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STRUCTURED PROGRAMMING

O.-J. Dahl. E. W . Dijkstra and C. A. R. Hoare

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STRUCTURED PROGRAMMING

O.-J. DAHL
*Universitet i Oslo,
Matematisk Institut,
Blindern, Oslo, Norway*

E. W. DIJKSTRA
*Department of Mathematics,
Technological University,
Eindhoven, The Netherlands*

C. A. R. HOARE
*Department of Computer Science,
The Queen's University of Belfast,
Belfast, Northern Ireland*



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PREFACE

In recent years there has been an increasing interest in the art of computer programming, the conceptual tools available for the design of programs, and the prevention of programming oversights and error. The initial outstanding contribution to our understanding of this subject was made by E. W. Dijkstra, whose Notes on Structured Programming form the first and major section of this book. They clearly expound the reflections of a brilliant programmer on the methods which he has hitherto unconsciously applied; there can be no programmer of the present day who could not increase his skills by a study and conscious application of these principles.

In the second monograph I have tried to describe how similar principles can be applied in the design of data structures. I have suggested that in analysing a problem and groping towards a solution, a programmer should take advantage of abstract concepts such as sets, sequences, and mappings; and judiciously postpone decisions on representation until he is constructing the more detailed code of the program. The monograph also describes a range of useful ideas for data representation, and suggests the criteria relevant for their selection.

The third monograph provides a synthesis of the previous two, and expounds the close theoretical and practical connections between the design of data and the design of programs. It introduces useful additional methods for program and data structuring which may be unfamiliar to many programmers. The examples show that structured programming principles can be equally applied in “bottom-up” as in “top-down” program design. The original inspiration, insight, and all the examples were contributed by O.-J. Dahl; I have only assembled the material, and added some additional explanations where I found it difficult to understand.

June 1972

C. A. R. HOARE

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I Notes on Structured Programming

EDSGER W. DIJKSTRA

1. TO MY READER

These notes have the status of “Letters written to myself”: I wrote them down because, without doing so, I found myself repeating the same arguments over and over again. When reading what I had written, I was not always too satisfied.

For one thing, I felt that they suffered from a marked verbosity. Yet I do not try to condense them (now), firstly because that would introduce another delay and I would like to “think on”, secondly because earlier experiences have made me afraid of being misunderstood: many a programmer tends to see his (sometimes rather specific) difficulties as the core of the subject and as a result there are widely divergent opinions as to what programming is really about.

I hope that, despite its defects, you will enjoy at least parts of it. If these notes prove to be a source of inspiration or to give you a new appreciation of the programmer’s trade, some of my goals will have been reached.

Prior to their publication in book form, the “Notes on Structured Programming” have been distributed privately. The interest then shown in them, for which I would like to express my gratitude here, has been one of the main incentives to supplement them with some additional material and to make them available to a wider public. In particular I would like to thank Bob Floyd, Ralph London and Mike Woodger for their encouraging comments and Peter Naur for the criticism he expressed. Finally I would like to express my gratitude to Mrs. E. L. Dijkstra-Tucker for her kind assistance in my struggles with the English language.

2. ON OUR INABILITY TO DO MUCH

I am faced with a basic problem of presentation. What I am really concerned about is the composition of large programs, the text of which may be, say, of the same size as the whole text of this booklet. Also I have to include examples to illustrate the various techniques. For practical reasons, the demonstration programs must be small, many times smaller than the “life-size pro-

grams” I have in mind. My basic problem is that precisely this difference in scale is one of the major sources of our difficulties in programming!

It would be very nice if I could illustrate the various techniques with small demonstration programs and could conclude with “...and when faced with a program a thousand times as large, you compose it in the same way.” This common educational device, however, would be self-defeating as one of my central themes will be that any two things that differ in some respect by a factor of already a hundred or more, are utterly incomparable.

History has shown that this truth is very hard to believe. Apparently we are too much trained to disregard differences in scale, to treat them as “gradual differences that are not essential”. We tell ourselves that what we can do once, we can also do twice and by induction we fool ourselves into believing that we can do it as many times as needed, but this is just not true! A factor of a thousand is already far beyond our powers of imagination!

