CS189/CS289A
Introduction to Machine Learning
Lecture 6:

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Outline

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- Recall: Gaussian class conditionals lead to a logistic posterior.

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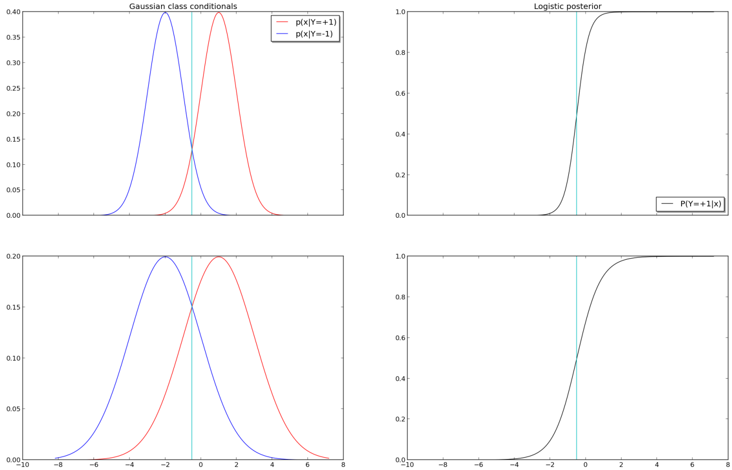
$$p(x|y=+1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu_+)^2}{2\sigma^2}\right).$$



Gaussian generative to logistic discriminative models

Class conditionals to posterior

For Gaussian class conditional densities with the same variance,

$$p(x|y = +1) = \mathcal{N}(\mu_+, \sigma^2), \qquad p(x|y = -1) = \mathcal{N}(\mu_-, \sigma^2),$$

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$$\theta = \frac{\mu_+ - \mu_-}{\sigma^2}, \qquad \theta_0 = \frac{\mu_-^2 - \mu_+^2}{2\sigma^2} - \log\left(\frac{P(-1)}{P(+1)}\right).$$

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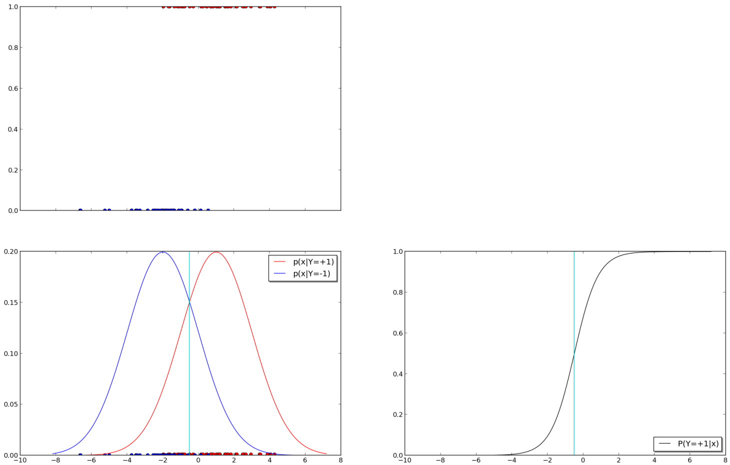
- How do we use the data to estimate the relevant probability distributions?

Estimating a Gaussian generative model



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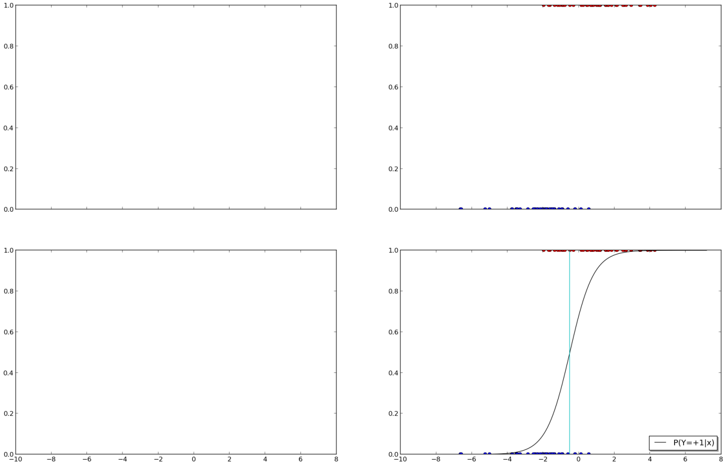
Estimating a logistic discriminative model



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- What is a good estimate of p?

