DISCRETE STRUCTURES
CHAPTER 2 PART 1

RELATIONS

Definition

- Let *A* and *B* be two sets.
- A binary relation, or simply a relation *R* from set *A* to set *B* is a subset of the cartesian product *A* x *B*

$$a \in A, b \in B, (a,b) \in A \times B \text{ and } R \subseteq A \times B$$

- If $(a,b) \in R$, we say *a* is related to *b* by *R*

write as *a R b* ($a\,R\,b \leftrightarrow (a,b) \in R$)

Example 1

Let $A = \{ 1, 2, 3, 4 \}$ *and* $B = \{ p, q, r \}$

$R = \{ (1, q), (2, r), (3, q), (4, p) \}$

$R \subseteq A \times B$

R is the relation from A to B

$1Rq$ (*1* is related to *q*)

$3\not R p$ (*1* is not related to *p*)

Relations

- Binary relations: xRy
  On sets $x \in X$ $y \in Y$ $R \subseteq X \times Y$
- Example:
  "less than" relation from $A=\{0,1,2\}$ to $B=\{1,2,3\}$

<u>Use traditional notation</u>

$0 < 1,\ 0 < 2,\ 0 < 3,\ 1 < 2,\ 1 < 3,\ 2 < 3$

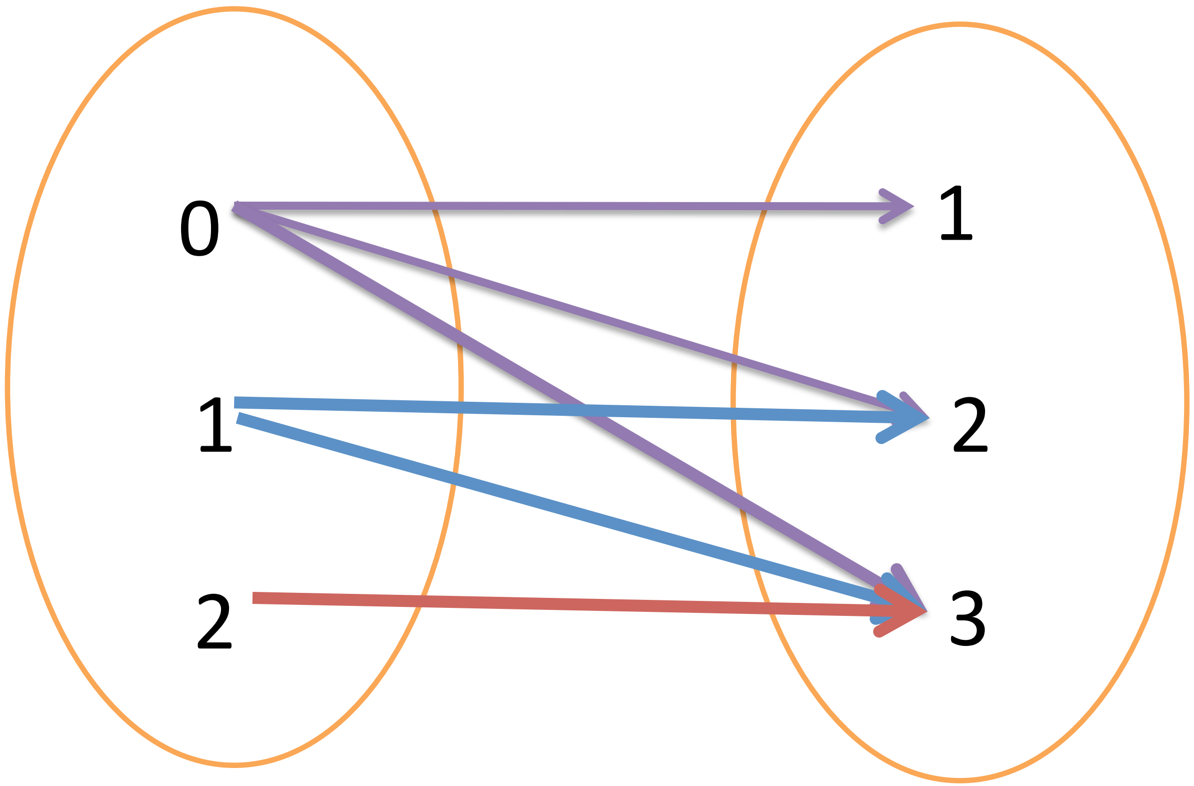
<u>Or use set notation</u>

$A \times B=\{(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$

$R=\{(0,1),(0,2),(0,3), (1,2),(1,3),(2,3)\}$

<u>Or use Arrow Diagrams</u>

Arrow Diagram



$R=\{(0,1),(0,2),(0,3), (1,2),(1,3),(2,3)\}$

Example 2

A = {New Delhi, Ottawa, London, Paris, Washington}
B = {Canada, England, India, France, United States}

Let $x \in A$, $y \in B$.

Define the relation between x and y by "x is the capital of y"

R = {(New Delhi, India), (Ottawa, Canada), (London, England), (Paris, France), (Washington, United States)}

Definition

- If R is a relation from set A into itself, we say that R is a relation on A.

  $a \in A$, $b \in A$ $(a,b) \in A \times A$ and $R \subseteq A \times A$

- Example

  Let A = (1,2,3,4,5) and R be defined by $a,b \in A$,

$$aRb \leftrightarrow b - a = 2$$

$R = \{(1,3),(2,4),(3,5)\}$

Exercise 1

Write the relation R as $(x,y) \in R$

(i) The relation R on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$

(ii) The relation R on {1, 2, 3, 4, 5} defined by $(x,y) \in R$ if 3 divides $x-y$

Domain and Range

Let R, a relation from A to B.

The set, **$\{ a \in A \mid (a,b) \in R$ for some $b \in B \}$** is called the **domain** of R.

The set, **$\{ b \in B \mid (a,b) \in R$ for some $a \in A \}$** is called **the range** of R.

Example 3

Let R be a relation on $X = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ *if* $x \le y$, *and* $x,y \in X$.

Then,
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

The **domain and range** of R are both equal to X.

