Lecture 10 - Lagrangian Equations of Motion

Topics:

A) Non-conservative terms.

How to systematically determine the righthand side of each equation of motion (EOM), with examples.

This involves breaking "rate of work" into a summation of terms, each with a "force" times a "velocity". (Or, for a rotational system, a "torque" times a rotational rate of velocity.)

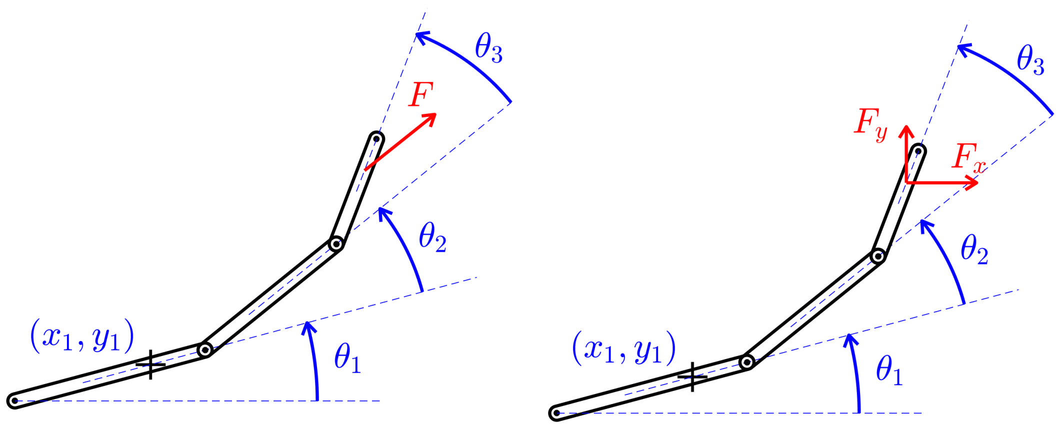
B) "How to cheat" on non-conservative terms.

The conservative terms are (often) easier to calculate (in terms of being a more systematic process) that the non-conservative terms.

However, there is a lot of symmetry in how a linear spring and a linear damper enter into the equations of motion. Because of this, you can sometimes (if you are careful, and are directly representing a linear damper as if it is a linear spring) "cheat" a bit to reason about the non-conservative forces by first calculating how a spring (placed in the same location) would affect each of the equations of motion.

Non-conservative forces and torques: Finding Xi

When deriving equations of motion through the Lagragian approach, conservative forces and torques are generated algorithmically. By contrast, non-conservative forces and torques can potentially be more challenging to calculate. In particular, the choice of Generalized Coordinates (GC's: ξ_1 through ξ_n), which are the set of independent and complete degrees of freedom used to described the dynamics of the system, must clearly affect what the non-conservative terms will be in a set of equations of motion (EOM's).



Example system. We will "walk through" calculation of the non-conservative forces using the system above. There are five GC's: x_1, y_1, θ_1, θ_2, θ_3, so there will be 5 EOM's. Conservative terms will appear on the lefthand side of the i^{th} equation, via the Lagragian approach, and all the non-conservative terms are lumped together as "Ξ_i" [i.e., "Xi" which sounds like "zigh", rhyming with "sigh", with sub-index "i" to match the equation in which it appears].

Note, x_1 and y_1 are arbitrarily-chosen coordinates on the first link. They aren't necessarily "center of mass" of the link, etc. θ_1 is an absolute angle, measure counter-clockwise (CCW) with respect to the x-axis, which θ_2 and θ_3 are relative angles, as illustrated above.

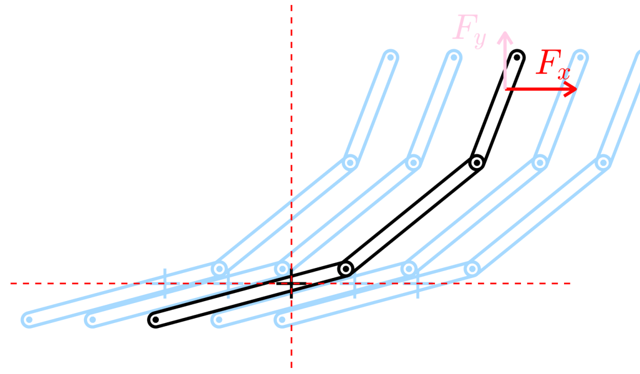
The only non-conservative force here is F, which is represented as a vector in the x-y plane on the lefthand side of the figure above. We can break F into its x and y direction components, F_x and F_y, as shown on the righthand figure; doing so will simplify our representation of Ξ later on.

