CS 5350/6350: Machine Learining Spring 2018

Homework 1 Solutions

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1 Hypothesis Space

1. [30 points] Conjunctions:

(a) **Answer:**

Φ (Always False)
x_1
$\neg x_1$
x_2
$\neg x_2$
x_3
$\neg x_3$
x_4
$\neg x_4$
$x_1 \wedge x_2$
$\neg x_1 \wedge x_2$
$x_1 \wedge \neg x_2$
$\neg x_1 \wedge \neg x_2$
$x_1 \wedge x_3$
$\neg x_1 \wedge x_3$
$x_1 \wedge \neg x_3$
$\neg x_1 \wedge \neg x_3$
$x_1 \wedge x_4$
$\neg x_1 \wedge x_4$
$x_1 \wedge \neg x_4$
$\neg x_1 \wedge \neg x_4$
$x_2 \wedge x_3$
$\neg x_2 \wedge x_3$
$x_2 \wedge \neg x_3$
$\neg x_2 \wedge \neg x_3$
$x_2 \wedge x_4$
$\neg x_2 \wedge x_4$
$x_2 \wedge \neg x_4$
$\neg x_2 \wedge \neg x_4$
$x_3 \wedge x_4$
$\neg x_3 \wedge x_4$
$x_3 \wedge \neg x_4$
$\neg x_3 \wedge \neg x_4$
$x_1 \wedge x_2 \wedge x_3$
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$x_1 \wedge \neg x_2 \wedge x_3$
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$\neg x_1 \wedge x_2 \wedge x_3 \wedge \neg x_4$
$x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4$
$x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4$

$x_1 \wedge x_2 \wedge \neg x_3 \wedge \neg x_4$ $\quad$ $\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4$ $\quad$ $\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4$

$\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4$ $\quad$ $x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4$

(b) **Answer:**
$\neg x_2 \wedge x_4$

(c) **Answer:**
3^n(each variable can be positive, negative or nothing)

(d) **Answer:**
$2^{2^n} = 4^n$

2. [30 points] m-of-n rules:

(a) **Answer:**

Always false
at least 1 of $\{x_1\}$ are 1
at least 1 of $\{x_2\}$ are 1
at least 1 of $\{x_3\}$ are 1
at least 1 of $\{x_4\}$ are 1
at least 1 of $\{x_1, x_2\}$ are 1
at least 2 of $\{x_1, x_2\}$ are 1
at least 1 of $\{x_1, x_3\}$ are 1
at least 2 of $\{x_1, x_3\}$ are 1
at least 1 of $\{x_1, x_4\}$ are 1
at least 2 of $\{x_1, x_4\}$ are 1
at least 1 of $\{x_2, x_3\}$ are 1
at least 2 of $\{x_2, x_3\}$ are 1
at least 1 of $\{x_2, x_4\}$ are 1
at least 2 of $\{x_2, x_4\}$ are 1
at least 1 of $\{x_3, x_4\}$ are 1
at least 2 of $\{x_3, x_4\}$ are 1
at least 1 of $\{x_1, x_2, x_3\}$ are 1
at least 2 of $\{x_1, x_2, x_3\}$ are 1
at least 3 of $\{x_1, x_2, x_3\}$ are 1
at least 1 of $\{x_1, x_2, x_4\}$ are 1
at least 2 of $\{x_1, x_2, x_4\}$ are 1
at least 3 of $\{x_1, x_2, x_4\}$ are 1
at least 1 of $\{x_1, x_3, x_4\}$ are 1
at least 2 of $\{x_1, x_3, x_4\}$ are 1
at least 3 of $\{x_1, x_3, x_4\}$ are 1
at least 1 of $\{x_2, x_3, x_4\}$ are 1
at least 2 of $\{x_2, x_3, x_4\}$ are 1
at least 3 of $\{x_2, x_3, x_4\}$ are 1
at least 1 of $\{x_1, x_2, x_3, x_4\}$ are 1
at least 2 of $\{x_1, x_2, x_3, x_4\}$ are 1
at least 3 of $\{x_1, x_2, x_3, x_4\}$ are 1
at least 4 of $\{x_1, x_2, x_3, x_4\}$ are 1
Always True

