CS189/CS289A
Introduction to Machine Learning
Lecture 5:

Peter Bartlett

February 3, 2015

Outline

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- Two facts from probability theory

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- Generative and discriminative models:
  Gaussian class conditionals lead to a logistic posterior.

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- **Two facts from probability theory**
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Two facts from probability theory

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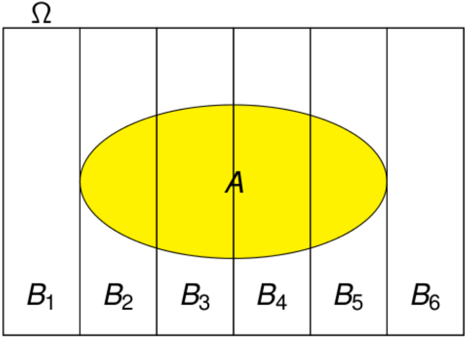
Some probability theory

Consider a discrete random variable X. Let $B_i = \{X = i\}$.



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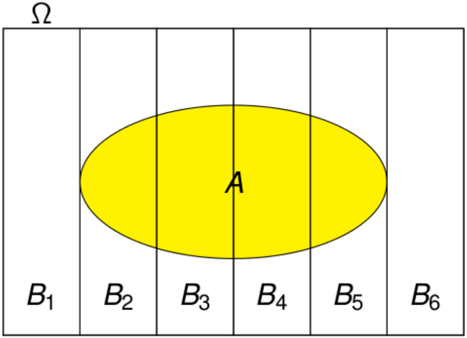

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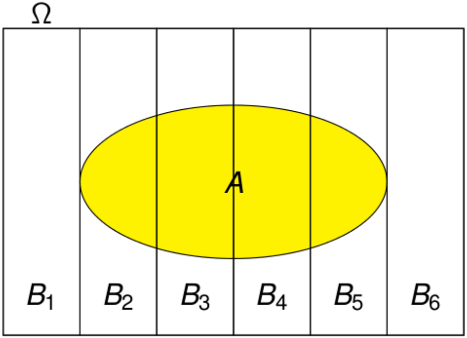

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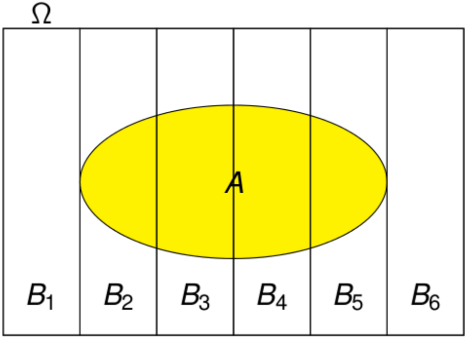

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- Suppose that, from a set of voters polled,
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- Suppose that, from a set of voters polled,
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- Suppose that, of these groups, the proportions who support free community college are:

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  - 86% of Democrat voters,

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- Suppose that, of these groups, the proportions who support free community college are:
  - 86% of Democrat voters,
  - 32% of Republican voters, and
  - 48% of Independent voters.
- What proportion of voters support free community college?

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Let P be the party that the random voter votes for.

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$$\begin{aligned}
\Pr(F) &= \mathbb{E}\Pr(F|P) \\
&= \sum_p \Pr(F|P=p)\Pr(P=p) \\
&= 0.86 \times 0.52 + 0.32 \times 0.46 + 0.48 \times 0.02.
\end{aligned}$$

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This is true not just for indicators of events, but for any (absolutely integrable) random variable Y:

$$\mathbb{E}Y = \mathbb{E}\left(\mathbb{E}(Y|X)\right).$$

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$$\mathbb{E}(S) = \mathbb{E}\mathbb{E}(S|V_P) = 86 \times 0.52 + 32 \times 0.46 + 48 \times 0.02.$$

Outline

- Two facts from probability theory
- **Generative and discriminative models:**
  Gaussian class conditionals lead to a logistic posterior.

