DIFFERENTIAL EQUATION NOTES

ON

1st Order Differential Equations
1st Order Numerical Methods

By

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Contents

1 Ordinary Differential Equation Applications

1.1 Basic definitions

DEFINITION 1.1 (ODE). *If $f(x)$ is a function defined at an interval $I: a < x < b$, then by ordinary differential equation (ODE) we mean an equation involving x, the function $f(x)$ and one or more of its derivatives.*

We often denote $y := f(x)$ so the following are ODEs:

$$\frac{dy}{dx} + y = 0$$

$$f''(x) = -4f'(x)$$

$$y^{(4)} + 2xy'' = x^3$$

1.2 Separable ODEs

DEFINITION 1.2 (separable ODE). *An ODE that can be written in the form*

$$N(y)\frac{dy}{dx} = M(x) \tag{1.1}$$

is called separable.

COROLLARY 1.1 (separable ODE). *A separable ODE can be rewritten in the form*

$$N(y)dy = M(x)dx \tag{1.2}$$

Proof. To solve Eq. (1.1), we integrate both sides w.r.t. x:

$$\int N(y)\frac{dy}{dx}\,dx = \int M(x)\,dx$$

Remember that by y we denote a function of x, i.e. $y = y(x)$. Therefore letting:

$$u := y(x) \Rightarrow du = y'(x)dx = \frac{dy}{dx}dx$$

, the integrals can be rewritten as:

$$\int N(u)\,du = \int M(x)\,dx$$

Because u is a dummy variable, this is equivalent to:

$$\int N(y)\,dy = \int M(x)\,dx$$

The takeaway is that to solve Eq. (1.1), we can pretend that the derivative $\frac{dy}{dx}$ is a fraction and that we can multiply both sides by dx. □

If an ODE is in the form:

$$f(x)dx + g(y)dy = 0$$

, then we can integrate to find either an implicit (e.g. $x\cos(y) + 4x^3 - 1 = 0$) or an explicit (e.g. $y = -ln\,x$) solution. Whether the solution is implicit or explicit depends on whether $f(x)$ and $g(x)$ can be integrated.

EXAMPLE 1.1 (no initial condition). *Solve the equation*

$$x\,dx + \frac{\sin y}{\cos x}\,dy = 0$$

SOLUTION 1.1. *Rewrite in the same form as Eq.* (1.2)*:*

$$x\cos x\,dx + \sin y\,dy = 0$$

Integrate both terms. Integrate the left term by parts:

$$\begin{aligned}
&\int x\cos x\,dx \\
&= \int x(\sin x)'\,dx \\
&= x\sin x + \int \sin x\,dx \\
&= x\sin x - \cos x + C
\end{aligned}$$

, where C *is an arbitrary constant. Integrate the right term:*

$$\int \sin y\,dy = -\cos y + C$$

, where C *is another arbitrary constant. Substitute both integrals to express the solution in a closed form:*

$$x\sin x - \cos x - \cos y + C = 0$$

Example 1.2 (initial condition)**.** *Solve the ODE*

$$\frac{dy}{dx} = \frac{x(x^{x^2}+2)}{6y^2}$$

given that $y(0) = 1$.

Solution 1.2. *Rewrite it as a separable equation:*

$$6y^2\,dy = x(e^{x^2}+2)dx$$

Integrate both sides:

$$\int 6y^2\,dy = \int x(e^{x^2}+2)\,dx \Rightarrow 2y^3 = \frac{1}{2}e^{x^2} + x^2 + C$$

Set $x = 0$, $y = 1$ *to determine* C*:*

$$2 = 1/2 + C \Rightarrow C = 3/2$$

Therefore the (implicit) solution is:

$$2y^3 = \frac{1}{2}e^{x^2} + x^2 + \frac{3}{2}$$

Example 1.3 (dividing by zero)**.** *Solve the ODE*

$$\frac{dy}{dx} = (1+e^{-x})(y^2-1)$$

Solution 1.3. *To separate the equation, we have to divide by* (y^2-1)*. However, that maybe be zero for certain values of* y*. Therefore before we divide by it, we have to exclude those values first and test them (manually) in a separate step.*

For starters, divide by (y^2-1) *for* $(y^2-1) \neq 0$*, i.e.* $y \neq 1$ *and* $y \neq -1$.

$$\frac{dy}{y^2-1} = (1+e^{-x})\,dx$$

Integrate it:

$$\begin{aligned}
\int \frac{dy}{y^2-1} &= \int (1+e^{-x})\,dx \Rightarrow \\
\int \left(\frac{-\frac{1}{2}}{y+1} + \frac{\frac{1}{2}}{y-1}\right)dy &= \int (1+e^{-x})\,dx \Rightarrow \\
\ln\left|\frac{y-1}{y+1}\right| &= 2(x - e^{-x} + C) \qquad (\star)
\end{aligned}$$

$Eq.$ (✯) describes a solution to the original ODE. However we have excluded $y = \pm 1$ so we need to test these too. For $y = \pm 1$, the original ODE yields a statement that is true for any x:

$$0 = (1 + e^{-x})(1 - 1)$$

Therefore the family

$$y = \pm 1 \tag{✯✯}$$

is also a solution.

1.2.1 Separable Equations and Substitution Methods

The theory of separable equations can be applies to solve first-order ODEs of the form $\frac{dy}{dx} = F(ax + by + c)$.

Corollary 1.2. *The substitution $v = ax + by + c$ transforms the equation*

$$\frac{dy}{dx} = F(ax + by + c) \tag{1.3}$$

into a separable equation.

Proof. Let's assume $b \neq 0$. Using the substitution $v = ax + by + c$ and differentiating w.r.t. $\frac{dv}{dx} = a + b\frac{dy}{dx}$, then the equation becomes:

$$\frac{dv}{dx} = a + bF(v) \Rightarrow \frac{dv}{a + F(v)} = dx$$

The latter equation is separable.

If $b = 0$, then the equation $\frac{dy}{dx} = F(ax + c)$ is already separable. □

1.2.2 Application: Newton's Law of Cooling

Newton's law of cooling describes the temperature of a hot body if it immersed in a medium of constant temperature [1], e.g. a thermos of warm coffee in a room. More concretely, "the time rate of change of the temperature $T(t)$ of a body of initial temperature T_0 immersed in a medium of constant temperature M is proportional to the difference $M - T(t)$". This translates into:

$$T'(t) = k(M - T(t))$$

Eq. (1.4) is separable as it can be rewritten as (assuming hot body, therefore $T(t) > M \; \forall t$):

$$\frac{T'(t)}{T(t) - M} = -k \Rightarrow$$

$$\ln|T(t) - M| = \ln(T(t) - M) = -kt + C \Rightarrow$$

$$T(t) = e^{-kt}e^{C} + M$$

We know $T(0) = T_0$, so plugging $t = 0$ in the latter equation we obtain:

$$e^{C} = T_0 - M$$

To summarise.

Corollary 1.3 (Newton's law of cooling)**.** *Assuming $T(t) > M$, the solution to the DE*

$$T'(t) = k(M - T(t)) \tag{1.4}$$

, where k and M are constants, is given by:

$$T(t) = M + e^{-kT}(T(0) - M) \tag{1.5}$$

1.3 Homogeneous ODEs

[1] Newton's law of cooling can be derived if we take into account the rate of *net* heat transfer by radiation equation (a.k.a. Stefan-Boltzmann law): $dQ = eA\sigma(T^4 - T_0^4)$ and the fundamental law of thermodynamics: $Q = msT \Rightarrow dQ = msdT$

DEFINITION 1.3 (homogeneous function). *A function $f(x,y)$ is said to be homogeneous of degree n if the equation*

$$f(zx,zy) = z^n f(x,y) \tag{1.6}$$

holds for all x, y and z (for which both sides are defined).