(b) **Answer:**
at least 2 of $\{x_1, x_3, x_4\}$ are 1

(c) **Answer:**
$\sum_{i=0}^{n} i\binom{n}{i} = n \cdot 2^{n-1}$

(d) **Answer:**
No, it's not. The m-of-n rule has a smaller hypothesis space than conjunctions.

2 Decision Tree

1. [20 points]

(a) **Answer:**

- **Step 1**
  Current Entropy,

$$H = -(\frac{2}{7}\log_2\frac{2}{7} + \frac{5}{7}\log_2\frac{5}{7}) = 0.8631$$

  Expected Entropy,

$$\begin{aligned}
H_{x_1} &= \frac{5}{7}\cdot[-(\frac{1}{5}\log_2\frac{1}{5} + \frac{4}{5}\log_2\frac{4}{5})] + \frac{2}{7}\cdot[-(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2})] \\
&= 0.8014 \\
H_{x_2} &= \frac{3}{7}\cdot[-(\frac{1}{3}\log_2\frac{1}{3} + \frac{2}{3}\log_2\frac{2}{3})] + \frac{4}{7}\cdot[-(1\log_2 1)] \\
&= 0.3936 \\
H_{x_3} &= \frac{4}{7}\cdot[-(\frac{1}{4}\log_2\frac{1}{4} + \frac{3}{4}\log_2\frac{3}{4})] + \frac{3}{7}\cdot[-(\frac{1}{3}\log_2\frac{1}{3} + \frac{2}{3}\log_2\frac{2}{3})] \\
&= 0.8571 \\
H_{x_4} &= \frac{4}{7}\cdot[-(1\log_2 1)] + \frac{3}{7}\cdot[-(\frac{2}{3}\log_2\frac{2}{3} + \frac{1}{3}\log_2\frac{1}{3})] \\
&= 0.3936
\end{aligned}$$

  Expected Information Gain,

$$Gain_{x_1} = H - H_{x_1} = 0.8631 - 0.8014 = 0.0617$$

$$Gain_{x_2} = H - H_{x_2} = 0.8631 - 0.3936 = 0.4695$$

$$Gain_{x_3} = H - H_{x_3} = 0.8631 - 0.8571 = 0.0060$$

$$Gain_{x_4} = H - H_{x_4} = 0.8631 - 0.3936 = 0.4695$$

  Thus, we split attribute x_2.

- **Step 2**
  For $x_2 = 0$, the data subset is as follows,

x_1	x_3	x_4	y
0	1	0	0
0	1	1	1
1	0	1	1

  Current Entropy,

$$H = -(\frac{1}{3}\log_2\frac{1}{3} + \frac{2}{3}\log_2\frac{2}{3}) = 0.9183$$

Expected Entropy,

$$\begin{aligned} H_{x_1} &= \frac{2}{3} \cdot [-(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2})] + \frac{1}{3} \cdot [-(1\log_2 1)] \\ &= 0.6667 \\ H_{x_3} &= \frac{1}{3} \cdot [-(1\log_2 1)] + \frac{2}{3} \cdot [-(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2})] \\ &= 0.6667 \\ H_{x_4} &= 2 \cdot (-1\log_2 1) \\ &= 0 \end{aligned}$$

Expected Information Gain,

$$Gain_{x_1} = H - H_{x_1} = 0.9183 - 0.6667 = 0.2516$$

$$Gain_{x_3} = H - H_{x_3} = 0.9183 - 0.6667 = 0.2516$$

$$Gain_{x_4} = H - H_{x_4} = 0.9183 - 0 = 0.9183$$

Thus, we split attribute x_4.
For $x_2 = 1$, the data subset is as follows,

x_1	x_3	x_4	y
0	0	0	0
0	1	0	0
1	0	0	0
0	0	1	0

Thus, the data is pure. We can create a leaf node and assign 0 as the label.