Let me give you two examples to rub this in. A one-year old child will crawl on all fours with a speed of, say, one mile per hour. But a speed of a thousand miles per hour is that of a supersonic jet. Considered as objects with moving ability the child and the jet are incomparable, for whatever one can do the other cannot and vice versa. Also: one can close one’s eyes and imagine how it feels to be standing in an open place, a prairie or a sea shore, while far away a big, reinless horse is approaching at a gallop, one can “see” it approaching and passing. To do the same with a phalanx of a thousand of these big beasts is mentally impossible: your heart would miss a number of beats by pure panic, if you could!

To complicate matters still further, problems of size do not only cause me problems of presentation, but they lie at the heart of the subject: widespread underestimation of the specific difficulties of size seems one of the major underlying causes of the current software failure. To all this I can see only one answer, viz. to treat problems of size as explicitly as possible. Hence the title of this section.

To start with, we have the “size” of the computation, i.e. the amount of information and the number of operations involved in it. It is essential that this size is large, for if it were really small, it would be easier not to use the computer at all and to do it by hand. The automatic computer owes its right to exist, its usefulness, precisely to its ability to perform large computations where we humans cannot. We want the computer to do what we could never do ourselves and the power of present-day machinery is such that even small

computations are by their very size already far beyond the powers of our unaided imagination.

Yet we must organize the computations in such a way that our limited powers are sufficient to guarantee that the computation will establish the desired effect. This organizing includes the composition of the program and here we are faced with the next problem of size, viz. the length of the program text, and we should give this problem also explicit recognition. We should remain aware of the fact that the extent to which we can read or write a text is very much dependent on its size. In my country the entries in the telephone directory are grouped by town or village and within each such group the subscribers are listed by name in alphabetical order. I myself live in a small village and given a telephone number I have only to scan a few columns to find out to whom the telephone number belongs, but to do the same in a large city would be a major data processing task!

It is in the same mood that I should like to draw the reader's attention to the fact that "clarity" has pronounced quantitative aspects, a fact many mathematicians, curiously enough, seem to be unaware of. A theorem stating the validity of a conclusion when ten pages full of conditions are satisfied is hardly a convenient tool, as all conditions have to be verified whenever the theorem is appealed to. In Euclidean geometry, Pythagoras' Theorem holds for any three points A , B and C such that through A and C a straight line can be drawn orthogonal to a straight line through B and C . How many mathematicians appreciate that the theorem remains applicable when some or all of the points A , B and C coincide? Yet this seems largely responsible for the convenience with which Pythagoras Theorem can be used.

Summarizing: as a slow-witted human being I have a very small head and I had better learn to live with it and to respect my limitations and give them full credit, rather than to try to ignore them, for the latter vain effort will be punished by failure.

3. ON THE RELIABILITY OF MECHANISMS

Being a programmer by trade, programs are what I am talking about and the true subject of this section really is the reliability of programs. That, nevertheless, I have mentioned "mechanisms" in its title is because I regard programs as specific instances of mechanisms, and that I wanted to express, at least once, my strong feeling that many of my considerations concerning software are, *mutatis mutandis*, just as relevant for hardware design.

Present-day computers are amazing pieces of equipment, but most amazing of all are the uncertain grounds on account of which we attach any validity to their output. It starts already with our belief that the hardware functions properly.

Let us restrict, for a moment, our attention to the hardware and let us wonder to what extent one can convince oneself of its being properly constructed. Some years ago a machine was installed on the premises of my University; in its documentation it was stated that it contained, among many other things, circuitry for the fixed-point multiplication of two 27-bit integers. A legitimate question seems to be: “Is this multiplier correct, is it performing according to the specifications?”.

The naïve answer to this is: “Well, the number of different multiplications this multiplier is claimed to perform correctly is finite, viz. 2^{54} , so let us try them all.” But, reasonable as this answer may seem, it is not, for although a single multiplication took only some tens of microseconds, the total time needed for this finite set of multiplications would add up to more than 10,000 years! We must conclude that exhaustive testing, even of a single component such as a multiplier, is entirely out of the question. (Testing a complete computer on the same basis would imply the established correct processing of all possible programs!)

