

1 Model

We begin with a general model that encapsulates many of the works we will discuss. For an input \mathbf{x} , we want to predict a response \mathbf{y}^* . Our prediction is a function f_θ of \mathbf{x} . The function f_θ is parameterized by θ . We will consider models of the form

$$f_\theta(\mathbf{x}) = y_\theta \left(\sum_i \alpha_\theta^{(i)} \phi_\theta^{(i)}(\mathbf{x}) \right). \quad (1)$$

Here, \mathbf{x} is an L -dimensional real vector and \mathbf{y}^* is an M -dimensional real vector. The ϕ_i are basis functions that map from \mathbb{R}^L to \mathbb{R}^K , and the α_i are real scalars. Additionally, the sum over i need not be finite. The coefficients α_i may depend on \mathbf{x} and our parameters θ , but we suppress the former dependence for notational simplicity. The function y_θ maps from \mathbb{R}^K to \mathbb{R}^M and is parameterized by θ as well.

We will consider a penalized mean squared error loss function

$$\mathcal{L}(\theta) = \frac{1}{2} \mathbb{E}_{\mathbf{x}, \mathbf{y}^*} [\|\mathbf{y}^* - f_\theta(\mathbf{x})\|_2^2] + \lambda p(\{\alpha^{(i)}\}_i, \theta). \quad (2)$$

Our penalty function p serves to induce sparseness in the coefficients α_i , and perhaps also regularize the parameters θ .

We will show how this model encapsulates the works we are interested in understanding.

2 Ingrosso et al. (2022)

The model in Ingrosso et al. (2022) sets $M = 1$, uses linear basis functions, and sets y_θ to be the mean function after applying a nonlinearity. That is,

$$\sum_i \alpha_\theta^{(i)} \phi_\theta^{(i)}(\mathbf{x}) = \Theta \mathbf{x} + b_\theta \quad (3)$$

$$y_\theta(\mathbf{x}) = \frac{1}{K} \mathbf{1}^\top \sigma(\mathbf{x}). \quad (4)$$

Here, Θ is a $K \times L$ matrix, b_θ is a K -dimensional vector, and σ is a nonlinear function applied elementwise. It seems that in much of their work, b_θ is fixed at -1 or 0 . **I still need to confirm experimentally that this does not affect the results.**

In this work, we also ignore the penalty term. I believe that our gradient update is of the form

$$a \quad (5)$$

3 Data Model

We assume that our data is generated by the following model. It is parameterized by the length of the input, $L \in \mathbb{N}$. It is also parameterized by a scale parameter $\xi \leq L$. We construct data $\{X_l\}_l \subseteq \{0, 1\}^L$ as follows:

1. Sample integers $l^* \sim \text{Uniform}[1, L]$ (starting position) and $T \sim \text{Uniform}[1, \xi]$ (length of pulse).
2. For $0 \leq i \leq T$, set $X_{l^*+i \pmod L} = 1$, and set all other X_l to 0.
3. Return the sequence $\{X_l\}_l$.

Now, we derive the conditional probability $p_{11} \triangleq \mathbb{P}(X_a = 1 \mid X_b = 1)$. For now, assume $d \triangleq b - a > 0$. If $d \geq \xi$, then $p_{11} = 0$. So, assume $d < \xi$.

Now, we count the number of values of T that result in both X_a and X_b being in the pulse for a given l^* . WLOG, assume $a = 0$. For $1 \leq l^* \leq d$, the range of values for T that results in both X_a and X_b being in the pulse is given by

$$L - l^* \leq T < \xi. \quad (6)$$

Thus, the number of values for T in this case is $\max(\xi - L + l^*, 0)$.

For $d < l^* \leq L$, the values of T that result in both X_a and X_b being in the pulse is given by

$$L - (l^* - d) \leq T < \xi. \quad (7)$$

So, the number of values for T in this case is $\max(\xi - (L - (l^* - d)), 0) = \max(\xi - L + l^* - d, 0)$.

Note that the first max condition is at least zero when $l^* \geq L - \xi$. This yields a sum over $\max(L - \xi, 1) \leq l^* \leq d$. Similarly, the second max condition is at least zero when $l^* \geq L - \xi + d$. This yields a sum over $\max(L - \xi + d, d + 1) \leq l^* \leq L$. Note that the first term dominates in both of these new max statements when $\xi \leq L - 1$. So, let us assume this is the case.

Now, we want to find the total number of values of T that result in both X_a and X_b being in the pulse.

$$T_1 = \sum_{L - \xi \leq l^* \leq d} (\xi - L + l^*) \quad (8)$$

$$T_2 = \sum_{L - \xi + d \leq l^* \leq L} (\xi - L + l^* - d) = \sum_{L - \xi \leq t \leq L - d} (\xi - L + t). \quad (9)$$

We compute T_1 as,

$$T_1 = \left[(\xi - L)(d - L + \xi + 1) + \frac{d(d+1)}{2} - \frac{(L - \xi)(L - \xi - 1)}{2} \right] \mathbb{1}(d \geq L - \xi).$$

Next, we compute T_2 as,

$$T_2 = \left[(\xi - d + 1)(\xi - L) + \frac{(L - d)(L - d + 1)}{2} - \frac{(L - \xi)(L - \xi - 1)}{2} \right] \mathbb{1}(d \leq \xi).$$