- **Step 3** For $x_2 = 0$ and $x_4 = 0$, the data subset is as follows,

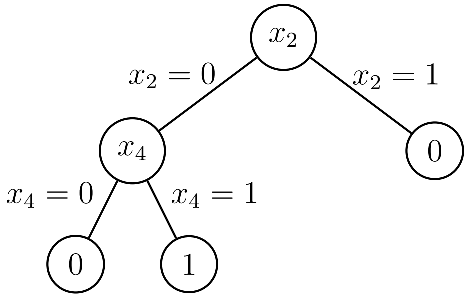
x_1	x_3	y
0	1	0

Thus, the data is pure. We can create a leaf node and assign 0 as the label.
For $x_2 = 0$ and $x_4 = 1$, the data subset is as follows,

x_1	x_3	y
0	1	1
1	0	1

Thus, the data is pure. We can create a leaf node and assign 1 as the label.

Since all the branches reach the leaf node, the tree is constructed successfully,



(b) **Answer:**

x_1	x_2	x_3	x_4	y
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

(c) **Answer:**
Because `ID3 Algorithm` is a greedy heuristic algorithm. It will not try to explore the entire hypothesis space and find the optimal result. Each step of the iteration will guarantee to find the purest way to split the data for that particular step. The overall solution may not be the best, but will be a pretty good one.

2. [20 points]

(a) **Answer:**

- **Step 1**
  Current Entropy,

$$H = -(\frac{2}{12}\log_2\frac{2}{12} + \frac{7}{12}\log_2\frac{7}{12} + \frac{3}{12}\log_2\frac{3}{12}) = 1.3844$$

Expected Entropy,

$$\begin{aligned}H_{shape} =&\frac{6}{12}\cdot[-(\frac{2}{6}\log_2\frac{2}{6}+\frac{2}{6}\log_2\frac{2}{6}+\frac{2}{6}\log_2\frac{2}{6})]+\frac{2}{12}\cdot[-(1\log_2 1)]+\\ &\frac{4}{12}\cdot[-(\frac{3}{4}\log_2\frac{3}{4}+\frac{1}{4}\log_2\frac{1}{4})]\\ =&1.0635\\ H_{color} =&\frac{3}{12}\cdot[-(1\log_2 1)]+\frac{4}{12}\cdot[-(\frac{2}{4}\log_2\frac{2}{4}+\frac{2}{4}\log_2\frac{2}{4})]+\\ &\frac{5}{12}\cdot[-(\frac{2}{5}\log_2\frac{2}{5}+\frac{2}{5}\log_2\frac{2}{5}+\frac{1}{5}\log_2\frac{1}{5})]\\ =&0.9675\end{aligned}$$

Expected Information Gain,

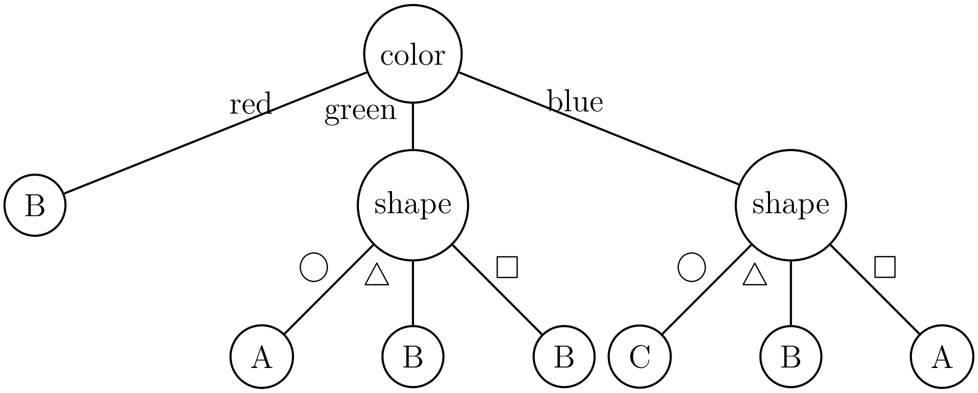
$$Gain_{shape} = H - H_{shape} = 1.3844 - 1.0635 = 0.3209$$

