt{\frac{\omega}{2a_n}}\,\delta, \qquad n = 1, 2 \tag{41}$$

determine the auto-oscillation frequency. Substitution of (41) in one of the expressions (37) yields the critical value of the generalized gain factor A_{c1} or A_{c2} depending on

† The wave $\exp i(\omega t + k_2 x)$ generated by the heater in phase *2* is superimposed by the reverse wave $\exp i(\omega t - k_2 x)$ produced through interface oscillations. In phase *1* the waves reverse wave $\exp i(\omega t - k_1 x)$ appears due to direct wave reflection from the thermostatic surface $\xi = 0$.

interface position. In its turn A_{cn} fixes phase l temperature gradient C_n (17), while C_n and $\bar{\xi}_b$ unambiguously define Θ_c. Now as three parameters Θ_c, T_c, and α in relation

$$\Theta_c = T_c/\beta = \alpha T_c/u_0$$

are known, it is possible to determine u_0. At last from relations

$$A_{c1} = -2\eta\kappa_{c1}[1 - T_{10}(\eta)] = -2\eta\kappa_{c1}[1 - C_1\eta], \qquad \bar{\xi}_b \in]\eta, 1[$$

$$A_{c2} = -2\eta\kappa_{c2}[1 - T_{20}(\eta)] = -2\eta\kappa_{c2}\left[1 - \Theta_c - \frac{\lambda_1}{\lambda_2} C_2(\eta - \bar{\xi}_b)\right], \qquad \bar{\xi}_b \in]0, \eta[$$

[see equations (14)] the critical value of amplifier gain is obtained as

$$K_c = \sqrt{\frac{\kappa_{cn}\lambda_2 SR}{\alpha u_0 \delta}}.$$

The resulting K_c and u_0 will be just these values of adjustable parameters with which the DS loss of stability and auto-oscillations excitation takes place. By K and u_0 tuning an auto-oscillation can be excited with any desired interface position.

5. HIGHER APPROXIMATIONS

The asymptotic method is applicable only when the auto-oscillation excitation mode is soft, or conformable to the present case when local temperature oscilation amplitude tends to zero with $\varepsilon \to 0$. Equation (12) relating auto-oscillation amplitude to supercriticity implies that soft excitation is possible only when the Lyapunov coefficient b_2 is a positive value. The latter is calculated in accordance with that of Andronov–Hopf bifurcation theorem formalism from resolvability condition for the forth problem of recurrent sequence. Thus to elucidate the mode the auto-oscillations are excited the higher approximations analysis is required. Though the analysis of higher approximations by analytic methods is hardly feasible by the reason of cumbrous computations, it is necessary at least to formulate the problem for investigation.

The deduction of boundary value problems for higher approximations comes to substitution of series (13) in (10) followed by grouping of the terms in accordance with ζ power. This procedure towards equations and boundary conditions free of $\bar{\xi}_b$ parameter presents no difficulty, but with respect to the Stefan and continuity conditions should be considered in more details. Let us start with the continuity condition, which regarding (13), can be written as

$$\begin{aligned} &T_{10}\left[\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau)\right] + \zeta T_{11}\left\{\left[\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau)\right], \tau\right\} \\ &\quad + \zeta^2 T_{12}\left\{\left[\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau)\right], \tau\right\} + \zeta^3 T_{13}\left\{\left[\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau)\right], \tau\right\} + \cdots \\ &= T_{20}\left[\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau)\right] + \zeta T_{21}\left\{\left[\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau)\right], \tau\right\} \\ &\quad + \zeta^2 T_{22}\left\{\left[\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau)\right], \tau\right\} + \zeta^3 T_{23}\left\{\left[\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau)\right], \tau\right\} + \cdots = \Theta_c. \end{aligned}$$

As with $\varepsilon \ll 1$ the interface oscillation amplitude is small, variables T_{1i}, T_{2i} can be expended in power series of $\xi_b(\tau) - \bar{\xi}_b = \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau)$