Approach. Here's how to calculate the Ξ's:

1. Calculate rate of work done by all non-conservative "nc" terms, i.e. dot products of "force times velocity" and/or "torque times angular velocity". (e.g., $\dot{W}_{nc} = F_x \dot{x}_f + ...$)
2. Imagine you "wiggle" the i^{th} G.C., ξ_i, while keeping all other ξ's constant.
3. Represent velocities in the work terms (from step 1) solely in terms of the velocity of the i^{th} G.C., dxi_i/dt, i.e., write $\dot{W}_{nc}$ as "$\dot{W}_{nc} = (something)$ times $\dot{\xi}_i$".
4. The "*something*" in the step above *is* Ξ_i, i.e., $\dot{W}_{nc} = \Xi_i \dot{\xi}_i$.

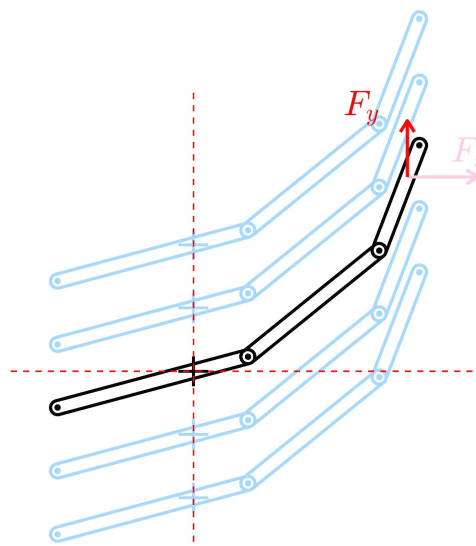
Note that one or more terms from step 1 may become zero for the i^{th} EOM, because holding (i.e., constraining) all other G.C.'s to remain constant (while only ξ_i is allowed to "wiggle") can in turn make velocity component(s) at a particular force (or torque) exactly zero.

For all five EOM's, we can use the same expression for work, from step 1: $\dot{W}_{nc} = F_x\dot{x}_F + F_y\dot{y}_f$. Here, $\dot{x}_f$ and $\dot{y}_f$ are simply the x and y velocities at the location where external force F is applied. If the direction of force has the same sign as the direction of velocity, the corresponding work term is positive.



The first G.C. is $\xi_1 = x_1$. If we wiggle only x_1, the entire system moves as a rigid unit, in left/right motion. In this case, $\dot{x}_f = \dot{x}_1$, since every point in the system has the same, identical velocity in the x direction. Also, $\dot{y}_f = 0$. Thus, we can write the work done by all non-conservative forces as: $\dot{W}_{nc} = F_x\dot{x}_1$. Since $\dot{\xi}_1 = \dot{x}_1$,

$$\Xi_1 = F_x.$$



If we reconsider the results for Ξ_1 and Ξ_2, above, the corresponding conservative terms (on the lefthand side of each EOM) would represent some (potentially messy-looking) representation of "mass time acceleration" minus whatever conservative forces are needed in the equation, for *a*ll the masses in the system (since all masses move when you wiggle $\xi_1 = x$. However, although these conservative terms may look messy, the Lagrangian approach gives a very systematic method for obtaining them, which can be automated in part by using MATLAB's symbolic toolbox for integration.

