SCSI1013: Discrete Structures

CHAPTER 1

Part 3:
Propositions, Conditional Propositions and Logical Equivalences

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Why Are We Studying Logic?

Some of the reasons:

- Logic is the foundation for computer operation
- Logical conditions are common in programs:

  Example:

  Selection: if (score <= max) { ... }

  Iteration: while (i<limit && list[i]!=sentinel) ...

- All manner of structures in computing have properties that need to be proven (and proofs that need to be understood).

Examples: Trees, Graphs, Recursive Algorithms, . . .

- Programs can be proven correct.
- Computational linguistics must represent and reason about human language, and language represents thought (and thus also logic).

PROPOSITION

A **statement** or a **proposition**, is a declarative sentence that is **either TRUE or FALSE, but not both.**

Example:

- 4 is less than 3.
- 7 is an even integer.
- Washington, DC, is the capital of United State.

Example

i) Why do we study mathematics?
ii) Study logic.
iii) What is your name?
iv) Quiet, please.

The above sentences are not propositions. Why ?

(i) & (iii) : is question, not a statement.
(ii) & (iv) : is a command.

Example

i) The temperature on the surface of the planet Venus is 800 F.

ii) The sun will come out tomorrow.

Propositions? Why?

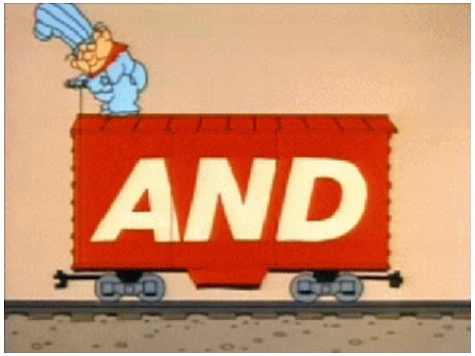
i) Is a statement since it is either true or false, but not both.

ii) However, we do not know at this time to determine whether it is true or false.

CONJUNCTIONS

Conjunctions are:

- Compound propositions formed in English with the word "**and**",
- Formed in logic with the caret symbol ("$\wedge$"), and
- True only when both participating propositions are true.



CONJUNCTIONS (cont.)

TRUTH TABLE: This tables aid in the evaluation of **compound propositions**.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example

p : 2 is an even integer
q : 3 is an odd number

propositions

$p \wedge q$

symbols

statements

: 2 is an even integer and 3 is an odd number

p : today is Monday ; q : it is hot

$p \wedge q$: today is Monday and it is hot

Example

Proposition:-

p : 2 *divides* 4

q : 2 *divides* 6

Symbol: Statement:-

$p \wedge q$: 2 divides 4 and 2 divides 6.

or,

$p \wedge q$: 2 divides both 4 and 6.

Example

Proposition:-

p : 5 is an integer

q : 5 is not an odd integer

Symbol: Statement:-

$p \land q$: 5 is an integer and 5 is not an odd integer.

or,

$p \land q$: 5 is an integer but 5 is not an odd integer.

DISJUNCTION

- Compound propositions formed in English with the word “**or**”,
- Formed in logic with the caret symbol (“ $\vee$ ”), and,
- True when one or both participating propositions are true.

DISJUNCTION (cont.)

Let ***p*** *and* ***q*** *be propositions.* The **disjunction** of ***p*** *and* ***q****, written* ***p*** $\vee$ ***q*** is the statement formed by putting statements ***p*** *and* ***q*** *together using the* word "**or**". The symbol $\vee$ is called "or"

DISJUNCTION (cont.)

The truth table for $p \vee q$:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example

i) ***p***: 2 is an integer ; ***q***: 3 is greater than 5

$p \vee q$ ➔ 2 is an integer or 3 is greater than 5

ii) ***p*** : 1+1=3 ; ***q*** *:* A decade is 10 years

$p \vee q$ ➔ 1+1=3 or a decade is 10 years

iii) ***p*** : 3 is an even integer ; ***q*** : 3 is an odd integer

$p \vee q$ ➔ 3 is an even integer or 3 is an odd integer ; or

3 is an even integer or an odd integer

NEGATION

Negating a proposition simply flips its value. Symbols representing negation include: $\neg x$, $\bar{x}$, ~x, x' (NOT)

Let *p* be a proposition.
The negation of *p, written* ¬ *p*
is the statement obtained by negating statement *p.*

NEGATION (cont.)