Example 4

Let X = {2, 3, 4} and Y = {3, 4, 5, 6, 7}

If we **define a relation R from X to Y by,**

(x,y) $\in$ R if y/x (*with zero* remainder)

We obtain,

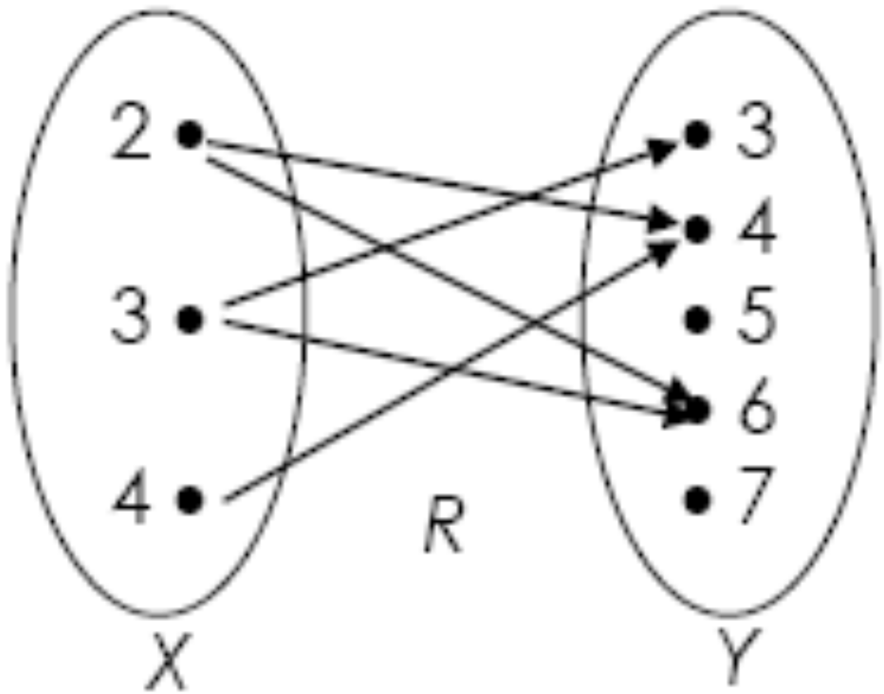
R = { (2,4), (2,6), (3,3), (3,6), (4,4) }

The domain of R is {2,3,4}

The **range** of R is {3,4,6}

Example 4(cont)

$R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$



Arrow diagram

Exercise 2

Find range and domain for:

(i) The relation R={(1,2), (2,1), (3,3), (1,1), (2,2)} on X={1, 2, 3}

(ii) The relation R on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$

Diagraph

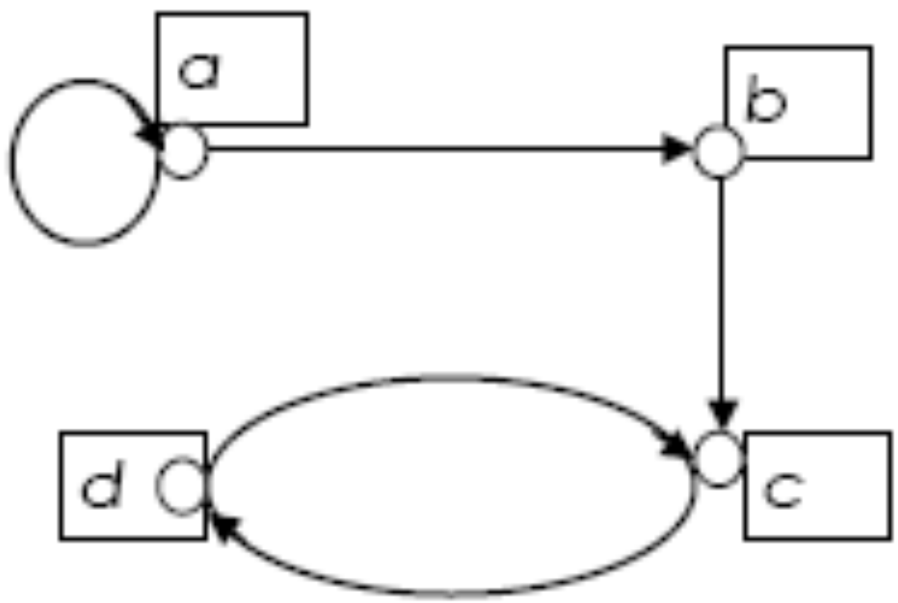
An informative way to picture a relation on a set is to draw its digraph.

- Let R be a relation on a finite set A.
- Draw dots (vertices) to represent the elements of A.
- If the element $(a,b) \in R$, draw an arrow (called a directed edge) from a to b

