CS189/CS289A
Introduction to Machine Learning
Lecture 13: A Little Convex Optimization and SVMs Revisited

Peter Bartlett

March 5, 2015

Outline

- Convex optimization ideas: primal, Lagrangian, dual.

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- Weak and strong duality

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  - Dual problem and kernels

A brief detour into optimization

SVM optimization problems

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Hard margin SVM

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Soft margin SVM

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Convexity

Convex and non-convex sets

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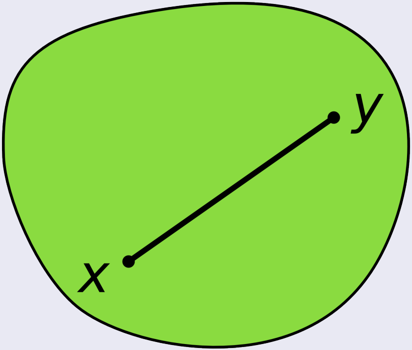

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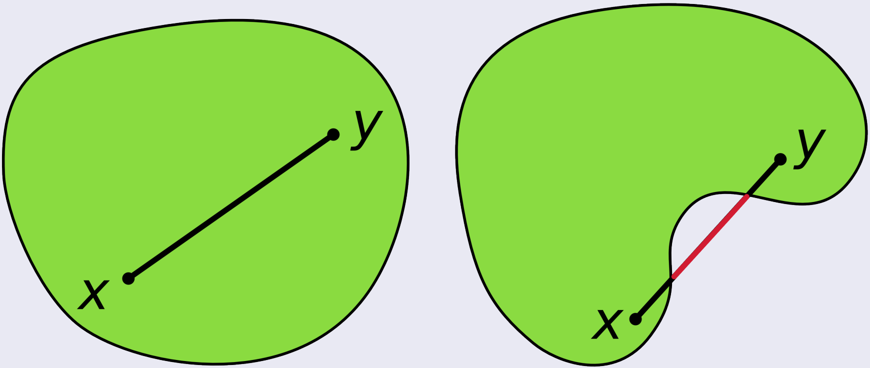

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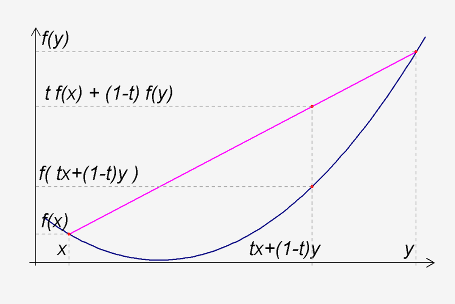

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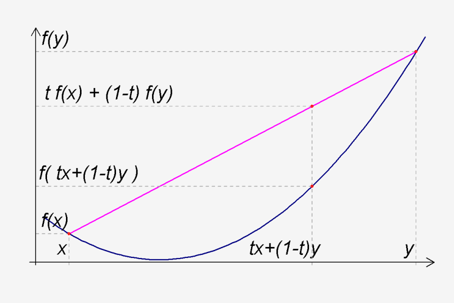

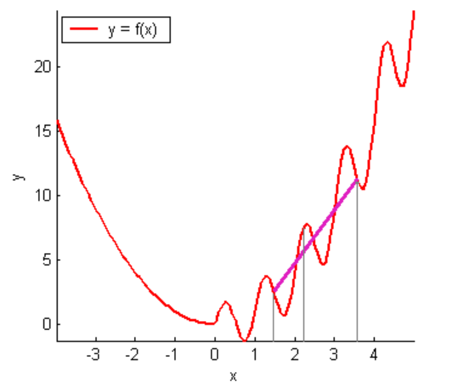

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Primal, Lagrangian, dual

Consider the convex optimization problem

$$\begin{aligned} p^* = \min_{x \in \mathbb{R}^n} & \ f_0(x) \\ \text{s.t.} & \ f_i(x) \leq 0, \qquad i = 1, 2, \ldots, m. \end{aligned}$$

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We can rewrite the constraints as penalties:

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Then replace that constraint penalty with something smaller:
Introduce Lagrange multipliers (dual variables) $\lambda_1, \ldots, \lambda_m \ge 0$, and define the Lagrangian $L : \mathbb{R}^{n+m} \to \mathbb{R}$ as

$$
L(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x).
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The Lagrangian

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Joseph-Louis Lagrange

1736-1813

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- Think of the λ_i as the cost of violating the constraint $f_i(x) \leq 0$.
- L defines a saddle point game: one player (MIN) chooses x to minimize L, the other player (MAX) chooses λ to maximize L. If MIN violates a constraint, $f_i(x) > 0$, then MAX can drive L to infinity.