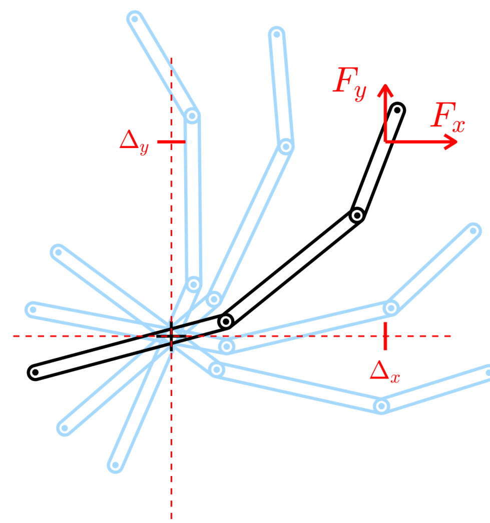
Said another way, each of these equations will be stating "$F = ma$" (force equals mass times acceleration). Imagine that "mass times acceleration" and the negative of any conservative terms within F will end up on the lefthand side of the equation. You are determining the non-conservative forces (or torque) due to active forces and torques and/or loss elements (dampers) in the system.

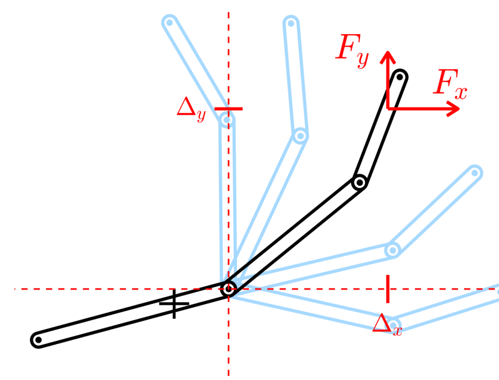
The next generalized coordinate is $\xi_2 = y_1$. When y_1 varies, while keeping x_1, θ_1, θ_2 and θ_3 all constant, the entire system more up/down, as illustrated at right. Here, $dx_f/dt = \dot{x}_f = 0$, and $\dot{y}_f = \dot{y}_1 = \dot{\xi}_2$.

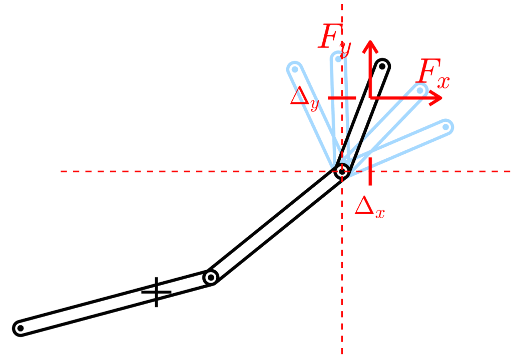
Rate of work is: $\dot{W}_{nc} = F_x\dot{x}_f + F_y\dot{y}_f = F_y\dot{\xi}_2$, since there is zero velocity in x, and velocity everywhere in the system is the same. Here,

$$\Xi_2 = F_y.$$

For the remaining three Generalized Coordinates (G.C.'s), here is the key:







Imagine all OTHER G.C.'s remain frozen at their respect values. In this example, we are using RELATIVE angles, so the if θ_1 moves, as shown in the topmost figure to the right, then imagine x and y remain fixed, as do the relative angles θ_2 and θ_3.

What is the virtual work done by all the non-conservative forces, when θ_1 (the 3^{rd} Generalized Coordinate) experiences a winy wiggle? Here, since θ_1 is an angle, the corresponding "big Xi", Ξ_3, must be a torque. Ξ_1 and Ξ_2 were forces, to match translational DOF's, while Ξ_3 through Ξ_5 will all have units of torque.

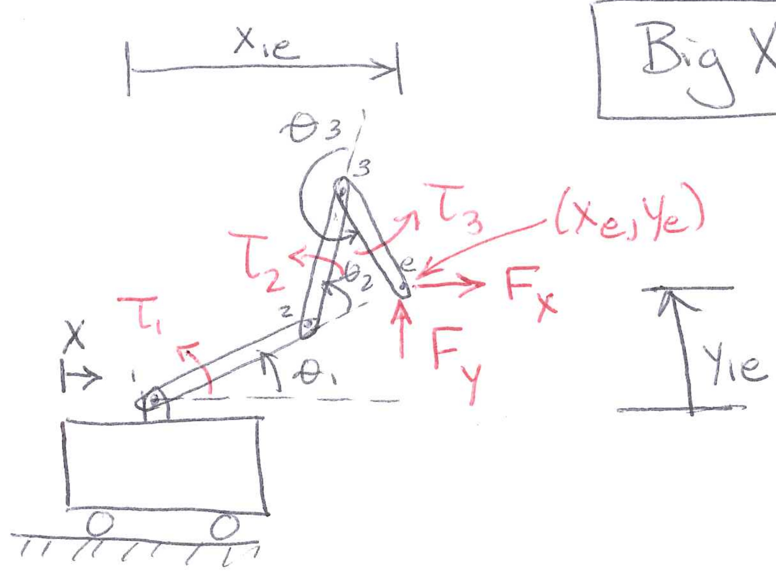
For each of the three remaining equations of motion, Ξ will be $\Xi_i = F_y \Delta x_i - F_x \Delta y_i$, where Δx_i and Δy_i will be distances from the point of rotation to the point where force is applied. Note that for each of the three cases at right, the point of rotation is just the axis that particular DOF rotates about.