Example 5

The relation R on $A = \{a, b, c, d\}$,

$R = \{(a, a), (a, b), (c, d), (d, c), (b, c)\}$



Example 6

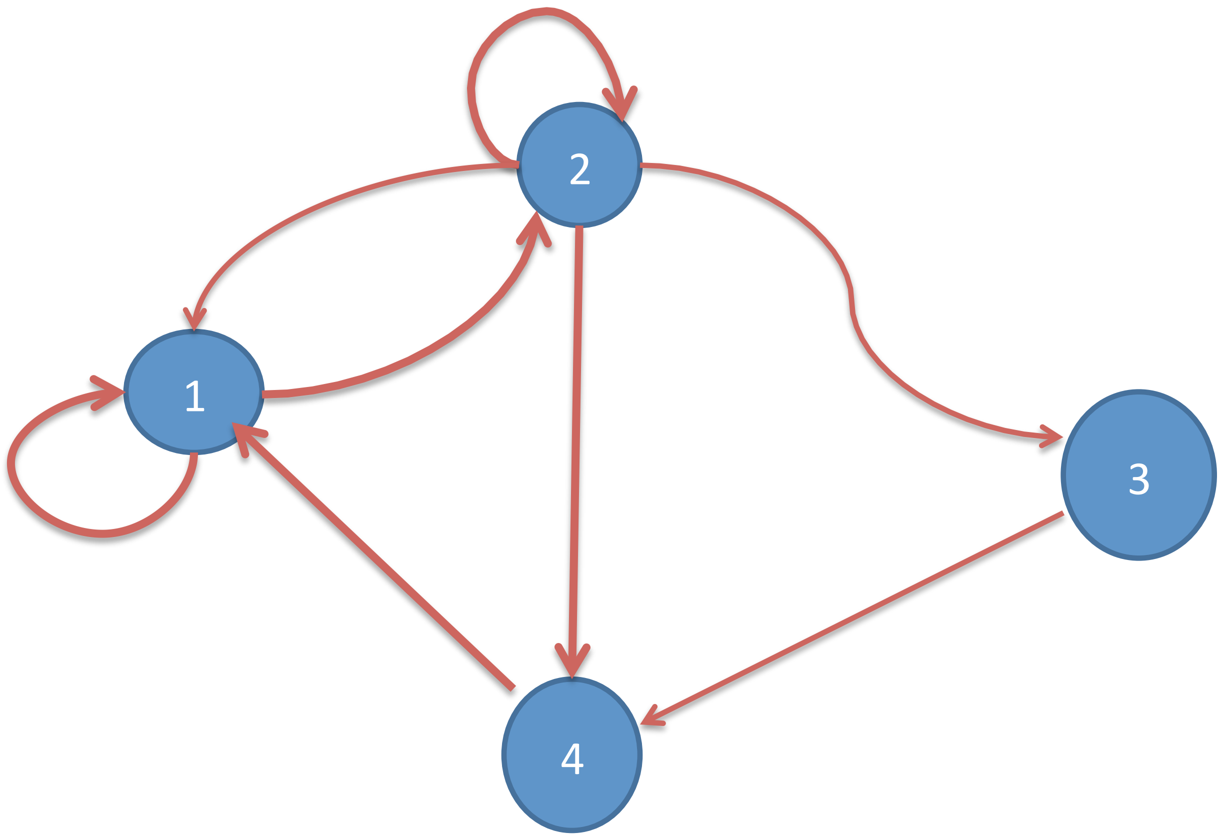
Let

$A = \{ 1,2,3,4\}$ and

$R = \{ (1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4) , (4,1)\}$

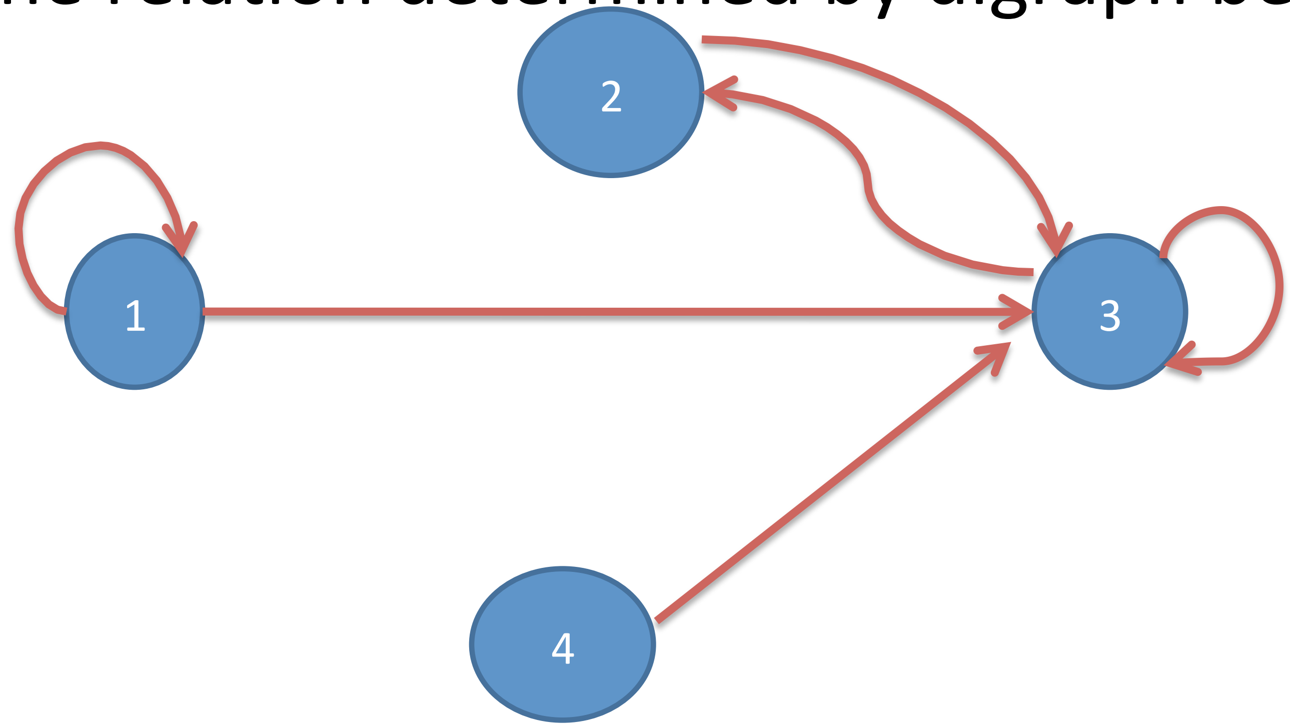
Draw the digraph of R

Digraph



Example 7

Find the relation determined by digraph below



Since $a_i\ R\ a_j$ if and only if there is an edge from a_i to a_j, so

R = { (1,1), (1,3), (2,3), (3,2), (3,3), (4,3)}

Exercise 3

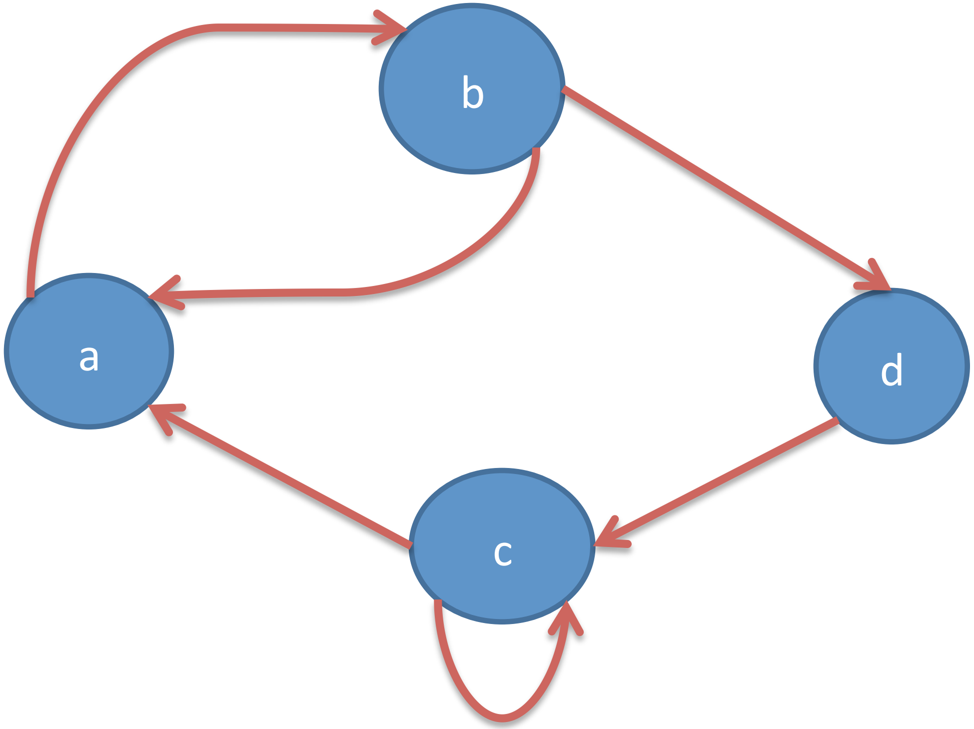
Draw the diagraph of the relation.

(i) R = {(a,c),(b,b),(b,c)}

(ii) The relation *R* on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$

Exercise 4

Write the relation as a set of ordered pair.



Matrices of Relations

A matrix is a convenient way to represent a relation *R* from *A* to *B.*

- Label the rows with the elements of *A* (in some arbitrary order)
- Label the columns with the elements of *B* (in some arbitrary order)

Matrices of Relations

- Let $M_R = [m_{ij}]_{n \times p}$ be the Boolean n x p matrix

$$M_R = \begin{bmatrix} m_{11} & m_{12} & .. & .. & m_{1p} \\ m_{21} & m_{22} & .. & .. & m_{2p} \\ \vdots & \vdots & .. & .. & \vdots \\ \vdots & \vdots & .. & .. & \vdots \\ m_{n1} & m_{n2} & .. & .. & m_{np} \end{bmatrix}$$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

Example 8

- The relation,
  $R = \{ (1,b),(1,d),(2,c),(3,c),(3,b),(4,a) \}$
  from, $X = \{ 1, 2, 3, 4 \}$ to $Y = \{ a, b, c, d \}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \text{ or } M_R = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Example 9

The matrix of the relation R from { 2, 3, 4 } to { 5, 6, 7, 8 } defined by

$x \; R \; y$ if x divides y

$$\begin{array}{c c} & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example

Let A={ a, b, c, d }

Let R be a relation on A.