A first consequence of the 10,000 years is that during its life-time the multiplier will be asked to perform only a negligible fraction of the vast number of all possible multiplications it could do: practically none of them! Funnily enough, we still require that it should do any multiplication correctly when ordered to do so. The reason underlying this fantastic quality requirement is that we do not know in advance, which are the negligibly few multiplications it will be asked to perform. In our reasoning about our programs we talk about “the product” and have abstracted from the specific values of the factors: we do not know them, we do not wish to know them, it is not our business to know them, it is our business not to know them! Our wish to think in terms of the concept “the product”, abstracted from the specific instances occurring in a computation is granted, but the price paid for this is precisely the reliability requirement that *any* multiplication of the vast set will be performed correctly. So much for the justification of our desire for a correct multiplier.

But how is the correctness established in a convincing manner? As long as the multiplier is considered as a black box, the only thing we can do is “testing by sampling”, i.e. offering to the multiplier a feasible amount of

factor pairs and checking the result. But in view of the 10,000 years, it is clear that we can only test a negligible fraction of the possible multiplications. Whole classes of in some sense “critical” multiplications may remain untested and in view of the reliability justly desired, our quality control is still most unsatisfactory. Therefore it is not done that way.

The straightforward conclusion is the following: a convincing demonstration of correctness being impossible as long as the mechanism is regarded as a black box, our only hope lies in not regarding the mechanism as a black box. I shall call this “taking the structure of the mechanism into account”.

From now onward the type of mechanisms we are going to deal with are programs. (In many respects, programs are mechanisms much easier to deal with than circuitry, which is really an analogue device and subject to wear and tear.) And also with programs it is fairly hopeless to establish the correctness beyond even the mildest doubt by testing, without taking their structure into account. In other words, we remark that the extent to which the program correctness can be established is not purely a function of the program’s external specifications and behavior but depends critically upon its internal structure.

Recalling that our true concern is with really large programs, we observe as an aside that the size itself requires a high confidence level for the individual program components. If the chance of correctness of an individual component equals p , the chance of correctness of a whole program, composed of N such components, is something like

$$P = p^N.$$

As N will be very large, p should be very, very close to 1 if we desire P to differ significantly from zero!

When we now take the position that it is not only the programmer’s task to produce a correct program but also to demonstrate its correctness in a convincing manner, then the above remarks have a profound influence on the programmer’s activity: the object he has to produce must be usefully structured.

The remaining part of this monograph will mainly be an exploration of what program structure can be used to good advantage. In what follows it will become apparent that program correctness is not my only concern, program adaptability or manageability will be another. This stress on program manageability is my deliberate choice, a choice that, therefore, I should like to justify.

While in the past the growth in power of the generally available equipment has mitigated the urgency of the efficiency requirements, this very same growth has created its new difficulties. Once one has a powerful machine at one's disposal one tries to use it and the size of the problems one tackles adjusts itself to the scope of the equipment: no one thinks about programming an algorithm that would take twenty years to execute. With processing power increased by a factor of a thousand over the last ten to fifteen years, Man has become considerably more ambitious in selecting problems that now should be "technically feasible". Size, complexity and sophistication of programs one should like to make have exploded and over the past years it has become patently clear that on the whole our programming ability has not kept pace with these exploding demands made on it.

The power of available equipment will continue to grow: we can expect manufacturers to develop still faster machines and even without that development we shall witness that the type of machine that is presently considered as exceptionally fast will become more and more common. The things we should like to do with these machines will grow in proportion and it is on this extrapolation that I have formed my picture of the programmer's task.

My conclusion is that it is becoming most urgent to stop to consider programming primarily as the minimization of a cost/performance ratio. We should recognize that already now programming is much more an intellectual challenge: the art of programming is the art of organizing complexity, of mastering multitude and avoiding its bastard chaos as effectively as possible.

My refusal to regard efficiency considerations as the programmer's prime concern is not meant to imply that I disregard them. On the contrary, efficiency considerations are recognized as one of the main incentives to modifying a logically correct program. My point, however, is that we can only afford to optimize (whatever that may be) provided that the program remains sufficiently manageable.

Let me end this section with a final aside on the significance of computers. Computers are extremely flexible and powerful tools and many feel that their application is changing the face of the earth. I would venture the opinion that as long as we regard them primarily as tools, we might grossly underestimate their significance. Their influence as tools might turn out to be but a ripple on the surface of our culture, whereas I expect them to have a much more profound influence in their capacity of intellectual challenge!

Corollary of the first part of this section:

Program testing can be used to show the presence of bugs, but never to show their absence!

4. ON OUR MENTAL AIDS

In the previous section we have stated that the programmer's duty is to make his product "usefully structured" and we mentioned the program structure in connection with a convincing demonstration of the correctness of the program.

