CS 5350/6350: Machine Learining Spring 2018

Homework 2 Solutions

Yulong Liang (u1143816)

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1 Expressiveness of Linear Classifiers

1. [60 points]

   (a) $f(x_1, x_2, x_3) = x_1 \vee x_2 \vee x_3$

   $$\mathbf{w} = (1, 1, 1) \quad bias = -1 \quad x_1 + x_2 + x_3 - 1 = 0$$

   (b) $f(x_1, x_2, x_3) = x_1 \wedge \neg x_2 \wedge \neg x_3$

   $$\mathbf{w} = (1, -1, -1) \quad bias = -1 \quad x_1 - x_2 - x_3 - 1 = 0$$

   (c) $f(x_1, x_2, x_3) = \neg x_1 \vee \neg x_2 \vee \neg x_3$

   $$\mathbf{w} = (-1, -1, -1) \quad bias = 2 \quad -x_1 - x_2 - x_3 + 2 = 0$$

   (d) $f(x_1, x_2, \ldots, x_n) = x_1 \vee x_2 \ldots \vee x_k$ (note that $k < n$)

   $$\mathbf{w} = (\underbrace{1, 1, \cdots, 1}_{\text{1 to k}}, \underbrace{0, 0, \cdots, 0}_{\text{k+1 to n}}) \in \mathbb{R}^n \quad bias = -1$$

   $$x_1 + x_2 + x_3 + \cdots + x_k - 1 = 0, \qquad i.e. \sum_{i=1}^{k} x_i - 1 = 0$$

   (e) $f(x_1, x_2, x_3, x_4) = (x_1 \vee x_2) \wedge (x_3 \vee x_4)$
   The above boolean function can be represented by,

   $$x_1 + x_2 \geq 1 \quad \textbf{and} \quad x_3 + x_4 \geq 1$$

   This is a space defined by two different hyperplanes in $\mathbb{R}^4$,

   $$x_1 + x_2 - 1 = 0 \quad and \quad x_3 + x_4 - 1 = 0$$

   Namely, it is the **intersection** of the positive sides of the **two hyperplanes**, which is not linear. So we CANNOT find such a linear classifier.

(f) $f(x_1, x_2, x_3, x_4) = (x_1 \wedge x_2) \vee (x_3 \wedge x_4)$
The above boolean function can be represented by,

$$x_1 + x_2 \geq 2 \quad \textbf{or} \quad x_3 + x_4 \geq 2$$

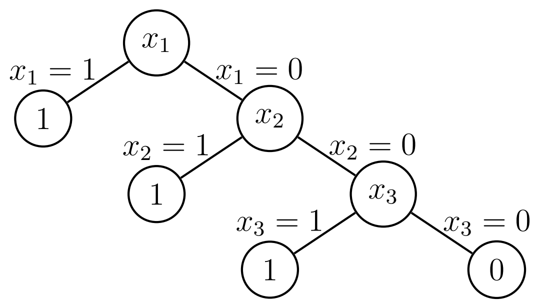
This is a space defined by two different hyperplanes in $\mathbb{R}^4$,

$$x_1 + x_2 - 2 = 0 \quad \textit{and} \quad x_3 + x_4 - 2 = 0$$

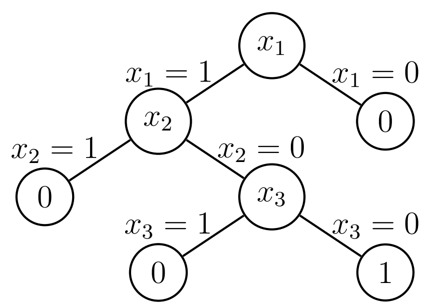
Namely, it is the **union** of the positive sides of the **two hyperplanes**, which is not linear. So we CANNOT find such a linear classifier.

2. [50 points]

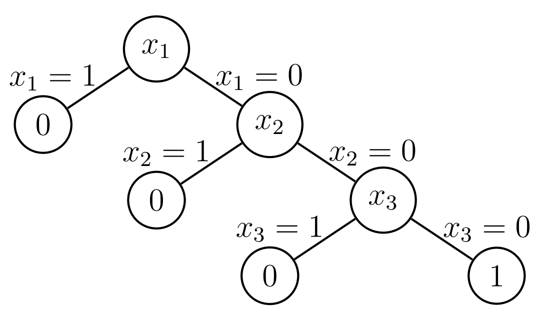
(a) $f(x_1, x_2, x_3) = x_1 \vee x_2 \vee x_3$



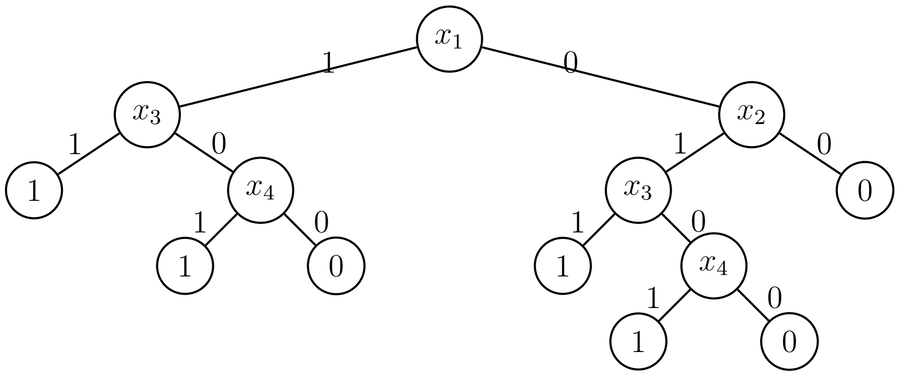
(b) $f(x_1, x_2, x_3) = x_1 \wedge \neg x_2 \wedge \neg x_3$