R = { (a,a),(b,b),(c,c),(d,d),(b,c),(c,b) }

$$M_R = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example 10

An airline services the five cities c_1, c_2, c_3, c_4 and c_5. Table below gives the cost (in dollars) of going from c_i to c_j. Thus the cost of going from c_1 to c_3 is RM100, while the cost of going from c_4 to c_2 is RM200

To from	c_1	c_2	c_3	c_4	c_5
c_1		140	100	150	200
c_2	190		200	160	220
c_3	110	180		190	250
c_4	190	200	120		150
c_5	200	100	200	150	

If the relation R on the set of cities $A = \{c_1, c_2, c_3, c_4, c_5\}$: $c_i \, R \, c_j$ if and only if the cost of going from c_i to c_j is defined and less than or equal to RM180.

i) Find R.

ii) Matrices of relations for R

Solution

R = $\{(c_1,c_2), (c_1,c_3), (c_1,c_4), (c_2,c_4),(c_3, c_1), (c_3,c_2), (c_4,c_3), (c_4,c_5), (c_5,c_2), (c_5,c_4)\}$

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Exercise 5

Let A = {1, 2, 3,4} and R is a relation from A to A.
Suppose R = {(1,2), (1,3), (1,4), (2,3), (2,4) , (3,4)}

- What is R (represent)?
- What is matrix representation of R?

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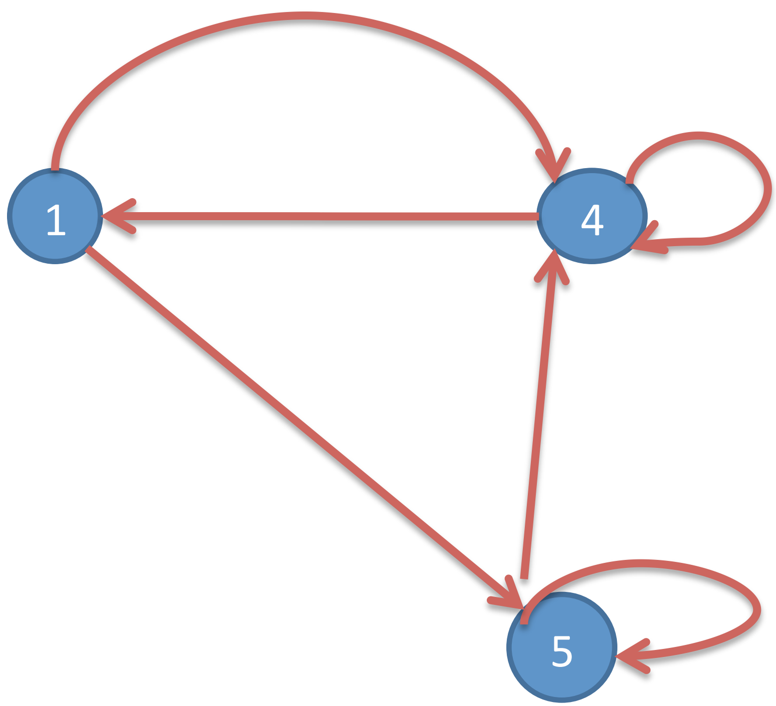
Exercise 6

Let A = {1, 4, 5} and let R be given by the digraph shown below. Find M_R and R



In degree and out degree

If R is a relation on a set A and $a \in A$, then the in-degree of a (relative to relation R) is the number of $b \in A$ such that $(b,a) \in R$.

The out degree of a is the number of $b \in A$ such that $(a,b) \in R$.

Meaning that, in terms of the digraph of R, is that the in-degree of a vertex is

"the number of edges terminating at the vertex"

The out-degree of a vertex is

" the number of edges leaving the vertex"

Example 11

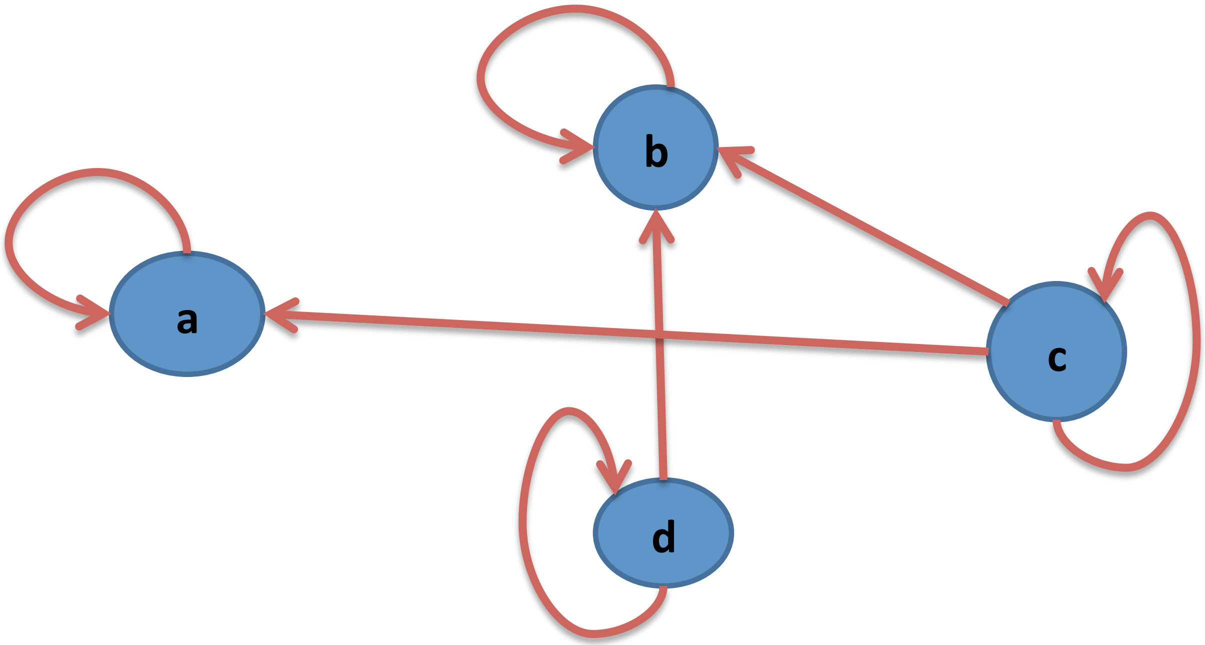
Let $A = \{a, b, c, d\}$, and let R be the relation on A that has the matrix (given below)

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Construct the digraph of R, and list in-degrees and out-degrees of all vertices.

