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Coupled spring equations

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Coupled spring equations for modelling the motion of two springs with weights attached, hung in series from the ceiling are described. For the linear model using Hooke's Law, the motion of each weight is described by a fourth-order linear differential equation. A nonlinear model is also described and damping and external forcing are considered. The model has many features that permit the meaningful introduction of many concepts including: accuracy of numerical algorithms, dependence on parameters and initial conditions, phase and synchronization, periodicity, beats, linear and nonlinear resonance, limit cycles, harmonic and subharmonic solutions. These solutions produce a wide variety of interesting motions and the model is suitable for study as a computer laboratory project in a beginning course on differential equations or as an individual or a small-group undergraduate research project.

1. Introduction

The classical syllabus for beginning differential equations is rapidly changing from emphasizing solution techniques for a variety of types of differential equations to emphasizing systems and more qualitative aspects of the theory of ordinary differential equations. In particular, there is an emphasis on nonlinear equations due largely to the wide availability of high powered numerical algorithms and almost effortless graphics capabilities that come with computer algebra systems such as *Mathematica* and *Maple*.

In this article, we investigate an old problem that appears now to be relegated to the exercises in texts, if it appears at all (see for example [1, pp. 220–221]). This is the problem of two springs and two weights attached in series, hanging from the ceiling. Under the assumption that the restoring forces behave according to Hooke's Law, this two degrees of freedom problem is modelled by a pair of coupled, second-order, linear differential equations. By differentiating and substituting one equation into the other, the motion of each weight can be shown to be determined by a linear, fourth-order differential equation. We like this example for this very reason, most models in elementary texts are only of second order. Moreover, the questions about phase now have a very nice physical interpretation;

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we can investigate when the motions of the two weights are synchronized (in phase) or opposing each other (180° out of phase). By tinkering with the spring constants, oscillatory motions can be produced that are much more interesting than those obtained from the classical single spring model. Moreover, there are other phenomena that can be investigated.

We also demonstrate, through examples, that interesting motions can arise when a slight nonlinearity is introduced in an attempt to make the restoring force more physically meaningful. In this situation, periodicity of the solutions becomes a more delicate matter. If forcing is introduced, subharmonic solutions of long periods can be found which again exhibit more interesting motions than for the classical linear case.

There is the opportunity here for many numerical and graphical investigations to be made by students. Periodicity, amplitude, phase, sensitivity to inital conditions, and many more concepts can be investigated by modifying the parameters in the model. And of course, it would not be difficult for students to derive a model for three springs and three weights (or more), and to investigate the various motions that could arise from both linear and nonlinear restoring forces.

2. The coupled spring model

The model consists of two springs and two weights. One spring, having spring constant k_1, is attached to the ceiling and a weight of mass m_1 is attached to the lower end of this spring. To this weight, a second spring is attached having spring constant k_2. To the bottom of this second spring, a weight of mass m_2 is attached and the entire system appears as illustrated in figure 1.

Allowing the system to come to rest in equilibrium, we measure the displacement of the centre of mass of each weight from equilibrium, as a function of time, and denote these measurements by $x_1(t)$ and $x_2(t)$ respectively.

2.1. *Assuming Hooke's Law*

Under the assumption of small oscillations, the restoring forces are of the form $-k_1 l_1$ and $-k_2 l_2$ where l_1 and l_2 are the elongations (or compressions) of

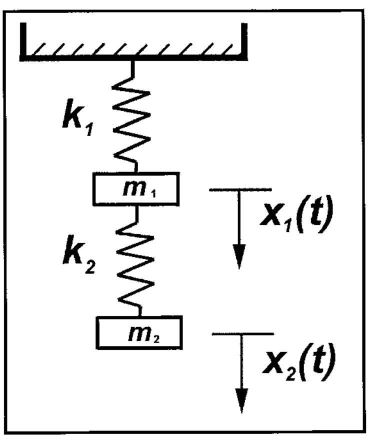


Figure 1. The coupled springs.

the two springs. Since the upper mass is attached to both springs, there are two restoring forces acting upon it: an upward restoring force $-k_1 x_1$ exerted by the elongation (or compression) x_1 of the first spring; an upward force $-k_2(x_2 - x_1)$ from the second spring's resistance to being elongated (or compressed) by the amount $x_2 - x_1$. The second mass only 'feels' the restoring force from the elongation (or compression) of the second spring. If we assume there are no damping forces present, then Newton's Law implies that the two equations representing the motions of the two weights are

$$\begin{aligned} m_1 \ddot{x}_1 &= -k_1 x_1 - k_2(x_1 - x_2) \\ m_2 \ddot{x}_2 &= -k_2(x_2 - x_1) \end{aligned} \tag{2.1}$$

Thus we have a pair of coupled second-order linear differential equations.

In order to find an equation for x_1 that doesn't involve x_2, we solve the first equation for x_2, obtaining

$$x_2 = \frac{m_1 \ddot{x}_1}{k_2} + \frac{k_1 + k_2}{k_2} x_1 \tag{2.2}$$

Substituting for x_2 in the second differential equation (2.1), and simplifying, we obtain

$$m_1 m_2 x_1^{(4)} + (m_2 k_1 + k_2(m_1 + m_2))\ddot{x}_1 + k_1 k_2 x_1 = 0 \tag{2.3}$$

Hence the motion of the first weight is determined by this fourth-order linear differential equation.

Now to find an equation that only involves x_2, we solve the second equation (2.1) for x_1:

$$x_1 = \frac{m_2}{k_2} \ddot{x}_2 + x_2 \tag{2.4}$$

and substitute into the first equation (2.1), producing the equation

$$m_1 m_2 x_2^{(4)} + (m_2 k_1 + k_2(m_1 + m_2))\ddot{x}_2 + k_1 k_2 x_2 = 0 \tag{2.5}$$

This is exactly the same fourth-order equation as that for the motion of the first weight. Thus the motions of each weight obey the same differential equation, and it is only the initial displacements and initial velocities that are needed in order to completely determine any specific case.

Typically, in a model of this sort, we would be given the initial displacements $x_1(0)$ and $x_2(0)$, and the initial velocities $\dot{x}_1(0)$ and $\dot{x}_2(0)$. But in order to solve equations (2.3) and (2.5), we need to know the values of $\ddot{x}_1(0)$, $\dddot{x}_1(0)$, $\ddot{x}_2(0)$ and $\dddot{x}_2(0)$. The values of the second derivatives are determined by evaluating equations (2.2) and (2.4) at time $t = 0$. The values of the third derivatives are obtained by evaluating the derivatives of equations (2.2) and (2.4) at time $t = 0$. Thus the motions for any set of initial conditions are determined by solving two fourth-order initial value problems.