Method of moments

Bernoulli estimate

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- We could choose p so that the distribution it defines has the same expectation as the average of the data.
- The number of $+1$s we see from a single coin toss is a random variable with a *Bernoulli distribution*, $\Pr(1) = p$, $\Pr(0) = 1 - p$.
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- To choose the parameter p of the distribution of the X_i so that the expectation is the average of the observed values, we choose $p = 0.6$.

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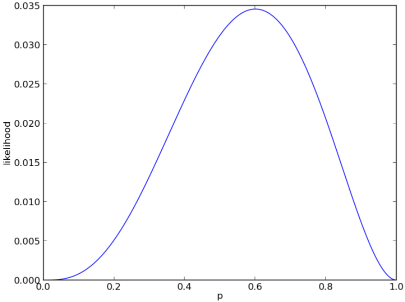

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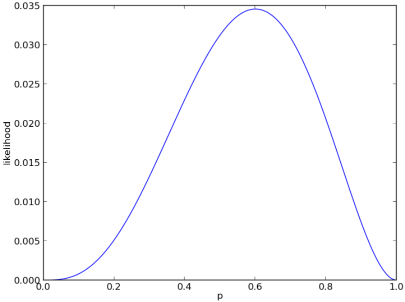

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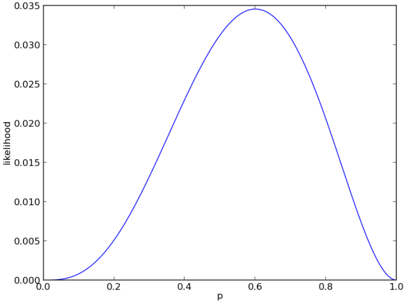

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- The maximizer of the likelihood in this case is $p = 0.6$.

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Set $0 = l'(p) = \frac{3}{p} - \frac{2}{1-p}$ and solve to get $p = 0.6$.

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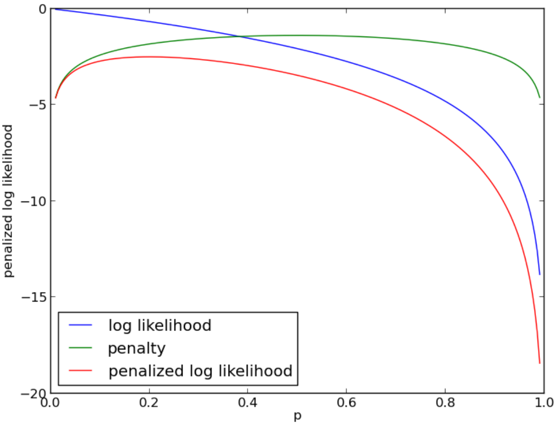
- Such estimators are particularly useful when the number of outcomes is not just 2 but large (for example, estimating the probability of words in a language).

Penalized maximum likelihood

0 successes in 3 trials:



Penalized maximum likelihood

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$$
p(n-k) = (1-p)k + 1 - 2p \qquad\qquad \hat{p} = \frac{k+1}{n+2}.
$$

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- For example, if we have no *a priori* preference for one value of p over another, we might model p as a uniformly distributed random variable.
- Each observation allows us to update our belief, via Bayes Theorem.

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Update

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prior distribution: $\pi(p) = 1$

Bayesian estimates

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$$\text{prior distribution:} \qquad \pi(p) = 1$$

$$\text{posterior distribution:} \qquad P(p|X_1 = 1) \propto \underbrace{P(X_1 = 1|p)}_{\text{likelihood}} \underbrace{\pi(p)}_{\text{prior}}$$