The truth table of $\neg p$:

p	$\neg p$
T	F
F	T

Example

p : 2 is positive

¬ ***p*** → 2 is not positive.

Exercise (1)

p: It will rain tomorrow ; ***q***: it will snow tomorrow

Give the negation of the following statement and write the symbol.

It will rain tomorrow or it will snow tomorrow.

Exercise (2)

In each of the following, form the conjunction and the disjunction of ***p*** and ***q*** by writing the symbol and the statements.

i) p: I will drive my car
q: I will be late

ii) p : NUM > 10
q : NUM $\leq$ 15

Exercise (3)

Suppose x is a particular real number. Let ***p***, ***q*** and ***r*** symbolize "$0 < x$", "$x < 3$" and "$x = 3$", respectively. Write the following inequalities symbolically:

a) $x \leq 3$

b) $0 < x < 3$

c) $0 < x \leq 3$

Solution:

a) $q \vee r$

b) $p \wedge q$

c) $p \wedge (q \vee r)$

Exercise (4)

State either TRUE or FALSE if ***p*** and ***r*** are TRUE and ***q*** is FALSE.

a) $\sim p \wedge (q \vee r)$

b) $(r \wedge \sim q) \vee (p \vee r)$

CONDITIONAL PROPOSITIONS

Let ***p*** *and* ***q*** be propositions.

"if ***p***, *then* ***q***"

is a statement called a **conditional proposition**, written as

$$\boxed{p \rightarrow q}$$

CONDITIONAL PROPOSITIONS(cont.)

The truth table of $p \rightarrow q$

=> Cause and effect relationship

FALSE if *p* = True and *q* =false

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

TRUE if both true or *p*=false for any value of *q*

Example

p : today is Sunday ; q : I will go for a walk

$p \rightarrow q$: If today is Sunday, then I will go for a walk.

p : I get a bonus ; q : I will buy a new car

$p \rightarrow q$: If I get a bonus, then I will buy a new car

Example

p : x/2 is an integer.

q : x is an even integer.

$p \rightarrow q$: if x/2 is an integer, then x is an even integer.

BICONDITIONAL

Let $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *be propositions.*

*"**p** if and only if **q**"*

is a statement called a **biconditional proposition**, written as

$$p \leftrightarrow q$$

BICONDITIONAL (cont.)

The truth table of $p \leftrightarrow q$:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example

p : my program will compile
q : it has no syntax error.

$p \leftrightarrow q$: My program will compile if and only if it has no syntax error.

p : x is divisible by 3
q : x is divisible by 9

$p \leftrightarrow q$**:** x is divisible by 3 if and only if x is divisible by 9.

LOGICAL EQUIVALENCE

- The compound propositions ***Q*** and ***R*** are made up of the propositions $p_1, \ldots, p_n$.

- ***Q*** and ***R*** are logically equivalent and write,

$$Q \equiv R$$

provided that given any truth values of $p_1, \ldots, p_n$, either ***Q*** and ***R*** are **both true** or ***Q*** and ***R*** are **both false**.

Example

$\boldsymbol{Q} = p \rightarrow q$

$\boldsymbol{R} = \neg q \rightarrow \neg p$

Show that, $\boldsymbol{Q} \equiv \boldsymbol{R}$

The truth table shows that, $Q \equiv R$

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Example

Show that,

$\neg (p \rightarrow q) \equiv p \wedge \neg q$

The truth table shows that,

$\neg (p \rightarrow q) \equiv p \wedge \neg q$

p	q	$\neg(p \rightarrow q)$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

PRECEDENCE OF LOGICAL CONNECTIVES

Precedence of logical connectives is as follows:			
not	$\neg$	$\uparrow$	Highest
and	$\wedge$		
or	$\vee$		
If…then	$\rightarrow$		
If and only if	$\leftrightarrow$		Lowest