Alternatively, we can turn this pair of coupled second-order equations into a system of four first-order equations by setting $\dot{x}_1 = u$ and $\dot{x}_2 = v$, so we have

$$
\begin{aligned}
\dot{x}_1 &= u \\
\dot{u} &= -\frac{k_1}{m_1}x_1 - \frac{k_2}{m_1}(x_1 - x_2) \\
\dot{x}_2 &= v \\
\dot{v} &= -\frac{k_2}{m_2}(x_2 - x_1)
\end{aligned}
\tag{2.6}
$$

and we need only consider the four initial conditions $x_1(0)$, $u(0)$, $x_2(0)$, and $v(0)$.

2.2. *Some examples with identical weights*

Let us consider the model having the two weights of the same mass. This model may be normalized by setting $m_1 = m_2 = 1$. In the case of no damping and no external forcing, the characteristic equation of the differential equations (2.3) and (2.5) is

$$m^4 + (k_1 + 2k_2)m^2 + k_1k_2 = 0 \tag{2.7}$$

which has the roots

$$\pm\sqrt{-\tfrac{1}{2}k_1 - k_2 \pm \tfrac{1}{2}\sqrt{k_1^2 + 4k_2^2}} \tag{2.8}$$

Example 2.1. Describe the motion for spring constants $k_1 = 6$ and $k_2 = 4$ with initial conditions $(x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)) = (1, 0, 2, 0)$.

It is easy to see that the roots of the characteristic equation are $\pm\sqrt{2}\mathrm{i}$ and $\pm 2\sqrt{3}\mathrm{i}$. Thus the general solution to equations (2.3) and (2.5) is

$$x(t) = c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t + c_3 \cos 2\sqrt{3}t + c_4 \sin 2\sqrt{3}t \tag{2.9}$$

It is also easy to compute that $(x_1(0), \dot{x}_1(0), \ddot{x}_1(0), \dddot{x}_1(0)) = (1, 0, -2, 0)$. This leads to the two pairs of simultaneous equations

$$
\begin{aligned}
c_1 + c_3 &= 1 & \sqrt{2}c_2 + 2\sqrt{3}c_4 &= 0 \\
-2c_1 - 12c_3 &= -2 & -2\sqrt{2}c_2 - 24\sqrt{3}c_4 &= 0
\end{aligned}
\tag{2.10}
$$

whose solutions are $c_1 = 1$, $c_3 = 0$, and $c_2 = c_4 = 0$. Thus the unique solution for $x_1(t)$ is

$$x_1(t) = \cos\sqrt{2}t \tag{2.11}$$

It is also easy to compute that $(x_2(0), \dot{x}_2(0), \ddot{x}_2(0), \dddot{x}_2(0)) = (2, 0, -2, 0)$. This leads to the two pairs of simultaneous equations

$$
\begin{aligned}
c_1 + c_3 &= 2 & \sqrt{2}c_2 + 2\sqrt{3}c_4 &= 0 \\
-2c_1 - 12c_3 &= -4 & -2\sqrt{2}c_2 - 24\sqrt{3}c_4 &= 0
\end{aligned}
\tag{2.12}
$$

whose solutions are $c_1 = 2$, $c_3 = 0$, and $c_2 = c_4 = 0$. The unique solution for $x_2(t)$ is

$$x_2(t) = 2\cos\sqrt{2}t \tag{2.13}$$

The motion here is synchronized and thus the weights move in phase with each other, having the same period of motion, merely having different amplitudes; this is shown in figure 2.1. Since the motion is simple periodic motion, the phase portraits for x_1 and x_2 are simple closed curves (ellipses) as shown in the left-hand frame in figure 2.2. Shown in the right-hand frame of figure 2.2 is a plot of x_1 against x_2; observe this plot is a straight line of slope 2.

The next example illustrates 180° out of phase motion.

Example 2.2. Describe the motion for spring constants $k_1 = 6$ and $k_2 = 4$ with initial conditions $(x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)) = (-2, 0, 1, 0)$.

In this example, for x_1, we have $c_1 = 0$, $c_3 = -2$, and $c_2 = c_4 = 0$, and accordingly

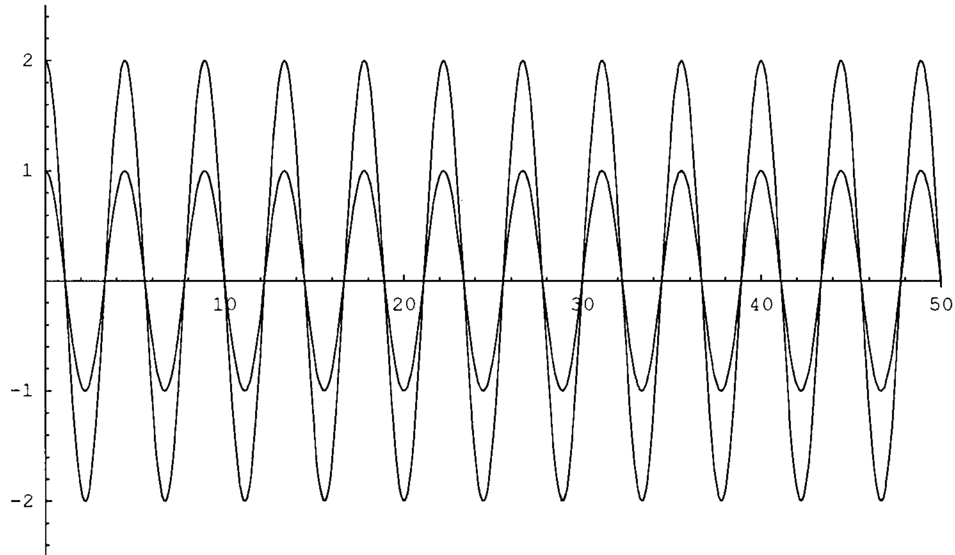


Figure 2.1. Plot of x_1 and x_2 showing synchronized motion.