Note that the direction of positive torque should be the same as the direction of position rotation. In general, we define counter-clockwise (CCW) rotations as "positive", as illustrated on page 1. Therefore, a torque is also positive when it acts in the CCW direction. F_x and F_y are defined to be positive when acting in the direction of their respective arrows. Correspondingly, when $\Delta x > 0$ and $F_y > 0$, a positive (CCW) torque is generated; however, when $\Delta y > 0$ and $F_x > 0$, a negative (clockwise) torque is generated. This explains the signs of the terms in $\Xi_i = F_y \Delta x_i - F_x \Delta y_i$.

Finally, generalize this approach to address any additional non-conservative torques or forces in the system. In ECE/ME 179D, these will simply include either (a) forces and/or torques actively applied by actuators or the external world, or (b) forces and/or torques passively due to the a damper (also sometimes called a dashpot). Recall that a damper is a dissipative mechanical impedance, with force (or torque) proportional to the velocity across the damper. To find each Xi_i, simply imagine whether there is any virtual work (force times displacement, or torque times angular displacement) due to a wiggle of that particular ξ_i.

ECE/ME 179D

$\boxed{\text{Big Xi}}$ examples...



1) Write out all "rates of work" by non-conservative forces and torques.

$$\tau_1\dot{\theta}_1 + \tau_2\dot{\theta}_2 + \tau_3\dot{\theta}_3 + F_x\dot{x}_e + F_y\dot{y}_e = \dot{W}$$

2) For each G.C., ξ_1 through ξ_n, "freeze" all other DOF's ($\dot{\xi}=0$), and then:

$$\Xi_i\,\dot{\xi}_i = \dot{W}$$

relative angles

$\boxed{\text{Case A}}$ ↳ Example. Say $\boxed{\xi_1=\theta_1,\ \xi_2=\theta_2,\ \xi_3=\theta_3,\ \xi_4=x.}$

For $i=1$, all velocities but $\dot{\xi}_1$ are "held frozen",

Now: $\dot{\theta}_1=\dot{\xi}_1,\ \dot{\theta}_2=0=\dot{\theta}_3=\dot{x}$

What are $\dot{x}_e$ and $\dot{y}_e$? Well,

$$x_e = x + L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3)$$

$$\dot{x}_e = \underbrace{\cancel{\dot{x}}}_{0} - L_1\sin\theta_1\dot{\theta}_1 - L_2\sin(\theta_1+\theta_2)(\dot{\theta}_1+\underbrace{\cancel{\dot{\theta}_2}}_{0}) - L_3\sin(\theta_1+\theta_2+\theta_3)(\dot{\theta}_1+\underbrace{\cancel{\dot{\theta}_2}}_{0}+\underbrace{\cancel{\dot{\theta}_3}}_{0})$$

$$\therefore \dot{x}_e = \left(-L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) - L_3\sin(\theta_1+\theta_2+\theta_3)\right)\dot{\theta}_1 = -y_{1e}\dot{\theta}_1$$

$$\Xi_1 = \frac{\tau_1\dot{\theta}_1 + \tau_2\cancel{\dot{\theta}_2}^{\,0} + \tau_3\cancel{\dot{\theta}_3}^{\,0} + F_x\dot{x}_e + F_y\dot{y}_e}{\dot{\theta}_1} \quad ?$$

