CS189/CS289A
Introduction to Machine Learning
Lecture 10: Regression and Regularization

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February 19, 2015

Outline

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- Review: Bias and variance

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- Subset selection

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\end{aligned}
$$

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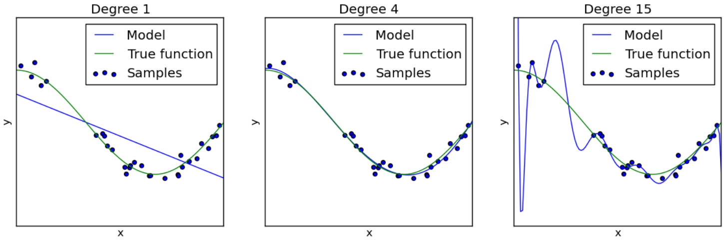

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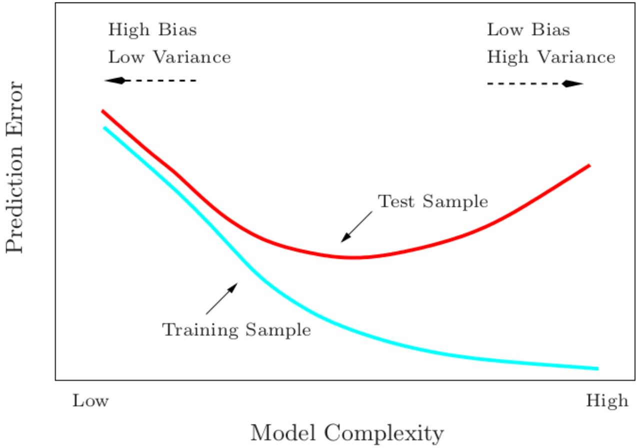

FIGURE 2.11. *Test and training error as a function of model complexity.*

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Restricting complexity corresponds to increasing bias and decreasing variance.

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- **Subset selection**
- Shrinkage:
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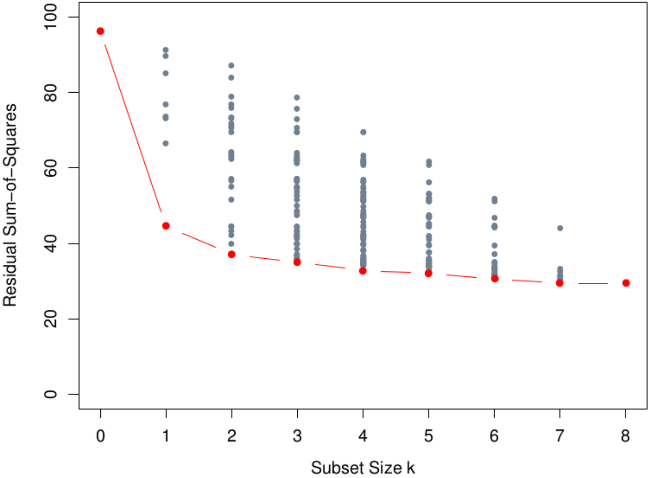

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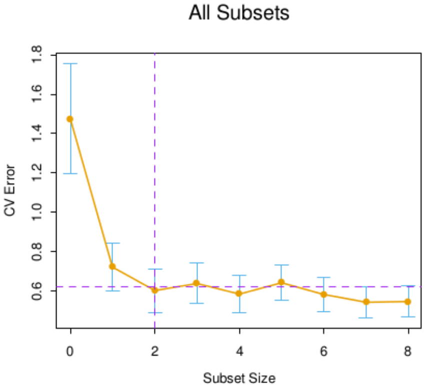

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\downarrow \\
\{x_0, x_3, x_{17}, x_5, x_1, x_{12}\} \\
\downarrow \\
\vdots
\end{array}
$$

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$\{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$ (Need $n > p$)

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\downarrow & \\
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- Shrinkage: encourage the linear predictor's coefficients to be small.

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(These problems are equivalent, in the sense that for given data, for every value of λ, there is a B such that the solutions are identical.)

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  (If $X'X$ is singular, the covariates are linearly dependent, so changing β in some direction will not change the linear map on the space spanned by the data. Adding the penalty eliminates these cancellations.)
- The solution depends on scaling of the covariates!
  It is common to standardize covariates (scale so variance is 1).