But how do we convince? And how do we convince ourselves? What are the typical patterns of thought enabling ourselves to understand? It is to a broad survey of such questions that the current section is devoted. It is written with my sincerest apologies to the professional psychologist, because it will be amateurishly superficial. Yet I hope (and trust) that it will be sufficient to give us a yardstick by which to measure the usefulness of a proposed structuring.

Among the mental aids available to understand a program (or a proof of its correctness) there are three that I should like to mention explicitly:

- (1) Enumeration
- (2) Mathematical induction
- (3) Abstraction.

4.1. ON ENUMERATION

I regard as an appeal to enumeration the effort to verify a property of the computations that can be evoked by an enumerated set of statements performed in sequence, including conditional clauses distinguishing between two or more cases. Let me give a simple example of what I call "enumerative reasoning".

It is asked to establish that the successive execution of the following two statements

$$\begin{aligned} & \text{"}dd := dd/2\text{";} \\ & \text{if } dd \leq r \text{ do } r := r - dd \end{aligned}$$

operating on the variables "r" and "dd" leaves the relations

$$0 \leq r < dd \tag{1}$$

invariant. One just "follows" the little piece of program assuming that (1) is satisfied to start with. After the execution of the first statement, which halves

the value of dd , but leaves r unchanged, the relations

$$0 \leq r < 2 \times dd \quad (2)$$

will hold. Now we distinguish two mutually exclusive cases.

(1) $dd \leq r$. Together with (2) this leads to the relations

$$dd \leq r < 2 \times dd \quad (3)$$

In this case the statement following **do** will be executed, ordering a decrease of r by dd , so that from (3) it follows that eventually

$$0 \leq r < dd,$$

i.e. (1) will be satisfied.

(2) **non** $dd \leq r$ (i.e. $dd > r$). In this case the statement following **do** will be skipped and therefore also r has its final value. In this case “ $dd > r$ ” together with (2), which is valid after the execution of the first statement leads immediately to

$$0 \leq r < dd$$

so that also in the second case (1) will be satisfied.

Thus we have completed our proof of the invariance of relations (1), we have also completed our example of enumerative reasoning, conditional clauses included.

4.2. ON MATHEMATICAL INDUCTION

I have mentioned mathematical induction explicitly because it is the only pattern of reasoning that I am aware of that eventually enables us to cope with loops (such as can be expressed by repetition clauses) and recursive procedures. I should like to give an example.

Let us consider the sequence of values

$$d_0, d_1, d_2, d_3, \dots \quad (1)$$

given by

$$\text{for } i = 0 \quad d_i = D \quad (2a)$$

$$\text{for } i > 0 \quad d_i = f(d_{i-1}) \quad (2b)$$

where D is a given value and f a given (computable) function. It is asked to make the value of the variable “ d ” equal to the first value d_k in the sequence

that satisfies a given (computable) condition “prop”. It is given that such a value exists for finite k . A more formal definition of the requirement is to establish the relation

$$d = d_k \quad (3)$$

where k is given by the (truth of the) expressions

$$\text{prop } (d_k) \quad (4)$$

$$\text{and } \mathbf{non} \text{ prop } (d_i) \quad \text{for all } i \text{ satisfying } 0 \leq i < k \quad (5)$$

We now consider the following program part:

$$\begin{aligned} & “d := D; \\ & \mathbf{while} \mathbf{non} \text{ prop } (d) \mathbf{do} d := f(d) ” \end{aligned} \quad (6)$$

in which the first line represents the initialization and the second one the loop, controlled by the (hopefully self-explanatory) repetition clause **while ... do**. (In terms of the conditional clause **if ... do**, used in our previous example, a more formal definition of the semantics of the repetition clause is by stating that

“**while** B **do** S ”

is semantically equivalent with

“**if** B **do**
 begin S ; **while** B **do** S **end**”

expressing that “**non** B ” is the necessary and sufficient condition for the repetition to terminate.)

Calling in the construction “**while** B **do** S ” the statement S “the repeated statement” we shall prove that in program (6):

after the n th execution of the repeated statement will hold (for $n \leq 0$)

$$d = d_k \quad (7a)$$

$$\text{and } \mathbf{non} \text{ prop } (d_1) \quad \text{for all } i \text{ satisfying } 0 \leq i < k \quad (7b)$$

The above statement holds for $n = 0$ (by enumerative reasoning); we have to prove (by enumerative reasoning) that when it holds for $n = N$ ($N \geq 0$), it will also hold for $n = N + 1$.