Now, for $i=1$, find $\dot{y}_e$

$$y_e = L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3)$$

$$\dot{y}_e = L_1\cos\theta_1\dot{\theta}_1 + L_2\cos(\theta_1+\theta_2)(\dot{\theta}_1+\cancelto{0}{\dot{\theta}_2}) + L_3\cos(\theta_1+\theta_2+\theta_3)(\dot{\theta}_1+\cancelto{0}{\dot{\theta}_2}+\cancelto{0}{\dot{\theta}_3})$$

$$\dot{y}_e = \left(L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3)\right)\dot{\theta}_1 = +x_{1e}\dot{\theta}_1$$

So,

$$\tilde{\Xi}_1 = \left(\frac{1}{\dot{\theta}_1}\right) * \left(\tau_1\dot{\theta}_1 - F_x y_{1e}\dot{\theta}_1 + F_y x_{1e}\dot{\theta}_1\right)$$

$$\boxed{\tilde{\Xi}_1 = \tau_1 - F_x y_{1e} + F_y x_{1e}}$$

For $i = 2 \to 5$,

$$\boxed{\tilde{\Xi}_2 = \tau_2 - F_x y_{2e} + F_y x_{2e}}$$

$$\boxed{\tilde{\Xi}_3 = \tau_3 - F_x y_{3e} + F_y x_{3e}}$$

$$\boxed{\tilde{\Xi}_4 = F_x}$$ ← Because $\dot{x}_e = \dot{x} - L_1\sin\theta_1\dot{\theta} - \dots$,

when $\dot{\xi}_1 = \dot{\xi}_2 = \dot{\xi}_3 = 0$,

then $\dot{x}_e = \dot{x}$,

so $\tilde{\Xi}_4 = F_x$

$\boxed{\text{Case B}}$ → Same system, w/ different G.C.'s...

$$\boxed{\underbrace{\xi_1 = \theta_1\ ,\ \xi_2 = \theta_1 + \theta_2\ ,\ \xi_3 = \theta_1 + \theta_2 + \theta_3}_{\text{absolute angles.}}\ ,\ \xi_4 = X}$$

As before: $\dot{W} = \tau_1\dot{\theta}_1 + \tau_2\dot{\theta}_2 + \tau_3\dot{\theta}_3 + F_x\dot{x}_e + F_y\dot{y}_e$

For $i=1$, $\dot{\xi}_1 = \dot{\theta}_1$

Other GC, hold frozen! → $\dot{\xi}_2 = 0 = \dot{\theta}_1 + \dot{\theta}_2$ → $\therefore\ \dot{\theta}_2 = -\dot{\theta}_1 = -\dot{\xi}_1$

$\dot{\xi}_3 = 0 = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$ → $\therefore\ \dot{\theta}_3 = 0$

$\dot{\xi}_4 = 0 = \dot{X}$, so $\dot{X} = 0$

Again look at $\dot{x}_e$ & $\dot{y}_e$, when $\dot{\xi}_2 = \dot{\xi}_3 = \dot{\xi}_4 = 0$:

$$\dot{x}_e = \underbrace{\cancel{\dot{X}}}_{0} - L_1 \sin\theta_1 \dot{\theta}_1 - L_2 \sin(\theta_1+\theta_2)\underbrace{\cancel{(\dot{\theta}_1+\dot{\theta}_2)}}_{0} - L_3 \sin(\theta_1+\theta_2+\theta_3)\cancel{(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)}$$

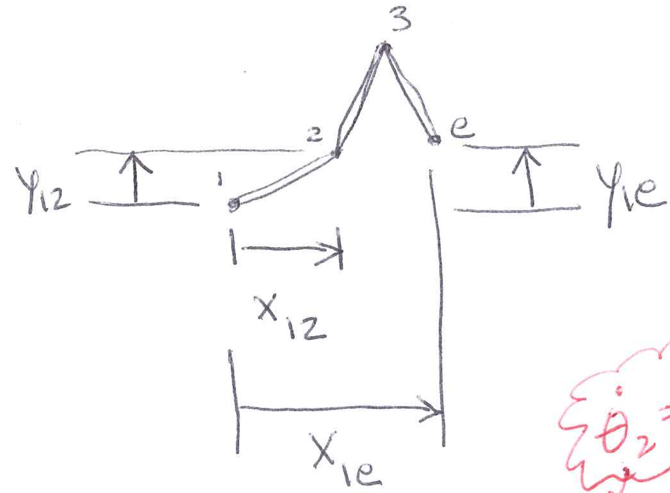
$\boxed{\dot{\theta}_2 = -\dot{\theta}_1}$ $\boxed{\begin{matrix}\dot{\theta}_2 = -\dot{\theta}_1 \\ \dot{\theta}_3 = 0\end{matrix}}$