$$\begin{aligned}
&\left[T_{10}(\bar{\xi}_b) + T'_{10}(\bar{\xi}_b) \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) + \cdots \right] \\
&\quad + \left[\zeta T_{11}(\bar{\xi}_b, \tau) + \zeta T'_{11}(\bar{\xi}_b, \tau) \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) + \frac{\zeta}{2} T''_{11}(\bar{\xi}_b, \tau) \left(\sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) \right)^2 + \cdots \right] \\
&\quad + \left[\zeta^2 T_{12}(\bar{\xi}_b, \tau) + \zeta^2 T'_{12}(\bar{\xi}_b, \tau) \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) + \cdots \right] + [\zeta^3 T_{13}(\bar{\xi}_b, \tau) + \cdots] \\
&= \left[T_{20}(\bar{\xi}_b) + T'_{20}(\bar{\xi}_b) \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) + \cdots \right] \\
&\quad + \left[\zeta T_{21}(\bar{\xi}_b, \tau) + \zeta T'_{21}(\bar{\xi}_b, \tau) \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) + \frac{\zeta}{2} T''_{21}(\bar{\xi}_b, \tau) \left(\sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) \right)^2 + \cdots \right] \\
&\quad + \left[\zeta^2 T_{22}(\bar{\xi}_b, \tau) + \zeta^2 T'_{22}(\bar{\xi}_b, \tau) \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) + \cdots \right] + [\zeta^3 T_{23}(\bar{\xi}_b, \tau) + \cdots] = \Theta_c .
\end{aligned}$$

Taking into account that $T_{10}(\bar{\xi}_b) = T_{10}(\bar{\xi}_b) = \Theta_c$, $T''_{10} = T''_{20} = 0$ and grouping the terms attached to ξ^2, ξ^3 we obtain the continuity condition for the third and the fourth problems of sequence

$$\begin{gathered}
T'_{10}(\bar{\xi}_b)\xi_2(\tau) + T'_{11}(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{12}(\bar{\xi}_b, \tau) = 0, \\
T'_{20}(\bar{\xi}_b)\xi_2(\tau) + T'_{21}(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{22}(\bar{\xi}_b, \tau) = 0, \\
T'_{10}(\bar{\xi}_b)\xi_3(\tau) + T'_{11}(\bar{\xi}_b, \tau)\xi_2(\tau) + T''_{11}(\bar{\xi}_b, \tau)\frac{\xi_1^2(\tau)}{2} + T'_{12}(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{13}(\bar{\xi}_b, \tau) = 0, \\
T'_{20}(\bar{\xi}_b)\xi_3(\tau) + T'_{21}(\bar{\xi}_b, \tau)\xi_2(\tau) + T''_{21}(\bar{\xi}_b, \tau)\frac{\xi_1^2(\tau)}{2} + T'_{22}(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{23}(\bar{\xi}_b, \tau) = 0.
\end{gathered} \tag{42}$$

The terms of the Stefan condition

$$\begin{aligned}
&\frac{\lambda_2}{\lambda_1}(1 + c_2\zeta^2 + \cdots)\left\{ T'_{20}\left[\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) \right] + \zeta T'_{21}\left[\left(\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) \right), \tau \right] \right. \\
&\quad \left. + \zeta^2 T'_{22}\left[\left(\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) \right), \tau \right] + \zeta^3 T'_{23}\left[\left(\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) \right), \tau \right] + \cdots \right\} \\
&= (1 + c_2\xi^2 + \cdots)\left\{ T'_{10}\left[\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) \right] + \zeta T'_{11}\left[\left(\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) \right), \tau \right] \right. \\
&\quad \left. + \zeta^2 T'_{12}\left[\left(\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) \right), \tau \right] + \zeta^3 T'_{13}\left[\left(\bar{\xi}_b + \sum_{n=1}^{\infty} \zeta^n \xi_n(\tau) \right), \tau \right] + \cdots \right\} \\
&\quad - \frac{\delta^2 Q}{a_1 \beta c_1}[\zeta\dot{\xi}_1(\tau) + \zeta^2\dot{\xi}_2(\tau) + \zeta^3\dot{\xi}_3(\tau) + \cdots]
\end{aligned}$$