Example

Construct the truth table for,

A = $\neg(p \lor q) \rightarrow (q \land p)$

Solution:

p	q	$(p \lor q)$	$\neg(p \lor q)$	$(q \land p)$	A
T	T	T	F	T	T
T	F	T	F	F	T
F	T	T	F	F	T
F	F	F	T	F	F

Exercise (5)

Construct the truth table for each of the following statements:

i) $\neg p \wedge q$

ii) $\neg(p \vee q) \rightarrow q$

iii) $\neg(\neg p \wedge q) \vee q$

iv) $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$

LOGIC & SET THEORY

Logic and set theory go very well togather. The previous definitions can be made very succinct:

$x \notin A$ if and only if $\neg(x \in A)$

$A \subseteq B$ if and only if $(x \in A \rightarrow x \in B)$ is True

$x \in (A \cap B)$ if and only if $(x \in A \wedge x \in B)$

$x \in (A \cup B)$ if and only if $(x \in A \vee x \in B)$

$x \in A - B$ if and only if $(x \in A \wedge x \notin B)$

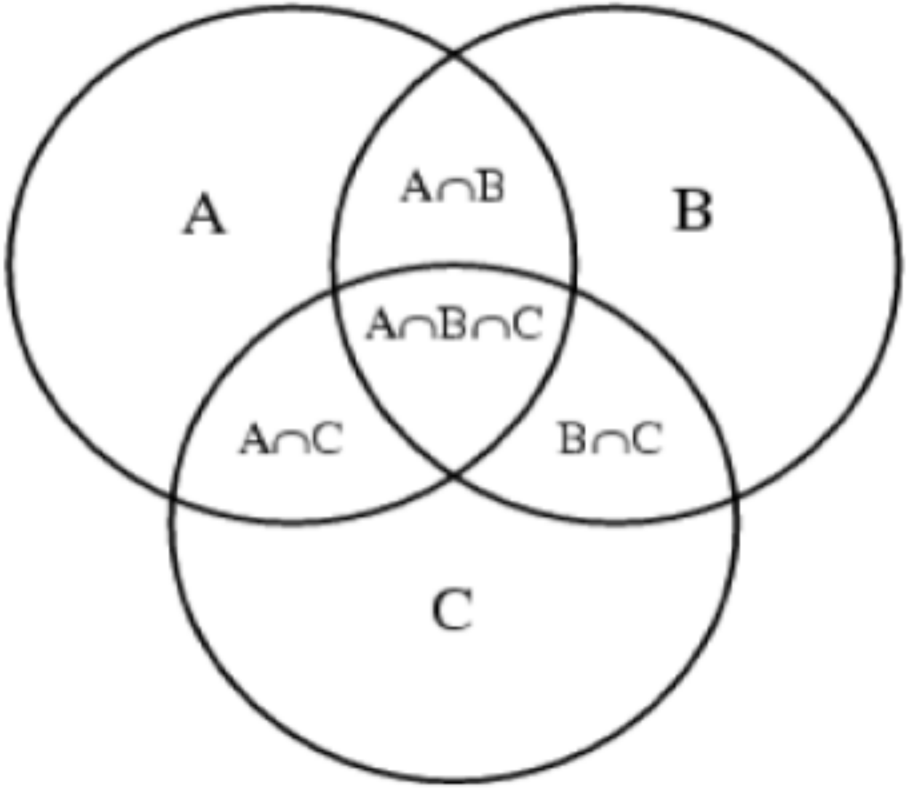
$x \in A \Delta B$ if and only if $(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$

$x \in A'$ if and only if $\neg(x \in A)$

$X \in P(A)$ if and only if $X \subseteq A$

Venn Diagrams

Venn Diagrams are used to depict the various unions, subsets, complements, intersections etc. of sets.