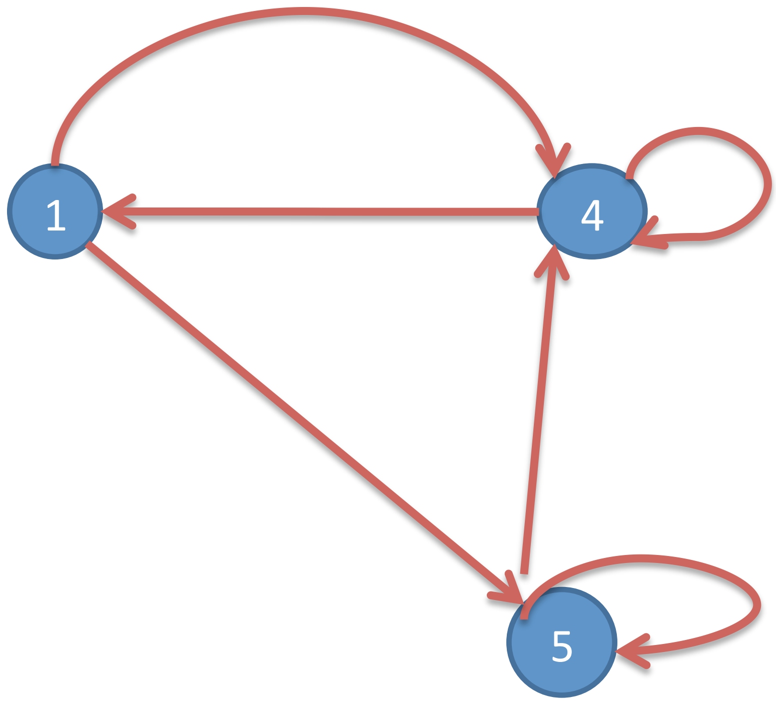
Solution

	a	b	c	d
In-degree	2	3	1	1
Out-degree	1	1	3	2



Exercise 7

Let A = {1, 4, 5} and let R be given by the digraph shown below. list in-degrees and out-degrees of all vertices.



Reflexive Relations

- Reflexive
  - A Relation *R* on set *A* is called **reflexive** if every $a \in A$ is related to itself OR
  - A relation *R* on a set *X* is called **reflexive** if all pair $(x,x) \in R;\ \forall x: x \in X$
- Irreflexive
  - A relation *R* on a set *A* is **irreflexive** if $x \cancel{R} x$ or $(x,x) \notin R;\ \forall x: x \in X$
- Not Reflexive
  - A Relation *R* is **not reflexive** if at least one pair of $(x,x) \in R,\ \forall x: x \in X$

Example 12

The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x \leq y$, $x,y \in X$ is a reflexive relation.

For each element $x \in X$, $(x,x) \in R$
(1,1), (2,2), (3,3), (4,4) are each in R.

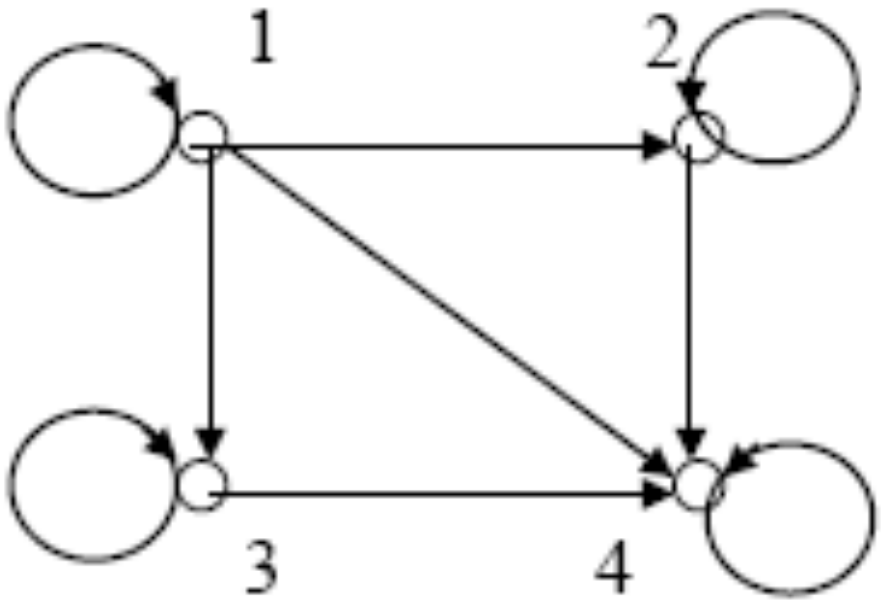
The relation $R = \{(a,a), (b,c), (c,b), (d,d)\}$ on $X=\{a, b, c, d\}$ is not reflexive.

This is because $b \in X$, *but* $(b,b) \notin R$. Also $c \in X$, *but* $(c,c) \notin R$

Reflexive Relations

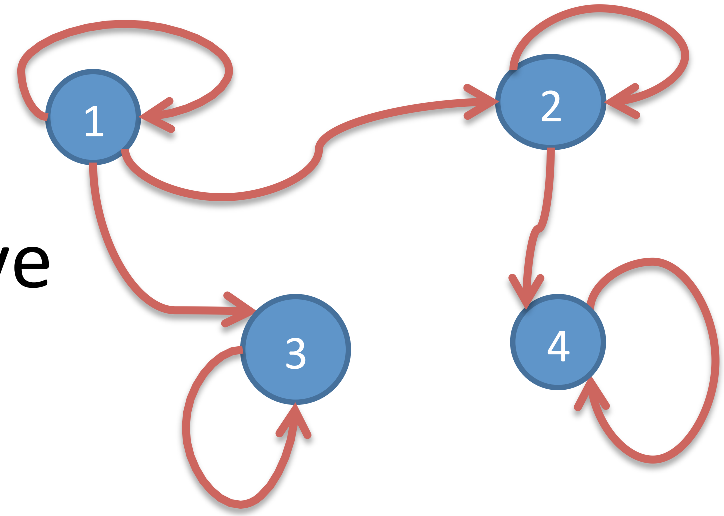
The digraph of a reflexive relation has a loop at every vertex.