$$\therefore\ \boxed{\dot{x}_e = -L_1 \sin\theta_1 \dot{\theta}_1}$$

$$\tilde{\tau}_1 = \frac{\tau_1\dot{\theta}_1 + \tau_2\overset{\dot{\theta}_2 = -\dot{\theta}_1}{\dot{\theta}_2} + \tau_3\overset{0}{\cancel{\dot{\theta}_3}} + F_x\dot{x}_e + F_y\dot{y}_e}{\dot{\theta}_1}$$

also → $\boxed{\dot{y}_e = L_1\cos\theta_1\dot{\theta}_1 = +X_{12}\dot{\theta}_1}$, $\boxed{\dot{x}_e = -Y_{12}\dot{\theta}_1}$

Here, x_{ab} means "from a to b":



$\dot\theta_2 = -\dot\theta_1$ $\dot\theta_3 = 0$ $\dot x_e = -y_{12}\dot\theta_1$ $\dot y_e = +x_{12}\dot\theta_1$

So: $$\left.\begin{aligned} \dot x_e &= -y_{12}\dot\theta_1 \\ \dot y_e &= +x_{12}\dot\theta_1 \end{aligned}\right\} \rightarrow \tilde{\Xi}_1 = \frac{\tau_1\dot\theta_1 + \tau_2\dot\theta_2 + \tau_3\dot\theta_3 + F_x\dot x_e + F_y\dot y_e}{\dot\theta_1}$$

$$\boxed{\tilde{\Xi}_1 = \tau_1 - \tau_2 - F_x y_{12} + F_y x_{12}}$$

For $i = 2 \rightarrow 5$,

$$\boxed{\tilde{\Xi}_2 = \tau_2 - \tau_3 - F_x y_{23} + F_y x_{23}}$$

$$\boxed{\tilde{\Xi}_3 = \tau_3 - F_x y_{3e} + F_y x_{3e}}$$

$$\boxed{\tilde{\Xi}_4 = F_x}$$

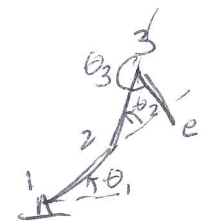
Final Comment(s)

- Although writing "x_{12}" or "y_{2e}", etc., is valid for one "snapshot" (one instant) in time, for an Equation of Motion (EOM), we would need an equation for each distance.

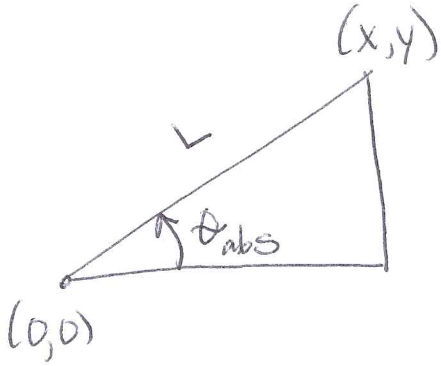
recall θ_1, θ_2 & θ_3 were relative angles,

$$X_{12} = L_1 \cos\theta_1$$

$$Y_{2e} = L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$



(et cetera)



$$\boxed{\begin{aligned} x &= L\cos(\theta_{abs}) \\ y &= L\sin(\theta_{abs}) \end{aligned}}$$

- Also, visualizing moment arms and torques can be a great "reality check" — and may be more simple than the math-via-equations approach here.

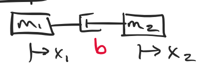
B) "Cheating" to find Xi. (Be careful! This is meant as a "double-check" or reality check. Part "A)" gives the real mathematical approach…

• "Cheating" to find Ξ_i:

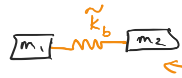
- Using the Lagrangian approach, conservative terms are generated automatically, e.g., from gravity or from spring forces.
- Non-conservative terms, in each Ξ_i, are more challenging to find.
- A "cheating" approach is to insert a "fake spring", to see its effect in the E.O.M.'s.