Logic and Sets

are closely related

Tautology	Set Operation Identity
$p \vee q \leftrightarrow q \vee p$	$A \cup B = B \cup A$
$p \wedge q \leftrightarrow q \wedge p$	$A \cap B = B \cap A$
$p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$	$A \cup (B \cup C) = (A \cup B) \cup C$
$p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r$	$A \cap (B \cap C) = (A \cap B) \cap C$
$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$p \wedge \neg q \leftrightarrow p \wedge \neg (p \wedge q)$	$A - B = A - (A \cap B)$
$p \wedge \neg (q \vee r) \leftrightarrow (p \wedge \neg q) \wedge (p \wedge \neg r)$	$A - (B \cap C) = (A - B) \cup (A - C)$
$p \wedge \neg (q \wedge r) \leftrightarrow (p \wedge \neg q) \vee (p \wedge \neg r)$	$A - (B \cup C) = (A - B) \cap (A - C)$
$p \wedge (q \wedge \neg r) \leftrightarrow (p \wedge q) \wedge \neg (p \wedge \neg r)$	$A \cap (B - C) = (A \cap B) - (A \cap C)$
$p \vee (q \wedge \neg r) \leftrightarrow (p \vee q) \wedge \neg (r \wedge \neg p)$	$A \cup (B - C) = (A \cup B) - (C - A)$
$p \wedge \neg \vee (q \wedge \neg r) \leftrightarrow (p \wedge \neg q) \vee (p \wedge r)$	$A - (B - C) = (A - B) \cup (A \cap C)$

The above identities serve as the basis for an "algebra of sets".

Logic and Sets

are closely related

Tautology	Set Operation Identity
$p \wedge p \leftrightarrow p$	$A \cap A = A$
$p \vee p \leftrightarrow p$	$A \cup A = A$
$p \wedge \neg(q \wedge \neg q) \leftrightarrow p$	$A - \varnothing = A$
$p \vee \neg(q \wedge \neg q) \leftrightarrow p$	$A \cup \varnothing = A$

Contradiction	Set Operation Identity
$p \wedge \neg p$	$A - A = \varnothing$
$p \wedge (q \wedge \neg q)$	$A \cap \varnothing = \varnothing$
$p \wedge \neg p$	$A - A = \varnothing$

The above identities serve as the basis for an "algebra of sets".

Theorem for Logic

Let ***p***, ***q*** and ***r*** be propositions.

Idempotent laws:

$$p \land p \equiv p$$

$$p \lor p \equiv p$$

Truth table:

p	$p \land p$	$p \lor p$
T	T	T
F	F	F

Theorem for Logic (cont.)

Double negation law:

$$\neg \neg p \equiv p$$

Commutative laws:

$$p \wedge q \equiv q \wedge p$$
$$p \vee q \equiv q \vee p$$

Theorem for Logic (cont.)

Associative laws:

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$
$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

Distributive laws:

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

PROVE

Absorption laws:

$$p \wedge (p \vee q) \equiv p$$
$$p \vee (p \wedge q) \equiv p$$

PROVE

Prove: Distributive Laws

p	q	r	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Prove: Absorption Laws

p	q	$p \wedge (p \vee q)$	$p \vee (p \wedge q)$
T	T	T	T
T	F	T	T
F	T	F	F
F	F	F	F

Theorem for Logic (cont.)

De Morgan's laws:

$\neg(p \land q) \equiv (\neg p) \lor (\neg q)$
$\neg(p \lor q) \equiv (\neg p) \land (\neg q)$

The truth table for $\neg(p \lor q) \equiv (\neg p) \land (\neg q)$

p	q	$\neg(p \lor q)$	$\neg p \land \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Exercise (6)

Given,

$$\boldsymbol{R} = p \wedge (\neg q \vee r)$$

$$\boldsymbol{Q} = p \vee (q \wedge \neg r)$$

State whether or not $\boldsymbol{R} \equiv \boldsymbol{Q}$.

Exercise (7)

Propositional functions ***p***, ***q*** and ***r*** are defined as follows:

- ***p*** is "$n = 7$"
- ***q*** is "$a > 5$"
- ***r*** is "$x = 0$"

Write the following expressions in terms of ***p***, ***q*** and ***r***, and show that each pair of expressions is **logically equivalent**. State carefully which of the above laws are used at each stage.