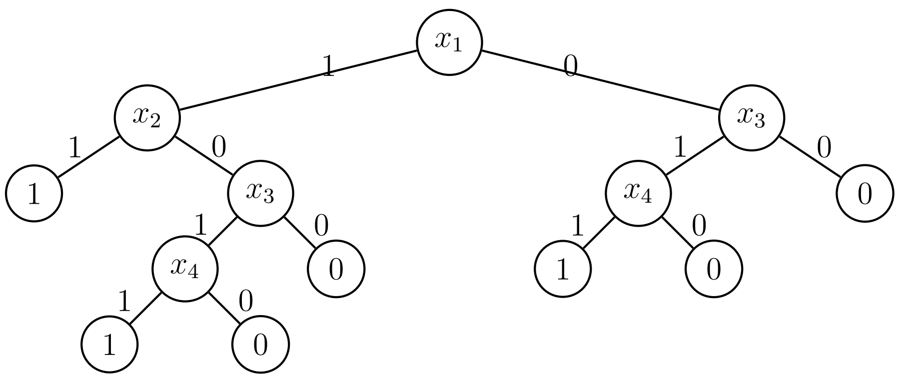
(c) $f(x_1, x_2, x_3) = \neg x_1 \vee \neg x_2 \vee \neg x_3$



(d) $f(x_1, x_2, x_3, x_4) = (x_1 \vee x_2) \wedge (x_3 \vee x_4)$



(e) $f(x_1, x_2, x_3, x_4) = (x_1 \wedge x_2) \vee (x_3 \wedge x_4)$



3. [10 points] Decision tree is **more expressive** than linear classifier. Decision tree can represent any boolean function but linear classifier cannot represent many non-trivial boolean functions such as parity.

4. [30 points]

   (a) $f(x_1, x_2) = (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2)$

$$z_1 = x_1^2 \quad z_2 = x_2^2 \quad z_3 = x_1 x_2$$

$$z_1 + z_2 - 2z_3 - 1 = 0$$

   (b) $f(x_1, x_2) = (x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2)$

$$z_1 = x_1^2 \quad z_2 = x_2^2 \quad z_3 = x_1 x_2$$

$$-z_1 - z_2 + 2z_3 + 1 = 0$$

   (c) $f(x_1, x_2, x_3)$

$$z_1 = x_1 \quad z_2 = x_2 \quad z_3 = x_3 \quad z_4 = [(x_1 - x_2)^2 - x_3]^2$$

$$z_4 - 1 = 0$$

5. [40 points]

(a) [10 points] $(\mathbf{x}^\top \mathbf{y})^2$

$$\phi(\mathbf{x}) = [x_1^2,\ \sqrt{2}x_1x_2,\ x_2^2]$$
$$\phi(\mathbf{y}) = [y_1^2,\ \sqrt{2}y_1y_2,\ y_2^2]$$

(b) [10 points] $(\mathbf{x}^\top \mathbf{y})^3$

$$\phi(\mathbf{x}) = [x_1^3,\ \sqrt{3}x_1^2x_2,\ \sqrt{3}x_1x_2^2,\ x_2^3]$$
$$\phi(\mathbf{y}) = [y_1^3,\ \sqrt{3}y_1^2y_2,\ \sqrt{3}y_1y_2^2,\ y_2^3]$$

(c) [20 points] $(\mathbf{x}^\top \mathbf{y})^\mathbf{k}$ where k is any positive integer.
According to `Yang Hui's triangle`, a.k.a. `Pascal's triangle`, the feature mappings are as follows,

$$\begin{aligned}&\phi(\mathbf{x})\\ =&\Big[\sqrt{\binom{t}{0}}x_1^t,\ \sqrt{\binom{t}{1}}x_1^{t-1}x_2,\ \sqrt{\binom{t}{2}}x_1^{t-2}x_2^2,\ \cdots,\ \sqrt{\binom{t}{t-1}}x_1x_2^{t-1},\ \sqrt{\binom{t}{t}}x_2^t\Big]\end{aligned}$$

$$\begin{aligned}&\phi(\mathbf{y})\\ =&\Big[\sqrt{\binom{t}{0}}y_1^t,\ \sqrt{\binom{t}{1}}y_1^{t-1}y_2,\ \sqrt{\binom{t}{2}}y_1^{t-2}y_2^2,\ \cdots,\ \sqrt{\binom{t}{t-1}}y_1y_2^{t-1},\ \sqrt{\binom{t}{t}}y_2^t\Big]\end{aligned}$$

2 Linear Regression

1. [10 points] Write down the LMS (least mean square) cost function $J(\mathbf{w}, b)$.

$$\begin{aligned}J(\mathbf{w}, b) =&\frac{1}{2}\sum_{x\in X}(y - (\mathbf{w}^\top\mathbf{x} + b))^2\\ =&\frac{1}{2}[(1 - [1, -1, 2]\mathbf{w} - b)^2 + (4 - [1, 1, 3]\mathbf{w} - b)^2 + (-1 - [-1, 1, 0]\mathbf{w} - b)^2 +\\ &(-2 - [1, 2, -4]\mathbf{w} - b)^2 + (0 - [3, -1, -1]\mathbf{w} - b)^2]\end{aligned}$$

2. [30 points] Calculate the gradient $\frac{\nabla J}{\nabla \mathbf{w}}$ and $\frac{\nabla J}{\nabla b}$

The gradient can be calculated according to the formula,

$$\frac{\nabla J}{\nabla \mathbf{w}} = \Big[\frac{\partial J}{\partial w_1}, \frac{\partial J}{\partial w_2}, \cdots, \frac{\partial J}{\partial w_d}\Big]$$
$$\frac{\partial J}{\partial w_j} = -\sum_{i=1}^{m}(y_i - \mathbf{w}^\top\mathbf{x}_i)x_{ij}$$