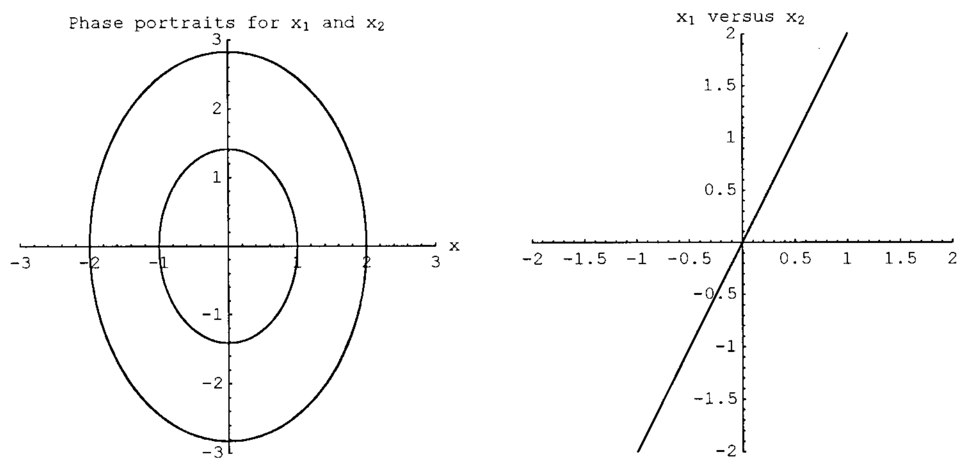


Figure 2.2. Plots for Example 2.1.

$$x_1(t) = -2\cos 2\sqrt{3}t \tag{2.14}$$

For x_2, we have $c_1 = 0$, $c_3 = 1$, and $c_2 = c_4 = 0$; thus

$$x_2(t) = \cos 2\sqrt{3}t \tag{2.15}$$

While the first weight is moving downward, the second weight is moving upward and when the first is moving upward, the second is moving downward. Again both motions have the same period, the motions are 180° out of phase. This is shown in the left-hand frame of figure 2.3. In the right-hand frame we plot x_1 against x_2 and observe a straight line of slope -1.

Generally speaking, the motions of the two weights will not be periodic, since the solutions for x_1 and x_2 are linear combinations of $\sin\sqrt{2}t$, $\cos\sqrt{2}t$, $\sin 2\sqrt{3}t$, and $\cos 2\sqrt{3}t$ and the ratio of the frequencies $\sqrt{2}$ and $2\sqrt{3}$ is not rational. For more on the period of such combinations we refer the reader to [2].

By tinkering with the parameters in this model, more interesting motions than those just described can be obtained.

Example 2.3. Describe the motion for spring constants $k_1 = 0.4$ and $k_2 = 1.808$ with initial conditions $(x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)) = (1/2, 0, -1/2, 7/10)$.

From equation (2.8), we see that it is the k_1 and k_2 values that completely determine the period and hence frequency of the response of this symmetric weight problem. Essentially the initial conditions only affect the amplitude and phase of the solutions. Thus, using a numerical solver and 'tweaking' the k values and experimenting with the initial conditions led to the choice of parameters in this example. This type of numerical and graphical interactive investigation can be carried out with a computer algebra system almost effortlessly. The phase portraits for the two solutions are shown in the top row of figure 2.4. These portraits show an interesting (almost?) period motion. Plots of the actual solutions are shown in the middle row. Plotting both solutions together on the same coordinate system also shows an interesting pattern. This is shown in the left-hand frame of the bottom row of figure 2.4. Finally, the plot of x_1 against x_2 shown in the right-hand frame of the bottom row of figure 2.4 appears to be a Lissajous type curve.

Student problem. Verify analytically the solutions to this initial value problem are periodic. Find the period. Use the analytic solution to produce other periodic motions.

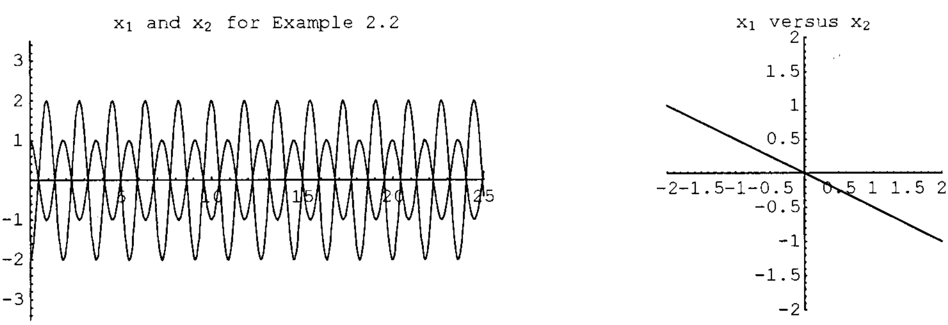


Figure 2.3. Plots for Example 2.2.