Gaussian generative to logistic discriminative models

Class conditionals to posterior

For Gaussian class conditional densities $P(X|Y=+1)$, $P(X|Y=-1)$ (with the same variance), the posterior probability is logistic

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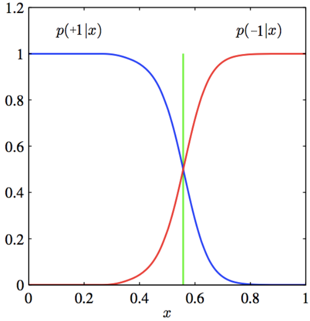

Gaussian generative to logistic discriminative models

- Suppose the class conditional distributions are Gaussian:

$$p(x|y=+1) = \mathcal{N}(\mu_+, \Sigma), \qquad p(x|y=-1) = \mathcal{N}(\mu_-, \Sigma)$$

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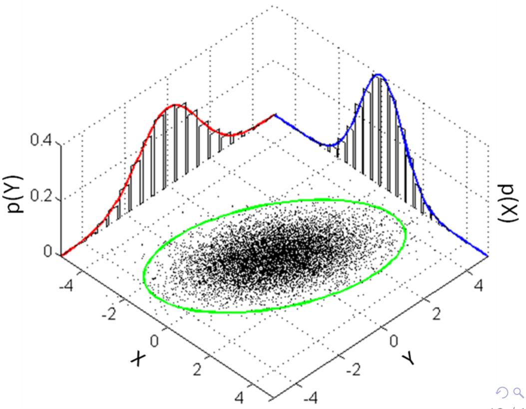

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- Class $+1$ has mean μ_+.
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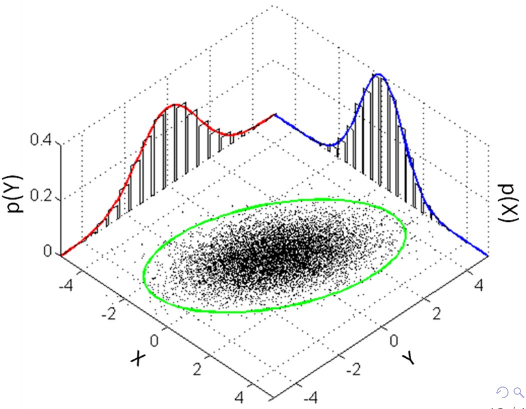

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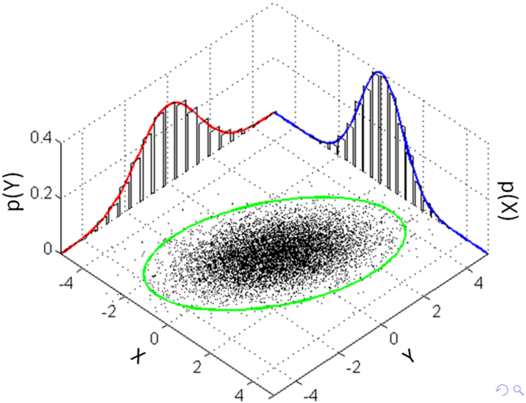
- Class $+1$ has mean μ_+. Class -1 has mean μ_-.
- Both have covariance matrix Σ.



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$$P(Y = +1|x) = \frac{p(x| + 1)P(+1)}{p(x| + 1)P(+1) + p(x| - 1)P(-1)}$$

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&= \frac{P(+1)\exp\left(-\frac{(x-\mu_+)^2}{2\sigma^2}\right)}{P(+1)\exp\left(-\frac{(x-\mu_+)^2}{2\sigma^2}\right)+P(-1)\exp\left(-\frac{(x-\mu_-)^2}{2\sigma^2}\right)}
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$$= \frac{1}{1 + \frac{P(-1)}{P(+1)}\exp\left(\frac{(x-\mu_+)^2-(x-\mu_-)^2}{2\sigma^2}\right)}$$

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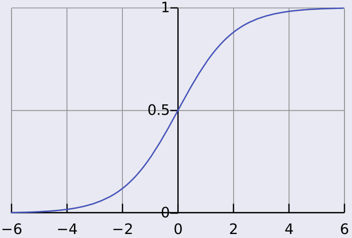

The logistic function $\dfrac{1}{1+e^{-\alpha}}$.

