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1.1

Yes. It is $-i$. No, just this.

1.2

No. k cannot be orthogonal to both vectors in a 2D plane.

1.3

Both will have an infinite amount of possible vectors.

2.

$$\det(A) = x_1y_2 + x_3y_1 + x_2y_3 - x_3y_2 - x_1y_3 - x_3y_2$$
$$Area = \frac{1}{2}|u \times v| = \frac{1}{2}\left|\left|\begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ 1 - 1 \end{bmatrix} \times \begin{bmatrix} x_3 - x_2 \\ y_3 - y_2 \\ 1 - 1 \end{bmatrix}\right|\right| = \frac{1}{2}|x_1y_2 + x_3y_1 + x_2y_3 - x_3y_2 - x_1y_3 - x_3y_2|$$

The value would be 0. Yes. Because the 0 is unique.

3. Plug in the two points on the matrix including a,b,c and set the determinant to 0. Solve.

4.

$$u \times v = \begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = (u_2v_3i + u_3v_1j + u_1v_2k) - (u_3v_2i + u_1v_3j + u_2v_1k)$$

As we can see, if i and j in $\{i, j, k\}$ matches the order of subscripts u_i and v_j, then it will hold its respective vector.

5.

$$\det(A) = \det\begin{bmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{bmatrix} - \det\begin{bmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_4 \end{bmatrix} + \det\begin{bmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{bmatrix} - \det\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$
$$V = \frac{\left|\left|\begin{bmatrix} x_1 - x_4 \\ y_1 - y_4 \\ z_1 - z_4 \\ 1 - 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x_2 - x_4 \\ y_2 - y_4 \\ z_2 - z_4 \\ 1 - 1 \end{bmatrix} \times \begin{bmatrix} x_3 - x_4 \\ y_3 - y_4 \\ z_3 - z_4 \\ 1 - 1 \end{bmatrix}\right)\right|\right|}{6}$$

The determinant is 0 when all points lie on a plane as they would not form a tetrahedron.

6. Plug in the 4 points on the matrix and set the determinant to 0. Solve.

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7. It is a vector orthogonal between the cross product two vector. It is also the area. The relationship of n and $p - p_0$ in terms of the dot product is the volume. As seen in questions 5 and 6, if the dot product of the cross product vector of the 3 points is 0, then the volume is 0 and thus, it must be on the plane.

8.

$$M(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$M^{-1}(\theta) = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$M^T(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$M(\theta_1 + \theta_2) = M(\theta_1)M(\theta_2)$$
$$\begin{bmatrix} \cos\theta_1 + \theta_2 & \sin\theta_1 + \theta_2 \\ -\sin\theta_1 + \theta_2 & \cos\theta_1 + \theta_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \\ -\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix}$$
$$M(-\theta) = M(\theta - \pi) = M(\theta)M(\pi) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

A sphere.

9.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$Mx_1 = b_1, \quad \begin{bmatrix} (a+b)(x_{1-1} + x_{1-2}) \\ (c+d)(x_{1-1} + x_{1-2}) \end{bmatrix} = \begin{bmatrix} b_{1-1} \\ b_{1-2} \end{bmatrix}$$
$$Mx_2 = b_2, \quad \begin{bmatrix} (a+b)(x_{2-1} + x_{2-2}) \\ (c+d)(x_{2-1} + x_{2-2}) \end{bmatrix} = \begin{bmatrix} b_{2-1} \\ b_{2-2} \end{bmatrix}$$

Solve the system of linear equations to find abcd. Then apply to Mx_3.

10.

$$(1-t)p_1 + tp_2$$
$$(1-s-t)p_1 + tp_2 + sp_3$$

For (a), all points include $t = [-\infty, \infty]$.

For (b), $t >$

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11.

$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \qquad x = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$x^T M x = ax^2 - 2bxy + cy^2$$

The matrix is an ellipsoid, thus it can be anywhere in the range between 0 and the max eigenvalue, which is the longest length.

$$Mx = \lambda x = 0, \qquad \lambda = 0$$

The matrix is a hyperboloid, so there exists a vector that is the center.

The both eigenvalues must both be positive.

12.

$$\det(M - \lambda I) = 0$$
$$\det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a\lambda + d\lambda) + (ac - cd) = \lambda^2 - trace(M)\lambda + \det(M)$$