should be also expended in Taylor series

$$
\begin{aligned}
&(1 + c_2\zeta^2 + \cdots)\frac{\lambda_2}{\lambda_1}\Bigg\{T'_{20}(\bar{\xi}_b) + \Bigg[\zeta T'_{21}(\bar{\xi}_b, \tau) \\
&\quad + \zeta T''_{21}(\bar{\xi}_b, \tau)\sum_{n=1}^{\infty}\zeta^n\xi_n(\tau) + \frac{1}{2}\zeta T'''_{21}(\bar{\xi}_b, \tau)\left(\sum_{n=1}^{\infty}\zeta^n\xi_n(\tau)\right)^2 + \cdots\Bigg] \\
&\quad + \Bigg[\zeta^2 T'_{22}(\bar{\xi}_b, \tau) + \zeta^2 T''_{22}(\bar{\xi}_b, \tau)\sum_{n=1}^{\infty}\zeta^n\xi_n(\tau) + \cdots\Bigg] + [\zeta^3 T'_{23}(\bar{\xi}_b, \tau) + \cdots] + \cdots\Bigg\} \\
&= (1 + c_2\zeta^2 + \cdots)\Bigg\{T'_{10}(\bar{\xi}_b) + \Bigg[\zeta T'_{11}(\bar{\xi}_b, \tau) + \zeta T''_{11}(\bar{\xi}_b, \tau)\sum_{n=1}^{\infty}\zeta^n\xi_n(\tau) + \frac{1}{2}\zeta T'''_{11}(\bar{\xi}_b, \tau) \\
&\quad \times\left(\sum_{n=1}^{\infty}\zeta^n\xi_n(\tau)\right)^2 + \cdots\Bigg] + \Bigg[\zeta^2 T'_{12}(\bar{\xi}_b, \tau) + \xi^2 T''_{12}(\bar{\xi}_b, \tau)\sum_{n=1}^{\infty}\zeta^n\xi_n(\tau) + \cdots\Bigg] \\
&\quad + [\zeta^3 T'_{13}(\bar{\xi}_b, \tau) + \cdots] + \cdots\Bigg\} - \frac{\delta^2 Q}{a_1\beta c_1}[\zeta\dot{\xi}_1(\tau) + \zeta^2\dot{\xi}_2(\tau) + \zeta^3\dot{\xi}_3(\tau) + \cdots].
\end{aligned}
$$

The terms with second and third ζ power gives the Stefan conditions of the third and the fourth problems

$$
\begin{aligned}
&\frac{\lambda_2}{\lambda_1}[T''_{21}(\bar{\xi}_b, \tau)\xi_1(\tau) + T'_{22}(\bar{\xi}_b, \tau) + c_2 T'_{20}(\bar{\xi}_b)] \\
&\quad = [T''_{11}(\bar{\xi}_b, \tau)\xi_1(\tau) + T'_{12}(\bar{\xi}_b, \tau) + c_2 T'_{10}(\bar{\xi}_b)] - \frac{\delta^2 Q}{a_1\beta c_1}\dot{\xi}_2(\tau), \\
&\frac{\lambda_2}{\lambda_1}\left[T''_{21}(\bar{\xi}_b, \tau)\xi_2(\tau) + \frac{1}{2}T'''_{21}(\bar{\xi}_b, \tau)\xi_1^2(\tau) + T''_{22}(\bar{\xi}_b, \tau)\xi_1(\tau) + T'_{23}(\bar{\xi}_b, \tau) + c_2 T'_{21}(\bar{\xi}_b)\right] \\
&\quad = \left[T''_{11}(\bar{\xi}_b, \tau)\xi_2(\tau) + \frac{1}{2}T'''_{11}(\bar{\xi}_b, \tau)\xi_1^2(\tau) + T''_{12}(\bar{\xi}_b, \tau)\xi_1(\tau) + T'_{13}(\bar{\xi}_b, \tau) + c_2 T'_{11}(\bar{\xi}_b)\right] \\
&\qquad - \frac{\delta^2 Q}{a_1\beta c_1}\dot{\xi}_3(\tau).
\end{aligned}
\tag{43}
$$

Conditions (42), (43), together with other conditions and equations corresponding to the order of approximation, form the third and fourth problems of recurrent sequence.

$$\dot{T}_{12} = \frac{a_1}{\delta^2}T''_{12},$$

$$\dot{T}_{22} = \frac{a_2}{\delta^2}T''_{22},$$

$$T_{12}(0, \tau) = 0,$$

$$T'_{10}(\bar{\xi}_b)\xi_2(\tau) + T'_{11}(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{12}(\bar{\xi}_b, \tau) = 0,$$

$$T'_{20}(\bar{\xi}_b)\xi_2(\tau) + T'_{21}(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{22}(\bar{\xi}_b, \tau) = 0,$$

$$\frac{\lambda_2}{\lambda_1}[T_{21}''(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{22}'(\bar{\xi}_b, \tau) + c_2 T_{20}'(\bar{\xi}_b)]$$

$$= [T_{11}''(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{12}'(\bar{\xi}_b, \tau) + c_2 T_{10}'(\bar{\xi}_b)] - \frac{\delta^2 Q}{a_1 \beta c_1}\dot{\xi}_2(\tau),$$

