

# CHAPTER 1

## **SET THEORY**

### **[Part 1: Set & Subset]**

# Introduction

- Why are we studying sets
  - The concept of set is basic to all of mathematics and mathematical applications.
  - Serves as a basis of description of higher concept and mathematical reasoning
  - Set is fundamental in many areas of Computer Science.
  - For us, **sets are useful to understand the principles of counting and probability theory**

# Sets

- The concept of set is basic to all of mathematics and mathematical applications.
- A set is a **well-defined collection of distinct objects**.
- These objects are called **members** or **elements** of the set.

# Sets

- Well-defined means that we can tell for certain whether an object is a member of the collection or not.
- If a set is finite and not too large, we can describe it by listing the elements in it.

# Example

- $A$  is a set of all positive integers less than 10,  
 $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $B$  is a set of first 5 positive odd integers,  
 $B = \{1, 3, 5, 7, 9\}$
- $C$  is a set of vowels,  $C = \{a, e, i, o, u\}$

# Defining Sets

- This can be done by:
  - Listing ALL elements of the set within braces.
  - Listing enough elements to show the pattern then an ellipsis.
  - Use set builder notation to define “rules” for determining membership in the set.

# Example

1. Listing ALL elements.  $A = \{1, 2, 3, 4\}$  explicitly
2. Demonstrating a pattern.  $\mathbb{N} = \{1, 2, 3, \dots\}$  implicitly
3. Using set builder notation.  $P = \{x \mid x \in \mathbb{R} \text{ and } x \notin \mathbb{C}\}$  implicitly

# Sets

- A set is determined by its elements and not by any particular order in which the element might be listed.

- **Example,**

$$A = \{1, 2, 3, 4\},$$

A might just as well be specified as

$$\{2, 3, 4, 1\} \text{ or } \{4, 1, 3, 2\}$$



# Sets

- The elements making up a set are assumed to be distinct, we may have duplicates in our list, only one occurrence of each element is in the set.
- **Example**

$\{a, b, c, a, c\} \longrightarrow \{a, b, c\}$

$\{1, 3, 3, 5, 1\} \longrightarrow \{1, 3, 5\}$

# Sets

- Use uppercase letters  $A, B, C \dots$  to denote sets, lowercase denote the elements of set.
- The symbol  $\in$  stands for 'belongs to'
- The symbol  $\notin$  stands for 'does not belong to'

# Example

- $X = \{ a, b, c, d, e \}$ ,  $b \in X$  and  $m \notin X$
- $A = \{ \{1\}, \{2\}, 3, 4 \}$ ,  $\{2\} \in A$  and  $1 \notin A$

# Sets

- If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for memberships
- Let  $S$  be a set, the notation,  
$$A = \{x \mid x \in S, P(x)\} \text{ or } A = \{x \in S \mid P(x)\}$$
means that  $A$  is the set of all elements  $x$  of  $S$  such that  $x$  satisfies the property  $P$ .

# Example

- Let  $A=\{1, 2, 3, 4, 5, 6\}$ , we can also write  $A$  as,

$$A=\{x \mid x \in \mathbb{Z}, 0 < x < 7\}$$

if  $\mathbb{Z}$  denotes the set of integers.

- Let  $B=\{x \mid x \in \mathbb{Z}, x > 0\}$ ,  $B=\{1, 2, 3, 4, \dots\}$

# Example

The set of natural numbers:  $\mathbb{N} = \{1, 2, 3, \dots\}$

The set of integers:  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

The set of positive integers:  $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$

The set of Rational Numbers (fractions):  $\frac{1}{2}, \frac{2}{3}, \frac{5}{7}, \text{etc} \in \mathbb{Q}$

More formally:  $\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{R}, b \neq 0 \right\}$

The set of Irrational Numbers:  $\sqrt{2}, \pi$ , or  $e$  are irrational

The Real numbers  $= \mathbb{R}$  = the union of the rational numbers  
with the irrational numbers

# Some Symbols Used With Set Builder Notation

The standard form of notation for this is called "set builder notation".

For instance,  $\{x \mid x \text{ is an odd positive integer}\}$  represents the set  $\{1, 3, 5, 7, 9, \dots\}$

$\{x \mid x \text{ is an odd positive integer}\}$  is read as

"the set consisting of all  $x$  such that  $x$  is an odd positive integer".