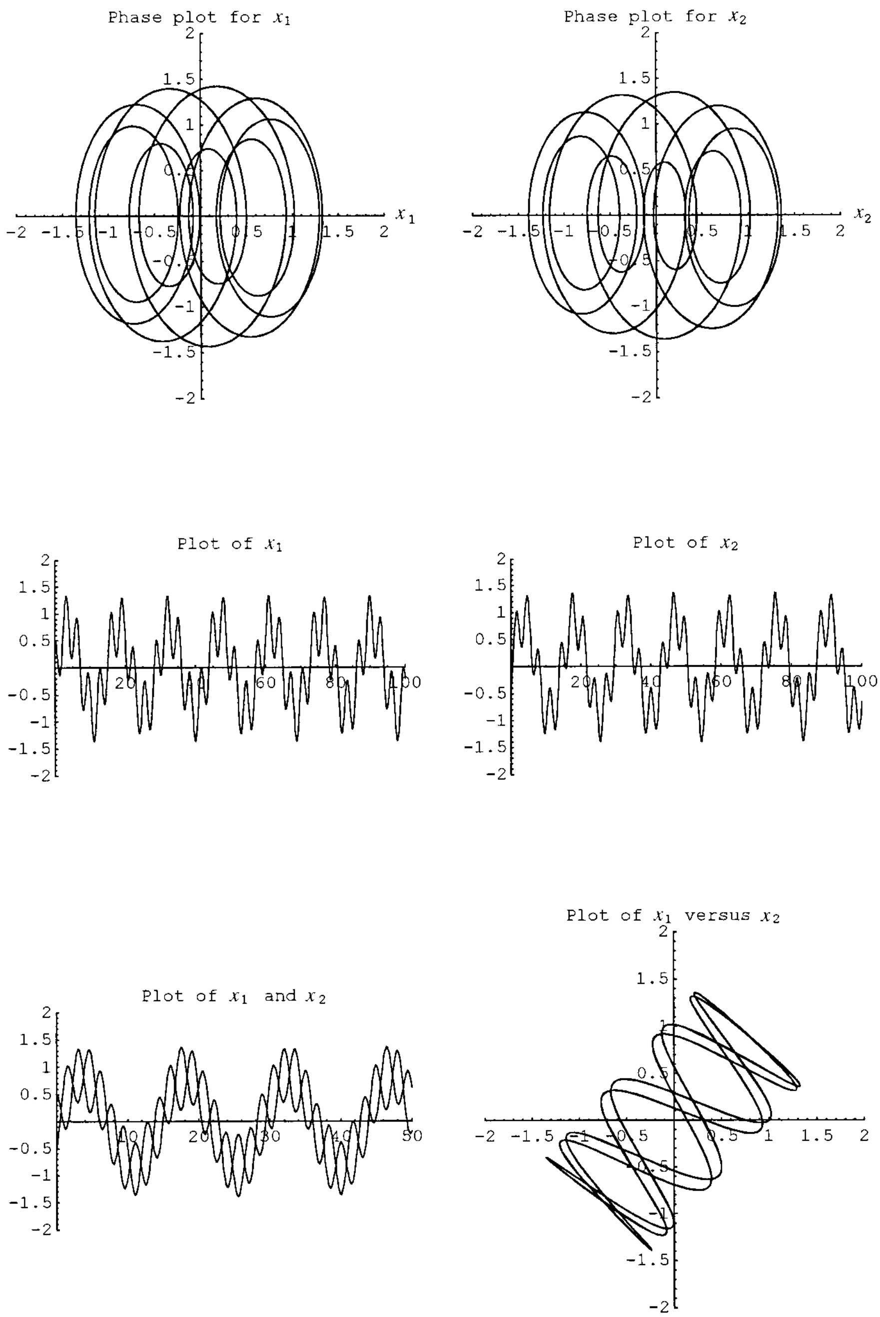


Figure 2.4. Plots for Example 2.3.

2.3. *Damping*

The most common type of damping encountered in beginning courses is that of viscous damping; the damping force is proportional to the velocity. The damping of the first weight depends solely on its velocity and not the velocity of the second weight, and vice versa. We add viscous damping to the model by adding the term

$-\delta_1\dot{x}_1$ to the first equation and $-\delta_2\dot{x}_2$ to the second equation (2.1). We assume that the damping coefficients δ_1 and δ_2 are small. The model becomes

$$\begin{aligned} m_1\ddot{x}_1 &= -\delta_1\dot{x}_1 - k_1x_1 - k_2(x_1 - x_2) \\ m_2\ddot{x}_2 &= -\delta_2\dot{x}_2 - k_2(x_2 - x_1) \end{aligned} \tag{2.16}$$

To obtain an equation for the motion x_1 which does not involve x_2, we solve the first equation (2.16) for x_2 and substitute into the second equation (2.16) to obtain

$$\begin{aligned} m_1m_2x_1^{(4)} + (m_1\delta_1 + m_2\delta_2)\dddot{x}_1 + (m_2k_1 + k_2(m_1 + m_2) + \delta_1\delta_2)\ddot{x}_1 \\ +(k_1\delta_2 + k_2(\delta_1 + \delta_2)\dot{x}_1 + k_1k_2x_1 = 0 \end{aligned} \tag{2.17}$$

In a similar manner, we may solve the second equation (2.16) for x_1 and substitute into the first equation to obtain a fourth-order equation that involves only x_2. We obtain

$$\begin{aligned} m_1m_2x_2^{(4)} + (m_1\delta_1 + m_2\delta_2)\dddot{x}_2 + (m_2k_1 + k_2(m_1 + m_2) + \delta_1\delta_2)\ddot{x}_2 \\ +(k_1\delta_2 + k_2(\delta_1 + \delta_2)\dot{x}_2 + k_1k_2x_2 = 0 \end{aligned} \tag{2.18}$$

So once again we have the same linear differential equation representing the motion of both weights.

The roots of the characteristic equation are now more complicated to discuss in general, but not surprisingly, we obtain damped oscillatory motion for both weights under simple assumptions on the parameters.

Example 2.4. Assume $m_1 = m_2 = 1$. Describe the motion for spring constants $k_1 = 0.4$ and $k_2 = 1.808$, damping coefficients $\delta_1 = 0.1$ and $\delta_2 = 0.2$, with initial conditions $(x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)) = (1, 1/2, 2, 1/2)$.

The phase portraits for x_1 and x_2 are shown in the top row of figure 2.5; one sees a regular pattern of motion with diminishing amplitude. Damped oscillatory motion is evident in plots of the solutions shown in the middle row; and in the left-hand frame of the bottom row, we plot both x_1 and x_2 and observe nearly synchronized motion. Finally, in the right-hand frame of the bottom row, we plot x_1 versus x_2 which also shows damped oscillatory motion of both weights.

Student problem. Investigate the model when $\delta_1 = 0$ and $\delta_2 > 0$. What happens if $\delta_1 > 0$ and $\delta_2 = 0$?

Student problem. What are the conditions on the parameters of the model that give rise to critically damped motion and over-critically damped motion, or do these concepts not apply to this model?

3. Adding nonlinearity

If we assume that the restoring forces are nonlinear, which they most certainly are for large vibrations, we can modify the model accordingly. Rather than assuming that the restoring force is of the form $-kx$ (Hooke's law), suppose we assume the restoring force has the form $-kx + \mu x^3$. Then our model becomes