After the N^{th} execution of the repeated statement relations (7a) and (7b) are satisfied for $n = N$. For the $N + 1^{\text{st}}$ execution to take place, the necessary and sufficient condition is the truth of

non prop (d)

which, thanks to (7a) for $n = N$ (i.e. $d = d_N$) means

non prop (d_N)

leading to condition (7b) being satisfied for $n = N + 1$. Furthermore, $d = d_N$ and (eq:induction-for-b) leads to

$$f(d) = d_{N+1}$$

so that the net effect of the $N + 1^{\text{st}}$ execution of the repeated statement

$$“d := f(d)”$$

established the relation

$$d = d_{N+1}$$

i.e. relation (7a) for $N = N + 1$ and thus the induction step (7) has been proved.

Now we shall show that the repetition terminates after the k th execution of the repeated statement. The n th execution cannot take place for $n > k$ for (on account of 7b) this would imply

non prop (d_k)

thereby violating (4). When the repetition terminates after the n th execution of the repeated statement, the necessary and sufficient condition for termination, viz.

non (non prop (d))

becomes, thanks to (7a)

$$\text{prop} (d_n). \tag{8}$$

This excludes termination for $n < k$, as this would violate (5). As a result the repetition will terminate with $n = k$, so that (3) follows from (7a), (4) follows from (8) and (5) follows from (7b). Which terminates our proof.

Before turning our attention away from this example illustrating the use of mathematical induction as a pattern of reasoning, I should like to add some remarks, because I have the uneasy feeling that by now some of my readers (in particular experienced and competent programmers) will be terribly irritated, viz. those readers for whom program (6) is so obviously correct that

they wonder what all the fuss is about: “Why his pompous restatement of the problem, as in (3), (4) and (5), because anyone knows what is meant by the first value in the sequence, satisfying a condition? Certainly he does not expect us, who have work to do, to supply such lengthy proofs, with all the mathematical dressing, whenever we use such a simple loop as that?” Etc.

To tell the honest truth: the pomp and length of the above proof infuriate me as well! But at present I cannot do much better if I really try to prove the correctness of this program. But it sometimes fills me with the same kind of anger as years ago the crazy proofs of the first simple theorems in plane geometry did, proving things of the same degree of “obviousness” as Euclid’s axioms themselves.

Of course I would not dare to suggest (at least at present!) that it is the programmer’s duty to supply such a proof whenever he writes a simple loop in his program. If so, he could never write a program of any size at all! It would be as impractical as reducing each proof in plane geometry explicitly and in extension to Euclid’s axioms. (Cf. Section “On our inability to do much.”)

My moral is threefold. Firstly, when a programmer considers a construction like (6) as obviously correct, he can do so because he is familiar with the construction. I prefer to regard his behavior as an unconscious appeal to a theorem he *knows*, although perhaps he has never bothered to formulate it; and once in his life he has convinced himself of its truth, although he has probably forgotten in which way he did it and although the way was (probably) unfit for print. But we could call our assertions about program (6), say, “The Linear Search Theorem” and knowing such a name it is much easier (and more natural) to appeal to it consciously.

Secondly, to the best of my knowledge, there is no set of theorems of the type illustrated above, whose usefulness has been generally accepted. But we should not be amazed about that, for the absence of such a set of theorems is a direct consequence of the fact that the type of object – i.e. programs – has not settled down. The kind of object the programmer is dealing with, viz. programs, is much less well-established than the kind of object that is dealt with in plane geometry. In the meantime the intuitively competent programmer is probably the one who confines himself, whenever acceptable, to program structures with which he is very familiar, while becoming very alert and careful whenever he constructs something unusual (for him). For an established style of programming, however, it might be a useful activity to

look for a body of theorems pertinent to such programs.

Thirdly, the length of the proof we needed in our last example is a warning that should not be ignored. There is of course the possibility that a better mathematician will do a much shorter and more elegant job than I have done. Personally I am inclined to conclude from this length that programming is more difficult than is commonly assumed: let us be honestly humble and interpret the length of the proof as an urgent advice to restrict ourselves to simple structures whenever possible and to avoid in all intellectual modesty “clever constructions” like the plague.