EXAMPLE 1.4. *Prove that the function $f(x,y) = x^2 + y^2 - xy$ is homogeneous and find its degree.*

SOLUTION 1.4.

$$f(zx,zy) = (zx)^2 + (zx)^2 - zx \cdot zy = z^2(x^2 + y^2 - xy) = z^2 f(x,y)$$

Therefore its degree is 2.

DEFINITION 1.4 (homogeneous ODE). *A first-order ODE is said to be* h o m o g e n e o u s *if $M(x,y)$ and $N(x,y)$ are both homogeneous functions of the same degree.*

Homogeneous ODEs of first order can be solved by setting

$$v = \frac{y}{x} \Rightarrow y = vx \tag{1.7}$$

$$\therefore \frac{dy}{dx} = \frac{d(vx)}{dx}$$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx} \tag{1.8}$$

EXAMPLE 1.5. *Solve $x^2 - y^2 + 2xyy' = 0$*

SOLUTION 1.5. *It can be rewritten as:*

$$\underbrace{(x^2 - y^2)}_{M(x,y)} dx + \underbrace{2xy}_{N(x,y)} dy = 0$$

$M(x,y)$ and $N(x,y)$ are both homogeneous functions of degree 2 as $M(zx,zy) = z^2 M(x,y)$ and $N(zx,zy) = z^2 N(x,y)$ therefore the DE is homogeneous of degree 2. To solve it by making it separable, the substitutions in Eq. (1.7), Eq. (1.8) can be applied. However before applying them, divide first by x^2 (for $x \neq 0$) to show up the ratio y/x:

$$1 - \left(\frac{y}{x}\right)^2 + 2\frac{y}{x}y' = 0 \tag{$\star$}$$

Now apply the transform in in Eq. (1.7), Eq. (1.8):

$$y = vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Eq. ($\star$) becomes:

$$1 - v^2 + 2v(v + xv') = 0 \Rightarrow$$

$$1 + v^2 + 2xvv' = 0 \Rightarrow$$

$$\frac{2vv'}{1+v^2} = -\frac{1}{x} \Rightarrow$$

$$\ln(1+v^2) = -\ln x + C = \ln(\frac{C}{x}) \Rightarrow$$

$$(1 + v^v 2)x = A$$

, where A is another arbitrary constant such that $\ln C = A$. Substituting back v with y/x:

$$x + \frac{y^2}{x} = A \Rightarrow$$

$$y^2 = Ax - x^2 \Rightarrow$$

$$y = \pm\sqrt{Ax - x^2} \tag{$\star\star$}$$

This solution holds for $x \neq 0$. Testing for $x = 0$:

$$y = 0$$

Hence the point $(0,0)$ and Eq. (✯✯) are all the solutions.

Example 1.6. *Find the solution(s) of the homogeneous ODE*

$$yy' + x = \sqrt{x^2 + y^2}, \quad x, y > 0$$

Solution 1.6. *The equation can be rewritten:*

$$y\,dy + x\,dx = \sqrt{x^2 + y^2}\,dx \Rightarrow$$

$$\underbrace{y}_{M(x)}\,dy = \underbrace{(\sqrt{x^2 + y^2} - x)}_{N(x,y)}\,dx$$

It's easy to see (Def'n 1.4) that the latter is homogeneous of degree 1, as both $M(x,y), N(x,y)$ have the same degree of 1. Set $y = vx \Rightarrow y' = v + x\frac{dv}{dx} = v + xv'$. Then the equation becomes (we eliminate y):

$$xv\left(v + xv'\right) + x = \sqrt{x^2(1+v^2)}\,dx = x\sqrt{1+v^2}\,dx \Rightarrow$$

$$v\left(v + xv'\right) = \sqrt{1+v^2} - 1 \Rightarrow$$

$$xvv' = \sqrt{1+v^2} - 1 - v^2$$

$y, x > 0 \therefore v > 0$ so we can divide:

$$\frac{vv'}{v^2 + 1 - \sqrt{1+v^2}} = -\frac{1}{x} \Rightarrow$$

$$\frac{vv'}{\sqrt{v^2+1}(\sqrt{v^2+1} - 1)}$$

We can multiply the LHS by the conjugate $(\sqrt{1+u^2} + 1)$ to simplify it:

$$\frac{v(\sqrt{1+v^2} + 1)v'}{v^2\sqrt{1+v^2}} = -\frac{1}{x}$$

This is already a separable DE but at the moment it cannot be integrated. Break the fraction to make the LHS integrateable:

$$\frac{v'}{v} + \frac{v'}{v\sqrt{1+v^2}} = -\frac{1}{x}$$

Integrate both sides w.r.t x:

$$\ln(v) + \int \frac{dv}{v\sqrt{1+v^2}} = -\ln x + C \tag{★}$$

To compute the remaining integral, change variables by setting $u = 1/v \Rightarrow dv = -du/u^2$:

$$\int \frac{dv}{v\sqrt{1+v^2}} = -\int \frac{u}{\sqrt{1+\frac{1}{u^2}}}\frac{du}{u^2} = -\int \frac{du}{\sqrt{1+u^2}}$$

$$= -\ln\left|u + \sqrt{1+u^2}\right| + C' = -\ln|u| + \ln\left|\sqrt{1+u^2} + 1\right| + C'$$

Plugging in the integral in Eq. (★) we get:

$$\ln\left|\sqrt{1+u^2} + 1\right| = \ln\frac{1}{x} + \ln K = \ln\frac{K}{x}$$

, where K is some other arbitrary non-negative constant such that $\ln K = C'$. Therefore, since $x > 0$:

$$\sqrt{1+u^2} + 1 = K/x \Rightarrow$$

$$\frac{xv^2}{\sqrt{1+v^2}+1} = K \Rightarrow$$

$$\frac{xv^2(\sqrt{1+v^2}-1)}{v^2} = x(\sqrt{1+v^2}-1) = K$$

Since $v = y/x$, we obtain:

$$x\left(\sqrt{1+\frac{y^2}{x^2}}-1\right) = \sqrt{x^2-y^2} - x = K \Rightarrow$$

Solving for $y(x)$:

$$y(x) = \sqrt{(2x+K)K}, \quad x \in (-K/2, \infty)$$

We obtained the last constraint since $(2x+K)$ is under the square root, therefore $2x+K \geq 0$. Additionally, the square root is positive as $y(x) > 0$.

1.4 Linear Differential Equations of First Order

DEFINITION 1.5 (linear non-homogeneous DE of first order)**.** *A differential equation of the type*

$$y' + a(x)y = f(x) \tag{1.9}$$

, where $a(x)$ and $f(x)$ are continuous functions of x, is called linear non-homogeneous (NH) DE of first order.