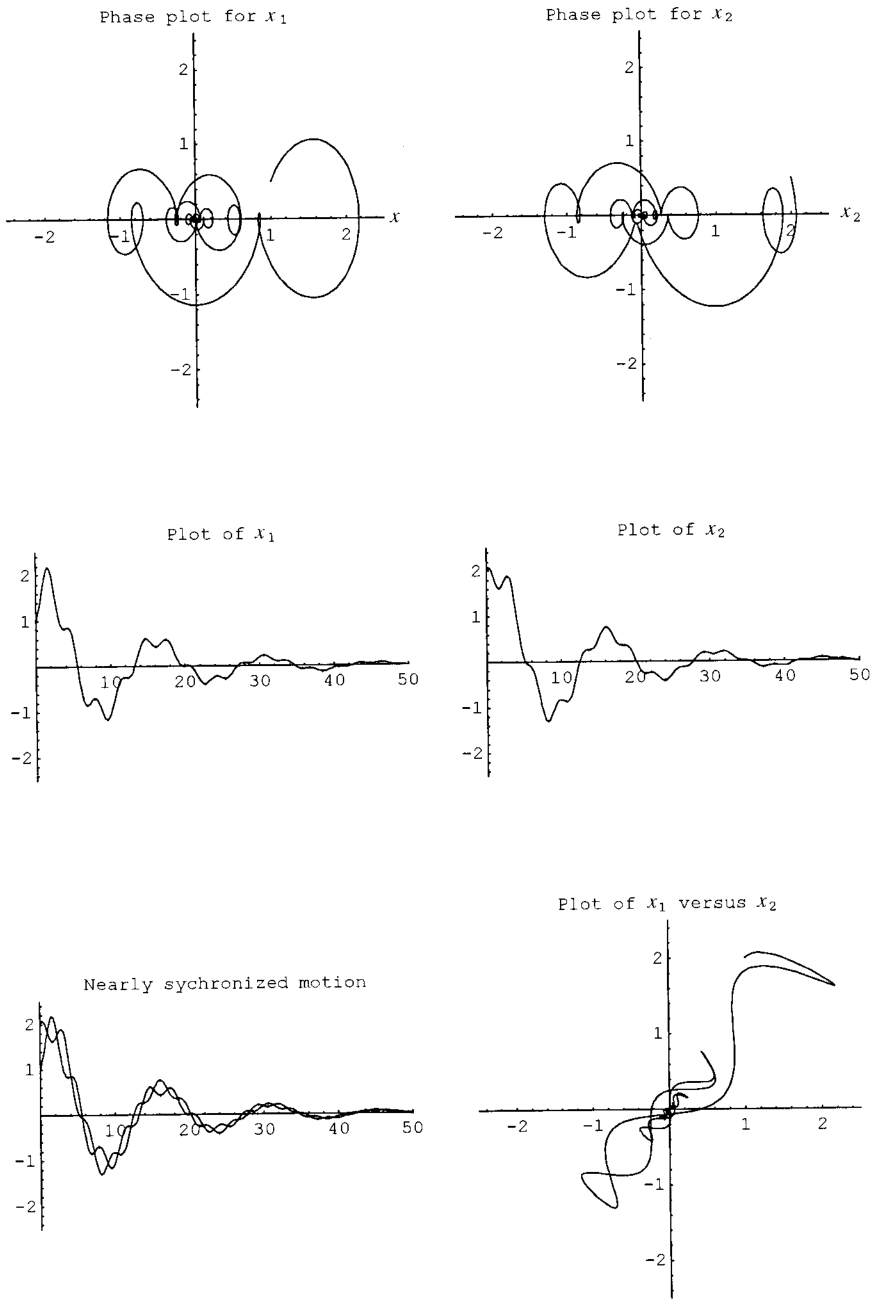


Figure 2.5. Plots for Example 2.4.

$$
\begin{aligned}
m_1\ddot{x}_1 &= -\delta_1\dot{x}_1 - k_1x_1 + \mu_1x_1^3 - k_2(x_1 - x_2) + \mu_2(x_1 - x_2)^3 \\
m_2\ddot{x}_2 &= -\delta_2\dot{x}_2 - k_2(x_2 - x_1) + \mu_2(x_2 - x_1)^3 \qquad (3.1)
\end{aligned}
$$

The range of motions for the nonlinear model is much more complicated than that for the linear model. To get an idea of this range of motions for a single spring model see [4]. Moreover, accuracy questions arise when solving these equations. No numerical solver can be expected to remain accurate over long time intervals. The accumulated local truncation error, algorithm error, roundoff error, propagation error, etc., eventually force the numerical solution to be inaccurate. This is discussed in some detail in the interesting paper by Knapp and Wagon [7], see also [3] and [5].

Example 3.1. Assume $m_1 = m_2 = 1$. Describe the motion for spring constants $k_1 = 0.4$ and $k_2 = 1.808$, damping coefficients $\delta_1 = 0$ and $\delta_2 = 0$, nonlinear coefficients $\mu_1 = -1/6$ and $\mu_2 = -1/10$, with initial conditions $(x_1(0)), \dot{x}_1(0), x_2(0), \dot{x}_2(0)) = (1, 0, -1/2, 0)$.

The motion in this example is quite nice. We have no damping, so the motions are oscillatory and appear to be periodic (for more on detecting and describing periodic solutions to nonlinear differential equations see [6]). Phase plane trajectories are shown in the top row of figure 3.1 for $0 \leqslant t \leqslant 50$; solutions are shown in the middle row. A plot of x_1 and x_2 shows the motions that appear to be 180° out of phase, see the left-hand frame of the bottom row. A plot of x_1 against x_2 is given in the right-hand frame.

Because of the nonlinearity, the model may exhibit sensitivity to initial conditions.

Example 3.2. Assume $m_1 = m_2 = 1$. Describe the motion for spring constants $k_1 = 0.4$ and $k_2 = 1.808$, damping coefficients $\delta_1 = 0$ and $\delta_2 = 0$, nonlinear coefficients $\mu_1 = -1/6$ and $\mu_2 = -1/10$, with initial conditions $(x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)) = (-0.5, 1/2, 3.001, 5.9)$.

Phase plots and a plot of x_1 versus x_2 are shown in figures 3.2 and 3.3. Very pleasing motions can be observed here. Next we change only the $x_1(0)$ value by 1/10, and we obtain quite different motions.

Example 3.3. Assume $m_1 = m_2 = 1$. Describe the motion for spring constants $k_1 = 0.4$ and $k_2 = 1.808$, damping coefficients $\delta_1 = 0$ and $\delta_2 = 0$, nonlinear coefficients $\mu_1 = -1/6$ and $\mu_2 = -1/10$, with initial conditions $(x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)) = (-0.6, 1/2, 3.001, 5.9)$.

The phase plots for this example are shown in figure 3.4 for $0 \leqslant t \leqslant 200$. A plot of x_1 versus x_2 is shown in figure 3.5.

4. Adding forcing

It is a simple matter to add external forcing to the model. Indeed, we can drive each weight differently. Suppose we assume simple sinusoidal forcing of the form $F \cos \omega t$. Then the model becomes

$$\begin{aligned} m_1\ddot{x}_1 &= -\delta_1\dot{x}_1 - k_1x_1 + \mu_1x_1^3 - k_2(x_1 - x_2) + \mu_2(x_1 - x_2)^3 + F_1 \cos \omega_1 t \\ m_2\ddot{x}_2 &= -\delta_2\dot{x}_2 - k_2(x_2 - x_1) + \mu_2(x_2 - x_1)^3 + F_2 \cos \omega_2 t \end{aligned} \tag{4.1}$$

The range of motions for nonlinear forced models is quite vast. We can expect to find bounded and unbounded solutions (nonlinear resonance), periodic solutions that share the period with the forcing (called harmonic solutions) and solutions that are periodic of period a multiple of the driving period (called subharmonic