(a) $((n = 7) \lor (a > 5))\ (x = 0)$

$((n = 7)\ (x = 0)) \lor ((a > 5)\ (x = 0))$

(b) $\neg((n = 7)\ (a \leq 5))$

$(n \neq 7) \lor (a > 5)$

(c) $(n = 7) \lor (\neg((a \leq 5)\ (x = 0)))$

$((n = 7) \lor (a > 5)) \lor (x \neq 0)$

Exercise (7a): Solution

p is "$n = 7$"
q is "$a > 5$"
r is "$x = 0$"

$((n = 7) \lor (a > 5))\ (x = 0) \Rightarrow (p \lor q) \land r$

$((n = 7)\ (x = 0)) \lor ((a > 5)\ (x = 0)) \Rightarrow (p \land r) \lor (q \land r)$

$(p \lor q) \land r \equiv (p \land r) \lor (q \land r)$?

→ $(p \lor q) \land r = r \land (p \lor q)$ ....*Commutative Law*

→ $(p \land r) \lor (q \land r) = (r \land p) \lor (r \land q)$

$= r \land (p \lor q)$ .....*Distributive Law*

Exercise (7b): Solution

$\neg((n = 7)\ (a \leq 5))$

$(n \neq 7) \vee (a > 5)$

p is "$n = 7$"
q is "$a > 5$"
r is "$x = 0$"

First, we note that,

$\neg q$ *is* "$a \leq 5$"; *and*

$\neg p$ *is* "$n \neq 7$".

So the expressions are:

$\neg(p \wedge \neg q)$

$\neg p \vee q$

$\neg(p \wedge \neg q) \quad = \neg p \vee \neg(\neg q)$ *De Morgan's Law*

$\qquad\qquad\quad = \neg p \vee q$ *Involution Law (Double negation)*

Exercise (7c): Solution

$(n = 7) \lor (\neg((a \leq 5)\ (x = 0)))$

$((n = 7) \lor (a > 5)) \lor (x \neq 0)$

p is "$n = 7$"
q is "$a > 5$"
r is "$x = 0$"

First, we note that,

$\neg r$ *is* "$x \neq 0$".

So the expressions are:

$p \lor (\neg(\neg q \land r))$

$(p \lor q) \lor \neg r$

$p \lor (\neg(\neg q \land r)) = p \lor (\neg(\neg q) \lor \neg r)$ *De Morgan's Law*

$= p \lor (q \lor \neg r)$ *Involution Law*

$= (p \lor q) \lor \neg r$ *Associative Law*

Exercise (8)

Propositions ***p***, ***q***, ***r*** and ***s*** are defined as follows:
p is "I shall finish my Coursework Assignment"
q is "I shall work for forty hours this week"
r is "I shall pass Maths"
s is "I like Maths"

Write each sentence in symbols:
(a) I shall not finish my Coursework Assignment.
(b) I don't like Maths, but I shall finish my Coursework Assignment.
(c) If I finish my Coursework Assignment, I shall pass Maths.
(d) I shall pass Maths only if I work for forty hours this week and finish my Coursework Assignment.

Write each expression as a sensible (if untrue!) English sentence:
(e) $q \vee p$
(f) $\neg p \Rightarrow \neg r$

Exercise (8): Solution

(a) $\neg p$

(b) $\neg s \wedge p$

(c) $p \rightarrow r$

(d) $r \leftrightarrow (q \wedge p)$

(e) I shall work for forty hours this week, or I'll finish my Coursework Assignment.

(f) If I shall not finish my Coursework Assignment, then I shouldn't pass Maths.

Exercise (9)

For each pair of expressions, construct truth tables to see if the two compound propositions are logically equivalent:

(a) $p \vee (q \wedge \neg p)$

$p \vee q$

(b) $(\neg p \wedge q) \vee (p \wedge \neg q)$

$(\neg p \wedge \neg q) \vee (p \wedge q)$

Exercise (9): Solution

(a) Yes; both results columns give

T, T, T, F

(b) No; first is

F, T, T, F

second is

T, F, F, T