Linear NH DEs of first order are typically solved by the method of integrating factor. If we multiply both sides of Eq. (1.9) by the factor $u(x) := \exp\left(\int a(x)dx\right)$, then it becomes:

$$\exp\left(\int a(x)dx\right)y' + \exp\left(\int a(x)dx\right)' y = f(x)\exp\left(\int a(x)dx\right) \Rightarrow$$

$$u(x)y' + u'(x)y = f(x)u(x) \Rightarrow$$

$$(u(x)y)' = f(x)u(x) \Rightarrow$$

$$y = \frac{\int f(x)u(x)dx + C}{u(x)}$$

, where C is an arbitrary constant. To summarise:

COROLLARY 1.4 (solution of NH DE of first 1 order by intergrating factor)**.** *The solution of the DE*

$$y' + a(x)y = f(x) \tag{1.10}$$

is given in terms of the integrating factor $u(x) = \exp\left(\int a(x)dx\right)$ as:

$$y = \frac{\int f(x)u(x)dx + C}{u(x)}, \quad u(x) = \exp\left(\int a(x)dx\right) \tag{1.11}$$

, where C is an arbitrary constant.

Constant C can be determined if we are given an initial condition $y(x_0) = y_0$.

1.5 Exact equations

Exact equations are a subset of the general first order DEs. They can easily be recognised and solved in some standard steps.

DEFINITION 1.6 (exact DE). *A DE of type*

$$P(x,y)dx + Q(x,y)dy = 0 \tag{1.12}$$

, or equivalently $P(x,y)+Q(x,y)y' = 0$ *is said to be* e x a c t *if there exists a function of two variables* $u(x,y)$ *with continuous partial derivatives such that:*

$$\frac{\partial u}{\partial x} = P(x,y), \quad \frac{\partial u}{\partial y} = Q(x,y) \tag{1.13}$$

COROLLARY 1.5. *Because of Eq.* (1.13)*, the general solution of an exact DE can be written in terms of the general solution as:*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} = 0$$

In other words, the total derivative of $u(x,y)$ *must be zero. Therefore any function* $u(x,y) = C$*, where* C *is a constant, satisfies the DE.*

From the interchangeability of the order of the partial derivatives, it also follows that the following condition is necessary for the Eq. (1.12) to be exact.

COROLLARY 1.6 (exactness conditions). *If*

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x} \tag{1.14}$$

, then the DE

$$P(x,y)dx + Q(x,y)dy = 0 \tag{1.15}$$

is exact.

Exact DEs of the form Eq. (1.12) can be solved in a certain recipe.

1. (Test for exactness) Check whether the condition in Eq. (1.15) is true:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

2. (Define the general solution function $u(x,y)$) Write the system

$$\begin{cases} \frac{\partial u}{\partial x} = P(x,y) \\ \frac{\partial u}{\partial y} = Q(x,y) \end{cases}$$

3. (Integrate w.r.t x) Integrate the first equation w.r.t. x or the second w.r.t. y (whatever is easier). Without loss of generality, we proceed in the first way. Because we integrate w.r.t. x, instead of constant C we have an unknown function of y:

$$u(x,y) = \int P(x,y)dx + \phi(y) \tag{2}$$

To find $u(x,y)$, we first need to determine $\phi(y)$.

When we integrate $u(x,y)$ w.r.t. x, we treat y as a constant, hence the int. constant ϕ will be a function y.

4. (Differentiate w.r.t. y to determine the int. constant) From the exactness condition, we know $u_y = Q(x,y)$. Differentiate Eq. (2) w.r.t. y to show up $\phi'(y)$:

$$\frac{du}{dy} = \frac{\partial}{\partial y}\left[\int P(x,y)dx + \phi(y)\right] = Q(x,y) \Rightarrow$$

$$\phi'(y) = Q(x,y) - \frac{\partial}{\partial y}\left(\int P(x,y)dx\right)$$

5. (Determine int. constant)Integrating the last equation w.r.t. y we can determine $\phi(y)$:

$$\phi(y) = \int Q(x,y)dy - \int P(x,y)dx \tag{3}$$

Plugging Eq. (3) in Eq. (2), we can determine $u(x,y)$.

Don't need an int. constant here – the general solution includes one.

6. (Obtain the general solution) The general solution is written in closed form as:

$$u(x,y) = C$$

EXAMPLE 1.7. *Consider the equation*

$$(2x+y)+(x+2y)y' = 0$$

Is it exact? If so, find the solution given the initial condition $y(1) = 1$.

SOLUTION 1.7. *We have $P(x,y) = 2x+y$, $Q(x,y) = x+2y$. $P_y = 1 = Q_x$ therefore is it exact. To solve it, we need to find a function $u(x,y)$ such that:*

$$u(x,y)_x = P(x,y) = 2x+y, \quad u(x,y)_y = Q(x,y) = x+2y \tag{✪}$$

Integrate the first equation (w.r.t.) x:

$$u(x,y) = \int u_x(x,y)\,dx = \int (2x+y)dx = x^2+xy+\phi(y) \tag{✪✪}$$

Differentiate Eq. (✪✪) w.r.t. y and use Eq. (✪):

$$u_x(x,y) = x+\phi'(y) = x+2y \Rightarrow$$

$$\phi'(y) = 2y \Rightarrow$$

$$\phi(y) = y^2$$

Therefore from Eq. (✪✪) we can determine $u(x,y)$:

$$u(x,y) = x^2+xy+\phi(y) = x^2+xy+y^2$$

The general solution is written in implicit form as:

$$u(x,y) = C \Leftrightarrow x^2+xy+y^2 = C$$

, where C is a constant that can be determined from the initial condition. The initial condition demands $y=1$ for $x=1$, therefore we can plug these in to obtain $C=3$. Therefore the implicit solution is:

$$x^2+xy+y^2 = 3$$

EXAMPLE 1.8. *Given the equation*

$$(xy^2+bx^2y)+(x+y)x^2y' = 0$$

1. *Find the values of b such that the equation is exact.*
2. *Solve it with that value of b.*

SOLUTION 1.8.

1. *We have*

$$P(x,y) = xy^2+bx^2y, \quad Q(x,y) = x^3+x^2y \therefore P_y = 2xy+bx^2, \quad Q_x = 3x^2+2xy$$

   P_y and Q_x must be equal for the equation to be exact therefore $b=3$.

2. *For $b=3$, the partial derivatives of the solution are:*

$$u_x = P(x,y) = xy^2+3x^2y, \quad u_y = Q(x,y) = x^3+x^2y$$

   Integrate the first equation w.r.t. x:

$$u(x,y) = \int (xy^2+3x^2y)dx + h(y) = \frac{1}{2}x^2y^2+x^3y+\phi(y)$$

   Differentiate the latter w.r.t. y and use the exactness condition (remember, we don't need a constant at this stage so in the end we take $h(y)=0$):

$$u_x = x^2y+x^3+\phi'(y) = x^3+x^2y \Rightarrow \phi'(y) = 0 \Rightarrow h(y) = 0$$

   The implicit solution is:

$$u(x,y) = C \Leftrightarrow \frac{1}{2}x^2y^2+x^3y = C$$

1.6 Application 2: Lake Pollution Model

First order differential equations can be applied to model the pollution in a lake in a simplistic way. In the lake pollution problem, a lake is polluted by a stream of water (input), has an outflow (output) and the goal is to find the concentration (mass/volume) of the pollutant in the lake at any time.

We make the following assumptions before we describe the model:

- The pollutant is flowing in the lake has a concentration of $c_{in}(t)$ (kg/m^3) and flows in at a volumetric flow (m^3/day) of $f(t)$.
- The lake is well-mixed, i.e. its pollutant concentration $c(t)$ (kg/m^3) is uniform.
- The lake maintains a constant volume V (m^3) by having outflow = inflow = $f(t)$ (m^3/day).