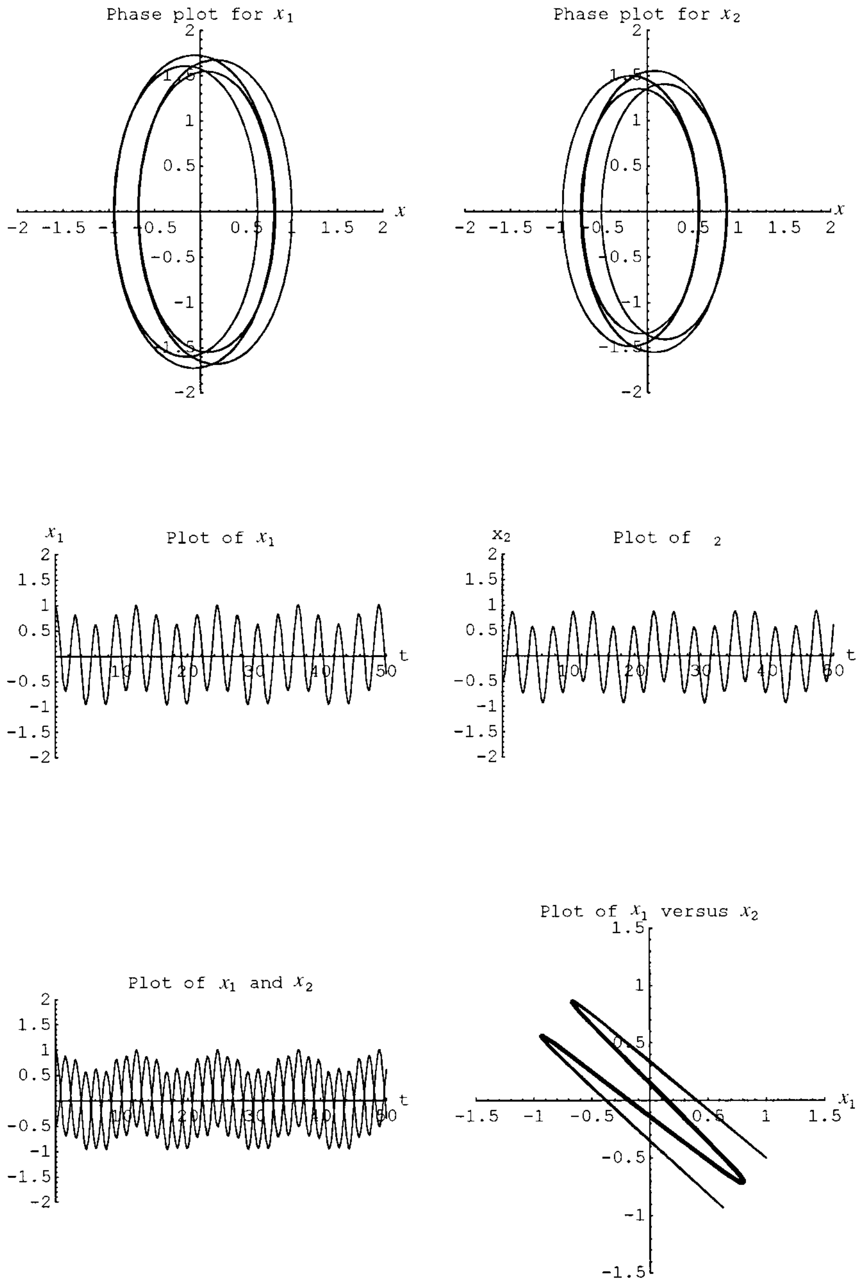


Figure 3.1. Plots for Example 3.1.

solutions), and steady state periodic solutions (limit cycles in the phase plane). The conditions under which these motions occur are by no means easy to state. We conclude with a simple forced example.

Example 4.1. Assume $m_1 = m_2 = 1$. Describe the motion for spring constants $k_1 = 2/5$ and $k_2 = 1$, damping coefficients $\delta_1 = 1/10$ and $\delta_2 = 1/5$, nonlinear

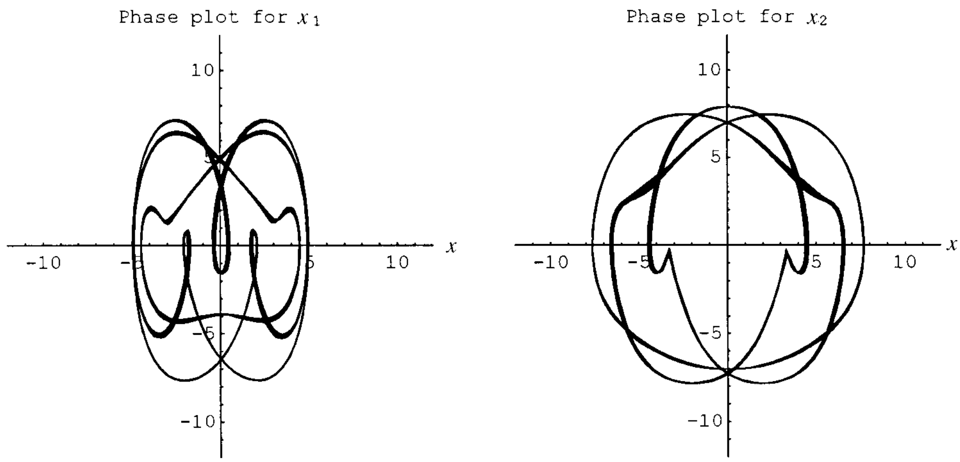


Figure 3.2. Phase plots for Example 3.2.