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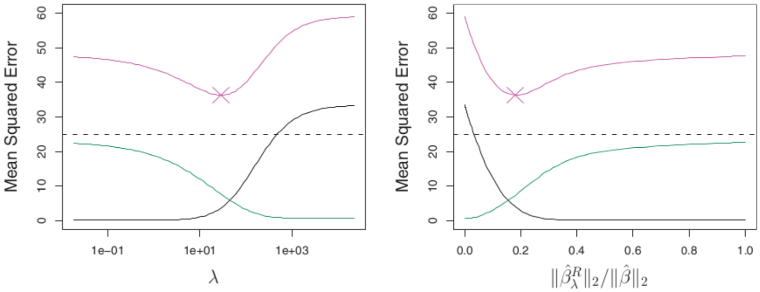

FIGURE 6.5. *Squared bias (black), variance (green), and test mean squared error (purple) for the ridge regression predictions on a simulated data set, as a function of* λ *and* $\|\hat{\beta}^R_\lambda\|_2/\|\hat{\beta}\|_2$. *The horizontal dashed lines indicate the minimum possible MSE. The purple crosses indicate the ridge regression models for which the MSE is smallest.*

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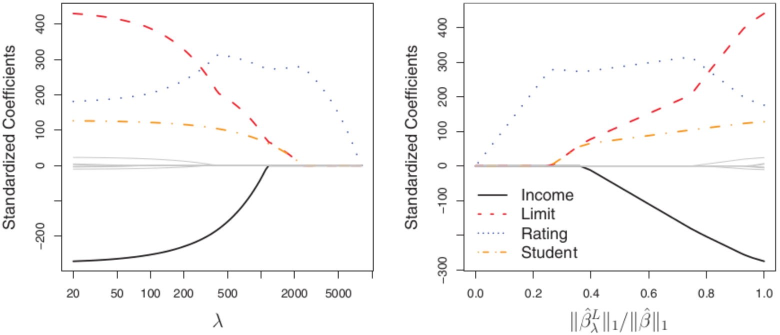


FIGURE 6.6. *The standardized lasso coefficients on the* Credit *data set are shown as a function of λ and $\|\hat{\beta}^L_\lambda\|_1/\|\hat{\beta}\|_1$.*

Shrinkage methods: Lasso

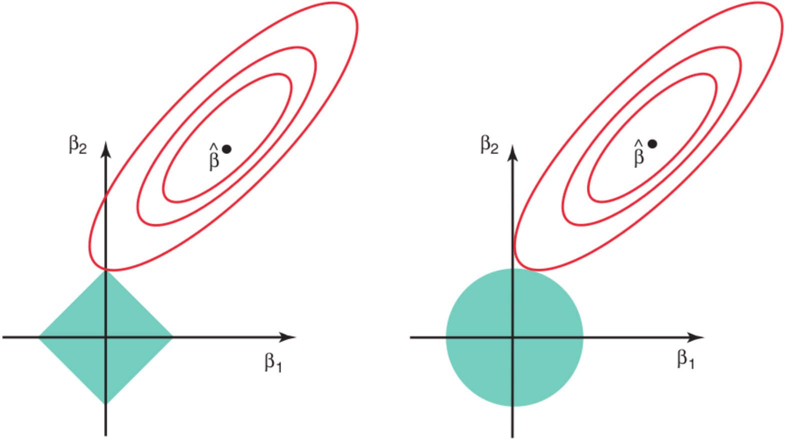
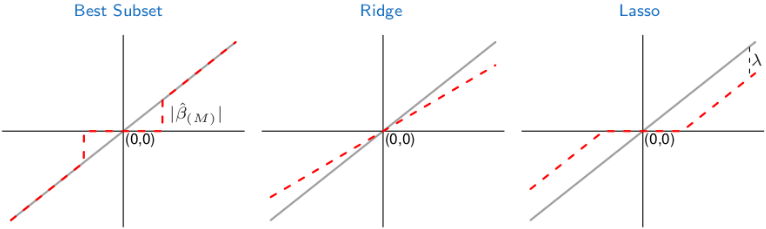


FIGURE 6.7. *Contours of the error and constraint functions for the lasso* (left) *and ridge regression* (right). *The solid blue areas are the constraint regions,* $|\beta_1| + |\beta_2| \leq s$ *and* $\beta_1^2 + \beta_2^2 \leq s$, *while the red ellipses are the contours of the RSS.*

Shrinkage methods: Lasso

TABLE 3.4. *Estimators of β_j in the case of orthonormal columns of $\mathbf{X}$. M and λ are constants chosen by the corresponding techniques;* sign *denotes the sign of its argument (± 1), and x_+ denotes "positive part" of x. Below the table, estimators are shown by broken red lines. The 45° line in gray shows the unrestricted estimate for reference.*

Estimator	Formula
Best subset (size M)	$\hat{\beta}_j \cdot I(\lvert\hat{\beta}_j\rvert \geq \lvert\hat{\beta}_{(M)}\rvert)$
Ridge	$\hat{\beta}_j/(1+\lambda)$
Lasso	$\text{sign}(\hat{\beta}_j)(\lvert\hat{\beta}_j\rvert - \lambda)_+$



Outline

- Review: Bias and variance
- Subset selection
- Shrinkage:
    - Ridge regression
    - Lasso