Sometimes c, c_{in} are measured in parts per m^3 but we'll stick to kg.

Consequently, the mass of the pollutant in the lake $M(t)$ at any time is equal to:

$$M(t) = c(t) \cdot V \tag{1}$$

From the conservation of the mass of the pollutant, the system can be described in high level as:

$$\left\{\begin{array}{c}\text{rate of change}\\\text{of mass}\\\text{of pollutant}\\\text{in lake}\end{array}\right\} = \left\{\begin{array}{c}\text{rate at}\\\text{which the}\\\text{pollutant enters}\\\text{the lake}\end{array}\right\} - \left\{\begin{array}{c}\text{rate at}\\\text{which the}\\\text{pollutant leaves}\\\text{the lake}\end{array}\right\}$$

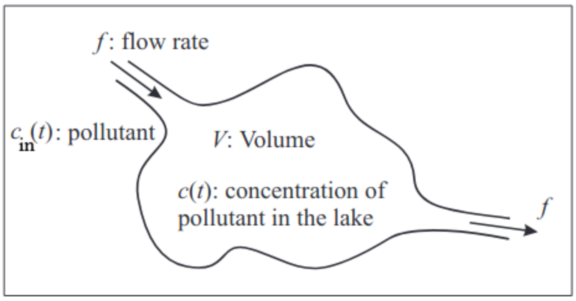

Fig. 1. The constant flow lake model.

Therefore the mass of pollutant entering the lake at any time is:

$$M_{in}(t) = f(t) c_{in}(t) \tag{2}$$

The concentration anywhere in the lake is $M(t)/V$ and the water flows out at (m^3/day) at $f(t)$. Therefore at any time, the mass leaving the lake is:

$$M_{out}(t) = \frac{f(t)}{V} M(t) = f(t) c_{in}(t) \tag{3}$$

Formulating the model and using Eq. (1)-(3):

$$\frac{dM}{dt} = M_{in} - M_out \Rightarrow$$
$$V \cdot c'(t) = f(t) \cdot c_i n(t) - f(t) c_{in}(t) \Rightarrow$$
$$c'(t) = \frac{f(t)}{V} c_{in} - \frac{f(t)}{V} c(t) \Rightarrow$$
$$c'(t) = \frac{f(t)}{V} \big(c_{in} - c(t) \big) \tag{4}$$

The flow $f(t)$ may or may not be constant. For instance, it may vary depending on the season, with the maximum reached at the peak of rainy season and the minimum in summer. $c(t)$ may also vary. However, the case where both $f(t)$ and $c(t)$ are constant leads to a simpler solution.

1. (*Constant flow with constant concentration*) If $f(t)$ and $c_{in}(t)$ respectively are constant, then the concentration equation Eq. (4) can be solved exactly the same way as Newton's law of cooling equation (Eq. (1.4)). According to Eq. (1.5), the solution is:

$$c(t) = c_{in} + e^{-\frac{f}{v}t}(c(0) - c_{in})$$

, where $c(0)$ is the initial condition, i.e. the concentration of pollutant in the lake at $t = 0$.

2. (*Variable flow, variable concentration*) In this case $f(t)$ is variable and periodical depending on the day of the year, e.g.

$$f(t) = f_0\left(1 + \epsilon \sin\left(\frac{2\pi}{T}t\right)\right)$$

, T is the period, e.g. 365 days, f_0 is the mean annual flow and ϵ is the normalised flow amplitude, therefore $|\epsilon| \leq 1$. This ensures that $f(t) \geq 0$. $c(t)$ may also be variable and depending on the day of the year in a similar manner. Then Eq. (1.8) is written as:

$$c'(t) = \frac{f(t)}{V}(c_{in}(t) - c(t)) = \frac{f_0\left(1 + \epsilon \sin\left(\frac{2\pi}{T}t\right)\right)}{V}(c_{in}(t) - c(t)) \Rightarrow$$

$$c'(t) + \frac{f(t)}{V}c(t) = \frac{f(t)c_{in}(t)}{V} \tag{5}$$

This is a non-homogeneous first order DE (Eq. (1.10)) and we know it can be solved by the integrating factor method. Let's be pedantic and solve it step-by-step.

$$c'(t) + \frac{f(t)}{V}c(t) = \frac{f(t)c_{in}(t)}{V}$$

Multiplying both sides by $\exp\left(\int \frac{f(t)}{V}dt\right)$:

$$c'(t)\underbrace{\exp\left(\int \frac{f(t)}{V}dt\right)}_{u(t)} + c(t)\underbrace{\exp\left(\int \frac{f(t)}{V}dt\right)'}_{u'(t)} = \frac{f(t)c_{in}(t)}{V}\underbrace{\exp\left(\int \frac{f(t)}{V}dt\right)}_{u(t)}$$

$$\left(c(t)u(t)\right)' = \frac{f(t)c_{in}(t)}{V}u(t)$$

Now, since we know the initial condition $c(t_0) = c_0$ and we want to find $c(t)$, we integrate from t_0 to t and replace dummy variable t with dummy variable s in the integral term:

$$c(t)u(t) - c(t_0)u(t_0) = \int_{t_0}^{t} \frac{f(s)c_{in}(s)}{V}u(s)ds$$

$$c(t) = \frac{1}{u(t)}\left\{\int_{t_0}^{t} \frac{f(s)c_{in}(s)}{V}u(s)ds + c(t_0)u(t_0)\right\}, \quad u(s) = \exp\left(\int \frac{f(s)}{V}ds\right) \tag{6}$$

Example 1.9. *The average summer flow into an out of a lake is* $4 \cdot 10^6\ m^3$/month. *The volume of that lake is constant and equal to* $28 \cdot 10^6\ m^3$. *When the measurements start at* $t = 0$, *the initial concentration of pollutant is* $c(0) = 10^7$ parts/m^3. *If there is no more pollution since the measurements have started, how long will it take for the lake pollution level to drop to 5% of its initial level?*

Solution 1.9. *When* $f(t)$ *and* $c_{in}(t)$ *are constant, we have derived that the pollutant concentration is given by:*

$$c(t) = c_{in} + e^{-\frac{f}{v}t}(c(0) - c_{in})$$

Plugging in the data:

$$c(t) = 0 + e^{-t/7}(c_0 - 0) = c_0 e^{-t/7}$$

We want a t_1 *s.t.* $c(t_1) = 0.05c_0 \Rightarrow c_0 e^{-t_1/7} = 0.05c_0 \Rightarrow t_1 = -7\ln(0.05) = 20.97$ months.

Example 1.10. *Consider the same lake (*$V = 28 \cdot 10^6\ m^3$*). Assume that* c_0 *is known but not necessarily zero. Plot the pollution mass over time* $t \in [0, 100]$ *in the lake* $c(t)$ *for the following cases:*

1. $f = 4 \cdot 10^6$, $c_{in}(t) = 3 \cdot 10^6$
2. $f = 4 \cdot 10^6$, $c_{in}(t) = 10^7(1+\cos(2\pi t))$
3. $f(t) = 10^6(1+6\sin(2\pi t))$, $c_{in} = 3 \cdot 10^6$
4. $f(t) = 10^6(1+6\sin(2\pi t))$, $c_{in} = 10^7(1+\cos(2\pi t))$

Consider the initial conditions for the concentration in the lake $c_0 \in \{1, 7/3, 11/3, 5\} \times 10^6$.

