CS 103: Mathematical Foundations of Computing
Problem Set #2

Zara Rutherford, Diya Ahuja

January 26, 2024

Due Friday, January 26 at 4:00 pm Pacific

Problems One through Six are to be answered by editing the appropriate files (see the Problem Set #2 instructions). You won't include your answers to those problems here.

Symbols Reference

Here are some symbols that may be useful for this PSet. If you are using LaTeX, view this section in the template file (the code in `cs103-ps2-template.tex`, not the PDF) and copy-paste math code snippets from the list below into your responses, as needed. If you are typing your Pset in another program such as Microsoft Word, you should be able to copy some of the symbols below from this PDF and paste them into your program. Unfortunately the symbols with a slash through them (for "not") and font formats such as exponents don't usually copy well from PDF, but you may be able to type them in your editor using its built-in tools.

- Logical AND: $\wedge$
- Logical OR: $\vee$
- Logical NOT: $\neg$
- Logical implies: $\to$
- Logical biconditional: $\leftrightarrow$
- Logical TRUE: $\top$
- Logical FALSE: $\bot$
- Universal quantifier: $\forall$
- Existential quantifier: $\exists$

LaTeXtyping tips:

- Set (curly braces need an escape character backslash): $1, 2, 3$ (incorrect), $\{1, 2, 3\}$ (correct)
- Exponents (use curly braces if exponent is more than 1 character): x^2, 2^{3x}
- Subscripts (use curly braces if subscript is more than 1 character): x_0, x_{10}

Problem Seven: Yablo's Paradox

i.

Theorem: For all natural numbers n, there is no statement S_n that is true.
Proof: Assume for the sake of contradiction that there exists a natural number n where the statement S_n is true. Since S_n is true, this means that all statements following S_n are false. Therefore, the statement S_{n+1} is false. Since S_{n+1} is false, there exists a statement after S_{n+1} that must be true. However, we know that all statements after S_n are false. We have reached a contradiction, so our assumption must have been wrong. Therefore, for all natural numbers n, there is no statement S_n that is true. ■

ii.

Theorem: For all natural numbers n, there is no statement S_n that is false.
Proof: Assume for the sake of contradiction that there exists a natural number n where the statement S_n is false. Since S_n is false, there exists some natural number k where $k > n$ and S_k is true.

Since S_k is true, all statements following S_k must be false. Therefore, the statement S_{k+1} is false. If S_{k+1} is false, there is at least one statement after S_{k+1} that must be true. However, we know that all statements after S_k are false. We have reached a contradiction, so our assumption must have been wrong. Therefore, for all natural numbers n, there is no statement S_n that is false.■

iii.

The last statement is true since there are no statements after it to cause a contradiction. Therefore, all the statements before the last one are false, because for any of those statements to hold true, they require the last statement to be false, however, we see that the last statement is vacuously true.

Problem Eight: Hereditary Sets

i.

To prove that there is at least one hereditary set, we will show that there exists a set S where every $T \in S$ is also a hereditary set. To do so, pick $S = \{\emptyset\}$. $T = \emptyset$ is the only element of set S. Therefore, we need to show that T is also a hereditary set.

Since T has no elements, the statement that "every element in T is also a hereditary set" is vacuously true. Therefore, we see that T is a hereditary set, which is what we needed to show. ■

ii.

Theorem: Given an arbitrary hereditary set S, then $\wp(S)$ is also a hereditary set.
Proof: Given an arbitrary hereditary set S, we need to show that $\wp(S)$ is also a hereditary set. To do so, consider an arbitrary set $T \in \wp(S)$. We need to show that T is also a hereditary set.

Choose another arbitrary set $X \in T$. Since $T \in \wp(S)$, this means that $T \subseteq S$, therefore we know that $X \in S$. Since all elements of S are hereditary, X must be hereditary. Therefore, T is hereditary, which is what we needed to show. ■

Problem Nine: Tournament Champions

i.

Player D: No, Player E: Yes

ii.

Theorem: In an arbitrary tournament T, if a player c won more games than anyone else in T or is tied for winning the greatest number of games, then c is a tournament champion of T.
Proof: Assume for the sake of contradiction that if a player c won more games than anyone else in an arbitrary tournament T or is tied for winning the greatest number of games, then c is not a tournament champion of T.

Pick an arbitrary player k in T who beats c. By our assumption, k is tied or has fewer wins than c. Let q be a natural number that represents the number of players that c has won against. If c is not a tournament champion, then none of the players q that c won against beat k. Therefore, it follows that all the players q that c beat are also beaten by k. However, if all the players q that c beats are also beaten by k, and k also beats c, then the number of players k has won against must be $q + 1$ or greater. This is impossible because we assumed that c has won the most games or is tied for winning the most games.

We have reached a contradiction, so our assumption must have been wrong. Therefore, if a player c won more games than anyone else in T or is tied for winning the greatest number of games, then c is a tournament champion of T. ■

Optional Fun Problem: Insufficient Connectives

Write your answer to the Optional Fun Problem here.