4.3. ON ABSTRACTION

At this stage I find it hard to be very explicit about the role of abstraction, partly because it permeates the whole subject. Consider an algorithm and all possible computations it can evoke: starting from the computations the algorithm is what remains when one abstracts from the specific values manipulated this time. The concept of “a variable” represents an abstraction from its current value. It has been remarked to me (to my great regret I cannot remember by whom and so I am unable to give credit where it seems due) that once a person has understood the way in which variables are used in programming, he has understood the quintessence of programming. We can find a confirmation for this remark when we return to our use of mathematical induction with regard to the repetition: on the one hand it is by abstraction that the concepts are introduced in terms of which the induction step can be formulated; on the other hand it is the repetition that really calls for the concept of “a variable”. (Without repetition one can restrict oneself to “quantities” the value of which has to be defined as most once but never has to be redefined as in the case of a variable.)

There is also an abstraction involved in naming an operation and using it on account of “what it does” while completely disregarding “how it works”. (In the same way one should state that a programming manual describes an abstract machine: the specific piece of hardware delivered by the manufacturer is nothing but a – usually imperfect! – mechanical model of this abstract machine.) There is a strong analogy between using a named operation in a program regardless of “how it works” and using a theorem regardless of how it has been proved. Even if its proof is highly intricate, it may be a very convenient theorem to use!

Here, again, I refer to our inability to do much. Enumerative reasoning is all right as far as it goes, but as we are rather slow-witted it does not go

very far. Enumerative reasoning is only an adequate mental tool under the severe boundary condition that we use it only very moderately. We should appreciate abstraction as our main mental technique to reduce the demands made upon enumerative reasoning.

(Here Mike Woodger, National Physical Laboratory, Teddington, England, made the following remark, which I insert in gratitude: “There is a parallel analogy between the unanalysed terms in which an axiom or theorem is expressed and the unanalysed operands upon which a named operation is expected to act.”)

5. AN EXAMPLE OF A CORRECTNESS PROOF

Let us consider the following program section, where the integer constants a and d satisfy the relations

```

 $a \geq 0$    and    $d > 0$ .
“integer  $r, dd$ ;
 $r := a; dd := d$ ;
while  $dd \leq r$  do  $dd := 2 \times dd$ ;
while  $dd \neq d$  do
  begin  $dd := dd/2$ ;
    if  $dd \leq r$  do  $r := r - dd$ 
  end”.

```

To apply the Linear Search Theorem (see Section “On our mental aids”, subsection “On mathematical induction”) we consider the sequence of values given by

$$\begin{aligned}
 &\text{for } i = 0 \quad dd_i = d \\
 &\text{for } i > 0 \quad dd_i = 2 \times dd_{i-1} \\
 &\text{from which} \quad dd_n = d \times 2^n
 \end{aligned} \tag{1}$$

can be derived by normal mathematical techniques, which also tell us that (because $d > 0$) for finite r

$$dd_k > r$$

will hold for some finite k , thus ensuring that the first repetition terminates with

$$dd = d \times 2^k$$

Solving the relation

$$d_i = 2 \times d_{i-1}$$

for d_{i-1} gives

$$d_{i-1} = d_i/2$$

and the Linear Search Theorem then tells us, that the second repetition will also terminate. (As a matter of fact the second repeated statement will be executed exactly the same number of times as the first one.)

At the termination of the first repetition,

$$dd = dd_k$$

and therefore,

$$0 \leq r < dd \tag{2}$$

holds. As shown earlier (Section “On our mental aids”, subsection “On enumeration”) the repeated statement of the second clause leaves this relation invariant. After termination (on account of “**while** $dd \neq d$ **do**”) we can conclude

$$dd = d$$

which together with (2) gives

$$0 \leq r < d \tag{3}$$

Furthermore we prove that after the initialization

$$dd \equiv 0 \pmod{d} \tag{4}$$

holds; this follows, for instance, from the fact that the possible values of dd are (see (1))

$$d \times 2^i \quad \text{for } 0 \leq i \leq k.$$

Our next step is to verify, that after the initial assignment to r the relation

$$a \equiv r \pmod{d} \tag{5}$$

holds.