SOLUTION 1.10. 1. *Since both $f(t)$ and $c_{in}(t)$ are constant, the solution is given by Newton's law of cooling:*

$$\begin{aligned} c(t) &= c_{in} + \exp(-ft/V) - c_{in}) \\ &= 3 \cdot 10^6 + \exp(-t/7) \cdot (c_0 - 3 \cdot 10^6) \end{aligned}$$

2. *We use Eq. (6) to find the general solution:*

$$u(s) = \exp\left(\int \frac{f(s)}{V} ds\right) = \exp\left(\int \frac{1}{7} ds\right) = \exp\left(\frac{s}{7}\right)$$

(We drop the constant). The general solution is then given by:

$$\begin{aligned} c(t) &= \frac{1}{u(t)} \left\{ \int_{t_0}^{t} \frac{f(s) c_{in}(s)}{V} u(s) ds + c(t_0) u(t_0) \right\}, \quad u(s) = \exp\left(\int \frac{f(s)}{V} ds\right) \\ &= \exp\left(-\frac{t}{7}\right) \left\{ \int_{t_0}^{t} \frac{4 \cdot 10^6}{28 \cdot 10^6} \cdot 10^7 \left(1 + \cos(2\pi s)\right) \cdot \exp\left(\frac{s}{7}\right) ds + c_0 u(t_0) \right\}, \quad t_0 = 0 \\ &= \exp\left(-\frac{t}{7}\right) \left\{ \frac{1}{2 + 392\pi^2} 2 \cdot 10^7 \left(-2 - 196\pi^2 + e^{t/7} \left(1 + 196\pi^2 + \cos(2\pi t) + 14\pi \sin(2\pi t)\right)\right) + c_0 u(0) \right\} \end{aligned}$$

3. *The general solution is given by the same formula. First. we find an integrating factor and then substitute it in the formula for $c(t)$.*

$$u(s) = \exp\left(\int \frac{f(s)}{V} ds\right) = \exp\left(\frac{1}{28} \int (1 + 6\sin(2\pi s)) ds\right) = \exp\left(\frac{1}{28}\left(s - \frac{3\cos(2\pi s)}{\pi}\right)\right)$$

$$\begin{aligned} \therefore c(t) &= \frac{1}{u(t)} \left\{ \int_0^t \frac{f(s) c_{in}(s)}{V} u(s) ds + c(0) u(0) \right\} \\ &= \exp\left(\frac{1}{28}\left(\frac{3\cos(2\pi s)}{\pi} - s\right)\right) \left\{ \int_0^t \frac{10^6(1 + 6\sin(2\pi s)) \cdot 30 \cdot 10^6}{28 \cdot 10^6} \exp\left(\frac{1}{28}\left(s - \frac{3\cos(2\pi s)}{\pi}\right)\right) ds + c(0) u(0) \right\} \end{aligned}$$

Such integrals can be extremely tedious or impossible to compute analytically, so it's preferred to solve this ODE in a software package, such as Matlab / Octave or Python. ODE solvers in Matlab (`ode23`) and Python (`scipy.integrate.odeint`) take the same arguments so here are these are a few things to remember before invoking them:

   (a) *The ODE to solve must be in the form $y' = a(x)y + b(x)$ (x is the free variable like t).*

   (b) *The solver works recursively by computing y' for different values of t, y.*

   (c) *Therefore the solver takes as arguments a function f that describes our ODE (the function itself, not its return $f(t, x)$!), the sampled free variable x, and the initial condtion $y_0 = y(t_0)$. It returns the particular solution for the said initial condition.*

   The Matlab solution given the data in this question is listed in A.1.

4. *Similarly to part (c), to plot the particular solutions $c(t)$ for various $c(0)$ and for the data $f(t) = 10^6(1 + 6\sin(2\pi t))$, $c_{in} = 10^7(1 + \cos(2\pi t))$ can be plotted in Matlab by the `lake` function in A.1.*

In the end, we obtain the following four figures for the concentrations given the data in each part.

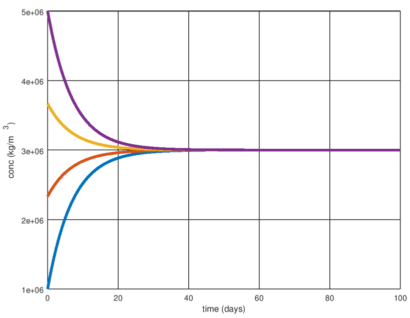

Fig. 2. Lake pollution solution for constant pollution and flow.

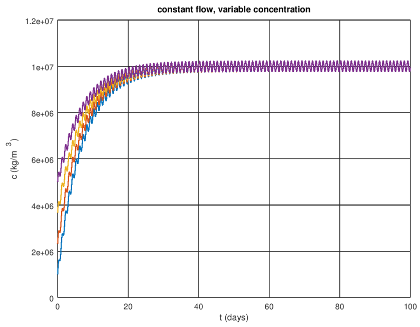

Fig. 3. Lake pollution solution for constant pollution and variable flow.

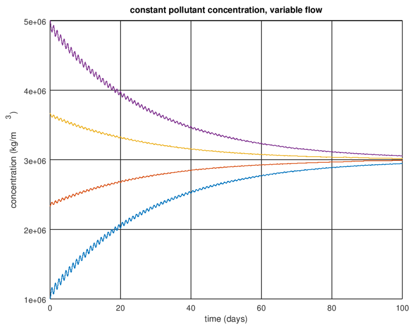

Fig. 4. Lake pollution solution for constant pollution and variable flow.

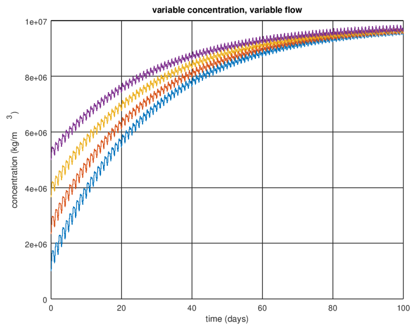

Fig. 5. Lake pollution solution for variable pollution and flow.

When c_{in} is constant, $c(t)$ converges to c_{in}

1.7 Directional Fields

A directional field (a.k.a. slope field) a set of tiny segments in the 2D space, each of which approximates the slope of a function at each point. The segment are draw in a sampled region of the 2D space. Slope fields are used to roughly illustrate the solution of an ODE of the form $\frac{dy}{dx} = f(x,y)$ by drawing $\frac{dy}{dx}$ without explicitly knowing them. If we connect such segments together so that they're all tangent to the same curve, we can graphically approximate a particular solution to $\frac{dy}{dx} = f(x,y)$.

This sounds vague so it's best to draw a slope field of an actual ODE.

Example 1.11. *Sketch the flow field of the solution to:*

$$\frac{dy}{dx} = y - x$$

Plot it over $[0,5] \times [0,6]$.

Solution 1.11. *The equation is already in a form where $\frac{dy}{dx}$ can be evaluated. Therefore we can iterate over all x's and y's with a certain step, e.g. 1, and compute $\frac{dy}{dx}$. Before sketching the actual field, let's think that the given ODE tells us about the derivative – the larger the absolute different between x and y, the steeper the magnitude. Along the line $y = x$, the line will be flat. Above $y = x$, the slope is positive and below it negative.*

The field is drawn by a slightly modified version of tamaskis's script function which is supported by Octave too. The source code is found in A.2. Two solutions are roughly sketched in magenta in Fig. 6.