Thus, the total number of values of T that result in both X_a and X_b being in the pulse is

$$T_1 + T_2 = \begin{cases} (\xi - d + 1)(\xi - L) + \frac{(L - d)(L - d + 1)}{2} - \frac{(L - \xi)(L - \xi - 1)}{2} & d \leq \xi, L - \xi + 1 \\ d^2 - dL + \frac{L^2}{2} + \frac{L}{2} - (L - \xi)(\xi + 1) & L - \xi \leq d \leq \xi \\ (\xi - L)(d - L + \xi + 1) + \frac{d(d+1)}{2} - \frac{(L - \xi)(L - \xi - 1)}{2} & d \geq L - \xi, \xi + 1 \end{cases}$$

There are $\xi - 1$ possible values for T and L values for l^* . Recall we sample T and l^* uniformly and independently. Thus, the probability that $X_b = 1$ given $X_a = 1$ is

$$p_{11} = \frac{T_1 + T_2}{(\xi - 1)L} = \frac{d^2 - dL + \frac{L^2}{2} + \frac{L}{2} - (L - \xi)(\xi + 1)}{(\xi - 1)L}. \quad (10)$$

Note that we assumed $\xi \leq L - 1$. However, we can check that the above expression is still valid when $\xi = L$. (Check Desmos. Otherwise, an exercise left to the reader.) It also satisfies the “sanity check” that it is minimized at $x = \frac{L}{2}$.

4 Nonlinear Gaussian Process

We now consider the nonlinear Gaussian process (NLGP) data model. Define a covariance matrix C with entries $C_{ij} = \exp(-(i-j)^2/\xi^2)$ for $i, j \in [L]$. Let $Z = [Z_1, \dots, Z_L] \sim \mathcal{N}(0, C)$. Then, Z is a Gaussian process with covariance C . Consider a nonlinearity ψ . We specifically consider the error function, $\psi(z) = \text{erf}(z/\sqrt{2})$. Define $X_i = \psi(gZ_i)$ for $i \in [L]$ and some $g > 0$. Then, X is a nonlinear Gaussian process (NLGP).

Now, we will compute the moments of X . Each X_i clearly has mean zero. To compute higher moments, we will need to employ some handy properties of the error function. Assume Z_1 and Z_2 have covariance ρ . First, note

$$\mathbb{E}[X_1 X_2] = \mathbb{E}_{X_1} [\mathbb{E}_{X_2} [X_1 X_2 \mid X_1]] \quad \text{Law of Total Expectation} \quad (11)$$

$$= \mathbb{E}_{X_1} [X_1 \mathbb{E}_{X_2} [X_2 \mid Z_1]]. \quad \psi \text{ invertible} \quad (12)$$

Now, we compute the inner expectation.

$$\mathbb{E}_{X_2} [X_2 \mid Z_1] = \mathbb{E}_{Z_2} [\psi(gZ_2) \mid Z_1] \quad (13)$$

$$= \int_{-\infty}^{\infty} \text{erf}\left(\frac{gz}{\sqrt{2}}\right) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(z-\mu)^2} dz \quad \mu = \rho Z_1, \sigma^2 = 1 - \rho^2 \quad (14)$$

$$= \text{erf}\left(\frac{g\mu/\sqrt{2}}{\sqrt{1+g^2\sigma^2}}\right) \quad (15)$$

$$= \text{erf}\left(\frac{\rho g}{\sqrt{2(1+g^2(1-\rho^2))}} Z_1\right). \quad (16)$$

Now, the outer expectation

$$\mathbb{E}_{X_1} \left[X_1 \text{erf}\left(\frac{\rho g}{\sqrt{2(1+g^2(1-\rho^2))}} Z_1\right) \right] \quad (17)$$

$$= \mathbb{E}_{Z_1} \left[\text{erf}\left(\frac{gZ_1}{\sqrt{2}}\right) \text{erf}\left(\frac{\rho g}{\sqrt{2(1+g^2(1-\rho^2))}} Z_1\right) \right] \quad (18)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \text{erf}\left(\frac{g}{\sqrt{2}}z\right) \text{erf}\left(\frac{\rho g}{\sqrt{2(1+g^2(1-\rho^2))}}z\right) dz \quad (19)$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}z^2} \text{erf}\left(\frac{g}{\sqrt{2}}z\right) \text{erf}\left(\frac{\rho g}{\sqrt{2(1+g^2(1-\rho^2))}}z\right) dz \quad \text{integrand is even} \quad (20)$$

$$= \frac{2}{\pi} \tan^{-1} \left(\frac{\frac{g}{\sqrt{2}} \cdot \frac{\rho g}{\sqrt{2(1+g^2(1-\rho^2))}}}{\sqrt{\frac{1}{2} \left(\frac{g^2}{2} + \frac{\rho^2 g^2}{2(1+g^2(1-\rho^2))} + \frac{1}{2} \right)}} \right) \quad \text{Prudnikov et al. (1990)} \quad (21)$$

$$= \frac{2}{\pi} \sin^{-1} \left(\frac{g^2}{1+g^2\rho} \right). \quad \text{I promise} \quad (22)$$

We use the same idea to compute a third-order moment,

$$\mathbb{E}[X_1 X_2 X_3] = \mathbb{E}_{X_1} [\mathbb{E}_{X_2} [\mathbb{E}_{X_3} [X_1 X_2 X_3 \mid X_2] \mid X_2]] \quad \text{Law of Total Expectation} \quad (23)$$

$$= \mathbb{E}_{X_1} [X_1 \mathbb{E}_{X_2} [X_2 \mathbb{E}_{X_3} [X_3 \mid X_2] \mid X_2]]. \quad (24)$$

Note that $Z_3 \mid Z_1, Z_2 \sim \mathcal{N}()$