- (1) It holds after the initial assignments.
- (2) The repeated statement of the first clause (“ $dd := 2 \times dd$ ”) maintains the invariance of (5) and therefore the whole first repetition maintains the validity of (5).

- (3) The second repeated statement consists of two statements. The first (“ $dd := dd/2$ ”) leaves (5) invariant, the second one also leaves (5) invariant for either it leaves r untouched or it decreases r by the current value of dd , an operation which on account of (4) also maintains the validity of (5). Therefore the whole second repeated statement leaves (5) invariant and therefore the whole repetition leaves (5) invariant. Combining (3) and (5), the final value therefore satisfies

$$0 \leq r < d \quad \text{and} \quad a \equiv r \pmod{d}$$

i.e. r is the smallest non-negative remainder of the division of a by d .

Remark 1. The program “integer $r, dd, q; r := a; dd := d; q := 0; \text{while } dd \leq r \text{ do } dd := 2 * dd; \text{while } dd = F \text{ d do begin } dd := dd/2; } q := 2 * q; \text{end if } dd \leq r \text{ do begin } r := r - dd; } q := q + 1 \text{ end}$

assigns to q the value of the corresponding quotient. The proof can be established by observing the invariance of the relation

$$a = q \times dd + r.$$

(I owe this example to my colleague N. G. de Bruijn.)

Remark 1. The program

```

“integer  $r, dd, q;$ 
 $r := a; dd := d; q := 0;$ 
while  $dd \leq r$  do  $dd := 2 \times dd;$ 
while  $dd \neq d$  do
  begin  $dd := dd/2; q := 2 \times q;$ 
    if  $dd \leq r$  do begin  $r := r - dd; q := q + 1$  end
  end”

```

assigns to q the value of the corresponding quotient. The proof can be established by observing the invariance of the relation

$$a = q \times dd + r.$$

(I owe this example to my colleague N. G. de Bruijn.)

Remark 2. In the subsection “On mathematical induction” we have proved the Linear Search Theorem. In the previous proof we have used another theorem about repetitions (a theorem that, obviously, can only be proved by mathematical induction, but the proof is so simple that we leave it as an exercise to the reader), viz. that if prior to entry of a repetition a certain relation P holds, whose truth is not destroyed by a single execution of the repeated statement, then relation P will still hold after termination of the repetition. This is a very useful theorem, often allowing us to bypass an explicit appeal to mathematical induction. (We can state the theorem a little more sharply; in the repetition

“while B do S ”

one has to show that S is such that the truth of

P and B

prior to the execution of S implies the truth of

P

after its execution.)

Remark 3. As an exercise for the reader (for which acknowledgment is due to James King, CMU, Pittsburgh, USA), prove that with integer A, B, x, y and z and

$$A > 0 \quad \text{and} \quad B \geq 0$$

after the execution of the program section

```

“ $x := A; y := B; z := 1;$ 
while  $y \neq 0$  do
    begin if  $\text{odd}(y)$  do begin  $y := y - 1; z := z \times x$  end;
       $y := y/2; x := x \times x$ 
    end”

```

finally $z = A^B$ will hold.

The proof has to show that (in spite of “ $y := y/2$ ”) all variables keep integer values; the method shows the invariance of

$$x > 0 \quad \text{and} \quad y \geq 0 \quad \text{and} \quad A^B = z \times x^y$$

6. ON THE VALIDITY OF PROOFS VERSUS THE VALIDITY OF IMPLEMENTATIONS

In the previous section I have assumed “perfect arithmetic” and in my experience the validity of such proofs often gets questioned by people who argue that in practice one never has perfect arithmetic at ones disposal: admissible integer values usually have an absolute upper bound, real numbers are only represented to a finite accuracy etc. So what is the validity of such proofs?

The answer to this question seems to be the following. If one proves the correctness of a program assuming an idealized, perfect world, one should not be amazed if something goes wrong when this ideal program gets executed by an “imperfect” implementation. Obviously! Therefore, if we wish to prove program correctness in a more realistic world, the thing to do is to acknowledge right at the start that all operations appealed to in the program (in particular all arithmetic operations) need not be perfect, provided we state – rather axiomatically – the properties they have to satisfy for the proper execution of the program, i.e. the properties on which the correctness proof relies. (In the example of the previous section this requirement is simply exact integer arithmetic in the range $[0, 2a]$.)