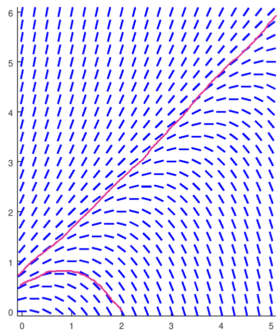

Fig. 6. Slope field of the solutions to $y' = y - x$ with two solutions roughly drawn.

1.8 Numerical Methods: Euler's Method

Often it's impossible to analytically solve a 1st order ODE with the techniques described. In this case it needs to be solved (approximated) numerically.

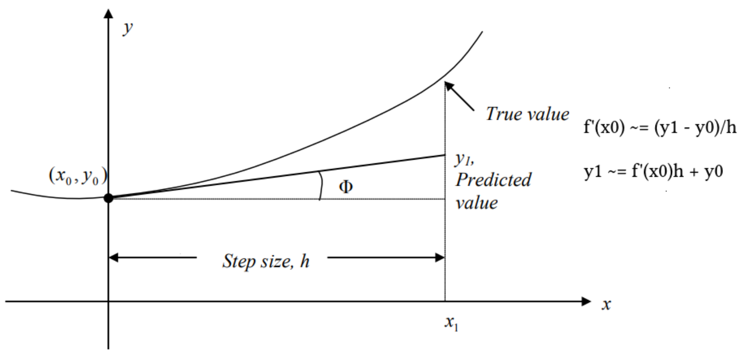

Fig. 7. Euler's method approximates the solution by a continuous set of segments tangent to the original curve.

Euler's method estimates a particular solution curve $y_p(x)$ to the ODE $y' = f(x, y)$ using some finite points $y_{p,i}$.It considers the curve as a connected set as some infinitesimal straight segments, each segment being defined by two points $(x_{i-1}, y_{p,i-1})$, $(x_i, y_{p,i})$. Point i is estimated given point $i-1$. As each segment is straight, point i is estimated via the derivative at $i-1$. Referring to Fig. 7, one can estimate y_1 given y_0 as:

Euler's method works for both linear and non-linear ODEs.

$$y'_0 = \frac{y_1 - y_0}{x - x_0} = \frac{y_1 - y_0}{h} \Rightarrow$$

$$y_1 = y_0 + h\, y'_0 \Rightarrow$$

$$y_1 = y_0 + h\, f(x_0, y_0)$$

The last equation comes from the fact that we are trying to solve the 1st order ODE $y' = f(x, y)$. h is the step size – a small fraction of the range of the x domain where we want to solve the equation on. Euler's method works iteratively, starting from y_0 to y_1 to y_n.

Corollary 1.7 (Euler's method (1st order ODEs)). *A particular solution y_p to a 1st order ODE*

$$\frac{dy}{dx} = f(x, y)$$

can iteratively be approximated in a domain $[x_{min}, x_{max}]$ by Euler's method as:

$$y_{p,i} = y_{p,i-1} + h\, f(x_{i-1}, y_{i-1}) \tag{1.16}$$

, where h is the step size $h := \frac{x_{max} - x_{min}}{N}$. Furthermore $y_{p,0} = y(t_0)$ is known from the initial condition.

Euler's method can easily be coded e.g. in Matlab (A.3).

EXAMPLE 1.12. *Solve the following ODE using Euler's method for $x \in [0.4, 0.8]$ for step sizes $h = 0.02, 0.01$. Solve it analytically and compare the numerical solution to the analytical one.*

$$\frac{dy}{dx} = (-4x + y)^2, \quad -4x + y > 2, \quad y(0.4) = 4$$

SOLUTION 1.12. *Let's solve it analytically first. It can be solved by change of variables. Let*

Remember y is a function of x!

$$z := -4x + y \Rightarrow$$

$$\frac{dz}{dx} = \frac{d(-4x+y)}{dx} = -4 + \frac{dy}{dx} \Rightarrow$$

$$\frac{dz}{dx} = -4 + z^2$$

This can be re-arranged as a separable equation ($M(x)dx = N(z)dz$ – see Eq. (1.2)):

$$4dx = \frac{dz}{(z-2)(z+2)} = \frac{dz}{z-2} - \frac{dz}{z+2}$$

Both sides can now be integrated and their integrals can be computed:

$$4x = \ln|z-2| - \ln|z+2| \Rightarrow$$

$$4x + C = \ln\left(\frac{|z-2|}{|z+2|}\right) = \ln\left(\frac{z-2}{z+2}\right)$$

Solve for z, hence for y:

$$e^{4x} = \frac{z-2}{z+2} \Rightarrow$$

$$z = 2\frac{1+Ce^{4x}}{1-Ce^{4x}} = 2\frac{1-Ce^{4x}+2Ce^{4x}}{1-Ce^{4x}} = \frac{4}{1-Ce^{4x}} - 2$$

Substitute back for y from $z = y - 4x$:

$$y = \frac{4}{1-Ce^{4x}} + 4x - 2 \tag{1}$$

Plugging in the initial condition $x, y = 0, 4$, we obtain $C \approx 0.0183$. This is the analytical ("true") solution.

The numerical approximation is computed using the code in A.3, e.g. from the following lines. Increasing the step size by 10 significantly reduces the error.

```
f = @(x,y) ((-4*x+y)^2)
C = 0.0183;
h = 0.01;
%       (f, [xmin, xmax], f(xmin), h)
[x,y] = myeuler(f, [0.4,0.8], 4, h);
```

Plotting the analytical solution, approximations for $h = 0.02$ and $h = 0.01$ we observe the error is significantly reduced by halving the step:

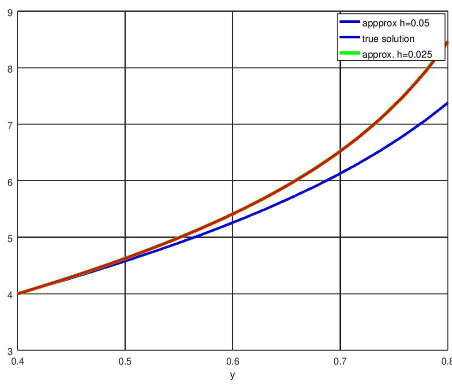

Fig. 8. The analytical solution to $y' = (-4x + y)^2$ vs two approximations.

1.9 Numerical Methods: Runge-Kutta Method

Euler's method is simple to implement but may be inaccurate, as it assumes that the slope is constant in interval $[x_n, x_{n+1}]$ and only takes into account the slope at x_n.

Runge-Kutta (RK) methods are a family of methods to solve an ODE $\frac{dy}{dx} = f(x,y)$ given an initial value $y(t_0) = y_0$. RK methods are

- iterative (when we know y_n, we compute y_{n+1} as $y_{n+1} = y_n + \text{increment}$)
- more accurate than Euler's since they compute the weighted average of the slopes at interval $[x_n, x_{n+1}]$,
- efficient as they do not rely on the derivatives $f_(x,y)$,
- more expensive than Euler's explicit method.

They rely on the Taylor series expansion of $f(x,y)$. RK2 require two evaluations of $f(x,y)$ within each sub-interval $[x_n, x_{n+1}]$, RK4 require four, etc. We shall derive the 2nd order RK method (RK2) and then formulate the 4th order one. In RK2, the increment is a weighted average of two quantities K_1, K_2, i.e.

$$y_{n+1} = y_n + \alpha K_1 + \beta K_2$$

, where α and β are constants. α, β, K_1, K_2 are chosen such that they satisfy a certain set of constraints.