(a) when $\mathbf{w} = [0, 0, 0]^\top$ and $b = 0$;

$$\frac{\nabla J}{\nabla \mathbf{w}} = [-4, 2, -22], \quad \frac{\nabla J}{\nabla b} = -2$$

(b) when $\mathbf{w} = [-1, 1, -1]^\top$ and $b = -1$;

$$\frac{\nabla J}{\nabla \mathbf{w}} = [-22, 16, -56], \quad \frac{\nabla J}{\nabla b} = -10$$

(c) when $\mathbf{w} = [1/2, -1/2, 1/2]^\top$ and $b = 1$.

$$\frac{\nabla J}{\nabla \mathbf{w}} = [7.5, -4, -5], \quad \frac{\nabla J}{\nabla b} = 4.5$$

The code I used is in the appendix.

3. [20 points] What are the optimal $\mathbf{w}$ and b that minimize the cost function?

$$\mathbf{w} = [1, 1, 1] \quad b = -1$$

The code I used is in the appendix.

4. [50 points]

(a) Step1

$$\frac{\nabla J}{\nabla \mathbf{w}} = [1, -1, 2], \quad \frac{\nabla J}{\nabla b} = 1$$
$$\mathbf{w} = [0.1000, -0.1000, 0.2000], \quad b = 0.1000$$

(b) Step2

$$\frac{\nabla J}{\nabla \mathbf{w}} = [3.3000, 3.3000, 9.9000], \quad \frac{\nabla J}{\nabla b} = 3.3000$$
$$\mathbf{w} = [0.4300, 0.2300, 1.1900], \quad b = 0.4300$$

(c) Step3

$$\frac{\nabla J}{\nabla \mathbf{w}} = [1.2300, -1.2300, 0], \quad \frac{\nabla J}{\nabla b} = -1.2300$$
$$\mathbf{w} = [0.5530, 0.1070, 1.1900], \quad b = 0.3070$$

(d) Step4

$$\frac{\nabla J}{\nabla \mathbf{w}} = [1.6860, 3.3720, -6.7440], \quad \frac{\nabla J}{\nabla b} = 1.6860$$
$$\mathbf{w} = [0.7216, 0.4442, 0.5156], \quad b = 0.4756$$

(e) Step5

$$\frac{\nabla J}{\nabla \mathbf{w}} = [-5.0418, 1.6806, 1.6806], \quad \frac{\nabla J}{\nabla b} = -1.6806$$
$$\mathbf{w} = [0.2174, 0.6123, 0.6837], \quad b = 0.3075$$

The code I used is in the appendix.

3 Mistake Driven Learning Algorithm

1. [10 points] Disjunction of n boolean variables.

   Taking negation into consideration,

   $$|C| = 3^n \qquad M_A(C) = \log_2(3^n) = O(n)$$

   $M_A(C)$ is polynomial to n. Thus Halving algorithm is a mistake bound algorithm for this concept class.

2. [10 points] Disjunction of k boolean variables out of the total n input variables.

   Taking negation into consideration,

   $$|C| = \binom{n}{k} 2^k \approx 2^k n^k \qquad M_A(C) = \log_2(2^k n^k) = k + k \log_2 n = O(k \log_2 n)$$

   $M_A(C)$ is logarithm of n, which is better than polynomial. Thus Halving algorithm is a mistake bound algorithm for this concept class.

3. [10 points] m-of-n rules. Note m is a constant and smaller than n.

   It is ambiguous whether m is a fixed number.
   If m is any constant $m \in [n]$,

   $$|C| = \sum_{i=1}^{n} i \binom{n}{k} = n \cdot 2^{n-1}$$

   $$M_A(C) = \log_2(n \cdot 2^{n-1}) = (n-1) \log_2 n = O(n \log_2 n)$$

   If m is a fixed constant $m \in [n]$,

   $$|C| = m \binom{n}{m} \approx m n^m$$

   $$M_A(C) = \log_2(m n^m) = \log_2 m + m \log_2 n = O(m \log_2 n)$$

   $M_A(C)$ is logarithm of n, which is better than polynomial. Thus Halving algorithm is a mistake bound algorithm for this concept class.

4. [20 points] All boolean function of n input boolean variables.

   $$|C| = 2^{2^n}$$

   $$M_A(C) = \log_2(2^{2^n}) = 2^n = O(2^n)$$

   $M_A(C)$ is exponential of n, which is worse than polynomial. Thus Halving algorithm is NOT a mistake bound algorithm for this concept class.

4 Perceptron

1. Let us review the Mistake Bound Theorem discussed in our lecture.

   (a) [10 points]

   Given that,

   $$\mathbf{u}^\top \mathbf{w}_t \geq t\gamma \qquad ||\mathbf{w}||^2 \leq tR^2$$

   We have,

   $$t\gamma \leq \mathbf{u}^\top \mathbf{w}_t \leq ||\mathbf{u}|| \cdot ||\mathbf{w}_t|| \leq ||\mathbf{u}||\sqrt{t}R$$

   Thus,

   $$(\frac{R}{\gamma})^2 \cdot ||\mathbf{u}||^2$$

   (b) [10 points]

   $$\begin{aligned}\gamma &= \min_{x_i \in X} \mathbf{dist}(x_i, h) \\ &= \min_{x_i \in X} \frac{\mathbf{u}^\top \mathbf{x}_i y_i}{||u||} \\ \gamma &\leq \frac{y_i(\mathbf{u}^\top \mathbf{x}_i)}{||\mathbf{u}||}\end{aligned}$$

   (c) [20 points]

   The conclusion is the upper bound is still $(\frac{R}{\gamma})^2$.