$$Gain_{color} = H - H_{color} = 1.3844 - 0.9675 = 0.4169$$

Thus, we split attribute **color**.

- **Step 2**
  For color=red, the data is pure. We can assign the label as B.
  For color=green, the data is not pure. We continue split with the remaining attribute **shape**.
  For color=blue, the data is not pure. We continue split with the remaining attribute **shape**.
- **Step 3**
  For color=green and shape=circle, all the data are labeled as A. Thus we can create a leaf node with label A.
  For color=green and shape=square, all the data are labeled as B. Thus we can create a leaf node with label B.
  For color=green and shape=triangle, there are no data. Thus we create a leaf node and assign the most common label in color=green. For this particular case, label A and label B are equally frequent. Thus we can assign the either of them. I will assign label B.
  For color=blue and shape=circle, all the data are labeled as C. Thus we can create a leaf node with label C.
  For color=blue and shape=square, all the data are labeled as A. Thus we can create a leaf node with label A.
  For color=blue and shape=triangle, all the data are labeled as B. Thus we can create a leaf node with label B.

Since all the branches reach the leaf node, the tree is constructed successfully,



(b) **Answer:**
Label B. (Label A is also reasonable because they are equally common.)

(c) **Answer:**
My tree take the corner case where the example set is empty into consideration. Namely, create a leaf node with the most common label in the upper level example set. However, the example tree does nothing when it encounter an empty example set.

3. [20 points]

(a) **Answer:**
The most common value for O(utlook) is S(unny) or R(ainy), both of which have 5 examples. I will pick S(unny) as the missing value.
Current Entropy,

$$H = -(\frac{5}{15}\log_2\frac{5}{15} + \frac{10}{15}\log_2\frac{10}{15}) = 0.9183$$

Expected Entropy,

$$\begin{aligned}
H_{Outlook} =& \frac{6}{15} \cdot [-(\frac{3}{6} \log_2 \frac{3}{6} + \frac{3}{6} \log_2 \frac{3}{6})] + \frac{4}{15} \cdot [-1 \log_2 1] + \\
& \frac{5}{15} \cdot [-(\frac{3}{5} \log_2 \frac{3}{5} + \frac{2}{5} \log_2 \frac{2}{5})] \\
=& 0.7237 \\
H_{Temperature} =& \frac{4}{15} \cdot [-(\frac{2}{4} \log_2 \frac{2}{4} + \frac{2}{4} \log_2 \frac{2}{4})] + \frac{7}{15} \cdot [-(\frac{2}{7} \log_2 \frac{2}{7} + \frac{5}{7} \log_2 \frac{5}{7})] + \\
& \frac{4}{15} \cdot [-(\frac{3}{4} \log_2 \frac{3}{4} + \frac{1}{4} \log_2 \frac{1}{4})] \\
=& 0.8858 \\
H_{Humidity} =& \frac{7}{15} \cdot [-(\frac{3}{7} \log_2 \frac{3}{7} + \frac{4}{7} \log_2 \frac{4}{7})] + \frac{8}{15} \cdot [-(\frac{1}{8} \log_2 \frac{1}{8} + \frac{7}{8} \log_2 \frac{7}{8})] \\
=& 0.7497 \\
H_{Wind} =& \frac{9}{15} \cdot [-(\frac{7}{9} \log_2 \frac{7}{9} + \frac{2}{9} \log_2 \frac{2}{9})] + \frac{6}{15} \cdot [-(\frac{3}{6} \log_2 \frac{3}{6} + \frac{3}{6} \log_2 \frac{3}{6})] \\
=& 0.8585
\end{aligned}$$