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- In a zero sum game, it's always better to play second:

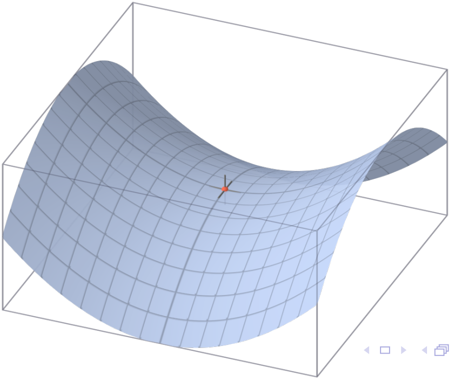
$$p^* = \min_x \max_{\lambda \geq 0} L(x, \lambda) \geq \max_{\lambda \geq 0} \min_x L(x, \lambda) = d^*.$$

This is called *weak duality*.

Strong duality

- If there is a *saddle point* (x^*, λ^*), so that for all x and $\lambda \geq 0$,

$$L(x^*, \lambda^*)$$



Strong duality

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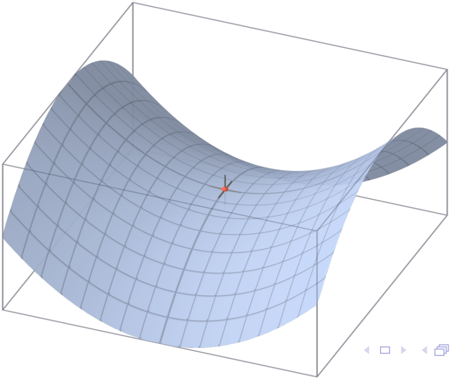

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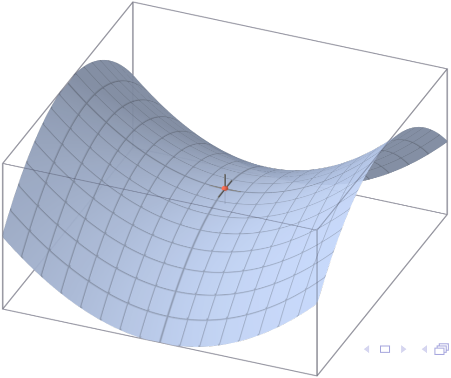


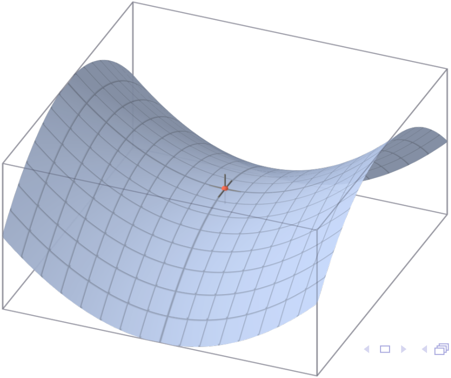
Strong duality

- If there is a *saddle point* (x^*, λ^*), so that for all x and $\lambda \geq 0$,

$$L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*),$$

then we have *strong duality*: the primal and dual have same value,

$$p^* = \min_x \max_{\lambda \geq 0} L(x, \lambda) = \max_{\lambda \geq 0} \min_x L(x, \lambda) = d^*.$$



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- **Complementary slackness**
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That is, if $f_i(x^*) < 0$ then $\lambda_i = 0$.
And if $\lambda_i > 0$ then $f_i(x^*) = 0$.

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$$f_0(x^*) = g(\lambda^*) = \min_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) \right)$$

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That is,

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But $\lambda_i^* \geq 0$ and $f_i(x^*) \leq 0$, so every term in the sum must be zero.