   i. Proof 1/3 - After t mistakes, $\frac{\mathbf{u}^\top \mathbf{w}_t}{||\mathbf{u}||} \geq t\gamma$

      For t = 0,

      $$\frac{\mathbf{u}^\top \mathbf{w}_t}{||\mathbf{u}||} = t\gamma = 0$$

      Assuming $t = t$ holds, for $t = t + 1$,

      $$\begin{aligned}\frac{\mathbf{u}^\top \mathbf{w}_t}{||\mathbf{u}||} &= \frac{\mathbf{u}^\top (\mathbf{w}_t + \mathbf{x}_i y_i)}{||\mathbf{u}||} \\ &= \underbrace{\frac{\mathbf{u}^\top \mathbf{w}_t}{||\mathbf{u}||}}_{\geq t\gamma} + \underbrace{\frac{\mathbf{u}^\top \mathbf{x}_i y_i}{||\mathbf{u}||}}_{\geq \gamma} \\ &\geq (t+1)\gamma\end{aligned}$$

      Thus, $\frac{\mathbf{u}^\top \mathbf{w}_t}{||\mathbf{u}||} \geq t\gamma$ holds.

ii. Proof 2/3 - After t mistakes, $||\mathbf{w}_t||^2 \leq tR^2$

PROOF IS SAME AS BEFORE

iii. Proof 3/3 - $t \leq (\frac{R}{\gamma})^2$

$$t\gamma \leq \frac{\mathbf{u}^\top \mathbf{w}_t}{||\mathbf{u}||} \leq \frac{||\mathbf{u}|| \cdot ||\mathbf{w}_t||}{||\mathbf{u}||} = ||\mathbf{w}_t|| \leq \sqrt{t}R$$

$$t \leq (\frac{R}{\gamma})^2$$

2. [20 points] A linear classifier for this disjunction is,

$$-x_1 - x_2 - \cdots - x_k + x_{k+1} + x_{k+2} + \cdots + x_{2k} + (k - \frac{1}{2}) = 0$$

Thus, the weight vector is,

$$\mathbf{w} = (\underbrace{-1, -1, \cdots, -1}_{\text{k}}, \underbrace{1, 1, \cdots, 1}_{\text{k}}, \underbrace{0, 0, \cdots, 0}_{\text{n-2k}}, k - \frac{1}{2})$$

$$||\mathbf{w}|| = \sqrt{2k + (k - \frac{1}{2})^2} = \sqrt{k^2 + k + \frac{1}{4}} = \sqrt{(k + \frac{1}{2})^2} = k + \frac{1}{2}$$

$$\mathbf{w}^\top \mathbf{x}_i y_i \geq \frac{1}{2}$$

$$\gamma = min \frac{\mathbf{w}^\top \mathbf{x}_i y_i}{||\mathbf{w}||} = \frac{\frac{1}{2}}{k + \frac{1}{2}} = \frac{1}{2k+1}$$

$$R = \sqrt{n+1}$$

The upper bound of the number of mistakes made by Perceptron in learning this disjunction is,

$$(\frac{R}{\gamma})^2 = \left(\frac{\sqrt{n+1}}{\frac{1}{2k+1}}\right)^2 = (2k+1)^2(n+1)$$

This number is polynomial to input dimensionality n. So Perceptron is a mistake bound algorithm.

5 Programming Assignments

1. Gradient Descent

(a) [90 points] Batch Gradient Descent

$$\mathbf{w} = (0.90022, 0.78594, 0.85067, 1.29862, 0.12983, 1.57179, 0.99835, -0.01520)$$

$$\gamma = 0.01$$

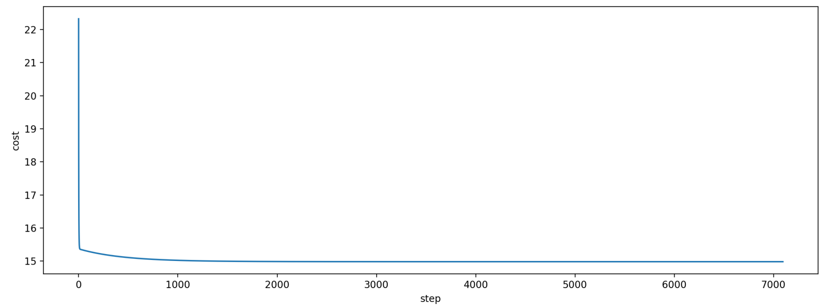


Figure 1: Cost Function Value of Training Data at Each Step

$$\mathcal{J}(\mathbf{w}) = 23.361305269196592$$

(b) [90 points] Stochastic Gradient Descent

$$\mathbf{w} = (-0.07214, -0.25999, -0.23955, 0.51804, -0.03139, 0.24585, 0.01012, -0.03060)$$

$$\gamma = 0.001$$

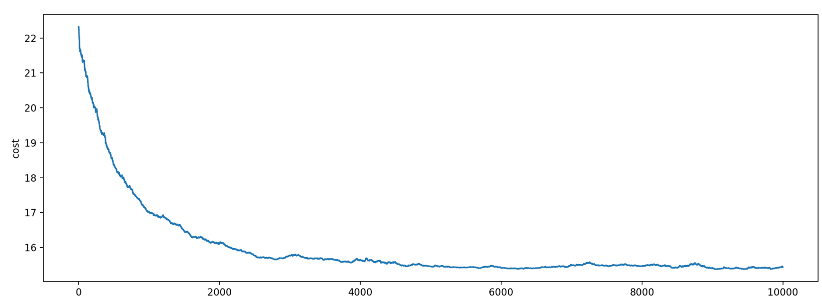


Figure 2: Cost Function Value of Training Data at Each Step

$$\mathcal{J}(\mathbf{w}) = 22.167107354823077$$

(c) [20 points] The optimal weight vector is,

$$\mathbf{w} = (0.90056, 0.78629, 0.85104, 1.29889, 0.12989, 1.57225, 0.99869, -0.01520)$$

- Batch gradient descent will always converge to a solution that is close to the optimal solution while stochastic gradient descent won't.
- Batch gradient descent requires less iteration to converge than stochastic gradient descent.
- Stochastic gradient descent takes much less time than batch gradient descent at each step calculating the gradient and updating the weight vector.