Expected Information Gain,

$$Gain_{Outlook} = H - H_{Outlook} = 0.9183 - 0.7237 = 0.1946$$

$$Gain_{Temperature} = H - H_{Temperature} = 0.9183 - 0.8858 = 0.0325$$

$$Gain_{Humidity} = H - H_{Humidity} = 0.9183 - 0.7497 = 0.1686$$

$$Gain_{Wind} = H - H_{Wind} = 0.9183 - 0.8585 = 0.0598$$

(b) **Answer:**
The most common value with the same label for O(utlook) is O(vercast), which has 4 examples.
Current Entropy $H = 0.9183$ and entropies $H_{Temperature} = 0.8858$, $H_{Humidity} = 0.7497$, $H_{Wind} = 0.7497$ remain the same as `answer(a)`.
Expected Entropy for Outlook,

$$\begin{aligned}
H_{Outlook} =& \frac{5}{15} \cdot [-(\frac{3}{5} \log_2 \frac{3}{5} + \frac{2}{5} \log_2 \frac{2}{5})] + \frac{5}{15} \cdot [-1 \log_2 1] + \\
& \frac{5}{15} \cdot [-(\frac{3}{5} \log_2 \frac{3}{5} + \frac{2}{5} \log_2 \frac{2}{5})] \\
=& \ 0.6473
\end{aligned}$$

Expected Information Gain,

$$Gain_{Outlook} = H - H_{Outlook} = 0.9183 - 0.6473 = 0.2710$$

$$Gain_{Temperature} = H - H_{Temperature} = 0.9183 - 0.8858 = 0.0325$$

$$Gain_{Humidity} = H - H_{Humidity} = 0.9183 - 0.7497 = 0.1686$$

$$Gain_{Wind} = H - H_{Wind} = 0.9183 - 0.8585 = 0.0598$$

(c) **Answer:**
For all the examples, there are $\frac{5}{14}$ S(unny), $\frac{4}{14}$ O(vercast), and $\frac{5}{14}$ R(ainy). We split the missing data and add fractional counts to the each possible value, namely $\frac{5}{14}$ S(unny) +, $\frac{4}{14}$ O(vercast) +, and $\frac{5}{14}$ R(ainy) +.
Current Entropy $H = 0.9183$ and entropies $H_{Temperature} = 0.8858$, $H_{Humidity} = 0.7497$, $H_{Wind} = 0.7497$ remain the same as `answer(a)`.
Expected Entropy for Outlook,

$$\begin{aligned} H_{Outlook} =& \frac{5+\frac{5}{14}}{15} \cdot [-(\frac{2+\frac{5}{14}}{5+\frac{5}{14}} \log_2 \frac{2+\frac{5}{14}}{5+\frac{5}{14}} + \frac{3}{5+\frac{5}{14}} \log_2 \frac{3}{5+\frac{5}{14}})]+ \\ & \frac{4+\frac{4}{14}}{15} \cdot [-1 \log_2 1]+ \\ & \frac{5+\frac{5}{14}}{15} \cdot [-(\frac{3+\frac{5}{14}}{5+\frac{5}{14}} \log_2 \frac{3+\frac{5}{14}}{5+\frac{5}{14}} + \frac{2}{5+\frac{5}{14}} \log_2 \frac{2}{5+\frac{5}{14}})]+ \\ =& 0.6939 \end{aligned}$$