$$T_{22}'(\xi, \tau)|_{\xi=1} = \frac{A_{c1}}{\eta} T_{12}(\eta, \tau) + K_{c1} T_{11}^2(\eta, \tau) + K_c b_2[1 - T_{10}(\eta)]^2, \qquad \bar{\xi}_b \in]\eta, 1[,$$

$$T_{22}'(\xi, \tau)|_{\xi=1} = \frac{A_{c2}}{\eta} T_{22}(\eta, \tau) + K_{c2} T_{21}^2(\eta, \tau) + K_c b_2[1 - T_{20}(\eta)]^2, \qquad \bar{\xi}_b \in]0, \eta[;$$

$$\dot{T}_{13} = \frac{a_1}{\delta^2}[T_{13}'' + c_2 T_{11}''],$$

$$\dot{T}_{23} = \frac{a_2}{\delta^2}[T_{23}'' + c_2 T_{21}''],$$

$$T_{13}(0, \tau) = 0,$$

$$T_{10}'(\bar{\xi}_b)\xi_3(\tau) + T_{11}'(\bar{\xi}_b, \tau)\xi_2(\tau) + T_{11}''(\bar{\xi}_b, \tau)\frac{\xi_1^2(\tau)}{2} + T_{12}'(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{13}(\bar{\xi}_b, \tau) = 0,$$

$$T_{20}'(\bar{\xi}_b)\xi_3(\tau) + T_{21}'(\bar{\xi}_b, \tau)\xi_2(\tau) + T_{21}''(\bar{\xi}_b, \tau)\frac{\xi_1^2(\tau)}{2} + T_{22}'(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{23}(\bar{\xi}_b, \tau) = 0.$$

$$\frac{\lambda_2}{\lambda_1}\left[T_{21}''(\bar{\xi}_b, \tau)\xi_2(\tau) + \frac{1}{2}T_{21}'''(\bar{\xi}_b, \tau)\xi_1^2(\tau) + T_{22}''(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{23}'(\bar{\xi}_b, \tau) + c_2 T_{21}'(\bar{\xi}_b)\right]$$

$$= \left[T_{11}''(\bar{\xi}_b, \tau)\xi_2(\tau) + \frac{1}{2}T_{11}'''(\bar{\xi}_b, \tau)\xi_1^2(\tau) + T_{12}''(\bar{\xi}_b, \tau)\xi_1(\tau) + T_{13}'(\bar{\xi}_b, \tau) + c_2 T_{11}'(\bar{\xi}_b)\right]$$

$$- \frac{\delta^2 Q}{a_1 \beta c_1}\dot{\xi}_3(\tau),$$

$$T_{23}'(\xi, \tau)|_{\xi=1} = \frac{A_{c1}}{\eta} T_{13}(\eta, \tau) + K_{c1} 2T_{11}(\eta, \tau)T_{12}(\eta, \tau) + \frac{A_{c1}}{\eta} b_2 T_{11}(\eta, \tau), \qquad \bar{\xi}_b \in]\eta, 1[,$$

$$T_{23}'(\xi, \tau)|_{\xi=1} = \frac{A_{c2}}{\eta} T_{23}(\eta, \tau) + \kappa_{c2} 2T_{21}(\eta, \tau)T_{22}(\eta, \tau) + \frac{A_{c2}}{\eta} b_2 T_{21}(\eta, \tau), \qquad \bar{\xi}_b \in]0, \eta[.$$

Solving these and subsequent problems against unknown $T_{ni}(\eta, \tau)$ one can obtain the solution of the entire problem with any desired accuracy. In practice the application of this algorithm comes across considerable computation difficulties. This problem presents just that very limit of complexity beyond which the utilization of analytical methods is no longer expedient and numerical simulation is required. The latter can be carried out either from the very beginning of the problem, or starting with its higher approximations. Because of their completeness and clarity of causal relationships the preference was given to analytical methods of DS description, though expressions necessary for numerical investigation are also provided.

6. SUMMARY

During recent years the application of the Andronov-Hopf bifurcation theorem has been spread over the wide class of evolutional systems such as equations with delayed argument, parabolic systems of relaxation-diffusive type and hyperbolic equations. From the mathematical point of view the main result of the present paper consists in extension of the sphere of the Andronov-Hopf theorem application by the new class of problems. Besides that, the work may prove to be useful for the purposes of practical application as it provides theoretical grounds for interface control and thermal properties measuring methods. In particular it may be the methods for thermal calorimetry, as this work's results promise considerable extension of their resolvability.

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