2. Perceptron

   (a) [60 points] Standard perceptron

$$\mathbf{w} = (-5.3207063, -3.563121, -4.4322626, -1.27922046, 5.7)$$

Average prediction error: 0.02000

   (b) [60 points] Voted perceptron

Please find the learnt weight vectors in the appendix.

Average prediction error: 0.01400

   (c) [60 points] Averaged perceptron

$$\mathbf{w} = (-40225.7178191, -26477.132171, -27533.9904514, -7870.9174287, 34571.2)$$

Average prediction error: 0.01400

Observation: Comparing to Voted Perceptron, Averaged Perceptron has the same prediction performance, but it does not have to store a large set of weight vectors, which dramatically reduces space consumption.

   (d) [20 points]

   - Generally speaking, Voted Perceptron and Average Perceptron have better prediction performance than Standard Perceptron.
   - Voted Perceptron and Average Perceptron have almost the same prediction performance, however, Averaged Perceptron takes much less space.

6 Appendix

6.1 Learnt Weight Vectors

Weight Vector	Count
(0.00000,0.00000,0.00000,0.00000,0.00000)	1
(-0.58818,0.76584,0.05558,-0.29155,0.10000)	3
(-0.87209,0.10284,1.10407,-0.33366,0.20000)	7
(-0.86959,0.44282,0.66080,-0.76021,0.30000)	2
(-0.89462,-0.48980,1.02953,-0.13478,0.20000)	4
(-1.35901,-0.15251,0.76977,-0.19004,0.10000)	3
(-1.34733,0.22099,0.32598,-0.62745,0.20000)	3
(-1.21121,-0.84841,0.15576,-0.33719,0.10000)	1
(-1.26202,-0.56161,-0.02532,-0.56331,0.20000)	4
(-1.24888,-0.38386,-0.85848,-0.59852,0.10000)	33
(-1.42964,-1.26517,0.01238,-0.62021,0.20000)	3
(-1.38458,-1.12839,-0.69620,-0.57990,0.10000)	13
(-1.23562,-0.78551,-1.09929,-0.72249,0.20000)	18
(-1.56254,-2.05957,0.45644,-0.73668,0.30000)	8
(-1.81757,-1.56439,-0.18085,-0.69508,0.20000)	10
(-1.91054,-1.18468,-0.64514,-0.66551,0.10000)	13
(-1.69111,-0.72965,-1.14274,-0.93805,0.20000)	13
(-2.04792,-1.55095,-0.13444,-0.84128,0.30000)	4
(-1.92513,-1.14786,-0.59879,-1.23253,0.40000)	8
(-1.82319,-1.03757,-0.82879,-1.17314,0.50000)	11
(-1.87514,-0.71124,-1.13774,-1.07465,0.40000)	2
(-2.21096,-1.43528,0.00645,-1.13176,0.50000)	9
(-2.14047,-1.41811,-0.17214,-1.09564,0.60000)	1
(-2.10613,-1.40569,-0.20087,-1.08099,0.70000)	13
(-2.24032,-0.96348,-1.00987,-0.90750,0.60000)	1
(-2.27324,-0.51796,-1.46705,-0.80862,0.50000)	5
(-2.39892,-0.66529,-1.17987,-0.76397,0.60000)	5
(-2.77395,-2.01115,0.57945,-1.04168,0.70000)	11
(-2.90062,-1.72932,0.33685,-1.23030,0.80000)	2
(-3.06094,-1.25069,-0.51508,-1.01827,0.70000)	3
(-3.00914,-1.22483,-0.59917,-0.92215,0.80000)	17
(-2.95833,-1.17703,-0.79721,-0.86443,0.90000)	11
(-2.90210,-1.07688,-1.02447,-0.86504,1.00000)	4
(-2.89907,-1.18200,-0.88423,-0.78767,1.10000)	44
(-2.95102,-0.85567,-1.19318,-0.68918,1.00000)	31
(-2.70629,-2.11814,-1.26675,0.07694,0.90000)	13
(-2.79926,-1.73843,-1.73104,0.10651,0.80000)	6
(-2.66408,-1.63248,-1.96541,0.14651,0.90000)	29
(-2.46231,-1.45266,-2.26122,0.16750,1.00000)	6