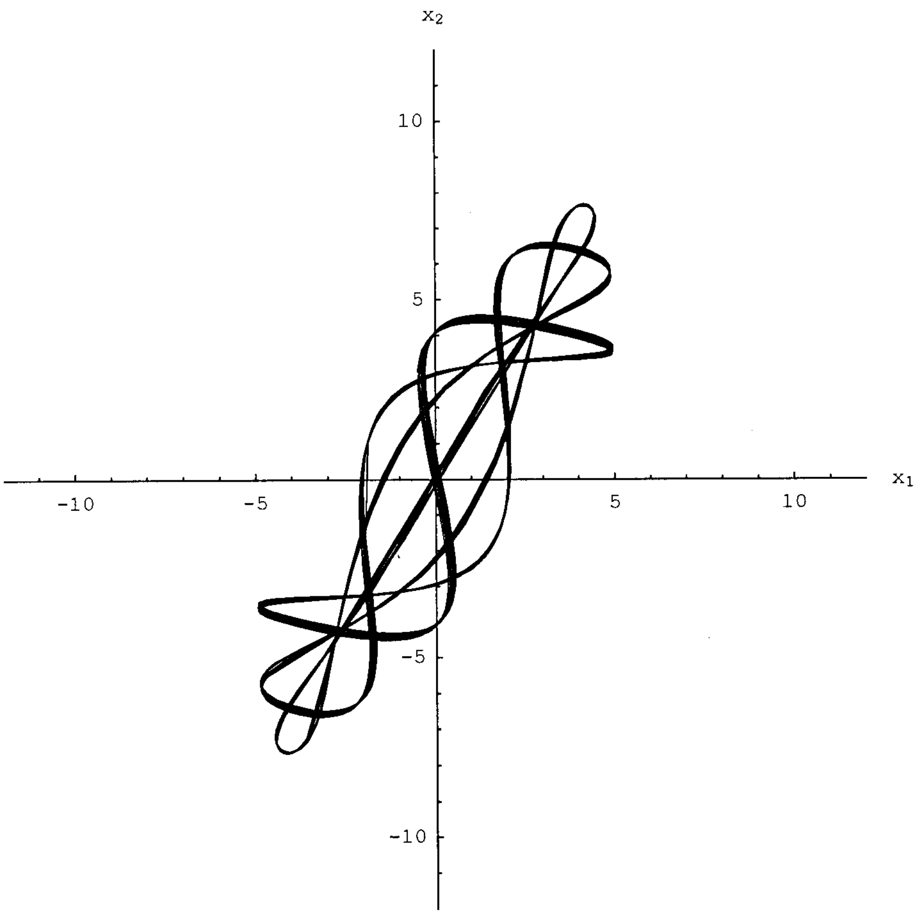


Figure 3.3. x_1 versus x_2, $0 \leqslant t \leqslant 200$, for Example 3.2.

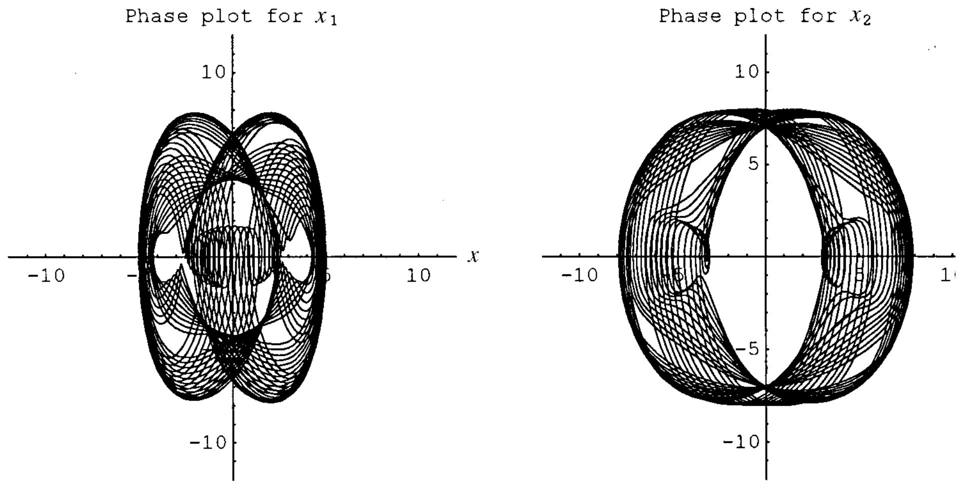


Figure 3.4. Phase plots for Example 3.3.

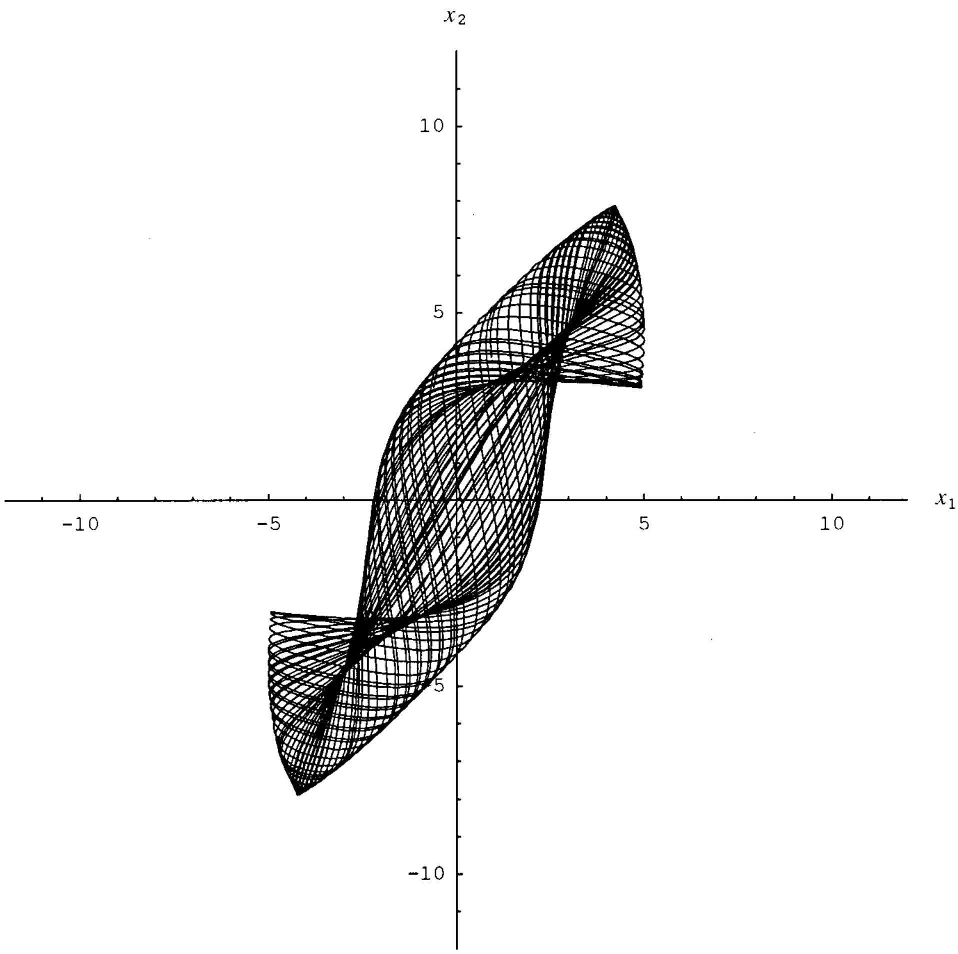


Figure 3.5. x_1 versus x_2, $0 \leqslant t \leqslant 200$, for Example 3.3.