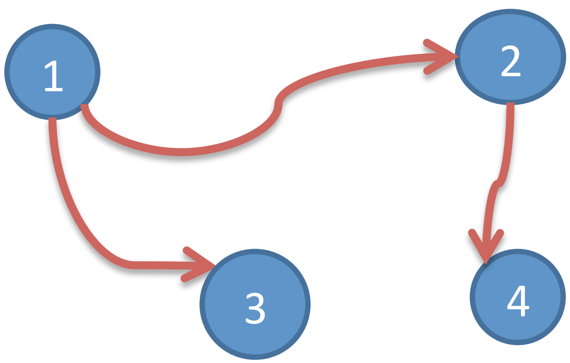
example



Reflexive Relations

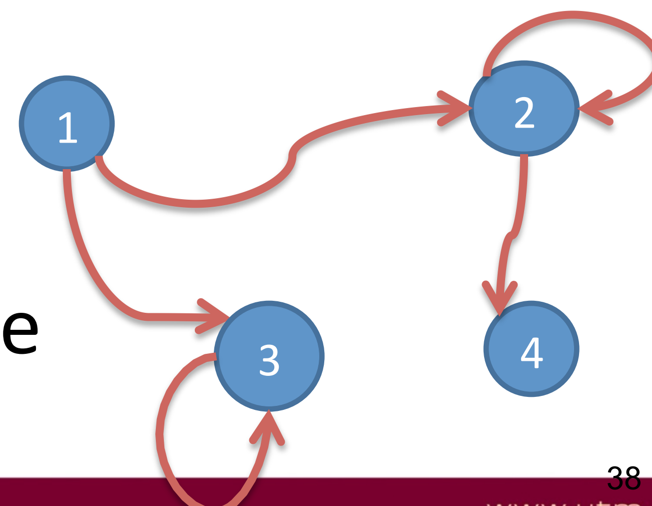
- Reflexive





- Irreflexive

- Not Reflexive



Reflexive Relations

The relation R is reflexive if and only if the matrix of relation has 1's on the main diagonal.

example

$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 1 & 0 & 0 & 0 \\
b & 0 & 1 & 1 & 0 \\
c & 0 & 1 & 1 & 0 \\
d & 0 & 0 & 0 & 1
\end{array}
$$

Reflexive Relations

The relation R is *irreflexive* if and only if the matrix relation have all 0's on its main diagonal

$$\begin{pmatrix} \boxed{0} & 1 & 0 \\ 1 & \boxed{0} & 0 \\ 1 & 1 & \boxed{0} \end{pmatrix}$$

Reflexive Relations

The relation *R* is not reflexive.

$$\begin{array}{c|cccc} & a & b & c & d \\ a & 1 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

$$\boxed{\begin{array}{l} b \in X \\ (b,b) \notin R \end{array}}$$

Exercise 8

- Consider the following relations on the set $\{1,2,3\}$:

$R_1=\{(1,1), (1,2), (2,1), (2,2), (3,1), (3,3)\}$

$R_2=\{(1,1), (1,3), (2,2), (3,1)\}$

$R_3=\{(2,3)\}$

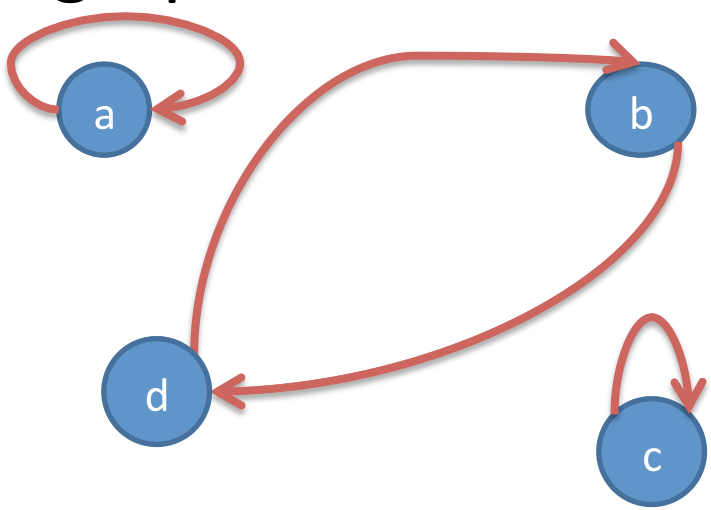
$R_4=\{(1,1)\}$

Which of them are reflexive?

Exercise 9

(i) Let R be the relation on $X=\{1,2,3,4\}$ defined by $(x,y) \in R$ if $x \leq y$, $x,y \in X$. Determine whether R is a reflexive relation.

(ii) The relation R on $X=\{a,b,c,d\}$ given by the below diagraph. Is R a reflexive relation?



Exercise 10

Let A = { 1,2,3,4). Construct the matrix of relation of *R*. Then, determine whether the relation is reflexive, not reflexive or irreflexive.

(i) R = { (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4))

(ii) R = { (1,3), (1,1), (3,1), (1,2), (3,3), (4,4)}

(iii) R ={ (1,2), (1,3), (3,1), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)}

(iv) R ={ (1,2), (1,3), (3,2), (1,4), (4,2), (3,4)}

Symmetric Relations

A relation R on a set X is called symmetric if for all $x,y \in X$, if $(x,y) \in R$, then $(y,x) \in R$.

$$\forall x,y \in X, (x,y) \in R \rightarrow (y,x) \in R$$

Let M be the matrix of relation R.

The relation R is symmetric if and only if for all i and j, the ij-th entry of M is equal to the ji-th entry of M.

Symmetric Relations

The matrix of relation M_R is symmetric if $M_R = M_R^T$

example

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = M_R^T$$

Symmetric Relations

The digraph of a symmetric relation has the property that whenever there is a directed edge from v to w, there is also a directed edge from w to v.

Example 13

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$$\boxed{\begin{array}{l}(b,c) \in R \\ (c,b) \in R\end{array}}$$

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

symmetric

Antisymmetric

A relation *R* on set *A* is antisymmetric if $a \neq b$, whenever *aRb*, then $b \not R a$. In other word if whenever *aRb*, then *bRa* then it implies that $a=b$

$$\forall a,b \in A, (a,b) \in R \wedge a \neq b \rightarrow (b, a) \notin R$$

Or

$$\forall a,b \in A, \ (a,b) \in R \wedge (b, a) \in R \rightarrow \ a = b$$

Antisymmetric Relations

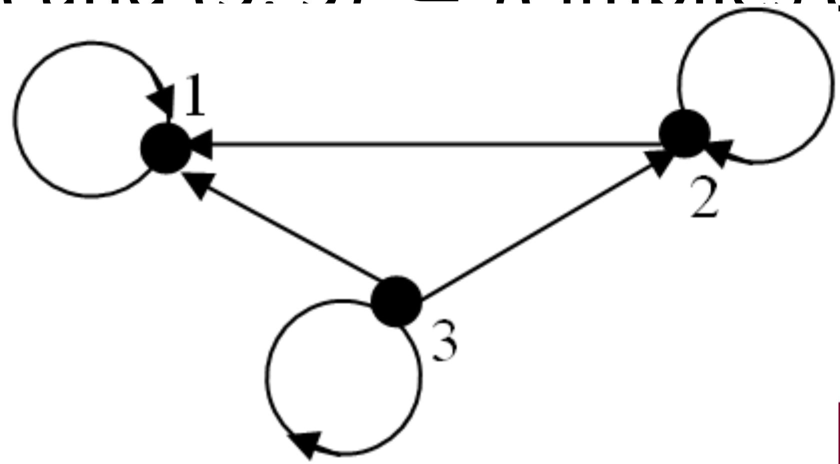
- Matrix $M_R = [M_{ij}]$ of an antisymmetric relation R satisfies the property that if $i \neq j$, then $m_{ij}=0$ or $m_{ji}=0$
- If R is antisymmetric relation, then for different vertices i and j there cannot be an edge from vertex i to vertex j and an edge from vertex j to vertex i
- At least one directed relation and one way