(-2.60118,-1.94039,-1.61348,0.20168,1.10000)	21
(-2.45041,-1.74443,-1.91932,0.18943,1.20000)	27
(-2.34404,-1.37486,-2.33526,-0.00436,1.30000)	2
(-2.70320,-1.99771,-1.31137,-0.11979,1.40000)	45
(-2.54510,-1.91080,-1.54275,-0.03737,1.50000)	13
(-2.37711,-1.49012,-1.99673,-0.27668,1.60000)	39
(-2.13794,-1.03447,-2.49561,-0.56655,1.70000)	2
(-2.24538,-1.66560,-1.96011,-0.48608,1.80000)	8
(-2.60523,-3.03153,-0.19959,-0.73535,1.90000)	1
(-2.70064,-2.83329,-0.43122,-0.85492,2.00000)	2
(-2.79005,-2.51338,-0.61341,-1.14944,2.10000)	7
(-2.78343,-2.26424,-0.90742,-1.21160,2.20000)	13
(-3.02829,-1.63249,-1.70374,-1.23220,2.10000)	56
(-2.76350,-2.64623,-1.57064,-0.68513,2.00000)	2
(-2.74740,-1.99999,-2.40637,-0.53297,1.90000)	11
(-3.21505,-2.56635,-1.30947,-0.56642,2.00000)	31
(-3.15500,-2.37308,-1.63835,-0.59883,2.10000)	52
(-3.03420,-1.96564,-2.11470,-0.86012,2.20000)	143
(-2.83110,-1.78044,-2.41591,-0.85982,2.30000)	26
(-2.88305,-1.45411,-2.72486,-0.76133,2.20000)	2
(-2.61826,-2.46785,-2.59176,-0.21426,2.10000)	9
(-3.25190,-1.53937,-2.59033,-0.89270,2.20000)	19
(-3.24887,-1.64449,-2.45009,-0.81533,2.30000)	32
(-3.06303,-2.43309,-2.28366,-0.63149,2.20000)	65
(-2.88972,-2.03765,-2.75778,-0.88166,2.30000)	28
(-3.19838,-2.70127,-1.70373,-0.97085,2.40000)	6
(-3.04028,-2.61436,-1.93511,-0.88843,2.50000)	15
(-2.83851,-2.43454,-2.23092,-0.86744,2.60000)	64
(-2.87144,-1.98902,-2.68810,-0.76856,2.50000)	1
(-3.02466,-2.49868,-2.02031,-0.75107,2.60000)	132
(-3.07661,-2.17235,-2.32926,-0.65258,2.50000)	107
(-3.10953,-1.72683,-2.78644,-0.55370,2.40000)	4
(-2.92369,-2.51543,-2.62001,-0.36986,2.30000)	23
(-2.72059,-2.33023,-2.92122,-0.36956,2.40000)	41
(-3.12232,-3.16146,-1.67575,-0.51331,2.50000)	10
(-3.77316,-2.28450,-1.65256,-0.90701,2.60000)	15
(-3.60517,-1.86382,-2.10654,-1.14632,2.70000)	41
(-3.44613,-1.64261,-2.41837,-1.15804,2.80000)	6
(-3.77305,-2.91667,-0.86264,-1.17222,2.90000)	3
(-3.71300,-2.81672,-1.08390,-1.16248,3.00000)	2
(-3.49357,-2.36169,-1.58150,-1.43502,3.10000)	5
(-3.41488,-3.31832,-1.20283,-0.68468,3.00000)	50
(-3.66991,-2.82314,-1.84012,-0.64309,2.90000)	17
(-3.72186,-2.49681,-2.14907,-0.54460,2.80000)	17
(-3.65739,-2.03619,-2.98377,-0.27361,2.70000)	26

(-3.85148,-2.90467,-2.06827,-0.17956,2.80000)	33
(-3.78330,-2.41963,-2.58960,-0.78999,2.90000)	90
(-3.83524,-2.09330,-2.89855,-0.69150,2.80000)	44
(-4.14947,-3.39695,-1.33082,-0.75766,2.90000)	13
(-4.01764,-3.20678,-1.66193,-0.75115,3.00000)	12
(-3.89684,-2.79934,-2.13828,-1.01244,3.10000)	69
(-3.74788,-2.45646,-2.54137,-1.15503,3.20000)	33
(-3.73178,-1.81022,-3.37710,-1.00287,3.10000)	10
(-3.98843,-2.49846,-2.62294,-0.93210,3.20000)	9
(-4.04038,-2.17213,-2.93189,-0.83361,3.10000)	41
(-4.11025,-2.50984,-2.51978,-0.68318,3.20000)	25
(-4.14317,-2.06432,-2.97696,-0.58430,3.10000)	5
(-3.94007,-1.87912,-3.27817,-0.58400,3.20000)	4
(-4.01736,-2.62385,-2.62897,-0.54788,3.30000)	46
(-4.06931,-2.29752,-2.93792,-0.44939,3.20000)	193
(-4.44434,-3.64338,-1.17860,-0.72710,3.30000)	9
(-4.47141,-3.31664,-1.53422,-1.03598,3.40000)	2
(-4.37470,-2.93238,-2.02736,-1.44921,3.50000)	35
(-4.42665,-2.60605,-2.33631,-1.35072,3.40000)	35
(-4.25334,-2.21061,-2.81043,-1.60089,3.50000)	50
(-3.98855,-3.22435,-2.67733,-1.05382,3.40000)	30
(-3.78678,-3.04453,-2.97314,-1.03283,3.50000)	133
(-3.56735,-2.58950,-3.47074,-1.30537,3.60000)	2
(-4.04066,-3.20739,-2.33194,-1.41278,3.70000)	4
(-4.13363,-2.82768,-2.79623,-1.38321,3.60000)	133
(-4.05494,-3.78431,-2.41756,-0.63287,3.50000)	38
(-3.98676,-3.29927,-2.93889,-1.24330,3.60000)	178
(-3.97065,-2.65303,-3.77462,-1.09114,3.50000)	10
(-4.40941,-3.42570,-2.57807,-1.23657,3.60000)	6
(-4.46136,-3.09937,-2.88702,-1.13808,3.50000)	47
(-4.30232,-2.87816,-3.19885,-1.14980,3.60000)	33
(-4.35427,-2.55183,-3.50780,-1.05131,3.50000)	10
(-4.55493,-3.22373,-2.60618,-1.04132,3.60000)	50
(-4.49046,-2.76311,-3.44088,-0.77033,3.50000)	155
(-4.54241,-2.43678,-3.74983,-0.67183,3.40000)	48
(-4.68209,-3.40376,-2.80331,-0.70671,3.50000)	37
(-4.50334,-2.92576,-3.31693,-1.03033,3.60000)	25
(-4.30157,-2.74594,-3.61274,-1.00934,3.70000)	29
(-4.58548,-3.40894,-2.56425,-1.05145,3.80000)	4
(-4.67845,-3.02923,-3.02854,-1.02188,3.70000)	2
(-4.73040,-2.70290,-3.33749,-0.92339,3.60000)	203
(-4.52863,-2.52308,-3.63330,-0.90240,3.70000)	45
(-4.96636,-3.07475,-2.53940,-0.94322,3.80000)	53
(-4.95025,-2.42851,-3.37513,-0.79106,3.70000)	12
(-4.71108,-1.97286,-3.87401,-1.08093,3.80000)	11