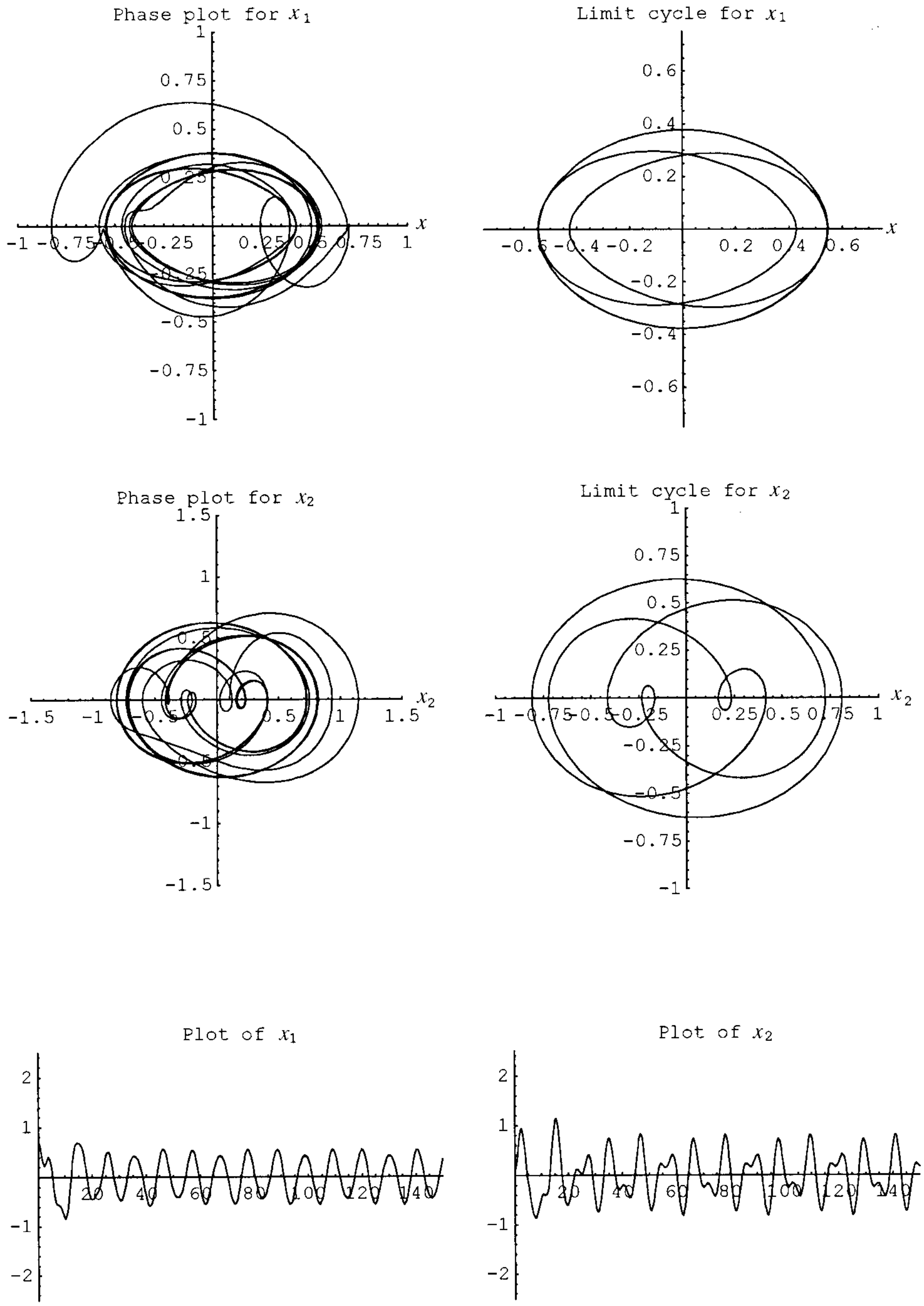
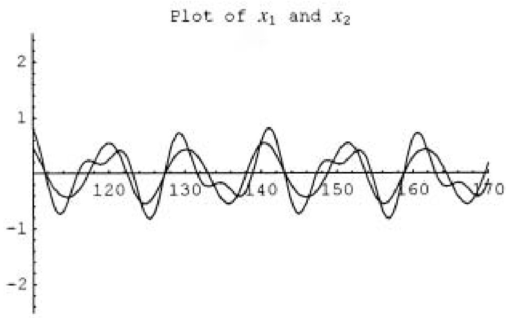


Figure 4.1. Plots for Example 4.1

coefficients $\mu_1 = 1/6$ and $\mu_2 = 1/10$, forcing amplitudes $F_1 = 1/3$ and $F_2 = 1/5$, and forcing frequencies $\omega_1 = 1$ and $\omega_2 = 3/5$, with initial conditions $(x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)) = (0.7, 0, 0.1, 0)$.

Because of the damping, we expect to see different behaviour for small values of t and steady-state behaviour for large values of t. Thus we can expect to see a limit



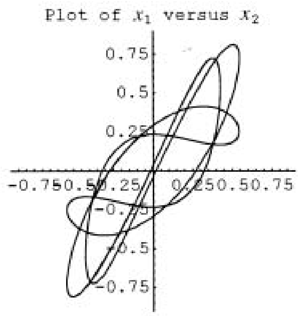


Figure 4.2. Plot of x_1 and x_2 for Example 4.1.

cycle in the phase plane for both x_1 and for x_2. The trajectories and limit cycles are shown in the top and middle rows of figures 4.1. In the bottom row we show plots of x_1 and x_2 for $0 \leqslant t \leqslant 150$. In the left-hand frame of figure 4.2, we plot both x_1 and x_2 for $110 \leqslant t \leqslant 170$ to show the steady-state solutions more clearly. We plot x_1 against x_2 in the steady state in the right-hand frame.

5. Conclusions

We have developed a simple model for two coupled springs, examined both the linear case and one possible form for the nonlinear case, and have included free motion, damped motion, and forced motion examples. These examples produced interesting solutions and are no more difficult for a student to produce than the elementary examples commonly encountered in beginning courses. This model reinforces much of the theory for linear equations and provides a nice elementary modelling example that can be used in a computer laboratory component of a beginning course or for an individual or a small-group undergraduate research project.

The model has many features that permit the meaningful introduction of many concepts including: accuracy of numerical algorithms, dependence on parameters and initial conditions, phase and synchronization, periodicity, beats, limit cycles, harmonic and subharmonic solutions. The use of a computer algebra system permits almost effortless numerical explorations, graphical interpretations, and motivation for analytical verifications.

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