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Logistic model: $$P(Y=+1|x) = \frac{1}{1+\exp\left(-\theta \cdot x - \theta_0\right)}.$$

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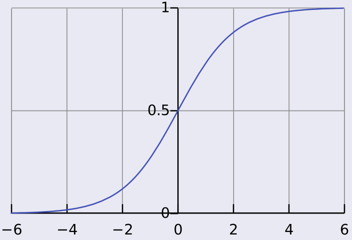

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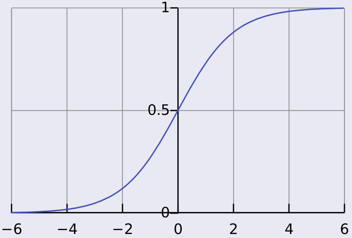

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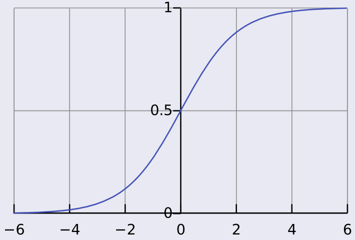

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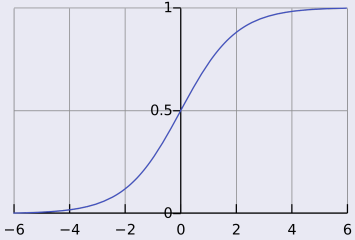
  - x moves from μ_- to μ_+,
  - $P(Y=1)$ increases.



The logistic function $\frac{1}{1+e^{-\alpha}}$.

Gaussian generative to logistic discriminative models

- Also true in d dimensions

Gaussian generative to logistic discriminative models

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$$\theta = \Sigma^{-1}(\mu_+ - \mu_-), \qquad \theta_0 = \frac{\mu_-' \Sigma^{-1} \mu_- - \mu_+' \Sigma^{-1} \mu_+}{2} - \log \frac{P(-1)}{P(1)}.$$

Gaussian generative to logistic discriminative models

- Also true in d dimensions:

$$\theta = \Sigma^{-1}(\mu_+ - \mu_-), \qquad \theta_0 = \frac{\mu'_-\Sigma^{-1}\mu_- - \mu'_+\Sigma^{-1}\mu_+}{2} - \log\frac{P(-1)}{P(1)}.$$

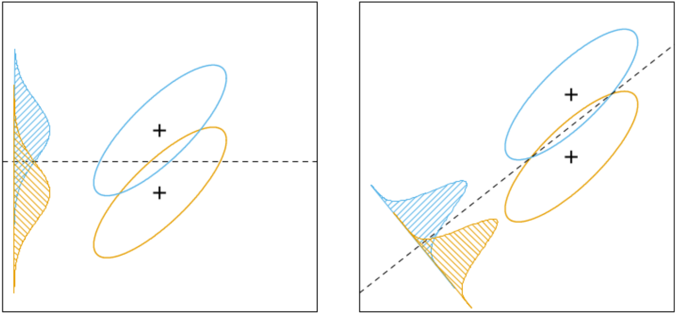

FIGURE 4.9. *Although the line joining the centroids defines the direction of greatest centroid spread, the projected data overlap because of the covariance (left panel). The discriminant direction minimizes this overlap for Gaussian data (right panel).*

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- The posterior $P(Y|X)$ is all we need for classification.
  To go from the posterior $P(Y|X)$ to the class conditionals $P(X|Y)$, we also need to know the marginal $P(X)$.

Outline

- Two facts from probability theory
- Generative and discriminative models:
  Gaussian class conditionals lead to a logistic posterior.