

$$2) \quad u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$$u(x, 0) = \sin(x)$$

This is the inviscid Burger's equation whose characteristics are given by the initial condition (as the speed of propagation is given by u at every time). When the characteristics intersect forwards in time, the solution develops a shock, hence at the time of the shock T^* , we have $u \notin C^1$ for $t \geq T^*$. Characteristics starting at position x are given by:

$$(t, X(t)) = (t, x + \sin(x) \cdot t)$$

Hence, intersections have to fulfill:

$$x + \sin(x)t = y + \sin(y)t$$

And we have

$$T^* = \inf_{t \in \mathbb{R}^+} \{ (x, y) \in [0, 2\pi]^2 \mid x + \sin(x)t = y + \sin(y)t, x \neq y \}$$

Rearranging for t gives

$$t = \frac{x - y}{\sin(y) - \sin(x)}$$

For any fixed x , minimizing over y yields:

$$-1(\sin(y) - \sin(x)) - \cos(y)(x - y) = 0$$

$$\Leftrightarrow \sin(y) = \cos(y)(y - x) + \sin(x)$$

$$\Rightarrow t = \frac{x - y}{\cos(y)(y - x)} = -\frac{1}{\cos(y)}$$

$$\Rightarrow T^* = \inf_{\gamma \in [0, 2\pi]} -\frac{1}{\cos(\gamma)} \quad \text{such that } T^* > 0.$$

$$\Rightarrow T^* = 1$$

Thus, we have a shock occurring at $T^* = 1$ and the solution is only C^1 for $t < T^*$.

1) The Lax-Wendroff method can be derived by

$$u(x, t + \Delta t) = u(x, t) + \Delta t \frac{\partial u}{\partial t}(x, t) + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2}(x, t) + O(\Delta t^3)$$

We can substitute for the first and second temporal derivative by:

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x}$$

and

$$\frac{\partial^2 u}{\partial t^2} = -a \frac{\partial^2 u}{\partial x \partial t} = -a \frac{\partial^2 u}{\partial t \partial x} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Approximating these with second-order central finite differences gives:

$$\begin{aligned} u_j^{n+1} &= u_j^n + \Delta t (-a) \frac{u_{j+1}^n - u_{j-1}^n}{2h} + \frac{a^2 \Delta t^2}{2} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \\ &= u_j^n (1 - a^2 \lambda^2) + u_{j+1}^n \left(-\frac{1}{2} a \lambda + \frac{1}{2} a^2 \lambda^2 \right) \\ &\quad + u_{j-1}^n \left(\frac{1}{2} a \lambda + \frac{1}{2} a^2 \lambda^2 \right) \end{aligned}$$

$$\begin{aligned} \Rightarrow \tilde{E}(\xi) &= (1 - a^2 \lambda^2) - \frac{1}{2} (a \lambda - a^2 \lambda^2) e^{i\xi} \\ &\quad + \frac{1}{2} (a \lambda + a^2 \lambda^2) e^{-i\xi} \\ &= 1 - a^2 \lambda^2 - a \lambda i \sin(\xi) + a^2 \lambda^2 \cos(\xi) \end{aligned}$$

$$\begin{aligned} \Rightarrow |\tilde{E}(\xi)|^2 &= 1 + a^4 \lambda^4 (\cos(\xi) - 1)^2 + 2a^2 \lambda^2 (\cos(\xi) - 1) \\ &\quad + a^2 \lambda^2 \sin(\xi)^2 \end{aligned}$$

$$\begin{aligned}
 &= 1 - a^2 \lambda^2 (1 - a^2 \lambda^2) (1 - \cos(\xi))^2 \\
 &\quad + a^2 \lambda^2 \underbrace{\left[(1 - \cos(\xi))^2 + 2\cos(\xi) - 2 + \sin(\xi)^2 \right]}_{=0}
 \end{aligned}$$

$$\leq 1$$

$\Leftrightarrow a^2 \lambda^2 \leq 1$, which guarantees the stability of the Lax-Wendroff method.