Definition 1.7 (Runge-Kutta 2). *Given the initial value problem*

$$\frac{dy}{dx} = f(x,y)$$

, the general form of 2nd order Runge-Kutta methods that solve it is:

$$y_{n+1} = y_n + \mathrm{a}K_1 + bK_2 \tag{1.17}$$

, where

$$\begin{cases} K_1 = hf(x_n, y_n) \\ K_2 = hf(x_n + \alpha h, y_n + \beta K_1) \end{cases} \tag{1.18}$$

h is the step size.

Before the RK2 method is applied, $\alpha, \beta, \mathrm{a}, b$ are chosen by the user. We will prove that certain constraints must hold for them, particularly:

$$\begin{cases} \mathrm{a} + b = 1 \\ b\beta = \frac{1}{2} \\ \alpha b = \frac{1}{2} \end{cases} \tag{1.19}$$

Proof. To determine the constants, we first expand (by Taylor series) y_{n+1} (denoting $y_n = y(x_n), y_{n+1} = y(x_{n+1})$) at a neighbourhood around x_n up to the h^2 term:

$$y_{n+1} = y_n + h\left.\frac{dy}{dx}\right|_{x_n} + \frac{h^2}{2}\left.\frac{d^2y}{dx^2}\right|_{x_n} + \mathcal{O}(h^3) \tag{1}$$

However, since we're trying to solve the problem $\frac{dy}{dx} = f(x,y)$, Eq. (1) can be rewritten as:

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2} f'(x_n, y_n) + \mathcal{O}(h^3) \tag{2}$$

However, from the total derivative/total differential formula, we know for $f'(x_n, y_n)$:

$$\begin{aligned} f'(x_n, y_n) &= \frac{\partial f(x_n, y_n)}{\partial x}\frac{dx}{dx} + \frac{\partial f(x_n, y_n)}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial f(x_n, y_n)}{\partial x} + \frac{\partial f(x_n, y_n)}{\partial y} f(x_n, y_n) \end{aligned}$$

Therefore Eq. (2) is rewritten in terms of the partial derivatives and $f(x,y)$ as:

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2}\left[\frac{\partial f(x_n, y_n)}{\partial x} + \frac{\partial f(x_n, y_n)}{\partial y} f(x_n, y_n)\right] \tag{3}$$

It turns out that we can also write the update step $y_{n+1} = y_n + \mathrm{a}K_1 + bK_2$ as a polynomial of h, whose each coefficient depends on f and its partial derivatives. First, expand $K_2 = f(x_n + \alpha h, y_n + \beta K_1)$ by multivariate Taylor series:

$$\begin{aligned} K_2 &= hf(x_n + \alpha h, y_n + \beta K_1) \\ &= h\left[f(x_n, y_n) + \frac{\partial f(x_n, y_n)}{\partial x}\alpha h + \frac{\partial f(x_n, y_n)}{\partial y}\beta h f(x_n, y_n)\right] \end{aligned}$$

Then the update equation is:

$$\begin{aligned} y_{+1} &= y_n + \mathrm{a}hf(x_n, y_n) + bh\left[f(x_n, y_n) + \frac{\partial f(x_n, y_n)}{\partial x}\alpha h + \frac{\partial f(x_n, y_n)}{\partial y}\beta h f(x_n, y_n)\right] \\ &= y_n + h(\mathrm{a}+b)f(x_n, y_n) + \frac{\partial f(x_n, y_n)}{\partial x}\alpha b h^2 + \frac{\partial f(x_n, y_n)}{\partial y} b\beta f(x_n, y_n)h^2 + \mathcal{O}(h^3) \end{aligned} \tag{4}$$

By matching terms in Eq. (3), (4), we can find the constants $\mathrm{a}, b, \alpha, \beta$. Comparing each polynomial coefficients and each derivative term we obtain:

$$\begin{cases} \mathrm{a} + b = 1 \\ b\beta = \frac{1}{2} \\ \alpha b = \frac{1}{2} \end{cases}$$

These equations summarise the possible choices of constants $\mathrm{a}, b, \alpha, \beta$ that satisfy RK2. □

Therefore there are infinitely many choices for $\mathrm{a}, b, \alpha, \beta$. A popular and fairly accurate RK2 method is Heun's, which selects $\mathrm{a} = 1/2, b = 1/2, \alpha = 1, \beta = 1$. Heun's method averages the two slopes at the ends x_n, x_{n+1} of each sub-interval $[x_n, x_{n+1}]$. Then the RK2 equations become:

$$\begin{aligned} y_{n+1} &= y_n + \frac{K_1}{2} + \frac{K_2}{2} \\ K_1 &= h(x_n, y_n) \\ K_2 &= hf(x_n + h, y_n + K_1) \end{aligned}$$

Visually, Heun's RK2 method is illustrated below. Note that we don't take the second slope at point $(x_n + h, y_n)$, which is on the graph, but at $(x_n + h, y_n + K_1)$.

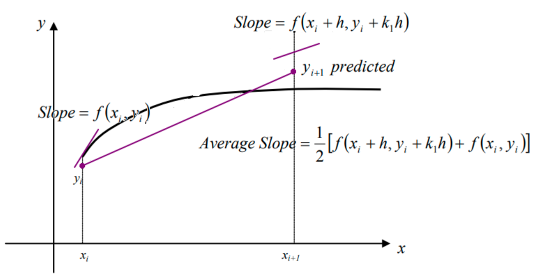

Fig. 9. The two slopes that Heun's (a RK2) method takes into account.

Runge-Kutta 4th order (RK4) method can be proven in the same fashion as RK2 but its proof is omitted here. It can be stated as follows.

DEFINITION 1.8 (Runge-Kutta 4). *The general form of the RK4 method is*

$$k_1 = hf(x_n, y_n) \tag{1.20}$$

$$k_2 = hf(x_n + \alpha_2 h, y_n + \beta_2 k_1) \tag{1.21}$$

$$k_3 = hf(x_n + \alpha_3 h, y_n + \beta_3 k_1 + \beta_3' k_2) \tag{1.22}$$

$$k_4 = hf(x_n + \alpha_4 h, y_n + \beta_4 k_1 + \beta_4' k_2 + \beta_4'' k_3) \tag{1.23}$$

$$y_{n+1} = y_n + \alpha_1 \mathrm{a}_1 k_1 + \mathrm{a}_2 d_2 + \mathrm{a}_3 k_3 + \mathrm{a}_4 k_4 \tag{1.24}$$

A popular RK4 method uses the values of $\alpha_2 = 1/2, \beta_2 = 1/2, b_3 = 0, b_3' = 1/2, a_4 = 1, b_4 = b_4' = 0, b_4'' = 1, \mathrm{a}_1 = 1/6, \mathrm{a}_2 = \mathrm{a}_3 = 2/6, \mathrm{a}_4 = 1/6$. In this case, the RK4 formulas as follows. Note that to find the new estimate of the solution (y_{n+1}), we increment y_n by the weighted sum of four slopes k_1, k_2, k_3, k_4.

$$k_1 = hf(x_n, y_n) \tag{1.25}$$

$$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \tag{1.26}$$

$$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \tag{1.27}$$

$$k_4 = hf(x_n + h, y_n + k_3) \tag{1.28}$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{1.29}$$

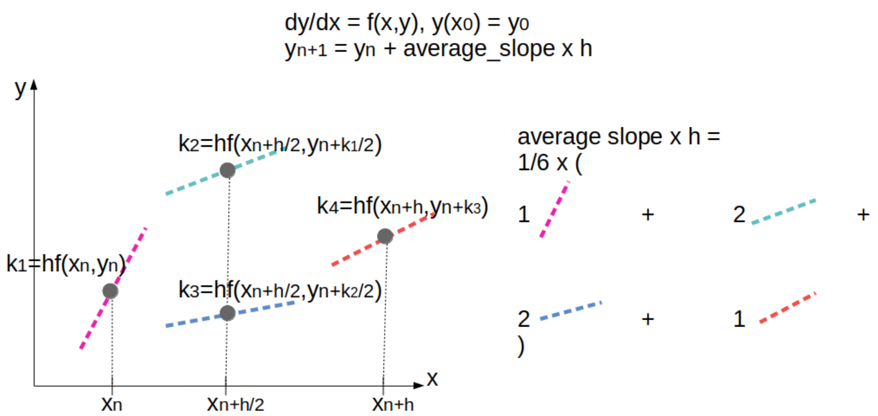

Fig. 10. RK4 uses a weighted average of four slopes $f(x, y)$ to update y_n.