When writing a program operating on real numbers with rounded operations, one must be aware of the assumptions one makes, such as

$$b > 0 \quad \text{implies} \quad a + b \geq a$$

$$a \times b = b \times a$$

$$-(a \times b) = (-a) \times b$$

$$0 \times x = 0$$

$$0 + x = x$$

$$1 * x = x \text{ etc. etc.}$$

Very often the validity of such relations is essential to the logic of the program. For the sake of compatibility, the programmer would be wise to be as undemanding as possible, whereas a good implementation should satisfy as many reasonable requirements as possible.

This is the place to confess one of my blunders. In implementing ALGOL 60 we decided that “ $x = y$ ” would deliver the value **true** not only in the case of exact equality, but also when the two values differed only in the least significant digit represented, because otherwise it was so very improbable that

the value *true* would ever be computed. We were thinking of converging iterations that could oscillate within rounding accuracy. While we had been generous (with the best of intentions!) in regarding real numbers as equal, it quickly turned out that the chosen operation was so weak as to be hardly of any use at all. What it boiled down to was that the established truth of $a = b$ **and** $b = c$ did not allow the programmer to conclude the truth of $a = c$. The decision was quickly changed. It is because of that experience that I know that the programmer can only use his tool by virtue of (a number of) its properties; conversely, the programmer must be able to state which properties he requires. (Usually programmers don't do so because, for lack of tradition as to what properties can be taken for granted, this would require more explicitness than is otherwise desirable. The proliferation of machines with lousy floating-point hardware – together with the misapprehension that the automatic computer is primarily the tool of the numerical analyst – has done much harm to the profession!)

7. ON UNDERSTANDING PROGRAMS

In my life I have seen many programming courses that were essentially like the usual kind of driving lessons, in which one is taught how to handle a car instead of how to use a car to reach one's destination.

My point is that a program is never a goal in itself; the purpose of a program is to evoke computations and the purpose of the computations is to establish a desired effect. Although the program is the final product made by the programmer, the possible computations evoked by it – the “making” of which is left to the machine! – are the true subject matter of his trade. For instance, whenever a programmer states that his program is correct, he really makes an assertion about the computations it may evoke.

The fact that the last stage of the total activity, viz. the transition from the (static) program text to the (dynamic) computation, is essentially left to the machine is an added complication. In a sense the making of a program is therefore more difficult than the making of a mathematical theory: both program and theory are structured, timeless objects. But while the mathematical theory makes sense as it stands, the program only makes sense via its execution.

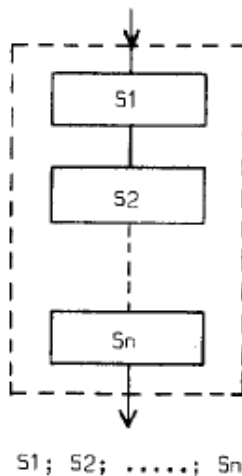
In the remaining part of this section I shall restrict myself to programs written for a sequential machine, and I shall explore some of the consequences of our duty to use our understanding of a program to make assertions

about the ensuing computations. It is my (unproven) claim that the ease and reliability with which we can do this depends critically upon the simplicity of the relation between the two, in particular upon the nature of sequencing control. In vague terms we may state the desirability that the structure of the program text reflects the structure of the computation. Or, in other terms, “What can we do to shorten the conceptual gap between the static program text (spread out in “text space”) and the corresponding computations (evolving in time)?”

It is the purpose of the computation to establish a certain desired effect. When it starts at a discrete moment t_0 it will be completed at a later discrete moment t_1 and we assume that its effect can be described by comparing “the state at t_0 ” with “the state at t_1 ”. If no intermediate states are taken into consideration the effect is regarded as being established by a primitive action.

When we do take a number of intermediate states into consideration this means that we have parsed the happening in time. We regard it as a sequential computation, i.e. the time-succession of a number of subactions and we have to convince ourselves that the cumulative effect of this time-succession of subactions indeed equals the desired net effect of the total computation.

The simplest case is a parsing, a decomposition, into a fixed number of subactions that can be enumerated. In flowchart form this can be represented as follows.



The validity of this decomposition has to be established by enumerative

reasoning. In this case, shortening of the conceptual gap between program and computation can be achieved by requiring that a linear piece of program text contains names or descriptions of the subactions in the order in which they have to take place. In our earlier example (invariance of $0 \leq r < dd$)

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