The vertical bar, " $\mid$ ", stands for "such that"

Other "short-hand" notation used in working with sets

" $\forall$ " stands for "for every"

" $\cup$ " stands for "union"

" $\subseteq$ " stands for "is a subset of"

" $\not\subset$ " stands for "is not a (proper) subset of"

" $\in$ " stands for "is an element of"

" $\times$ " stands for "cartesian cross product"

" $\exists$ " stands for "there exists"

" $\cap$ " stands for "intersection"

" $\subset$ " stands for "is a (proper) subset of"

" $\emptyset$ " stands for the "empty set"

" $\notin$ " stands for "is not an element of"

" $=$ " stands for "is equal to"

# Subset

- If every element of  $A$  is an element of  $B$ , we say that  $A$  is a subset of  $B$  and write  $A \subseteq B$ .

$$A=B, \text{ if } A \subseteq B \text{ and } B \subseteq A$$

- The empty set ( $\emptyset$ ) is a subset of every set.



# Example

$$A = \{1, 2, 3\}$$

Subset of A,

$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

*Note:*

A is a subset of A

# Exercise

Answer true or false

a)  $\{x\} \subseteq \{x\}$

b)  $\{x\} \in \{x\}$

c)  $\{x\} \in \{x, \{x\}\}$

d)  $\{x\} \subseteq \{x, \{x\}\}$

# Proper Subsets

If  $A \subseteq B$  and  $B$  contains an element that is not in  $A$ , then we say “ $A$  is a **proper subset** of  $B$ ”:  $A \subset B$  or  $B \supset A$ .

Formally:  $A \subseteq B$  means  $\forall x [x \in A \rightarrow x \in B]$ .

For all sets:  $A \subseteq A$ .

**Note:** If  $A$  is a subset of  $B$  and  $A$  does not equal  $B$ , we say that  $A$  is a proper subset of  $B$  ( $A \subseteq B$  and  $A \neq B$  ( $B \not\subseteq A$ ))

# Example

- $A = \{1, 2, 3\}$
- Proper subset of  $A$ ,  
 $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$

## ***Note:***

A proper subset of a set  $A$  is a subset of  $A$  that is not equal to  $A$  (  $\{1,2,3\} \not\subseteq A$  )

# Example

Let a set,  $B = \{1, 2, 3, 4, 5, 6\}$  and the subset,  
 $A = \{1, 2, 3\}$ .

$\Rightarrow A$  is proper subset of  $B$ .

# Example

$A = \{a, b, c, d, e, f, g, h\}$

$B = \{b, d, e\}$

$C = \{a, b, c, d, e\}$

$D = \{r, s, d, e\}$

- Proper subset of  $A$  ??  $B$  and  $C$

# Empty Sets

The **empty set**  $\emptyset$  or  $\{\}$  **but not**  $\{\emptyset\}$   
is the set without elements.

Note:

- Empty set has no elements
- Empty set is a subset of any set
- There is exactly one empty set
- Properties of empty set:

$$A \cup \emptyset = A, A \cap \emptyset = \emptyset$$

$$A \cap A' = \emptyset, A \cup A' = U$$

$$U' = \emptyset, \emptyset' = U$$

# Example

$$\emptyset = \{x \mid x \text{ is a real number and } x^2 = -3\}$$

$$\emptyset = \{x \mid x \text{ is positive integer and } x^3 < 0\}$$



# Equal Sets

The sets  $A$  and  $B$  are **equal** ( $A=B$ ) if and only if each element of  $A$  is an element of  $B$  and vice versa.

Formally:  $A=B$  means  $\forall x [x \in A \leftrightarrow x \in B]$ .

# Example

$$A=\{a, b, c\}, B=\{b, c, a\}, \quad A=B$$

$$C=\{1, 2, 3, 4\}$$

$$D=\{x \mid x \text{ is a positive integer and } 2x < 10\},$$

$$C=D$$

# Exercise

Determine whether each pair of sets is equal

a)  $\{1, 2, 2, 3\}, \{1, 3, 2\}$

b)  $\{x \mid x^2 + x = 2\}, \{1, -2\}$

c)  $\{x \mid x \text{ is a real number and } 0 < x \leq 2\}, \{1, 2\}$

# Equivalent Sets

Two sets, A and B, are **equivalent** if there exists a **one-to-one correspondence** between them.

When we say sets “have the same size”, we mean that they are **equivalent**.

# Example

**Set A:** {A, B, C, D, E} and **Set B:** {1, 2, 3, 4, 5}

## Note:

- An **equivalent set** is simply a **set** with an **equal** number of elements.
- The **sets** do not have to have the same exact elements, just the same number of elements.