Algorithmically, RK4 with the parameter selection in Eq. (1.25) to Eq. (1.29) can be implemented as follows.

Algorithm 1 Runge-Kutta 4th order

```
procedure RK4(f(x,y), [x_0, x_n], f(x_0), h)
    x ← x_0
    y ← f(x_0)
    while x < x_n do
        x ← x + h
        k_1 ← hf(x, y)
        k_2 ← hf(x_n + h/2, y_n + k_1/2)
        k_3 ← hf(x_n + h/2, y_n + k_2/2)
        k_4 ← hf(x_n + h, y_n + k_3)
        y ← y + 1/6 (k_1 + 2k_2 + 2k_3 + k_4)                ▷ y_{n+1} ← y_n + ...
```

EXAMPLE 1.13. *For the initial value problem*

$$\frac{dy}{dx} = y - x^2 + 1, \quad y(0) = 0.5$$

estimate $y(1.5)$ *in steps of* $h = 0.5$ *using the specific RK4 method in Eq.* (1.25) *to Eq.* (1.29).

SOLUTION 1.13. *By following Alg. 1, we obtain the following values and estimate* $y(1.5) \approx 4.0134$.

x_i	k_1	k_2	k_3	k_4	y_n	y_{n+1}
0.0	0.7500	0.9062	0.9453	1.0976	1.0000	1.4251
0.5	1.0875	1.2032	1.2321	1.3286	1.4251	2.6396
1.0	1.3198	1.3685	1.3806	1.4251	2.6396	4.0134
1.5					4.0134	

A Appendices

A.1 Lake Pollution Model Solution in Matlab

Listing 1: A code listing (src/lake_pollution/lake.m).

```
% Description
%   First order ODE model for the pollution in a lake assuming constant volume
%   y'(t) = f/V * (cin - y)
% Example :
%   y0 = 1e6;
%   timespan = 0:dt:100;
%   [t, y] = ode23(@lake, timespan, y0); plot(t,y);
% Hints:
%   Feel free to change f, cin, and V according to your flow, pollution, and
%   volume data
function yp = lake(t,y)
    f = 1e6*(1 + 6*sin(2*pi*t));
    cin = 3e6;
    V = 28e6;
    yp = f/V * (cin - y);
end
```

The particular solutions y for various initial concentrations c0 can be plotted by the following snippet:

```
dt = 0.01;
timespan = 0:dt:100;
c0 = 1e6 * [1, 7/3, 11/3, 5]
hold on;

for i = 1:length(c0)
  [_, y] = ode23(@lake, timespan, c0(i));
  plot(t, y);
end

xlabel('t (days)'); ylabal('pol. concentration (kg/m^3)'); grid on;
% reset figure holding and close windows
hold off;
close all;
```

A.2 Lake Pollution Model Solution in Matlab

Listing 2: A code listing (src/slope_fields/slope_field.m).

```
%==========================================================================
%
% slope_field  Draws the slope field of a first-order, univariate, ordinary
% differential equation.
%
%   slope_field(f,[xmin,xmax],[ymin,ymax])
%   slope_field(f,[xmin,xmax],[ymin,ymax],density,width)
%   fig = slope_field(__)
%
% Copyright (c) 2021 Tamas Kis
% Last Update: 2021-08-28
% Website: https://tamaskis.github.io
% Contact: tamas.a.kis@outlook.com
%
%--------------------------------------------------------------------------
%
% ------
% INPUT:
% ------
%   f            - (function_handle) dy/dx = f(x,y)
%   [xmin,xmax] - (1x2 double) lower and upper bounds of independent var.
%   [ymin,ymax] - (1x2 double) lower and upper bounds of dependent variable
%   density     - (OPTIONAL) (1x1 double) line density
%   width       - (OPTIONAL) (1x1 double) line width
%
% -------
% OUTPUT:
% -------
%   fig     - (Figure) slope field plot
%
% -----
% NOTE:
% -----
%   --> "density" defines the number of lines to draw in the horizontal
%       direction (effectively controlling how many lines are drawn to
%       create the slope field)
%
%==========================================================================
function fig = slope_field(f,x_domain,y_domain,density,width)

    % sets default values of density, color, and width if not specified
    if nargin == 3
        density = 20;
        width = 1.25;
    elseif nargin == 4
        width = 1.25;
    end

    % domain limits (rounds values in case non-integers are entered)
    xmin = floor(x_domain(1));
    xmax = ceil(x_domain(2));
    ymin = floor(y_domain(1));
    ymax = ceil(y_domain(2));

    % creates mesh
    x = xmin:((xmax-xmin)/density):xmax;
    y = ymin:((xmax-xmin)/density):ymax;

    % length of lines
    L = 0.75*(xmax-xmin)/density;

```

```
    % initializes figure and sets axes limits
    fig = figure;
    axis equal;
    xlim([xmin-L/2,xmax+L/2]);
    ylim([ymin-L/2,ymax+L/2]);

    % plots lines (slopes)
    hold on;
    for i = 1:length(x)
        for j = 1:length(y)
            % initially assumes the slope will not be indeterminate
            indeterminate = false;

            % calculates slope and avoids division by 0 errors
            try
                slope = f(x(i),y(j));
            catch
                indeterminate = true;
            end

            % angle formed by slope
            if indeterminate
                angle = pi/2;
            else
                angle = atan(slope);
            end

            % calculates components of line
            dx = L*cos(angle)/2;
            dy = L*sin(angle)/2;

            % plots line (but only if slope is real)
            if isreal(slope)
                plot([x(i)-dx,x(i)+dx],[y(j)-dy,y(j)+dy],'color', 'blue', 'linewidth',1.5);
            end

        end
    end
    hold off;
end
```

A.3 Lake Pollution Model Solution in Matlab

Listing 3: A code listing (src/euler/myeuler.m).

```
% Description:
%   Solves the 1st order ODE dy/dx = f(x,y) using Euler's methods
%
% Example:
%
% f = @(x,y) (x-y)
% [x,y] = myeuler(f, [0,100], 1, 0.001);
function [x, yest] = myeuler(f,x_domain, y0, h)
  yest = [];
  xmin = x_domain(1);
  xmax = x_domain(length(x_domain));
  x = linspace(xmin, xmax, abs(xmin - xmax)/h);
  yest = [y0];
  for i = 2:length(x)
    yest = [yest yest(i-1) + h * f(x(i-1), yest(i-1))];
  end
 end
```