(-4.52524,-2.76146,-3.70758,-0.89709,3.70000)	10
(-4.85216,-4.03552,-2.15185,-0.91127,3.80000)	45
(-4.70139,-3.83956,-2.45769,-0.92351,3.90000)	24
(-4.55243,-3.49668,-2.86078,-1.06610,4.00000)	13
(-4.37912,-3.10124,-3.33490,-1.31627,4.10000)	130
(-4.43107,-2.77491,-3.64385,-1.21778,4.00000)	24
(-4.80610,-4.12077,-1.88453,-1.49549,4.10000)	2
(-5.00086,-3.64339,-2.73723,-1.30881,4.00000)	40
(-4.79776,-3.45819,-3.03844,-1.30851,4.10000)	74
(-4.84971,-3.13186,-3.34739,-1.21002,4.00000)	48
(-4.90166,-2.80553,-3.65634,-1.11153,3.90000)	19
(-5.28369,-4.11104,-1.96051,-1.34205,4.00000)	5
(-5.18252,-4.02082,-2.19557,-1.29934,4.10000)	1
(-5.21544,-3.57530,-2.65275,-1.20046,4.00000)	12
(-5.08026,-3.46935,-2.88712,-1.16046,4.10000)	26
(-4.92216,-3.38245,-3.11850,-1.07805,4.20000)	129
(-4.95508,-2.93693,-3.57568,-0.97917,4.10000)	116
(-4.87639,-3.89356,-3.19701,-0.22883,4.00000)	42
(-4.65696,-3.43853,-3.69461,-0.50137,4.10000)	73
(-4.47821,-2.96053,-4.20823,-0.82499,4.20000)	22
(-4.82881,-4.21720,-2.69217,-0.90020,4.30000)	45
(-5.01097,-3.56972,-3.49731,-0.85835,4.20000)	130
(-4.86596,-3.20905,-3.90288,-1.01801,4.30000)	117
(-4.89888,-2.76353,-4.36006,-0.91913,4.20000)	27
(-5.36653,-3.32989,-3.26316,-0.95258,4.30000)	11
(-5.19322,-2.93445,-3.73728,-1.20275,4.40000)	8
(-4.99012,-2.74925,-4.03849,-1.20245,4.50000)	39
(-5.05613,-3.07185,-3.65791,-1.08409,4.60000)	38
(-5.10807,-2.74552,-3.96686,-0.98559,4.50000)	3
(-4.92223,-3.53412,-3.80043,-0.80175,4.40000)	7
(-4.68306,-3.07847,-4.29931,-1.09162,4.50000)	32
(-5.04222,-3.70132,-3.27542,-1.20705,4.60000)	16
(-5.02612,-3.05508,-4.11115,-1.05489,4.50000)	95
(-5.34035,-4.35873,-2.54342,-1.12106,4.60000)	2
(-5.39230,-4.03240,-2.85237,-1.02257,4.50000)	56
(-5.44425,-3.70607,-3.16132,-0.92408,4.40000)	51
(-5.49620,-3.37974,-3.47027,-0.82559,4.30000)	26
(-5.33716,-3.15853,-3.78210,-0.83731,4.40000)	3
(-5.09799,-2.70288,-4.28098,-1.12718,4.50000)	1
(-5.45715,-3.32573,-3.25709,-1.24261,4.60000)	78
(-5.29905,-3.23882,-3.48847,-1.16020,4.70000)	138
(-5.07962,-2.78379,-3.98607,-1.43274,4.80000)	17
(-5.14538,-3.06397,-3.61492,-1.33300,4.90000)	50
(-5.19733,-2.73764,-3.92387,-1.23451,4.80000)	12
(-5.39799,-3.40954,-3.02225,-1.22452,4.90000)	88