Bayesian estimates

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$$
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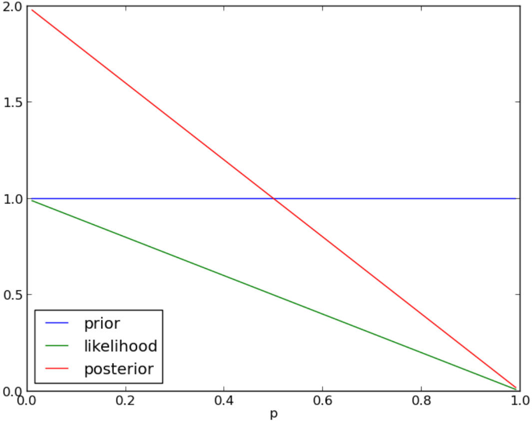
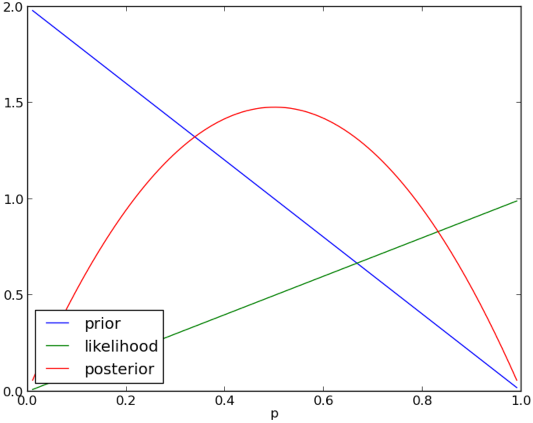
\text{prior distribution:} \quad & \pi(p) = 1 \\
\text{posterior distribution:} \quad & P(p|X_1 = 1) \propto \underbrace{P(X_1 = 1|p)}_{\text{likelihood}} \underbrace{\pi(p)}_{\text{prior}} \\
& = \frac{P(X_1 = 1|p)\pi(p)}{\int P(X_1 = 1|q)\, d\pi(q)}
\end{aligned}
$$

Prior $\pi(p) = 1$ on $[0, 1]$. $X_1 = 0$



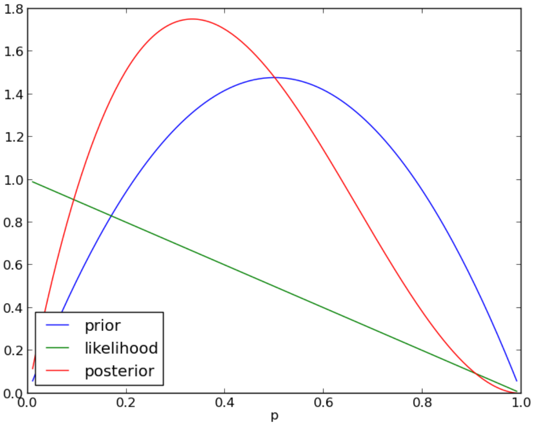
Bayesian estimates

$$X_2 = 1$$



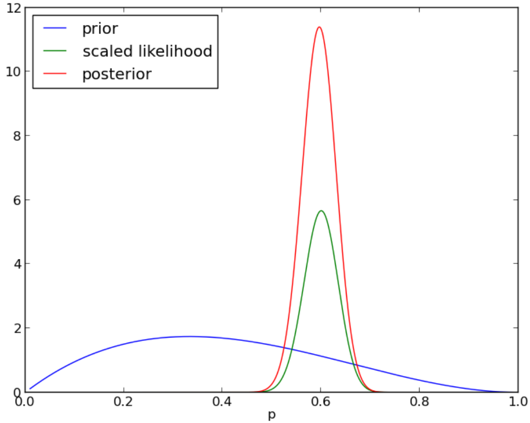
Bayesian estimates

$$X_3 = 0$$



Bayesian estimates

$$\frac{1}{200}\sum_{i=4}^{204} X_i = \frac{120}{200}$$



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- The Bayesian approach: assume that the parameters are randomly chosen with a fixed, known distribution.
- Then everything in our prediction problem is a random variable with a known distribution. In that sense, there are no unknowns, just unobserved random variables with known distributions.
- Bayesian estimation is just a computation: compute a conditional probability distribution.