Example 14

- Let R be a relation on A = {1, 2, 3} defined as $(a, b) \in R$ if $a \geq b$, $a, b \in A$ is an antisymmetric relation because for all $a, b \in A$, $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$, for example

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

$$(3, 3) \in R \text{ and } (3, 3) \in R \text{ implies } a = b$$



Example 15

- The relation R on $X = \{1, 2, 3, 4\}$ defined by,

$$(x,y) \in R \quad \text{if } x \le y,\ x,y \in X$$

$$\boxed{\begin{array}{l}(1,2) \in R \\ (2,1) \notin R\end{array}}$$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

antisymmetric

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Example 16

The relation

$R = \{ (a,a), (b,b), (c,c) \}$

on $X = \{ a, b, c \}$

R has no members of the form (x,y) *with* $x \neq y$, then R is antisymmetric

$$\begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Asymmetric

Arelation is asymmetric if and only if it is both antisymmetric and irreflexive.

The matrix $M_{R} = [m_{ij}]$ of an asymmetric relation R satisfies the property that

If $m_{ij} = 1$ then $m_{ji} = 0$

$m_{ii} = 0$ for all i (the main diagonal of matrix M_R consists entirely of 0's or otherwise)

Digraph

- If *R* is asymmetric relation, then the digraph of *R* cannot simultaneously have an edge from vertex *i* to vertex *j* and an edge from vertex *j* to vertex *i*
- All edges are “one way street” and no loop at every vertex

Example 17

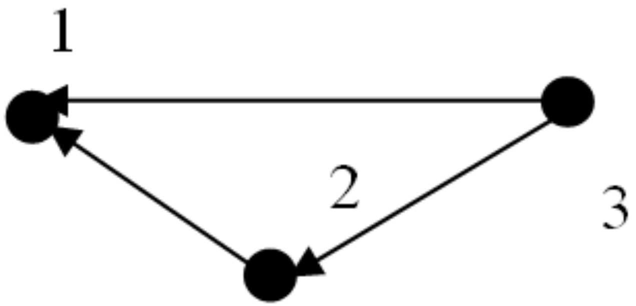
- Let R be the relation on A = {1, 2, 3} defined by $(a, b) \in R$ if $a > b$, $a,b \in A$ is an asymmetric relation because,

$$(2, 1) \in R \text{ but } (1, 2) \notin R$$

$$(3, 1) \in R \text{ but } (1, 3) \notin R$$

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

$$(1, 1) \notin R, (2, 2) \notin R, (3, 3) \notin R$$



Not Symmetric

- Let R be a relation on a set A. Then R is called ***not symmetric***, if for all a, $b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$.

$$\exists\, a,b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

Not Symmetric AND not antisymmetric

- Let R be a relation on a set A. Then R is called ***not symmetric*** and ***not antisymmetric***, if for all $a, b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$ and if $(a, b) \in R$, there exist $(b, a) \in R$.

$$\exists\, a,b \in A, (a, b) \in R \rightarrow (b, a) \in R$$

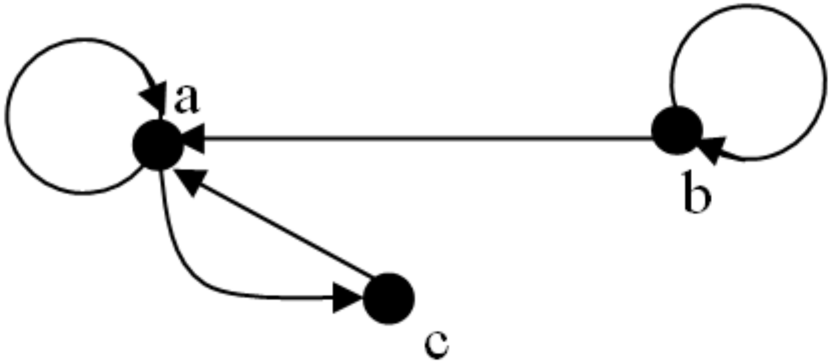
AND

$$\exists\, a,b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

Example 18

- Relation $R = \{(a, c), (b, b), (c, a), (b, a), (a, a)\}$ on $A = \{a, b, c\}$ is not symmetric and not antisymmetric relation because there is,

$(a,c), (c,a) \in R$ and also $(b,a) \in R$ but $(a,b) \notin R$



Example 19

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$$\boxed{\begin{array}{l} (b,c) \in R \\ (c,b) \in R \end{array}}$$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

Symmetric and

not antisymmetric

Example 20

1. Let $A=Z$, the set of integers and let $R=\{(a,b)\in A\times A \mid a<b\}$. So that R is the relation "less than".

Is R symmetric, asymmetric or antisymmetric?

2. Let $A=\{1,2,3,4\}$ and let $R = \{(1,2), (2,2), (3,4), (4,1)\}$

Determine whether R symmetric, asymmetric or antisymmetric.

Solution

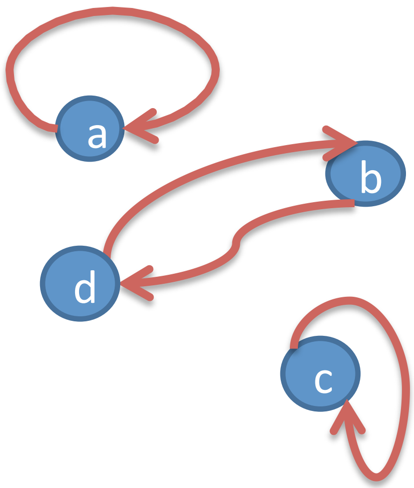
Question 1

- Symmetric : If $a<b$, then it is not true that $b<a$, so R is not symmetric
- Assymetric : If $a<b$ then $b>a$ (b is greater than a), so R is assymetric
- Antisymmetric : If $a\neq b$, then either $a>b$ or $b>a$, so R is antisymmetric

Question 2

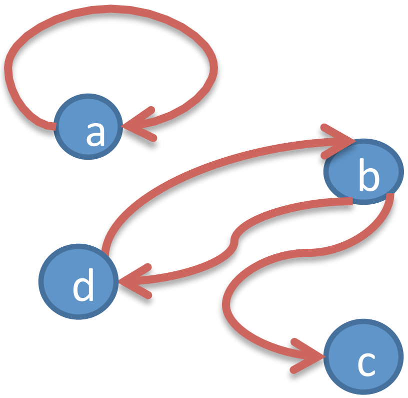
- R is not symmetric since $(1,2)\in R$, but $(2,1)\notin$ R
- R is not asymmetric , since $(2,2)\in R$
- R is antisymmetric, since $a\neq b$, either $(a,b)\notin$ R or $(b,a)\notin R$