Expected Information Gain,

$$Gain_{Outlook} = H - H_{Outlook} = 0.9183 - 0.6939 = 0.2244$$

$$Gain_{Temperature} = H - H_{Temperature} = 0.9183 - 0.8858 = 0.0325$$

$$Gain_{Humidity} = H - H_{Humidity} = 0.9183 - 0.7497 = 0.1686$$

$$Gain_{Wind} = H - H_{Wind} = 0.9183 - 0.8585 = 0.0598$$

4. **Answer:**

$$\begin{aligned} Gain(S, A) =& H(S) - H(S|A) && (1) \\ =& \sum_{s \in S} -p(s) \log_2 p(s) - \sum_{a \in A} p(a) \sum_{s \in S} -p(s|a) \log_2 p(s|a) && (2) \\ =& \sum_{a \in A} p(a) \sum_{s \in S} p(s|a) \log_2 p(s|a) - \sum_{s \in S} p(s) \log_2 p(s) && (3) \\ =& \sum_{a \in A} \sum_{s \in S} p(a) p(s|a) \log_2 p(s|a) - \sum_{s \in S} \sum_{a \in A} p(s, a) \log_2 p(s) && (4) \\ =& \sum_{a \in A} \sum_{s \in S} p(s, a) \log_2 p(s|a) - \sum_{s \in S} \sum_{a \in A} p(s, a) \log_2 p(s) && (5) \\ =& \sum_{a \in A} \sum_{s \in S} p(s, a) \big[\log_2 p(s|a) - \log_2 p(s) \big] && (6) \\ =& - \sum_{a \in A} \sum_{s \in S} p(s, a) \big[\log_2 \frac{p(s)}{p(s|a)} \big] && (7) \\ =& - \sum_{a \in A} p(a) \sum_{s \in S} p(s|a) \big[\log_2 \frac{p(s)}{p(s|a)} \big] && (8) \end{aligned}$$

Let $f(x) = \log_2 x$, then $f'(x) = \dfrac{1}{x \ln 2} > 0$. $\quad \therefore f(x) = \log 2_x$ is convex.
According to Jensen's inequality,

$$\sum_x p(x) f(x) \leq f(\sum_x p(x) x) \tag{9}$$

$$\begin{aligned}
Gain(S, A) &= -\sum_{a \in A} p(a) \sum_{s \in S} p(s|a) \big[\log_2 \frac{p(s)}{p(s|a)} \big] \tag{10} \\
&\geq -\sum_{a \in A} p(a) \big[\log_2 \sum_{s \in S} \frac{p(s|a) p(s)}{p(s|a)} \big] \tag{11} \\
&\geq -\log_2 \big[\sum_{a \in A} \sum_{s \in S} \frac{p(s|a) p(s) p(a)}{p(s|a)} \big] \tag{12} \\
&= -\log_2 \big[\sum_{a \in A} p(a) \sum_{s \in S} p(s) \big] \tag{13} \\
&= -\log_2 1 \tag{14} \\
&=0 \tag{15}
\end{aligned}$$

3 Programming Assignments

2. **Answer:**

Tree Depth	Average Prediction Error			
	Entropy		Majority Error	
	Training	Testing	Training	Testing
1	0.3020	0.2967	0.3020	0.2967
2	0.2220	0.2225	0.2920	0.3132
3	0.1810	0.1964	0.1800	0.1923
4	0.0820	0.1470	0.0860	0.1511
5	0.0270	0.0838	0.0290	0.0879
6	0.0000	0.0838	0.0000	0.0879
7	0.0000	0.0838	0.0000	0.0879

3. **Answer:**

- With the depth of the tree increasing, the prediction error for training data will drop until reaches zero while the prediction error for testing data will first decrease and then remain unchanged at a number greater than zero. If there are some noises in the data (not in this case), the error may increase again because of overfitting.

- When decision tree is shallow, the prediction error for training data might be larger than for testing data. If the tree grows deeper, the prediction error of training data will drop faster and be smaller than that of testing data.

- Entropy and Majority Error are both good for calculating information gain. The prediction errors in this case are almost the same with these two mechanisms.