2) $$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$$u(x,0) = \sin(x)$$

This is the inviscid Burger's equation who's characteristics are given by the initial condition (as the speed of propagation is given by u at every time). When the characteristics intersect forwards in time, the solution develops a shock, hence at the time of the shock T^*, we have $u \notin C^1$ for $t \geq T^*$.
Characteristics starting at position x are given by:

$$(t, X(t)) = (t, x + \sin(x) \cdot t)$$

Hence, intersections have to fulfill:

$$x + \sin(x) t = y + \sin(y) t$$

And we have

$$T^* = \inf_{t \in \mathbb{R}^+} \{ (x,y) \in [0,2\pi]^2 \mid x + \sin(x) \cdot t = y + \sin(y) t, x \neq y \}$$

Rearranging for t gives

$$t = \frac{x-y}{\sin(y) - \sin(x)}$$

For any fixed x, minimizing over y yields:

$$-1(\sin(y) - \sin(x)) - \cos(y)(x-y) = 0$$

$$\Leftrightarrow \sin(y) = \cos(y)(y-x) + \sin(x)$$

$$\Rightarrow t = \frac{x-y}{\cos(y)(y-x)} = -\frac{1}{\cos(y)}$$

$$\Rightarrow \quad T^* = \inf_{y \in [0, 2\pi]} -\frac{1}{\cos(y)} \quad \text{such that } T^* > 0.$$

$$\Rightarrow \quad T^* = 1$$

Thus, we have a shock occurring at $T^* = 1$ and the solution is only C^1 for $t < T^*$.

1) The Lax-Wendroff method can be derived by

$$u(x, t+\Delta t) = u(x,t) + \Delta t \frac{\partial u}{\partial t}(x,t) + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2}(x,t) + O(\Delta t^3)$$

We can substitute for the first and second temporal derivative by: $\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x}$

and $\frac{\partial^2 u}{\partial t^2} = -a \frac{\partial^2 u}{\partial x \partial t} = -a \frac{\partial^2 u}{\partial t \partial x} = a^2 \frac{\partial^2 u}{\partial x^2}$

Approximating these with second-order central finite differences gives:

$$u_j^{n+1} = u_j^n + \Delta t(-a) \frac{u_{j+1}^n - u_{j-1}^n}{2h} + \frac{a^2 \Delta t^2}{2} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}$$

$$= u_j^n \left(1 - a^2 \lambda^2\right) + u_{j+1}^n \left(-\frac{1}{2} a\lambda + \frac{1}{2} a^2 \lambda^2\right) + u_{j-1}^n \left(\frac{1}{2} a\lambda + \frac{1}{2} a^2 \lambda^2\right)$$

$$\Rightarrow \tilde{E}(\xi) = \left(1 - a^2\lambda^2\right) - \frac{1}{2}\left(a\lambda - a^2\lambda^2\right) e^{i\xi} + \frac{1}{2}\left(a\lambda + a^2\lambda^2\right) e^{-i\xi}$$

$$= 1 - a^2\lambda^2 - a\lambda i \sin(\xi) + a^2\lambda^2 \cos(\xi)$$

$$\Rightarrow |\tilde{E}(\xi)|^2 = 1 + a^4\lambda^4 (\cos(\xi) - 1)^2 + 2a^2\lambda^2 (\cos(\xi) - 1) + a^2\lambda^2 \sin(\xi)^2$$

$$= 1 - a^2\lambda^2(1-a^2\lambda^2)(1-\cos(\xi))^2 + a^2\lambda^2\underbrace{[(1-\cos(\xi))^2 + 2\cos(\xi) - 2 + \sin(\xi)^2]}_{=0}$$

$$\overset{!}{\leq} 1$$

$\Leftrightarrow$ $a^2\lambda^2 \leq 1$, which guarantees the stability of the Lax-Wendroff method.