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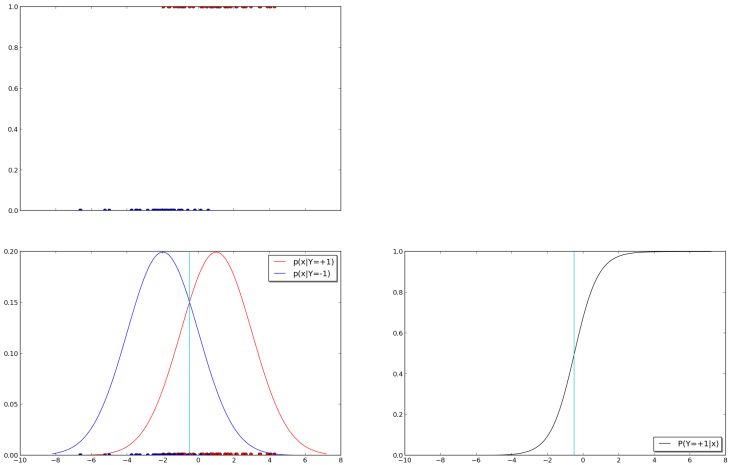
- The MAP estimate with a uniform prior corresponds to the maximum likelihood estimate.
- The MAP estimate with any other prior corresponds to a penalized maximum likelihood estimate. For instance, the penalty we considered earlier corresponds to a prior proportional to $p(1-p)$

Estimating a Gaussian generative model



Estimation

- Suppose we want to use data to solve a classification problem.
- How do we use the data to estimate the relevant probability distributions?
- How do we estimate the class probabilities? **Estimate a Bernoulli.**
- How do we estimate the class conditional distributions?

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- For instance, it might be reasonable to assume that the class conditional distribution is a Gaussian.
- Then we need to estimate the mean and variance.
- How can we do that?

Outline

- Recall: Gaussian class conditionals lead to a logistic posterior.
- Estimation.
  - Estimating the parameter of a Bernoulli random variable.
  - **Estimating the parameters of a Gaussian random variable.**
- Parameter estimation methods:
  - Method of moments.
  - Maximum likelihood.
  - Penalized maximum likelihood.
  - Bayesian estimates.

Gaussian Estimation

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- What is a good estimate of $\theta = (\mu, \sigma^2)$?

Method of moments

Gaussian Estimation

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- What about σ^2?

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- In general, if we have p parameters, we can solve p equations, and so need to match the corresponding number of moments.

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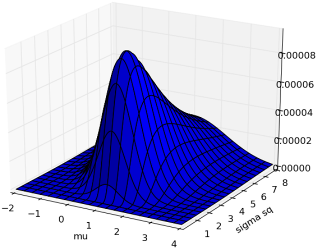

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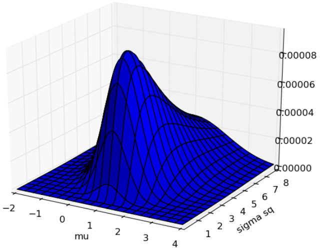

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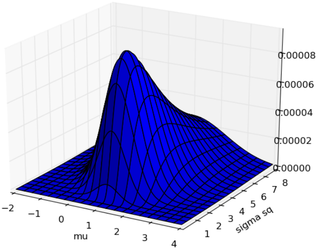

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- The maximizer of the likelihood is $\mu = 0.996$, $\sigma^2 = 2.38$.

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- Again, it's more convenient to work with the log likelihood.

$$l(\mu, \sigma^2) = \log \left(\prod_{i=1}^{n} \left(\frac{1}{(2\pi\sigma^2)^{1/2}} \exp \left(-\frac{(x_i - \mu)^2}{2\sigma^2} \right) \right) \right)$$

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Set $\nabla l(\theta) = 0$ and solve.

Maximum likelihood

Gaussian maximum likelihood estimation

$\nabla l(\theta) = 0$ for

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \frac{n}{2\sigma^2} = \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2.$$

Other estimators for Gaussian parameters

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- We can also use penalized maximum likelihood estimators:

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\end{aligned}
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- And Bayesian estimators:

$$
\begin{aligned}
\text{prior distribution:} \quad & \pi(\theta) = 1 \\
\text{posterior distribution:} \quad & p(\theta | X_1 = x_1) \propto \underbrace{p(X_1 = x_1 | \theta)}_{\text{likelihood}} \underbrace{\pi(\theta)}_{\text{prior}} \\
& = \frac{p(X_1 = x_1 | \theta) \pi(\theta)}{\int p(X_1 = x_1 | q) \, d\pi(q)}
\end{aligned}
$$

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Gaussian Estimation

- Penalized maximum likelihood estimators and Bayesian estimators are particularly effective in the high-dimensional setting, when the number of parameters is large compared to the amount of data.

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