# Finite Sets

A set  $A$  is **finite**

if it is empty

or

if there is a natural number  $n$   
such that set  $A$  is equivalent to  
 $\{1, 2, 3, \dots n\}$ .

# Example

$$A = \{1, 2, 3, 4\}$$

$$B = \{x \mid x \text{ is an integer, } 1 \leq x \leq 4\}$$

## Note:

**There exists a nonnegative integer  $n$  such that  $A$  has  $n$  elements ( $A$  is called a finite set with  $n$  elements)**

# Infinite Sets

A set  $A$  is **infinite**

if there is **NOT** a natural number  $n$  such that set  $A$  is equivalent to  $\{1, 2, 3, \dots n\}$ .

Infinite sets are **uncountable**.

**Are all infinite sets equivalent?**

**A set is infinite if it is equivalent to a proper subset of itself!**



# Example

$Z = \{x \mid x \text{ is an integer}\}$

or  $Z = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

$S = \{x \mid x \text{ is a real number and } 1 \leq x \leq 4\}$

# Example

$$C = \{5, 6, 7, 8, 9, 10\} \text{ (finite set)}$$

$$B = \{x \mid x \text{ is an integer, } 10 < x < 20\} \text{ (finite set)}$$

$$D = \{x \mid x \text{ is an integer, } x > 0\} \text{ (infinite set)}$$

# Universal Set

- Sometimes we are dealing with sets all of which are subsets of a set  $U$ .
- This set  $U$  is called a universal set or a universe.
- The set  $U$  must be explicitly given or inferred from the context.

# Universal Set

Typically we consider a set  $A$   
a part of a **universal set**  $\mathcal{U}$ ,  
which consists of all possible elements.

To be entirely correct we should say

$$\forall x \in \mathcal{U} [x \in A \leftrightarrow x \in B]$$

instead of

$$\forall x [x \in A \leftrightarrow x \in B] \text{ for } A=B.$$

Note that  $\{x \mid 0 < x < 5\}$  is can be ambiguous.

Compare  $\{x \mid 0 < x < 5, x \in \mathbb{N}\}$  with  $\{x \mid 0 < x < 5, x \in \mathbb{Q}\}$

# Example

- The sets  $A=\{1,2,3\}$ ,  $B=\{2,4,6,8\}$  and  $C=\{5,7\}$
- One may choose  $U=\{1,2,3,4,5,6,7,8\}$  as a universal set.
- Any superset of  $U$  can also be considered a universal set for these sets  $A$ ,  $B$ , and  $C$ .  
For example,  $U=\{x \mid x \text{ is a positive integer}\}$

# Cardinality of Set

- Let  $S$  be a finite set with  $n$  distinct elements, where  $n \geq 0$ .
- Then we write  $|S|=n$  and say that the **cardinality** (or **the number of elements**) of  $S$  is  $n$ .
- **Example**

$$A = \{1, 2, 3\}, \quad |A| = 3$$

$$B = \{a, b, c, d, e, f, g\}, \quad |B| = 7$$

# Exercise

- If  $M$  is finite, determine the  $|M|$ 
  - If  $M = \{1, 2, 3, 4\}$
  - If  $M = \{4, 4, 4\}$
  - If  $M = \{\}$
  - If  $M = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$

# Power Set

- The set of all subsets of a set  $A$ , denoted  $P(A)$ , is called the **power set of  $A$** .

$$P(A) = \{X \mid X \subseteq A\}$$

- If  $|A| = n$ , then  $|P(A)| = 2^n$



# Example

- $A = \{1, 2, 3\}$
- The power set of  $A$ ,  
 $P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
- Notice that  $|A| = 3$ , and  $|P(A)| = 2^3 = 8$

# Exercise

- Let  $X = \{1, 2, 2, \{1\}, a\}$
- Find:
  - $|X|$
  - Proper subset of  $X$
  - Power set of  $X$

# Exercise

- List the member of  $P(\{a, b, c, d\})$ . Which are proper subset of  $\{a, b, c, d\}$ ?

## Summary

# How to Think of Sets

The elements of a set do not have an ordering,  
hence  $\{a,b,c\} = \{b,c,a\}$

The elements of a set do not have multitudes,  
hence  $\{a,a,a\} = \{a,a\} = \{a\}$

All that matters is: “Is  $x$  an element of  $A$  or not?”

The size of  $A$  is thus the number of *different* elements