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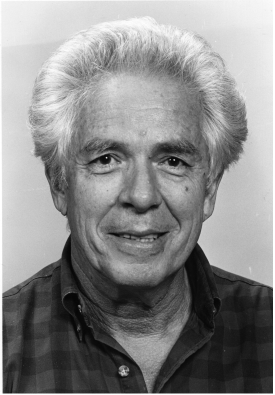
1. Primal feasibility: $f_i(x) \leq 0$.
2. Dual feasibility: $\lambda_i \geq 0$.
3. Complementary slackness: $\lambda_i f_i(x) = 0$.
4. Stationarity: $\nabla f_0(x) + \sum_i \lambda_i \nabla f_i(x) = 0$.

Optimality conditions

William Karush

1917-1997



Harold Kuhn

1925-2014

Albert W. Tucker

1905-1995

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$$g(\alpha) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j y_i y_j x_i' x_j.$$

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It turns out that, if there is a feasible θ (that is, the data are separable), we have strong duality.
We can express the optimal θ^* in terms of the solution, α^*, to the dual problem:

$$\begin{aligned} \max_{\alpha} \quad & \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j y_i y_j x_i' x_j \\ \text{s.t.} \quad & \alpha_i \geq 0, \qquad i = 1, 2, \ldots, n. \end{aligned}$$

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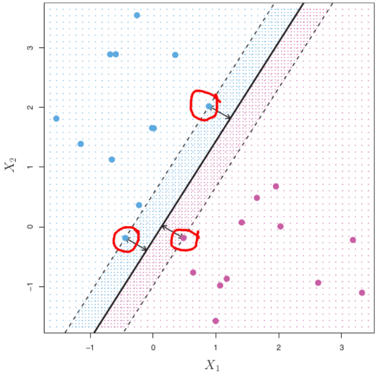

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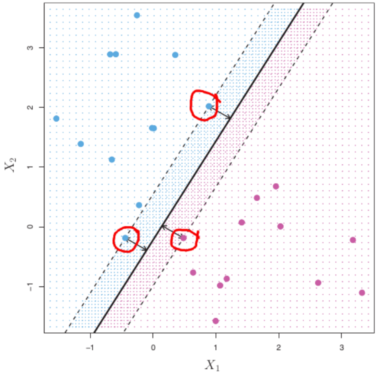

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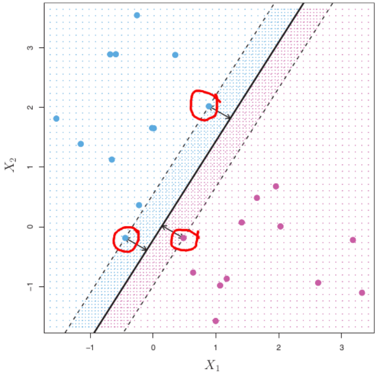
That is, only the points for which the constraints are tight (support vectors) appear in the sum defining θ^*.



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&= \operatorname{sign}\left(\sum_{i=1}^{n} \alpha_i y_i \left\langle \Phi(x_i), \Phi(x) \right\rangle\right) \\
&= \operatorname{sign}\left(\sum_{i=1}^{n} \alpha_i y_i k(x_i, x)\right),
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The optimal slack variables ξ_i satisfy $\xi_i = \left(1 - y_i\theta^T x_i\right)_+$.

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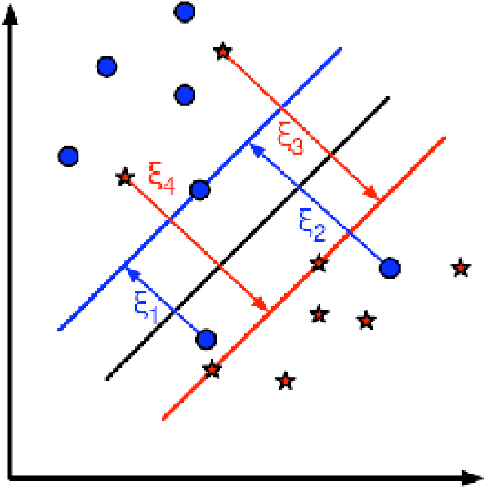
Support vectors



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- In the dual, decreasing C forces the α_is to be small. So the solution is not strongly influenced by a single outlier.

Outline

- Optimization ideas: primal, Lagrangian, dual.
- Weak and strong duality
- Complementary slackness
- Karush-Kuhn-Tucker optimality conditions
- SVMs
    - Complementary slackness and support vectors
    - Dual problem and kernels