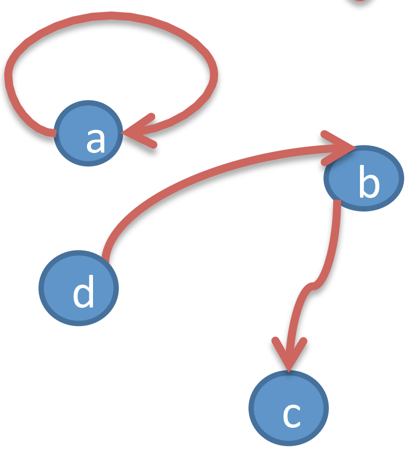
Summary on Symmetric



Symmetric

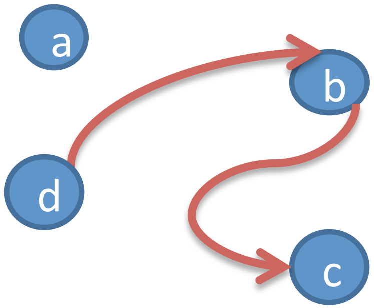
Not Symmetric





Antisymmetric

Asymmetric



Exercise 14

- Consider the following relations on the set {1,2,3}:

R_1={(1,1), (1,2), (2,1), (2,2), (3,1), (3,3)}

R_2={(1,1), (1,3), (2,2), (3,1)}

R_3={(2,3)}

R_4={(1,1)}

Which of them are symmetric?

Which of them are antisymmetric?

Exercise 15

Let A = { 1,2,3,4). Construct the matrix of relation of *R*. Then, determine whether the relation is symmetric, assymetric, antisymmetric, not symmetric or not antisymmetric.

(i) *R* = { (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4))

(ii) *R* = { (1,3), (1,1), (3,1), (1,2), (3,3), (4,4)}

(iii) *R* ={ (1,2), (1,3), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)}

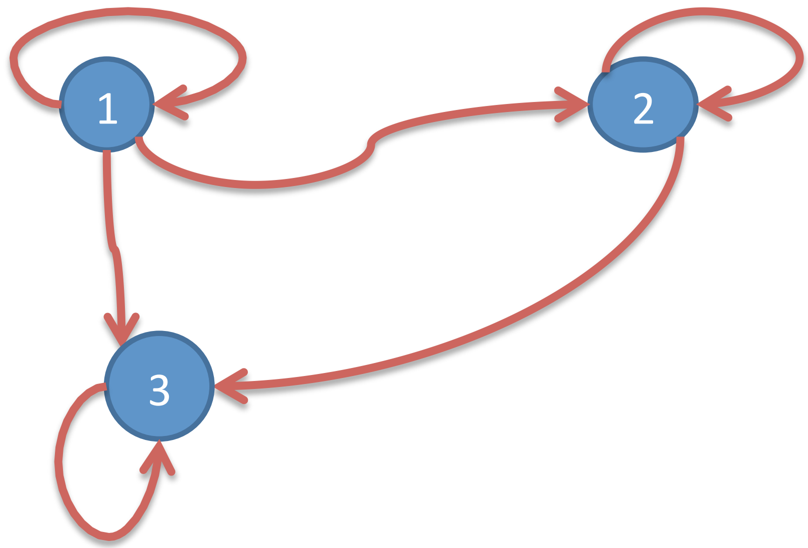
Transitive Relations

- A relation *R* on set *A* is transitive if for all $a,b \in A$, $(a,b) \in R$ and $(b,c) \in R$ implies that $(a,c) \in R$
- In the diagraph of *R*, *R* is a transitive relation if and only if there is a directed edge from one vertex *a* to another vertex *b*, an if there exits a directed edge from vertex *b* to vertex *c*, then there must exists a directed edge from *a* to *c*

Example 21

R={(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)}

The diagraph:



Exercise 16

- Consider the following relations on the set {1,2,3}:

R_1={(1,1),(1,2),(2,3)}

R_2={(1,2),(2,3),(1,3) }

Which of them is transitive?

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Transitive Relations

If $M_R \otimes M_R$ has a 1 in any position then M_R must have a 1 in the same position.

$\otimes$ is a product of Boolean

+	1	0
1	1	1
0	1	0

.	1	0
1	1	0
0	0	0

Example 22

The relation R on A={1,2,3} defined by $(a,b)\in R$ if $a\leq b$, $a,b\in A$, is a transitive. The matrix of relation M_R,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that,(1,2) and $(2,3)\in R$, $(1,3)\in R$

Example (Exercise 16)

- The relation R on A={1,2,3} is R={(1,2),(2,3),(1,3)} is transitive. The matrix of relation M_R,

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_R \otimes M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that, $M_R \otimes M_R$ has a 1 in position 1,3 then M_R also have a 1 in the same position.

Example 23

The relation R on A={a,b,c,d} IS r={(a,a), (b,b), (c,c), (d,d), (a,c), (c,b)} is not transitive. The matrix of relation M_R,

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$M_R \otimes M_R \neq M_R$

The product of boolean,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \otimes \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note that,(a,c) and (c,b)∈R, (a,b)∉R

Example 24

Let R be a relation on $A=\{1,2,3\}$ is defined by $(a,b)\in R$ if $a\leq b$, $a,b\in A$. Find R. Is R a transitive relation?

Solution:

$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

R is a transitive relation because

(1,2) and (2,2)$\in R$, (1,2)$\in R$

(1,2) and (2,3)$\in R$, (1,3)$\in R$

(1,3) and (3,3)$\in R$, (1,3)$\in R$

(2,2) and (2,3)$\in R$, (2,3)$\in R$

(2,3) and (3,3)$\in R$, (2,3)$\in R$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Equivalence Relations

Relation R on set A is called an equivalence relation if it is a **reflexive, symmetric and transitive**.

Example 25

Let R={(1,1), (1,3), (2,2), (3,1), (3,3)} on {1,2,3}, the matirx of the relation M_R,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

All the main diagonal matrix elements are 1 and the matris is reflexive.

Equivalence Relations

Example 25(Cont.)

The transpose matrix M_R,M_R^T is equal to M_R, so R is symmetric

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \qquad M_R^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

The product of boolean show that the matrix is transitive.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Partial Order Relations

Relation R on set A is called a partial order relation if it is a reflexive, antisymmetric and transitive.

Example 26

Let R be a relation on a set $A=\{1,2,3\}$ defined by $(a,b)\in R$ if $a\leq b$, $a,b\in R$.

$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

R is reflexive, antisymmetric and transitive.

So R is a partial order relation.

Exercise 17

The relation R on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if $x+y \leq 6$

(i) List the elements of R

(ii) Find the domain of R

(iii) Find the range of R

(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?

Exercise 18

The relation R on the set {1,2,3,4,5} defined by the rule $(x,y)\in R$ if 3 divides $x-y$

(i) List the elements of R

(ii) Find the domain of R

(iii) Find the range of R

(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?

Exercise 19

The relation R on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if $x=y-1$

(i) List the elements of R

(ii) Find the domain of R

(iii) Find the range of R

(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?