→ Two cases are typical

1) Damper: BETWEEN 2 DOF's

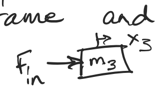


temporarily model as:

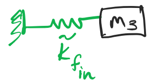


← $V = \frac{1}{2}\left(\tilde{k}_b \cdot (x_2 - x_1)^2\right)$

2) External force: between reference frame and a mass:



put spring on a wall…



← $V = \frac{1}{2}\left(\tilde{k}_{f_{in}} x_3^2\right)$

Case 1) $\xi_1 = x_1$, $\xi_2 = x_2$ (absolute coordinates)

$$T^* = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$\boxed{V = 0}$, then, pretend $\boxed{\tilde{V} = \frac{1}{2}\left(k_b \cdot (x_2 - x_1)^2\right)}$

$\mathcal{L} = T^* - V$, but pretend: $\tilde{\mathcal{L}} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \tilde{V}$

pretending: $\tilde{\mathcal{L}} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_b \left(x_1^2 - 2 x_1 x_2 + x_2^2\right)$

Also, $\boxed{\text{pretend } \Xi_1 = \Xi_2 = 0}$, ∴ only do "lefthand sides"

For ξ_1: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_1}\right) - \frac{\partial \mathcal{L}}{\partial \xi_1} = 0 \rightarrow \frac{d}{dt}\left(m_1 \dot{x}_1\right) + k_b (x_1 - x_2) = 0$

Cheat! Really, this was a DAMPER, so…

a) replace $\tilde{k}_b$ with b

b) replace positions (x_1 & x_2) with velocities ($\dot{x}_1$ & $\dot{x}_2$)

$$m_1 \ddot{x}_1 + k_b (x_1 - x_2) = 0$$

is Really: $\boxed{m_1 \ddot{x}_1 + b (\dot{x}_1 - \dot{x}_2) = 0}$

(Truly: $V = 0$, & $\Xi_1 = -b(\dot{x}_1 - \dot{x}_2)$ here…)

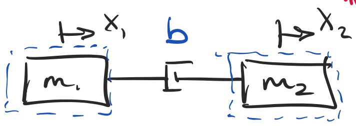
- Again this is <u>cheating</u>.

- The better approach is to use earlier notes to write:

force (or torque) — velocity at this force (or torque)

$$\dot{W} = \sum_{\forall j} F_j \cdot V_j$$

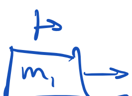
then, $$\boxed{\tilde{\Xi}_i = \sum_{\forall j} F_j \cdot \frac{\partial v_j}{\partial \dot{\xi}_j}}.$$

For: 

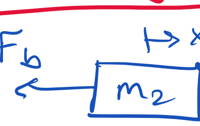
, w/ $\xi_1 = x_1$ and $\xi_2 = x_2$,

Do <u>NOT</u> differentiate the "F_j", <u>above</u>!!

$$\boxed{\dot{W} = F_b \dot{x}_1 - F_b \dot{x}_2}$$



$F_b = b(\dot{x}_2 - \dot{x}_1)$



$-F_b = -b(\dot{x}_2 - \dot{x}_1)$

$F_b = b(\dot{x}_2 - \dot{x}_1)$

$$\therefore \boxed{\tilde{\Xi}_1 = F_b = b(\dot{x}_2 - \dot{x}_1)}$$

no spring in REAL system...

and: $$\boxed{\mathcal{L} = T^* - V = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 - 0}$$

$\therefore$ E.O.M. #1 is: $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_1}\right) - \frac{\partial \mathcal{L}}{\partial \xi} = \tilde{\Xi}_1 = b(\dot{x}_2 - \dot{x}_1)$$

$$\boxed{m_1 \ddot{x}_1 = b\dot{x}_2 - b\dot{x}_1}$$

which <u>is</u>, as predicted on last page: $\boxed{m_1 \ddot{x}_1 + b(\dot{x}_1 - \dot{x}_2) = 0}$