# The HJB equation for controlled diffusions

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These notes are a summary of material for the class. The essential theoretical tool is Ito's formula. The theory of stochastic differential equations (SDE) or of the nonlinear Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) is not needed. What follows is just the so-called verification lemma without the theoretical details.

Consider the controlled diffusion process $X(t) = X^U(t)$ that satisfies the Ito stochastic differential equation

$$dX(t) = b(X(t), U(t))dt + \sigma(X(t), U(t))dB(t)$$

with $X(0) = x$. Here $U(t) \in \mathcal{U}$ is the control process, a non-anticipating function with values in the set $\mathcal{U}$, to be determined. We assume that the coefficients $b(x, u)$ and $\sigma(x, u)$ satisfy the Ito conditions as functions of $x$, uniformly in $u \in \mathcal{U}$. The Hamilton-Jacobi-Bellman (HJB) equation for the value function

$$V(t, x) = \inf_U E_{t,x}\{g(X(T))\} \ , \quad t \le T$$

has the form

$$V_t(t, x) + \inf_u \{\mathcal{L}_u V(t, x)\} = 0 \ , \quad t < T$$

with terminal conditions $V(T, x) = g(x)$. The infimum in this PDE is pointwise in $(t, x)$, and $\mathcal{L}_u$ is the generator of the controlled diffusion with a fixed constant control $u$

$$\mathcal{L}_u = \frac{1}{2}\sigma^2(x, u)\frac{\partial^2}{\partial x^2} + b(x, u)\frac{\partial}{\partial x}$$

We will assume that the HJB equation has a classical solution, that is, with one time and two space derivatives, and denote the unique pointwise minimal control $u$ by $u^* = u^*(t, x)$, assumed differentiable. The optimal, Markovian, control is then $U^*(t) = u^*(t, X^*(t))$ where $X^*(t)$ is the optimally controlled diffusion satisfying the Ito SDE

$$dX^*(t) = b(X^*(t), u^*(t, X^*(t)))dt + \sigma(X^*(t), u^*(t, X^*(t)))dB(t)$$

with $X^*(0) = x$, which is assumed to have a solution as an Ito diffusion.

Let $U(t)$ be any admissible control, let $X^U(t)$ be the solution of the Ito SDE (assuming it exists in the usual way). We will apply Ito's formula to $V(t, X^U(t))$ and deduce that

$$E_{t,x}\{g(X^U(T))\} \geq V(t,x),$$

and since the right side is independent of $U$,

$$\inf_U E_{t,x}\{g(X^U(T))\} \geq V(t,x)$$

From Itô's formula and after integrating we have

$$V(T, X^U(T)) = V(t, X^U(t)) + \int_t^T [V_t(s, X^U(s)) + \mathcal{L}_u V(s, X^U(s))] ds$$
$$+ \int_t^T \sigma(X^U(s), U(s)) V_x(s, X^U(s)) dB_s$$

Using the terminal condition and taking expectation given $X(t) = x$ we have further

$$E_{t,x}[g(X^U(T))] = E_{t,x}[V(T, X^U(T))] = V(t,x) + E_{t,x}\left[\int_t^T [V_t(s, X^U(s)) + \mathcal{L}_u V(s, X^U(s))] ds\right].$$

For suboptimal $u$, $V_t + \mathcal{L}_u V \geq 0$, pointwise in $(t,x)$ so

$$E_{t,x}[g(X^U(T))] \geq V(t,x)$$

and, as noted, since $V$ does not depend on $U$ we have

$$\tilde{V}(t,x) = \inf_U E_{t,x}[g(X^U(T))] \geq V(t,x).$$

This says that the solution of the HJB equation $V(t,x)$ is a lower bound for the function $\tilde{V}(t,x)$, the value function of the control problem.

In fact, $\tilde{V}(t,x) = V(t,x)$, so the solution of the HJB equation is the value function of the optimal control. To see this we apply Ito's formula to $V(t, X^*(t))$ and the optimal control $U^*(t) = u^*(t, X^*(t))$. We have

$$E_{t,x}[g(X^*(T)))] = V(t,x) + E_{t,x}\left[\int_t^T [V_t(s, X^*(s)) + \mathcal{L}_{U^*(s)} V(s, X^*(s))] ds\right],$$

assuming that the martingale term is well defined so that its expectation is zero. Since the control $U^*(t) = u^*(t, X^*(t))$ is minimal for the generator we have $V_t(t, X^*(t)) + \mathcal{L}_{u^*(t,X^*(t))} V(t, X^*(t)) = 0$. Therefore,

$$E_{t,x}[g(X^*(T))] = V(t,x).$$

So, $U^*$ is the optimal control since it attains the lower bound, and we have the desired identification

$$V(t,x) = \tilde{V}(t,x) = \inf_U E_{t,x}[g(X(T))].$$

Keep in mind that this identification or verification process (Lemma or Theorem) relies heavily on the regularity of the solution $V$ of the HJB equation and then on the construction of the optimal process $X^*$. This can be a very difficult issue in practice.

If there is a constraint, for example $X(t) \geq 0$ for all time then we need a stopping time and a boundary condition for the value function. Let $\tau$ be the exit time from zero, $\tau = \inf\{s > t \mid X(s) = 0\}$. Also, assume that the value function $V$ is zero if $X(s) = 0$. We can apply Ito's formula up to the minimum of $\tau$ and $T$

$$V(\tau \wedge T, X^U(\tau \wedge T)) = V(t, X^U(t)) + \int_t^{\tau \wedge T} [V_t(s, X^U(s)) + \mathcal{L}_u V(s, X^U(s))] ds$$

$$+ \int_t^{\tau \wedge T} \sigma(X^U(s), U(s)) V_x(s, X^U(s)) dB_s$$

Taking expectations (and using the optional stopping theorem) we have

$$E_{t,x}[V(\tau \wedge T, X^U(\tau \wedge T))] = V(t,x) + E_{t,x}\left[\int_t^{\tau \wedge T} [V_t(s, X^U(s)) + \mathcal{L}_u V(s, X^U(s))] ds\right].$$

As above, for suboptimal $U$, we have

$$E_{t,x}[V(\tau \wedge T, X^U(\tau \wedge T))] \geq V(t,x)$$

We now note that

$$E_{t,x}[V(\tau \wedge T, X^U(\tau \wedge T))] = E_{t,x}[V(\tau, X^U(\tau))\chi_{\tau \leq T}] + E_{t,x}[V(T, X^U(T))\chi_{\tau > T}]$$

and using the terminal and boundary conditions we get the left side is $E_{t,x}[g(X^U(T))\chi_{\tau > T}]$ and therefore

$$E_{t,x}[g(X^U(T))\chi_{\tau > T}] \geq V(t,x)$$

As above, again, we get equality when using the optimal control so that

$$\inf_U E_{t,x}[g(X^U(T))\chi_{\tau > T}] = V(t,x)$$