(-5.43092,-2.96402,-3.47943,-1.12564,4.80000)	6
(-5.48286,-2.63769,-3.78838,-1.02714,4.70000)	125
(-5.81868,-3.36173,-2.64419,-1.08426,4.80000)	25
(-5.91165,-2.98202,-3.10848,-1.05469,4.70000)	11
(-5.83296,-3.93865,-2.72981,-0.30435,4.60000)	3
(-5.80609,-3.43995,-3.24489,-0.94348,4.70000)	71
(-5.85804,-3.11362,-3.55384,-0.84499,4.60000)	19
(-5.69900,-2.89241,-3.86567,-0.85671,4.70000)	85
(-5.49590,-2.70721,-4.16688,-0.85641,4.80000)	17
(-5.85575,-4.07314,-2.40636,-1.10568,4.90000)	7
(-5.73495,-3.66570,-2.88271,-1.36697,5.00000)	35
(-5.56696,-3.24502,-3.33669,-1.60628,5.10000)	16
(-5.59988,-2.79950,-3.79387,-1.50740,5.00000)	2
(-5.52119,-3.75613,-3.41520,-0.75706,4.90000)	27
(-5.31942,-3.57631,-3.71101,-0.73607,5.00000)	162
(-5.08025,-3.12066,-4.20989,-1.02594,5.10000)	127
(-5.21912,-3.60839,-3.56215,-0.99176,5.20000)	16
(-5.27107,-3.28206,-3.87110,-0.89327,5.10000)	31
(-5.32302,-2.95573,-4.18005,-0.79478,5.00000)	22
(-5.52368,-3.62763,-3.27843,-0.78478,5.10000)	45
(-5.45921,-3.16701,-4.11313,-0.51379,5.00000)	116
(-5.78613,-4.44107,-2.55740,-0.52798,5.10000)	2
(-5.83808,-4.11474,-2.86635,-0.42949,5.00000)	40
(-5.89002,-3.78841,-3.17530,-0.33099,4.90000)	96
(-5.76922,-3.38097,-3.65165,-0.59228,5.00000)	37
(-5.80215,-2.93545,-4.10883,-0.49340,4.90000)	40
(-6.16200,-4.30138,-2.34831,-0.74267,5.00000)	33
(-6.13467,-3.81365,-2.84025,-1.32465,5.10000)	3
(-5.93290,-3.63383,-3.13606,-1.30366,5.20000)	13
(-5.78394,-3.29095,-3.53915,-1.44625,5.30000)	41
(-5.76783,-2.64471,-4.37488,-1.29409,5.20000)	4
(-6.14286,-3.99057,-2.61556,-1.57180,5.30000)	29
(-5.99785,-3.62990,-3.02113,-1.73146,5.40000)	93
(-5.79608,-3.45008,-3.31694,-1.71047,5.50000)	4
(-5.84802,-3.12375,-3.62589,-1.61198,5.40000)	12
(-5.66218,-3.91235,-3.45946,-1.42814,5.30000)	12
(-5.69511,-3.46683,-3.91664,-1.32926,5.20000)	9
(-5.61642,-4.42346,-3.53797,-0.57892,5.10000)	41
(-5.49363,-4.02037,-4.00232,-0.97017,5.20000)	27
(-5.52655,-3.57485,-4.45950,-0.87129,5.10000)	115
(-5.57850,-3.24852,-4.76845,-0.77280,5.00000)	11
(-5.98023,-4.07975,-3.52298,-0.91655,5.10000)	41
(-5.91576,-3.61913,-4.35768,-0.64556,5.00000)	85
(-5.96771,-3.29280,-4.66663,-0.54707,4.90000)	3
(-6.25162,-3.95580,-3.61814,-0.58918,5.00000)	62

(-6.09258,-3.73459,-3.92997,-0.60091,5.10000)	57
(-5.91383,-3.25659,-4.44359,-0.92453,5.20000)	61
(-6.24075,-4.53065,-2.88786,-0.93871,5.30000)	31
(-6.09574,-4.16998,-3.29343,-1.09837,5.40000)	24
(-5.92243,-3.77454,-3.76755,-1.34854,5.50000)	43
(-5.71933,-3.58934,-4.06876,-1.34824,5.60000)	41
(-5.77128,-3.26301,-4.37771,-1.24975,5.50000)	13
(-6.14089,-4.63080,-2.61976,-1.51156,5.60000)	4
(-5.97290,-4.21012,-3.07374,-1.75087,5.70000)	8
(-5.73373,-3.75447,-3.57262,-2.04074,5.80000)	80
(-5.65504,-4.71110,-3.19395,-1.29040,5.70000)	34
(-5.43561,-4.25607,-3.69155,-1.56294,5.80000)	16
(-5.52858,-3.87636,-4.15584,-1.53337,5.70000)	239
(-5.28941,-3.42071,-4.65472,-1.82324,5.80000)	24
(-5.69159,-4.25111,-3.39922,-1.97423,5.90000)	14
(-5.74354,-3.92478,-3.70817,-1.87574,5.80000)	86
(-5.58450,-3.70357,-4.02000,-1.88747,5.90000)	3
(-5.63645,-3.37724,-4.32895,-1.78898,5.80000)	7
(-5.66937,-2.93172,-4.78613,-1.69010,5.70000)	5
(-6.13275,-4.20681,-3.11447,-2.01178,5.80000)	13
(-6.18470,-3.88048,-3.42342,-1.91329,5.70000)	34
(-6.16859,-3.23424,-4.25915,-1.76113,5.60000)	118
(-6.08990,-4.19087,-3.88048,-1.01079,5.50000)	23
(-5.91659,-3.79543,-4.35460,-1.26096,5.60000)	77
(-5.94951,-3.34991,-4.81178,-1.16208,5.50000)	15
(-5.68472,-4.36365,-4.67868,-0.61501,5.40000)	23
(-5.73667,-4.03732,-4.98763,-0.51651,5.30000)	179
(-6.05972,-4.75867,-3.82330,-0.61113,5.40000)	11

6.2 Code for Linear Regression Question

```
function result = gradient(w,x,y)
    result = zeros(1,length(w));
    for j=1:length(w)
        sum = 0;
        for i=1:length(x)
            disp(x(i,:))
            sum = sum + (y(i)-dot(w,x(i,:)))*x(i,j);
        end
        result(j) = -sum;
    end
end

function w = gradientDescent(x,y)
    w = [0 0 0 0];
```

```
    t = 0;
    while t < 1000
        w = w - 0.05 * gradient(w,x,y);
        t = t + 1;
    end
end

function w = stochasticGradientDescent(x,y)
    w = [0 0 0 0];
    t = 0;
    while t < 1
        for i=1:length(y)
            grad = zeros(1,length(w));
            for j=1:length(w)
                grad(j) = (y(i)-dot(w,x(i,:)))*x(i,j);
            end
            disp(grad)
            w = w + 0.1 * grad;
            disp(w)
        end
        t = t + 1;
    end
end
```